TOPOLOGICAL GEOMETRODYNAMICS: PHYSICS AS INFINITE-DIMENSIONAL GEOMETRY

Matti Pitkänen

Rinnekatu 2-4 A 8, Karkkila, 03620, Finland

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0.1 PREFACE

This book belongs to a series of online books summarizing the recent state Topological Geometrodynamics (TGD) and its applications. TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students at seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 37 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of imbedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, main stream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to same multiplet of the gauge group implying instability of proton.

There have been also longstanding problems.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. Furthermore, the imbeddings of Robertson-Walker cosmologies turned out to be vacuum extremals with respect to the inertial energy. About 25 years was needed to realize that the sign of the inertial energy can be also negative and in cosmological scales the density of inertial energy vanishes: physically acceptable universes are creatable from vacuum. Eventually this led to the notion of zero energy ontology (ZEO) which deviates dramatically from the standard ontology being however consistent with the crossing symmetry of quantum field theories. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of so called causal diamonds defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle as it is expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields.

  It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to dark matter. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem, and 32 years was needed to discover that what I call weak form of electric-magnetic duality gives a satisfactory solution of the problem and provides also surprisingly powerful insights to the mathematical structure of quantum TGD.

  The latest development was the realization that the well- definedness of electromagnetic charge as quantum number for the modes of the induced spinors field requires that the CP_2 projection of the region in which they are non-vanishing carries vanishing W boson field and is 2-D. This implies in the generic case their localization to 2-D surfaces: string world sheets and possibly also partonic 2-surfaces. This localization applies to all modes except covariantly constant right handed neutrino generating supersymmetry and mplies that string model in 4-D space-time is part of TGD. Localization is possible only for Kähler-Dirac assigned with Kähler action defining the dynamics of space-time surfaces. One must however leave open the question whether W field might vanish for the space-time of GRT if related to many-sheeted space-time in the proposed manner even when they do not vanish for space-time sheets.

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but the first discovery made already during first year of TGD was that these formalisms might be useless due to the extreme non-linearity and enormous vacuum degeneracy of the theory. This turned out to be the case.

- It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as 3-dimensional surfaces. Still few years had to pass before I understood that general coordinate invariance leads to a more or less unique solution of the problem and in positive energyontology implies that space-time surfaces are analogous to Bohr orbits. This in positive energy ontology in which space-like 3-surface is basic object. It is not clear whether Bohr orbitology is necessary also in ZEO in which space-time surfaces connect space-like 3-surfaces at the light-like boundaries of causal diamond CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). The reason is that the pair of 3-surfaces replaces the boundary conditions at single 3-surface involving also time derivatives. If one assumes Bohr orbitology then strong correlations between the 3-surfaces at the ends of CD follow. Still a couple of years and I discovered that quantum states of the Universe can be identified as classical spinor fields in WCW. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

- TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Pronciple with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

- One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

  During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

  Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

  The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. It has indeed turned out that the dream about explicit formula is unrealistic before one has understood what happens in quantum jump. Symmetries and general physical principles have turned out to be the proper guide line here. To give some impressions about what is required some highlights are in order.

- With the emergence of ZEO the notion of S-matrix was replaced with M-matrix defined between positive and negative energy parts of zero energy states. M-matrix can be interpreted as a complex square root of density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally. M-matrices in turn bombine to form the rows of unitary U-matrix defined between zero energy states.

- A decisive step was the strengthening of the General Coordinate Invariance to the requirement that the formulations of the theory in terms of light-like 3-surfaces identified as 3-surfaces at which the induced metric of space-time surfaces changes its signature and in terms of space-like 3-surfaces are equivalent. This means effective 2-dimensionality in the sense that partonic 2-surfaces defined as intersections of these two kinds of surfaces plus 4-D tangent space data at partonic 2-surfaces code for the physics. Quantum classical correspondence requires the coding of the quantum numbers characterizing quantum states assigned to the partonic 2-surfaces to the geometry of space-time surface. This is achieved by adding to the modified Dirac action a measurement interaction term assigned with light-like 3-surfaces.

- The replacement of strings with light-like 3-surfaces equivalent to space-like 3-surfaces means enormous generalization of the super conformal symmetries of string models. A further generalization of these symmetries to non-local Yangian symmetries generalizing the recently discovered Yangian symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is highly suggestive. Here the replacement of point like particles with partonic 2-surfaces means the replacement of conformal symmetry of Minkowski space with infinite-dimensional super-conformal algebras. Yangian symmetry provides also a further refinement to the notion of conserved quantum numbers allowing to define them for bound states using non-local energy conserved currents.

- A further attractive idea is that quantum TGD reduces to almost topological quantum field theory. This is possible if the Kähler action for the preferred extremals defining WCW Kähler function reduces to a 3-D boundary term. This takes place if the conserved currents are so called Beltrami fields with the defining property that the coordinates associated with flow lines extend to single global coordinate variable. This ansatz together with the weak form of electric-magnetic duality reduces the Kähler action to Chern-Simons term with the condition that the 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric magnetic duality. It is the latter constraint which prevents the trivialization of the theory to a topological quantum field theory. Also the identification of the Kähler function of WCW as Dirac determinant finds support as well as the description of the scattering amplitudes in terms of braids with interpretation in terms of finite measurement resolution coded to the basic structure of the solutions of field equations.

- In standard QFT Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

  In TGD framework this generalization of Feynman diagrams indeed emerges unavoidably. Light-like 3-surfaces replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. The approximate localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In TGD framework fermionic variant of twistor Grassmann formalism leads to a stringy variant of twistor diagrammatics in which basic fermions can be said to be on mass-shell but carry non-physical helicities in the internal lines. This suggests the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras.

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 32 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks.

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Matti Pitkänen

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Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family and Kalevi and Ritva Tikkanen and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 45 lonely years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During the last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss my work. Pertti Kärkkäinen is my old physicist friend and has provided continued economic support for a long time. I have also had stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Tommi Ullgren has provided both economic support and encouragement during years. Pekka Rapinoja has offered his help in this respect and I am especially grateful to him for my Python skills.

During the last five years I have had inspiring discussions with many people in Finland interested in TGD. We have had video discussions with Sini Kunnas and had podcast discussions with Marko Manninen related to the TGD based view of physics and consciousness. Marko has also helped in the practical issues related to computers and quite recently he has done a lot of testing of chatGPT helping me to get an overall view of what it is. The discussions in a Zoom group involving Marko Manninen, Tuomas Sorakivi and Rode Majakka have given me the valuable opportunity to clarify my thoughts.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation in CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. I am grateful to Mark McWilliams, Paul Kirsch, Gary Ehlenberg, and Ulla Matfolk and many others for providing links to possibly interesting websites and articles. We have collaborated with Peter Gariaev and Reza Rastmanesh. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the public through the iron wall of academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as an individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without direct support from power holders- even in archives like arXiv.org.

Situation changed as Andrew Adamatsky proposed the writing of a book about TGD when I had already gotten used to the thought that my work would not be published during my lifetime. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loophole. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy.

And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his 73th birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During the period when the right wing held political power this trend was steadily strengthening and the situation is the same as I am writing this. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

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Matti Pitkänen

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Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K1].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A74]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

   M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of

electromagnetic fields are nonvanishing. The correlations functions for weak fields are non-vanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question: super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A36] [B33, B24, B25]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B22]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4\times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n\times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A61, A73, A50, A68].

The identification of the space-time as a sub-manifold [A62, A88] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. ??** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

[1] There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.1.5 Construction of scattering amplitudes

Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A80, A92, A100]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \to B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K53]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, $M^8 - H$ duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and $M^8 - H$ duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and $M^8 - H$ duality would map the algebraic physics in M^8 to the ordinary physics in $M^4 \times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their M^8-H duals in M^8_c are defined for both reals and various p-adic number fields. This is true if they are defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p=3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired "Universe as Computer" vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.
2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since M^8-H duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K49].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to "mind stuff", the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M_c^8$, where M_c^8 is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M_c^8, equivalently E_c^8, can be regarded as complexified octonions. M_c^8 has a subspace M_c^4 containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M_c^8.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M_c^8

$Y^4 \subset M_c^8$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. M^8-H duality assigns to the normal space $N(y)$ a point of CP_2. M_c^4 point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M_c^4 \subset M_c^8$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{(Re(E)^2 - Im(E)^2)}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{(Re(E)^2 - Im(E)^2)}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \quad . \tag{1.1.1}$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M_c^8$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M_c^4 \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same.

The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M_c^8 and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M_c^8$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H_i^3, where the observations are made. Is this condition too strong?
2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.
3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X_i^3.

1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that

Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K76].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K68, K69, K66]) support the view that dark matter might be a key player in living matter.

Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [K86]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A74]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   $M^8 - H$ duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L55].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L47]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t-m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

1.2 Bird's Eye of View about the Topics of the Book

The topics of this book are the purely geometric aspects of the vision about physics as an infinite-dimensional Kähler geometry of the "world of classical worlds", with classical world identified either as light-like 3-D surface of the unique Bohr orbit like 4-surface traversing through it. The non-determinism of Kähler action forces to generalize the notion of 3-surface so that unions of space-like surfaces with time like separations must be allowed. Zero energy ontology allows to formulate this picture elegantly in terms of causal diamonds defined as intersections of future and past directed light-cones. Also a a geometric realization of coupling constant evolution and finite measurement resolution emerges.

There are two separate tasks involved.

1. Provide WCW of 3-surfaces with Kähler geometry which is consistent with 4-dimensional general coordinate invariance so that the metric is Diff^4 degenerate. General coordinate invariance implies that the definition of metric must assign to a given light-like 3-surface X^3 a 4-surface as a kind of Bohr orbit $X^4(X^3)$.
2. Provide the WCW with a spinor structure. The great idea is to identify WCW gamma matrices in terms of super algebra generators expressible using second quantized fermionic oscillator operators for induced free spinor fields at the space-time surface assignable to a given 3-surface. The isometry generators and contractions of Killing vectors with gamma matrices would thus form a generalization of Super Kac-Moody algebra.

The condition of mathematical existence poses surprisingly strong conditions on WCW metric and spinor structure.

1. From the experience with loop spaces one can expect that there is no hope about existence of well-defined Riemann connection unless this space is union of infinite-dimensional symmetric spaces with constant curvature metric and simple considerations requires that vacuum Einstein equations are satisfied by each component in the union. The coordinates labeling these symmetric spaces are zero modes having interpretation as genuinely classical variables which do not quantum fluctuate since they do not contribute to the line element of the WCW.

2. The construction of the Kähler structure involves also the identification of complex structure. Direct construction of Kähler function as action associated with a preferred Bohr orbit like extremal for some physically motivated action action leads to a unique result. Second approach is group theoretical and is based on a direct guess of isometries of the infinite-dimensional symmetric space formed by 3-surfaces with fixed values of zero modes. The group of isometries is generalization of Kac-Moody group obtained by replacing finite-dimensional Lie group with the group of symplectic transformations of $\delta M^4_+ \times CP_2$, where δM^4_+ is the boundary of 4-dimensional future light-cone. A crucial role is played by the generalized conformal invariance assignable to light-like 3-surfaces and to the boundaries of causal diamond. Contrary to the original belief, the coset construction does not provide a realization of Equivalence Principle at quantum level. The proper realization of EP at quantum level seems to be based on the identification of classical Noether charges in Cartan algebra with the eigenvalues of their quantum counterparts assignable to Kähler-Dirac action. At classical level EP follows at GRT limit obtained by lumping many-sheeted space-time to M^4 with effective metric satisfying Einstein's equations as a reflection of the underlying Poincare invariance.

3. Fermionic statistics and quantization of spinor fields can be realized in terms of WCW spinors structure. Quantum criticality and the idea about space-time surfaces as analogs of Bohr orbits have served as basic guiding lines of Quantum TGD. These notions can be formulated more precisely in terms of the modified Dirac equation for induced spinor fields allowing also realization of super-conformal symmetries and quantum gravitational holography. A rather detailed view about how WCW Kähler function emerges as Dirac determinant allowing a tentative identification of the preferred extremals of Kähler action as surface for which second variation of Kähler action vanishes for some deformations of the surface. The catastrophe theoretic analog for quantum critical space-time surfaces are the points of space spanned by behavior and control variables at which the determinant defined by the second derivatives of potential function with respect to behavior variables vanishes. Number theoretic vision leads to rather detailed view about preferred extremals of Kähler action. In particular, preferred extremals should be what I have dubbed as hyper-quaternionic surfaces. It it still an open question whether this characterization is equivalent with quantum criticality or not.

1.2.1 The organization of "Quantum Physics as Infinite-Dimensional Geometry"

The book is divided into 2 parts.

1. The 1st part begins with a summary of the picture about preferred extremals. The picture is somewhat out-of-date since the action used is Kähler action. The twistor lift of TGD adds to the action a volume term having interpretation in terms of length scale dependent cosmological constant. All known extremals of Kähler action having non-vanishing induced Kähler form are however minimal surfaces so that twistor lift means only the loss of these vacuum extremals and for vanishing dynamically determined value of cosmological constant (also possible) also they are obtained: this limit corresponds to infinite size scale for the space-time surfaces.

   Two chapters are devoted to the construction of the Kähler geometry of WCW either from a proposal for Kähler function (note that the volume term for twistor lift implies a modification) or from symmetry principles. Next chapter is about spin structure of WCW and the last chapter summarizes Kähler geometry and spin structure for WCW.

2. 2nd part of the book involves loosely related topics. There are chapters about twistor lift of TGD and number theoretic vision. Since the braiding of flux tubes plays key role in TGD, the last chapter is devoted to the knot theoretic aspects of TGD.

1.3 Sources

The eight online books about TGD [K92, K87, K72, K58, K17, K54, K38, K79] and nine online books about TGD inspired theory of consciousness and quantum biology [K84, K13, K65, K12, K34, K45, K47, K78, K83] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (http://tinyurl.com/ybv8dt4n) contains a lot of material about TGD. In particular, a TGD glossary at http://tinyurl.com/yd6jf3o7).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (http://tinyurl.com/ycyrxj4o founded by Lian Sidorov and in *Prespacetime Journal* (http://tinyurl.com/ycvktjhn), *Journal of Consciousness Research and Exploration* (http://tinyurl.com/yba4f672), and *DNA Decipher Journal* (http://tinyurl.com/y9z52khg), all of them founded by Huping Hu. One can find the list about the articles published at http://tinyurl.com/ybv8dt4n. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.4 The contents of the book

In the following abstracts of various chapters of the book are given in order to provide overall view.

Part I

PHYSICS AS GEOMETRY OF “WORD OF CLASSICAL WORLDS”

Chapter 2

About Identification of the Preferred extremals of Kähler Action

2.1 Introduction

Preferred extremal of Kähler action have remained one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what the attribute "preferred" really means. Symmetries give a clue to the problem. The conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [K86]. Preferred extremal property should rely on this symmetry.

In Zero Energy Ontology (ZEO) preferred extremals are space-time surfaces connecting two space-like 3-surfaces at the ends of space-time surfaces at boundaries of causal diamond (CD). A natural looking condition is that the symplectic Noether charges associated with a sub-algebra of symplectic algebra with conformal weights n-multiples of the weights of the entire algebra vanish for preferred extremals. These conditions would be classical counterparts the condition that super-symplectic sub-algebra annihilates the physical states. This would give a hierarchy of super-symplectic symmetry breakings and quantum criticalities having interpretation in terms of hierarchy of Planck constants $h_{eff} = n \times h$ identified as a hierarchy of dark matter. n could be interpreted as the number of space-time conformal gauge equivalence classes for space-time sheets connecting the 3-surfaces at the ends of space-time surface.

There are also many other proposals for what preferred extremal property could mean or imply. The weak form of electric-magnetic duality combined with the assumption that the contraction of the Kähler current with Kähler gauge potential vanishes for preferred extremals implies that Kähler action in Minkowskian space-time regions reduces to Chern-Simons terms at the light-like orbits of wormhole throats at which the signature of the induced metric changes its signature from Minkowskian to Euclidian. In regions with 4-D CP_2 projection (wormhole contacts) also a 3-D contribution not assignable to the boundary of the region might be possible. These conditions pose strong physically feasible conditions on extremals and might be true for preferred extremals too.

Number theoretic vision leads to a proposal that either the tangent space or normal space of given point of space-time surface is associative and thus quaternionic. Also the formulation in terms of quaternion holomorphy and quaternion-Kähler property is an attractive possibility. So called $M^8 - H$ duality is a variant of this vision and would mean that one can map associative/co-associative space-time surfaces from M^8 to H and also iterate this mapping from H to H to generate entire category of preferred extremals. The signature of M^4 is a general technical problem. For instance, the holomorphy in 2 complex variables could correspond to what I have called Hamilton-Jacobi property. Associativity/co-associativity of the tangent space makes sense also in Minkowskian signature.

In this chapter various views about preferred extremal property are discussed.

2.1.1 Preferred Extremals As Critical Extremals

The study of the Kähler-Dirac equation leads to a detailed view about criticality. Quantum criticality [D2] fixes the values of Kähler coupling strength as the analog of critical temperature. Quantum criticality implies that second variation of Kähler action vanishes for critical deformations and the existence of conserved current except in the case of Cartan algebra of isometries. Quantum criticality allows to fix the values of couplings appearing in the measurement interaction by using the condition $K \rightarrow K + f + \overline{f}$. p-Adic coupling constant evolution can be understood also and corresponds to scale hierarchy for the sizes of causal diamonds (CDs).

The discovery that the hierarchy of Planck constants realized in terms of singular covering spaces of $CD \times CP_2$ can be understood in terms of the extremely non-linear dynamics of Kähler action implying 1-to-many correspondence between canonical momentum densities and time derivatives of the embedding space coordinates led to a further very concrete understanding of the criticality at space-time level and its relationship to zero energy ontology [K39].

Criticality is accompanied by conformal invariance and this leads to the proposal that critical deformations correspond to Kac-Moody type conformal algebra respecting the light-likeness of the partonic orbits and acting trivially at partonic 2-surfaces. Sub-algebras of conformal algebras with conformal weights divisible by integer n would act as gauge symmetries and these algebras would form an inclusion hierarchy defining hierarchy of symmetry breakings. n would also characterize the value of Planck constant $h_{eff} = n \times h$ assignable to various phases of dark matter.

2.1.2 Construction Of Preferred Extremals

There has been considerable progress in the understanding of both preferred extremals and Kähler-Dirac equation.

1. For preferred extremals the generalization of conformal invariance to 4-D situation is very attractive idea and leads to concrete conditions formally similar to those encountered in string model [K10]. In particular, Einstein's equations with cosmological constant would solve consistency conditions and field equations would reduce to a purely algebraic statements analogous to those appearing in equations for minimal surfaces if one assumes that space-time surface has Hermitian structure or its Minkowskian variant Hamilton-Jacobi structure (Appendix).
2. The older approach based on basic heuristics for massless equations, on effective 3-dimensionality, weak form of electric magnetic duality, and Beltrami flows is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic embedding space [K82].

The basic step of progress was the realization that the known extremals of Kähler action - certainly limiting cases of more general extremals - can be deformed to more general extremals having interpretation as preferred extremals.

1. The generalization boils down to the condition that field equations reduce to the condition that the traces $Tr(TH^k)$ for the product of energy momentum tensor and second fundamental form vanish. In string models energy momentum tensor corresponds to metric and one obtains minimal surface equations. The equations reduce to purely algebraic conditions stating that T and H^k have no common components. Complex structure of string world sheet makes this possible.

   Stringy conditions for metric stating $g_{zz} = g_{\overline{z}\overline{z}} = 0$ generalize. The condition that field equations reduce to $Tr(TH^k) = 0$ requires that the terms involving Kähler gauge current in field equations vanish. This is achieved if Einstein's equations hold true (one can consider also more general way to satisfy the conditions). The conditions guaranteeing the vanishing of the trace in turn boil down to the existence of Hermitian structure in the case of Euclidian signature and to the existence of its analog - Hamilton-Jacobi structure - for Minkowskian signature (Appendix). These conditions state that certain components of the induced metric vanish in complex coordinates or Hamilton-Jacobi coordinates.

2. In string model the replacement of the embedding space coordinate variables with quantized ones allows to interpret the conditions on metric as Virasoro conditions. In the recent case a generalization of classical Virasoro conditions to four-dimensional ones would be in question. An interesting question is whether quantization of these conditions could make sense also in TGD framework at least as a useful trick to deduce information about quantum states in WCW degrees of freedom.

3. The interpretation of the extended algebra as Yangian [A36] [B24] suggested previously [K86] to act as a generalization of conformal algebra in TGD Universe is attractive. There is also the conjecture that preferred extremals could be interpreted as quaternionic of co-quaternionic 4-surface of the octonionic embedding space with octonionic representation of the gamma matrices defining the notion of tangent space quanternionicity.

2.2 Weak Form Electric-Magnetic Duality And Its Implications

The notion of electric-magnetic duality [B4] was proposed first by Olive and Montonen and is central in $\mathcal{N} = 4$ supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for CP_2 geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [K21] . What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

1. The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

2. This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be $(2, -1, -1)$ and could be proportional to color hyper charge.

3. The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects: this could become manifest at LHC energies.

4. The weak form electric-magnetic duality together with Beltrami flow property of Kähler leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable. This has enormous impact concerning practical calculability of the theory.

5. One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT. The solution ansatz is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. The proposed

solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow). A general ansatz satisfying the integrability conditions is found.

The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current. Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the gradients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural.

2.2.1 Could A Weak Form Of Electric-Magnetic Duality Hold True?

Holography means that the initial data at the partonic 2-surfaces should fix the WCW metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. A stronger condition allows all partonic 2-surfaces in the slicing of space-time sheet to partonic 2-surfaces and string world sheets. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the embedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

Definition of the weak form of electric-magnetic duality

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

1. The expression of the matrix elements of the metric and Kähler form of WCW in terms of the Kähler fluxes weighted by Hamiltonians of $\delta M^4_\pm$ at the partonic 2-surface X^2 looks very attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of $X^2 \subset X^4$.
2. Electric-magnetic duality of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the identification of the flux Hamiltonians as sums of of the magnetic and electric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.
3. A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of CP_2 type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.
4. To formulate a weaker form of the condition let us introduce coordinates (x^0, x^3, x^1, x^2) such (x^1, x^2) define coordinates for the partonic 2-surface and (x^0, x^3) define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} = KJ_{12} \ . \tag{2.2.1}$$

A more general form of this duality is suggested by the considerations of [K39] reducing the hierarchy of Planck constants to basic quantum TGD and also reducing Kähler function for preferred extremals to Chern-Simons terms [B1] at the boundaries of CD and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta}J_{\gamma\delta}\sqrt{g_4} \ . \tag{2.2.2}$$

Here the index n refers to a normal coordinate for the space-like 3-surface at either boundary of CD or for light-like wormhole throat. ϵ is a sign factor which is opposite for the two ends of CD. It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

5. Information about the tangent space of the space-time surface can be coded to the WCW metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and K is symplectic invariant. Using the sum

$$J_e + J_m = (1+K)J_{12} \ , \tag{2.2.3}$$

where J denotes the Kähler magnetic flux, , makes it possible to have a non-trivial WCW metric even for $K = 0$, which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then K could be a non-constant function of X^2 depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of CD.

Electric-magnetic duality physically

What could the weak duality condition mean physically? For instance, what constraints are obtained if one assumes that the quantization of electro-weak charges reduces to this condition at classical level?

1. The first thing to notice is that the flux of J over the partonic 2-surface is analogous to magnetic flux

$$Q_m = \frac{e}{\hbar}\oint BdS = n \ .$$

n is non-vanishing only if the surface is homologically non-trivial and gives the homology charge of the partonic 2-surface.

2. The expressions of classical electromagnetic and Z^0 fields in terms of Kähler form [L2] , [L2] read as

$$\begin{aligned} \gamma &= \frac{eF_{em}}{\hbar} = 3J - sin^2(\theta_W)R_{03} \ , \\ Z^0 &= \frac{g_Z F_Z}{\hbar} = 2R_{03} \ . \end{aligned} \tag{2.2.4}$$

Here R_{03} is one of the components of the curvature tensor in vielbein representation and F_{em} and F_Z correspond to the standard field tensors. From this expression one can deduce

$$J = \frac{e}{3\hbar}F_{em} + sin^2(\theta_W)\frac{g_Z}{6\hbar}F_Z \ . \qquad (2.2.5)$$

3. The weak duality condition when integrated over X^2 implies

$$\begin{aligned} \frac{e^2}{3\hbar}Q_{em} + \frac{g_Z^2 p}{6}Q_{Z,V} &= K\oint J = Kn \ , \\ Q_{Z,V} &= \frac{I_V^3}{2} - Q_{em} \ , \ p = sin^2(\theta_W) \ . \end{aligned} \qquad (2.2.6)$$

Here the vectorial part of the Z^0 charge rather than as full Z^0 charge $Q_Z = I_L^3 + sin^2(\theta_W)Q_{em}$ appears. The reason is that only the vectorial isospin is same for left and right handed components of fermion which are in general mixed for the massive states.

The coefficients are dimensionless and expressible in terms of the gauge coupling strengths and using $\hbar = r\hbar_0$ one can write

$$\begin{aligned} \alpha_{em}Q_{em} &+ p\frac{\alpha_Z}{2}Q_{Z,V} = \frac{3}{4\pi}\times rnK \ , \\ \alpha_{em} &= \frac{e^2}{4\pi\hbar_0} \ , \ \alpha_Z = \frac{g_Z^2}{4\pi\hbar_0} = \frac{\alpha_{em}}{p(1-p)} \ . \end{aligned} \qquad (2.2.7)$$

4. There is a great temptation to assume that the values of Q_{em} and Q_Z correspond to their quantized values and therefore depend on the quantum state assigned to the partonic 2-surface. The linear coupling of the Kähler-Dirac operator to conserved charges implies correlation between the geometry of space-time sheet and quantum numbers assigned to the partonic 2-surface. The assumption of standard quantized values for Q_{em} and Q_Z would be also seen as the identification of the fine structure constants α_{em} and α_Z. This however requires weak isospin invariance.

The value of K from classical quantization of Kähler electric charge

The value of K can be deduced by requiring classical quantization of Kähler electric charge.

1. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K would give the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant which should invariant under coupling constant evolution by quantum criticality. Within experimental uncertainties one has $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

2. The quantization of Planck constants makes the condition highly non-trivial. The most general quantization of r is as rationals but there are good arguments favoring the quantization as integers corresponding to the allowance of only singular coverings of CD andn CP_2. The point is that in this case a given value of Planck constant corresponds to a finite number pages of the "Big Book". The quantization of the Planck constant implies a further quantization of K and would suggest that K scales as $1/r$ unless the spectrum of values of Q_{em} and Q_Z allowed by the quantization condition scales as r. This is quite possible and the interpretation would be that each of the r sheets of the covering carries (possibly same) elementary charge. Kind of discrete variant of a full Fermi sphere would be in question. The interpretation in terms of anyonic phases [K64] supports this interpretation.

3. The identification of J as a counterpart of $eB/\hbar$ means that Kähler action and thus also Kähler function is proportional to $1/\alpha_K$ and therefore to $\hbar$. This implies that for large values of $\hbar$ Kähler coupling strength $g_K^2/4\pi$ becomes very small and large fluctuations are suppressed in the functional integral. The basic motivation for introducing the hierarchy of Planck constants was indeed that the scaling $\alpha \rightarrow \alpha/r$ allows to achieve the convergence of perturbation theory: Nature itself would solve the problems of the theoretician. This of

course does not mean that the physical states would remain as such and the replacement of single particles with anyonic states in order to satisfy the condition for K would realize this concretely.

4. The condition $K = g_K^2/\hbar$ implies that the Kähler magnetic charge is always accompanied by Kähler electric charge. A more general condition would read as

$$K = n \times \frac{g_K^2}{\hbar}, n \in Z \ . \tag{2.2.8}$$

This would apply in the case of cosmic strings and would allow vanishing Kähler charge possible when the partonic 2-surface has opposite fermion and anti-fermion numbers (for both leptons and quarks) so that Kähler electric charge should vanish. For instance, for neutrinos the vanishing of electric charge strongly suggests $n = 0$ besides the condition that abelian Z^0 flux contributing to em charge vanishes.

It took a year to realize that this value of K is natural at the Minkowskian side of the wormhole throat. At the Euclidian side much more natural condition is

$$K = \frac{1}{hbar} \ . \tag{2.2.9}$$

In fact, the self-duality of CP_2 Kähler form favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for CP_2 type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of CP_2 radius and α_K the effective replacement $g_K^2 \rightarrow 1$ would spoil the argument.

The boundary condition $J_E = J_B$ for the electric and magnetic parts of Kählwer form at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded CP_2 is such that in CP_2 coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for CP_2 type vacuum extremals since by the light-likeness of M^4 projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole.

Reduction of the quantization of Kähler electric charge to that of electromagnetic charge

The best manner to learn more is to challenge the form of the weak electric-magnetic duality based on the induced Kähler form.

1. Physically it would seem more sensible to pose the duality on electromagnetic charge rather than Kähler charge. This would replace induced Kähler form with electromagnetic field, which is a linear combination of induced Kahler field and classical Z^0 field

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{2.2.10}$$

Here $Z_0 = 2R_{03}$ is the appropriate component of CP_2 curvature form [L2]. For a vanishing Weinberg angle the condition reduces to that for Kähler form.

2. For the Euclidian space-time regions having interpretation as lines of generalized Feynman diagrams Weinberg angle should be non-vanishing. In Minkowskian regions Weinberg angle could however vanish. If so, the condition guaranteeing that electromagnetic charge of the partonic 2-surfaces equals to the above condition stating that the em charge assignable to the

fermion content of the partonic 2-surfaces reduces to the classical Kähler electric flux at the Minkowskian side of the wormhole throat. One can argue that Weinberg angle must increase smoothly from a vanishing value at both sides of wormhole throat to its value in the deep interior of the Euclidian region.

3. The vanishing of the Weinberg angle in Minkowskian regions conforms with the physical intuition. Above elementary particle length scales one sees only the classical electric field reducing to the induced Kähler form and classical Z^0 fields and color gauge fields are effectively absent. Only in phases with a large value of Planck constant classical Z^0 field and other classical weak fields and color gauge field could make themselves visible. Cell membrane could be one such system [K71]. This conforms with the general picture about color confinement and weak massivation.

The GRT limit of TGD suggests a further reason for why Weinberg angle should vanish in Minkowskian regions.

1. The value of the Kähler coupling strength mut be very near to the value of the fine structure constant in electron length scale and these constants can be assumed to be equal.
2. GRT limit of TGD with space-time surfaces replaced with abstract 4-geometries would naturally correspond to Einstein-Maxwell theory with cosmological constant which is non-vanishing only in Euclidian regions of space-time so that both Reissner-Nordström metric and CP_2 are allowed as simplest possible solutions of field equations [K90]. The extremely small value of the observed cosmological constant needed in GRT type cosmology could be equal to the large cosmological constant associated with CP_2 metric multiplied with the 3-volume fraction of Euclidian regions.
3. Also at GRT limit quantum theory would reduce to almost topological QFT since Einstein-Maxwell action reduces to 3-D term by field equations implying the vanishing of the Maxwell current and of the curvature scalar in Minkowskian regions and curvature scalar + cosmological constant term in Euclidian regions. The weak form of electric-magnetic duality would guarantee also now the preferred extremal property and prevent the reduction to a mere topological QFT.
4. GRT limit would make sense only for a vanishing Weinberg angle in Minkowskian regions. A non-vanishing Weinberg angle would make sense in the deep interior of the Euclidian regions where the approximation as a small deformation of CP_2 makes sense.

The weak form of electric-magnetic duality has surprisingly strong implications for the basic view about quantum TGD as following considerations show.

2.2.2 Magnetic Confinement, The Short Range Of Weak Forces, And Color Confinement

The weak form of electric-magnetic duality has surprisingly strong implications if one combines it with some very general empirical facts such as the non-existence of magnetic monopole fields in macroscopic length scales.

How can one avoid macroscopic magnetic monopole fields?

Monopole fields are experimentally absent in length scales above order weak boson length scale and one should have a mechanism neutralizing the monopole charge. How electroweak interactions become short ranged in TGD framework is still a poorly understood problem. What suggests itself is the neutralization of the weak isospin above the intermediate gauge boson Compton length by neutral Higgs bosons. Could the two neutralization mechanisms be combined to single one?

1. In the case of fermions and their super partners the opposite magnetic monopole would be a wormhole throat. If the magnetically charged wormhole contact is electromagnetically neutral but has vectorial weak isospin neutralizing the weak vectorial isospin of the fermion only the electromagnetic charge of the fermion is visible on longer length scales. The distance of this wormhole throat from the fermionic one should be of the order weak boson Compton length. An interpretation as a bound state of fermion and a wormhole throat state with the quantum

numbers of a neutral Higgs boson would therefore make sense. The neutralizing throat would have quantum numbers of $X_{-1/2} = \nu_L\overline{\nu}_R$ or $X_{1/2} = \overline{\nu}_L\nu_R$. $\nu_L\overline{\nu}_R$ would not be neutral Higgs boson (which should correspond to a wormhole contact) but a super-partner of left-handed neutrino obtained by adding a right handed neutrino. This mechanism would apply separately to the fermionic and anti-fermionic throats of the gauge bosons and corresponding space-time sheets and leave only electromagnetic interaction as a long ranged interaction.

2. One can of course wonder what is the situation situation for the bosonic wormhole throats feeding gauge fluxes between space-time sheets. It would seem that these wormhole throats must always appear as pairs such that for the second member of the pair monopole charges and I_V^3 cancel each other at both space-time sheets involved so that one obtains at both space-time sheets magnetic dipoles of size of weak boson Compton length. The proposed magnetic character of fundamental particles should become visible at TeV energies so that LHC might have surprises in store!

Well-definedness of electromagnetic charge implies stringiness

Well-definedness of electromagnetic charged at string world sheets carrying spinor modes is very natural constraint and not trivially satisfied because classical W boson fields are present. As a matter fact, all weak fields should be effectively absent above weak scale. How this is possible classical weak fields identified as induced gauge fields are certainly present.

The condition that em charge is well defined for spinor modes implies that the space-time region in which spinor mode is non-vanishing has 2-D CP_2 projection such that the induced W boson fields are vanishing. The vanishing of classical Z^0 field can be poses as additional condition - at least in scales above weak scale. In the generic case this requires that the spinor mode is restricted to 2-D surface: string world sheet or possibly also partonic 2-surface. This implies that TGD reduces to string model in fermionic sector. Even for preferred extremals with 2-D projecting the modes are expected to allow restriction to 2-surfaces. This localization is possible only for Kähler-Dirac action.

A word of warning is however in order. The GRT limit or rather limit of TGD as Einstein Yang-Mills theory replaces the sheets of many-sheeted space-time with Minkowski space with effective metric obtained by summing to Minkowski metric the deviations of the induced metrics of space-time sheets from Minkowski metric. For gauge potentials a similar identification applies. YM-Einstein equations coupled with matter and with non-vanishing cosmological constant are expected on basis of Poincare invariance. One cannot exclude the possibility that the sums of weak gauge potentials from different space-time sheet tend to vanish above weak scale and that well-definedness of em charge at classical level follows from the effective absence of classical weak gauge fields.

Magnetic confinement and color confinement

Magnetic confinement generalizes also to the case of color interactions. One can consider also the situation in which the magnetic charges of quarks (more generally, of color excited leptons and quarks) do not vanish and they form color and magnetic singles in the hadronic length scale. This would mean that magnetic charges of the state $q_{\pm 1/2} - X_{\mp 1/2}$ representing the physical quark would not vanish and magnetic confinement would accompany also color confinement. This would explain why free quarks are not observed. To how degree then quark confinement corresponds to magnetic confinement is an interesting question.

For quark and antiquark of meson the magnetic charges of quark and antiquark would be opposite and meson would correspond to a Kähler magnetic flux so that a stringy view about meson emerges. For valence quarks of baryon the vanishing of the net magnetic charge takes place provided that the magnetic net charges are $(\pm 2, \mp 1, \mp 1)$. This brings in mind the spectrum of color hyper charges coming as $(\pm 2, \mp 1, \mp 1)/3$ and one can indeed ask whether color hypercharge correlates with the Kähler magnetic charge. The geometric picture would be three strings connected to single vertex. Amusingly, the idea that color hypercharge could be proportional to color hyper charge popped up during the first year of TGD when I had not yet discovered CP_2 and believed on $M^4 \times S^2$.

p-Adic length scale hypothesis and hierarchy of Planck constants defining a hierarchy of dark variants of particles suggest the existence of scaled up copies of QCD type physics and weak physics. For p-adically scaled up variants the mass scales would be scaled by a power of $\sqrt{2}$ in the most general case. The dark variants of the particle would have the same mass as the original one. In particular, Mersenne primes $M_k = 2^k - 1$ and Gaussian Mersennes $M_{G,k} = (1+i)^k - 1$ has been proposed to define zoomed copies of these physics. At the level of magnetic confinement this would mean hierarchy of length scales for the magnetic confinement.

One particular proposal is that the Mersenne prime M_{89} should define a scaled up variant of the ordinary hadron physics with mass scaled up roughly by a factor $2^{(107-89)/2} = 512$. The size scale of color confinement for this physics would be same as the weal length scale. It would look more natural that the weak confinement for the quarks of M_{89} physics takes place in some shorter scale and M_{61} is the first Mersenne prime to be considered. The mass scale of M_{61} weak bosons would be by a factor $2^{(89-61)/2} = 2^{14}$ higher and about 1.6×10^4 TeV. M_{89} quarks would have virtually no weak interactions but would possess color interactions with weak confinement length scale reflecting themselves as new kind of jets at collisions above TeV energies.

In the biologically especially important length scale range 10 nm -2500 nm there are as many as four scaled up electron Compton lengths $L_e(k) = \sqrt{5}L(k)$: they are associated with Gaussian Mersennes $M_{G,k}$, $k = 151, 157, 163, 167$. This would suggest that the existence of scaled up scales of magnetic-, weak- and color confinement. An especially interesting possibly testable prediction is the existence of magnetic monopole pairs with the size scale in this range. There are recent claims about experimental evidence for magnetic monopole pairs [D1] .

Magnetic confinement and stringy picture in TGD sense

The connection between magnetic confinement and weak confinement is rather natural if one recalls that electric-magnetic duality in super-symmetric quantum field theories means that the descriptions in terms of particles and monopoles are in some sense dual descriptions. Fermions would be replaced by string like objects defined by the magnetic flux tubes and bosons as pairs of wormhole contacts would correspond to pairs of the flux tubes. Therefore the sharp distinction between gravitons and physical particles would disappear.

The reason why gravitons are necessarily stringy objects formed by a pair of wormhole contacts is that one cannot construct spin two objects using only single fermion states at wormhole throats. Of course, also super partners of these states with higher spin obtained by adding fermions and anti-fermions at the wormhole throat but these do not give rise to graviton like states [?] . The upper and lower wormhole throat pairs would be quantum superpositions of fermion anti-fermion pairs with sum over all fermions. The reason is that otherwise one cannot realize graviton emission in terms of joining of the ends of light-like 3-surfaces together. Also now magnetic monopole charges are necessary but now there is no need to assign the entities $X_\pm$ with gravitons.

Graviton string is characterized by some p-adic length scale and one can argue that below this length scale the charges of the fermions become visible. Mersenne hypothesis suggests that some Mersenne prime is in question. One proposal is that gravitonic size scale is given by electronic Mersenne prime M_{127}. It is however difficult to test whether graviton has a structure visible below this length scale.

What happens to the generalized Feynman diagrams is an interesting question. It is not at all clear how closely they relate to ordinary Feynman diagrams. All depends on what one is ready to assume about what happens in the vertices. One could of course hope that zero energy ontology could allow some very simple description allowing perhaps to get rid of the problematic aspects of Feynman diagrams.

1. Consider first the recent view about generalized Feynman diagrams which relies ZEO. A highly attractive assumption is that the particles appearing at wormhole throats are on mass shell particles. For incoming and outgoing elementary bosons and their super partners they would be positive it resp. negative energy states with parallel on mass shell momenta. For virtual bosons they the wormhole throats would have opposite sign of energy and the sum of on mass shell states would give virtual net momenta. This would make possible twistor description of virtual particles allowing only massless particles (in 4-D sense usually and in 8-D sense in TGD framework). The notion of virtual fermion makes sense only if one assumes

in the interaction region a topological condensation creating another wormhole throat having no fermionic quantum numbers.

2. The addition of the particles $X^{\pm}$ replaces generalized Feynman diagrams with the analogs of stringy diagrams with lines replaced by pairs of lines corresponding to fermion and $X_{\pm 1/2}$. The members of these pairs would correspond to 3-D light-like surfaces glued together at the vertices of generalized Feynman diagrams. The analog of 3-vertex would not be splitting of the string to form shorter strings but the replication of the entire string to form two strings with same length or fusion of two strings to single string along all their points rather than along ends to form a longer string. It is not clear whether the duality symmetry of stringy diagrams can hold true for the TGD variants of stringy diagrams.

3. How should one describe the bound state formed by the fermion and $X^{\pm}$? Should one describe the state as superposition of non-parallel on mass shell states so that the composite state would be automatically massive? The description as superposition of on mass shell states does not conform with the idea that bound state formation requires binding energy. In TGD framework the notion of negentropic entanglement has been suggested to make possible the analogs of bound states consisting of on mass shell states so that the binding energy is zero [K49] . If this kind of states are in question the description of virtual states in terms of on mass shell states is not lost. Of course, one cannot exclude the possibility that there is infinite number of this kind of states serving as analogs for the excitations of string like object.

4. What happens to the states formed by fermions and $X_{\pm 1/2}$ in the internal lines of the Feynman diagram? Twistor philosophy suggests that only the higher on mass shell excitations are possible. If this picture is correct, the situation would not change in an essential manner from the earlier one.

The highly non-trivial prediction of the magnetic confinement is that elementary particles should have stringy character in electro-weak length scales and could behaving to become manifest at LHC energies. This adds one further item to the list of non-trivial predictions of TGD about physics at LHC energies [K50] .

2.2.3 Could Quantum TGD Reduce To Almost Topological QFT?

There seems to be a profound connection with the earlier unrealistic proposal that TGD reduces to almost topological quantum theory in the sense that the counterpart of Chern-Simons action assigned with the wormhole throats somehow dictates the dynamics. This proposal can be formulated also for the Kähler-Dirac action action. I gave up this proposal but the following argument shows that Kähler action with weak form of electric-magnetic duality effectively reduces to Chern-Simons action plus Coulomb term.

1. Kähler action density can be written as a 4-dimensional integral of the Coulomb term $j_K^{\alpha} A_{\alpha}$ plus and integral of the boundary term $J^{n\beta} A_{\beta} \sqrt{g_4}$ over the wormhole throats and of the quantity $J^{0\beta} A_{\beta} \sqrt{g_4}$ over the ends of the 3-surface.

2. If the self-duality conditions generalize to $J^{n\beta} = 4\pi\alpha_K \epsilon^{n\beta\gamma\delta} J_{\gamma\delta}$ at throats and to $J^{0\beta} = 4\pi\alpha_K \epsilon^{0\beta\gamma\delta} J_{\gamma\delta}$ at the ends, the Kähler function reduces to the counterpart of Chern-Simons action evaluated at the ends and throats. It would have same value for each branch and the replacement $h \rightarrow n \times h$ would effectively describe this. Boundary conditions would however give $1/n$ factor so that $\hbar$ would disappear from the Kähler function! It is somewhat surprising that Kähler action gives Chern-Simons action in the vacuum sector defined as sector for which Kähler current is light-like or vanishes.

Holography encourages to ask whether also the Coulomb interaction terms could vanish. This kind of dimensional reduction would mean an enormous simplification since TGD would reduce to an almost topological QFT. The attribute "almost" would come from the fact that one has non-vanishing classical Noether charges defined by Kähler action and non-trivial quantum dynamics in M^4 degrees of freedom. One could also assign to space-time surfaces conserved four-momenta which is not possible in topological QFTs. For this reason the conditions guaranteeing the vanishing of Coulomb interaction term deserve a detailed analysis.

1. For the known extremals j_K^α either vanishes or is light-like ("massless extremals" for which weak self-duality condition does not make sense [K10]) so that the Coulomb term vanishes identically in the gauge used. The addition of a gradient to A induces terms located at the ends and wormhole throats of the space-time surface but this term must be cancelled by the other boundary terms by gauge invariance of Kähler action. This implies that the M^4 part of WCW metric vanishes in this case. Therefore massless extremals as such are not physically realistic: wormhole throats representing particles are needed.

2. The original naïve conclusion was that since Chern-Simons action depends on CP_2 coordinates only, its variation with respect to Minkowski coordinates must vanish so that the WCW metric would be trivial in M^4 degrees of freedom. This conclusion is in conflict with quantum classical correspondence and was indeed too hasty. The point is that the allowed variations of Kähler function must respect the weak electro-magnetic duality which relates Kähler electric field depending on the induced 4-metric at 3-surface to the Kähler magnetic field. Therefore the dependence on M^4 coordinates creeps via a Lagrange multiplier term

$$\int \Lambda_\alpha(J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma}J_{\beta\ gamma})\sqrt{g_4}d^3x \quad . \tag{2.2.11}$$

The (1,1) part of second variation contributing to M^4 metric comes from this term.

3. This erratic conclusion about the vanishing of M^4 part WCW metric raised the question about how to achieve a non-trivial metric in M^4 degrees of freedom. The proposal was a modification of the weak form of electric-magnetic duality. Besides CP_2 Kähler form there would be the Kähler form assignable to the light-cone boundary reducing to that for $r_M = constant$ sphere - call it J^1. The generalization of the weak form of self-duality would be $J^{n\beta} = \epsilon^{n\beta\gamma\delta}K(J_{\gamma\delta}+\epsilon J^1_{\gamma\delta})$. This form implies that the boundary term gives a non-trivial contribution to the M^4 part of the WCW metric even without the constraint from electric-magnetic duality. Kähler charge is not affected unless the partonic 2-surface contains the tip of CD in its interior. In this case the value of Kähler charge is shifted by a topological contribution. Whether this term can survive depends on whether the resulting vacuum extremals are consistent with the basic facts about classical gravitation.

4. The Coulombic interaction term is not invariant under gauge transformations. The good news is that this might allow to find a gauge in which the Coulomb term vanishes. The vanishing condition fixing the gauge transformation ϕ is

$$j_K^\alpha\partial_\alpha\phi = -j^\alpha A_\alpha \quad . \tag{2.2.12}$$

This differential equation can be reduced to an ordinary differential equation along the flow lines j_K by using $dx^\alpha/dt = j_K^\alpha$. Global solution is obtained only if one can combine the flow parameter t with three other coordinates- say those at the either end of CD to form space-time coordinates. The condition is that the parameter defining the coordinate differential is proportional to the covariant form of Kähler current: $dt = \phi j_K$. This condition in turn implies $d^2t = d(\phi j_K) = d(\phi j_K) = d\phi\wedge j_K + \phi dj_K = 0$ implying $j_K\wedge dj_K = 0$ or more concretely,

$$\epsilon^{\alpha\beta\gamma\delta}j^K_\beta\partial_\gamma j^K_{delta} \quad = \quad 0 \quad . \tag{2.2.13}$$

j_K is a four-dimensional counterpart of Beltrami field [B9] and could be called generalized Beltrami field.

The integrability conditions follow also from the construction of the extremals of Kähler action [K10] . The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires $j_K\wedge J = 0$. One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current: $j_K = \phi j_I$, where $j_I = {}^*(J\wedge A)$ is the instanton current, which is not conserved for 4-D CP_2 projection. The conservation of j_K implies the condition $j_I^\alpha\partial_\alpha\phi = \partial_\alpha j^\alpha\phi$ and from this ϕ can be integrated if the integrability condition $j_I\wedge dj_I = 0$ holds true implying the same condition for j_K. By introducing at least 3 or CP_2 coordinates as space-time coordinates, one finds that the contravariant form of j_I is purely topological so that the integrability condition fixes the dependence on M^4 coordinates and this selection is coded into the scalar function ϕ. These functions define families of conserved currents $j_K^\alpha\phi$ and $j_I^\alpha\phi$ and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

5. There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations $A \rightarrow A + \nabla\phi$ for which the scalar function the integral $\int j_K^\alpha \partial_\alpha \phi$ reduces to a total divergence a giving an integral over various 3-surfaces at the ends of CD and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_\alpha(j^\alpha\phi) = 0 . \tag{2.2.14}$$

As a consequence Coulomb term reduces to a difference of the conserved charges $Q_\phi^e = \int j^0\phi\sqrt{g_4}d^3x$ at the ends of the CD vanishing identically. The change of the Chern-Simons type term is trivial if the total weighted Kähler magnetic flux $Q_\phi^m = \sum \int J\phi dA$ over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

6. The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the Kähler-Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of CP_2. It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action. The gauge transformed Kähler gauge potential couples to the Kähler-Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since K would transform only by an addition of a real part of a holomorphic function.

7. A first guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a $U(1)$ gauge transformation induced by a transformation of $\delta CD \times CP_2$ generating the gauge transformation represented by ϕ. This interpretation makes sense if the fluxes defined by Q_ϕ^m and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.

8. Later a simpler proposal assuming Kähler action with Chern-Simons term at partonic orbits and Kähler-Dirac action with Chern-Simons Dirac term at partonic orbits emerged. Measurement interaction terms would correspond to Lagrange multiplier terms at the ends of space-time surface fixing the values of classical conserved charges to their quantum values. Super-symmetry requires the assignment of this kind of term also to Kähler-Dirac action as boundary term.

   Kähler-Dirac equation gives rise to a boundary condition at space-like ends of the space-time surface stating that the action of the Kähler-Dirac gamma matrix in normal direction annihilates the spinor modes. The normal vector would be light-like and the value of the incoming on mass shell four-momentum would be coded to the geometry of the space-time surface and string world sheet.

   One can assign to partonic orbits Chern-Simons Dirac action and now the condition would be that the action of C-S-D operator equals to that of massless M^4 Dirac operator. C-S-D Dirac action would give rise to massless Dirac propagator. Twistor Grassmann approach suggests that also the virtual fermions reduce effectively to massless on-shell states but have non-physical helicity.

To sum up, one could understand the basic properties of WCW metric in this framework. Effective 2-dimensionality would result from the existence of an infinite number of conserved charges in two different time directions (genuine conservation laws plus gauge fixing). The infinite-dimensional symmetric space for given values of zero modes corresponds to the Cartesian product of the WCWs associated with the partonic 2-surfaces at both ends of CD and the generalized Chern-Simons term decomposes into a sum of terms from the ends giving single particle Kähler functions and to the terms from light-like wormhole throats giving interaction term between positive and negative energy parts of the state. Hence Kähler function could be calculated without any knowledge about the interior of the space-time sheets and TGD would reduce to almost topological QFT as speculated earlier. Needless to say this would have immense boost to the program of constructing WCW Kähler geometry.

2.3 An attempt to understand preferred extremals of Kähler action

Preferred extremal of Kähler action is one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what "preferred" really means. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [L41]. The problem is however how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

2.3.1 What "preferred" could mean?

The first question is what preferred extremal could mean.

1. In positive energy ontology preferred extremal would be a space-time surface assignable to given 3-surface and unique in the ideal situation: since one cannot pose conditions to the normal derivatives of embedding space coordinates at 3-surface, there is infinity of extremals. Some additional conditions are required and space-time surface would be analogous to Bohr orbit: hence the attribute "preferred". The problem would be to understand what "preferred" could mean. The non-determinism of Kähler action however destroyed this dream in its original form and led to zero energy ontology (ZEO).

2. In ZEO one considers extremals as space-time surfaces connecting two space-like 3-surfaces at the boundaries. One might hope that these 4-surfaces are unique. The non-determinism of Kähler action suggests that this is not the case. At least there is conformal invariance respecting the light-likeness of the 3-D parton orbits at which the signature of the induced metric changes: the conformal transformations would leave the space-like 3-D ends or at least partonic 2-surfaces invariant. This non-determinism would correspond to quantum criticality.

3. Effective 2-dimensionality follows from strong form of general coordinate invariance (GCI) stating that light-like partonic orbits and space-like 3-surfaces at the ends of space-time surface are equivalent physically: partonic 2-surfaces and their 4-D tangent space data would determine everything. One can however worry about how effective 2-dimensionality relates to the fact that the modes of the induced spinor field are localized at string world sheets and partonic 2-surface. Are the tangent space data equivalent with the data characterizing string world sheets as surfaces carrying vanishing electroweak fields?

   There is however a problem: the hierarchy of Planck constants (dark matter) requires that the conformal equivalence classes of light-like surfaces must be counted as physical degrees of freedom so that either space-like or light-like surfaces do not seem to be quite enough.

   Should one then include also the light-like partonic orbits to the what one calls 3-surface? The resulting connected 3-surfaces would define analogs of Wilson loops. Could the conformal equivalence class of the preferred extremal be unique without any additional conditions? If so, one could get rid of the attribute "preferred". The fractal character of the many-sheeted space-time however suggests that one can have this kind of uniqueness only in given length scale resolution and that "radiative corrections" due to the non-determinism are always present.

These considerations show that the notion of preferred extremal is still far from being precisely defined and it is not even clear whether the attribute "preferred" is needed. If not then the question is what are the extremals of Kähler action.

2.3.2 What is known about extremals?

A lot is is known about properties of extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

1. Hamilton-Jacobi coordinates for M^4 (discussed in this chapter) define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for X^4 as those for M^4. Hamilton-Jacobi coordinates consist of light-like coordinate m and its dual defining local 2-plane $M^2 \subset M^4$ and complex transversal complex coordinates $(w, \overline{w})$ for a plane E^2_x orthogonal to M^2_x at each point of M^4. Clearly, hyper-complex analyticity and complex analyticity are in question.
2. Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).
3. The quaternionic planes of octonion space containing preferred hyper-complex plane are labelled by CP_2, which might be called CP_2^{mod} [K82]. The identification $CP_2 = CP_2^{mod}$ motivates the notion of $M^8 - -M^4 \times CP_2$ duality [K20]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space $G_2/SU(3)$. The group G_2 of octonion automorphisms has already earlier appeared in TGD framework.
4. The duality between partonic 2-surfaces and string world sheets in turn suggests that the $CP_2 = CP_2^{mod}$ conditions reduce to string model for partonic 2-surfaces in $CP_2 = SU(3)/U(2)$. String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

1. To begin with express octonions in the form $o = q_1 + Iq_2$, where q_i is quaternion and I is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of $H = M^4 \times CP_2$ to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of H to get a map $H \to H$. This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.
2. Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of embedding space and would emerge naturally. The ends of braid strands at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

2.3.3 Basic ideas about preferred extremals

The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

1. Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function.
2. The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [B9] so that corresponding 1-forms J satisfy the condition $J \wedge dJ = 0$. These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

hold true for conserved currents. From this one obtains that Ψ defines global coordinate varying along flow lines of J.
3. A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of Ψ and Φ are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \ ,$$

and that the Ψ satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

as a consequence of current conservation. If Ψ defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \ ,$$

the light-like dual of Φ -call it Φ_c- defines a light-like like coordinate and Φ and Φ_c defines a light-like plane at each point of space-time sheet.

If also Φ satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \ ,$$

also the current

$$K = \Psi \nabla \Phi$$

is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-lik plane defined by local light-like momentum direction.

If Φ allows a continuation to an analytic function of the transversal complex coordinate, one obtains a coordinatization of space-time surface by Ψ and its dual (defining hyper-complex coordinate) and $w, \overline{w}$. Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of M^4.

This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of J defined Beltrami flow it seems that the distribution of momentum planes is integrable.
4. General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

Hamilton-Jacobi coordinates for M^4

The earlier attempts to construct preferred extremals [K10] led to the realization that so called Hamilton-Jacobi coordinates (m, w) for M^4 define its slicing by string world sheets parametrized by partonic 2-surfaces. m would be pair of light-like conjugate coordinates associated with an integrable distribution of planes M^2 and w would define a complex coordinate for the integrable distribution of 2-planes E^2 orthogonal to M^2. There is a great temptation to assume that these coordinates define preferred coordinates for M^4.

1. The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane M^2 can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points z of sphere S^2 telling the direction of the line $M^2 \cap E^3$, when one assigns rest frame and therefore S^2 with the preferred time coordinate defined by the line connecting the tips of CD. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor $(u, \overline{u}) \rightarrow \lambda u, \overline{u}/\lambda)$ define the same plane. Projective twistor like entities defining CP_1 having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of E^2 could serve as a pair of complex coordinates (z, w) for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [L41].
2. The coordinate Ψ appearing in Beltrami flow defines the light-like vector field defining M^2 distribution. Its hyper-complex conjugate would define Ψ_c and conjugate light-like direction. An attractive possibility is that Φ allows analytic continuation to a holomorphic function of w. In this manner one would have four coordinates for M^4 also for space-time sheet.
3. The general vision is that at each point of space-time surface one can decompose the tangent space to $M^2(x) \subset M^4 = M^2_x \times E^2_x$ representing momentum plane and polarization plane $E^2 \subset E^2_x \times T(CP_2)$. The moduli space of planes $E^2 \subset E^6$ is 8-dimensional and parametrized by $SO(6)/SO(2) \times SO(4)$ for a given E^2_x. How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

Space-time surfaces as associative/co-associative surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time. It took several trials before the recent form of this hypothesis was achieved.

1. Octonionic structure is defined in terms of the octonionic representaton of gamma matrices of the embedding space existing only in dimension $D = 8$ since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds to a preferred time axes (and rest frame) identified naturally as that connecting the tips of CD. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.
2. Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of H with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [K96, K73]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex subspace of octonions.
3. The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane M^2.

The obvious questions are following.

1. Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space $M^2 \subset M^4$ for preferred extremals? For massless extremals [K10] this condition would be true. The orthogonal decomposition $T(X^4) = M^2 \oplus_\perp E^2$ can be defined at each point if this is true. For massless extremals also the functions Ψ and Φ can be identified.

2. One should answer also the following delicate question. Can M^2 really depend on point x of space-time? CP_2 as a moduli space of quaternionic planes emerges naturally if M^2 is *same* everywhere. It however seems that one should allow an integrable distribution of M^2_x such that M^2_x is same for all points of a given partonic 2-surface.

   How could one speak about fixed CP_2 (the embedding space) at the entire space-time sheet even when M^2_x varies?

   (a) Note first that G_2 (see `http://tinyurl.com/y9rrs7un`) defines the Lie group of octonionic automorphisms and G_2 action is needed to change the preferred hyper-octonionic sub-space. Various $SU(3)$ subgroups of G_2 are related by G_2 automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of G_2. One would have Minkowskian string model with G_2 as a target space. As a matter fact, this string model is defined in the target space $G_2/SU(3)$ having dimension $D = 6$ since $SU(3)$ automorphisms leave given $SU(3)$ invariant.

   (b) This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit q_1 with "color isospin" $I_3 = 1/2$ and "color hypercharge" $Y = -1/3$ and its conjugate $\overline{q}_1$ with opposite color isospin and hypercharge.

   (c) The CP_2 point assigned with the quaternionic basis would correspond to the $SU(3)$ rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of $SU(3)$ rotation matrix. Hyper-complex analyticity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analycitity is enough-since Kähler action already defines it.

3. The WZW model (see `http://tinyurl.com/ydxcvfhv`) inspired approach to the situation would be following. The parameterization corresponds to a map $g : X^2 \to G_2$ for which g defines a flat G_2 connection at string world sheet. WZW type action would give rise to this kind of situation. The transition $G_2 \to G_2/SU(3)$ would require that one gauges $SU(3)$ degrees of freedom by bringing in $SU(3)$ connection. Similar procedure for $CP_2 = SU(3)/U(2)$ would bring in $SU(3)$ valued chiral field and $U(2)$ gauge field. Instead of introducing these connections one can simply introduce $G_2/SU(3)$ and $SU(3)/U(2)$ valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

The two interpretations of CP_2

An old observation very relevant for what I have called $M^8 - H$ duality [K20] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as M^8) containing preferred hyper-complex plane is CP_2. Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by CP_2. This CP_2 can be called it CP_2^{mod} to avoid confusion. In the recent case this would mean that the space $E^2(x) \subset E^2_x \times T(CP_2)$ is represented by a point of CP_2^{mod}. On the other hand, the embedding of space-time surface to H defines a point of "real" CP_2. This gives two different CP_2s.

1. The highly suggestive idea is that the identification $CP_2^{mod} = CP_2$ (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to CP_2 would fix the local polarization plane completely. This condition for $E^2(x)$ would be purely local and depend on the values of CP_2 coordinates only. Second condition

for $E^2(x)$ would involve the gradients of embedding space coordinates including those of CP_2 coordinates.

2. The conditions that the planes M^2_x form an integrable distribution at space-like level and that M^2_x is determined by the modified gamma matrices. The integrability of this distribution for M^4 could imply the integrability for X^2. X^4 would differ from M^4 only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of M^2s.

   Does this mean that one can begin from vacuum extremal with constant values of CP_2 coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which CP_2 coordinates depend on transversal coordinates defined by $\epsilon \cdot m$ and $\epsilon \cdot k$. One could however allow dependence of CP_2 coordinates on light-like M^4 coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

   Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of CP_2 points on the light-like coordinates assignable to the distribution of M^2_x would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

2.3.4 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes $E^2(x)$ determined by the point of $CP_2 = CP_2^{mod}$ identification and by the tangent space of $E^2_x \times CP_2$ are same. The challenge is to transform this condition to an explicit form. $CP_2 = CP_2^{mod}$ identification should be general coordinate invariant. This requires that also the representation of E^2 as (e^2, e^3) plane is general coordinate invariant suggesting that the use of preferred CP_2 coordinates - presumably complex Eguchi-Hanson coordinates - could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of X^4 but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation $T^m_x(X^4)$ about the modified tangent space and call the vectors of $T^m_x(X^4)$ modified tangent vectors. I hope that this would not cause confusion.

$CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of $T^m_x(X^4)$ allows an explicit formulation using the tangent vectors of $T^m_x(X^4)$.

1. The unit vector pair (e_2, e_3) should correspond to a unique tangent vector of H defined by the coordinate differentials dh^k in some natural coordinates used. Complex Eguchi-Hanson coordinates [L2] are a natural candidate for CP_2 and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of H uniquely, this is possible.

2. The pair (e_2, e_3) as also its complexification $(q_1 = e_2 + ie_3, \overline{q}_1 = e_2 - ie_3)$ is expressible as a linear combination of octonionic units $I_2, ...I_7$ should be mapped to a point of $CP_2^{mod} = CP_2$ in canonical manner. This mapping is what should be expressed explicitly. One should express given (e_2, e_3) in terms of $SU(3)$ rotation applied to a standard vector. After that one should define the corresponding CP_2 point by the bundle projection $SU(3) \rightarrow CP_2$.

3. The tangent vector pair

$$(\partial_w h^k, \partial_{\overline{w}} h^k)$$

   defines second representation of the tangent space of $E^2(x)$. This pair should be equivalent with the pair $(q_1, \overline{q}_1)$. Here one must be however very cautious with the choice of coordinates. If the choice of w is unique apart from constant the gradients should be unique. One can use also real coordinates (x, y) instead of $(w = x + iy, \overline{w} = x - iy)$ and the pair (e_2, e_3). One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonionic basis

$$(\partial_x h^k, \partial_y h^k) \rightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A) e_A) \leftrightarrow (e_2, e_3) \ ,$$

where the e_A denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of (e_2, e_3) derived from the knowledge of CP_2 projection.

Formulation of quaternionicity condition in terms of octonionic structure constants

One can consider also a formulation of the quaternionic tangent planes in terms of (e_2, e_3) expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic (see `http://tinyurl.com/5m5lqr`) *resp.* quaternionic (see `http://tinyurl.com/3rr79p9`) structure constants can be found at [A25] *resp.* [A27].

1. The ansatz is

$$\begin{aligned}
\{E_k\} &= \{1, I_1, E_2, E_3\} \ , \\
E_2 &= E_{2k}e^k \equiv \sum_{k=2}^{7} E_{2k}e^k \ , \ E_3 = E_{3k}e^k \equiv \sum_{k=2}^{7} E_{3k}e^k \ , \\
|E_2| &= 1 \ , \ |E_3| = 1 \ .
\end{aligned} \tag{2.3.1}$$

2. The multiplication table for octonionic units expressible in terms of octonionic triangle (see `http://tinyurl.com/5m5lqr`) [A25] gives

$$f^{1kl}E_{2k} = E_{3l} \ , \ f^{1kl}E_{3k} = -E_{2l} \ , \ f^{klr}E_{2k}E_{3l} = \delta_1^r \ . \tag{2.3.2}$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

3. The conditions are linear and quadratic in the coefficients E_{2k} and E_{3k} and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on (E_2, E_3) is of the form

$$\begin{pmatrix} f_1 & 1 \\ -1 & f_1 \end{pmatrix} \ ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by I_1 analogous to color hyper charge. Both values of color hyper charged are obtained.

Explicit expression for the $CP_2 = CP_2^{mod}$ conditions

The symmetry under $SU(3)$ allows to construct the solutions of the above equations directly.

1. One can introduce complexified basis of octonion units transforming like $(1, 1, 3, \overline{3})$ under SU(3). Note the analogy of triplet with color triplet of quarks. One can write complexified basis as $(1, e_1, (q_1, q_2, q_3), (\overline{q}_1\overline{q}_2, \overline{q}_3))$. The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind of line can be used to form pair of complexified unit and its conjugate. In the tangent space of $M^4 \times CP_2$ the basis vectors q_1, and q_2 are mixtures of E^2_x and CP_2 tangent vectors. q_3 involves only CP_2 tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

2. The quaternionic basis is real and must transform like $(1, 1, q_1, \overline{q}_1)$, where q_1 is any quark in the triplet and $\overline{q}_1$ its conjugate in antitriplet. Having fixed some basis one can perform SU(3) rotations to get a new basis. The action of the rotation is by 3×3 special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in (e_2, e_3) plane not affecting the plane itself. The action of $SU(3)$ on q_1 is simply the action of its first row on (q_1, q_2, q_3) triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \tag{2.3.3}$$

The triplets (z_1, z_2, z_3) defining a complex unit vector and point of S^5. Since overall phase does not matter a point of CP_2 is in question. The new real octonion units are given by the formulas

$$\begin{aligned} e_2 &\rightarrow Re(z_1)e_2 + Re(z_2)e_4 + Re(z_3)e_6 - Im(z_1)e_3 - Im(z_2)e_5 - Im(z_3)e_7 \ , \\ e_3 &\rightarrow Im(z_1)e_2 + Im(z_2)e_4 + Im(z_3)e_6 + Re(z_1)e_3 + Re(z_2)e_5 + Re(z_3)e_7 \ . \end{aligned} \tag{2.3.4}$$

For instance the CP_2 coordinates corresponding to the coordinate patch (z_1, z_2, z_3) with $z_3 \neq 0$ are obtained as $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$.

Using these expressions the equations expressing the conjecture $CP_2 = CP_2^{mod}$ equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \leftrightarrow (\partial_x h^k e^A_k e_A, \partial_y h^k e^A_k e_A) \ , \tag{2.3.5}$$

where e_A denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to expresses the contractions of the partial derivatives with vielbein vectors with the 6 components of e_2 and e_3. Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overal normalization factor for the right hand side. The equations are invariant under scalings of (x, y). The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamilton-Jacobi coordinates for M^4 and Eguchi-Hanson complex coordinates in which $SU(2) \times U(1)$ is represented linearly for CP_2. These coordinates are preferred because they carry deep physical meaning.

Does TGD boil down to two string models?

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and $CP_2 = CP_2^{mod}$ conditions one has what one might call string model with 6-dimensional $G_2/SU(3)$ as targent space. The orbit of string in $G_2/SU(3)$ allows to deduce the G_2 rotation identifiable as a point of $G_2/SU(3)$ defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [K41, K42, K80, K48]. This duality suggests that the solutions to the $CP_2 = CP_2^{mod}$ conditions could reduce to holomorphy with respect to the coordinate w for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regarded as dual string models in $G_2/SU(3)$ and $SU(3)/U(2)$ and also to string model in M^4 and X^4! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

2.4 In What Sense TGD Could Be An Integrable Theory?

During years evidence supporting the idea that TGD could be an integrable theory in some sense has accumulated. The challenge is to show that various ideas about what integrability means form pieces of a bigger coherent picture. Of course, some of the ideas are doomed to be only partially correct or simply wrong. Since it is not possible to know beforehand what ideas are wrong and what are right the situation is very much like in experimental physics and it is easy to claim (and has been and will be claimed) that all this argumentation is useless speculation. This is the price that must be paid for real thinking.

Integrable theories allow to solve nonlinear classical dynamics in terms of scattering data for a linear system. In TGD framework this translates to quantum classical correspondence. The solutions of Kähler-Dirac equation define the scattering data. This data should define a real analytic function whose octonionic extension defines the space-time surface as a surface for which its imaginary part in the representation as bi-quaternion vanishes. There are excellent hopes about this thanks to the reduction of the Kähler-Dirac equation to geometric optics.

In the following I will first discuss briefly what integrability means in (quantum) field theories, list some bits of evidence for integrability in TGD framework, discuss once again the question whether the different pieces of evidence are consistent with other and what one really means with various notions. An an outcome I represent what I regard as a more coherent view about integrability of TGD. The notion of octonion analyticity developed in the previous section is essential for the for what follows.

2.4.1 What Integrable Theories Are?

The following is an attempt to get some bird's eye of view about the landscape of integrable theories.

Examples of integrable theories

Integrable theories are typically non-linear 1+1-dimensional (quantum) field theories. Solitons and various other particle like structures are the characteristic phenomenon in these theories. Scattering matrix is trivial in the sense that the particles go through each other in the scattering and suffer only a phase change. In particular, momenta are conserved. Korteveg- de Vries equation (see `http://tinyurl.com/3cyt8hk`) [B2] was motivated by the attempt to explain the experimentally discovered shallow water wave preserving its shape and moving with a constant velocity. Sine-Gordon equation (see `http://tinyurl.com/yafl243x`) [B7] describes geometrically constant curvature surfaces and defines a Lorentz invariant non-linear field theory in 1+1-dimensional space-time, which can be applied to Josephson junctions (in TGD inspired quantum biology it is encountered in the model of nerve pulse [K71]). Non- linear Schrödinger equation (see `http://tinyurl.com/y88efbo7`) [B5] having applications to optics and water waves represents a further example. All these equations have various variants.

From TGD point of view conformal field theories represent an especially interesting example of integrable theories. (Super-)conformal invariance is the basic underlying symmetry and by its infinite-dimensional character implies infinite number of conserved quantities. The construction of the theory reduces to the construction of the representations of (super-)conformal algebra. One can solve 2-point functions exactly and characterize them in terms of (possibly anomalous) scaling dimensions of conformal fields involved and the coefficients appearing in 3-point functions can be solved in terms of fusion rules leading to an associative algebra for conformal fields. The basic applications are to 2-dimensional critical thermodynamical systems whose scaling invariance generalizes to conformal invariance. String models represent second application in which a collection of super-conformal field theories associated with various genera of 2-surface is needed to describe loop corrections to the scattering amplitudes. Also moduli spaces of conformal equivalence classes become important.

Topological quantum field theories (see http://tinyurl.com/lsvx7g3) are also examples of integrable theories. Because of its independence on the metric Chern-Simons action (see http://tinyurl.com/ydgsqm2c) is in 3-D case the unique action defining a topological quantum field theory. The calculations of knot invariants (for TGD approach see [K41]), topological invariants of 3-manifolds and 4-manifolds, and topological quantum computation (see http://tinyurl.com/dkpo4y) (for a model of DNA as topological quantum computer see [K2]) represent applications of this approach. TGD as almost topological QFT means that the Kähler action for preferred extremals reduces to a surface term by the vanishing of Coulomb term in action and by the weak form of electric-magnetic duality reduces to Chern-Simons action. Both Euclidian and Minkowskian regions give this kind of contribution.

$\mathcal{N} = 4$ SYM is the a four-dimensional and very nearly realistic candidate for an integral quantum field theory. The observation that twistor amplitudes allow also a dual of the 4-D conformal symmetry motivates the extension of this symmetry to its infinite-dimensional Yangian variant [A36]. Also the enormous progress in the construction of scattering amplitudes suggests integrability. In TGD framework Yangian symmetry would emerge naturally by extending the symplectic variant of Kac-Moody algebra from light-cone boundary to the interior of causal diamond and the Kac-Moody algebra from light-like 3-surface representing wormhole throats at which the signature of the induced metric changes to the space-time interior [K86].

About mathematical methods

The mathematical methods used in integrable theories are rather refined and have contributed to the development of the modern mathematical physics. Mention only quantum groups, conformal algebras, and Yangian algebras.

The basic element of integrability is the possibility to transform the non-linear classical problem for which the interaction is characterized by a potential function or its analog to a linear scattering problem depending on time. For instance, for the ordinary Schrödinger function one can solve potential once single solution of the equation is known. This does not work in practice. One can however gather information about the asymptotic states in scattering to deduce the potential. One cannot do without information about bound state energies too.

In TGD framework asymptotic states correspond to partonic 2-surfaces at the two light-like boundaries of CD (more precisely: the largest CD involved and defining the IR resolution for momenta). From the scattering data coding information about scattering for various values of energy of the incoming particle one deduced the potential function or its analog.

1. The basic tool is inverse scattering transform known as Gelfand-Marchenko-Levitan (GML) transform (see http://tinyurl.com/y9f7ybln) described in simple terms in [B8].

   (a) In 1+1 dimensional case the S-matrix characterizing scattering is very simple since the only thing that can take place in scattering is reflection or transmission. Therefore the S-matrix elements describe either of these processes and by unitarity the sum of corresponding probabilities equals to 1. The particle can arrive to the potential either from left or right and is characterized by a momentum. The transmission coefficient can have a pole meaning complex (imaginary in the simplest case) wave vector serving as a signal for the formation of a bound state or resonance. The scattering data are represented by the reflection and transmission coefficients as function of time.

   (b) One can deduce an integral equation for a propagator like function $K(t,x)$ describing how delta pulse moving with light velocity is scattered from the potential and is expressible in terms of time integral over scattering data with contributions from both scattering states and bound states. The derivation of GML transform [B8] uses time reversal and time translational invariance and causality defined in terms of light velocity. After some tricks one obtains the integral equation as well as an expression for the time independent potential as $V(x) = K(x,x)$. The argument can be generalized to more complex problems to deduce the GML transform.

2. The so called Lax pair (see `http://tinyurl.com/yc93nw53`) is one manner to describe integrable systems [B3]. Lax pair consists of two operators L and M. One studies what might be identified as "energy" eigenstates satisfying $L(x,t)\Psi = \lambda\Psi$. λ does not depend on time and one can say that the dynamics is associated with x coordinate whereas as t is time coordinate parametrizing different variants of eigenvalue problem with the same spectrum for L. The operator $M(t)$ does not depend on x at all and the independence of λ on time implies the condition

$$\partial_t L = [L, M] \ .$$

This equation is analogous to a quantum mechanical evolution equation for an operator induced by time dependent "Hamiltonian" M and gives the non-linear classical evolution equation when the commutator on the right hand side is a multiplicative operator (so that it does not involve differential operators acting on the coordinate x). Non-linear classical dynamics for the time dependent potential emerges as an integrability condition.

One could say that $M(t)$ introduces the time evolution of $L(t,x)$ as an automorphism which depends on time and therefore does not affect the spectrum. One has $L(t,x) = U(t)L(0,x)U^{-1}(t)$ with $dU(t)/dt = M(t)U(t)$. The time evolution of the analog of the quantum state is given by a similar equation.

3. A more refined view about Lax pair is based on the observation that the above equation can be generalized so that M depends also on x. The generalization of the basic equation for $M(x,t)$ reads as

$$\partial_t L - \partial_x M - [L, M] = 0 \ .$$

The condition has interpretation as a vanishing of the curvature of a gauge potential having components $A_x = L, A_t = M$. This generalization allows a beautiful geometric formulation of the integrability conditions and extends the applicability of the inverse scattering transform. The monodromy of the flat connection becomes important in this approach. Flat connections in moduli spaces are indeed important in topological quantum field theories and in conformal field theories.

4. There is also a connection with the so called Riemann-Hilbert problem (see `http://tinyurl.com/ybay4qjg`) [A29]. The monodromies of the flat connection define monodromy group and Riemann-Hilbert problem concerns the existence of linear differential equations having a given monodromy group. Monodromy group emerges in the analytic continuation of an analytic function and the action of the element of the monodromy group tells what happens for the resulting many-valued analytic function as one turns around a singularity once ("mono-"). The linear equations obviously relate to the linear scattering problem. The flat connection (M, L) in turn defines the monodromy group. What is needed is that the functions involved are analytic functions of (t, x) replaced with a complex or hyper-complex variable. Again Wick rotation is involved. Similar approach generalizes also to higher dimensional moduli spaces with complex structures.

In TGD framework the effective 2-dimensionality raises the hope that this kind of mathematical apparatus could be used. An interesting possibility is that finite measurement resolution could be realized in terms of a gauge group or Kac-Moody type group represented by trivial gauge potential defining a monodromy group for n-point functions. Monodromy invariance would hold for the full n-point functions constructed in terms of analytic n-point functions and their conjugates. The ends of braid strands are natural candidates for the singularities around which monodromies are defined.

2.4.2 Why TGD Could Be Integrable Theory In Some Sense?

There are many indications that TGD could be an integrable theory in some sense. The challenge is to see which ideas are consistent with each other and to build a coherent picture where everything finds its own place.

1. 2-dimensionality or at least effective 2-dimensionality seems to be a prerequisite for integrability. Effective 2-dimensionality is suggested by the strong form of General Coordinate Invariance implying also holography and generalized conformal invariance predicting infinite number of conservation laws. The dual roles of partonic 2-surfaces and string world sheets supports a four-dimensional generalization of conformal invariance. Twistor considerations [K86, L33] indeed suggest that Yangian invariance and Kac-Moody invariances combine to a 4-D analog of conformal invariance induced by 2-dimensional one by algebraic continuation.
2. Octonionic representation of embedding space Clifford algebra and the identification of the space-time surfaces as quaternionic space-time surfaces would define a number theoretically natural generalization of conformal invariance. The reason for using gamma matrix representation is that vector field representation for octonionic units does not exist. The problem concerns the precise meaning of the octonionic representation of gamma matrices.

   Space-time surfaces could be quaternionic also in the sense that conformal invariance is analytically continued from string curve to 8-D space by octonion real-analyticity. The question is whether the Clifford algebra based notion of tangent space quaternionicity is equivalent with octonionic real-analyticity based notion of quaternionicity.

   The notions of co-associativity and co-quaternionicity make also sense and one must consider seriously the possibility that associativity-co-associativity dichotomy corresponds to Minkowskian-Euclidian dichotomy.
3. Field equations define hydrodynamic Beltrami flows satisfying integrability conditions of form $J \wedge dJ = 0$.
   (a) One can assign local momentum and polarization directions to the preferred extremals and this gives a decomposition of Minkowskian space-time regions to massless quanta analogous to the 1+1-dimensional decomposition to solitons. The linear superposition of modes with 4-momenta with different directions possible for free Maxwell action does not look plausible for the preferred extremals of Kähler action. This rather quantal and solitonic character is in accordance with the quantum classical correspondence giving very concrete connection between quantal and classical particle pictures. For 4-D volume action one does not obtain this kind of decomposition. In 2-D case volume action gives superposition of solutions with different polarization directions so that the situation is nearer to that for free Maxwell action and is not like soliton decomposition.
   (b) Beltrami property in strong sense allows to identify 4 preferred coordinates for the space-time surface in terms of corresponding Beltrami flows. This is possible also in Euclidian regions using two complex coordinates instead of hyper-complex coordinate and complex coordinate. The assumption that isometry currents are parallel to the same light-like Beltrami flow implies hydrodynamic character of the field equations in the sense that one can say that each flow line is analogous to particle carrying some quantum numbers. This property is not true for all extremals (say cosmic strings).
   (c) The tangent bundle theoretic view about integrability is that one can find a Lie algebra of vector fields in some manifold spanning the tangent space of a lower-dimensional manifolds and is expressed in terms of Frobenius theorem (see `http://tinyurl.com/of6vfz5`) [A10]. The gradients of scalar functions defining Beltrami flows appearing in the ansatz for preferred exremals would define these vector fields and the slicing. Partonic 2-surfaces would correspond to two complex conjugate vector fields (local polarization direction) and string world sheets to light-like vector field and its dual (light-like momentum directions). This slicing generalizes to the Euclidian regions.
4. Infinite number of conservation laws is the signature of integrability. Classical field equations follow from the condition that the vector field defined by Kähler-Dirac gamma matrices has vanishing divergence and can be identified an integrability condition for the Kähler-Dirac equation guaranteeing also the conservation of super currents so that one obtains an infinite number of conserved charges.
5. Quantum criticality is a further signal of integrability. 2-D conformal field theories describe critical systems so that the natural guess is that quantum criticality in TGD framework relates to the generalization of conformal invariance and to integrability. Quantum criticality implies that Kähler coupling strength is analogous to critical temperature. This condition

does affects classical field equations only via boundary conditions expressed as weak form of electric magnetic duality at the wormhole throats at which the signature of the metric changes.

For finite-dimensional systems the vanishing of the determinant of the matrix defined by the second derivatives of potential is similar signature and applies in catastrophe theory. Therefore the existence of vanishing second variations of Kähler action should characterize criticality and define a property of preferred extremals. The vanishing of second variations indeed leads to an infinite number of conserved currents [K10] following the conditions that the deformation of Kähler-Dirac gamma matrix is also divergenceless and that the Kähler-Dirac equation associated with it is satisfied.

2.4.3 Could TGD Be An Integrable Theory?

Consider first the abstraction of integrability in TGD framework. Quantum classical correspondence could be seen as a correspondence between linear quantum dynamics and non-linear classical dynamics. Integrability would realize this correspondence. In integrable models such as Sine-Gordon equation particle interactions are described by potential in 1+1 dimensions. This too primitive for the purposes of TGD. The vertices of generalized Feynman diagrams take care of this. At lines one has free particle dynamics so that the situation could be much simpler than in integrable models if one restricts the considerations to the lines or Minkowskian space-time regions surrounding them.

The non-linear dynamics for the space-time sheets representing incoming lines of generalized Feynman diagram should be obtainable from the linear dynamics for the induced spinor fields defined by Kähler-Dirac operator. There are two options.

1. Strong form of the quantum classical correspondence states that each solution for the linear dynamics of spinor fields corresponds to space-time sheet. This is analogous to solving the potential function in terms of a single solution of Schrödinger equation. Coupling of space-time geometry to quantum numbers via measurement interaction term is a proposal for realizing this option. It is however the quantum numbers of positive/negative energy parts of zero energy state which would be visible in the classical dynamics rather than those of induced spinor field modes.

2. Only overall dynamics characterized by scattering data- the counterpart of S-matrix for the Kähler-Dirac operator- is mapped to the geometry of the space-time sheet. This is much more abstract realization of quantum classical correspondence.

3. Can these two approaches be equivalent? This might be the case since quantum numbers of the state are not those of the modes of induced spinor fields.

What the scattering data could be for the induced spinor field satisfying Kähler-Dirac equation?

1. If the solution of field equation has hydrodynamic character, the solutions of the Kähler-Dirac equation can be localized to light-like Beltrami flow lines of hydrodynamic flow. These correspond to basic solutions and the general solution is a superposition of these. There is no dispersion and the dynamics is that of geometric optics at the basic level. This means geometric optics like character of the spinor dynamics.

   Solutions of the Kähler-Dirac equation are completely analogous to the pulse solutions defining the fundamental solution for the wave equation in the argument leading from wave equation with external time independent potential to Marchenko-Gelfand-Levitan equation allowing to identify potential in terms of scattering data. There is however no potential present now since the interactions are described by the vertices of Feynman diagram where the particle lines meet. Note that particle like regions are Euclidian and that this picture applies only to the Minkowskian exteriors of particles.

2. Partonic 2-surfaces at the ends of the line of generalized Feynman diagram are connected by flow lines. Partonic 2-surfaces at which the signature of the induced metric changes are in a special position. Only the imaginary part of the bi-quaternionic value of the octonion valued map is non-vanishing at these surfaces which can be said to be co-complex 2-surfaces. By

geometric optics behavior the scattering data correspond to a diffeomorphism mapping initial partonic 2-surface to the final one in some preferred complex coordinates common to both ends of the line.

3. What could be these preferred coordinates? Complex coordinates for S^2 at light-cone boundary define natural complex coordinates for the partonic 2-surface. With these coordinates the diffeomorphism defining scattering data is diffeomorphism of S^2. Suppose that this map is real analytic so that maps "real axis" of S^2 to itself. This map would be same as the map defining the octonionic real analyticity as algebraic extension of the complex real analytic map. By octonionic analyticity one can make large number of alternative choices for the coordinates of partonic 2-surface.

4. There can be non-uniqueness due to the possibility of $G_2/SU(3)$ valued map characterizing the local octonionic units. The proposal is that the choice of octonionic imaginary units can depend on the point of string like orbit: this would give string model in $G_2/SU(3)$. Conformal invariance for this string model would imply analyticity and helps considerably but would not probably fix the situation completely since the element of the coset space would constant at the partonic 2-surfaces at the ends of CD. One can of course ask whether the $G_2/SU(3)$ element could be constant for each propagator line and would change only at the 2-D vertices?

This would be the inverse scattering problem formulated in the spirit of TGD. There could be also dependence of space-time surface on quantum numbers of quantum states but not on individual solution for the induced spinor field since the scattering data of this solution would be purely geometric.

2.5 Do Geometric Invariants Of Preferred Extremals Define Topological Invariants Of Space-time Surface And Code For Quantumphysics?

The recent progress in the understanding of preferred extremals [K10] led to a reduction of the field equations to conditions stating for Euclidian signature the existence of Kähler metric. The resulting conditions are a direct generalization of corresponding conditions emerging for the string world sheet and stating that the 2-metric has only non-diagonal components in complex/hypercomplex coordinates. Also energy momentum of Kähler action and has this characteristic (1, 1) tensor structure. In Minkowskian signature one obtains the analog of 4-D complex structure combining hyper-complex structure and 2-D complex structure.

The construction lead also to the understanding of how Einstein's equations with cosmological term follow as a consistency condition guaranteeing that the covariant divergence of the Maxwell's energy momentum tensor assignable to Kähler action vanishes. This gives $T = kG + \Lambda g$. By taking trace a further condition follows from the vanishing trace of T:

$$R = \frac{4\Lambda}{k} . \tag{2.5.1}$$

That any preferred extremal should have a constant Ricci scalar proportional to cosmological constant is very strong prediction. Note that the accelerating expansion of the Universe would support positive value of Λ. Note however that both Λ and $k \propto 1/G$ are both parameters characterizing one particular preferred extremal. One could of course argue that the dynamics allowing only constant curvature space-times is too simple. The point is however that particle can topologically condense on several space-time sheets meaning effective superposition of various classical fields defined by induced metric and spinor connection.

The following considerations demonstrate that preferred extremals can be seen as canonical representatives for the constant curvature manifolds playing central role in Thurston's geometrization theorem (see `http://tinyurl.com/y8bbzlnr`) [A32] known also as hyperbolization theorem implying that geometric invariants of space-time surfaces transform to topological invariants. The generalization of the notion of Ricci flow to Maxwell flow in the space of metrics and further to

Kähler flow for preferred extremals in turn gives a rather detailed vision about how preferred extremals organize to one-parameter orbits. It is quite possible that Kähler flow is actually discrete. The natural interpretation is in terms of dissipation and self organization.

Quantum classical correspondence suggests that this line of thought could be continued even further: could the geometric invariants of the preferred extremals could code not only for space-time topology but also for quantum physics? How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge of quantum TGD. Could the correlation functions be reduced to statistical geometric invariants of preferred extemals? The latest (means the end of 2012) and perhaps the most powerful idea hitherto about coupling constant evolution is quantum classical correspondence in statistical sense stating that the statistical properties of a preferred extremal in quantum superposition of them are same as those of the zero energy state in question. This principle would be quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This principle would allow to deduce correlation functions and S-matrix from the statistical properties of single preferred extremal alone using classical intuition. Also coupling constant evolution would be coded by the statistical properties of the representative preferred extremal.

2.5.1 Preferred Extremals Of Kähler Action As Manifolds With Constant Ricci Scalar Whose Geometric Invariants Are Topological-Invariants

An old conjecture inspired by the preferred extremal property is that the geometric invariants of space-time surface serve as topological invariants. The reduction of Kähler action to 3-D Chern-Simons terms (see `http://tinyurl.com/ybp86sho`) [K10] gives support for this conjecture as a classical counterpart for the view about TGD as almost topological QFT. The following arguments give a more precise content to this conjecture in terms of existing mathematics.

1. It is not possible to represent the scaling of the induced metric as a deformation of the space-time surface preserving the preferred extremal property since the scale of CP_2 breaks scale invariance. Therefore the curvature scalar cannot be chosen to be equal to one numerically. Therefore also the parameter $R = 4\Lambda/k$ and also Λ and k separately characterize the equivalence class of preferred extremals as is also physically clear.

   Also the volume of the space-time sheet closed inside causal diamond CD remains constant along the orbits of the flow and thus characterizes the space-time surface. Λ and even $k \propto 1/G$ can indeed depend on space-time sheet and p-adic length scale hypothesis suggests a discrete spectrum for Λ/k expressible in terms of p-adic length scales: $\Lambda/k \propto 1/L_p^2$ with $p \simeq 2^k$ favored by p-adic length scale hypothesis. During cosmic evolution the p-adic length scale would increase gradually. This would resolve the problem posed by cosmological constant in GRT based theories.

2. One could also see the preferred extremals as 4-D counterparts of constant curvature 3-manifolds in the topology of 3-manifolds. An interesting possibility raised by the observed negative value of Λ is that most 4-surfaces are constant negative curvature 4-manifolds. By a general theorem coset spaces (see `http://tinyurl.com/y8d3udpr`) H^4/Γ, where $H^4 = SO(1,4)/SO(4)$ is hyperboloid of M^5 and Γ a torsion free discrete subgroup of $SO(1,4)$ [A16]. It is not clear to me, whether the constant value of Ricci scalar implies constant sectional curvatures and therefore hyperbolic space property. It could happen that the space of spaces with constant Ricci curvature contain a hyperbolic manifold as an especially symmetric representative. In any case, the geometric invariants of hyperbolic metric are topological invariants.

   By Mostow rigidity theorem (see `http://tinyurl.com/yacbu8sk`) [A24] finite-volume hyperbolic manifold is unique for $D > 2$ and determined by the fundamental group of the manifold. Since the orbits under the Kähler flow preserve the curvature scalar the manifolds at the orbit must represent different embeddings of one and hyperbolic 4-manifold. In 2-D case the moduli space for hyperbolic metric for a given genus $g > 0$ is defined by Teichmueller parameters and has dimension $6(g-1)$. Obviously the exceptional character of $D = 2$ case relates to conformal invariance. Note that the moduli space in question (see `http://tinyurl.com/ybowqm5v`) plays a key role in p-adic mass calculations [K18].

In the recent case Mostow rigidity theorem could hold true for the Euclidian regions and maybe generalize also to Minkowskian regions. If so then both "topological" and "geometro" in "Topological GeometroDynamics" would be fully justified. The fact that geometric invariants become topological invariants also conforms with "TGD as almost topological QFT" and allows the notion of scale to find its place in topology. Also the dream about exact solvability of the theory would be realized in rather convincing manner.

These conjectures are the main result independent of whether the generalization of the Ricci flow discussed in the sequel exists as a continuous flow or possibly discrete sequence of iterates in the space of preferred extremals of Kähler action. My sincere hope is that the reader could grasp how far reaching these result really are.

2.5.2 Is There A Connection Between Preferred Extremals And AdS$_4$/CFT Correspondence?

The preferred extremals satisfy Einstein Maxwell equations with a cosmological constant and have negative scalar curvature for negative value of Λ. 4-D space-times with hyperbolic metric provide canonical representation for a large class of four-manifolds and an interesting question is whether these spaces are obtained as preferred extremals and/or vacuum extremals.

4-D hyperbolic space with Minkowski signature is locally isometric with AdS$_4$. This suggests at connection with AdS$_4$/CFT correspondence of M-theory. The boundary of AdS would be now replaced with 3-D light-like orbit of partonic 2-surface at which the signature of the induced metric changes. The metric 2-dimensionality of the light-like surface makes possible generalization of 2-D conformal invariance with the light-like coordinate taking the role of complex coordinate at light-like boundary. AdS could represent a special case of a more general family of space-time surfaces with constant Ricci scalar satistying Einstein-Maxwell equations and generalizing the AdS$_4$/CFT correspondence. There is however a strong objection from cosmology: the accelerated expansion of the Universe requires positive value of Λ and favors De Sitter Space dS_4 instead of AdS_4.

These observations provide motivations for finding whether AdS$_4$ and/or dS_4 allows an embedding as a vacuum extremal to $M^4\times S^2\subset M^4\times CP_2$, where S^2 is a homologically trivial geodesic sphere of CP_2. It is easy to guess the general form of the embedding by writing the line elements of, M^4, S^2, and AdS$_4$.

1. The line element of M^4 in spherical Minkowski coordinates (m,r_M,θ,ϕ) reads as

$$ds^2 = dm^2-dr_M^2-r_M^2d\Omega^2 \ . \tag{2.5.2}$$

2. Also the line element of S^2 is familiar:

$$ds^2 = -R^2(d\Theta^2+sin^2(\theta)d\Phi^2) \ . \tag{2.5.3}$$

3. By visiting in Wikipedia (see `http://tinyurl.com/y9hw95ql`) one learns that in spherical coordinate the line element of AdS$_4$/dS_4 is given by

$$\begin{aligned} ds^2 &= A(r)dt^2-\frac{1}{A(r)}dr^2-r^2d\Omega^2 \ , \\ A(r) &= 1+\epsilon y^2 \ , \ y=\frac{r}{r_0} \ , \\ \epsilon &= 1 \ \text{for} \ AdS_4 \ , \ \epsilon=-1 \ \text{for} \ dS_4 \ . \end{aligned} \tag{2.5.4}$$

4. From these formulas it is easy to see that the ansatz is of the same general form as for the embedding of Schwartschild-Nordstöm metric:

$$\begin{array}{ll} m=\Lambda t+h(y) \ , & r_M=r \ , \\ \Theta=s(y) \ , & \Phi=\omega(t+f(y)) \ . \end{array} \tag{2.5.5}$$

The non-trivial conditions on the components of the induced metric are given by

$$\begin{aligned}
g_{tt} &= \Lambda^2 - x^2 sin^2(\Theta) = A(r) \ , \\
g_{tr} &= \frac{1}{r_0}\left[\Lambda\frac{dh}{dy} - x^2 sin^2(\theta)\frac{df}{dr}\right] = 0 \ , \\
g_{rr} &= \frac{1}{r_0^2}\left[(\frac{dh}{dy})^2 - 1 - x^2 sin^2(\theta)(\frac{df}{dy})^2 - R^2(\frac{d\Theta}{dy})^2\right] = -\frac{1}{A(r)} \ , \\
x &= R\omega \ .
\end{aligned} \tag{2.5.6}$$

By some simple algebraic manipulations one can derive expressions for $sin(\Theta)$, df/dr and dh/dr.

1. For $\Theta(r)$ the equation for g_{tt} gives the expression

$$\begin{aligned}
sin(\Theta) &= \pm\frac{P^{1/2}}{x} \ , \\
P &= \Lambda^2 - A = \Lambda^2 - 1 - \epsilon y^2 \ .
\end{aligned} \tag{2.5.7}$$

The condition $0 \leq sin^2(\Theta) \leq 1$ gives the conditions

$$\begin{array}{lll}
(\Lambda^2 - x^2 - 1)^{1/2} \leq & y & \leq (\Lambda^2 - 1)^{1/2} \quad \text{for } \epsilon = 1 \ (AdS_4) \ , \\
(-\Lambda^2 + 1)^{1/2} \leq & y & \leq (x^2 + 1 - \Lambda^2)^{1/2} \quad \text{for } \epsilon = -1 \ (dS_4) \ .
\end{array} \tag{2.5.8}$$

Only a spherical shell is possible in both cases. The model for the final state of star considered in [K90] predicted similar layer layer like structure and inspired the proposal that stars quite generally have an onion-like structure with radii of various shells characterize by p-adic length scale hypothesis and thus coming in some powers of $\sqrt{2}$. This brings in mind also Titius-Bode law.

2. From the vanishing of g_{tr} one obtains

$$\frac{dh}{dy} = \frac{P}{\Lambda}\frac{df}{dy} \ . \tag{2.5.9}$$

3. The condition for g_{rr} gives

$$(\frac{df}{dy})^2 = \frac{r_0^2}{AP}[A^{-1} - R^2(\frac{d\Theta}{dy})^2] \ . \tag{2.5.10}$$

Clearly, the right-hand side is positive if $P \geq 0$ holds true and $Rd\Theta/dy$ is small. One can express $d\Theta/dy$ using chain rule as

$$(\frac{d\Theta}{dy})^2 = \frac{x^2y^2}{P(P-x^2)} \ . \tag{2.5.11}$$

One obtains

$$(\frac{df}{dy})^2 = \Lambda r_0^2\frac{y^2}{AP}\left[\frac{1}{1+y^2} - x^2(\frac{R}{r_0})^2\frac{1}{P(P-x^2)}\right] \ . \tag{2.5.12}$$

The right hand side of this equation is non-negative for certain range of parameters and variable y. Note that for $r_0 \gg R$ the second term on the right hand side can be neglected. In this case it is easy to integrate $f(y)$.

The conclusion is that both AdS_4 and dS^4 allow a local embedding as a vacuum extremal. Whether also an embedding as a non-vacuum preferred extremal to $M^4 \times S^2$, S^2 a homologically non-trivial geodesic sphere is possible, is an interesting question.

2.5.3 Generalizing Ricci Flow To Maxwell Flow For 4-Geometries And Kähler Flow For Space-Time Surfaces

The notion of Ricci flow has played a key part in the geometrization of topological invariants of Riemann manifolds. I certainly did not have this in mind when I choose to call my unification attempt "Topological Geometrodynamics" but this title strongly suggests that a suitable generalization of Ricci flow could play a key role in the understanding of also TGD.

Ricci flow and Maxwell flow for 4-geometries

The observation about constancy of 4-D curvature scalar for preferred extremals inspires a generalization of the well-known volume preserving Ricci flow (see `http://tinyurl.com/2cwlzh9l`) [A28] introduced by Richard Hamilton. Ricci flow is defined in the space of Riemann metrics as

$$\frac{dg_{\alpha\beta}}{dt} = -2R_{\alpha\beta} + 2\frac{R_{avg}}{D}g_{\alpha\beta} \ . \tag{2.5.13}$$

Here R_{avg} denotes the average of the scalar curvature, and D is the dimension of the Riemann manifold. The flow is volume preserving in average sense as one easily checks ($\langle g^{\alpha\beta}dg_{\alpha\beta}/dt\rangle = 0$). The volume preserving property of this flow allows to intuitively understand that the volume of a 3-manifold in the asymptotic metric defined by the Ricci flow is topological invariant. The fixed points of the flow serve as canonical representatives for the topological equivalence classes of 3-manifolds. These 3-manifolds (for instance hyperbolic 3-manifolds with constant sectional curvatures) are highly symmetric. This is easy to understand since the flow is dissipative and destroys all details from the metric.

What happens in the recent case? The first thing to do is to consider what might be called Maxwell flow in the space of all 4-D Riemann manifolds allowing Maxwell field.

1. First of all, the vanishing of the trace of Maxwell's energy momentum tensor codes for the volume preserving character of the flow defined as

$$\frac{dg_{\alpha\beta}}{dt} = T_{\alpha\beta} \ . \tag{2.5.14}$$

   Taking covariant divergence on both sides and assuming that d/dt and D_α commute, one obtains that $T^{\alpha\beta}$ is divergenceless.

   This is true if one assumes Einstein's equations with cosmological term. This gives

$$\frac{dg_{\alpha\beta}}{dt} = kG_{\alpha\beta} + \Lambda g_{\alpha\beta} = kR_{\alpha\beta} + (-\frac{kR}{2} + \Lambda)g_{\alpha\beta} \ . \tag{2.5.15}$$

   The trace of this equation gives that the curvature scalar is constant. Note that the value of the Kähler coupling strength plays a highly non-trivial role in these equations and it is quite possible that solutions exist only for some critical values of α_K. Quantum criticality should fix the allow value triplets (G, Λ, α_K) apart from overall scaling

$$(G, \Lambda, \alpha_K) \to (xG, \Lambda/x, x\alpha_K \ .$$

   Fixing the value of G fixes the values remaining parameters at critical points. The rescaling of the parameter t induces a scaling by x.

2. By taking trace one obtains the already mentioned condition fixing the curvature to be constant, and one can write

$$\frac{dg_{\alpha\beta}}{dt} = kR_{\alpha\beta} - \Lambda g_{\alpha\beta} \ . \tag{2.5.16}$$

Note that in the recent case $R_{avg} = R$ holds true since curvature scalar is constant. The fixed points of the flow would be Einstein manifolds (see http://tinyurl.com/ybrnakuu) [A8, A75] satisfying

$$R_{\alpha\beta} = \frac{\Lambda}{k} g_{\alpha\beta} \tag{2.5.17}$$

.

3. It is by no means obvious that continuous flow is possible. The condition that Einstein-Maxwell equations are satisfied might pick up from a completely general Maxwell flow a discrete subset as solutions of Einstein-Maxwell equations with a cosmological term. If so, one could assign to this subset a sequence of values t_n of the flow parameter t.

4. I do not know whether 3-dimensionality is somehow absolutely essential for getting the topological classification of closed 3-manifolds using Ricci flow. This ignorance allows me to pose some innocent questions. Could one have a canonical representation of 4-geometries as spaces with constant Ricci scalar? Could one select one particular Einstein space in the class four-metrics and could the ratio Λ/k represent topological invariant if one normalizes metric or curvature scalar suitably. In the 3-dimensional case curvature scalar is normalized to unity. In the recent case this normalization would give $k = 4\Lambda$ in turn giving $R_{\alpha\beta} = g_{\alpha\beta}/4$. Does this mean that there is only single fixed point in local sense, analogous to black hole toward which all geometries are driven by the Maxwell flow? Does this imply that only the 4-volume of the original space would serve as a topological invariant?

Maxwell flow for space-time surfaces

One can consider Maxwell flow for space-time surfaces too. In this case Kähler flow would be the appropriate term and provides families of preferred extremals. Since space-time surfaces inside CD are the basic physical objects are in TGD framework, a possible interpretation of these families would be as flows describing physical dissipation as a four-dimensional phenomenon polishing details from the space-time surface interpreted as an analog of Bohr orbit.

1. The flow is now induced by a vector field $j^k(x,t)$ of the space-time surface having values in the tangent bundle of embedding space $M^4 \times CP_2$. In the most general case one has Kähler flow without the Einstein equations. This flow would be defined in the space of all space-time surfaces or possibly in the space of all extremals. The flow equations reduce to

$$h_{kl} D_\alpha j^k(x,t) D_\beta h^l = \frac{1}{2} T_{\alpha\beta} \ . \tag{2.5.18}$$

   The left hand side is the projection of the covariant gradient $D_\alpha j^k(x,t)$ of the flow vector field $j^k(x,t)$ to the tangent space of the space-time surface. D_{alpha} is covariant derivative taking into account that j^k is embedding space vector field. For a fixed point space-time surface this projection must vanish assuming that this space-time surface reachable. A good guess for the asymptotia is that the divergence of Maxwell energy momentum tensor vanishes and that Einstein's equations with cosmological constant are well-defined.

   Asymptotes corresponds to vacuum extremals. In Euclidian regions CP_2 type vacuum extremals and in Minkowskian regions to any space-time surface in any 6-D sub-manifold $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2 having therefore vanishing induced Kähler form. Symplectic transformations of CP_2 combined with diffeomorphisms of M^4 give new Lagrangian manifolds. One would expect that vacuum extremals are approached but never reached at second extreme for the flow.

   If one assumes Einstein's equations with a cosmological term, allowed vacuum extremals must be Einstein manifolds. For CP_2 type vacuum extremals this is the case. It is quite possible that these fixed points do not actually exist in Minkowskian sector, and could be replaced with more complex asymptotic behavior such as limit, chaos, or strange attractor.

2. The flow could be also restricted to the space of preferred extremals. Assuming that Einstein Maxwell equations indeed hold true, the flow equations reduce to

$$h_{kl}D_\alpha j^k(x,t)\partial_\beta h^l = \frac{1}{2}(kR_{\alpha\beta} - \Lambda g_{\alpha\beta}) \ . \tag{2.5.19}$$

   Preferred extremals would correspond to a fixed sub-manifold of the general flow in the space of all 4-surfaces.

3. One can also consider a situation in which $j^k(x,t)$ is replaced with $j^k(h,t)$ defining a flow in the entire embedding space. This assumption is probably too restrictive. In this case the equations reduce to

$$(D_r j_l(x,t) + D_l j_r)\partial_\alpha h^r \partial_\beta h^l = kR_{\alpha\beta} - \Lambda g_{\alpha\beta} \ . \tag{2.5.20}$$

   Here D_r denotes covariant derivative. Asymptotia is achieved if the tensor $D_k j_l + D_k j_l$ becomes orthogonal to the space-time surface. Note for that Killing vector fields of H the left hand side vanishes identically. Killing vector fields are indeed symmetries of also asymptotic states.

It must be made clear that the existence of a continuous flow in the space of preferred extremals might be too strong a condition. Already the restriction of the general Maxwell flow in the space of metrics to solutions of Einstein-Maxwell equations with cosmological term might lead to discretization, and the assumption about reprentability as 4-surface in $M^4 \times CP_2$ would give a further condition reducing the number of solutions. On the other hand, one might consiser a possibility of a continuous flow in the space of constant Ricci scalar metrics with a fixed 4-volume and having hyperbolic spaces as the most symmetric representative.

Dissipation, self organization, transition to chaos, and coupling constant evolution

A beautiful connection with concepts like dissipation, self-organization, transition to chaos, and coupling constant evolution suggests itself.

1. It is not at all clear whether the vacuum extremal limits of the preferred extremals can correspond to Einstein spaces except in special cases such as CP_2 type vacuum extremals isometric with CP_2. The imbeddability condition however defines a constraint force which might well force asymptotically more complex situations such as limit cycles and strange attractors. In ordinary dissipative dynamics an external energy feed is essential prerequisite for this kind of non-trivial self-organization patterns.

   In the recent case the external energy feed could be replaced by the constraint forces due to the imbeddability condition. It is not too difficult to imagine that the flow (if it exists!) could define something analogous to a transition to chaos taking place in a stepwise manner for critical values of the parameter t. Alternatively, these discrete values could correspond to those values of t for which the preferred extremal property holds true for a general Maxwell flow in the space of 4-metrics. Therefore the preferred extremals of Kähler action could emerge as one-parameter (possibly discrete) families describing dissipation and self-organization at the level of space-time dynamics.

2. For instance, one can consider the possibility that in some situations Einstein's equations split into two mutually consistent equations of which only the first one is independent

$$\begin{aligned} xJ^\alpha_{\ \nu}J^{\nu\beta} &= R^{\alpha\beta} \ , \\ L_K &= xJ^\alpha_{\ \nu}J^{\nu\beta} = 4\Lambda \ , \\ x &= \frac{1}{16\pi\alpha_K} \ . \end{aligned} \tag{2.5.21}$$

   Note that the first equation indeed gives the second one by tracing. This happens for CP_2 type vacuum extremals.

Kähler action density would reduce to cosmological constant which should have a continuous spectrum if this happens always. A more plausible alternative is that this holds true only asymptotically. In this case the flow equation could not lead arbitrary near to vacuum extremal, and one can think of situation in which $L_K = 4\Lambda$ defines an analog of limiting cycle or perhaps even strange attractor. In any case, the assumption would allow to deduce the asymptotic value of the action density which is of utmost importance from calculational point of view: action would be simply $S_K = 4\Lambda V_4$ and one could also say that one has minimal surface with Λ taking the role of string tension.

3. One of the key ideas of TGD is quantum criticality implying that Kähler coupling strength is analogous to critical temperature. Second key idea is that p-adic coupling constant evolution represents discretized version of continuous coupling constant evolution so that each p-adic prime would correspond a fixed point of ordinary coupling constant evolution in the sense that the 4-volume characterized by the p-adic length scale remains constant. The invariance of the geometric and thus geometric parameters of hyperbolic 4-manifold under the Kähler flow would conform with the interpretation as a flow preserving scale assignable to a given p-adic prime. The continuous evolution in question (if possible at all!) might correspond to a fixed p-adic prime. Also the hierarchy of Planck constants relates to this picture naturally. Planck constant $\hbar_{eff} = n\hbar$ corresponds to a multi-furcation generating n-sheeted structure and certainly affecting the fundamental group.
4. One can of course question the assumption that a continuous flow exists. The property of being a solution of Einstein-Maxwell equations, imbeddability property, and preferred extremal property might allow allow only discrete sequences of space-time surfaces perhaps interpretable as orbit of an iterated map leading gradually to a fractal limit. This kind of discrete sequence might be also be selected as preferred extremals from the orbit of Maxwell flow without assuming Einstein-Maxwell equations. Perhaps the discrete p-adic coupling constant evolution could be seen in this manner and be regarded as an iteration so that the connection with fractality would become obvious too.

Does a 4-D counterpart of thermodynamics make sense?

The interpretation of the Kähler flow in terms of dissipation, the constancy of R, and almost constancy of L_K suggest an interpretation in terms of 4-D variant of thermodynamics natural in zero energy ontology (ZEO), where physical states are analogs for pairs of initial and final states of quantum event are quantum superpositions of classical time evolutions. Quantum theory becomes a "square root" of thermodynamics so that 4-D analog of thermodynamics might even replace ordinary thermodynamics as a fundamental description. If so this 4-D thermodynamics should be qualitatively consistent with the ordinary 3-D thermodynamics.

1. The first naïve guess would be the interpretation of the action density L_K as an analog of energy density $e = E/V_3$ and that of R as the analog to entropy density $s = S/V_3$. The asymptotic states would be analogs of thermodynamical equilibria having constant values of L_K and R.
2. Apart from an overall sign factor ϵ to be discussed, the analog of the first law $de = Tds - pdV/V$ would be

$$dL_K = kdR + \Lambda\frac{dV_4}{V_4} \quad .$$

One would have the correspondences $S \to \epsilon RV_4$, $e \to \epsilon L_K$ and $k \to T$, $p \to -\Lambda$. $k \propto 1/G$ indeed appears formally in the role of temperature in Einstein's action defining a formal partition function via its exponent. The analog of second law would state the increase of the magnitude of ϵRV_4 during the Kähler flow.

3. One must be very careful with the signs and discuss Euclidian and Minkowskian regions separately. Concerning purely thermodynamic aspects at the level of vacuum functional Euclidian regions are those which matter.
   (a) For CP_2 type vacuum extremals $L_K \propto E^2 + B^2$, $R = \Lambda/k$, and Λ are positive. In thermodynamical analogy for $\epsilon = 1$ this would mean that pressure is negative.

(b) In Minkowskian regions the value of $R = \Lambda/k$ is negative for $\Lambda < 0$ suggested by the large abundance of 4-manifolds allowing hyperbolic metric and also by cosmological considerations. The asymptotic formula $L_K = 4\Lambda$ considered above suggests that also Kähler action is negative in Minkowskian regions for magnetic flux tubes dominating in TGD inspired cosmology: the reason is that the magnetic contribution to the action density $L_K \propto E^2 - B^2$ dominates.

Consider now in more detail the 4-D thermodynamics interpretation in Euclidian and Minkowskian regions assuming that the evolution by quantum jumps has Kähler flow as a space-time correlate.

1. In Euclidian regions the choice $\epsilon = 1$ seems to be more reasonable one. In Euclidian regions $-\Lambda$ as the analog of pressure would be negative, and asymptotically (that is for CP_2 type vacuum extremals) its value would be proportional to $\Lambda \propto 1/GR^2$, where R denotes CP_2 radius defined by the length of its geodesic circle.

   A possible interpretation for negative pressure is in terms of string tension effectively inducing negative pressure (note that the solutions of the Kähler-Dirac equation indeed assign a string to the wormhole contact). The analog of the second law would require the increase of RV_4 in quantum jumps. The magnitudes of L_K, R, V_4 and Λ would be reduced and approach their asymptotic values. In particular, V_4 would approach asymptotically the volume of CP_2.

2. In Minkowskian regions Kähler action contributes to the vacuum functional a phase factor analogous to an imaginary exponent of action serving in the role of Morse function so that thermodynamics interpretation can be questioned. Despite this one can check whether thermodynamic interpretation can be considered. The choice $\epsilon = -1$ seems to be the correct choice now. $-\Lambda$ would be analogous to a negative pressure whose gradually decreases. In 3-D thermodynamics it is natural to assign negative pressure to the magnetic flux tube like structures as their effective string tension defined by the density of magnetic energy per unit length. $-R \geq 0$ would entropy and $-L_K \geq 0$ would be the analog of energy density.

   $R = \Lambda/k$ and the reduction of Λ during cosmic evolution by quantum jumps suggests that the larger the volume of CD and thus of (at least) Minkowskian space-time sheet the smaller the negative value of Λ.

   Assume the recent view about state function reduction explaining how the arrow of geometric time is induced by the quantum jump sequence defining experienced time [K4]. According to this view zero energy states are quantum superpositions over CDs of various size scales but with common tip, which can correspond to either the upper or lower light-like boundary of CD. The sequence of quantum jumps the gradual increase of the average size of CD in the quantum superposition and therefore that of average value of V_4. On the other hand, a gradual decrease of both $-L_K$ and $-R$ looks physically very natural. If Kähler flow describes the effect of dissipation by quantum jumps in ZEO then the space-time surfaces would gradually approach nearly vacuum extremals with constant value of entropy density $-R$ but gradually increasing 4-volume so that the analog of second law stating the increase of $-RV_4$ would hold true.

3. The interpretation of $-R > 0$ as negentropy density assignable to entanglement is also possible and is consistent with the interpretation in terms of second law. This interpretation would only change the sign factor ϵ in the proposed formula. Otherwise the above arguments would remain as such.

2.5.4 Could Correlation Functions, S-Matrix, And Coupling Constant Evolution Be Coded The Statistical Properties Of Preferred Extremals?

How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge. Generalized Feynman diagrams provide a powerful vision which however does not help in practical calculations. Some big idea has been lacking.

Quantum classical correspondence states that all aspects of quantum states should have correlates in the geometry of preferred extremals. In particular, various elementary particle propagators should have a representation as properties of preferred extremals. This would allow to realize the old dream about being able to say something interesting about coupling constant evolution although it is not yet possible to calculate the M-matrices and U-matrix. The general structure of U-matrix is however understood [K53]. Hitherto everything that has been said about coupling constant evolution has been rather speculative arguments except for the general vision that it reduces to a discrete evolution defined by p-adic length scales. General first principle definitions are however much more valuable than ad hoc guesses even if the latter give rise to explicit formulas.

In quantum TGD and also at its QFT limit various correlation functions in given quantum state should code for its properties. By quantum classical correspondence these correlation functions should have counterparts in the geometry of preferred extremals. Even more: these classical counterparts for a given preferred extremal ought to be identical with the quantum correlation functions for the superposition of preferred extremals. This correspondence could be called quantum ergodicity by its analogy with ordinary ergodicity stating that the member of ensemble becomes representative of ensemble.

This principle would be a quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This symmetry principle analogous to holography might allow to fix S-matrix uniquely even in the case that the hermitian square root of the density matrix appearing in the M-matrix would lead to a breaking of quantum ergodicity as also 4-D spin glass degeneracy suggests.

This principle would allow to deduce correlation functions from the statistical properties of single preferred extremal alone using just classical intuition. Also coupling constant evolution would be coded by the statistical properties of preferred extremals. Quantum ergodicity would mean an enormous simplification since one could avoid the horrible conceptual complexities involved with the functional integrals over WCW .

This might of course be too optimistic guess. If a sub-algebra of symplectic algebra acts as gauge symmmetries of the preferred extremals in the sense that corresponding Noether charges vanish, it can quite well be that correlations functions correspond to averages for extremals belonging to single conformal equivalence class.

1. The marvellous implication of quantum ergodicity would be that one could calculate everything solely classically using the classical intuition - the only intuition that we have. Quantum ergodicity would also solve the paradox raised by the quantum classical correspondence for momentum eigenstates. Any preferred extremal in their superposition defining momentum eigenstate should code for the momentum characterizing the superposition itself. This is indeed possible if every extremal in the superposition codes the momentum to the properties of classical correlation functions which are identical for all of them.

2. The only manner to possibly achieve quantum ergodicity is in terms of the statistical properties of the preferred extremals. It should be possible to generalize the ergodic theorem stating that the properties of statistical ensemble are represented by single space-time evolution in the ensemble of time evolutions. Quantum superposition of classical worlds would effectively reduce to single classical world as far as classical correlation functions are considered. The notion of finite measurement resolution suggests that one must state this more precisely by adding that classical correlation functions are calculated in a given UV and IR resolutions meaning UV cutoff defined by the smallest CD and IR cutoff defined by the largest CD present.

3. The skeptic inside me immediately argues that TGD Universe is 4-D spin glass so that this quantum ergodic theorem must be broken. In the case of the ordinary spin classes one has not only statistical average for a fixed Hamiltonian but a statistical average over Hamiltonians. There is a probability distribution over the coupling parameters appearing in the Hamiltonian. Maybe the quantum counterpart of this is needed to predict the physically measurable correlation functions.

   Could this average be an ordinary classical statistical average over quantum states with different classical correlation functions? This kind of average is indeed taken in density matrix formalism. Or could it be that the square root of thermodynamics defined by ZEO actually gives automatically rise to this average? The eigenvalues of the “hermitian square root” of the density matrix would code for components of the state characterized by different classical correlation functions. One could assign these contributions to different “phases”.

4. Quantum classical correspondence in statistical sense would be very much like holography (now individual classical state represents the entire quantum state). Quantum ergodicity would pose a rather strong constraint on quantum states. This symmetry principle could actually fix the spectrum of zero energy states to a high degree and fix therefore the M-matrices given by the product of hermitian square root of density matrix and unitary S-matrix and unitary U-matrix constructible as inner products of M-matrices associated with CDs with various size scales [K53].
5. In TGD inspired theory of consciousness the counterpart of quantum ergodicity is the postulate that the space-time geometry provides a symbolic representation for the quantum states and also for the contents of consciousness assignable to quantum jumps between quantum states. Quantum ergodicity would realize this strongly self-referential looking condition. The positive and negative energy parts of zero energy state would be analogous to the initial and final states of quantum jump and the classical correlation functions would code for the contents of consciousness like written formulas code for the thoughts of mathematician and provide a sensory feedback.

How classical correlation functions should be defined?

1. General Coordinate Invariance and Lorentz invariance are the basic constraints on the definition. These are achieved for the space-time regions with Minkowskian signature and 4-D M^4 projection if linear Minkowski coordinates are used. This is equivalent with the contraction of the indices of tensor fields with the space-time projections of M^4 Killing vector fields representing translations. Accepting ths generalization, there is no need to restrict oneself to 4-D M^4 projection and one can also consider also Euclidian regions identifiable as lines of generalized Feynman diagrams.

   Quantum ergodicity very probably however forces to restrict the consideration to Minkowskian and Euclidian space-time regions and various phases associated with them. Also CP_2 Killing vector fields can be projected to space-time surface and give a representation for classical gluon fields. These in turn can be contracted with M^4 Killing vectors giving rise to gluon fields as analogs of graviton fields but with second polarization index replaced with color index.
2. The standard definition for the correlation functions associated with classical time evolution is the appropriate starting point. The correlation function $G_{XY}(\tau)$ for two dynamical variables $X(t)$ and $Y(t)$ is defined as the average $G_{XY}(\tau) = \int_T X(t)Y(t+\tau)dt/T$ over an interval of length T, and one can also consider the limit $T \to \infty$. In the recent case one would replace τ with the difference $m_1 - m_2 = m$ of M^4 coordinates of two points at the preferred extremal and integrate over the points of the extremal to get the average. The finite time interval T is replaced with the volume of causal diamond in a given length scale. Zero energy state with given quantum numbers for positive and negative energy parts of the state defines the initial and final states between which the fields appearing in the correlation functions are defined.
3. What correlation functions should be considered? Certainly one could calculate correlation functions for the induced spinor connection given electro-weak propagators and correlation functions for CP_2 Killing vector fields giving correlation functions for gluon fields using the description in terms of Killing vector fields. If one can uniquely separate from the Fourier transform uniquely a term of form $Z/(p^2 - m^2)$ by its momentum dependence, the coefficient Z can be identified as coupling constant squared for the corresponding gauge potential component and one can in principle deduce coupling constant evolution purely classically. One can imagine of calculating spinorial propagators for string world sheets in the same manner. Note that also the dependence on color quantum numbers would be present so that in principle all that is needed could be calculated for a single preferred extremal without the need to construct QFT limit and to introduce color quantum numbers of fermions as spin like quantum numbers (color quantum numbers corresponds to CP_2 partial wave for the tip of the CD assigned with the particle).

Many detailed speculations about coupling constant evolution to be discussed in the sections below must be taken as innovative guesses doomed to have the eventual fate of guesses. The notion of quantum ergodicity could however be one of the really deep ideas about coupling constant evolution comparable to the notion of p-adic coupling constant evolution. Quantum Ergodicity (briefly

QE) would also state something extremely non-trivial also about the construction of correlation functions and S-matrix. Because this principle is so new, the rest of the chapter does not yet contain any applications of QE. This should not lead the reader to under-estimate the potential power of QE.

2.6 About Deformations Of Known Extremals Of Kähler Action

I have done a considerable amount of speculative guesswork to identify what I have used to call preferred extremals of Kähler action. The difficulty is that the mathematical problem at hand is extremely non-linear and that I do not know about existing mathematical literature relevant to the situation. One must proceed by trying to guess the general constraints on the preferred extremals which look physically and mathematically plausible. The hope is that this net of constraints could eventually chrystallize to Eureka! Certainly the recent speculative picture involves also wrong guesses. The need to find explicit ansatz for the deformations of known extremals based on some common principles has become pressing. The following considerations represent an attempt to combine the existing information to achieve this.

2.6.1 What Might Be The Common Features Of The Deformations Of Known Extremals

The dream is to discover the deformations of all known extremals by guessing what is common to all of them. One might hope that the following list summarizes at least some common features.

Effective three-dimensionality at the level of action

1. Holography realized as effective 3-dimensionality also at the level of action requires that it reduces to 3-dimensional effective boundary terms. This is achieved if the contraction $j^\alpha A_\alpha$ vanishes. This is true if j^α vanishes or is light-like, or if it is proportional to instanton current in which case current conservation requires that CP_2 projection of the space-time surface is 3-dimensional. The first two options for j have a realization for known extremals. The status of the third option - proportionality to instanton current - has remained unclear.
2. As I started to work again with the problem, I realized that instanton current could be replaced with a more general current $j = *B \wedge J$ or concretely: $j^\alpha = \epsilon^{\alpha\beta\gamma\delta} B_\beta J_{\gamma\delta}$, where B is vector field and CP_2 projection is 3-dimensional, which it must be in any case. The contractions of j appearing in field equations vanish automatically with this ansatz.
3. Almost topological QFT property in turn requires the reduction of effective boundary terms to Chern-Simons terms: this is achieved by boundary conditions expressing weak form of electric magnetic duality. If one generalizes the weak form of electric-magnetic duality to $J = \Phi * J$ one has $B = d\Phi$ and j has a vanishing divergence for 3-D CP_2 projection. This is clearly a more general solution ansatz than the one based on proportionality of j with instanton current and would reduce the field equations in concise notation to $Tr(TH^k) = 0$.
4. Any of the alternative properties of the Kähler current implies that the field equations reduce to $Tr(TH^k) = 0$, where T and H^k are shorthands for Maxwellian energy momentum tensor and second fundamental form and the product of tensors is obvious generalization of matrix product involving index contraction.

Could Einstein's equations emerge dynamically?

For j^α satisfying one of the three conditions, the field equations have the same form as the equations for minimal surfaces except that the metric g is replaced with Maxwell energy momentum tensor T.

1. This raises the question about dynamical generation of small cosmological constant Λ: $T = \Lambda g$ would reduce equations to those for minimal surfaces. For $T = \Lambda g$ Kähler-Dirac gamma matrices would reduce to induced gamma matrices and the Kähler-Dirac operator would be

proportional to ordinary Dirac operator defined by the induced gamma matrices. One can also consider weak form for $T = \Lambda g$ obtained by restricting the consideration to a sub-space of tangent space so that space-time surface is only "partially" minimal surface but this option is not so elegant although necessary for other than CP_2 type vacuum extremals.

2. What is remarkable is that $T = \Lambda g$ implies that the divergence of T which in the general case equals to $j^\beta J^\alpha_\beta$ vanishes. This is guaranteed by one of the conditions for the Kähler current. Since also Einstein tensor has a vanishing divergence, one can ask whether the condition to $T = \kappa G + \Lambda g$ could the general condition. This would give Einstein's equations with cosmological term besides the generalization of the minimal surface equations. GRT would emerge dynamically from the non-linear Maxwell's theory although in slightly different sense as conjectured [K90] ! Note that the expression for G involves also second derivatives of the embedding space coordinates so that actually a partial differential equation is in question. If field equations reduce to purely algebraic ones, as the basic conjecture states, it is possible to have $Tr(GH^k) = 0$ and $Tr(gH^k) = 0$ separately so that also minimal surface equations would hold true.

   What is amusing that the first guess for the action of TGD was curvature scalar. It gave analogs of Einstein's equations as a definition of conserved four-momentum currents. The recent proposal would give the analog of ordinary Einstein equations as a dynamical constraint relating Maxwellian energy momentum tensor to Einstein tensor and metric.

3. Minimal surface property is physically extremely nice since field equations can be interpreted as a non-linear generalization of massless wave equation: something very natural for non-linear variant of Maxwell action. The theory would be also very "stringy" although the fundamental action would not be space-time volume. This can however hold true only for Euclidian signature. Note that for CP_2 type vacuum extremals Einstein tensor is proportional to metric so that for them the two options are equivalent. For their small deformations situation changes and it might happen that the presence of G is necessary. The GRT limit of TGD discussed in [K90] [L20] indeed suggests that CP_2 type solutions satisfy Einstein's equations with large cosmological constant and that the small observed value of the cosmological constant is due to averaging and small volume fraction of regions of Euclidian signature (lines of generalized Feynman diagrams).

4. For massless extremals and their deformations $T = \Lambda g$ cannot hold true. The reason is that for massless extremals energy momentum tensor has component T^{vv} which actually quite essential for field equations since one has $H^k_{vv} = 0$. Hence for massless extremals and their deformations $T = \Lambda g$ cannot hold true if the induced metric has Hamilton-Jacobi structure meaning that g^{uu} and g^{vv} vanish. A more general relationship of form $T = \kappa G + \Lambda G$ can however be consistent with non-vanishing T^{vv} but require that deformation has at most 3-D CP_2 projection (CP_2 coordinates do not depend on v).

5. The non-determinism of vacuum extremals suggest for their non-vacuum deformations a conflict with the conservation laws. In, also massless extremals are characterized by a non-determinism with respect to the light-like coordinate but like-likeness saves the situation. This suggests that the transformation of a properly chosen time coordinate of vacuum extremal to a light-like coordinate in the induced metric combined with Einstein's equations in the induced metric of the deformation could allow to handle the non-determinism.

Are complex structure of CP_2 and Hamilton-Jacobi structure of M^4 respected by the deformations?

The complex structure of CP_2 and Hamilton-Jacobi structure of M^4 could be central for the understanding of the preferred extremal property algebraically.

1. There are reasons to believe that the Hermitian structure of the induced metric ((1, 1) structure in complex coordinates) for the deformations of CP_2 type vacuum extremals could be crucial property of the preferred extremals. Also the presence of light-like direction is also an essential elements and 3-dimensionality of M^4 projection could be essential. Hence a good guess is that allowed deformations of CP_2 type vacuum extremals are such that (2, 0) and (0, 2) components the induced metric and/or of the energy momentum tensor vanish. This gives rise to the conditions implying Virasoro conditions in string models in quantization:

$$g_{\xi^i\xi^j} = 0 \ , \quad g_{\overline{\xi}^i\overline{\xi}^j} = 0 \ , \quad i,j = 1,2 \ . \tag{2.6.1}$$

Holomorphisms of CP_2 preserve the complex structure and Virasoro conditions are expected to generalize to 4-dimensional conditions involving two complex coordinates. This means that the generators have two integer valued indices but otherwise obey an algebra very similar to the Virasoro algebra. Also the super-conformal variant of this algebra is expected to make sense.

These Virasoro conditions apply in the coordinate space for CP_2 type vacuum extremals. One expects similar conditions hold true also in field space, that is for M^4 coordinates.

2. The integrable decomposition $M^4(m) = M^2(m) + E^2(m)$ of M^4 tangent space to longitudinal and transversal parts (non-physical and physical polarizations) - Hamilton-Jacobi structure- could be a very general property of preferred extremals and very natural since non-linear Maxwellian electrodynamics is in question. This decomposition led rather early to the introduction of the analog of complex structure in terms of what I called Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ for M^4. (u, v) defines a pair of light-like coordinates for the local longitudinal space $M^2(m)$ and $(w, \overline{w})$ complex coordinates for $E^2(m)$. The metric would not contain any cross terms between $M^2(m)$ and $E^2(m)$: $g_{uw} = g_{vw} = g_{u\overline{w}} = g_{v\overline{w}} = 0$.

   A good guess is that the deformations of massless extremals respect this structure. This condition gives rise to the analog of the constraints leading to Virasoro conditions stating the vanishing of the non-allowed components of the induced metric. $g_{uu} = g_{vv} = g_{ww} = g_{\overline{w}\overline{w}} = g_{uw} = g_{vw} = g_{u\overline{w}} = g_{v\overline{w}} = 0$. Again the generators of the algebra would involve two integers and the structure is that of Virasoro algebra and also generalization to super algebra is expected to make sense. The moduli space of Hamilton-Jacobi structures would be part of the moduli space of the preferred extremals and analogous to the space of all possible choices of complex coordinates. The analogs of infinitesimal holomorphic transformations would preserve the modular parameters and give rise to a 4-dimensional Minkowskian analog of Virasoro algebra. The conformal algebra acting on CP_2 coordinates acts in field degrees of freedom for Minkowskian signature.

Field equations as purely algebraic conditions

If the proposed picture is correct, field equations would reduce basically to purely algebraically conditions stating that the Maxwellian energy momentum tensor has no common index pairs with the second fundamental form. For the deformations of CP_2 type vacuum extremals T is a complex tensor of type (1, 1) and second fundamental form H^k a tensor of type (2, 0) and (0, 2) so that $Tr(TH^k) =$ is true. This requires that second light-like coordinate of M^4 is constant so that the M^4 projection is 3-dimensional. For Minkowskian signature of the induced metric Hamilton-Jacobi structure replaces conformal structure. Here the dependence of CP_2 coordinates on second light-like coordinate of $M^2(m)$ only plays a fundamental role. Note that now T^{vv} is non-vanishing (and light-like). This picture generalizes to the deformations of cosmic strings and even to the case of vacuum extremals.

2.6.2 What Small Deformations Of CP_2 Type Vacuum Extremals Could Be?

I was led to these arguments when I tried find preferred extremals of Kähler action, which would have 4-D CP_2 and M^4 projections - the Maxwell phase analogous to the solutions of Maxwell's equations that I conjectured long time ago. It however turned out that the dimensions of the projections can be $(D_{M^4} \leq 3, D_{CP_2} = 4)$ or $(D_{M^4} = 4, D_{CP_2} \leq 3)$. What happens is essentially breakdown of linear superposition so that locally one can have superposition of modes which have 4-D wave vectors in the same direction. This is actually very much like quantization of radiation field to photons now represented as separate space-time sheets and one can say that Maxwellian superposition corresponds to union of separate photonic space-time sheets in TGD.

Approximate linear superposition of fields is fundamental in standard physics framework and is replaced in TGD with a linear superposition of effects of classical fields on a test particle topologically condensed simultaneously to several space-time sheets. One can say that linear superposition is replaced with a disjoint union of space-time sheets. In the following I shall restrict the consideration to the deformations of CP_2 type vacuum extremals.

Solution ansatz

I proceed by the following arguments to the ansatz.

1. Effective 3-dimensionality for action (holography) requires that action decomposes to vanishing $j^\alpha A_\alpha$ term + total divergence giving 3-D "boundary" terms. The first term certainly vanishes (giving effective 3-dimensionality) for

$$D_\beta J^{\alpha\beta} = j^\alpha = 0 \ .$$

   Empty space Maxwell equations, something extremely natural. Also for the proposed GRT limit these equations are true.

2. How to obtain empty space Maxwell equations $j^\alpha = 0$? The answer is simple: assume self duality or its slight modification:

$$J = *J$$

   holding for CP_2 type vacuum extremals or a more general condition

$$J = k * J \ ,$$

   In the simplest situation k is some constant not far from unity. * is Hodge dual involving 4-D permutation symbol. $k = \mathit{constant}$ requires that the determinant of the induced metric is apart from constant equal to that of CP_2 metric. It does not require that the induced metric is proportional to the CP_2 metric, which is not possible since M^4 contribution to metric has Minkowskian signature and cannot be therefore proportional to CP_2 metric.

   One can consider also a more general situation in which k is scalar function as a generalization of the weak electric-magnetic duality. In this case the Kähler current is non-vanishing but divergenceless. This also guarantees the reduction to $Tr(TH^k) = 0$. In this case however the proportionality of the metric determinant to that for CP_2 metric is not needed. This solution ansatz becomes therefore more general.

3. Field equations reduce with these assumptions to equations differing from minimal surfaces equations only in that metric g is replaced by Maxwellian energy momentum tensor T. Schematically:

$$Tr(TH^k) = 0 \ ,$$

   where T is the Maxwellian energy momentum tensor and H^k is the second fundamental form - asymmetric 2-tensor defined by covariant derivative of gradients of embedding space coordinates.

How to satisfy the condition $Tr(TH^k) = 0$?

It would be nice to have minimal surface equations since they are the non-linear generalization of massless wave equations. It would be also nice to have the vanishing of the terms involving Kähler current in field equations as a consequence of this condition. Indeed, $T = \kappa G + \Lambda g$ implies this. In the case of CP_2 vacuum extremals one cannot distinguish between these options since CP_2 itself is constant curvature space with $G \propto g$. Furthermore, if G and g have similar tensor structure the algebraic field equations for G and g are satisfied separately so that one obtains minimal surface property also now. In the following minimal surface option is considered.

1. The first opton is achieved if one has

$$T = \Lambda g \ .$$

Maxwell energy momentum tensor would be proportional to the metric! One would have dynamically generated cosmological constant! This begins to look really interesting since it appeared also at the proposed GRT limit of TGD [L20] (see `http://tinyurl.com/hzkldnb`). Note that here also non-constant value of Λ can be considered and would correspond to a situation in which k is scalar function: in this case the the determinant condition can be dropped and one obtains just the minimal surface equations.

2. Very schematically and forgetting indices and being sloppy with signs, the expression for T reads as

$$T = JJ - g/4Tr(JJ) \ .$$

Note that the product of tensors is obtained by generalizing matrix product. This should be proportional to metric.

Self duality implies that $Tr(JJ)$ is just the instanton density and does not depend on metric and is constant.

For CP_2 type vacuum extremals one obtains

$$T = -g + g = 0 \ .$$

Cosmological constant would vanish in this case.

3. Could it happen that for deformations a small value of cosmological constant is generated? The condition would reduce to

$$JJ = (\Lambda - 1)g \ .$$

Λ must relate to the value of parameter k appearing in the generalized self-duality condition. For the most general ansatz Λ would not be constant anymore.

This would generalize the defining condition for Kähler form

$$JJ = -g \ \ (i^2 = -1 \ \ geometrically)$$

stating that the square of Kähler form is the negative of metric. The only modification would be that index raising is carried out by using the induced metric containing also M^4 contribution rather than CP_2 metric.

4. Explicitly:

$$J_{\alpha\mu}J^{\mu}_{\ \beta} = (\Lambda - 1)g_{\alpha\beta} \ .$$

Cosmological constant would measure the breaking of Kähler structure. By writing $g = s + m$ and defining index raising of tensors using CP_2 metric and their product accordingly, this condition can be also written as

$$Jm = (\Lambda - 1)mJ \ .$$

If the parameter k is constant, the determinant of the induced metric must be proportional to the CP_2 metric. If k is scalar function, this condition can be dropped. Cosmological constant would not be constant anymore but the dependence on k would drop out from the field equations and one would hope of obtaining minimal surface equations also now. It however seems that the dimension of M^4 projection cannot be four. For 4-D M^4 projection the contribution of the M^2 part of the M^4 metric gives a non-holomorphic contribution to CP_2 metric and this spoils the field equations.

For $T = \kappa G + \Lambda g$ option the value of the cosmological constant is large - just as it is for the proposed GRT limit of TGD [K90] [L20]. The interpretation in this case is that the average value of cosmological constant is small since the portion of space-time volume containing generalized Feynman diagrams is very small.

More detailed ansatz for the deformations of CP_2 type vacuum extremals

One can develop the ansatz to a more detailed form. The most obvious guess is that the induced metric is apart from constant conformal factor the metric of CP_2. This would guarantee self-duality apart from constant factor and $j^\alpha = 0$. Metric would be in complex CP_2 coordinates tensor of type (1, 1) whereas CP_2 Riemann connection would have only purely holomorphic or anti-holomorphic indices. Therefore CP_2 contributions in $Tr(TH^k)$ would vanish identically. M^4 degrees of freedom however bring in difficulty. The M^4 contribution to the induced metric should be proportional to CP_2 metric and this is impossible due to the different signatures. The M^4 contribution to the induced metric breaks its Kähler property but would preserve Hermitian structure.

A more realistic guess based on the attempt to construct deformations of CP_2 type vacuum extremals is following.

1. Physical intuition suggests that M^4 coordinates can be chosen so that one has integrable decomposition to longitudinal degrees of freedom parametrized by two light-like coordinates u and v and to transversal polarization degrees of freedom parametrized by complex coordinate w and its conjugate. M^4 metric would reduce in these coordinates to a direct sum of longitudinal and transverse parts. I have called these coordinates Hamilton-Jacobi coordinates.
2. w would be holomorphic function of CP_2 coordinates and therefore satisfy the analog of massless wave equation. This would give hopes about rather general solution ansatz. u and v cannot be holomorphic functions of CP_2 coordinates. Unless wither u or v is constant, the induced metric would receive contributions of type (2, 0) and (0, 2) coming from u and v which would break Kähler structure and complex structure. These contributions would give no-vanishing contribution to all minimal surface equations. Therefore either u or v is constant: the coordinate line for non-constant coordinate -say u- would be analogous to the M^4 projection of CP_2 type vacuum extremal.
3. With these assumptions the induced metric would remain $(1, 1)$ tensor and one might hope that $Tr(TH^k)$ contractions vanishes for all variables except u because the there are no common index pairs (this if non-vanishing Christoffel symbols for H involve only holomorphic or anti-holomorphic indices in CP_2 coordinates). For u one would obtain massless wave equation expressing the minimal surface property.
4. If the value of k is constant the determinant of the induced metric must be proportional to the determinant of CP_2 metric. The induced metric would contain only the contribution from the transversal degrees of freedom besides CP_2 contribution. Minkowski contribution has however rank 2 as CP_2 tensor and cannot be proportional to CP_2 metric. It is however enough that its determinant is proportional to the determinant of CP_2 metric with constant proportionality coefficient. This condition gives an additional non-linear condition to the solution. One would have wave equation for u (also w and its conjugate satisfy massless wave equation) and determinant condition as an additional condition.

   The determinant condition reduces by the linearity of determinant with respect to its rows to sum of conditions involved 0, 1, 2 rows replaced by the transversal M^4 contribution to metric given if M^4 metric decomposes to direct sum of longitudinal and transversal parts. Derivatives with respect to derivative with respect to particular CP_2 complex coordinate appear linearly in this expression they can depend on u via the dependence of transversal metric components on u. The challenge is to show that this equation has (or does not have) non-trivial solutions.
5. If the value of k is scalar function the situation changes and one has only the minimal surface equations and Virasoro conditions.

What makes the ansatz attractive is that special solutions of Maxwell empty space equations are in question, equations reduces to non-linear generalizations of Euclidian massless wave equations, and possibly space-time dependent cosmological constant pops up dynamically. These properties are true also for the GRT limit of TGD [L20] (see `http://tinyurl.com/hzkldnb`).

2.6.3 Hamilton-Jacobi Conditions In Minkowskian Signature

The maximally optimistic guess is that the basic properties of the deformations of CP_2 type vacuum extremals generalize to the deformations of other known extremals such as massless extremals, vacuum extremals with 2-D CP_2 projection which is Lagrangian manifold, and cosmic strings characterized by Minkowskian signature of the induced metric. These properties would be following.

1. The recomposition of M^4 tangent space to longitudinal and transversal parts giving Hamilton-Jacobi structure. The longitudinal part has hypercomplex structure but the second light-like coordinate is constant: this plays a crucial role in guaranteeing the vanishing of contractions in $Tr(TH^k)$. It is the algebraic properties of g and T which are crucial. T can however have light-like component T^{vv}. For the deformations of CP_2 type vacuum extremals $(1,1)$ structure is enough and is guaranteed if second light-like coordinate of M^4 is constant whereas w is holomorphic function of CP_2 coordinates.
2. What could happen in the case of massless extremals? Now one has 2-D CP_2 projection in the initial situation and CP_2 coordinates depend on light-like coordinate u and single real transversal coordinate. The generalization would be obvious: dependence on single light-like coordinate u and holomorphic dependence on w for complex CP_2 coordinates. The constraint is $T = \Lambda g$ cannot hold true since T^{vv} is non-vanishing (and light-like). This property restricted to transversal degrees of freedom could reduce the field equations to minimal surface equations in transversal degrees of freedom. The transversal part of energy momentum tensor would be proportional to metric and hence covariantly constant. Gauge current would remain light-like but would not be given by $j = *d\phi \wedge J$. $T = \kappa G + \Lambda g$ seems to define the attractive option.

It therefore seems that the essential ingredient could be the condition

$$T = \kappa G + \lambda g \ ,$$

which has structure (1, 1) in both $M^2(m)$ and $E^2(m)$ degrees of freedom apart from the presence of T^{vv} component with deformations having no dependence on v. If the second fundamental form has (2, 0)+(0, 2) structure, the minimal surface equations are satisfied provided Kähler current satisfies on of the proposed three conditions and if G and g have similar tensor structure.

One can actually pose the conditions of metric as complete analogs of stringy constraints leading to Virasoro conditions in quantization to give

$$g_{uu} = 0 \ , \quad g_{vv} = 0 \ , \quad g_{ww} = 0 \ , \quad g_{\overline{w}\overline{w}} = 0 \ . \tag{2.6.2}$$

This brings in mind the generalization of Virasoro algebra to four-dimensional algebra for which an identification in terms of non-local Yangian symmetry [A36] [B33, B24, B25] has been proposed [K86]. The number of conditions is four and the same as the number of independent field equations. One can consider similar conditions also for the energy momentum tensor T but allowing non-vanishing component T^{vv} if deformations has no v-dependence. This would solve the field equations if the gauge current vanishes or is light-like. On this case the number of equations is 8. First order differential equations are in question and they can be also interpreted as conditions fixing the coordinates used since there is infinite number of ways to choose the Hamilton-Jacobi coordinates.

One can can try to apply the physical intuition about general solutions of field equations in the linear case by writing the solution as a superposition of left and right propagating solutions:

$$\xi^k = f_+^k(u,w) + f_+^k(v,w) \ . \tag{2.6.3}$$

This could guarantee that second fundamental form is of form (2, 0)+(0, 2) in both M^2 and E^2 part of the tangent space and these terms if $Tr(TH^k)$ vanish identically. The remaining terms involve contractions of T^{uw}, $T^{u\overline{w}}$ and T^{vw}, $T^{v\overline{w}}$ with second fundamental form. Also these terms should sum up to zero or vanish separately. Second fundamental form has components coming from f_+^k and f_-^k

Second fundamental form H^k has as basic building bricks terms $\hat{H}^k$ given by

$$\hat{H}^k_{\alpha\beta} = \partial_\alpha\partial_\beta h^k + \left({}_{l}{}^{k}{}_{m}\right) \partial_\alpha h^l \partial_\beta h^m \ . \tag{2.6.4}$$

For the proposed ansatz the first terms give vanishing contribution to H^k_{uv}. The terms containing Christoffel symbols however give a non-vanishing contribution and one can allow only f^k_+ or f^k_- as in the case of massless extremals. This reduces the dimension of CP_2 projection to $D = 3$.

What about the condition for Kähler current? Kähler form has components of type $J_{w\overline{w}}$ whose contravariant counterpart gives rise to space-like current component. J_{uw} and $J_{u\overline{w}}$ give rise to light-like currents components. The condition would state that the $J^{w\overline{w}}$ is covariantly constant. Solutions would be characterized by a constant Kähler magnetic field. Also electric field is represent. The interpretation both radiation and magnetic flux tube makes sense.

2.6.4 Deformations Of Cosmic Strings

In the physical applications it has been assumed that the thickening of cosmic strings to Kähler magnetic flux tubes takes place. One indeed expects that the proposed construction generalizes also to the case of cosmic strings having the decomposition $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 a complex homologically non-trivial sub-manifold of CP_2. Now the starting point structure is Hamilton-Jacobi structure for $M^2_m \times Y^2$ defining the coordinate space.

1. The deformation should increase the dimension of either CP_2 or M^4 projection or both. How this thickening could take place? What comes in mind that the string orbits X^2 can be interpreted as a distribution of longitudinal spaces $M^2(x)$ so that for the deformation w coordinate becomes a holomorphic function of the natural Y^2 complex coordinate so that M^4 projection becomes 4-D but CP_2 projection remains 2-D. The new contribution to the X^2 part of the induced metric is vanishing and the contribution to the Y^2 part is of type $(1,1)$ and the ansatz $T = \kappa G + \Lambda g$ might be needed as a generalization of the minimal surface equations The ratio of κ and G would be determined from the form of the Maxwellian energy momentum tensor and be fixed at the limit of undeformed cosmic strong to $T = (ag(Y^2) - bg(Y^2)$. The value of cosmological constant is now large, and overall consistency suggests that $T = \kappa G + \Lambda g$ is the correct option also for the CP_2 type vacuum extremals.
2. One could also imagine that remaining CP_2 coordinates could depend on the complex coordinate of Y^2 so that also CP_2 projection would become 4-dimensional. The induced metric would receive holomorphic contributions in Y^2 part. As a matter fact, this option is already implied by the assumption that Y^2 is a complex surface of CP_2.

2.6.5 Deformations Of Vacuum Extremals?

What about the deformations of vacuum extremals representable as maps from M^4 to CP_2?

1. The basic challenge is the non-determinism of the vacuum extremals. One should perform the deformation so that conservation laws are satisfied. For massless extremals there is also non-determinism but it is associated with the light-like coordinate so that there are no problems with the conservation laws. This would suggest that a properly chosen time coordinate consistent with Hamilton-Jacobi decomposition becomes light-like coordinate in the induced metric. This poses a conditions on the induced metric.
2. Physical intuition suggests that one cannot require $T = \Lambda g$ since this would mean that the rank of T is maximal whereas the original situation corresponds to the vanishing of T. For small deformations rank two for T looks more natural and one could think that T is proportional to a projection of metric to a 2-D subspace. The vision about the long length scale limit of TGD is that Einstein's equations are satisfied and this would suggest $T = kG$ or $T = \kappa G + \Lambda g$. The rank of T could be smaller than four for this ansatz and this conditions binds together the values of κ and G.
3. These extremals have CP_2 projection which in the generic case is 2-D Lagrangian sub-manifold Y^2. Again one could assume Hamilton-Jacobi coordinates for X^4. For CP_2 one could assume Darboux coordinates (P_i, Q_i), $i = 1,2$, in which one has $A = P_i dQ^i$, and that $Y^2 \subset CP_2$ corresponds to $Q_i = constant$. In principle P_i would depend on arbitrary manner on M^4 coordinates. It might be more convenient to use as coordinates (u,v) for M^2 and (P_1, P_2) for Y^2. This covers also the situation when M^4 projection is not 4-D. By its 2-dimensionality

Y^2 allows always a complex structure defined by its induced metric: this complex structure is not consistent with the complex structure of CP_2 (Y^2 is not complex sub-manifold).

Using Hamilton-Jacobi coordinates the pre-image of a given point of Y^2 is a 2-dimensional sub-manifold X^2 of X^4 and defines also 2-D sub-manifold of M^4. The following picture suggests itself. The projection of X^2 to M^4 can be seen for a suitable choice of Hamilton-Jacobi coordinates as an analog of Lagrangian sub-manifold in M^4 that is as surface for which v and $Im(w)$ vary and u and $Re(w)$ are constant. X^2 would be obtained by allowing u and $Re(w)$ to vary: as a matter fact, (P_1, P_2) and $(u, Re(w))$ would be related to each other. The induced metric should be consistent with this picture. This would requires $g_{uRe(w)} = 0$.

For the deformations Q_1 and Q_2 would become non-constant and they should depend on the second light-like coordinate v only so that only g_{uu} and g_{uw} and $gu\overline{w}$ $g_{w,w}$ and $g_{\overline{w},\overline{w}}$ receive contributions which vanish. This would give rise to the analogs of Virasoro conditions guaranteeing that T is a tensor of form $(1,1)$ in both M^2 and E^2 indices and that there are no cross components in the induced metric. A more general formulation states that energy momentum tensor satisfies these conditions. The conditions on T might be equivalent with the conditions for g and G separately.

4. Einstein's equations provide an attractive manner to achieve the vanishing of effective 3-dimensionality of the action. Einstein equations would be second order differential equations and the idea that a deformation of vacuum extremal is in question suggests that the dynamics associated with them is in directions transversal to Y^2 so that only the deformation is dictated partially by Einstein's equations.

5. Lagrangian manifolds do not involve complex structure in any obvious manner. One could however ask whether the deformations could involve complex structure in a natural manner in CP_2 degrees of freedom so that the vanishing of g_{ww} would be guaranteed by holomorphy of CP_2 complex coordinate as function of w.

   One should get the complex structure in some natural manner: in other words, the complex structure should relate to the geometry of CP_2 somehow. The complex coordinate defined by say $z = P_1 + iQ^1$ for the deformation suggests itself. This would suggest that at the limit when one puts $Q_1 = 0$ one obtains $P_1 = P_1(Re(w))$ for the vacuum extremals and the deformation could be seen as an analytic continuation of real function to region of complex plane. This is in spirit with the algebraic approach. The vanishing of Kähler current requires that the Kähler magnetic field is covariantly constant: $D_z J^{z\overline{z}} = 0$ and $D_{\overline{z}} J^{z\overline{z}} = 0$.

6. One could consider the possibility that the resulting 3-D sub-manifold of CP_2 can be regarded as contact manifold with induced Kähler form non-vanishing in 2-D section with natural complex coordinates. The third coordinate variable- call it s- of the contact manifold and second coordinate of its transversal section would depend on time space-time coordinates for vacuum extremals. The coordinate associated with the transversal section would be continued to a complex coordinate which is holomorphic function of w and u.

7. The resulting thickened magnetic flux tubes could be seen as another representation of Kähler magnetic flux tubes: at this time as deformations of vacuum flux tubes rather than cosmic strings. For this ansatz it is however difficult to imagine deformations carrying Kähler electric field.

2.6.6 About The Interpretation Of The Generalized Conformal Algebras

The long-standing challenge has been finding of the direct connection between the super-conformal symmetries assumed in the construction of the geometry of the "world of classical worlds" (WCW) and possible conformal symmetries of field equations. 4-dimensionality and Minkowskian signature have been the basic problems. The recent construction provides new insights to this problem.

1. In the case of string models the quantization of the Fourier coefficients of coordinate variables of the target space gives rise to Kac-Moody type algebra and Virasoro algebra generators are quadratic in these. Also now Kac-Moody type algebra is expected. If one were to perform a quantization of the coefficients in Laurents series for complex CP_2 coordinates, one would obtain interpretation in terms of $su(3) = u(2) + t$ decomposition, where t corresponds to CP_3:

the oscillator operators would correspond to generators in t and their commutator would give generators in $u(2)$. SU(3)/SU(2) coset representation for Kac-Moody algebra would be in question. Kac-Moody algebra would be associated with the generators in both M^4 and CP_2 degrees of freedom. This kind of Kac-Moody algebra appears in quantum TGD.

2. The constraints on induced metric imply a very close resemblance with string models and a generalization of Virasoro algebra emerges. An interesting question is how the two algebras acting on coordinate and field degrees of freedom relate to the super-conformal algebras defined by the symplectic group of $\delta M^4_+ \times CP_2$ acting on space-like 3-surfaces at boundaries of CD and to the Kac-Moody algebras acting on light-like 3-surfaces. It has been conjectured that these algebras allow a continuation to the interior of space-time surface made possible by its slicing by 2-surfaces parametrized by 2-surfaces. The proposed construction indeed provides this kind of slicings in both M^4 and CP_2 factor.

3. In the recent case, the algebras defined by the Fourier coefficients of field variables would be Kac-Moody algebras. Virasoro algebra acting on preferred coordinates would be expressed in terms of the Kac-Moody algebra in the standard Sugawara construction applied in string models. The algebra acting on field space would be analogous to the conformal algebra assignable to the symplectic algebra so that also symplectic algebra is present. Stringy pragmatist could imagine quantization of symplectic algebra by replacing CP_2 coordinates in the expressions of Hamiltonians with oscillator operators. This description would be counterpart for the construction of spinor harmonics in WCW and might provide some useful insights.

4. For given type of space-time surface either CP_2 or M^4 corresponds to Kac-Moody algebra but not both. From the point of view of quantum TGD it looks as that something were missing. An analogous problem was encountered at GRT limit of TGD [L20]. When Euclidian space-time regions are allowed Einstein-Maxwell action is able to mimic standard model with a surprising accuracy but there is a problem: one obtains either color charges or M^4 charges but not both. Perhaps it is not enough to consider either CP_2 type vacuum extremal or its exterior but both to describe particle: this would give the direct product of the Minkowskian and Euclidian algebras acting on tensor product. This does not however seem to be consistent with the idea that the two descriptions are duality related (the analog of T-duality).

2.7 About TGD counterparts of classical field configurations in Maxwell's theory

Classical physics is an exact part of TGD so that the study of extremals of dimensionally reduces 6-D Kähler action can provide a lot of intuition about quantum TGD and see how quantum-classical correspondence is realized. In the following I will try to develop further understanding about TGD counterparts of the simplest field configurations in Maxwell's theory.

In the sequel CP_2 type extremals will be considered from the point of view of quantum criticality and the view about string world sheets, their lightlike boundaries as carriers of fermion number, and the ends as point like particles as singularities acting as sources for minimal surfaces satisfying non-linear generalization of d'Alembert equation.

I will also discuss the delicacies associated with M^4 Kähler structure and its connection with what I call Hamilton-Jacobi structure and with M^8 approach based on classical number fields. I will argue that the breaking of CP symmetry associated with M^4 Kähler structure is small without any additional assumptions: this is in contrast with the earlier view.

The difference between TGD and Maxwell's theory and consider the TGD counterparts of simple em field configurations will be also discussed. Topological field quantization provides a geometric view about formation of atoms as bound states based on flux tubes as correlates for binding, and allows to identify space-time correlates for second quantization. These considerations force to take seriously the possibility that preferred extremals besides being minimal surfaces also possess generalized holomorphy reducing field equations to purely algebraic conditions and that minimal surfaces without this property are not preferred extremals. If so, at microscopic level only CP_2 type extremals, massless extremals, and string like objects and their deformations would exist as preferred extremals and serve as building bricks for the counterparts of Maxwellian field configurations and the counterparts of Maxwellian field configurations such as Coulomb potential would emerge only at the QFT limit.

2.7.1 About differences between Maxwell's ED and TGD

TGD differs from Maxwell's theory in several important aspects.

1. The TGD counterparts of classical electroweak gauge potentials are induced from component of spinor connection of CP_2. Classical color gauge potentials corresponds to the projections of Killing vector fields of color isometries.
2. Also M^4 has Kähler potential, which is induced to space-time surface and gives rise to an additional $U(1)$ force. The couplings of M^4 gauge potential to quarks and leptons are of same sign whereas the couplings of CP_2 Kähler potential to B and L are of opposite sign so that the contributions to 6-D Kähler action reduce to separate terms without interference term.

   Coupling to induced M^4 Kähler potential implies CP breaking. This could explain the small CP breaking in hadronic systems and also matter antimatter asymmetry in which there are opposite matter-antimatter asymmetries inside cosmic strings and their exteriors respectively. A priori it is however not obvious that the CP breaking is small.
3. General coordinate invariance implies that there are only 4 local field like degrees of freedom so that for extremals with 4-D M^4 projection corresponding to GRT space-time both metric, electroweak and color gauge potentials can be expressed in terms four CP_2 coordinates and their gradients. Preferred extremal property realized as minimal surface condition means that field equations are satisfied separately for the 4-D Kähler and volume action reduces the degrees of freedom further.

   If the CP_2 part of Kähler form is non-vanishing, minimal surface conditions can be guaranteed by a generalization of holomorphy realizing quantum criticality (satisfied by known extremals). One can say that there is no dependence on coupling parameters. If CP_2 part of Kähler form vanishes identically, the minimal surface condition need not be guaranteed by holomorphy. It is not at all clear whether quantum criticality and preferred extremal property allow this kind of extremals.
4. Supersymplectic symmetries act as isometries of "world of classical worlds" (WCW). In a well-defined sense supersymplectic symmetry generalizes 2-D conformal invariance to 4-D context. The key observation here is that light-like 3-surfaces are metrically 2-D and therefore allow extended conformal invariance.

   Preferred extremal property realizing quantum criticality boils down to a condition that sub-algebra of SSA and its commutator with SSA annihilate physical states and that corresponding Noether charges vanish. These conditions could be equivalent with minimal surface property. This implies that the set of possible field patterns is extremely restricted and one might talk about "archetypal" field patterns analogous to partial waves or plane waves in Maxwell's theory.
5. Linear superposition of the archetypal field patterns is not possible. TGD however implies the notion of many-sheeted space-time and each sheet can carry its own field pattern. A test particle which is space-time surface itself touches all these sheets and experiences the sum of the effects caused by fields at various sheets. Effects are superposed rather than fields and this is enough. This means weakening of the superposition principle of Maxwell's theory and the linear superposition of fields at same space-time sheet is replaced with set theoretic union of space-time sheets carrying the field patterns whose effects superpose.

   This observation is also essential in the construction of QFT limit of TGD. The gauge potentials in standard model and gravitational field in general relativity are superpositions of those associated with space-time sheets idealized with slightly curved piece of Minkowski space M^4.
6. An important implication is that each system has field identity - field body or magnetic body (MB). In Maxwell's theory superposition of fields coming from different sources leads to a loss of information since one does not anymore now which part of field came from particular source. In TGD this information loss does not happen and this is essential for TGD inspired quantum biology.

Remark: An interesting algebraic analog is the notion of co-algebra. Co-product is analogous to reversal of product AB= C in the sense that it assigns to C and a linear combination of products $\sum A_i \otimes B_i$ such that $A_i B_i = C$. Quantum groups and co-algebras are indeed important in TGD and it might be that there is a relationship. In TGD inspired quantum biology magnetic body plays a key role as an intentional agent receiving sensory data from biological body and using it as motor instrument.

7. I have already earlier considered a space-time correlate for second quantization in terms of sheets of covering for $h_{eff} = nh_0$. In [L56] it is proposed that n factorizes as $n = n_1 n_2$ such that n_1 (n_2) is the number sheets for space-time surface as covering of CP_2 (M^4). One could have quantum mechanical linear superposition of space-time sheets, each with a particular field pattern. This kind state would correspond to single particle state created by quantum field in QFT limit. For instance, one could have spherical harmonic for orientations of magnetic flux tube or electric flux tube.

   One could also have superposition of configurations containing several space-time sheets simultaneously as analogs of many-boson states. Many-sheeted space-time would correspond to this kind many-boson states. Second quantization in quantum field theory (QFT) could be seen as an algebraic description of many-sheetedness having no obvious classical correlate in classical QFT.

8. Flux tubes should be somehow different for gravitational fields, em fields, and also weak and color gauge fields. The value of $n = n_1 n_2$ [L56] for gravitational flux tubes is very large by Nottale formula $\hbar_{eff} = \hbar_{gr} = GMm/v_0$. The value of n_2 for gravitational flux tubes is $n_2 \sim 10^7$ if one accepts the formula $G = R^2/n_2\hbar$. For em fields much smaller values of n and therefore of n_2 are suggestive. There the value of n measuring in adelic physics algebraic complexity and evolutionary level would distinguish between gravitational and em flux tubes.

   Large value of n would mean quantum coherence in long scales. For gravitation this makes sense since screening is absent unlike for gauge interactions. Note that the large value of $h_{eff} = h_{gr}$ implies that $\alpha_{em} = e^2/4\pi\hbar_{eff}$ is extremely small for gravitational flux tubes so that they would indeed be gravitational in an excellent approximation.

   n would be the dimension of extension of rationals involved and n_2 would be the number space-time sheets as covering of M^4. If this picture is correct, gravitation would correspond to much larger algebraic complexity and much larger value of Planck constant. This conforms with the intuition that gravitation plays essential role in the quantum physics of living matter.

   There are also other number theoretic characteristics such as ramified primes of the extension identifiable as preferred p-adic primes in turn characterizing elementary particle. Also flux tubes mediating weak and strong interactions should allow characterization in terms of number theoretic parameters. There are arguments that in atomic physics one has $h = 6h_0$. Since the quantum coherence scale of hadrons is smaller than atomic scale, one can ask whether one could have $h_{eff} < h$.

2.7.2 CP_2 type extremals as ultimate sources of fields and singularities

CP_2 type extremals have Euclidian signature of induced metric and therefore represent the most radical deviation from Maxwell's ED, gauge theories, and GRT. CP_2 type extremal with light-like geodesic as M^4 projection represents a model for wormhole contact. The light-like orbit of partonic 2-surface correspond to boundary between wormhole contact and Minkowskian region and is associated with both throats of wormhole contact. The throats of wormhole contact can carry part of a boundary of string world sheet connecting the partonic orbits associated with different particles. These light-like lines can carry fermion number and would correspond to lines of TGD counterparts of twistor diagrams.

These world lines would correspond to singularities for the minimal surface equations analogous to sources of massless vector fields carrying charge [L55, L61]. These singularities would serve as ultimate sources of classical em fields. Various currents would consist of wormhole throat pairs representing elementary particle and carrying charges at the partonic orbits. Two-sheetedness is essential and could be interpreted in terms of a double covering formed by space-time sheet glued along their common boundary. This necessary since space-time sheet has a finite size being not continuable beyond certain minimal size as preferred extremal since some of the real coordinates would become complex.

Quantum criticality for CP_2 type extremals

TGD predicts a hierarchy of quantum criticalities. The increase in criticality means that some space-time sheets for space-time surface regarded as a covering with sheets related by Galois group of extension of rationals degenerate to single sheet. The action of Galois group would reduce to that for its subgroup.

This is analogous to the degeneration of some roots of polynomial to single root and in M^8 representation space-time sheets are indeed quite concretely roots of octonionic polymomial defined by vanishing of real or imaginary part in the decomposition $o = q_1 + iq_2$ of octonion to a sum quaternionic real and imaginary parts.

The hierarchy of criticalities is closely related to the hierarchy of Planck constants $h_{eff}/h_0 = n = n_1n_2$, where n_1 corresponds to number of sheets as covering over CP_2 and n_2 as covering over M^4. One can also consider special cases in which M^4 projection has dimension $D < 4$. The proposal is that n corresponds to the dimension of Galois group for extension of rationals defining the level of dark matter hierarchy. If n is prime, one has either $n_1 = 1$ or $n_2 = 1$.

It seems that the range of n_2 is rather limited since the expression for Newton's constant as $G = R^2/n_2\hbar$ varies in rather narrow range. If the covering has symmetries assignable to some discrete subgroup of $SU(3)$ acting as isometries of CP_2 this could be understood. The increase of criticality could mean that n_1 or n_2 or both are reduced.

What is the position of CP_2 type extremals in the hierarchies of Planck constants and quantum criticalities?

1. Consider first n_2. CP_2 type extremal have 1-D geodesic line as M^4 projection. The light-like geodesic as 1-D structure could be interpreted as covering for which two geodesic lines along the orbits of opposite throats of wormhole contact form a kind of time loop. In this case one would have $n_2 = 2$ and one could have $n = 2p$, p prime.

   In this sense CP_2 type extremal or at least its core would be maximally critical. Deformations replacing the light-like geodesic as projection with higher-D region of M^4 presumably reduce criticality and one has $n_2 > 2$ is obtained. Whether this is possible inside wormhole contact is not clear. One can imagine that as one approaches partonic 2-surface, the criticality and degeneration increase in CP_2 degrees of freedom step by step and reach maximum in its core. This would be like realization of Thom's catastrophe involving parts with various degrees of criticalities.

   At the flux tubes mediating gravitational interaction $n_2 \sim 10^7$ would hold true in the exterior of associated CP_2 type extremals. This would suggests that CP_2 type extremals have maximal criticality in M^4 degrees of freedom and M^4 covering reduces to 2-fold covering for wormhole contacts.

2. What about criticality as n_1-fold covering of CP_2. This covering corresponds to a situation in which CP_2 coordinates as field in M^4 have given values of CP_2 coordinates n_1 times. A lattice like structure formed by n_1 wormhole contacts is suggestive. n_1 can be arbitrary large in principle and the gravitational Planck constant $h_{gr}/h_0 = n_1n_2$ would correspond to this situation. Singularities would now correspond to a degeneration of some wormhole contacts to single wormhole contact and could have interpretation in terms of fusion of particles to single particle. One might perhaps interpret elementary particle reaction vertices as catastrophes.

Wormhole contacts can be regarded as CP_2 type extremals having two holes corresponding to the 3-D orbits of wormhole contacts. Mathematician would probably speak of a blow up. CP_2 type extremals is glued to surrounding Minkowskian space-time sheets at the 3-D boundaries of these holes. At the orbit of partonic 2-surface the induced 4-metric degenerates to 3-D metric and 4-D tangent space becomes metrically 3-D. Light-likeness of the M^4 projection would correspond to this. For CP_2 type extremal 3 space-like M^4 directions of Minkowskian region would transmute to CP_2 directions at the light-like geodesic and time direction would become light-like. This is like graph of function for which tangent becomes vertical. For deformations of CP_2 type extremals this process could take place in several steps, one dimension in given step. This process could take place inside CP_2 or outside it depending on which order the transmutation of dimensions takes place.

2.7.3 Delicacies associated with M^4 Kähler structure

Twistor lift forces to assume that also M^4 possesses the analog of Kähler form, and Minkowskian signature does not prevent this [K14]. M^4 Kähler structure breaks CP symmetry and provides a very attractive manner to break CP symmetry and explain generation of matter antimatter symmetry and CP breaking in hadron physics. The CP breaking is very small characterized by a dimensionless number of order 10^{-9} identifiable as photon/baryon ratio. Can one understand the smallness of CP breaking in TGD framework?

Hamilton-Jacobi structure

Hamilton-Jacobi structure [L114] can be seen as a generalization of complex structure and involves a local but integrable selection of subspaces of various dimension for the tangent space of M^4. Integrability means that the selected subspaces are tangent spaces of a sub-manifold of M^4. M^8-H duality allows to interpret this selection as being induced by a global selection of a hierarchy of real, complex, and quaternionic subspaces associated with octonionic structure mapped to M^4 in such a way that this global selection becomes local at the level of H.

1. The 4-D analog of conformal invariance is due to very special conformal properties of light-like 3-surfaces and light-cone boundary of M^4. This raises hopes about construction of general solution families by utilizing the generalized form of conformal invariance. Massless extremals (MEs) in fact define extremely general solution family of this kind and involve light-like direction vector k and polarization vector ϵ orthogonal to it defining decomposition $M^4 = M^2 \times E^2$. I have proposed that this decomposition generalizes to local but integrable decomposition so that the distributions for M^2 and E^2 integrate to string world sheets and partonic 2-surfaces.

2. One can have decomposition $M^4 = M^2 \times E^2$ such that one has Minkowskian analog of conformal symmetry in M^2. This decomposition is defined by the vectors k and ϵ. An unproven conjecture is that these vectors can depend on point and the proposed Hamilton-Jacobi structure would mean a *local* decomposition of tangent space of M^4 , which is integrable meaning that local M^2s integrate to string world sheet in M^4 and local E^2s integrate to closed 2-surface as special case corresponds to partonic 2-surface. Generalizing the terminology, one could talk about family of partonic surfaces. These decompositions could define families of exremals.

   An integrable decomposition of M^4 to string world sheets and partonic 2-surfaces would characterize the preferred extremals with 4-D M^4 projection. Integrable distribution would mean assignment of partonic 2-surface to each point of string world sheet and vice versa.

3. M^4 Kähler form defines unique decomposition $M^2 \times E^2$. This is however not consistent Lorentz invariance. To cure this problem one must allow moduli space for M^4 Kähler forms such that one can assign to each Hamilton-Jacobi structure M^4 Kähler form defining the corresponding integrable surfaces in terms of light-like vector and polarization vector whose directions depend on point of M^4.

   This looks strange since the very idea is that the embedding space if unique. However, this local decomposition could be secondary being associated only with $H = M^4 \times CP_2$ and emerge in $M^8 - H$ duality mapping of space-time surfaces $X^4 \subset M^8$ to surfaces in $M^4 \times CP_2$. There is a moduli space for octonion structures in M^8 defined as a choice of preferred time axis M^1 (rest system), preferred M^2 defining hypercomplex place and preferred direction (light-like vector), and quaternionic plane $M^2 \times E^2$ (also polarization direction is included). Lorentz boosts mixing the real and imaginary octonion coordinates and changing the direction of time axis give rise to octonion structures not equivalent with the original one.

   Thus the choice $M^1 \subset M^2 \subset M^4 = M^2 \times E^2 \subset M^8$ is involved with the definition of octonion structure and quaternionion structure. The image of this decomposition under M^8-H duality mapping quaternionic tangent space of $X^4 \subset M^8$ containing M^1 and M^2 as sub-spaces would be such that the image of $M^1 \subset M^2 \subset M^2 \times E^2$ depends on point of $M^4 \subset H$ in integrable manner so that Hamilton-Jacobi structure in H is obtained.

Also CP_2 allows the analog of Hamilton-Jacobi structure as a local decomposition integrating to a family of geodesic spheres S^2_I as analog of partonic 2-surfaces with complex structure and having at each point as a fiber different S^2_I - these spheres necessary intersect at single point. This decomposition could correspond to the 4-D complex structure of CP_2 and complex coordinates of CP_2 would serve as coordinates for the two geodesic spheres.

Could one imagine decompositions in which fiber is 2-D Lagrangian manifold - say S^2_{II} - with vanishing induced Kähler form and not possessing induced complex structure? S^2_{II} does not have complex structure as induced complex structure and is therefore analogous to M^2. S^2_{II} coordinates would be functions of string world sheet coordinates (in special as analytic in hypercomplex sense and describing wave propagating with light-velocity). S^2_I coordinates would be analytic functions of complex coordinates of partonic 2-surface.

CP breaking and M^4 Kähler structure

The CP breaking induce by M^4 Kähler structure should be small. Is this automatically true or must one make some assumptions to achieve this.

Could one guarantee this by brute force by assuming M^4 and CP_2 parts of Kähler action to have different normalizations. The proposal for the length scale evolution of cosmological constant however relies on almost cancellation M^4 induced Kähler forms of M^4 and CP_2 parts due to the fact that the induced forms differ from each other by a rotation of the twistor sphere S^2. The S^2 part $M^4 \times S^2$ Kähler for can have opposite with respect to $T(CP_2) = SU(3)/U(1) \times U(1)$ Kähler so that for trivial rotation the forms cancel completely. If the normalizations of Kähler actions differ this cannot happen at the level of 4-D Kähler action.

To make progress, it is useful to look at the situation more concretely.

1. Kähler action is dimensionless. The square of Kähler form is metric so that $J_{kl}J^{kl}$ is dimensionless. One must include to the 4-D Kähler action a dimensional factor $1/L^4$ to make it dimensionless. The natural choice for L is as the radius R of CP_2 geodesic sphere to radius of twistor spheres for M^4 and CP_2. Note however that there is numerical constant involved and if it is changed there must be a compensating change of Kähler coupling strength. Therefore M^4 contribution to action is proportional to the volume of M^4 region using R^4 as unit. This contribution is very large for macroscopic regions of M^4 unless self-duality of M^4 Kähler form would not cause cancellation ($E^2 - B^2 = 0$).

2. What about energy density? The naïve expectation based on Maxwell's theory is that the energy density assignable to M^4 Kähler form is by self-duality proportional to $E^2 + B^2 = 2E^2$ and non-vanishing. By naïve order of magnitude estimate using Maxwellian formula for the energy of this kind extremal is proportional to Vol_3/R^4 and very large. Does this exclude these extremals or should one assume that they have very small volume? For macroscopic lengths of one should assume extremely thin MEs with thickness smaller than R. Could one have 2-fold covering formed by gluing to copies of very thin MEs together along their boundaries. This does not look feasible.

   Luckily, the Maxwellian intuition fails in TGD framework. The Noether currents associated in presence of M^4 Kähler action involve also a term coming from the variation of the induced M^4 Kähler form. This term guarantees that canonical momentum currents as H-vector fields are orthogonal to the space-time surface. In the case of CP_2 type extremals this causes the cancellation of the canonical momentum currents associated with Kähler action and corresponding contributions to conserved charges. The complete symmetry between M^4 and CP_2 and also physical intuition demanding that canonically imbedded M^4 os vacuum require that cancellation takes place also for M^4 part so that only the term corresponding to cosmological constant remains.

M^4 Kähler form and CP breaking for various kinds of extremals

I have considered already earlier the proposal that CP breaking is due to M^4 Kähler form [K14]. CP breaking is very small and the proposal inspired by the Cartesian product structure of the embedding space and its twistor bundle and also by the similar decomposition of $T(M^4) = M^4 \times S^2$ was that the coefficient of M^4 part of Kähler action can be chosen to be much smaller than the coefficient of CP_2 part. The proposed mechanism giving rise to p-adic length scale evolution of cosmological constant however requires that the coefficients of are identical. Luckily, the CP breaking term is automatically very small as the following arguments based on the examination of various kinds of extremals demonstrate.

1. For CP_2 type extremals with light-like M^4 geodesics as M^4 projection the induced M^4 Kähler form vanishes so that there is no CP breaking. For small deformations CP_2 type extremals thickening the M^4 projection the induced M^4 Kähler form is non-vanishing. An attractive hypothesis is that the small CP breaking parameter quantifies the order of magnitude of the induced M^4 Kähler form. This picture could allow to understand CP breaking of hadrons.
2. Canonically imbedded M^4 is a minimal surface. A small breaking of CP symmetry is generated in small deformations of M^4. In particular, for massless extremals (MEs) having 4-D M^4 projection the action associated with M^4 part of Kähler action vanishes at the M^4 limit when the local polarization vector characterizing ME approaches zero. The small CP breaking is characterized by the size of the polarization vector ϵ giving a contribution to the induced metric. This conforms with the perturbative CP breaking.
3. String like objects of type $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 is 2-surface in CP_2. The M^4 projection contains only electric part but no magnetic part. The M^4 part of action is proportional to the volume Y^2 and therefore very small. This in turn guarantees smallness of CP breaking effects.
   (a) If Y^2 is homologically non-trivial (magnetic flux tube carries monopole flux), CP_2 part of action is large since action density is proportional $1/\sqrt{det(g_2)}$ for Y^2 and therefore large. The thickening of the flux tube however reduces the value of the action by flux conservation as discussed already earlier.
   M^4 and CP_2 contributions to the actions are of opposite sign but M^4 contribution os however very small as compared to CP_2 contribution. One can look the situation in $M^2 \times S^2$ coordinates. The transverse deformation would correspond to the dependence of E^2 coordinates on S^2 coordinates. The induced Kähler form would give a contribution to the S^2 part of induced Kähler form whose size would characterize CP breaking.
   (b) Y^2 can be also homologically trivial. In particular, for $Y^2 = S^2_{II}$ the CP_2 contribution to the total Kähler action vanishes and only the small M^4 contribution proportional to the area of Y^2 remains.

2.7.4 About TGD counterparts for the simplest classical field patterns

What could be the TGD counterparts of typical configurations of classical fields? Since minimal surface equation is a nonlinear generalization of massless field equations, one can hope that the simplest solutions of Maxwell's equations have TGD analogs. The strong non-linearity poses a strong constraint, which can be solved if the extremal allows generalization of holomorphic structure so that field equations are trivially true since they involve in complex coordinates a contraction of tensors of type (1,1) with tensors of type (2,0) or (0,2). It is not clear whether minimal surface property reducing to holomorphy is equivalent with preferred extremal property.

Can one have the basic field patterns such as multipoles as structures with 4-D M^4 projection or could it be that flux tube picture based on spherical harmonics for the orientation of flux tube is all that one can have? Same question can be made for radiation fields having MEs as archetypal representatives in TGD framework. What about the possible consistency problems produced by M^4 Kähler form breaking Lorentz invariance?

I have considered these questions already earlier. The following approach is just making questions and guesses possibly helping to develop general ideas about the correspondence.

1. In QFT approach one expresses fields as superpositions of partial waves, which are indeed very simple field patterns and the coefficients in the superposition become oscillator operators. What could be the analogs of partial waves in TGD? Simultaneous extremals of Kähler action and volume strongly suggest themselves as carriers of field archetypes but the non-linearity of field equations does not support the idea that partial waves could be realized at classical level as extremals with 4-D M^4 projection. A more plausible option is that they correspond

to spherical harmonics for the orientation of flux tube carrying say electric flux. Could the flux tubes of various kinds serve as building of all classical fields?

2. String-like objects $X^2 \times Y^2 \subset M^4 \times CP_2$, where string world sheet X^2 is minimal surface and Y^2 is sub-manifold of CP_2 and their deformations in M^4 degrees of freedom transversal to X^2 and depending on the coordinates Y^2 are certainly good candidates for archetypal field configurations.

   Y^2 can be homologically trivial and could correspond to Lagrangian sub-manifold. Y^2 can also carry homology charge n identifiable as Kähler magnetic charge and correspond to complex sub-manifold of CP_2 with complex structure induced from that of CP_2.

The simplest option corresponds to geodesic sphere $Y^2 = S^2$. There are two geodesic spheres in CP_2 and they correspond to simplest string like objects.

1. S^2_I has Kähler magnetic charge of one unit and the cosmic and its deformations carry monopole flux. These field configurations are not possible in Maxwell's electrodynamics and the proposal is that they appear in all length scales. The model for the formation of galaxies solving also the problem of galactic dark matter relies on long cosmic strings. They are proposed to appear also in biology.

2. S^2_{II} is homologically trivial so that magnetic flux over it vanishes although magnetic field is non-vanishing. Note that although the Kähler magnetic field is vanishing, the electromagnetic ordinary magnetic field is non-vanishing because em field is a combination of Kähler form and component of CP_2 curvature form with vanishing weak isospin. The total flux of ordinary magnetic field over S^2_{II} vanishes whereas electric flux can be non-vanishing.

Coulomb fields

By the vanishing of magnetic flux flux tubes for S^2_{II} cannot represent ordinary magnetic field. They can however serve as radial flux tubes carrying electromagnetic flux. Magnetic flux tubes indeed allow time dependent deformations for which the phase angles of CP_2 coordinates depend linearly of M^4 time coordinate. This would give rise to an archetypal flux tube representation of the electric field created by point charge. Also gravitational flux tubes should correspond to this kind flux tubes emanating radially from the source.

Charge quantization suggests that these flux tubes carry unit charge. In the case of charged elementary particle there would be only single flux tubes but there would be wave function for its orientation having no angular dependence. In principle, this wave function can any spherical harmonic.

Does the orientation angle dependence of flux distribution have any counterpart in Maxwell's theory. One would have the analog of $1/r$ Coulomb potential with the modulus squared of spherical harmonic Y_{lm} modulating it. Could one consider the possibility that in atoms the spherical harmonics for excited states correspond to this kind of distribution for the electric flux coming from nucleus. The probability amplitude for electrons touching the flux tube would inherit this distribution.

For many particle system with large em charge there would be large number of radial flux tubes and the approximation of electric field with Coulomb field becomes natural. In the case of atoms this limit is achieved for large enough nuclear charges. This does not exclude the possibility of having space-time surfaces carrying Coulomb potential in Maxwellian sense: in this case however the field equations cannot solved by holomorphy and quantum criticality might exclude these configurations.

What about gravitation? The notion of gravitational Planck constant requires that Planck mass replaced in TGD framework by CP_2 mass defining the unit of gravitational flux - $h_{gr}0GMm/v_0$ cannot be smaller than h_0. What happens in systems possessing mass smaller than CP_2 mass? Are gravitational flux tubes absent. Is gravitational interaction absent in this kind of systems or is its description analogous to string model description meaning that $h_{gr} = h_0$ for masses smaller than CP_2 mass?

Magnetic fields

As such S^2_{II} flux tubes cannot serve as counterparts of ordinary magnetic fields. The flux tubes have now boundary and the current at boundary creates the magnetic field inside the tube. This would mean cutting of a disk D^2 from S^2_{II} so that the net magnetic flux becomes non-vanishing.

The assumption has been that genuine boundaries are not possible since conservation laws very probably prevent them (the normal components of canonical momentum currents should vanish at boundaries but this is not possible). This requires that this flux tube must be glued along the boundary of $D^2 \times D^1$ to surrounding space-time surface X^4, which has a similar hole. At the boundary of this hole the space-time surface must turn to the direction of CP_2 meaning that the dimension of M^4 projection is reduced to $D = 2$. Algebraic geometer would talk about blow-up.

Ordinary multipole magnetic field could correspond to spherical harmonic for the orientation of this kind flux tubes. They could also carry electric flux but the em charge could be fractionized. These flux tubes might relate to anyons carrying fractional em charge. Also the fractional charges of quarks could classically correspond to flux tubes mediating both color magnetic field and em flux. The spherical harmonic in question corresponds to that associated with electron in atoms.

Magnetic and electric fields associated with straight current wire

Magnetic and electric fields associated with straight current wire need not allow representation as archetypes since they are obviously macroscopic entities.

1. Is the magnetic field associated with straight current wire representable in terms of extremal with 4-D M^4 projection. The magnetic field lines rotate around the current and it is does not seem natural to model it the field in terms of flux tubes. Forget the presence of M^4 Kähler form. One can imbed this kind of magnetic field as a surface with 4-D M^4 projection and possessing cylindrical symmetry. Line current would correspond to a source of the magnetic field and could be realized as a flux tube carrying em current and topologically condensed to the space-time sheet in question.

   The embedding however fails at certain critical radius and the assumption is that no boundaries are allowed by conservation laws. Should one glue the structure to the surrounding space-time surface at this radius. In Maxwell's theory one would have surface current in direction opposite to the source cancelling the magnetic field outside. Could this current have interpretation as a return current?

   One can also imagine glueing its copy to it along the boundary at critical radius. It would seem that the magnetic fields must have same direction at the boundary and therefore also in interior.

2. What about current ring? Separation of variables is essential for the simplest embeddings implying a reduction of partial different equations to differential equation. There is rather small number of coordinates system in E^3 in which Laplacian allows separation of variables. The metric is diagonal in these coordinates. One example is toroidal coordinates assignable with a current ring having toroidal geometry. This would allow a construction of minimal surface solution in some finite volume. Minimal surface property would *not* reduce to complex analyticity for these extremals and they would be naturally associated $M^4 \times S^2_{II}$.

Remark: This kind of extremals are not holomorphic and could be excluded by quantum criticality and preferred extremal property. GRT space-time would be idealization making sense only at the QFT limit of TGD.

Time dependent fields

What about time dependent fields such as the field created by oscillating dipole and radiation fields? One can imagine quantal and classical option.

1. The simplest possibility is reduction to quantum description at single particle level. The dipole current corresponds to a wave function for the source particle system consisting of systems with opposite total charge.

Spherical harmonics representing multipoles would induce wave function for the orientations of MEs (topological light ray) carrying radial wave. This is certainly the most natural options as far radiation field at large distances from sources is considered. One can also have second quantization in the proposed sense giving rise to multi-photon states and one can also define coherent states.

One should also understand time dependent fields near sources having also non-radiative part. This requires a model for source such as oscillating dipole. The simplest possibility is that in the case of dipole there are charges of opposite sign with oscillating distance creating Coulomb fields represented in the proposed manner. It is however not obvious that preferred extremals of this kind exist.

2. One can consider also classical description. The model of elementary particle as consisting of two wormhole contacts, whose throats effectively serve as end of monopole flux tubes at the two sheets involved suggests a possible model. If the wormhole contacts carry opposite em charges realized in terms of fermion and antifermions an oscillating dipole could correspond to flux tube whose length oscillates. This means generation of radiation and for elementary particles this would suggest instability against decay. One can however consider excitation which decay to ground states - say for hadrons. For scaled up variants of this structure this would not mean instability although energy is lost and the system must end up to non-oscillating state.

   One possibility is that there are two charges at different space-time sheets connected by wormhole contacts and oscillating by their mutual interaction in harmonic oscillator state. Ground state would be stable and have not dipole moment.

Effectively 2-D systems

In classical electrodynamics effectively 2-D systems are very special in that they allow conformal invariance assignable to 2-D Laplacian.

1. Since minimal surface equation is generalization of massless d'Alambertian and since field equations are trivially true for analytic solutions, one can hope that the basic solutions of 4-D d'Alembertian generalize in TGD framework. This would conform with the universality of quantum criticality meaning that coupling parameters disappear from field equations. Conformal invariance or its generalization would mean huge variety of field patterns. This suggests that effectively 2-D systems serve as basic building bricks of more complex field configurations. Flux tubes of various kinds would represent basic examples of this kind of surfaces. Also the magnetic end electric fields associated with straight current wire would serve as an example.

2. Are there preferred extremals analogous to the solutions of field equations of general relativity in faraway regions, where they become simple and might allow an analog in TGD framework? If our mathematical models reflect the preferred extremals as archetypal structures, this could be the case.

   Forget for a moment the technicalities related to M^4 Kähler form. One can construct a spherically symmetric ansatz in $M^4 \times S^2_{II}$ as a minimal surface for which Φ depends linearly on time t and u is function of r. The ansatz reduces to a highly non-linear differential equation for u. In this case hyper-complex analyticity is obviously not satisfied. This ansatz could give the analog of Schwartschild metric giving also the electric field of point charge appearing as source of the non-linear variant of d'Alembertian. It is however far from clear whether this kind extremals is allowed as preferred extremals.

   Under which conditions spherically symmetric ansatz is consistent with M^4 Kähler form? Obviously, the M^4 Kähler form must be spherically symmetric as also the Hamilton-Jacobi structure it. Suppose local Hamilton-Jacobi structures for which M^2s integrate to t, r coordinate planes and E^2s integrate to (θ, ϕ) sphere are allowed and that M^4 Kähler form defines this decomposition. In this case there are hopes that consistency conditions can be satisfied. Note however that M^4 Kähler form defines in this case orthogonal magnetic and electric monopole fields defining an analog of instanton. Can one really allow this or should one exclude the time line with $r = 0$?

Similar M^4 Kähler structure can be associated with cylindrical coordinates and other separable coordinates system. M^4 Kähler structure would define Hamilton-Jacobi structure.

2.8 Minimal surfaces and TGD

The twistor lift of TGD [K86, L50, L57] meant a revolution in the understanding of TGD and led to a new view about what preferred extremal property means physically and why it is needed.

1. The construction of twistor lift of TGD replaces space-time surfaces with 6-D surfaces but requires that they are dynamically effectively 4-D as the analogs of twistor space having the structure of S^2 bundle with space-time surface as the base. This requires dimensional reduction making S^2 fiber of the twistor space non-dynamical.

   One can say that twistor structure is induced from that for 12-D product of the geometric 6-D twistor spaces of M^4 and CP_2. The condition that 6-D Kähler action exists requires that the twistor spaces of M^4 and CP_2 have Kähler structure. This condition allows only $H = M^4 \times CP_2$ [A74]. The condition that one obtains standard model symmetries leads to the same conclusion.

2. The dimensionally reduced Kähler action decomposes to a sum of 4-D Kähler action and volume term. The interaction is as analog of Maxwell action plus action of point-like particle replaced with 3-D surface. The coefficient of the volume term has an interpretation as cosmological constant having a discrete spectrum [L61]. The natural proposal it that it depends on p-adic length scale approaching zero in long length scales. This solves the cosmological constant problem.

3. I had actually known for decades that all non-vacuum extremals of 4-D Kähler action are minimal surfaces thus minimizing the space-time volume in the induced metric. This is because the field equations for Kähler action for known non-vacuum extremals were reduced essentially to algebraic conditions realizing holomorphy. Also so called CP_2 type vacuum extremals of 4-D Kähler action are minimal surfaces. This finding conforms with the fact that in $M^8 - H$ duality [L42] one has regard field equations as purely algebraic conditions at M^8 side of the duality.

   This inspired the proposal that preferred extremal property of space-time surface is realized by requiring that space-time surfaces as base spaces of these 6-D twistor spaces are quite generally minimal surfaces, and therefore represent a non-linear geometrization for the notion of massless field in accordance with conformal invariance forced by quantum criticality.

   Also a more general proposal that space-time contains regions inside which there is an exchange of canonical momenta between Kähler action and volume term was considered. Minimal surface regions would correspond to incoming particles and non-minimal ones to interaction regions.

   Later this proposal was simplified by requiring that interaction regions are 2-D string world sheets as singularities: this implied that string world sheets required by general considerations [K96] indeed emerge from 4-D action. This could happen also at the 1-D boundaries of string world sheets at 3-D light-like boundaries between Minkowskian and Euclidian regions behaving like ordinary point-like particles and carrying fermion number, and in the most general case also at these 3-D light-like 3-surfaces.

2.8.1 Space-time surfaces as singular minimal surfaces

From the physics point this is not surprising since minimal surface equations are the geometric analog for massless field equations.

1. The boundary value problem in TGD is analogous to that defining soap films spanned by frames: space-time surface is thus like a 4-D soap film. Space-time surface has 3-D ends at the opposite boundaries of causal diamond of M^4 with points replaced with CP_2: I call this 8-D object just causal diamond (CD). Geometrically CD brings in mind big-bang followed by big crunch.

These 3-D ends are like the frame of a soap film. This and the Minkowskian signature guarantees the existence of minimal surface extremals. Otherwise one would expect that the non-compactness does not allow minimal surfaces as non-self-intersecting surfaces.

2. Space-time is a 4-surface in 8-D $H = M^4 \times CP_2$ and is a minimal surface, which can have 2-D or 1-D singularities identifiable as string world sheets having 1-D singularities as light-like orbits - they could be geodesics of space-time surface.

   Remark: I considered in [L54] the possibility that the minimal surface property could fail only at the reaction vertices associated with partonic 2-surfaces defining the ends of string world sheet boundaries. This condition however seems to be too strong. It is essential that the singular surface defines a sub-manifold giving deltafunction like contribution to the action density and that one can assign conserved quantities to this surface. This requires that the singular contributions to energy momentum tensor and canonical momentum currents as spacetime vectors are parallel to the singular surface. Singular points do not satisfy this condition.

   String boundaries represent orbits of fundamental point-like fermions located at 3-D light-like surfaces which represent orbits of partonic 2-surfaces. String world sheets are minimal surfaces and correspond to stringy objects associated with say hadrons. There are also degrees of freedom associated with space-time interior. One have objects of various dimension which all are minimal surfaces. Modified Dirac equation extends the field equations to supersymmetric system and assigns fermionic degrees of freedom to these minimal surfaces of varying dimension.

   From the physics point of view, the singular surfaces are analogous to carriers of currents acting as point- and string-like sources of massless field equations.

3. Geometrically string world sheets are analogous to folds of paper sheet. Space-time surfaces are extremals of an action which is sum of volume term having interpretation in terms of cosmological constant and what I call Kähler action - analogous to Maxwell action. Outside singularities one has minimal surfaces stationary with respect to variations of both volume term and Kähler action - note the analogy with free massless field. At singularities there is an exchange of conserved quantities between volume and Kähler degrees of freedom analogous to the interaction of charged particle with electromagnetic field. One can see TGD as a generalization of a dynamics of point-like particle coupled to Maxwell field by making particle 3-D surface.

4. The condition that the exchange of conserved charges such as four-momentum is restricted to lower-D surfaces realizes preferred extremal property as a consequence of quantum criticality demanding a universal dynamics independent of coupling parameters [L61]. Indeed, outside the singularities the minimal surfaces dynamics has no explicit dependence on coupling constants provided local minimal surface property guarantees also the local stationarity of Kähler action.

   Preferred extremal property has also other formulations. What is essential is the generalization of super-conformal symmetry playing key role in super string models and in the theory of 2-D critical systems so that field equations reduce to purely algebraic conditions just like for analytic functions in 2-D space providing solutions of Laplace equations.

5. TGD provides a large number of specific examples about closed minimal surfaces [K5]. Cosmic strings are objects, which are Cartesian products of minimal surfaces (string world sheets) in M^4 and of complex algebraic curves (2-D surfaces). Both are minimal surfaces and extremize also Kähler action. These algebraic surfaces are non-contractible and characterized by homology charge having interpretation as Kähler magnetic charge. These surfaces are genuine minima just like the geodesics at torus.

   CP_2 contains two kinds of geodesic spheres, which are trivially minimal surfaces. The reason is that the second fundamental form defining as its trace the analogs of external curvatures in the normal space of the surfaces vanishes identically. The geodesic sphere of the first kind is non-contractible minimal surface and absolute minimum. Geodesic spheres of second kind is contractible and one has Minimax type situation.

   These geodesic spheres are analogous to 2-planes in flat 3-space with vanishing external curvatures. For a generic minimal surface in 3-space the principal curvatures are non-vanishing and sum up to zero. This implies that minimal surfaces look locally like saddles. For 2-plane the curvatures vanish identically so that saddle is not formed.

2.8.2 Kähler action as Morse function in the space of minimal 4-surfaces

It was found that surface volume could define a Morse function in the space of surfaces. What about the situation in TGD, where volume is replaced with action which is sum of volume term and Kähler action [L57, L55, L61]?

Morse function interpretation could appear in two ways. The first possibility is that the action defines an analog of Morse function in the space of 4-surfaces connecting given 3-surfaces at the boundaries of CD. Could it be that there is large number of preferred extremals connecting given 3-surfaces at the boundaries of CD? This would serve as analogy for the existence of infinite number of closed surfaces in the case of compact embedding space. The fact that preferred extremals extremize almost everywhere two different actions suggests that this is not the case but one must consider also this option.

1. The simplest realization of general coordinate invariance would allow only single preferred extremal but I have considered also the option for which one has several preferred extremals. In this case one encounters problem with the definition of Kähler function which would become many-valued unless one is ready to replace 3-surfaces with its covering so that each preferred extremal associated with the given 3-surface gives rise to its own 3-surface in the covering space. Note that analogy with the definition of covering space of say circle by replacing points with the set of homologically equivalence classes of closed paths at given point (rotating arbitrary number of times around circle).
2. Number theoretic vision [K94, K31] suggests that these possibly existing different preferred extremals are analogous to same algebraic computation but performed in different ways or theorem proved in different ways. There is always the shortest manner to do the computation and an attractive idea is that the physical predictions of TGD do not depend on what preferred extremal is chosen.
3. An interesting question is whether the "drum theorem" could generalize to TGD framework. If there exists infinite series of preferred extremals which are singular minimal surfaces, the volume of space-time surface for surfaces in the series would depend only on the volume of the CD containing it. The analogy with the high frequencies and drum suggests that the surfaces in the series have more and more local details. In number theoretic vision this would correspond to emergence of more and more un-necessary pieces to the computation. One cannot exclude the possibility that these details are analogs for what is called loop corrections in quantum field theory.
4. If the action defines Morse action, the preferred extremals give information about its topology. Note however that the requirement that one has extremum of both volume term and Kähler action almost everywhere is an extremely strong additional condition and corresponds physically to quantum criticality.

   Remark: The original assumption was that the space-time surface decomposes to critical regions which are minimal surfaces locally and to non-critical regions inside which there is flow of canonical momentum currents between volume and Kähler degrees of freedom. The stronger hypothesis is that this flow occurs at 2-D and 1-D surfaces only.

2.8.3 Kähler function as Morse function in the space of 3-surfaces

The notion of Morse function can make sense also in the space of 3-surfaces - the world of classical worlds which in zero energy ontology consists of pairs of 3-surfaces at opposite boundaries of CD connected by preferred extremal of Kähler action [K21, K73, L57, L55]. Kähler action for the preferred extremal is assumed to define Kähler function defining Kähler metric of WCW via its second derivates $\partial_K\partial_{\overline{L}}K$. Could Kähler function define a Morse function?

1. First of all, Morse function must be a genuine function. For general Kähler metric this is not the case. Rather, Kähler function K is a section in a $U(1)$ bundle consisting of patches transforming by real part of a complex gradient as one moves between the patches of the

bundle. A good example is CP_2, which has non-trivial topology, and which decomposes to 3 coordinate patches such that Kähler functions in overlapping patches are related bythe analog of U(1) gauge transformation.

Kähler action for preferred extremal associated with given 3-surface is however uniquely defined unless one includes Chern-Simons term which changes in U(1) gauge transformation for Kähler gauge potential of CP_2.

2. What could one conclude about the topology of WCW if the action for preferred extremal defines a Morse function as a functional of 3-surface? This function cannot have saddle points: in a region of WCW around saddle point the WCW metric depending on the second derivatives of Morse function would not be positive definite, and this is excluded by the positivity of Hilbert space inner product defined by the Kähler metric essential for the unitarity of the theory. This would suggest that the space of 3-surfaces has very simple topology if Kähler function.

   This is too hasty conclusion! WCW metric is expected to depend also on zero modes, which do not contribute to the WCW line element. What suggests itself is bundle structure. Zero modes define the base space and dynamical degrees of freedom contributing to WCW line element as fiber. The space of zero modes can be topologically complex.

There is a fascinating open problem related to the metric of WCW.

1. The conjecture is that WCW metric possess the symplectic symmetries of $\Delta M^4_+ \times CP_2$ as isometries. In infinite dimensional case the existence of Riemann/Kähler geometry is not at all obvious as the work of Dan Freed demonstrated in the case of loops spaces [A54], and the maximal group of isometries would guarantee the existence of WCW Kähler geometry. Geometry would be determined by symmetries alone and all points of the space would be metrically equivalent. WCW would be an infinite-dimensional analog of symmetric space.
2. Isometry group property does not require that symplectic symmetries leave Kähler action, and even less volume term for preferred extremal, invariant. Just the opposite: if the action would remain invariant, Kähler function and Kähler metric would be trivial!
3. The condition for the existence of symplectic isometries must fix the ratio of the coefficients of Kähler action and volume term highly uniquely. The physical interpretation is in terms of quantum criticality realized mathematically in terms of the symplectic symmetry serving as analog of ordinary conformal symmetry characterizing 2-D critical systems. Note that at classical level quantum criticality realized as minimal surface property says nothing outside singular surfaces since the field equations in this regions are algebraic. At singularities the situation changes. Note also that the minimal surface property is a geometric analog of masslessness which in turn is a correlate of criticality.
4. Twistor lift of TGD [?]eads to a proposal for the spectra of Kähler coupling strength and cosmological constant allowed by quantum criticality [L55]. What is surprising that cosmological constant identified as the coefficient of the volume term takes the role of cutoff mass in coupling constant evolution in TGD framework. Coupling constant evolution discretizes in accordance with quantum criticality which must give rise to infinite-D group of WCW isometries. There is also a connection with number theoretic vision in which coupling constant evolution has interpretation in terms of extensions of rationals [K94, L47, L42].

2.8.4 Kähler calibrations: an idea before its time?

While updating book introductions I was surprised to find that I had talked about so called calibrations of sub-manifolds as something potentially important for TGD and later forgotten the whole idea! A closer examination however demonstrated that I had ended up with the analog of this notion completely independently later as the idea that preferred extremals are minimal surfaces apart form 2-D singular surfaces, where there would be exchange of Noether charges between Kähler and volume degrees of freedom.

1. The original idea that I forgot too soon was that the notion of calibration (see `http://tinyurl.com/y3lyead3`) generalizes and could be relevant for TGD. A calibration in Riemann

manifold M means the existence of a k-form ϕ in M such that for any orientable k-D sub-manifold the integral of ϕ over M equals to its k-volume in the induced metric. One can say that metric k-volume reduces to homological k-volume.

Calibrated k-manifolds are minimal surfaces in their homology class, in other words their volume is minimal. Kähler calibration is induced by the k^{th} power of Kähler form and defines calibrated sub-manifold of real dimension $2k$. Calibrated sub-manifolds are in this case precisely the complex sub-manifolds. In the case of CP_2 they would be complex curves (2-surfaces) as has become clear.

2. By the Minkowskian signature of M^4 metric, the generalization of calibrated sub-manifold so that it would apply in $M^4 \times CP_2$ is non-trivial. Twistor lift of TGD however forces to introduce the generalization of Kähler form in M^4 (responsible for CP breaking and matter antimatter asymmetry) and calibrated manifolds in this case would be naturally analogs of string world sheets and partonic 2-surfaces as minimal surfaces. Cosmic strings are Cartesian products of string world sheets and complex curves of CP_2. Calibrated manifolds, which do not reduce to Cartesian products of string world sheets and complex surfaces of CP_2 should also exist and are minimal surfaces.

   One can also have 2-D calibrated surfaces and they could correspond to string world sheets and partonic 2-surfaces which also play key role in TGD. Even discrete points assignable to partonic 2-surfaces and representing fundamental fermions play a key role and would trivially correspond to calibrated surfaces.

3. Much later I ended up with the identification of preferred extremals as minimal surfaces by totally different route without realizing the possible connection with the generalized calibrations. Twistor lift and the notion of quantum criticality led to the proposal that preferred extremals for the twistor lift of Kähler action containing also volume term are minimal surfaces. Preferred extremals would be separately minimal surfaces and extrema of Kähler action and generalization of complex structure to what I called Hamilton-Jacobi structure would be an essential element. Quantum criticality outside singular surfaces would be realized as decoupling of the two parts of the action. May be all preferred extremals be regarded as calibrated in generalized sense.

   If so, the dynamics of preferred extremals would define a homology theory in the sense that each homology class would contain single preferred extremal. TGD would define a generalized topological quantum field theory with conserved Noether charges (in particular rest energy) serving as generalized topological invariants having extremum in the set of topologically equivalent 3-surfaces.

   It is interesting to recall that the original proposal for the preferred extremals as absolute minima of Kähler action has transformed during years to a proposal that they are absolute minima of volume action within given homology class and having fixed ends at the boundaries of CD.

4. The experience with CP_2 would suggest that the Kähler structure of M^4 defining the counterpart of form ϕ is unique. There is however infinite number of different closed self-dual Kähler forms of M^4 defining what I have called Hamilton-Jacobi structures. These forms can have subgroups of Poincare group as symmetries. For instance, magnetic flux tubes correspond to given cylindrically symmetry Kähler form. The problem disappears as one realizes that Kähler structures characterize families of preferred extremals rather than M^4 itself.

If the notion of calibration indeed generalizes, one ends up with the same outcome - preferred extremals as minimal surfaces with 2-D string world sheets and partonic 2-surfaces as singularities - from many different directions.

1. Quantum criticality requires that dynamics does not depend on coupling parameters so that extremals must be separately extremals of both volume term and Kähler action and therefore minimal surfaces for which these degrees of freedom decouple except at singular 2-surfaces, where the necessary transfer of Noether charges between two degrees of freedom takes place at these. One ends up with string picture but strings alone are of course not enough. For instance, the dynamical string tension is determined by the dynamics for the twistor lift.
2. Almost topological QFT picture implies the same outcome: topological QFT property fails only at the string world sheets.

3. Discrete coupling constant evolution, vanishing of loop corrections, and number theoretical condition that scattering amplitudes make sense also in p-adic number fields, requires a representation of scattering amplitudes as sum over resonances realized in terms of string world sheets.
4. In the standard QFT picture about scattering incoming states are solutions of free massless field equations and interaction regions the fields have currents as sources. This picture is realized by the twistor lift of TGD in which the volume action corresponds to geodesic length and Kähler action to Maxwell action and coupling corresponds to a transfer of Noether charges between volume and Kähler degrees of freedom. Massless modes are represented by minimal surfaces arriving inside causal diamond (CD) and minimal surface property fails in the scattering region consisting of string world sheets.
5. Twistor lift forces M^4 to have generalize Kähler form and this in turn strongly suggests a generalization of the notion of calibration. At physics side the implication is the understanding of CP breaking and matter anti-matter asymmetry.
6. M^8-H duality requires that the dynamics of space-time surfaces in H is equivalent with the algebraic dynamics in M^8. The effective reduction to almost topological dynamics implied by the minimal surface property implies this. String world sheets (partonic 2-surfaces) in H would be images of complex (co-complex sub-manifolds) of $X^4 \subset M^8$ in H. This should allows to understand why the partial derivatives of embedding space coordinates can be discontinuous at these edges/folds but there is no flow between interior and singular surface implying that string world sheets are minimal surfaces (so that one has conformal invariance).

The analogy with foams in 3-D space deserves to be noticed.

1. Foams can be modelled as 2-D minimal surfaces with edges meeting at vertices. TGD space-time could be seen as a dynamically generated foam in 4-D many-sheeted space-time consisting of 2-D minimal surfaces such that also the 4-D complement is a minimal surface. The counterparts for vertices would be light-like curves at light like orbits of partonic 2-surfaces from which several string world sheets can emanate.
2. Can one imagine something more analogous to the usual 3-D foam? Could the light-like orbits of partonic 2-surfaces define an analog of ordinary foam? Could also partonic 2-surfaces have edges consisting of 2-D minimal surfaces joined along edges representing strings connecting fermions inside partonic 2-surface?

   For years ago I proposed what I called as symplectic QFT (SQFT) as an analog of conformal QFT and as part of quantum TGD [K16]. SQFT would have symplectic transformations as symmetries, and provide a description for the symplectic dynamics of partonic 2-surfaces. SQFT involves an analog of triangulation at partonic 2-surfaces and Kähler magnetic fluxes associated with them serve as observables. The problem was how to fix this kind of network. Partonic foam could serve as a concrete physical realization for the symplectic network and have fundamental fermions at vertices. The edges at partonic 2-surfaces would be space-like geodesics. The outcome would be a calibration involving objects of all dimensions $0 \leq D \leq 4$ - a physical analog of homology theory.

2.9 Are space-time boundaries possible in the TGD framework?

One of the key ideas of TGD from the very beginning was that the space-time surface has boundaries and we see them directly as boundaries of physical objects.

It however turned out that it is not at all clear whether the boundary conditions stating that no isometry currents flow out of the boundary, can be satisfied. Therefore the cautious conclusion was that perhaps the boundaries are only apparent. For instance, the space-time regions correspond to maps $M^4 \to CP_2$, which are many-valued and have as turning points, which have 3-D projections to M^4. The boundary surfaces between regions with Minkowskian and Euclidean signatures of the induced metric seem to be unavoidable, at least those assignable to deformations of CP_2 type extremals assignable to wormhole contacts.

There are good reasons to expect that the possible boundaries are light-like and possibly also satisfy the $det(g_4) = 0$ condition and I have considered the boundary conditions but have not been able to make definite conclusions about how they could be realized.

1. The action principle defining space-times as 4-surfaces in $H = M^4 \times CP_2$ as preferred extremals contains a 4-D volume term and the Kähler action plus possible boundary term if boundaries are possible at all. This action would give rise to a boundary term representing a normal flow of isometry currents through the boundary. These currents should vanish.
2. There could also be a 3-D boundary part in the action but if the boundary is light-like, it cannot depend on the induced metric. The Chern-Simons term for the Kähler action is the natural choice. Twistor lift suggests that it is present also in M^4 degrees of freedom. Topological field theories utilizing Chern-Simons type actions are standard in condensed matter physics, in particular in the description of anyonic systems, so that the proposal is not so radical as one might think. One might even argue that in anyonic systems, the fundamental dynamics of the space-time surface is not masked by the information loss caused by the approximations leading to the field theory limit of TGD.

   Boundary conditions would state that the normal components of the isometry currents are equal to the divergences of Chern-Simons currents and in this way guarantee conservation laws. In CP_2 degrees of freedom the conditions would be for color currents and in M^4 degrees of freedom for 4-momentum currents.
3. This picture would conform with the general view of TGD. In zero energy ontology (ZEO) [L68, L98] phase transitions would be induced by macroscopic quantum jumps at the level of the magnetic body (MB) of the system. In ZEO, they would have as geometric correlates classical deterministic time evolutions of space-time surface leading from the initial to the final state [L60]. The findings of Minev et al provide [L60] lend support for this picture.

2.9.1 Light-like 3-surfaces from $det(g_4) = 0$ condition

How the light-like 3- surfaces could be realized?

1. A very general condition considered already earlier is the condition $det(g_4) = 0$ at the light-like 4-surface. This condition means that the tangent space of X^4 becomes metrically 3-D and the tangent space of X^3 becomes metrically 2-D. In the local light-like coordinates, $(u, v, W, \overline{W})$ $guv = g_{vu}$) would vanish (g_{uu} and g_{vv} vanish by definition.

   Could $det(g_4) = 0$ and $det(g_3) = 0$ condition implied by it allow a universal solution of the boundary conditions? Could the vanishing of these dimensional quantities be enough for the extended conformal invariance?
2. 3-surfaces with $det(g_4) = 0$ could represent boundaries between space-time regions with Minkowskian and Euclidean signatures or genuine boundaries of Minkowskian regions.

   A highly attractive option is that what we identify the boundaries of physical objects are indeed genuine space-time boundaries so that we would directly see the space-time topology. This was the original vision. Later I became cautious with this interpretation since it seemed difficult to realize, or rather to understand, the boundary conditions.

   The proposal that the outer boundaries of different phases and even molecules make sense and correspond to 3-D membrane like entities [L101], served as a partial inspiration for this article but this proposal is not equivalent with the proposal that light-like boundaries defining genuine space-time boundaries can carry isometry charges and fermions.
3. How does this relate to $M^8 - H$ duality [L80, L81]? At the level of rational polynomials P determined 4-surfaces at the level of M^8 as their "roots" and the roots are mass shells. The points of M^4 have interpretation as momenta and would have values, which are algebraic integers in the extension of rationals defined by P.

   Nothing prevents from posing the additional condition that the region of $H^3 \subset M^4 \subset M^8$ is finite and has a boundary. For instance, fundamental regions of tessellations defining hyperbolic manifolds (one of them appears in the model of the genetic code [L95]) could be considered. $M^8 - H$ duality would give rise to holography associating to these 3-surfaces space-time surfaces in H as minimal surfaces with singularities as 4-D analogies to soap films with frames.

The generalization of the Fermi torus and its boundary (usually called Fermi sphere) as the counterpart of unit cell for a condensed matter cubic lattice to a fundamental region of a tessellation of hyperbolic space H^3 acting is discussed is discussed in [L103]. The number of tessellations is infinite and the properties of the hyperbolic manifolds of the "unit cells" are fascinating. For instance, their volumes define topological invariants and hyperbolic volumes for knot complements serve as knot invariants.

This picture resonates with an old guiding vision about TGD as an almost topological quantum field theory (QFT) [K39, K5, K98], which I have even regarded as a third strand in the 3-braid formed by the basic ideas of TGD based on geometry-number theory-topology trinity.

1. Kähler Chern-Simons form, also identifiable as a boundary term to which the instanton density of Kähler form reduces, defines an analog of topological QFT.
2. In the recent case the metric is however present via boundary conditions and in the dynamics in the interior of the space-time surface. However, the preferred extremal property essential for geometry-number theory duality transforms geometric invariants to topological invariants. Minimal surface property means that the dynamics of volume and Kähler action decouple outside the singularities, where minimal surface property fails. Coupling constants are present in the dynamics only at these lower-D singularities defining the analogs of frames of a 4-D soap film.

   Singularities also include string worlds sheets and partonic 2-surfaces. Partonic two-surfaces play the role of topological vertices and string world sheets couple partonic 2-orbits to a network. It is indeed known that the volume of a minimal surface can be regarded as a homological invariant.
3. If the 3-surfaces assignable to the mass shells H^3 define unit cells of hyperbolic tessellations and therefore hyperbolic manifolds, they also define topological invariants. Whether also string world sheets could define topological invariants is an interesting question.

2.9.2 Can one allow macroscopic Euclidean space-time regions

Euclidean space-time regions are not allowed in General Relativity. Can one allow them in TGD?

1. CP_2 extremals with a Euclidean induced metric and serving as correlates of elementary particles are basic pieces of TGD vision. The quantum numbers of fundamental fermions would reside at the light-like orbit of 2-D wormhole throat forming a boundary between Minkowskian space-time sheet and Euclidean wormhole contact- parton as I have called it. More precisely, fermionic quantum numbers would flow at the 1-D ends of 2-D string world sheets connecting the orbits of partonic 2-surfaces. The signature of the 4-metric would change at it.
2. It is difficult to invent any mathematical reason for excluding even macroscopic surfaces with Euclidean signature or even deformations of CP_2 type extremals with a macroscopic size. The simplest deformation of Minkowski space is to a flat Euclidean space as a warping of the canonical embedding $M^4 \subset M^4 \times S^1$ changing its signature.
3. I have wondered whether space-time sheets with an Euclidean signature could give rise to black-hole like entities. One possibility is that the TGD variants of blackhole-like objects have a space-time sheet which has, besides the counterpart of the ordinary horizon, an additional inner horizon at which the signature changes to the Euclidean one. This could take place already at Schwarzschild radius if g_{rr} component of the metric does not change its sign.

2.9.3 But are the normal components of isometry currents finite?

Whether this scenario works depends on whether the normal components for the isometry currents are finite.

1. $det(g_4) = 0$ condition gives boundaries of Euclidean and Minkowskian regions as 3-D light-like minimal surfaces. There would be no scales in accordance with generalized conformal invariance. g_{uv} in light-cone coordinates for M^2 vanishes and implies the vanishing of $det(g_4)$ and light-likeness of the 3-surface.

   What is important is that the formation of these regions would be unavoidable and they would be stable against perturbations.

2. $g^{uv}\sqrt{|g_4|}$ is finite if $det(g_4) = 0$ condition is satisfied, otherwise it diverges. The terms $g^{ui}\partial_i h^k\sqrt{|g_4|}$ must be finite. $g^{ui} = cof(g_{iu})/det(g_4)$ is finite since $g_{uv}g_{vu}$ in the cofactor cancels it from the determinant in the expression of g^{ui}. The presence of $\sqrt{|g_4|}$ implies that the these contributions to the boundary conditions vanish. Therefore only the condition boundary condition for g^{uv} remains.

3. If also Kähler action is present, the conditions are modified by replacing $T^{uk} = g^{u\alpha}\partial_\alpha h^k\sqrt{|g_4|}$ with a more general expression containing also the contribution of Kähler action. I have discussed the details of the variational problem in [K10, K5].

   The Kähler contribution involves the analogy of Maxwell's energy momentum tensor, which comes from the variation of the induced metric and involves sum of terms proportional to $J_{\alpha\mu}J_\mu^{beta}$ and $g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu}$.

   In the first term, the dangerous index raisings by g^{uv} appear 3 times. The most dangerous term is given by $J^{uv}J_v^v\sqrt{|g|} = g^{u\mu}g^{v\nu}J_{\alpha\beta}g^{vu}J_{vu}\sqrt{|g|}$. The divergent part is $g^{uv}g^{vu}J_{uv}g^{vu}J_{vu}\sqrt{|g|}$. The diverging g^{uv} appears 3 times and $J_{uv} = 0$ condition eliminates two of these. $g^{vu}\sqrt{|g|}$ is finite by $\sqrt{|g|} = 0$ condition. $J_{uv} = 0$ guarantees also the finiteness of the most dangerous part in $g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu}\sqrt{|g|}$.

   There is also an additional term coming from the variation of the induced Kähler form. This to the normal component of the isometry current is proportional to the quantity $J^{n\alpha}J_l^k\partial_\beta h^l\sqrt{|g|}$. Also now, the most singular term in $J^{u\beta} = g^{u\mu}g^{\beta\nu}J_{\mu\nu}$ corresponds to J^{uv} giving $g^{uv}g^{vu}J^{uv}\sqrt{|g|}$. This term is finite by $J_{uv} = 0$ condition.

   Therefore the boundary conditions are well-defined but only because $det(g_4) = 0$ condition is assumed.

4. Twistor lift strongly suggests that the assignment of the analogy of Kähler action also to M^4 and also this would contribute. All terms are finite if $det(g_4) = 0$ condition is satisfied.

5. The isometry currents in the normal direction must be equal to the divergences of the corresponding currents assignable to the Chern-Simons action at the boundary so that the flow of isometry charges to the boundary would go to the Chern-Simons isometry charges at the boundary.

   If the Chern-Simons term is absent, one expects that the boundary condition reduces to $\partial_v h^k = 0$. This would make X^3 2-dimensional so that Chern-Simons term is necessary. Note that light-likeness does not force the M^4 projection to be light-like so that the expansion of X^2 need not take with light-velocity. If CP_2 complex coordinates are holomorphic functions of W depending also on $U = v$ as a parameter, extended conformal invariance is obtained.

2.9.4 $det(g_4) = 0$ condition as a realization of quantum criticality

Quantum criticality is the basic dynamical principle of quantum TGD. What led to its discovery was the question "How to make TGD unique?". TGD has a single coupling constant, Kähler couplings strength, which is analogous to a critical temperature. The idea was obvious: require quantum criticality. This predicts a spectrum of critical values for the Kähler coupling strength. Quantum criticality would make the TGD Universe maximally complex. Concerning living matter, quantum critical dynamics is ideal since it makes the system maximally sensitive and maximallt reactive.

Concerning the realization of quantum criticality, it became gradually clear that the conformal invariance accompanying 2-D criticality, must be generalized. This led to the proposal that super symplectic symmetries, extended isometries and conformal symmetries of the metrically 2-D boundary of lightcone of M^4, and the extension of the Kac-Moody symmetries associated with the light-like boundaries of deformed CP_2 type extremals should act as symmetries of TGD extending the conformal symmetries of 2-D conformal symmetries. These huge infinite-D symmetries are also required by the existence of the Kähler geometry of WCW [K39, K21, K73] [L99, L109].

However, the question whether light-like boundaries of 3-surfaces with scale larger than CP_2 are possible, remained an open question. On the basis of preceding arguments, the answer seems to be affirmative and one can ask for the implications.

1. At M^8 level, the concrete realization of holography would involve two ingredients. The intersections of the space-time surface with the mass shells H^3 with mass squared value determined as the roots of polynomials P and the tlight-like 3-surfaces as $det(g_4) = 0$ surfaces as boundaries (genuine or between Minkowskian and Euclidean regions) associated by $M^8 - H$ duality to 4-surface of M^8 having associative normal space, which contains commutative 2-D subspace at each point. This would make possible both holography and $M^8 - H$ duality.

   Note that the identification of the algebraic geometric characteristics of the counterpart of $det(g_4) = 0$ surface at the level of H remains still open.

   Since holography determines the dynamics in the interior of the space-time surface from the boundary conditions, the classical dynamics can be said to be critical also in the interior.

2. Quantum criticality means ability to self-organize. Number theoretical evolution allows us to identify evolution as an increase of the algebraic complexity. The increase of the degree n of polynomial P serves as a measure for this. $n = h_{eff}/h_0$ also serves as a measure for the scale of quantum coherence, and dark matter as phases of matter would be characterized by the value of n.

3. The 3-D boundaries would be places where quantum criticality prevails. Therefore they would be ideal seats for the development of life. The proposal that the phase boundaries between water and ice serve as seats for the evolution of prebiotic life, is discussed from the point of TGD based view of quantum gravitation involving huge value of gravitational Planck constant $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ making possible quantum coherence in astrophysical scales [L105]. Density fluctuations would play an essential role, and this would mean that the volume enclosed by the 2-D M^4 projection of the space-time boundary would fluctuate. Note that these fluctuations are possible also at the level of the field body and magnetic body.

4. It has been said that boundaries, where the nervous system is located, distinguishes living systems from inanimate ones. One might even say that holography based on $det(g_4) = 0$ condition realizes nervous systems in a universal manner.

5. I have considered several variants for the holography in the TGD framework, in particular strong form of holography (SH). SH would mean that either the light-like 3-surfaces or the 3-surfaces at the ends of the causal diamond (CD) determine the space-time surface so that the 2-D intersections of the 3-D ends of the space-time surface with its light-like boundaries would determine the physics.

   This condition is perhaps too strong but a fascinating, weaker, possibility is that the internal consistency requires that the intersections of the 3-surface with the mass shells H^3 are identifiable as fundamental domains for the coset spaces $SO(1,3)/\Gamma$ defining tessellations of H^3 and hyperbolic manifolds. This would conform nicely with the TGD inspired model of genetic code [L95].

Chapter 3

Identification of WCW Kähler Function

3.1 Introduction

The topics of this chapter are the purely geometric aspects of the vision about physics as an infinite-dimensional Kähler geometry of the "world of classical worlds", with " classical world" identified either as light-like 3-D surface of the unique Bohr orbit like 4-surface traversing through it. The non-determinism of Kähler action forces to generalize the notion of 3-surface so that unions of space-like surfaces with time like separations must be allowed. Zero energy ontology allows to formulate this picture elegantly in terms of causal diamonds defined as intersections of future and past directed light-cones. Also a a geometric realization of coupling constant evolution and finite measurement resolution emerges.

There are two separate but closely related tasks involved.

1. Provide WCW with Kähler geometry which is consistent with 4-dimensional general coordinate invariance so that the metric is Diff^4 degenerate. General coordinate invariance implies that the definition of metric must assign to a given light-like 3-surface X^3 a 4-surface as a kind of Bohr orbit $X^4(X^3)$.
2. Provide WCW with a spinor structure. The great idea is to identify WCW gamma matrices in terms of super algebra generators expressible using second quantized fermionic oscillator operators for induced free spinor fields at the space-time surface assignable to a given 3-surface. The isometry generators and contractions of Killing vectors with gamma matrices would thus form a generalization of Super Kac-Moody algebra.

In this chapter a summary about basic ideas related to the construction of the Kähler geometry of infinite-dimensional configuration of 3-surfaces (more or less-equivalently, the corresponding 4-surfaces defining generalized Bohr orbits) or "world of classical worlds" (WCW).

3.1.1 The Quantum States Of Universe As Modes Of Classical Spinor Field In The "World Of Classical Worlds"

The vision behind the construction of WCW geometry is that physics reduces to the geometry of classical spinor fields in the infinite-dimensional WCW of 3-surfaces of $M^4_+ \times CP_2$ or $M^4 \times CP_2$, where M^4 and M^4_+ denote Minkowski space and its light cone respectively. This WCW might be called the "world of classical worlds".

Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that WCW possesses Kähler geometry. One of the basic features of the Kähler geometry is that it is solely determined by the so called. which defines both the J and the components of the g in complex coordinates via the general formulas [A61]

$$\begin{aligned} J &= i\partial_k\partial_{\bar{l}}K dz^k \wedge d\bar{z}^l \ . \\ ds^2 &= 2\partial_k\partial_{\bar{l}}K dz^k d\bar{z}^l \ . \end{aligned} \tag{3.1.1}$$

Kähler form is covariantly constant two-form and can be regarded as a representation of imaginary unit in the tangent space of the WCW

$$J_{mr}J^{rn} = -g_m^{\ n} \ . \tag{3.1.2}$$

As a consequence Kähler form defines also symplectic structure in WCW.

3.1.2 WCW Kähler Metric From Kähler Function

The task of finding Kähler geometry for the WCW reduces to that of finding Kähler function and identifying the complexification. The main constraints on the Kähler function result from the requirement of Diff^4 symmetry and degeneracy. requires that the definition of the Kähler function assigns to a given 3-surface X^3, which in Zero Energy Ontology is union of 3-surfaces at the opposite boundaries of causal diamond CD, a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with X^3. The natural guess is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of CP_2 coordinates.

Absolute minimization was the first guess for how to fix $X^4(X^3)$ uniquely. It has however become clear that this option might well imply that Kähler is negative and infinite for the entire Universe so that the vacuum functional would be identically vanishing. This condition can make sense only inside wormhole contacts with Euclidian metric and positive definite Kähler action.

Quantum criticality of TGD Universe suggests the appropriate principle to be the criticality, that is vanishing of the second variation of Kähler action. This principle now follows from the conservation of Noether currents the Kähler-Dirac action. This formulation is still rather abstract and if spinors are localized to string world sheets, it it is not satisfactory. A further step in progress was the realization that preferred extremals could carry vanishing super-conformal Noether charges for sub-algebras whose generators have conformal weight vanishing modulo n with n identified in terms of effective Planck constant $h_{eff}/h = n$.

If Kähler action would define a strictly deterministic variational principle, Diff^4 degeneracy and general coordinate invariance would be achieved by restricting the consideration to 3-surfaces Y^3 at the boundary of M^4_+ and by defining Kähler function for 3-surfaces X^3 at $X^4(Y^3)$ and diffeo-related to Y^3 as $K(X^3) = K(Y^3)$. The classical non-determinism of the Kähler action however introduces complications. As a matter fact, the hierarchy of Planck constants has nice interpretation in terms of non-determinism: the space-time sheets connecting the 3-surface at the ends of CD form n conformal equivalence classes. This would correspond to the non-determinism of quantum criticality accompanied by generalized conformal invariance

3.1.3 WCW Kähler Metric From Symmetries

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan [A54] [A54] has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that WCW is a union of symmetric spaces labelled by zero modes not appearing in the line element as differentials. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and embedding space which is a product of four-dimensional Minkowski space or its future light cone with CP_2.

The detailed formulas for the matrix elements of the Kähler metric however remain educated guesses so that this approach is not entirely satisfactory.

3.1.4 WCW Kähler Metric As Anti-commutators Of Super-Symplectic Super Noether Charges

The third approach identifies the Kähler metric of WCW as anti-commutators of WCW gamma matrices. This is not yet enough to get concrete expressions but the identification of WCW gamma matrices as Noether super-charges for super-symplectic algebra assignable to the boundary of WCW changes the situation. One also obtains a direct connection with elementary particle physics.

The super charges are linear in the mode of induced spinor field and second quantized spinor field itself, and involve the infinitesimal action of symplectic generator on the spinor field. One can fix fermionic anti-commutation relations by second quantization of the induced spinor fields (as a matter fact, here one can still consider two options). Hence one obtains explicit expressions for the matrix elements of WCW metric.

If the induced spinor fields are localized at string world sheets - as the well-definedness of em charge and number theoretic arguments suggest - one obtains an expression for the matrix elements of the metric in terms of 1-D integrals over strings connecting partonic 2-surfaces. If spinors are localized to string world sheets also in the interior of CP_2, the integral is over a closed circle and could have a representation analogous to a residue integral so that algebraic continuation to p-adic number fields might become straightforward.

The matrix elements of WCW metric are labelled by the conformal weights of spinor modes, those of symplectic vector fields for light-like CD boundaries and by labels for the irreducible representations of $SO(3)$ acting on light-cone boundary $\delta M^4_\pm = R_+ \times S^2$ and of $SU(3)$ acting in CP_2. The dependence on spinor modes and their conformal weights could not be guessed in the approach based on symmetries only. The presence of two rather than only one conformal weights distinguishes the metric from that for loop spaces [A54] and reflects the effective 2-dimensionality. The metric codes a rather scarce information about 3-surfaces. This is in accordance with the notion of finite measurement resolution. By increasing the number of partonic 2-surfaces and string world sheets the amount of information coded - measurement resolution - increases. Fermionic quantum state gives information about 3-geometry. The alternative expression for WCW metric in terms of Kähler function means analog of AdS/CFT duality: Kähler metric can be expressed either in terms of Kähler action associated with the Euclidian wormhole contacts defining Kähler function or in terms of the fermionic oscillator operators at string world sheets connecting partonic 2-surfaces.

3.1.5 What Principle Selects The Preferred Extremals?

In positive energy ontology space-time surfaces should be analogous to Bohr orbits in order to make possible possible realization of general coordinate invariance. The first guess was that absolute minimization of Kähler action might be the principle selecting preferred extremals. One can criticize the assumption that extremals correspond to the absolute minima of Kähler action for entire spacetime surface, as too strong since Kähler action from Minkowskian regions is proportional to imaginary unit and corresponds to ordinary QFT action defining a phase factor of vacuum functional. Furthermore, the notion of absolute minimization does not make sense in p-adic context unless one manages to reduce it to purely algebraic conditions. Absolute minimization could however make sense for Euclidian space-time regions defining the lines of generalized Feynman diagras, where Kähler action has definite sign. Kähler function is indeed the Kähler action for these regions.

What is needed is the association of a unique space-time surface to a given 3-surface defined as union of 3-surfaces at opposite boundaries of CD. One can imagine many way to achieve this. "Unique" is too much to demand: for the proposal unique space-time surface is replaced with finite number of conformal gauge equivalence classes of space-time surfaces. In any case, it is better to talk just about preferred extremals of Kähler action and accept as the fact that there are several proposals for what this notion could mean.

1. For instance, one can consider the identification of space-time surface as associative (co-associative) sub-manifold meaning that tangent space of space-time surface can be regarded as associative (co-associative) sub-manifold of complexified octonions defining tangent space of embedding space. One way to define "associative sub-manifold" is by introducing octonionic representation of embedding space gamma matrices identified as tangent space vectors. It must be also assumed that the tangent space contains a preferred commutative (co-commutative) sub-space at each point and defining an integrable distribution having identification as string world sheet (also slicing of space-time sheet by string world sheets can be

considered). Associativity and commutativity would define the basic dynamical principle. A closely related approach is based on so called Hamilton-Jacobi structure [K10] defining also this kind of slicing and the approaches could be equivalent.

2. In zero energy ontology (ZEO) 3-surfaces become pairs of space-like 3-surfaces at the boundaries of causal diamond (CD). Even the light-like partonic orbits could be included to give the analog of Wilson loop. In absence of non-determinism of Kähler action this forces to ask whether the attribute "preferred" is un-necessary. There are however excellent reasons to expect that there is an infinite gauge degeneracy assignable to quantum criticality and represented in terms of Kac-Moody type transformations of partonic orbits respecting their light-likeness and giving rise to the degeneracy behind hierarchy of Planck constants $h_{eff} = n \times h$. n would give the number of conformal equivalence classes of space-time surfaces with same ends. In given measurement resolution one might however hope that the "preferred" could be dropped away.

   The already mentioned vanishing of Noether charges for sub-algebras of conformal algebras with conformal weights coming as multiples of n at the ends of space-time surface would be a concrete realization of this picture.

3. The construction of quantum TGD in terms of the Kähler- Dirac action associated with Kähler action led to a possible answer to the question about the principle selecting preferred extremals. The Noether currents associated with Kähler-Dirac action are conserved if second variations of Kähler action vanish. This is nothing but space-time correlate for quantum criticality and it is amusing that I failed to realize this for so long time. A further very important result is that in generic case the modes of induced spinor field are localized at 2-D surfaces from the condition that em charge is well-defined quantum number (W fields must vanish and also Z^0 field above weak scale in order to avoid large parity breaking effects). The criticality conditions are however rather complicated and it seems that the vanishing of the symplectic Noether charges is the practical way to formulate what "preferred" does mean.

In this chapter I will first consider the basic properties of the WCW, briefly discuss the various approaches to the geometrization of the WCW, and introduce the alternative strategies for the construction of Kähler metric based on a direct guess of Kähler function, on the group theoretical approach assuming that WCW can be regarded as a union of symmetric spaces, and on the identification of Kähler metric as anti-commutators of gamma matrices identified as Noether super charges for the symplectic algebra. After these preliminaries a definition of the Kähler function is proposed and various physical and mathematical motivations behind the proposed definition are discussed. The key feature of the Kähler action is classical non-determinism, and various implications of the classical non-determinism are discussed.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L22].

3.2 WCW

The view about configuration space ("world of classical worlds", WCW) has developed considerably during the last two decades. Here only the recent view is summarized in order to not load reader with unessential details.

3.2.1 Basic Notions

The notions of embedding space, 3-surface (and 4-surface), and WCW or "world of classical worlds" (WCW), are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$ (see **Figs.** `http://tgdtheory.fi/appfigures/Hoo.jpg`, `http://tgdtheory.fi/appfigures/cp2.jpg`, `http://tgdtheory.fi/appfigures/Hoo.futurepast`, `http://tgdtheory.fi/appfigures/penrose.jpg`, which are also in the appendix of this book), and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize GCI. During years these notions have however evolved considerably.

The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K81, K82, K80].

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book. As matter fact, this gluing idea generalizes to the level of WCW .

2. With the discovery of zero energy ontology [K96, K20] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [L55] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K29] led to a further generalization of the notion of embedding space. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and possibly also factor spaces of CD and CP_2 to form a book like structure. There are good physical and mathematical arguments suggesting that only the singular coverings should be allowed [K80]. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW .

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial and the receont view is an outcome of a long and tedious process involving many hastily done mis-interpretations.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence implied by GCI. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the GCI in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. Light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces. Therefore it seems that one must choose between light-like and space-like 3-surfaces or assume generalized GCI requiring that equivalently either space-like 3-surfaces or light-like 3-surfaces at the ends of CDs can be identified as the fundamental geometric objects. General GCI requires that the basic objects correspond to the partonic 2-surfaces identified as intersections of these 3-surfaces plus common 4-D tangent space distribution.

   At the level of WCW metric this suggests that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. Since the information about normal space of the 2-surface is needed one has only effective 2-dimensionality. Weak form of self-duality [K21] however implies that the normal data (flux Hamiltonians associated with Kähler electric field) reduces to magnetic flux Hamiltonians. This is essential for conformal symmetries and also simplifies the construction enormously.

It however turned out that this picture is too simplistic. It turned out that the solutions of the Kähler-Dirac equation are localized at 2-D string world sheets, and this led to a generalization of the formulation of WCW geometry: given point of partonic 2-surface is effectively replaced with a string emanating from it and connecting it to another partonic 2-surface. Hence the formulation becomes 3-dimensional but thanks to super-conformal symmetries acting like gauge symmetries one obtains effective 2-dimensionality albeit in weaker sense [K73].

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further but inessential complication relates to the hierarchy of Planck constants forcing to generalize the notion of embedding space and also to the fact that for non-standard values of Planck constant there is symmetry breaking due to preferred plane M^2 preferred homologically trivial geodesic sphere of CP_2 having interpretation as geometric correlate for the selection of quantization axis. For given sector of CH this means union over choices of this kind.

The basic vision forced by the generalization of GCI has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

The study of the Kähler-Dirac equation led to the realization that classical field equations for Kähler action can be seen as consistency conditions for the Kähler-Dirac action and led to the identification of preferred extremals in terms of criticality. This identification which follows naturally also from quantum criticality.

1. The condition that electromagnetic charge is well-defined for the modes of Kähler-Dirac operator implies that in the generic case the modes are restricted to 2-D surfaces (string world sheets or possibly also partonic 2-surfaces) with vanishing W fields [K96]. Above weak scale at least one can also assume that Z^0 field vanishes. Also for space-time surfaces with 2-D CP_2 projection (cosmic strongs would be examples) the localization is expected to be possible. This localization is possible only for Kähler action and the set of these 2-surfaces is discrete except for the latter case. The stringy form of conformal invariance allows to solve Kähler-Dirac equation just like in string models and the solutions are labelled by integer valued conformal weights.

2. The next step of progress was the realization that the requirement that the conservation of the Noether currents associated with the Kähler-Dirac equation requires that the second variation of the Kähler action vanishes. In strongest form this condition would be satisfied for all variations and in weak sense only for those defining dynamical symmetries. The interpretation is as a space-time correlate for quantum criticality and the vacuum degeneracy of Kähler action makes the criticality plausible.

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig. ??** in the appendix of this book).

Weak form of electric-magnetic duality gives a precise formulation for how Kähler coupling strength is visible in the properties of preferred extremals. A generalization of the ideas of the catastrophe theory to infinite-dimensional context results. These conditions make sense also in p-adic context and have a number theoretical universal form.

The notion of number theoretical compactication led to important progress in the understanding of the preferred extremals and the conjectures were consistent with what is known about the known extremals.

1. The conclusion was that one can assign to the 4-D tangent space $T(X^4(X_l^3)) \subset M^8$ a subspace $M^2(x) \subset M^4$ having interpretation as the plane of non-physical polarizations. This in the case that the induced metric has Minkowskian signature. If not, and if co-hyper-quaternionic surface is in question, similar assigned should be possible in normal space. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW . This is as it must be since complexification does not make sense in M^2 degrees of freedom.
2. In number theoretical framework $M^2(x)$ has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. The condition $M^2(x) \subset T(X^4(X_l^3)))$ in principle fixes the tangent space at X_l^3, and one has good hopes that the boundary value problem is well-defined and could fix $X^4(X^3)$ at least partially as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2(x) \subset M^4$ plays also other important roles.
3. At the level of H the counterpart for the choice of $M^2(x)$ seems to be following. Suppose that $X^4(X_l^3)$ has Minkowskian signature. One can assign to each point of the M^4 projection $P_{M^4}(X^4(X_l^3))$ a sub-space $M^2(x) \subset M^4$ and its complement $E^2(x)$, and the distributions of these planes are integrable and define what I have called Hamilton-Jacobi coordinates which can be assigned to the known extremals of Kähler with Minkowskian signature. This decomposition allows to slice space-time surfaces by string world sheets and their 2-D partonic duals. Also a slicing to 1-D light-like surfaces and their 3-D light-like duals Y_l^3 parallel to X_l^3 follows under certain conditions on the induced metric of $X^4(X_l^3)$. This decomposition exists for known extremals and has played key role in the recent developments. Physically it means that 4-surface (3-surface) reduces effectively to 3-D (2-D) surface and thus holography at space-time level. A physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K41] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.
4. The weakest form of number theoretic compactification [K82] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

If one takes M^-H duality seriously, one must conclude that one can choose any partonic 2-surface in the slicing of X^4 as a representative. This means gauge invariance reflect in the definition of Kähler function as $U(1)$ gauge transformation $K \to K + f + \overline{f}$ having no effect on Kähler metric and Kähler form.

Although the details of this vision might change it can be defended by its ability to fuse together all great visions about quantum TGD. In the sequel the considerations are restricted to 3-surfaces in $M^4_\pm \times CP_2$. The basic outcome is that Kähler metric is expressible using the data at partonic 2-surfaces $X^2 \subset \delta M^4_+ \times CP_2$. The generalization to the actual physical situation requires the replacement of $X^2 \subset \delta M^4_+ \times CP_2$ with unions of partonic 2-surfaces located at light-like boundaries of CDs and sub-CDs.

The notions of space-time sheet and many-sheeted space-time are basic pieces of TGD inspired phenomenology (see **Fig. ??** in the appendix of this book). Originally the space-time sheet was understood to have a boundary as "sheet" strongly suggests. It has however become clear that genuine boundaries are not allowed. Rather, space-time sheet is typically double (at least) covering of M^4. The light-like 3-surfaces separating space-time regions with Euclidian and

Minkowskian signature are however very much like boundaries and define what I call generalized Feynman diagrams. A fascinating possibility is that every material object is accompanied by an Euclidian region representing the interior of the object and serving as TGD analog for blackhole like object. Space-time sheets suffer topological condensation (gluing by wormhole contacts or topological sum in more mathematical jargon) at larger space-time sheets. Space-time sheets form a length scale hierarchy. Quantitative formulation is in terms of p-adic length scale hypothesis and hierarchy of Planck constants proposed to explain dark matter as phases of ordinary matter.

The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the basis question is "M^4_+ or M^4?" and that this question had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.
2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW .
3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW . The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW (WCW) is a union of WCW s associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. It must be however emphasized that Kähler function depends on partonic 2-surfaces at both ends of space-time surface so that WCW is topologically Cartesian product of corresponding symmetric spaces. WCW metric must therefore have parts corresponding to the partonic 2-surfaces (free part) and also an interaction term depending on the partonic 2-surface at the opposite ends of the light-like 3-surface. The conclusion is that geometrization reduces to that for single like of generalized Feynman diagram containing partonic 2-surfaces at its ends. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case corresponding to a line of generalized Feynman diagram. One can also deduce the free part of the metric by restricting the consideration to partonic 2-surfaces at single end of generalized Feynman diagram.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.
2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of

$\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub- WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!

3.2.2 Constraints On WCW Geometry

The constraints on the WCW result both from the infinite dimension of WCW and from physically motivated symmetry requirements. There are three basic physical requirements on the WCW geometry: namely four-dimensional GCI in strong form, Kähler property and the decomposition of WCW into a union $\cup_i G/H_i$ of symmetric spaces G/H_i, each coset space allowing G-invariant metric such that G is subgroup of some "universal group" having natural action on 3-surfaces. Together with the infinite dimensionality of WCW these requirements pose extremely strong constraints on WCW geometry. In the following we shall consider these requirements in more detail.

Diff^4 invariance and Diff^4 degeneracy

Diff^4 plays fundamental role as the gauge group of General Relativity. In string models $Diff^2$ invariance ($Diff^2$ acts on the orbit of the string) plays central role in making possible the elimination of the time like and longitudinal vibrational degrees of freedom of string. Also in the present case the elimination of the tachyons (time like oscillatory modes of 3-surface) is a physical necessity and Diff^4 invariance provides an obvious manner to do the job.

In the standard path l integral formulation the realization of Diff^4 invariance is an easy task at the formal level. The problem is however that path integral over four-surfaces is plagued by divergences and doesn't make sense. In the present case WCW consists of 3-surfaces and only $Diff^3$ emerges automatically as the group of re-parameterizations of 3-surface. Obviously one should somehow define the action of Diff^4 in the space of 3-surfaces. Whatever the action of Diff^4 is it must leave the WCW metric invariant. Furthermore, the elimination of tachyons is expected to be possible only provided the time like deformations of the 3-surface correspond to zero norm vector fields of WCW so that 3-surface and its Diff^4 image have zero distance. The conclusion is that WCW metric should be both Diff^4 invariant and Diff^4 degenerate.

The problem is how to define the action of Diff^4 in $C(H)$. Obviously the only manner to achieve Diff^4 invariance is to require that the very definition of the WCW metric somehow associates a unique space time surface to a given 3-surface for Diff^4 to act on. The obvious physical interpretation of this space time surface is as "classical space time" so that "Classical Physics" would be contained in WCW geometry. In fact, this space-time surface is analogous to Bohr orbit so that semiclassical quantization rules become an exact part of the quantum theory. It is this requirement, which has turned out to be decisive concerning the understanding of the WCW geometry.

Decomposition of WCW into a union of symmetric spaces G/H

The extremely beautiful theory of finite-dimensional symmetric spaces constructed by Elie Cartan suggests that WCW should possess decomposition into a union of coset spaces $CH = \cup_i G/H_i$ such that the metric inside each coset space G/H_i is left invariant under the infinite dimensional isometry group G. The metric equivalence of surfaces inside each coset space G/H_i does not mean that 3-surfaces inside G/H_i are physically equivalent. The reason is that the vacuum functional is exponent of Kähler action which is not isometry invariant so that the 3-surfaces, which correspond to maxima of Kähler function for a given orbit, are in a preferred position physically. For instance, one can imagine of calculating functional integral around this maximum perturbatively. Symmetric space property actually allows also much more powerful non-perturbative approach based on harmonic analysis [K96]. The sum of over i means actually integration over the zero modes of the metric (zero modes correspond to coordinates not appearing as coordinate differentials in the metric tensor).

The coset space G/H is a symmetric space only under very special Lie-algebraic conditions. Denoting the decomposition of the Lie-algebra g of G to the direct sum of H Lie-algebra h and its complement t by $g = h \oplus t$, one has

$$[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ .$$

This decomposition turn out to play crucial role in guaranteeing that G indeed acts as isometries and that the metric is Ricci flat.

The four-dimensional $Diff$ invariance indeed suggests to a beautiful solution of the problem of identifying G. The point is that any 3-surface X^3 is $Diff^4$ equivalent to the intersection of $X^4(X^3)$ with the light cone boundary. This in turn implies that 3-surfaces in the space $\delta H = \delta M^4_+ \times CP_2$ should be all what is needed to construct WCW geometry. The group G can be identified as some subgroup of diffeomorphisms of δH and H_i contains that subgroup of G, which acts as diffeomorphisms of the 3-surface X^3. Since G preserves topology, WCW must decompose into union $\cup_i G/H_i$, where i labels 3-topologies and various zero modes of the metric. For instance, the elements of the Lie-algebra of G invariant under WCW complexification correspond to zero modes.

The reduction to the light cone boundary, identifiable as the moment of big bang, looks perhaps odd at first. In fact, it turns out that the classical non-determinism of Kähler action does not allow the complete reduction to the light cone boundary: physically this is a highly desirable implication but means a considerable mathematical challenge.

Kähler property

Kähler property implies that the tangent space of the configuration space allows complexification and that thereexists a covariantly constant two-form J_{kl}, which can be regarded as a representation of the imaginary unit in the tangent space of the WCW :

$$J_k^{\ r} J_{rl} = -G_{kl} \ . \tag{3.2.1}$$

There are several physical and mathematical reasons suggesting that WCW metric should possess Kähler property in some generalized sense.

1. The deepest motivation comes from the need to geometrize hermitian conjugation which is basic mathematical operation of quantum theory.
2. Kähler property turns out to be a necessary prerequisite for defining divergence free WCW integration. We will leave the demonstration of this fact later although the argument as such is completely general.
3. Kähler property very probably implies an infinite-dimensional isometry loop groups $Map(S^1, G)$ [A54] shows that loop group allows only

   Riemann connection and this metric allows local G as its isometries!

   To see this consider the construction of Riemannian connection for $Map(X^3, H)$. The defining formula for the connection is given by the expression

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y,Z) + Y(Z,X) - Z(X,Y) \\ &+ ([X,Y],Z) + ([Z,X],Y) - ([Y,Z],X) \end{aligned} \tag{3.2.2}$$

   X, Y, Z are smooth vector fields in $Map(X^3, G)$. This formula defines $\nabla_X Y$ uniquely provided the tangent space of Map is complete with respect to Riemann metric. In the finite-dimensional case completeness means that the inverse of the covariant metric tensor exists so that one can solve the components of connection from the conditions stating the covariant constancy of the metric. In the case of the loop spaces with Kähler metric this is however not the case.

Now the symmetry comes into the game: if X, Y, Z are left (local gauge) invariant vector fields defined by the Lie-algebra of local G then the first three terms drop away since the scalar products of left invariant vector fields are constants. The expression for the covariant derivative is given by

$$\nabla_X Y = (Ad_X Y - Ad_X^* Y - Ad_Y^* X)/2 \tag{3.2.3}$$

where Ad_X^* is the adjoint of Ad_X with respect to the metric of the loop space.

At this point it is important to realize that Freed's argument does not force the isometry group of WCW to be $Map(X^3, M^4 \times SU(3))$! Any symmetry group, whose Lie algebra is complete with respect to the WCW metric (in the sense that any tangent space vector is expressible as superposition of isometry generators modulo a zero norm tangent vector) is an acceptable alternative.

The Kähler property of the metric is quite essential in one-dimensional case in that it leads to the requirement of left invariance as a mathematical consistency condition and we expect that dimension three makes no exception in this respect. In 3-dimensional case the degeneracy of the metric turns out to be even larger than in 1-dimensional case due to the four-dimensional Diff degeneracy. So we expect that the metric ought to possess some infinite-dimensional isometry group and that the above formula generalizes also to the 3-dimensional case and to the case of local coset space. Note that in M^4 degrees of freedom $Map(X^3, M^4)$ invariance would imply the flatness of the metric in M^4 degrees of freedom.

The physical implications of the above purely mathematical conjecture should not be underestimated. For example, one natural looking manner to construct physical theory would be based on the idea that configuration space geometry is dynamical and this approach is followed in the attempts to construct string theories [B19]. Various physical considerations (in particular the need to obtain oscillator operator algebra) seem to imply that WCW geometry is necessarily Kähler. The above result however states that WCW Kähler geometry cannot be dynamical quantity and is dictated solely by the requirement of internal consistency. This result is extremely nice since it has been already found that the definition of the WCW metric must somehow associate a unique classical space time and "classical physics" to a given 3-surface: uniqueness of the geometry implies the uniqueness of the "classical physics".

4. The choice of the embedding space becomes highly unique. In fact, the requirement that WCW is not only symmetric space but also (contact) Kähler manifold inheriting its (degenerate) Kähler structure from the embedding space suggests that spaces, which are products of four-dimensional Minkowski space with complex projective spaces CP_n, are perhaps the only possible candidates for H. The reason for the unique position of the four-dimensional Minkowski space turns out to be that the boundary of the light cone of D-dimensional Minkowski space is metrically a sphere S^{D-2} despite its topological dimension $D-1$: for $D = 4$ one obtains two-sphere allowing Kähler structure and infinite parameter group of conformal symmetries!

5. It seems possible to understand the basic mathematical structures appearing in string model in terms of the Kähler geometry rather nicely.

   (a) The projective representations of the infinite-dimensional isometry group (not necessarily Map!) correspond to the ordinary representations of the corresponding centrally extended group [A69]. The representations of Kac Moody group indeed play central role in string models [B45, B43] and WCW approach would explain their occurrence, not as a result of some quantization procedure, but as a consequence of symmetry of the underlying geometric structure.

   (b) The bosonic oscillator operators of string models would correspond to centrally extended Lie-algebra generators of the isometry group acting on spinor fields of the WCW .

   (c) The "fermionic" fields (Ramond fields, Schwartz, Green) should correspond to gamma matrices of the WCW . Fermionic oscillator operators would correspond simply to contractions of isometry generators j_A^k with complexified gamma matrices of WCW

$$\begin{aligned}\Gamma_A^{\pm} &= j_A^k \Gamma_k^{\pm} \\ \Gamma_k^{\pm} &= (\Gamma^k \pm J_l^k \Gamma^l)/\sqrt{2}\end{aligned} \quad (3.2.4)$$

(J_l^k is the Kähler form of WCW) and would create various spin excitations of WCW spinor field. $\Gamma_k^{\pm}$ are the complexified gamma matrices, complexification made possible by the Kähler structure of the WCW .

This suggests that some generalization of the so called Super Kac Moody algebra of string models [B45, B43] should be regarded as a spectrum generating algebra for the solutions of field equations in configuration space.

Although the Kähler structure seems to be physically well motivated there is a rather heavy counter argument against the whole idea. Kähler structure necessitates complex structure in the tangent space of WCW . In CP_2 degrees of freedom no obvious problems of principle are expected: WCW should inherit in some sense the complex structure of CP_2.

In Minkowski degrees of freedom the signature of the Minkowski metric seems to pose a serious obstacle for complexification: somehow one should get rid of two degrees of freedom so that only two Euclidian degrees of freedom remain. An analogous difficulty is encountered in quantum field theories: only two of the four possible polarizations of gauge boson correspond to physical degrees of freedom: mathematically the wrong polarizations correspond to zero norm states and transverse states span a complex Hilbert space with Euclidian metric. Also in string model analogous situation occurs: in case of D-dimensional Minkowski space only $D-2$ transversal degrees of freedom are physical. The solution to the problem seems therefore obvious: WCW metric must be degenerate so that each vibrational mode spans effectively a 2-dimensional Euclidian plane allowing complexification.

We shall find that the definition of Kähler function to be proposed indeed provides a solution to this problem and also to the problems listed before.

1. The definition of the metric doesn't differentiate between 1- and N-particle sectors, avoids spin statistics difficulty and has the physically appealing property that one can associate to each 3-surface a unique classical space time: classical physics is described by the geometry of WCW and d the geometry of WCW is determined uniquely by the requirement of mathematical consistency.

2. Complexification is possible only provided the dimension of the Minkowski space equals to four and is due to the effective 3-dimensionality of light-cone boundary.

3. It is possible to identify a unique candidate for the necessary infinite-dimensional isometry group G. G is subgroup of the diffeomorphism group of $\delta M_+^4 \times CP_2$. Essential role is played by the fact that the boundary of the four-dimensional light cone, which, despite being topologically 3-dimensional, is metrically two-dimensional Euclidian sphere, and therefore allows infinite-parameter groups of isometries as well as conformal and symplectic symmetries and also Kähler structure unlike the higher-dimensional light cone boundaries. Therefore WCW metric is Kähler only in the case of four-dimensional Minkowski space and allows symplectic $U(1)$ central extension without conflict with the no-go theorems about higher dimensional central extensions.

   The study of the vacuum degeneracy of Kähler function defined by Kähler action forces to conclude that the isometry group must consist of the symplectic transformations of $\delta H = \delta M_+^4 \times CP_2$. The corresponding Lie algebra can be regarded as a loop algebra associated with the symplectic group of $S^2 \times CP_2$, where S^2 is $r_M = constant$ sphere of light cone boundary. Thus the finite-dimensional group G defining loop group in case of string models extends to an infinite-dimensional group in TGD context. This group has a monstrous size. The radial Virasoro localized with respect to $S^2 \times CP_2$ defines naturally complexification for both G and H. The general form of the Kähler metric deduced on basis of this symmetry has same qualitative properties as that deduced from Kähler function identified as preferred extremal of Kähler action. Also the zero modes, among them isometry invariants, can be identified.

4. The construction of the WCW spinor structure is based on the identification of the WCW gamma matrices as linear superpositions of the oscillator operators associated with the second quantized induced spinor fields. The extension of the symplectic invariance to super symplectic

invariance fixes the anti-commutation relations of the induced spinor fields, and WCW gamma matrices correspond directly to the super generators. Physics as number theory vision suggests strongly that WCW geometry exists for 8-dimensional embedding space only and that the choice $M^4_+ \times CP_2$ for the embedding space is the only possible one.

3.3 Identification Of The Kähler Function

There are three approaches to the construction of the WCW geometry: a direct physics based guess of the Kähler function, a group theoretic approach based on the hypothesis that CH can be regarded as a union of symmetric spaces, and the approach based on the construction of WCW spinor structure first by second quantization of induced spinor fields. Here the first approach is discussed.

3.3.1 Definition Of Kähler Function

Consider first the basic definitions related to Kähler metric and Kähler function.

Kähler metric in terms of Kähler function

Quite generally, Kähler function K defines Kähler metric in complex coordinates via the following formula

$$J_{k\bar{l}} = ig_{k\bar{l}} = i\partial_k\partial_{\bar{l}}K \ . \tag{3.3.1}$$

Kähler function is defined only modulo a real part of holomorphic function so that one has the gauge symmetry

$$K \rightarrow K + f + \overline{f} \ . \tag{3.3.2}$$

Let X^3 be a given 3-surface and let X^4 be any four-surface containing X^3 as a sub-manifold: $X^4 \supset X^3$. The 4-surface X^4 possesses in general boundary. If the 3-surface X^3 has nonempty boundary δX^3 then the boundary of X^3 belongs to the boundary of X^4: $\delta X^3 \subset \delta X^4$.

Induced Kähler form and its physical interpretation

Induced Kähler form defines a Maxwell field and it is important to characterize precisely its relationship to the gauge fields as they are defined in gauge theories. Kähler form J is related to the corresponding Maxwell field F via the formula

$$J = xF \ , \ x = \frac{g_K}{\hbar} \ . \tag{3.3.3}$$

Similar relationship holds true also for the other induced gauge fields. The inverse proportionality of J to $\hbar$ does not matter in the ordinary gauge theory context where one routinely choses units by putting $\hbar = 1$ but becomes very important when one considers a hierarchy of Planck constants [K29].

Unless one has $J = (g_K/\hbar_0)$, where $\hbar_0$ corresponds to the ordinary value of Planck constant, $\alpha_K = g_K^2/4\pi\hbar$ together the large Planck constant means weaker interactions and convergence of the functional integral defined by the exponent of Kähler function and one can argue that the convergence of the functional integral is what forces the hierarchy of Planck constants. This is in accordance with the vision that Mother Nature likes theoreticians and takes care that the perturbation theory works by making a phase transition increasing the value of the Planck constant in the situation when perturbation theory fails. This leads to a replacement of the M^4 (or more precisely, causal diamond CD) and CP_2 factors of the embedding space ($CD \times CP_2$) with its $r = \hbar/\hbar_0$-fold singular covering (one can consider also singular factor spaces). If the components of the space-time surfaces at the sheets of the covering are identical, one can interpret r-fold value of Kähler action as a sum of r identical contributions from the sheets of the covering with ordinary value of Planck constant and forget the presence of the covering. Physical states are however different even in the case that one assumes that sheets carry identical quantum states and anyonic phase could correspond to this kind of phase [K64].

Kähler action

One can associate to Kähler form Maxwell action and also Chern-Simons anomaly term proportional to $\int_{X^4} J \wedge J$ in well known manner. Chern Simons term is purely topological term and well defined for orientable 4-manifolds, only. Since there is no deep reason for excluding non-orientable space-time surfaces it seems reasonable to drop Chern Simons term from consideration. Therefore Kähler action $S_K(X^4)$ can be defined as

$$S_K(X^4) = k_1 \int_{X^4;X^3 \subset X^4} J \wedge (*J) \ . \tag{3.3.4}$$

The sign of the square root of the metric determinant, appearing implicitly in the formula, is defined in such a way that the action density is negative for the Euclidian signature of the induced metric and such that for a Minkowskian signature of the induced metric Kähler electric field gives a negative contribution to the action density.

The notational convention

$$k_1 \equiv \frac{1}{16\pi\alpha_K} \ , \tag{3.3.5}$$

where α_K will be referred as Kähler coupling strength will be used in the sequel. If the preferred extremals minimize/maximize [K82] the absolute value of the action in each region where action density has a definite sign, the value of α_K can depend on space-time sheet.

Kähler function

One can define the Kähler function in the following manner. Consider first the case $H = M^4_+ \times CP_2$ and neglect for a moment the non-determinism of Kähler action. Let X^3 be a 3-surface at the light-cone boundary $\delta M^4_+ \times CP_2$. Define the value $K(X^3)$ of Kähler function K as the value of the Kähler action for some preferred extremal in the set of four-surfaces containing X^3 as a sub-manifold:

$$K(X^3) = K(X^4_{pref}) \ , \ X^4_{pref} \subset \{X^4 | X^3 \subset X^4\} \ . \tag{3.3.6}$$

The most plausible identification of preferred extremals is in terms of quantum criticality in the sense that the preferred extremals allow an infinite number of deformations for which the second variation of Kähler action vanishes. Combined with the weak form of electric-magnetic duality forcing appearance of Kähler coupling strength in the boundary conditions at partonic 2-surfaces this condition might be enough to fix preferred extremals completely.

The precise formulation of Quantum TGD has developed rather slowly. Only quite recently-33 years after the birth of TGD - I have been forced to reconsider the question whether the precise identification of Kähler function. Should Kähler function actually correspond to the Kähler action for the space-time regions with Euclidian signature having interpretation as generalized Feynman graphs? If so what would be the interpretation for the Minkowskian contribution?

1. If one accepts just the formal definition for the square root of the metric determinant, Minkowskian regions would naturally give an imaginary contribution to the exponent defining the vacuum functional. The presence of the phase factor would give a close connection with the path integral approach of quantum field theories and the exponent of Kähler function would make the functional integral well-defined.

2. The weak form of electric magnetic duality would reduce the contributions to Chern-Simons terms from opposite sides of wormhole throats with degenerate four-metric with a constraint term guaranteeing the duality.

The motivation for this reconsideration came from the applications of ideas of Floer homology to TGD framework [K48]: the Minkowskian contribution to Kähler action for preferred extremals would define Morse function providing information about WCW homology. Both Kähler and Morse would find place in TGD based world order.

One of the nasty questions about the interpretation of Kähler action relates to the square root of the metric determinant. If one proceeds completely straightforwardly, the only reason conclusion is that the square root is imaginary in Minkowskian space-time regions so that Kähler action would be complex. The Euclidian contribution would have a natural interpretation as positive definite Kähler function but how should one interpret the imaginary Minkowskian contribution? Certainly the path integral approach to quantum field theories supports its presence. For some mysterious reason I was able to forget this nasty question and serious consideration of the obvious answer to it. Only when I worked between possibile connections between TGD and Floer homology [K48] I realized that the Minkowskian contribution is an excellent candidate for Morse function whose critical points give information about WCW homology. This would fit nicely with the vision about TGD as almost topological QFT.

Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. Minkowskian contribution would give the quantal interference effects and stationary phase approximation. The analog of Floer homology would represent quantum superpositions of critical points identifiable as ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function. One would have maxima also for the Kähler function but only in the zero modes not contributing to the WCW metric.

There is a further question related to almost topological QFT character of TGD. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

1. All arguments for this have been represented for Minkowskian regions [K96] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of CP_2 bounded by wormhole throats: for CP_2 itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time supersymmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the Kähler-Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.
2. If the reduction occurs in Euclidian regions, it gives in the case of CP_2 two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for CP_2 so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.
3. There is also an argument stating that Dirac determinant for Chern-Simons Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

CP breaking and ground state degeneracy

The Minkowskian contribution of Kähler action is imaginary due to the negativity of the metric determinant and gives a phase factor to vacuum functional reducing to Chern-Simons terms at wormhole throats. Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define 2×2 matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full CP_2 type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.
2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K - \overline{K}$ and of CKM matrix should reduce to this mixing. K^0 mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of CP_2 type extremals representing wormhole throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for B^0 mesons.
3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and shortlived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral measons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only K^0 but not for the environment in the construction of states. One can probably imagine also alternative interpretations.

3.3.2 The Values Of The Kähler Coupling Strength?

Since the vacuum functional of the theory turns out to be essentially the exponent $exp(K)$ of the Kähler function, the dynamics depends on the normalization of the Kähler function. Since the Theory of Everything should be unique it would be highly desirable to find arguments fixing the normalization or equivalently the possible values of the Kähler coupling strength α_K.

Quantization of α_K follow from Dirac quantization in WCW?

The quantization of Kähler form of WCW could result in the following manner. It will be found that Abelian extension of the isometry group results by coupling spinors of WCW to a multiple of Kähler potential. This means that Kähler potential plays role of gauge connection so that Kähler form must be integer valued by Dirac quantization condition for magnetic charge. So, if Kähler form is co-homologically nontrivial the value of α_K is quantized.

Quantization from criticality of TGD Universe?

Mathematically α_K is analogous to temperature and this suggests that α_K is analogous to critical temperature and therefore quantized. This analogy suggests also a physical motivation for the unique value or value spectrum of α_K. Below the critical temperature critical systems suffer something analogous to spontaneous magnetization. At the critical point critical systems are characterized by long range correlations and arbitrarily large volumes of magnetized and non-magnetized phases are present. Spontaneous magnetization might correspond to the generation of Kähler magnetic fields: the most probable 3-surfaces are Kähler magnetized for subcritical values of α_K. At the critical values of α_K the most probable 3-surfaces contain regions dominated by either Kähler electric and or Kähler magnetic fields: by the compactness of CP_2 these regions have in general outer boundaries.

This suggests that 3-space has hierarchical, fractal like structure: 3-surfaces with all sizes (and with outer boundaries) are possible and they have suffered topological condensation on each other. Therefore the critical value of α_K allows the richest possible topological structure for the most probable 3-space. In fact, this hierarchical structure is in accordance with the basic ideas about renormalization group invariance. This hypothesis has highly nontrivial consequences even at the level of ordinary condensed matter physics.

Unfortunately, the exact definition of renormalization group concept is not at all obvious. There is however a much more general but more or less equivalent manner to formulate the condition fixing the value of α_K. Vacuum functional $exp(K)$ is analogous to the exponent $exp(-H/T)$ appearing in the definition of the partition function of a statistical system and S-matrix elements and other interesting physical quantities are integrals of type $\langle O\rangle = \int exp(K)O\sqrt{G}dV$ and therefore analogous to the thermal averages of various observables. α_K is completely analogous to temperature. The critical points of a statistical system correspond to critical temperatures T_c for which the partition function is non-analytic function of $T - T_c$ and according RGE hypothesis critical systems correspond to fixed points of renormalization group evolution. Therefore, a mathematically more precise manner to fix the value of α_K is to require that some integrals of type $\langle O\rangle$ (not necessary S-matrix elements) become non-analytic at $1/\alpha_K - 1/\alpha_K^c$.

Renormalization group invariance is closely related with criticality. The self duality of the Kähler form and Weyl tensor of CP_2 indeed suggest RG invariance. The point is that in $N = 1$ super-symmetric field theories duality transformation relates the strong coupling limit for ordinary particles with the weak coupling limit for magnetic monopoles and vice versa. If the theory is self dual these limits must be identical so that action and coupling strength must be RG invariant quantities. The geometric realization of the duality transformation is easy to guess in the standard complex coordinates ξ_1, ξ_2 of CP_2 (see Appendix of the book). In these coordinates the metric and Kähler form are invariant under the permutation $\xi_1 \leftrightarrow \xi_2$ having Jacobian -1.

Consistency requires that the fundamental particles of the theory are equivalent with magnetic monopoles. The deformations of so called CP_2 type vacuum extremals indeed serve as building bricks of a elementary particles. The vacuum extremals are are isometric embeddings of CP_2 and can be regarded as monopoles. Elementary particle corresponds to a pair of wormhole contacts and monopole flux runs between the throats of of the two contacts at the two space-time sheets and through the contacts between space-time sheets. The magnetic flux however flows in internal degrees of freedom (possible by nontrivial homology of CP_2) so that no long range $1/r^2$ magnetic field is created. The magnetic contribution to Kähler action is positive and this suggests that ordinary magnetic monopoles are not stable, since they do not minimize Kähler action: a cautious conclusion in accordance with the experimental evidence is that TGD does not predict magnetic monopoles. It must be emphasized that the prediction of monopoles of practically all gauge theories and string theories and follows from the existence of a conserved electromagnetic charge.

Does α_K have spectrum?

The assumption about single critical value of α_K is probably too strong.

1. The hierarchy of Planck constants which would result from non-determinism of Kähler action implying n conformal equivalences of space-time surface connecting 3-surfaces at the boundaries of causal diamond CD would predict effective spectrum of α_K as $\alpha_K = g_K^2/4\pi\hbar_{eff}$, $\hbar_{eff}/h = n$. The analogs of critical temperatures would have accumulation point at zero temperature.
2. p-Adic length scale hierarchy together with the immense vacuum degeneracy of the Kähler action leads to ask whether different p-adic length scales correspond to different critical values of α_K, and that ordinary coupling constant evolution is replaced by a piecewise constant evolution induced by that for α_K.

3.3.3 What Conditions Characterize The Preferred Extremals?

The basic vision forced by the generalization of General Coordinate Invariance has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

In positive energy ontology space-time surfaces should be analogous to Bohr orbits in order to make possible possible realization of general coordinate invariance. The first guess was that absolute minimization of Kähler action might be the principle selecting preferred extremals. One can criticize the assumption that extremals correspond to the absolute minima of Kähler action for entire space-time surface as too strong since the Kähler action from Minkowskian regions is proportional to imaginary unit and corresponds to ordinary QFT action defining a phase factor of vacuum functional. Absolute minimization could however make sense for Euclidian space-time regions defining the lines of generalized Feynman diagras, where Kähler action has definite sign. Kähler function is indeed the Kähler action for these regions. Furthermore, the notion of absolute minimization does not make sense in p-adic context unless one manages to reduce it to purely algebraic conditions.

Is preferred extremal property needed at all in ZEO?

It is good to start with a critical question. Could it be that the notion of preferred extremal might be un-necessary in ZEO (ZEO)? The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is unique.

Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and same Kähler action and same conserved quantities. One must be however cautious: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal gauge invariance is associated with 2-D criticality and is expected to be present also now. and this is the recent view.

One can of course ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated - this the starting point in ZEO. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations might be present and correspond to the Bohr orbit property, space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics. This indeed seems to be the correct conclusion.

How to identify preferred extremals?

What is needed is the association of a unique space-time surface to a given 3-surface defined as union of 3-surfaces at opposite boundaries of CD. One can imagine many ways to achieve this. "Unique" is too much to demand: for the proposal unique space-time surface is replaced with finite number of conformal gauge equivalence classes of space-time surfaces. In any case, it is better to talk just about preferred extremals of Kähler action and accept as the fact that there are several proposals for what this notion could mean.

1. For instance, one can consider the identification of space-time surface as associative (co-associative) sub-manifold meaning that tangent space of space-time surface can be regarded as associative (co-associative) sub-manifold of complexified octonions defining tangent space of embedding space. One manner to define "associative sub-manifold" is by introducing octonionic representation of embedding space gamma matrices identified as tangent space vectors. It must be also assumed that the tangent space contains a preferred commutative (co-commutative) sub-space at each point and defining an integrable distribution having identification as string world sheet (also slicing of space-time sheet by string world sheets can be considered). Associativity and commutativity would define the basic dynamical principle. A closely related approach is based on so called Hamilton-Jacobi structure [K10] defining also this kind of slicing and the approaches could be equivalent.

2. In ZEO 3-surfaces become pairs of space-like 3-surfaces at the boundaries of causal diamond (CD). Even the light-like partonic orbits could be included to give the analog of Wilson loop. In absence of non-determinism of Kähler action this forces to ask whether the attribute "preferred" is un-necessary. There are however excellent reasons to expect that there is an

infinite gauge degeneracy assignable to quantum criticality and represented in terms of Kac-Moody type transformations of partonic orbits respecting their light-likeness and giving rise to the degeneracy behind hierarchy of Planck constants $h_{eff} = n \times h$. n would give the number of conformal equivalence classes of space-time surfaces with same ends. In given measurement resolution one might however hope that the "preferred" could be dropped away.

The vanishing of Noether charges for sub-algebras of conformal algebras with conformal weights coming as multiples of n at the ends of space-time surface would be a concrete realization of this picture and looks the most feasible option at this moment since it is direct classical correlated for broken super-conformal gauge invariance at quantum level.

3. The construction of quantum TGD in terms of the Kähler-Dirac action associated with Kähler action suggested a possible answer to the question about the principle selecting preferred extremals. The Noether currents associated with Kähler-Dirac action are conserved if second variations of Kähler action vanish. This is nothing but space-time correlate for quantum criticality and it is amusing that I failed to realize this for so long time. A further very important result is that in generic case the modes of induced spinor field are localized at 2-D surfaces from the condition that em charge is well-defined quantum number (W fields must vanish and also Z^0 field above weak scale in order to avoid large parity breaking effects).

   The localization at string world sheets means that quantum criticality as definition of "preferred" works only if there selection of string world sheets, partonic 2-surfaces, and their light-like orbits fixes the space-time surface completely. The generalization of AdS/CFT correspondence (or strong form of holography) suggests that this is indeed the case. The criticality conditions are however rather complicated and it seems that the vanishing of the symplectic Noether charges is the practical manner to formulate what "preferred" does mean.

3.3.4 Why Non-Local Kähler Function?

Kähler function is non-local functional of 3-surface. Non-locality of the Kähler function seems to be at odds with basic assumptions of local quantum field theories. Why this rather radical departure from the basic assumptions of local quantum field theory? The answer is shortly given: WCW integration appears in the definition of the inner product for WCW spinor fields and this inner product must be free from perturbative divergences. Consider now the argument more closely.

In the case of finite-dimensional symmetric space with Kähler structure the representations of the isometry group necessitate the modification of the integration measure defining the inner product so that the integration measure becomes proportional to the exponent $exp(K)$ of the Kähler function [B26]. The generalization to infinite-dimensional case is obvious. Also the requirement of Kac-Moody symmetry leads to the presence of this kind of vacuum functional as will be found later. The exponent is in fact uniquely fixed by finiteness requirement. WCW integral is of the following form

$$\int \bar{S}_1 exp(K) S_1 \sqrt{g} dX \quad . \tag{3.3.7}$$

One can develop perturbation theory using local complex coordinates around a given 3-surface in the following manner. The $(1,1)$-part of the second variation of the Kähler function defines the metric and therefore propagator as contravariant metric and the remaining $(2,0)-$ and $(0,2)$-parts of the second variation are treated perturbatively. The most natural choice for the 3-surface are obviously the 3-surfaces, which correspond to extrema of the Kähler function.

When perturbation theory is developed around the 3-surface one obtains two ill-defined determinants.

1. The Gaussian determinant coming from the exponent, which is just the inverse square root for the matrix defined by the metric defining $(1,1)$-part of the second variation of the Kähler function in local coordinates.
2. The metric determinant. The matrix representing covariant metric is however same as the matrix appearing in Gaussian determinant by the defining property of the Kähler metric: in local complex coordinates the matrix defined by second derivatives is of type $(1,1)$. Therefore

these two ill defined determinants (recall the presence of Diff degeneracy) cancel each other exactly for a unique choice of the vacuum functional!

Of course, the cancellation of the determinants is not enough. For an arbitrary local action one encounters the standard perturbative divergences. Since most local actions (Chern-Simons term is perhaps an exception [B55]) for induced geometric quantities are extremely nonlinear there is no hope of obtaining a finite theory. For non-local action the situation is however completely different. There are no local interaction vertices and therefore no products of delta functions in perturbation theory.

A further nice feature of the perturbation theory is that the propagator for small deformations is nothing but the contravariant metric of WCW . Also the various vertices of the theory are closely related to the metric of WCW since they are determined by the Kähler function so that perturbation theory would have a beautiful geometric interpretation. Furthermore, since four-dimensional Diff degeneracy implies that the propagator doesn't couple to un-physical modes.

It should be noticed that divergence cancellation arguments do not necessarily exclude Chern Simons term from vacuum functional defined as imaginary exponent of $exp(ik_2 \int_{X^4} J \wedge J)$. The term is not well defined for non-orientable space-time surfaces and one must assume that k_2 vanishes for these surfaces. The presence of this term might provide first principle explanation for CP breaking. If k_2 is integer multiple of $1/(8\pi)$ Chern Simons term gives trivial contribution for closed space-time surfaces since instanton number is in question. By adding a suitable boundary term of form $exp(ik_3 \int_{\delta X^3} J \wedge A)$ it is possible to guarantee that the exponent is integer valued for 4-surfaces with boundary, too.

There are two arguments suggesting that local Chern Simons term would not introduce divergences. First, 3-dimensional Chern Simons term for ordinary Abelian gauge field is known to define a divergence free field theory [B55]. The term doesn't depend at all on the induced metric and therefore contains no dimensional parameters (CP_2 radius) and its expansion in terms of CP_2 coordinate variables is of the form allowed by renormalizable field theory in the sense that only quartic terms appear. This is seen by noticing that there always exist symplectic coordinates, where the expression of the Kähler potential is of the form

$$A = \sum_k P_k dQ^k \quad . \tag{3.3.8}$$

The expression for Chern-Simons term in these coordinates is given by

$$k_2 \int_{X^3} \sum_{k,l} P_l dP_k \wedge dQ^k \wedge dQ^l \quad , \tag{3.3.9}$$

and clearly quartic CP_2 coordinates. A further nice property of the Chern Simons term is that this term is invariant under symplectic transformations of CP_2, which are realized as $U(1)$ gauge transformation for the Kähler potential.

The expressibility of WCW Kähler metric as anti-commutators of super-symplectic Noether super-charges localized at 2-D string world sheets inspires an even stronger conjecture about Kähler action. The super-symmetry between Kähler-Dirac action and Kähler action suggests that Kähler action is expressible as sum of string world sheet areas in the effective metric defined by the anti-commutators of K-D gamma matrices. This would conform with the strong form of holography in turn implies by strong form of General Coordinate Invariance, and could be seen as analog of AdS/CFT correspondence, which as such is not enough in TGD possessing super-conformal symmetries, which are gigantic as compared to those of super string models.

3.4 Some Properties Of Kähler Action

In this section some properties of Kähler action and Kähler function are discussed in light of experienced gained during about 15 years after the introduction of the notion.

3.4.1 Vacuum Degeneracy And Some Of Its Implications

The vacuum degeneracy is perhaps the most characteristic feature of the Kähler action. Although it is not associated with the preferred extremals of Kähler action, there are good reasons to expect that it has deep consequences concerning the structure of the theory.

Vacuum degeneracy of the Kähler action

The basic reason for choosing Kähler action is its enormous vacuum degeneracy, which makes long range interactions possible (the well known problem of the membrane theories is the absence of massless particles [B54]). The Kähler form of CP_2 defines symplectic structure and any 4-surface for which CP_2 projection is so called Lagrangian manifold (at most two dimensional manifold with vanishing induced Kähler form), is vacuum extremal due to the vanishing of the induced Kähler form. More explicitly, in the local coordinates, where the vector potential A associated with the Kähler form reads as $A = \sum_k P_k dQ^k$. Lagrangian manifolds are expressible locally in the following form

$$P_k = \partial_k f(Q^i) \ . \tag{3.4.1}$$

where the function f is arbitrary. Notice that for the general YM action surfaces with one-dimensional CP_2 projection are vacuum extremals but for Kähler action one obtains additional degeneracy.

There is also a second kind of vacuum degeneracy, which is relevant to the elementary particle physics. The so called CP_2 type vacuum extremals are warped embeddings X^4 of CP_2 to H such that Minkowski coordinates are functions of a single CP_2 coordinate, and the one-dimensional projection of X^4 is random light like curve. These extremals have a non-vanishing action but vanishing Poincare charges. Their small deformations are identified as space-time counterparts of fermions and their super partners. Wormhole throats identified as pieces of these extremals are identified as bosons and their super partners.

The conditions stating light likeness are equivalent with the Virasoro conditions of string models and this actually led to the eventualo realization that conformal invariance is a basic symmetry of TGD and that WCW can be regarded as a union of symmetric spaces with isometry groups having identification as symplectic and Kac-Moody type groups assignable to the partonic 2-surfaces.

Approximate symplectic invariance

Vacuum extremals have diffeomorphisms of M^4_+ and M^4_+ local symplectic transformations as symmetries. For non-vacuum extremals these symmetries leave induced Kähler form invariant and only induced metric breaks these symmetries. Symplectic transformations of CP_2 act on the Maxwell field defined by the induced Kähler form in the same manner as ordinary $U(1)$ gauge symmetries. They are however not gauge symmetries since gauge invariance is still present. In fact, the construction of WCW geometry relies on the assumption that symplectic transformations of $\delta M^4_+ \times CP_2$ which infinitesimally correspond to combinations of M^4_+ local CP_2 symplectic and CP_2-local M^4_+ symplectic transformations act as isometries of WCW . In zero energy ontology these transformations act simultaneously on all partonic 2-surfaces characterizing the space-time sheet representing a generalized Feynman diagram inside CD.

The fact that CP_2 symplectic transformations do not act as genuine gauge transformations means that $U(1)$ gauge invariance is effectively broken. This has non-trivial implications. The field equations allow purely geometric vacuum 4-currents not possible in Maxwell's electrodynamics [K10]. For the known extremals (massless extremals) they are light-like and a possible interpretation is in terms of Bose-Einstein condensates of collinear massless bosons.

Spin glass degeneracy

Vacuum degeneracy means that all surfaces belonging to $M^4_+ \times Y^2$, Y^2 any Lagrangian sub-manifold of CP_2 are vacua irrespective of the topology and that symplectic transformations of CP_2 generate new surfaces Y^2. If preferred extremals are obtained as small deformations of vacuum extremals (for which the criticality is maximal), one expects therefore enormous ground state degeneracy, which could be seen as 4-dimensional counterpart of the spin glass degeneracy. This degeneracy corresponds to the hypothesis that WCW is a union of symmetric spaces labeled by zero modes which do not appear at the line-element of the WCW metric.

Zero modes define what might be called the counterpart of spin glass energy landscape and the maxima Kähler function as a function of zero modes define a discrete set which might be called reduced configuration space. Spin glass degeneracy turns out to be crucial element for understanding how macro-temporal quantum coherence emerges in TGD framework. One of the basic ideas about p-adicization is that the maxima of Kähler function define the TGD counterpart of spin glass energy landscape [K81, K32]. The hierarchy of discretizations of the symmetric spaces corresponding to a hierarchy of measurement resolutions [K96] could allow an identification in terms of a hierarchy spin glass energy landscapes so that the algebraic points of the WCW would correspond to the maxima of Kähler function. The hierarchical structure would be due to the failure of strict non-determinism of Kähler action allowing in zero energy ontology to add endlessly details to the space-time sheets representing zero energy states in shorter scale.

Generalized quantum gravitational holography

The original naïve belief was that the construction of the configuration space geometry reduces to $\delta H = \delta M^4_+ \times CP_2$. An analogous idea in string model context became later known as quantum gravitational holography. The basic implication of the vacuum degeneracy is classical non-determinism, which is expected to reflect itself as the properties of the Kähler function and WCW geometry. Obviously classical non-determinism challenges the notion of quantum gravitational holography.

The hope was that a generalization of the notion of 3-surface is enough to get rid of the degeneracy and save quantum gravitational holography in its simplest form. This would mean that one just replaces space-like 3-surfaces with "association sequences" consisting of sequences of space-like 3-surfaces with time like separations as causal determinants. This would mean that the absolute minima of Kähler function would become degenerate: same space-like 3-surface at δH would correspond to several association sequences with the same value of Kähler function.

The life turned out to be more complex than this. CP_2 type extremals have Euclidian signature of the induced metric and therefore CP_2 type extremals glued to space-time sheet with Minkowskian signature of the induced metric are surrounded by light like surfaces X^3_l, which might be called elementary particle horizons. The non-determinism of the CP_2 type extremals suggests strongly that also elementary particle horizons behave non-deterministically and must be regarded as causal determinants having time like projection in M^4_+. Pieces of CP_2 type extremals are good candidates for the wormhole contacts connecting a space-time sheet to a larger space-time sheet and are also surrounded by an elementary particle horizons and non-determinism is also now present. That this non-determinism would allow the proposed simple description seems highly implausible.

Zero energy ontology realized in terms of a hierarchy of CDs seems to provide the most plausible treatment of the non-determinism and has indeed led to a breakthrough in the construction and understanding of quantum TGD. At the level of generalized Feynman diagrams sub-CDs containing zero energy states represent a hierarchy of radiative corrections so that the classical determinism is direct correlate for the quantum non-determinism. Determinism makes sense only when one has specified the length scale of measurement resolution. One can always add a CD containing a vacuum extremal to get a new zero energy state and a preferred extremal containing more details.

Classical non-determinism saves the notion of time

Although classical non-determinism represents a formidable mathematical challenge it is a must for several reasons. Quantum classical correspondence, which has become a basic guide line in the development of TGD, states that all quantum phenomena have classical space-time correlates. This is not new as far as properties of quantum states are considered. What is new that also quantum jumps and quantum jump sequences which define conscious existence in TGD Universe, should have classical space-time correlates: somewhat like written language is correlate for the contents of consciousness of the writer. Classical non-determinism indeed makes this possible. Classical non-determinism makes also possible the realization of statistical ensembles as ensembles formed by strictly deterministic pieces of the space-time sheet so that even thermodynamics has space-time representations. Space-time surface can thus be seen as symbolic representations for the quantum existence.

In canonically quantized general relativity the loss of time is fundamental problem. If quantum gravitational holography would work in the most strict sense, time would be lost also in TGD since all relevant information about quantum states would be determined by the moment of big bang. More precisely, geometro-temporal localization for the contents of conscious experience would not be possible. Classical non-determinism together with quantum-classical correspondence however suggests that it is possible to have quantum jumps in which non-determinism is concentrated in space-time region so that also conscious experience contains information about this region only.

3.4.2 Four-Dimensional General Coordinate Invariance

The proposed definition of the Kähler function is consistent with GCI and implies also 4-dimensional Diff degeneracy of the Kähler metric. Zero energy ontology inspires strengthening of the GCI in the sense that space-like 3-surfaces at the boundaries of CD are physically equivalent with the light-like 3-surfaces connecting the ends. This implies that basic geometric objects are partonic 2-surfaces at the boundaries of CDs identified as the intersections of these two kinds of surfaces. Besides this the distribution of 4-D tangent planes at partonic 2-surfaces would code for physics so that one would have only effective 2-dimensionality. The failure of the non-determinism of Kähler action in the standard sense of the word affects the situation also and one must allow a fractal hierarchy of CDs inside CDs having interpretation in terms of radiative corrections.

Resolution of tachyon difficulty and absence of Diff anomalies

In TGD as in string models the tachyon difficulty is potentially present: unless the time like vibrational excitations possess zero norm they contribute tachyonic term to the mass squared operator of Super Kac Moody algebra. This difficulty is familiar already from string models [B45, B43].

The degeneracy of the metric with respect to the time like vibrational excitations guarantees that time like excitations do not contribute to the mass squared operator so that mass spectrum is tachyon free. It also implies the decoupling of the tachyons from physical states: the propagator of the theory corresponds essentially to the inverse of the Kähler metric and therefore decouples from time like vibrational excitations. The experience with string model suggests that if metric is degenerate with respect to diffeomorphisms of $X^4(X^3)$ there are indeed good hopes that time like excitations possess vanishing norm with respect to WCW metric.

The four-dimensional Diff invariance of the Kähler function implies that Diff invariance is guaranteed in the strong sense since the scalar product of two Diff vector fields given by the matrix associated with $(1,1)$ part of the second variation of the Kähler action vanishes identically. This property gives hopes of obtaining theory, which is free from Diff anomalies: in fact loop space metric is not Diff degenerate and this might be the underlying reason to the problems encountered in string models [B45, B43].

Complexification of WCW

Strong form of GCI plays a fundamental role in the complexification of WCW . GCI in strong form reduces the basic building brick of WCW to the pairs of partonic 2-surfaces and their 4-D tangent space data associated with ends of light-like 3-surface at light-like boundaries of CD. At boths end the embedding space is effectively reduces to $\delta M^4_+ \times CP_2$ (forgetting the complications due to non-determinism of Kähler action). Light cone boundary in turn is metrically 2-dimensional Euclidian sphere allowing infinite-dimensional group of conformal symmetries and Kähler structure. Therefore one can say that in certain sense configuration space metric inherits the Kähler structure of $S^2 \times CP_2$. This mechanism works in case of four-dimensional Minkowski space only: higher-dimensional spheres do not possess even Kähler structure. In fact, it turns out that the quantum fluctuating degrees of freedom can be regarded in well-defined sense as a local variant of $S^2 \times$

CP_2 and thus as an infinite-dimensional analog of symmetric space as the considerations of [K21] demonstrate.

The details of the complexification were understood only after the construction of WCW geometry and spinor structure in terms of second quantized induced spinor fields [K96]. This also allows to make detailed statements about complexification [K21].

Contravariant metric and Diff^4 degeneracy

Diff degeneracy implies that the definition of the contravariant metric, which corresponds to the propagator associated to small deformations of minimizing surface is not quite straightforward. We believe that this problem is only technical. Certainly this problem is not new, being encountered in both GRT and gauge theories [B59, B44]. In TGD a solution of the problem is provided by the existence of infinite-dimensional isometry group. If the generators of this group form a complete set in the sense that any vector of the tangent space is expressible as as sum of these generators plus some zero norm vector fields then one can restrict the consideration to this subspace and in this subspace the matrix $g(X,Y)$ defined by the components of the metric tensor indeed indeed possesses well defined inverse $g^{-1}(X,Y)$. This procedure is analogous to gauge fixing conditions in gauge theories and coordinate fixing conditions in General Relativity.

It has turned that the representability of WCW as a union of symmetric spaces makes possible an approach to WCW integration based on harmonic analysis replacing the perturbative approach based on perturbative functional integral. This approach allows also a p-adic variant and leads an effective discretization in terms of discrete variants of WCW for which the points of symmetric space consist of algebraic points. There is an infinite number of these discretizations [K81] and the interpretation is in terms of finite measurement resolution. This gives a connection with the p-adicization program, infinite primes, inclusions of hyper-finite factors as representation of the finite measurement resolution, and the hierarchy of Planck constants [K80] so that various approaches to quantum TGD converge nicely.

General Coordinate Invariance and WCW spinor fields

GCI applies also at the level of quantum states. WCW spinor fields are Diff^4 invariant. This in fact fixes not only classical but also quantum dynamics completely. The point is that the values of the WCW spinor fields must be essentially same for all Diff^4 related 3-surfaces at the orbit X^4 associated with a given 3-surface. This would mean that the time development of Diff^4 invariant configuration spinor field is completely determined by its initial value at the moment of the big bang!

This is of course a naïve over statement. The non-determinism of Kähler action and zero energy ontology force to take the causal diamond (CD) defined by the intersection of future and past directed light-cones as the basic structural unit of WCW , and there is fractal hierarchy of CDs within CDs so that the above statement makes sense only for giving CD in measurement resolution neglecting the presence of smaller CDs. Strong form of GCI also implies factorization of WCW spinor fields into a sum of products associated with various partonic 2-surfaces. In particular, one obtains time-like entanglement between positive and negative energy parts of zero energy states and entanglement coefficients define what can be identified as M-matrix expressible as a "complex square root" of density matrix and reducing to a product of positive definite diagonal square root of density matrix and unitary S-matrix. The collection of orthonormal M-matrices in turn define unitary U-matrix between zero energy states. M-matrix is the basic object measured in particle physics laboratory.

3.4.3 WCW Geometry, Generalized Catastrophe Theory, And Phase Transitions

The definition of WCW geometry has nice catastrophe theoretic interpretation. To understand the connection consider first the definition of the ordinary catastrophe theory [A60].

1. In catastrophe theory one considers extrema of the potential function depending on dynamical variables x as function of external parameters c. The basic space decomposes locally into cartesian product $E = C \times X$ of control variables c, appearing as parameters in potential

function $V(c,x)$ and of state variables x appearing as dynamical variables. Equilibrium states of the system correspond to the extrema of the potential $V(x,c)$ with respect to the variables x and in the absence of symmetries they form a sub-manifold of M with dimension equal to that of the parameter space C. In some regions of C there are several extrema of potential function and the extremum value of x as a function of c is multi-valued. These regions of $C \times X$ are referred to as catastrophes. The simplest example is cusp catastrophe (see **Fig. ??**) with two control parameters and one state variable.

2. In catastrophe regions the actual equilibrium state must be selected by some additional physical requirement. If system obeys flow dynamics defined by first order differential equations the catastrophic jumps take place along the folds of the cusp catastrophe (delay rule). On the other hand, the Maxwell rule obeyed by thermodynamic phase transitions states that the equilibrium state corresponds to the absolute minimum of the potential function and the state of system changes in discontinuous manner along the Maxwell line in the middle between the folds of the cusp (see **Fig. 3.1**).
3. As far as discontinuous behavior is considered, fold catastrophe is the basic catastrophe: all catastrophes contain folds as there "satellites" and one aim of the catastrophe theory is to derive all possible ways for the stable organization of folds into higher catastrophes. The fundamental result of the catastrophe theory is that for dimensions d of C smaller than 5 there are only 7 basic catastrophes and polynomial potential functions provide a canonical representation for the catastrophes: fold catastrophe corresponds to third order polynomial (in fold the two real roots become a pair of complex conjugate roots), cusp to fourth order polynomial, etc.

Consider now the TGD counterpart of this. TGD allows allows two kinds of catastrophe theories.

1. The first one is related to Kähler action as a local functional of 4-surface. The nature of this catastrophe theory depends on what one means with the preferred extremals.
2. Second catastrophe theory corresponds to Kähler function a non-local functional of 3-surface. The maxima of the vacuum functional defined as the exponent of Kähler function define what might called effective space-times, and discontinuous jumps changing the values of the parameters characterizing the maxima are possible.

Consider first the option based on Kähler action.

1. Potential function corresponds to Kähler action restricted to the solutions of Euler Lagrange equations. Catastrophe surface corresponds to the four-surfaces found by extremizing Kähler action with respect to the variables of X (time derivatives of coordinates of C specifying X^3 in H_a) keeping the variables of C specifying 3-surface X^3 fixed. Preferred extremal property is analogous to the Bohr quantization since canonical momenta cannot be chosen freely as in the ordinary initial value problems of the classical physics. Preferred extremals are by definition at criticality. Behavior variables correspond to the deformations of the 4-surface keeping partonic 2-surfaces and 3-D tangent space data fixed and preserving extremal property. Control variables would correspond to these data.
2. At criticality the rank of the infinite-dimensional matrix defined by the second functional derivatives of the Kähler action is reduced. Catastrophes form a hierarchy characterized by the reduction of the rank of this matrix and Thom's catastrophe theory generalizes to infinite-dimensional context. Criticality in this sense would be one aspect of quantum criticality having also other aspects. No discrete jumps would occur and system would only move along the critical surface becoming more or less critical.
3. There can exist however several critical extremals assignable to a given partonic 2-surface but have nothing to do with the catastrophes as defined in Thom's approach. In presence of degeneracy one should be able to choose one of the critical extremals or replace this kind of regions of WCW by their multiple coverings so that single partonic 2-surface is replaced with its multiple copy. The degeneracy of the preferred extremals could be actually a deeper reason for the hierarchy of Planck constants involving in its most plausible version n-fold singular coverings of CD and CP_2. This interpretation is very satisfactory since the generalization of

the embedding space and hierarchy of Planck constants would follow naturally from quantum criticality rather than as separate hypothesis.

4. The existence of the catastrophes is implied by the vacuum degeneracy of the Kähler action. For example, for pieces of Minkowski space in $M^4_+ \times CP_2$ the second variation of the Kähler action vanishes identically and only the fourth variation is non-vanishing: these 4-surfaces are analogous to the tip of the cusp catastrophe. There are also space-time surfaces for which the second variation is non-vanishing but degenerate and a hierarchy of subsets in the space of extremal 4-surfaces with decreasing degeneracy of the second variation defines the boundaries of the projection of the catastrophe surface to the space of 3-surfaces. The space-times for which second variation is degenerate contain as subset the critical and initial value sensitive preferred extremal space-times.

Consider next the catastrophe theory defined by Kähler function.

1. In this case the most obvious identification for the behavior variables would be in terms of the space of all 3-surfaces in $CD \times CP_2$ - and if one believes in holography and zero energy ontology - the 2-surfaces assignable the boundaries of causal diamonds (CDs).
2. The natural control variables are zero modes whereas behavior variables would correspond to quantum fluctuating degrees of freedom contributing to the WCW metric. The induced Kähler form at partonic 2-surface would define infinitude of purely classical control variables. There is also a correlation between zero modes identified as degrees of freedom assignable to the interior of 3-surface and quantum fluctuating degrees of freedom assigned to the partonic 2-surfaces. This is nothing but holography and effective 2-dimensionality justifying the basic assumption of quantum measurement theory about the correspondence between classical and quantum variables. The absence of several maxima implies also the presence of saddle surfaces at which the rank of the matrix defined by the second derivatives is reduced. This could lead to a non-positive definite metric. It seems that it is possible to have maxima of Kähler function without losing positive definiteness of the metric since metric is defined as (1, 1)-type derivatives with respect to complex coordinates. In case of CP_2 however Kähler function has single degenerate maximum corresponding to the homologically trivial geodesic sphere at $r = \infty$. It might happen that also in the case of infinite-D symmetric space finite maxima are impossible.
3. The criticality of Kähler function would be analogous to thermodynamical criticality and to the criticality in the sense of catastrophe theory. In this case Maxwell's rule is possible and even plausible since quantum jump replaces the dynamics defined by a continuous flow.

Cusp catastrophe provides a simple concretization of the situation for the criticality of Kähler action (as distinguished from that for Kähler function).

1. The set M of the critical 4-surfaces corresponds to the V-shaped boundary of the 2-D cusp catastrophe in 3-D space to plane. In general case it forms codimension one set in WCW . In TGD Universe physical system would reside at this line or its generalization to higher dimensional catastrophes. For the criticality associated with Kähler action the transitions would be smooth transitions between different criticalities characterized by the rank defined above: in the case of cusp (see **Fig. 3.1**) from the tip of cusp to the vertex of cusp or vice versa. Evolution could mean a gradual increase of criticality in this sense. If preferred extremals are not unique, cusp catastrophe does not provide any analogy. The strong form of criticality would mean that the system would be always "at the tip of cusp" in metaphoric sense. Vacuum extremals are maximally critical in trivial sense, and the deformations of vacuum extremals could define the hierarchy of criticalities.
2. For the criticality of Kähler action Maxwell's rule stating that discontinuous jumps occur along the middle line of the cusp is in conflict with catastrophe theory predicting that jumps occurs along at criticality. For the criticality of Kähler function - if allowed at all by symmetric space property - Maxwell's rule can hold true but cannot be regarded as a fundamental law. It is of course known that phase transitions can occur in different ways (super heating and super cooling).

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. Conformal symmetry would

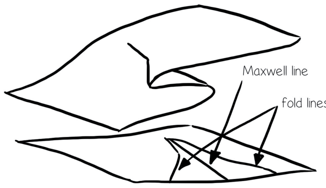

Figure 3.1: Cusp catastrophe

be naturally associated with the super-symplectic algebra of $\delta M^4_\pm$ for which the light-like radial coordinate plays the role of complex coordinate z for ordinary 2-D conformal symmetry. At criticality the symplectic subalgebra represented as gauge symmetries would change to its isomorphic subalgebra or which versa and having conformal weights are multiples of integer n. One would have fractal hierarchies of sub-algebras characterized by integers $n_{i+1} = \prod_{k<i+1} m_k$.

In each transition to lower criticality the gauge sub-algebra of the symplectic algebra would become a sub-algebra of the original one. These transitions would occur spontaneously. The transitions in the reverse direction would not take place spontaneously. The proposal is that these phase transitions take place in both directions in living matter and that the phase transitions reducing criticality require metabolic energy.

The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg` or **Fig. ??** in the appendix of this book). The hierarchy of Planck constants in turn is identified as dark phases of matter [K29].

Chapter 4

Construction of WCW Kähler Geometry from Symmetry Principles

4.1 Introduction

The most general expectation is that configuration space ("world of classical worlds" (WCW)) can be regarded as a union of coset spaces which are infinite-dimensional symmetric spaces with Kähler structure: $C(H) = \cup_i G/H(i)$. Index i labels 3-topology and zero modes. The group G, which can depend on 3-surface, can be identified as a subgroup of diffeomorphisms of $\delta M^4_+ \times CP_2$ and H must contain as its subgroup a group, whose action reduces to $Diff(X^3)$ so that these transformations leave 3-surface invariant.

In zero energy ontology (ZEO) 3-surface corresponds to a pair of space-like 3-surfaces at the opposide boundaries of causal diamond (CD) and thus to a more or less unique extremal of Kähler action. The interpretation would be in terms of holography. One can also consider the inclusion of the light-like 3-surfaces at which the signature of the induced metric changes to the 3-surface so that it would become connected.

The task is to identify plausible candidate for G and H and to show that the tangent space of the WCW allows Kähler structure, in other words that the Lie-algebras of G and $H(i)$ allow complexification. One must also identify the zero modes and construct integration measure for the functional integral in these degrees of freedom. Besides this one must deduce information about the explicit form of WCW metric from symmetry considerations combined with the hypothesis that Kähler function is Kähler action for a preferred extremal of Kähler action. One must of course understand what "preferred" means.

4.1.1 General Coordinate Invariance And Generalized Quantum Gravitational Holography

The basic motivation for the construction of WCW geometry is the vision that physics reduces to the geometry of classical spinor fields in the infinite-dimensional WCW of 3-surfaces of $M^4_+ \times CP_2$ or of $M^4 \times CP_2$. Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that WCW possesses Kähler geometry. Kähler geometry is coded into Kähler function.

The original belief was that the four-dimensional general coordinate invariance of Kähler function reduces the construction of the geometry to that for the boundary of configuration space consisting of 3-surfaces on $\delta M^4_+ \times CP_2$, the moment of big bang. The proposal was that Kähler function $K(Y^3)$ could be defined as a preferred extremal of so called Kähler action for the unique space-time surface $X^4(Y^3)$ going through given 3-surface Y^3 at $\delta M^4_+ \times CP_2$. For Diff^4 transforms of Y^3 at $X^4(Y^3)$ Kähler function would have the same value so that Diff^4 invariance and degeneracy would be the outcome. The proposal was that the preferred extremals are absolute minima of Kähler action.

This picture turned out to be too simple.

1. I have already described the recent view about light-like 3-surfaces as generalized Feynman diagrams and space-time surfaces as preferred extremals of Kähler action and will not repeat what has been said. Note that the inclusion of space-like ends at boundaries of CD gives analog of Wilson loop.
2. It has also become obvious that the gigantic symmetries associated with $\delta M^4_\pm \times CP_2 \subset CD \times CP_2$ manifest themselves as the properties of propagators and vertices. Cosmological considerations, Poincare invariance, and the new view about energy favor the decomposition of the WCW to a union of configuration spaces assignable to causal diamonds CDs defined as intersections of future and past directed light-cones. The minimum assumption is that CDs label the sectors of CH: the nice feature of this option is that the considerations of this chapter restricted to $\delta M^4_+ \times CP_2$ generalize almost trivially. This option is beautiful because the center of mass degrees of freedom associated with the different sectors of CH would correspond to M^4 itself and its Cartesian powers.

The definition of the Kähler function requires that the many-to-one correspondence $X^3 \rightarrow X^4(X^3)$ must be replaced by a bijective correspondence in the sense that X^3_l as light-like 3-surface is unique among all its Diff^4 translates. This also allows physically preferred "gauge fixing" allowing to get rid of the mathematical complications due to Diff4 degeneracy. The internal geometry of the space-time sheet must define the preferred 3-surface X^3_l.

The realization of this vision means a considerable mathematical challenge. The effective metric 2-dimensionality of 3-dimensional light-like surfaces X^3_l of M^4 implies generalized conformal and symplectic symmetries allowing to generalize quantum gravitational holography from light like boundary so that the complexities due to the non-determinism can be taken into account properly.

4.1.2 Light Like 3-D Causal Determinants And Effective 2-Dimensionality

The light like 3-surfaces X^3_l of space-time surface appear as 3-D causal determinants. Basic examples are boundaries and elementary particle horizons (parton orbits) at which Minkowskian signature of the induced metric transforms to Euclidian one. This brings in a second conformal symmetry related to the metric 2-dimensionality of the 3-D light-like 3-surface. This symmetry is analogous to TGD counterpart of the Kac Moody symmetry of string models and seems to be associated with quantum criticality implying non-uniqueness of the space-time surface with given space-like ends at boundaries of CD. Critical deformations would be Kac-Moody type transformation preserving the light-likeness of the parton orbits. The challenge is to understand the relationship of this symmetry to WCW geometry and the interaction between the two conformal symmetries.

1. Field-particle duality is realized. Light-like 3-surfaces X^3_l -generalized Feynman diagrams - correspond to the particle aspect of field-particle duality whereas the physics in the interior of space-time surface $X^4(X^3_l)$ would correspond to the field aspect. Generalized Feynman diagrams in 4-D sense could be identified as regions of space-time surface having Euclidian signature.
2. One could also say that light-like 3-surfaces X^3_l and the space-like 3-surfaces X^3 in the intersections of $X^4(X^3_l) \cap CD \times CP_2$ where the causal diamond CD is defined as the intersections of future and past directed light-cones provide dual descriptions.
3. Generalized coset construction implies that the differences of super-symplectic and Super Kac-Moody type Super Virasoro generators annihilated physical states. This construction in turn led to the realization that WCW for fixed values of zero modes - in particular the values of the induced Kähler form of $\delta M^4_\pm \times CP_2$ - allows identification as a coset space obtained by dividing the symplectic group of $\delta M^4_\pm \times CP_2$ with Kac-Moody group, whose generators vanish at $X^2 = X^3_l \times \delta M^4_\pm \times CP_2$. One can say that quantum fluctuating degrees of freedom in a very concrete sense correspond to the local variant of $S^2 \times CP_2$.

The analog of conformal invariance in the light-like direction of X^3_l and in the light-like radial direction of $\delta M^4_\pm$ suggests that the data at either X^3 or X^3_l should be enough to determine WCW geometry. This implies that the relevant data is contained to their intersection X^2 at least for finite regions of X^3. This is the case if the deformations of X_l^3 not affecting X^2 and preserving light likeness corresponding to zero modes or gauge degrees of freedom and induce deformations of X^3 also acting as zero modes. The outcome is effective 2-dimensionality. One must be however cautious in order to not make over-statements. The reduction to 2-D theory in global sense would trivialize the theory and the reduction to 2-D theory must takes places for finite region of X^3 only so one has in well defined sense three-dimensionality in discrete sense. A more precise formulation of this vision is in terms of hierarchy of CDs containing CDs containing.... The introduction of sub-CD:s brings in improved measurement resolution and means also that effective 2-dimensionality is realized in the scale of sub-CD only.

Experience has however taught to be extremely cautious: it could also be that in ZEO the unions of the space-like 3-surfaces at the ends of CD and of the light-like partonic orbits at which the signature of the induced metric changes are the basic objects analogous to Wilson loops. In this case the notion of effective 2-dimensionality is not so clear. Also in this case the Kac-Moody type symmetry preserving the light-likeness of partonic orbits could reduce the additional degrees of freedom to a finite number of conformal equivalence classes of partonic orbits for given pair of 3-surfaces.

One cannot over-emphasize the importance of the effective 2-dimensionality. It indeed simplifies dramatically the earlier formulas for WCW metric involving 3-dimensional integrals over $X^3 \subset M^4_+ \times CP_2$ reducing now to 2-dimensional integrals. Note that X^3 is determined by preferred extremal property of $X^4(X^3_l)$ once X^3_l is fixed and one can hope that this mapping is one-to-one.

4.1.3 Magic Properties Of Light Cone Boundary And Isometries Of WCW

The special conformal, metric and symplectic properties of the light cone of four-dimensional Minkowski space: δM^4_+, the boundary of four-dimensional light cone is metrically 2-dimensional(!) sphere allowing infinite-dimensional group of conformal transformations and isometries(!) as well as Kähler structure. Kähler structure is not unique: possible Kähler structures of light cone boundary are parameterized by Lobatchevski space $SO(3,1)/SO(3)$. The requirement that the isotropy group $SO(3)$ of S^2 corresponds to the isotropy group of the unique classical 3-momentum assigned to $X^4(Y^3)$ defined as a preferred extremum of Kähler action, fixes the choice of the complex structure uniquely. Therefore group theoretical approach and the approach based on Kähler action complement each other.

1. The allowance of an infinite-dimensional group of isometries isomorphic to the group of conformal transformations of 2-sphere is completely unique feature of the 4-dimensional light cone boundary. Even more, in case of $\delta M^4_+ \times CP_2$ the isometry group of δM^4_+ becomes localized with respect to CP_2! Furthermore, the Kähler structure of δM^4_+ defines also symplectic structure.

   Hence any function of $\delta M^4_+ \times CP_2$ would serve as a Hamiltonian transformation acting in both CP_2 and δM^4_+ degrees of freedom. These transformations obviously differ from ordinary local gauge transformations. This group leaves the symplectic form of $\delta M^4_+ \times CP_2$, defined as the sum of light cone and CP_2 symplectic forms, invariant. The group of symplectic transformations of $\delta M^4_+ \times CP_2$ is a good candidate for the isometry group of the WCW.

2. The approximate symplectic invariance of Kähler action is broken only by gravitational effects and is exact for vacuum extremals. If Kähler function were exactly invariant under the symplectic transformations of CP_2, CP_2 symplectic transformations wiykd correspond to zero modes having zero norm in the Kähler metric of WCW. This does not make sense since symplectic transformations of $\delta M^4 \times CP_2$ actually parameterize the quantum fluctuation degrees of freedom.

3. The groups G and H, and thus WCW itself, should inherit the complex structure of the light cone boundary. The diffeomorphims of M^4 act as dynamical symmetries of vacuum extremals. The radial Virasoro localized with respect to $S^2 \times CP_2$ could in turn act in zero modes perhaps inducing conformal transformations: note that these transformations lead out from the symmetric space associated with given values of zero modes.

4.1.4 Symplectic Transformations Of $\Delta M^4_+ \times CP_2$ As Isometries Of WCW

The symplectic transformations of $\delta M^4_+ \times CP_2$ are excellent candidates for inducing symplectic transformations of the WCW acting as isometries. There are however deep differences with respect to the Kac Moody algebras.

1. The conformal algebra of the WCW is gigantic when compared with the Virasoro + Kac Moody algebras of string models as is clear from the fact that the Lie-algebra generator of a symplectic transformation of $\delta M^4_+ \times CP_2$ corresponding to a Hamiltonian which is product of functions defined in δM^4_+ and CP_2 is sum of generator of δM^4_+-local symplectic transformation of CP_2 and CP_2-local symplectic transformations of δM^4_+. This means also that the notion of local gauge transformation generalizes.
2. The physical interpretation is also quite different: the relevant quantum numbers label the unitary representations of Lorentz group and color group, and the four-momentum labeling the states of Kac Moody representations is not present. Physical states carrying no energy and momentum at quantum level are predicted. The appearance of a new kind of angular momentum not assignable to elementary particles might shed some light to the longstanding problem of baryonic spin (quarks are not responsible for the entire spin of proton). The possibility of a new kind of color might have implications even in macroscopic length scales.
3. The central extension induced from the natural central extension associated with $\delta M^4_+ \times CP_2$ Poisson brackets is anti-symmetric with respect to the generators of the symplectic algebra rather than symmetric as in the case of Kac Moody algebras associated with loop spaces. At first this seems to mean a dramatic difference. For instance, in the case of CP_2 symplectic transformations localized with respect to δM^4_+ the central extension would vanish for Cartan algebra, which means a profound physical difference. For $\delta M^4_+ \times CP_2$ symplectic algebra a generalization of the Kac Moody type structure however emerges naturally.

   The point is that δM^4_+-local CP_2 symplectic transformations are accompanied by CP_2 local δM^4_+ symplectic transformations. Therefore the Poisson bracket of two δM^4_+ local CP_2 Hamiltonians involves a term analogous to a central extension term symmetric with respect to CP_2 Hamiltonians, and resulting from the δM^4_+ bracket of functions multiplying the Hamiltonians. This additional term could give the entire bracket of the WCW Hamiltonians at the maximum of the Kähler function where one expects that CP_2 Hamiltonians vanish and have a form essentially identical with Kac Moody central extension because it is indeed symmetric with respect to indices of the symplectic group.

The most natural option is that symplectic and Kac-Moody algebras together generate the isometry algebra and that the corresponding transformations leaving invariant the partonic 2-surfaces and their 4-D tangent space data act as gauge transformations and affect only zero modes.

4.1.5 Does The Symmetric Space Property Reduce To Coset Construction For Super Virasoro Algebras?

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ . \quad [t, t] \subset h \ . \quad [h, t] \subset t \ . \tag{4.1.1}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough.

WCW geometry allows two super-conformal symmetries assignable the coset space decomposition G/H for a sector of WCW with fixed values of zero moes. One can assign to the tangent space algebras g resp. h of G resp. H analogous to Kac-Moody algebras super Virasoro algebras and construct super-conformal representation as a coset representation meaning that the differences of super Virasoro generators annihilate the physical states. This obviously generalizes Goddard-Olive-Kent construction [A72].

The identification of the two algeras is not a mechanical task and has involved a lot of trial and erroring. The algebra g should be be spanned by the generators of super-symplectic algebra of light-cone boundary and by the Kac-Moody algebra acting on light-like orbits of partonic 2-surfaces. The sub-algebra h should be spanned by generators which vanish for a preferred point of WCW analogous to origin of $CP_2 = SU(3)/U(2)$. Now this point would correspond to maximum or minimum of Kähler function (no saddle points are allowed if the WCW metric has definite signature). In hindsight it is obvious that the generators of both symplectic and Kac-Moody algebras are needed to generate g and h: already the effective 2-dimensionality meaning that 4-D tangent space data of partonic surface matters requires this.

The maxima of Kähler function could correspond to this kind of points (pairs formed by 3-surfaces at different ends of CD in ZEO) and could play also an essential role in the integration over WCW by generalizing the Gaussian integration of free quantum field theories. It took quite a long time to realize that Kähler function must be identified as Kähler action for the Euclidian region of preferred extremal. Kähler action for Minkowskian regions gives imaginary contribution to the action exponential and has interpretation in terms of Morse function. This part of Kähler action can have and is expected to have saddle points and to define Hessian with signature which is not positive definite.

4.1.6 What Effective 2-Dimensionality And Holography Really Mean?

Concerning the interpretation of Kac-Moody algebra there are some poorly understood points, which directly relate to what one means with holography.

1. Holography suggests that light-like 3-surfaces with fixed ends give rise to same WCW metric and the deformations of these surfaces by Kac-Moody algebra correspond to zero modes just like the interior degrees of freedom for space-like 3-surface do. The same would be true for space-like 3-surfaces at the ends of space-time surface with respect to symplectic transformations.
2. The non-trivial action of Kac-Moody algebra in the interior of X_l^3 together with effective 2-dimensionality and holography would encourage the interpretation of Kac-Moody symmetries acting trivially at X^2 as gauge symmetries. Light-like 3-surfaces having fixed partonic 2-surfaces at their ends would be equivalent physically and effective 2-dimensionality and holography would be realized modulo gauge transformations. As a matter fact, the action on WCW metric would be a change of zero modes so that one could identify it as analog of conformal scaling. The action of symplectic transformations vanishing in the interior of space-like 3-surface at the end of space-time surface affects only zero modes.

4.1.7 For The Reader

Few words about the representation of ideas are in order. For a long time the books about TGD served as kind of lab note books - a bottom-up representation providing kind of a ladder making clear the evolution of ideas. This led gradually to a rather chaotic situation in which it was difficult for me to control the internal consistency and for the possible reader to distinguish between the big ideas and ad hoc guesses, most of them related to the detailed realization of big visions. Therefore I have made now and the decision to clean up a lot of the ad hoc stuff. In this process I have also changed the representation so that it is more top-down and tries to achieve over-all views.

There are several visions about what TGD is and I have worked hardly to achieve a fusion of these visions. Hence simple linear representation in which reader climbs to a tree of wisdom is impossible. I must summarize overall view from the beginning and refer to the results deduced in chapters towards the end of the book and also to ideas discussed in other books. For instance, the construction of WCW ("world of classical worlds" (WCW)) spinor structure discussed in chapters [K96] provides the understanding necessary to make the construction of configuration space geometry more detailed. Also number theoretical vision discussed in another book [K58] is necessary. Somehow it seems that a graphic representation emphasizing visually the big picture should be needed to make the representation more comprehensible.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.`

`fi/tgdglossary.pdf` [L22].

4.2 How To Generalize The Construction Of WCW Geometry To Take Into Account The Classical Non-Determinism?

If the embedding space were $H_+ = M^4_+ \times CP_2$ and if Kähler action were deterministic, the construction of WCW geometry reduces to $\delta M^4_+ \times CP_2$. Thus in this limit quantum holography principle [B23, B46] would be satisfied also in TGD framework and actually reduce to the general coordinate invariance. The classical non-determinism of Kähler action however means that this construction is not quite enough and the challenge is to generalize the construction.

4.2.1 Quantum Holography In The Sense Of Quantum GravityTheories

In string theory context quantum holography is more or less synonymous with Maldacena conjecture Maldacena which (very roughly) states that string theory in Anti-de-Sitter space AdS is equivalent with a conformal field theory at the boundary of AdS. In purely quantum gravitational context [B23] , quantum holography principle states that quantum gravitational interactions at high energy limit in AdS can be described using a topological field theory reducing to a conformal (and non-gravitational) field theory defined at the *time like* boundary of the AdS. Thus the time like boundary plays the role of a dynamical hologram containing all information about correlation functions of $d+1$ dimensional theory. This reduction also conforms with the fact that black hole entropy is proportional to the horizon area rather than the volume inside horizon.

Holography principle reduces to general coordinate invariance in TGD. If the action principle assigning space-time surface to a given 3-surface X^3 at light cone boundary were completely deterministic, four-dimensional general coordinate invariance would reduce the construction of the configuration geometry for the space of 3-surfaces in $M^4_+ \times CP_2$ to the construction of the geometry at the boundary of WCW consisting of 3-surfaces in $\delta M^4_+ \times CP_2$ (moment of big bang). Also the quantum theory would reduce to the boundary of the future light cone.

The classical non-determinism of Kähler action however implies that quantum holography in this strong form fails. This is very desirable from the point of view of both physics and consciousness theory. Classical determinism would also mean that time would be lost in TGD as it is lost in GRT. Classical non-determinism is also absolutely essential for quantum consciousness and makes possible conscious experiences with contents localized into finite time interval despite the fact that quantum jumps occur between WCW spinor fields defining what I have used to call quantum histories. Classical non-determinism makes it also possible to generalize quantum-classical correspondence in the sense that classical non-determinism at the space-time level provides correlate for quantum non-determinism. The failure of classical determinism is a difficult challenge for the construction of WCW geometry. One might however hope that the notion of quantum holography generalizes.

4.2.2 How Does The Classical Determinism Fail In TGD?

One might hope that determinism in a generalized sense might be achieved by generalizing the notion of 3-surface by allowing unions of space-like 3-surfaces with time like separations with very strong but not complete correlations between the space-like 3-surfaces. In this case the non-determinism would mean that the 3-surfaces Y^3 at light cone boundary correspond to at most enumerable number of preferred extremals $X^4(Y^3)$ of Kähler action so that one would get finite or at most enumerably infinite number of replicas of a given WCW region and the construction would still reduce to the light cone boundary.

1. This is probably quite too simplistic view. Any 4-surface which has CP_2 projection which belongs to so called Lagrange manifold of CP_2 having by definition vanishing induced Kähler form is vacuum extremal. Thus there is an infinite variety of 6-dimensional sub-manifolds of H for which all extremals of Kähler action are vacua.
2. CP_2 type vacuum extremals are different since they possess non-vanishing Kähler form and Kähler action. They are identifiable as classical counterparts of elementary particles have M^4_+ projection which is a random light like curve (this in fact gives rise to conformal invariance

identifiable as counterpart of quaternion conformal invariance). Thus there are good reasons to suspect that classical non-determinism might destroy the dream about complete reduction to the light cone boundary.

3. The wormhole contacts connecting different space-time sheets together can be seen as pieces of CP_2 type extremals and one expects that the non-determinism is still there and that the metrically 2-dimensional elementary particle horizons (light like 3-surfaces of H surrounding wormhole contacts and having time-like M^4_+ projection) might be a crucial element in the understanding of quantum TGD. The non-determinism of CP_2 type extremals is absolutely crucial for the ordinary elementary particle physics. It seems that the conformal symmetries responsible for the ordinary elementary particle quantum numbers acting in these degrees of freedom do not contribute to the WCW metric line element.

The treatment of the non-determinism in a framework in which the prediction of time evolution is seen as initial value problem, seems to be difficult. Also the notion of WCW becomes a messy concept. ZEO changes the situation completely. Light-like 3-surfaces become representations of generalized Feynman diagrams and brings in the notion of finite time resolution. One obtains a direct connection with the concepts of quantum field theory with path integral with cutoff replaced with a sum over various preferred extremals with cutoff in time resolution.

4.2.3 The Notions Of Embedding Space, 3-Surface, And Configuration Space

The notions of embedding space, 3-surface (and 4-surface), and configuration space ("world of classical worlds", WCW) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K81, K82, K80] .

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of ZEO [K96, K20] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K57] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K29] led to a further generalization of the notion of embedding space - at least as a convenient auxialiary structure. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry

of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

It seems that the covering of embedding space is only a convenient auxiliary structure. The space-time surfaces in the n-fold covering correspond to the n conformal equivalence classes of space-time surfaces connecting fixed 3-surfaces at the ends of CD: the space-time surfaces are branched at their ends. The situation can be interpreted at the level of WCW in several ways. There is single 3-surface at both ends but by non-determinism there are n space-time branches of the space-time surface connecting them so that the Kähler action is multiplied by factor n. If one forgets the presence of the n branches completely, one can say that one has $h_{eff} = n \times h$ giving $1/\alpha_K = n/\alpha_K(n = 1)$ and scaling ofKähler action. One can also imagine that the 3-surfaces at the ends of CD are actually surfaces in the n-fold covering space consisting of n identical copies so that Kähler action is multiplied by n. One could also include the light-like partonic orbits to the 3-surface so that 3-surfaces would not have boundaries: in this case the n-fold degeneracy would come out very naturally.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [K64] .

The notion of 3-surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of General Coordinate Invariance since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces and their 4-D tangent spaces. It is however essential that information about normal space of the 2-surface is needed.

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further complication relates to the hierarchy of Planck constants. At “microscopic” level this means that there number of conformal equivalence classes of space-time surfaces connecting the 3-surfaces at boundaries of CD matters and this information is coded by the value of $h_{eff} = n \times h$. One can divide WCW to sectors corresponding to different values of h_{eff} and conformal symmetry breakings connect these sectors: the transition $n_1 \rightarrow n_2$ such that n_1 divides n_2 occurs spontaneously since it reduces the quantum criticality by transforming super-generators acting as gauge symmetries to dynamical ones.

The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.
2. With the discovery of ZEO (with motivation coming from the non-determinism of Kähler action) it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.
3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW (WCW) is a union of WCWs associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.
2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).
3. This leads to the identification of the coset space structure of the sub-WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!
4. Generalized coset construction and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level. Contrary to the original belief, this construction does not provide a realization of Equivalence Principle at quantum level. The proper realization of EP at quantum level seems to be based on the identification of classical Noether charges in Cartan algebra with the eigenvalues of their quantum counterparts assignable to Kähler-Dirac action. At classical level EP follows at GRT limit obtained by lumping many-sheeted space-time to M^4 with effective metric satisfying Einstein's equations as a reflection of the underlying Poincare invariance.

5. Now it has become clear that EP in the sense of quantum classical correspondence allows a concrete realization for the fermion lines defined by the light-like boundaries of string world sheets at light-like orbits of partonic 2-surfaces. Fermion lines are always light-like or space-like locally. Kähler-Dirac equation reducing to its algebraic counterpart with light-like 8-momentum defined by the tangent of the boundary curve. 8-D light-likeness means the possibility of massivation in M^4 sense and gravitational mass is defined in an obvious manner. The M^4-part of 8-momentum is by quantum classical correspondence equal to the 4-momentum assignable to the incoming fermion. EP generalizes also to CP_2 degrees of freedom and relates $SO(4)$ acting as symmetries of Eucldian part of 8-momentum to color $SU(3)$. $SO(4)$ can be assigned to hadrons and $SU(3)$ to quarks and gluons.

   The 8-momentum is light-like with respect to the effective metric defined by K-D gamma matrices. Is it also light-like with respect to the induced metric and proportional to the tangent vector of the fermion line? If this is not the case, the boundary curve is locally space-like in the induced metric. Could this relate to the still poorly understand question how the necessariy tachyonic ground state conformal weight of super-conformal representations needed in padic mass calculations [K46] emerges? Could it be that "empty" lines carrying no fermion number are tachyonic with respect to the induced metric?

4.2.4 The Treatment Of Non-Determinism Of Kähler Action In Zero Energy Ontology

The non-determinism of Kähler action means that the reduction of the construction of WCW geometry to the light cone boundary fails. Besides degeneracy of the preferred extrema of Kähler action, the non-determinism should manifest itself as a presence of causal determinants also other than light cone boundary.

One can imagine two kinds of causal determinants.

1. Elementary particle horizons and light-like boundaries $X_l^3 \subset X^4$ of 4-surfaces representing wormhole throats act as causal determinants for the space-time dynamics defined by Kähler action. The boundary values of this dynamics have been already considered.
2. At embedding space level causal determinants correspond to light like CD forming a fractal hierarchy of CDs within CDs. These causal determinants determine the dynamics of zero energy states having interpretation as pairs of initial and final states in standard quantum theory.

The manner to treat the classical non-determinism would be roughly following.

1. The replacement of space-like 3-surface X^3 with X_l^3 transforms initial value problem for X^3 to a boundary value problem for X_l^3. In principle one can also use the surfaces $X^3 \subset \delta CD \times CP_2$ if X_l^3 fixes $X^4(X_l^3)$ and thus X^3 uniquely. For years an important question was whether both X^3 and X_l^3 contribute separately to WCW geometry or whether they provide descriptions, which are in some sense dual.
2. Only Super-Kac-Moody type conformal algebra makes sense in the interior of X_l^3. In the 2-D intersections of X_l^3 with the boundary of causal diamond (CD) defined as intersection of future and past directed light-cones super-symplectic algebra makes sense. This implies effective two-dimensionality which is broken by the non-determinism represented using the hierarchy of CDs meaning that the data from these 2-D surfaces and their normal spaces at boundaries of CDs in various scales determine the WCW metric.
3. An important question has been whether Kac-Moody and super-symplectic algebras provide descriptions which are dual in some sense. At the level of Super-Virasoro algebras duality seems to be satisfied in the sense of generalized coset construction meaning that the differences of Super Virasoro generators of super-symplectic and super Kac-Moody algebras annihilate physical states. Among other things this means that four-momenta assignable to the two Super Virasoro representations are identical. T he interpretation is in terms of a generalization of Equivalence Principle [K96, K20] . This gives also a justification for p-adic thermodynamics applying only to Super Kac-Moody algebra.

4. Light-like 3-surfaces can be regarded also as generalized Feynman diagrams. The finite length resolution mean means also a cutoff in the number of generalized Feynman diagrams and this number remains always finite if the light-like 3-surfaces identifiable as maxima of Kähler function correspond to the diagrams. The finiteness of this number is also essential for number theoretic universality since it guarantees that the elements of M-matrix are algebraic numbers if momenta and other quantum numbers have this property. The introduction of new sub-CDs means also introduction of zero energy states in corresponding time scale.

5. The notion of finite measurement resolution expressed in terms of hierarchy of CDs within CDs is important for the treatment of classical non-determinism. In a given resolution the non-determinism of Kähler action remains invisible below the time scale assigned to the smallest CDs. One could also say that complete non-determinism characterized in terms path integral with cutoff is replaced in TGD framework with the partial failure of classical non-determinism leading to generalized Feynman diagrams. This gives rise to discrete coupling constant evolution and avoids the mathematical ill-definedness and infinities plaguing path integral formalism since the functional integral over 3-surfaces is well defined.

4.2.5 Category Theory And WCW Geometry

Due the effects caused by the classical non-determinism even classical TGD universes are very far from simple Cartesian clockworks, and the understanding of the general structure of WCW is a formidable challenge. Category theory is a branch of mathematics which is basically a theory about universal aspects of mathematical structures. Thus category theoretical thinking might help in disentangling the complexities of WCW geometry and the basic ideas of category theory are discussed in this spirit and as an innocent layman. It indeed turns out that the approach makes highly non-trivial predictions.

In ZEO the effects of non-determinism are taken into account in terms of causal diamonds forming a hierarchical fractal structure. One must allow also the unions of CDs, CDs within CDs, and probably also overlapping of CDs, and there are good reasons to expert that CDs and corresponding algebraic structures could define categories. If one does not allow overlapping CDs then set theoretic inclusion map defines a natural arrow. If one allows both unions and intersections then CDs would form a structure analogous to the set of open sets used in set theoretic topology. One could indeed see CDs (or rather their Cartesian products with CP_2) as analogs of open sets in Minkowskian signature.

So called ribbon categories seem to be tailor made for the formulation of quantum TGD and allow to build bridge to topological and conformal field theories. This discussion based on standard ontology. In [K16] rather detailed category theoretical constructions are discussed. Important role is played by the notion of operad operad,operads : this structure can be assigned with both generalized Feynman diagrams and with the hierarchy of symplectic fusion algebras realizing symplectic analogs of the fusion rules of conformal field theories.

4.3 Identification Of The Symmetries And Coset Space Structure Of WCW

In this section the identification of the isometry group of the configuration ("world of classical worlds" or briefly WCW) will be discussed at general level.

4.3.1 Reduction To The Light Cone Boundary

The reduction to the light cone boundary would occur exactly if Kähler action were strictly deterministic. This is not the case but it is possible to generalize the construction at light cone boundary to the general case if causal diamonds define the basic structural units of the WCW .

Old argument

The identification of WCW follows as a consequence of 4-dimensional Diff invariance. The right question to ask is the following one. How could one coordinatize the physical(!) vibrational degrees of freedom for 3-surfaces in Diff^4 invariant manner: coordinates should have same values for all Diff^4 related 3-surfaces belonging to the orbit of X^3? The answer is following:

1. Fix some 3-surface (call it Y^3) on the orbit of X^3 in Diff^4 invariant manner.
2. Use as WCW coordinates of X^3 and all its diffeomorphs the coordinates parameterizing small deformations of Y^3. This kind of replacement is physically acceptable since metrically the WCW is equivalent with $Map/Diff^4$.
3. Require that the fixing procedure is Lorentz invariant, where Lorentz transformations in question leave light M^4_+ invariant and thus act as isometries.

The simplest choice of Y^3 is the intersection of the orbit of 3-surface (X^4) with the set $\delta M^4_+ \times CP_2$, where δM^4_+ denotes the boundary of the light cone (moment of big bang):

$$Y^3 \quad = \quad X^4 \cap \delta M^4_+ \times CP_2 \tag{4.3.1}$$

Lorentz invariance allows also the choice $X \times CP_2$, where X corresponds to the hyperboloid $a = \sqrt{(m^0)^2 - r_M^2} = constant$ but only the proposed choice ($a = 0$) leads to a natural complexification in M^4 degrees of freedom. This choice is also cosmologically very natural and completely analogous to the quantum gravitational holography of string theories.

WCW has a fiber space structure. Base space consists of 3-surfaces $Y^3 \subset \delta M^4_+ \times CP_2$ and fiber consists of 3-surfaces on the orbit of Y^3, which are Diff^4 equivalent with Y^3. The distance between the surfaces in the fiber is vanishing in WCW metric. An elegant manner to avoid difficulties caused by Diff^4 degeneracy in WCW integration is to *define* integration measure as integral over the reduced WCW consisting of 3-surfaces Y^3 at the light cone boundary.

Situation is however quite not so simple. The vacuum degeneracy of Kähler action suggests strongly classical non-determinism so that there are several, possibly, infinite number of preferred extremals $X^4(Y^3)$ associated with given Y^3 on light cone boundary. This implies additional degeneracy.

One might hope that the reduced WCW could be replaced by its covering space so that given Y^3 corresponds to several points of the covering space and WCW has many-sheeted structure. Obviously the copies of Y^3 have identical geometric properties. WCW integral would decompose into a sum of integrals over different sheets of the reduced WCW . Note that WCW spinor fields are in general different on different sheets of the reduced WCW .

Even this is probably not enough: it is quite possible that all light like surfaces of M^4 possessing Hamilton Jacobi structure (and thus interpretable as light fronts) are involved with the construction of the WCW geometry. Because of their metric two-dimensionality the proposed construction should generalize. This would mean that WCW geometry has also local laboratory scale aspects and that the general ideas might allow testing.

New version of the argument

The above summary was the basic argument for two decades ago. A more elegant formulation would in terms of light-like 3-surfaces connecting the boundaries of causal diamond taken as basic geometric objects and identified as generalized Feynman diagrams so that they are singular as manifolds at the vertices.

If both formulations are required to be correct, the only conclusion is that effective 2-dimensionality must hold true in the scale of given CD. In other words, the intersection $X^2 = X^3_l \cap X^3$ at the boundary of CD is effectively the basic dynamical unit. The failure of strict non-determinism however forces to introduce entire hierarchy of CDs responsible also for coupling constant evolution defined in terms of the measurement resolution identified as the size of the smallest CD present.

4.3.2 WCW As A Union Of Symmetric Spaces

In finite-dimensional context globally symmetric spaces are of form G/H and connection and curvature are independent of the metric, provided it is left invariant under G. The hope is that same holds true in infinite-dimensional context. The most one can hope of obtaining is the decomposition

$C(H) = \cup_i G/H_i$ over orbits of G. One could allow also symmetry breaking in the sense that G and H depend on the orbit: $C(H) = \cup_i G_i/H_i$ but it seems that G can be chosen to be same for all orbits. What is essential is that these groups are infinite-dimensional. The basic properties of the coset space decomposition give very strong constraints on the group H, which certainly contains the subgroup of G, whose action reduces to diffeomorphisms of X^3.

Consequences of the decomposition

If the decomposition to a union of coset spaces indeed occurs, the consequences for the calculability of the theory are enormous since it suffices to find metric and curvature tensor for single representative 3-surface on a given orbit (contravariant form of metric gives propagator in perturbative calculation of matrix elements as functional integrals over the WCW). The representative surface can be chosen to correspond to the maximum of Kähler function on a given orbit and one obtains perturbation theory around this maximum (Kähler function is not isometry invariant).

The task is to identify the infinite-dimensional groups G and H and to understand the zero mode structure of the WCW . Almost twenty (seven according to long held belief!) years after the discovery of the candidate for the Kähler function defining the metric, it became finally clear that these identifications follow quite nicely from $Diff^4$ invariance and $Diff^4$ degeneracy as well as special properties of the Kähler action.

The guess (not the first one!) would be following. G corresponds to the symplectic transformations of $\delta M^4_\pm \times CP_2$ leaving the induced Kähler form invariant. If G acts as isometries the values of Kähler form at partonic 2-surfaces (remember effective 2-dimensionality realized in simplistic manner) are zero modes and WCW allows slicing to symplectic orbits of the partonic 2-surface with fixed induced Kähler form. Quantum fluctuating degrees of freedom would correspond to symplectic group and to the fluctuations of the induced metric. The group H dividing G would act as diffeomorphisms at the preferred 3-surface X^3 and leaving X^3 itself invariant. Therefore the identification of g and h would be in terms of tangent space algebra of WCW sector realized as coset space G/H.

Coset space structure of WCW and Equivalence Principle

The realization of WCW sectors with fixed values of zero modes as symmetric spaces G/H (analogous to $CP_2 = SU(3)/U(2)$) suggests that one can assign super-Virasoro algebras with G. What the two algebras g and h are is however difficult question. The following vision is only one of the many (the latest one).

1. Symplectic algebra g generates isometries and h is identified as algebra, whose generators generate diffeormorphisms at preferred X^3.

2. The original long-held belief was that the Super Kac-Moody symmetry corresponds to local embedding space isometries for light-like 3-surfaces X^3_l, which might be boundaries of X^4 (probably not: it seems that boundary conditions cannot be satisfied so that space-time surfaces must consists of regions defining at least double coverings of M^4) and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry would be identifiable as the counterpart of the Kac Moody symmetry of string models.

   It has turned out that one can assume Kac-Moody algebra to be sub-algebra of symplectic algebra consisting of the symplectic isometries of embedding space. This Super Kac-Moody algebra is generated by super-currents assignable to the modes of induced spinor fields other than right-handed neutrino and localized at string world sheets. The entire symplectic algebra would correspond to the modes of right-handed neutrino and the entire algebra one would be direct sum of these two algebras so that the number of tensor factors would be indeed 5. The beauty of this option is that localization would be for both algebras inherent and with respect to the light-like coordinate of light-cone boundary rather than forced by hand.

3. p-Adic mass calculations require that symplectic and Kac-Moody algebras together generate the entire algebra. In this situation strong form of holography implies that transformations located to the interior of space-like 3-surface and light-like partonic orbit define zero modes and act like gauge symmetries. The physically non-trivial transformations correspond to

transformations acting non-trivially at partonic 2-surfaces. g corresponds to the algebra generated by these transformations and for preferred 3-surface - identified as (say) maximum of Kähler function - h corresponds to the elements of this algebra generating diffeomorphisms of X^3. Super-conformal representation has five tensor factors corresponding to color algebra, two factors from electroweak $U(2)$, one factor from transversal M^4 translations and one factor from symplectic algebra (note that also Hamiltonians which are products of δM^4_+ and CP_2 Hamiltonians are possible.

Equivalence Principle (EP) has been a longstanding problem for TGD although the recent stringy view about graviton mediated scattering makes it can be argued to reduce to a tautology. I have considered several explanations for EP and coset representation has been one of them.

1. Coset representation associated with the super Virasoro algebra is defined by the condition that the differences of super Virasoro generators for g and h annihilate the physical. The original proposal for the realization of EP was that this condition implies that the four-momenta associated with g and h are identical and identifiable as inertial and gravitational four-momenta. Translations however lead out from CD boundary and cannot leave 3-surface invariant. Hence the Virasoro generators for h should not carry four-momentum. Therefore EP cannot be understood in terms of coset representations.
2. The equivalence of classical Noether momentum associated with Kähler action with eigenvalues of the corresponding quantal momentum for Kähler-Dirac action certainly realizes quantum classical correspondence (QCC) EP could correspond to QCC.
3. A further option is that EP reduces to the identification of the four momenta for Super Virasoro representations assignable to space-like and light-like 3-surfaces and therefore become part of strong form of holography in turn implied by strong form of GCI! It seems that this option is the most plausible one found hitherto.

WCW isometries as a subgroup of $Diff(\delta M^4_+ \times CP_2)$

The reduction to light cone boundary leads to the identification of the isometry group as some subgroup of for the group G for the diffeomorphisms of $\delta M^4_+ \times CP_2$. These diffeomorphisms indeed act in a natural manner in δCH, the space of 3-surfaces in $\delta M^4_+ \times CP_2$. WCW is expected to decompose to a union of the coset spaces G/H_i, where H_i corresponds to some subgroup of G containing the transformations of G acting as diffeomorphisms for given X^3. Geometrically the vector fields acting as diffeomorphisms of X^3 are tangential to the 3-surface. H_i could depend on the topology of X^3 and since G does not change the topology of 3-surface each 3-topology defines separate orbit of G. Therefore, the union involves sum over all topologies of X^3 plus possibly other "zero modes". Different topologies are naturally glued together since singular 3-surfaces intermediate between two 3-topologies correspond to points common to the two sectors with different topologies.

Isometries of WCW geometry as symplectic transformations of $\delta M^4_+ \times CP_2$

During last decade I have considered several candidates for the group G of isometries of WCW as the sub-algebra of the subalgebra of $Diff(\delta M^4_+ \times CP_2)$. To begin with let us write the general decomposition of $diff(\delta M^4_+ \times CP_2)$:

$$diff(\delta M^4_+ \times CP_2) = S(CP_2) \times diff(\delta M^4_+) \oplus S(\delta M^4_+) \times diff(CP_2) \ . \quad (4.3.2)$$

Here $S(X)$ denotes the scalar function basis of space X. This Lie-algebra is the direct sum of light cone diffeomorphisms made local with respect to CP_2 and CP_2 diffeomorphisms made local with respect to light cone boundary.

The idea that entire diffeomorphism group would act as isometries looks unrealistic since the theory should be more or less equivalent with topological field theory in this case. Consider now the various candidates for G.

1. The fact that symplectic transformations of CP_2 and M^4_+ diffeomorphisms are dynamical symmetries of the vacuum extremals suggests the possibility that the diffeomorphisms of the light cone boundary and symplectic transformations of CP_2 could leave Kähler function invariant and thus correspond to zero modes. The symplectic transformations of CP_2 localized with respect to light cone boundary acting as symplectic transformations of CP_2 have interpretation as local color transformations and are a good candidate for the isometries. The fact that local color transformations are not even approximate symmetries of Kähler action is not a problem: if they were exact symmetries, Kähler function would be invariant and zero modes would be in question.

2. CP_2 local conformal transformations of the light cone boundary act as isometries of δM^4_+. Besides this there is a huge group of the symplectic symmetries of $\delta M^4_+ \times CP_2$ if light cone boundary is provided with the symplectic structure. Both groups must be considered as candidates for groups of isometries. $\delta M^4_+ \times CP_2$ option exploits fully the special properties of $\delta M^4_+ \times CP_2$, and one can develop simple argument demonstrating that $\delta M^4_+ \times CP_2$ symplectic invariance is the correct option. Also the construction of WCW gamma matrices as super-symplectic charges supports $\delta M^4_+ \times CP_2$ option.

WCW as a union of symmetric spaces

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ , \quad [t, t] \subset h \ , \quad [h, t] \subset t \ . \tag{4.3.3}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough. $[t, t] \subset h$ condition is highly nontrivial and equivalent with the existence of involution. Inversion in the light-like radial coordinate of δM^4 is a natural guess for this involution and induces complex conjugation in super-conformal algebras mapping positive and negative conformal weights to each other.

WCW geometry allows two super-conformal symmetries. The first one corresponds to super-symplectic transformations acting at the level of embedding space. The second one corresponds to super Kac-Moody symmetry. The original identification of Kac-Moody was in terms of deformations of light-like 3-surfaces respecting their light-likeness. This not wrong as such: also entire symplectic algebra can be assigned with light-like surfaces and the theory can be constructed using also these conformal algebras. This identification however makes it very difficult to see how Kac-Moody could act as isometry: in particular, the localization with respect to internal coordinates of 3-surface produces technical problems since symplectic algebra is localized with respect to the light-like radial coordinate of light-cone boundary.

The more plausible identification is as the sub-algebra of symplectic algebra realized as isometries of δCD so that localization is inherent and in terms of the radial light-like coordinate of light-like boundary [K73]. This identification is made possible by the wisdom gained from the solutions of the Kähler-Dirac equations predicting the localization of its modes (except right-handed neutrino) to string world sheets.

1. g would thus correspond to a direct sum of super-symplectic algebra and super Kac-Moody algebra defined by its isometry sub-algebra but represented in different manner (this is absolutely essential!). More concretely, neutrino modes defined super Hamiltonians associated with the super symplectic algebra and other modes of induced spinor field the super Hamiltonians associated with the super Kac-Moody algebra. The maxima of Kähler function could be chosen as natural candidates for the preferred points and could play also an essential role in WCW integration by generalizing the Gaussian integration of free quantum field theories.

2. These super-conformal algebra representations form a direct sum. p-Adic mass calculations require five super-conformal tensor factors and the number of tensor factors would be indeed this.

3. This algebra has as sub-algebra the algebra for which generators leave 3-surface invariant - in other words, induce its diffeomorphism. Quantum states correspond to the coset representations for entire algebra and this algebra so that differences of the corresponding super-Virasoro generators annihilate physical states. This obviously generalizes Goddard-Olive-Kent construction [A72]. It seems now clear that coset representation does not imply EP: the four-momentum simply does not appear in the representation of the isotropy sub-algebra since translations lead out of CD boundary.

To minimize confusions it must be emphasized that only the contribution of the symplectic algebra realized in terms of single right-handed neutrino mode is discussed in this chapter and the WCW Hamiltonians have 2-dimensional representation. Also the direct connection with the dynamics of Kähler action is lacking. A more realistic construction [K73] uses 3-dimensional representations of Hamiltonians and requires all modes of right-handed neutrino for symplectic algebra and the modes of induced spinor field carrying electroweak quantum numbers in the case of Kac-Moody algebra.

4.4 Complexification

A necessary prerequisite for the Kähler geometry is the complexification of the tangent space in vibrational degrees of freedom. What this means in recent context is non-trivial.

4.4.1 Why Complexification Is Needed?

The Minkowskian signature of M^4 metric seems however to represent an insurmountable obstacle for the complexification of M^4 type vibrational degrees of freedom. On the other hand, complexification seems to have deep roots in the actual physical reality.

1. In the perturbative quantization of gauge fields one associates to each gauge field excitation polarization vector e and massless four-momentum vector p ($p^2 = 0$, $p \cdot e = 0$). These vectors define the decomposition of the tangent space of M^4: $M^4 = M^2 \times E^2$, where M^2 type polarizations correspond to zero norm states and E^2 type polarizations correspond to physical states with non-vanishing norm. Same type of decomposition occurs also in the linearized theory of gravitation. The crucial feature is that E^2 allows complexification! The general conclusion is that the modes of massless, linear, boson fields define always complexification of M^4 (or its tangent space) by effectively reducing it to E^2. Also in string models similar situation is encountered. For a string in D-dimensional space only D-2 transversal Euclidian degrees of freedom are physical.
2. Since symplectically extended isometry generators are expected to create physical states in TGD approach same kind of physical complexification should take place for them, too: this indeed takes place in string models in critical dimension. Somehow one should be able to associate polarization vector and massless four momentum vector to the deformations of a given 3-surface so that these vectors define the decomposition $M^4 = M^2 \times E^2$ for each mode. Configuration space metric should be degenerate: the norm of M^2 deformations should vanish as opposed to the norm of E^2 deformations.

Consider now the implications of this requirement.

1. In order to associate four-momentum and polarization (or at least the decomposition $M^4 = M^2 \times E^2$) to the deformations of the 3-surface one should have field equations, which determine the time development of the 3-surface uniquely. Furthermore, the time development for small deformations should be such that it makes sense to associate four momentum and polarization or at least the decomposition $M^4 = M^2 \times E^2$ to the deformations in suitable basis.

   The solution to this problem is afforded by the proposed definition of the Kähler function. The definition of the Kähler function indeed associates to a given 3-surface a unique four-surface as the preferred extremal of the Kähler action. Therefore one can associate a unique time development to the deformations of the surface X^3 and if TGD describes the observed world this time development should describe the evolution of photon, gluon, graviton, etc. states and so we can hope that tangent space complexification could be defined.

2. We have found that M^2 part of the deformation should have zero norm. In particular, the time like vibrational modes have zero norm in WCW metric. This is true if Kähler function is not only $Diff^3$ invariant but also Diff^4 invariant in the sense that Kähler function has same value for all 3-surfaces belonging to the orbit of X^3 and related to X^3 by diffeomorphism of X^4. This is indeed the case.
3. Even this is not enough. One expects the presence of massive modes having also longitudinal polarization and for these states the number of physical vibrational degrees of freedom is 3 so that complexification seems to be impossible by odd dimension.

The reduction to the light cone boundary implied by $Diff^4$ invariance makes possible to identify the complexification. Crucial role is played by the special properties of the boundary of 4-dimensional light cone, which is metrically two-sphere and thus allows generalized complex and Kähler structure.

4.4.2 The Metric, Conformal And Symplectic Structures Of The Light Cone Boundary

The special metric properties of the light cone boundary play a crucial role in the complexification. The point is that the boundary of the light cone has degenerate metric: although light cone boundary is topologically 3-dimensional it is metrically 2-dimensional: effectively sphere. In standard spherical Minkowski coordinates light cone boundary is defined by the equation $r_M = m^0$ and induced metric reads

$$ds^2 = -r_M^2 d\Omega^2 = -r_M^2 dz d\bar{z}/(1+z\bar{z})^2 \ , \tag{4.4.1}$$

and has Euclidian signature. Since S^2 allows complexification and thus also Kähler structure (and as a by-product also symplectic structure) there are good hopes of obtaining just the required type of complexification in non-degenerate M^4 degrees of freedom: WCW would effectively inherit its Kähler structure from $S^2 \times CP_2$.

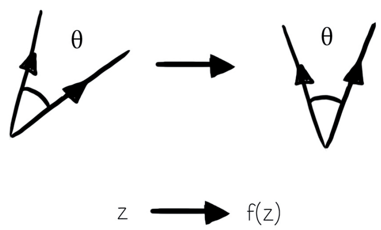

Figure 4.1: Conformal symmetry preserves angles in complex plane

By its effective two-dimensionality the boundary of the four-dimensional light cone has infinite-dimensional group of (local) conformal transformations. Using complex coordinate z for S^2 the general local conformal transformation reads (see **Fig. 4.1**)

$$\begin{aligned} r &\rightarrow f(r_M, z, \bar{z}) \ , \\ z &\rightarrow g(z) \ , \end{aligned} \tag{4.4.2}$$

where f is an arbitrary real function and g is an arbitrary analytic function with a finite number of poles. The infinitesimal generators of this group span an algebra, call it C, analogous to Virasoro algebra. This algebra is semidirect sum of two algebras L and R given by

$$\begin{aligned} C &= L \oplus R \ , \\ [L, R] \subset R \ , \end{aligned} \tag{4.4.3}$$

where L denotes standard Virasoro algebra of the two- sphere generated by the generators

$$L_n = z^{n+1} d/dz \tag{4.4.4}$$

and R denotes the algebra generated by the vector fields

$$R_n = f_n(z, \bar{z}, r_M)\partial_{r_M} \ , \tag{4.4.5}$$

where $f(z, \bar{z}, r_M)$ forms complete real scalar function basis for light cone boundary. The vector fields of R have the special property that they have vanishing norm in M^4 metric.

This modification of conformal group implies that the Virasoro generator L_0 becomes $L_0 = zd/dz - r_M d/dr_M$ so that the scaling momentum becomes the difference $n - m$ or S^2 and radial scaling momenta. One could achieve conformal invariance by requiring that S^2 and radial scaling quantum numbers compensate each other.

Of crucial importance is that light cone boundary allows infinite dimensional group of isometries! An arbitrary conformal transformation $z \rightarrow f(z)$ induces to the metric a conformal factor given by $|df/dz|^2$. The compensating radial scaling $r_M \rightarrow r_M/|df/dz|$ compensates this factor so that the line element remains invariant.

The Kähler structure of light cone boundary defines automatically symplectic structure. The symplectic form is degenerate and just the area form of S^2 given by

$$J = r_M^2 sin(\theta) d\theta \wedge d\phi,$$

in standard spherical coordinates, there is infinite-dimensional group of symplectic transformations leaving the symplectic form of the light cone boundary (that is S^2) invariant. These transformations are local with respect to the radial coordinate r_M. The symplectic and Kähler structures of light cone boundary are not unique: different structures are labeled by the coset space $SO(3,1)/SO(3)$. One can however associate with a given 3-surface Y^3 a unique structure by requiring that the corresponding subgroup $SO(3)$ of Lorentz group acts as the isotropy group of the conserved classical four-momentum assigned to Y^3 by the preferred extremal property.

In the case of $\delta M^4_+ \times CP_2$ both the conformal transformations, isometries and symplectic transformations of the light cone boundary can be made local also with respect to CP_2. The idea that the infinite-dimensional algebra of symplectic transformations of $\delta M^4_+ \times CP_2$ act as isometries of WCW and that radial vector fields having zero norm in the metric of light cone boundary possess zero norm also in WCW metric, looks extremely attractive.

In the case of $\delta M^4_+ \times CP_2$ one could combine the symplectic and Kähler structures of δM^4_+ and CP_2 to single symplectic/Kähler structure. The symplectic transformations leaving this symplectic structure invariant would be generated by the function algebra of $\delta M^4_+ \times CP_2$ such that a arbitrary function serves as a Hamiltonian of a symplectic transformation. This group serves as a candidate for the isometry group of WCW . An alternative identification for the isometry algebra is as symplectic symmetries of CP_2 localized with respect to the light cone boundary. Hamiltonians would be also now elements of the function algebra of $\delta M^4_+ \times CP_2$ but their Poisson brackets would be defined using only CP_2 symplectic form.

The problem is to decide which option is correct. There is a simple argument fixing the latter option. The symplecticly imbedded CP_2 would be left invariant under δM^4_+ local symplectic transformations of CP_2. This seems strange. Under symplectic algebra of $\delta M^4_+ \times CP_2$ also symplecticly imbedded CP_2 is deformed and this sounds more realistic. The isometry algebra is therefore assumed to be the group $can(\delta M^4_+ \times CP_2)$ generated by the scalar function basis $S(\delta M^4_+ \times CP_2) = S(\delta M^4_+) \times S(CP_2)$ of the light cone boundary using the Poisson brackets to be discussed in more detail later.

There are some no-go theorems associated with higher-dimensional Abelian extensions [A78], and although the contexts are quite different, it is interesting to consider the recent situation in light of these theorems.

1. Conformal invariance is an essentially 2-dimensional notion. Light cone boundary is however metrically and conformally 2-sphere, and therefore the conformal algebra is effectively that associated with the 2-sphere. In the same manner, the quaternion conformal algebra associated with the metrically 2-dimensional elementary particle horizons surrounding wormhole contacts allows the usual Kac Moody algebra and actually also contributes to the WCW metric.
2. In dimensions $D > 2$ Abelian extensions of the gauge algebra are extensions by an infinite-dimensional Abelian group rather than central extensions by the group $U(1)$. This result has an analog at the level of WCW geometry. The extension associated with the symplectic algebra of CP_2 localized with respect to the light cone boundary is analogous a symplectic extension defined by Poisson bracket $\{p, q\} = 1$. The central extension is the function space associated with δM^4_+ and indeed infinite-dimensional if only only CP_2 symplectic structure induces the Poisson bracket but one-dimensional if $\delta M^4_+ \times CP_2$ Poisson bracket induces the extension. In the latter case the symmetries fix the metric completely at the point corresponding to the origin of symmetric space (presumably the maximum of Kähler function for given values of zero modes).
3. $D > 2$ extensions possess no unitary faithful representations (satisfying certain well motivated physical constraints) [A78]. It might be that the degeneracy of the WCW metric is the analog for the loss of faithful representations.

4.4.3 Complexification And The Special Properties Of The Light Cone Boundary

In case of Kähler metric G and H Lie-algebras must allow complexification so that the isometries can act as holomorphic transformations. Since G and H can be regarded as subalgebras of the vector fields of $\delta M^4_+ \times CP_2$, they inherit in a natural manner the complex structure of the light cone boundary.

There are two candidates for WCW complexification. The simplest, and also the correct, alternative is that complexification is induced by natural complexification of vector field basis on $\delta M^4_+ \times CP_2$. In CP_2 degrees of freedom there is natural complexification

$$\xi \rightarrow \bar{\xi} \ .$$

In δM^4_+ degrees of freedom this could involve the transformation

$$z \rightarrow \bar{z}$$

and certainly involves complex conjugation for complex scalar function basis in the radial direction:

$$f(r_M) \rightarrow \overline{f(r_M)} \ ,$$

which turns out to play same role as the function basis of circle in the Kähler geometry of loop groups [A54].

The requirement that the functions are eigen functions of radial scalings favors functions $(r_M/r_0)^k$, where k is in general a complex number. The function can be expressed as a product of real power of r_M and logarithmic plane wave. It turns out that the radial complexification alternative is the correct manner to obtain Kähler structure. The reason is that symplectic transformations leave the value of r_M invariant. Radial Virasoro invariance plays crucial role in making the complexification possible.

One could consider also a second alternative assumed in the earlier formulation of the WCW geometry. The close analogy with string models and conformal field theories suggests that for Virasoro generators the complexification must reduce to the hermitian conjugation of the conformal field theories: $L_n \rightarrow L_{-n} = L_n^\dagger$. Clearly this complexification is induced from the transformation $z \rightarrow \frac{1}{z}$ and differs from the complexification induced by complex conjugation $z \rightarrow \bar{z}$. The basis would be polynomial in z and $\bar{z}$. Since radial algebra could be also seen as Virasoro algebra localized with respect to $S^2 \times CP_2$ one could consider the possibility that also in radial direction the inversion $r_M \rightarrow \frac{1}{r_M}$ is involved.

In fact, the complexification changing the signs of radial conformal weights is induced from inversion $r_M/r_0 \rightarrow r_0/r_M$. This transformation is also an excellent candidate for the involution necessary for obtaining the structure of symmetric space implying among other things the covariant constancy of the curvature tensor, which is of special importance in infinite-D context.

The essential prerequisite for the Kähler structure is that both G and H allow same complexification so that the isometries in question can be regarded as holomorphic transformations. In finite-dimensional case this essentially what is needed since metric can be constructed by parallel translation along the orbit of G from H-invariant Kähler metric at a representative point. The requirement of H-invariance forces the radial complexification based on complex powers r_M^k: radial complexification works since symplectic transformations leave r_M invariant.

Some comments on the properties of the proposed complexification are in order.

1. The proposed complexification, which is analogous to the choice of gauge in gauge theories is not Lorentz invariant unless one can fix the coordinates of the light cone boundary apart from $SO(3)$ rotation not affecting the value of the radial coordinate r_M (if the imaginary part of k in r_M^k is always non-vanishing). This is possible as will be explained later.

2. It turns out that the function basis of light-cone boundary multiplying CP_2 Hamiltonians corresponds to unitary representations of the Lorentz group at light cone boundary so that the Lorentz invariance is rather manifest.

3. There is a nice connection with the proposed physical interpretation of the complexification. At the moment of the big bang all particles move with the velocity of light and therefore behave as massless particles. To a given point of the light cone boundary one can associate a unique direction of massless four-momentum by semiclassical considerations: at the point $m^k = (m^0, m^i)$ momentum is proportional to the vector $(m^0, -m^i)$. Since the particles are massless only two polarization vectors are possible and these correspond to the tangent vectors to the sphere $m^0 = r_M$. Of course, one must always fix polarizations at some point of tangent space but since massless polarization vectors are not physical this doesn't imply difficulties: different choices correspond to different gauges.

4. Complexification in the proposed manner is not possible except in the case of four-dimensional Minkowski space. Non-zero norm deformations correspond to vector fields of the light cone boundary acting on the sphere S^{D-2} and the decomposition to $(1,0)$ and $(0,1)$ parts is possible only when the sphere in question is two-dimensional since other spheres do allow neither complexification nor Kähler structure.

4.4.4 How To Fix The Complex And Symplectic Structures In A Lorentz Invariant Manner?

One can assign to light-cone boundary a symplectic structure since it reduces effectively to S^2. The possible symplectic structures of δM^4_+ are parameterized by the coset space $SO(3,1)/SO(3))$, where H is the isotropy group $SO(3)$ of a time like vector. Complexification also fixes the choice of the spherical coordinates apart from rotations around the quantization axis of angular momentum.

The selection of some preferred symplectic structure in an ad hoc manner breaks manifest Lorentz invariance but is possible if physical theory remains Lorentz invariant. The more natural possibility is that 3-surface Y^3 itself fixes in some natural manner the choice of the symplectic structure so that there is unique subgroup $SO(3)$ of $SO(3,1)$ associated with Y^3. If WCW Kähler function corresponds to a preferred extremal of Kähler action, this is indeed the case. One can associate unique conserved four-momentum $P^k(Y^3)$ to the preferred extremal $X^4(Y^3)$ of the Kähler action and the requirement that the rotation group $SO(3)$ leaving the symplectic structure invariant leaves also $P^k(Y^3)$ invariant, fixes the symplectic structure associated with Y^3 uniquely.

Therefore WCW decomposes into a union of symplectic spaces labeled by $SO(3,1)/SO(3)$ isomorphic to $a = constant$ hyperboloid of light cone. The direction of the classical angular momentum vector $w^k = \epsilon^{klmn} P_l J_{mn}$ determined by the classical angular momentum tensor of associated with Y^3 fixes one coordinate axis and one can require that $SO(2)$ subgroup of $SO(3)$ acting as rotation around this coordinate axis acts as phase transformation of the complex coordinate z of S^2. Other rotations act as nonlinear holomorphic transformations respecting the complex structure.

Clearly, the coordinates are uniquely fixed modulo $SO(2)$ rotation acting as phase multiplication in this case. If $P^k(Y^3)$ is light like, one can only require that the rotation group $SO(2)$ serving as the isotropy group of 3-momentum belongs to the group $SO(3)$ characterizing the symplectic structure and it seems that symplectic structure cannot be uniquely fixed without additional constraints in this case. Probably this has no practical consequences since the 3-surfaces considered have actually infinite size and 4-momentum is most probably time like for them. Note however that the direction of 3-momentum defines unique axis such that $SO(2)$ rotations around this axis are represented as phase multiplication.

Similar almost unique frame exists also in CP_2 degrees of freedom and corresponds to the complex coordinates transforming linearly under $U(2)$ acting as isotropy group of the Lie-algebra element defined by classical color charges Q_a of Y^3. One can fix unique Cartan subgroup of $U(2)$ by noticing that $SU(3)$ allows completely symmetric structure constants d_{abc} such that $R_a = {d_a}^{bc}Q_bQ_c$ defines Lie-algebra element commuting with Q_a. This means that R_a and Q_a span in generic case $U(1) \times U(1)$ Cartan subalgebra and there are unique complex coordinates for which this subgroup acts as phase multiplications. The space of nonequivalent frames is isomorphic with $CP(2)$ so that one can say that cm degrees of freedom correspond to Cartesian product of $SO(3,1)/SO(3)$ hyperboloid and CP_2 whereas coordinate choices correspond to the Cartesian product of $SO(3,1)/SO(2)$ and $SU(3)/U(1) \times U(1)$.

Symplectic transformations leave the value of δM^4_+ radial coordinate r_M invariant and this implies large number of additional zero modes characterizing the size and shape of the 3-surface. Besides this Kähler magnetic fluxes through the $r_M = constant$ sections of X^3 as a function of r_M provide additional invariants, which are functions rather than numbers. The Fourier components for the magnetic fluxes provide infinite number of symplectic invariants. The presence of these zero modes imply that 3-surfaces behave much like classical objects in the sense that neither their shape nor form nor classical Kähler magnetic fields, are subject to Gaussian fluctuations. Of course, quantum superpositions of 3-surfaces with different values of these invariants are possible.

There are reasons to expect that at least certain infinitesimal symplectic transformations correspond to zero modes of the Kähler metric (symplectic transformations act as dynamical symmetries of the vacuum extremals of the Kähler action). If this is indeed the case, one can ask whether it is possible to identify an integration measure for them.

If one can associate symplectic structure with zero modes, the symplectic structure defines integration measure in a standard manner (for 2n-dimensional symplectic manifold the integration measure is just the n-fold wedge power $J \wedge J ... \wedge J$ of the symplectic form J). Unfortunately, in infinite-dimensional context this is not enough since divergence free functional integral analogous to a Gaussian integral is needed and it seems that it is not possible to integrate in zero modes and that this relates in a deep manner to state function reduction. If all symplectic transformations of $\delta M^4_+ \times CP_2$ are represented as symplectic transformations of the configuration space, then the existence of symplectic structure decomposing into Kähler (and symplectic) structure in complexified degrees of freedom and symplectic (but not Kähler) structure in zero modes, is an automatic consequence.

4.4.5 The General Structure Of The Isometry Algebra

There are three options for the isometry algebra of WCW .

1. Isometry algebra as the algebra of CP_2 symplectic transformations leaving invariant the symplectic form of CP_2 localized with respect to δM^4_+.
2. Certainly the WCW metric in δM^4_+ must be non-trivial and actually given by the magnetic flux Hamiltonians defining symplectic invariants. Furthermore, the super-symplectic generators constructed from quarks automatically give as anti-commutators this part of the WCW metric. One could interpret these symplectic invariants as WCW Hamiltonians for δM^4_+ symplectic transformations obtained when CP_2 Hamiltonian is constant.
3. Isometry algebra consists of $\delta M^4_+ \times CP_2$ symplectic transformations. In this case a local color transformation involves necessarily a local S^2 transformation. Unfortunately, it is difficult to decide at this stage which of these options is correct.

The eigen states of the rotation generator and Lorentz boost in the same direction defining a unitary representation of the Lorentz group at light cone boundary define the most natural function basis for the light cone boundary. The elements of this bases have also well defined scaling quantum numbers and define also a unitary representation of the conformal algebra. The product of the basic functions is very simple in this basis since various quantum numbers are additive.

Spherical harmonics of S^2 provide an alternative function basis for the light cone boundary:

$$H^m_{jk} \equiv Y_{jm}(\theta,\phi)r^k_M \ . \tag{4.4.6}$$

One can criticize this basis for not having nice properties under Lorentz group.

The product of basis functions is given by Glebch-Gordan coefficients for symmetrized tensor product of two representation of the rotation group. Poisson bracket in turn reduces to the Glebch-Gordans of anti-symmetrized tensor product. The quantum numbers m and k are additive. The basis is eigen-function basis for the imaginary part of the Virasoro generator L_0 generating rotations around quantization axis of angular momentum. In fact, only the imaginary part of the Virasoro generator $L_0 = zd/dz = \rho\partial_\rho - \frac{2}{2}\partial_\phi$ has global single valued Hamiltonian, whereas the corresponding representation for the transformation induced by the real part of L_0, with a compensating radial scaling added, cannot be realized as a global symplectic transformation.

The Poisson bracket of two functions $H^m_{j_1k_1}$ and $H^m_{j_2k_2}$ can be calculated and is of the general form

$$\{H^{m_1}_{j_1k}, H^{m_2}_{j_2k_2}\} \equiv C(j_1m_1j_2m_2|j,m_1+m_2)_A H^{m_1+m_2}_{j,k_1+k_2} \ . \tag{4.4.7}$$

The coefficients are Glebch-Gordan coefficients for the anti-symmetrized tensor product for the representations of the rotation group.

The isometries of the light cone boundary correspond to conformal transformations accompanied by a local radial scaling compensating the conformal factor coming from the conformal transformations having parametric dependence of radial variable and CP_2 coordinates. It seems however that isometries cannot in general be realized as symplectic transformations. The first difficulty is that symplectic transformations cannot affect the value of the radial coordinate. For rotation algebra the representation as symplectic transformations is however possible.

In CP_2 degrees of freedom scalar function basis having definite color transformation properties is desirable. Scalar function basis can be obtained as the algebra generated by the Hamiltonians of color transformations by multiplication. The elements of basis can be typically expressed as monomials of color Hamiltonians H^A_c

$$H^A_D = \sum_{\{B_j\}} C^A_{DB_1B_2....B_N} \prod_{B_i} H^{B_i}_c \ , \tag{4.4.8}$$

where summation over all index combinations $\{B_i\}$ is understood. The coefficients $C^A_{DB_1B_2....B_N}$ are Glebch-Gordan coefficients for completely symmetric N: th power $8\otimes 8...\otimes 8$ of octet representations. The representation is not unique since $\sum_A H^A_c H^A_c = 1$ holds true. One can however find for each representation D some minimum value of N.

The product of Hamiltonians $H^{D_1}_A$ and $H^B_{D_2}$ can be decomposed by Glebch-Gordan coefficients of the symmetrized representation $(D_1\otimes D_2)_S$ as

$$H^A_{D_1}H^B_{D_2} = C^{ABD}_{D_1D_2DC}(S)H^C_D \ , \tag{4.4.9}$$

where $'S'$ indicates that the symmetrized representation is in question. In the similar manner one can decompose the Poisson bracket of two Hamiltonians

$$\{H^A_{D_1}, H^B_{D_2}\} = C^{ABD}_{D_1D_2DC}(A)H^C_D \ . \tag{4.4.10}$$

Here $'A'$ indicates that Glebch-Gordan coefficients for the anti-symmetrized tensor product of the representations D_1 and D_2 are in question.

One can express the infinitesimal generators of CP_2 symplectic transformations in terms of the color isometry generators J_c^B using the expansion of the Hamiltonian in terms of the monomials of color Hamiltonians:

$$\begin{aligned} j_{DN}^A &= F_{DB}^A J_c^B \ , \\ F_{DB}^A &= N \sum_{\{B_j\}} C_{DB_1B_2...B_{N-1}B}^A \prod_j H_c^{B_j} \ , \end{aligned} \tag{4.4.11}$$

where summation over all possible $\{B_j\}$: s appears. Therefore, the interpretation as a color group localized with respect to CP_2 coordinates is valid in the same sense as the interpretation of space-time diffeomorphism group as local Poincare group. Thus one can say that TGD color is localized with respect to the entire $\delta M_+^4 \times CP_2$.

A convenient basis for the Hamiltonians of $\delta M_+^4 \times CP_2$ is given by the functions

$$H_{jkD}^{mA} = H_{jk}^m H_D^A \ .$$

The symplectic transformation generated by H_{jkD}^{mA} acts both in M^4 and CP_2 degrees of freedom and the corresponding vector field is given by

$$J^r = H_D^A J^{rl}(\delta M_+^4)\partial_l H_{jk}^m + H_{jk}^m J^{rl}(CP_2)\partial_l H_D^A \ . \tag{4.4.12}$$

The general form for their Poisson bracket is:

$$\begin{aligned} &\{H_{j_1k_1D_1}^{m_1A_1}, H_{j_2k_2D_2}^{m_2A_2}\} = H_{D_1}^{A_1} H_{D_2}^{A_2}\{H_{j_1k_1}^{m_1}, H_{j_2k_2}^{m_2}\} + H_{j_1k_1}^{m_1} H_{j_2k_2}^{m_2}\{H_{D_1}^{A_1}, H_{D_2}^{A_2}\} \\ &= \left[C_{D_1D_2D}^{A_1A_2A}(S) C(j_1m_1j_2m_2|jm)_A + C_{D_1D_2D}^{A_1A_2A}(A) C(j_1m_1j_2m_2|jm)_S \right] H_{j,k_1+k_2,D}^{mA} \ . \end{aligned} \tag{4.4.13}$$

What is essential that radial "momenta" and angular momentum are additive in δM_+^4 degrees of freedom and color quantum numbers are additive in CP_2 degrees of freedom.

4.4.6 Representation Of Lorentz Group And Conformal Symmetries At Light Cone Boundary

A guess deserving testing is that the representations of the Lorentz group at light cone boundary might provide natural building blocks for the construction of the WCW Hamiltonians. In the following the explicit representation of the Lorentz algebra at light cone boundary is deduced, and a function basis giving rise to the representations of Lorentz group and having very simple properties under modified Poisson bracket of δM_+^4 is constructed.

Explicit representation of Lorentz algebra

It is useful to write the explicit expressions of Lorentz generators using complex coordinates for S^2. The expression for the $SU(2)$ generators of the Lorentz group are

$$\begin{aligned} J_x &= (z^2-1)d/dz + c.c. = L_1 - L_{-1} + c.c. \ , \\ J_y &= (iz^2+1)d/dz + c.c. = iL_1 + iL_{-1} + c.c. \ , \\ J_z &= iz\frac{d}{dz} + c.c. = iL_z + c.c. \ . \end{aligned} \tag{4.4.14}$$

The expressions for the generators of Lorentz boosts can be derived easily. The boost in m^3 direction corresponds to an infinitesimal transformation

$$\begin{aligned}
\delta m^3 &= -\varepsilon r_M \ , \\
\delta r_M &= -\varepsilon m^3 = -\varepsilon\sqrt{r_M^2-(m^1)^2-(m^2)^2} \ .
\end{aligned} \tag{4.4.15}$$

The relationship between complex coordinates of S^2 and M^4 coordinates m^k is given by stereographic projection

$$\begin{aligned}
z &= \frac{(m^1+im^2)}{(r_M-\sqrt{r_M^2-(m^1)^2-(m^2)^2})} \\
&= \frac{sin(\theta)(cos\phi+isin\phi)}{(1-cos\theta)} \ , \\
cot(\theta/2) &= \rho=\sqrt{z\bar{z}} \ , \\
tan(\phi) &= \frac{m^2}{m^1} \ .
\end{aligned} \tag{4.4.16}$$

This implies that the change in z coordinate doesn't depend at all on r_M and is of the following form

$$\delta z = -\frac{\varepsilon}{2}(1+\frac{z(z+\bar{z})}{2})(1+z\bar{z}) \ . \tag{4.4.17}$$

The infinitesimal generator for the boosts in z-direction is therefore of the following form

$$L_z = [\frac{2z\bar{z}}{(1+z\bar{z})}-1]r_M\frac{\partial}{\partial_{r_M}}-iJ_z \ . \tag{4.4.18}$$

Generators of L_x and L_y are most conveniently obtained as commutators of $[L_z,J_y]$ and $[L_z,J_x]$. For L_y one obtains the following expression:

$$L_y = 2\frac{(z\bar{z}(z+\bar{z})+i(z-\bar{z}))}{(1+z\bar{z})^2}r_M\frac{\partial}{\partial_{r_M}}-iJ_y \ , \tag{4.4.19}$$

For L_x one obtains analogous expressions. All Lorentz boosts are of the form $L_i = -iJ_i +$ *local radial scaling* and of zeroth degree in radial variable so that their action on the general generator $X^{klm}\propto z^k\bar{z}^l r_M^m$ doesn't change the value of the label m being a mere local scaling transformation in radial direction. If radial scalings correspond to zero norm isometries this representation is metrically equivalent with the representations of Lorentz boosts as Möbius transformations.

Representations of the Lorentz group reduced with respect to $SO(3)$

The ordinary harmonics of S^2 define in a natural manner infinite series of representation functions transformed to each other in Lorentz transformations. The inner product defined by the integration measure $r_M^2 d\Omega dr_M/r_M$ remains invariant under Lorentz boosts since the scaling of r_M induced by the Lorentz boost compensates for the conformal scaling of $d\Omega$ induced by a Lorentz transformation represented as a Möbius transformation. Thus unitary representations of Lorentz group are in question.

The unitary main series representations of the Lorentz group are characterized by half-integer m and imaginary number $k_2 = i\rho$, where ρ is any real number [A67]. A natural guess is that $m = 0$ holds true for all representations realizable at the light cone boundary and that radial waves are of form r_M^k, $k = k_1 + ik_2 = -1 + i\rho$ and thus eigen states of the radial scaling so that the action of Lorentz boosts is simple in the angular momentum basis. The inner product in radial degrees of freedom reduces to that for ordinary plane waves when $log(r_M)$ is taken as a new integration variable. The complexification is well-defined for non-vanishing values of ρ.

It is also possible to have non-unitary representations of the Lorentz group and the realization of the symmetric space structure suggests that one must have $k = k_1 + ik_2$, k_1 half-integer. For these representations unitarity fails because the inner product in the radial degrees of freedom is non-unitary. A possible physical interpretation consistent with the general ideas about conformal invariance is that the representations $k = -1 + i\rho$ correspond to the unitary ground state representations and $k = -1 + n/2 + i\rho$, $n = \pm 1, \pm 2, ...$, to non-unitary representations. The general view about conformal invariance suggests that physical states constructed as tensor products satisfy the condition $\sum_i n_i = 0$ completely analogous to Virasoro conditions.

Representations of the Lorentz group with $E^2 \times SO(2)$ as isotropy group

One can construct representations of Lorentz group and conformal symmetries at the light cone boundary. Since $SL(2, C)$ is the group generated by the generators L_0 and $L_\pm$ of the conformal algebra, it is clear that infinite-dimensional representations of Lorentz group can be also regarded as representations of the conformal algebra. One can require that the basis corresponds to eigen functions of the rotation generator J_z and corresponding boost generator L_z. For functions which do not depend on r_M these generators are completely analogous to the generators L_0 generating scalings and iL_0 generating rotations. Also the generator of radial scalings appears in the formulas and one must consider the possibility that it corresponds to the generator L_0.

In order to construct scalar function eigen basis of L_z and J_z, one can start from the expressions

$$\begin{aligned} L_3 &\equiv i(L_z + L_{\bar{z}}) = 2i[\frac{2z\bar{z}}{(1+z\bar{z})} - 1]r_M \frac{\partial}{\partial_{r_M}} + i\rho\partial_\rho \ , \\ J_3 &\equiv iL_z - iL_{\bar{z}} = i\partial_\phi \ . \end{aligned} \tag{4.4.20}$$

If the eigen functions do not depend on r_M, one obtains the usual basis z^n of Virasoro algebra, which however is not normalizable basis. The eigenfunctions of the generators L_3, J_3 and $L_0 = ir_M d/dr_M$ satisfying

$$\begin{aligned} J_3 f_{m,n,k} &= m f_{m,n,k} \ , \\ L_3 f_{m,n,k} &= n f_{m,n,k} \ , \\ L_0 f_{m,n,k} &= k f_{m,n,k} \ . \end{aligned} \tag{4.4.21}$$

are given by

$$f_{m,n,k} = e^{im\phi} \frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \ . \tag{4.4.22}$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2 The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{(1+\rho^2)^2} d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n - k)$. This suggests that the condition

$$|m| \leq 2(n - k) \tag{4.4.23}$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3,1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra.

The representations of the Lorentz group are characterized by one half-integer valued and one complex parameter. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

The nice feature of the function basis is that various quantum numbers are additive under multiplication:

$$f(m_a, n_a, k_a) \times f(m_b, n_b, k_b) = f(m_a + m_b, n_a + n_b, k_a + k_b) \ .$$

These properties allow to cast the Poisson brackets of the symplectic algebra of WCW into an elegant form.

The Poisson brackets for the δM^4_+ Hamiltonians defined by f_{mnk} can be written using the expression $J^{\rho\phi} = (1 + \rho^2)/\rho$ as

$$\begin{aligned} \{f_{m_a,n_a,k_a}, f_{m_b,n_b,k_b}\} &= i\left[(n_a - k_a)m_b - (n_b - k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-2,k_a+k_b} \\ &+ 2i\left[(2 - k_a)m_b - (2 - k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-1,k_a+k_b-1} \ . \end{aligned} \tag{4.4.24}$$

Can one find unitary light-like representations of Lorentz group?

It is interesting to compare the representations in question to the unitary representations Gelfand.

1. The unitary representations discussed in [A67] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labeled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labeled by three different quantum numbers.

2. The representations of [A67] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labeling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ formal unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves for $k_1 = -1$. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteeing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity is possible only for the function basis consisting of spherical harmonics.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, WCW spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Hermitian form for light cone Hamiltonians involves also the integration over S^2 degrees of freedom and the non-unitarity of the inner product reflects itself as non-orthogonality of the eigen function basis. Introducing the variable $u = \rho^2 + 1$ as a new integration variable, one can express the inner product in the form

$$\begin{aligned}
\langle m_a, n_a, k_a | m_b, n_b, k_b \rangle &= \pi\delta(k_{2a} - k_{2b}) \times \delta_{m_1,m_2} \times I \ , \\
I &= \int_1^\infty f(u)du \ , \\
f(u) &= \frac{(u-1)^{\frac{(N-K)+i\Delta}{2}}}{u^{K+2}} \ .
\end{aligned} \tag{4.4.25}$$

The integrand has cut from $u = 1$ to infinity along real axis. The first thing to observe is that for $N = K$ the exponent of the integral reduces to very simple form and integral exists only for $K = k_{1a} + k_{1b} > -1$. For $k_{1i} = -1/2$ the integral diverges.

The discontinuity of the integrand due to the cut at the real axis is proportional to the integrand and given by

$$\begin{aligned}
f(u) - f(e^{i2\pi}u) &= \left[1 - e^{-\pi\Delta}\right] f(u) \ , \\
\Delta &= n_{1a} - k_{1a} - n_{1b} + k_{1b} \ .
\end{aligned} \tag{4.4.26}$$

This means that one can transform the integral to an integral around the cut. This integral can in turn completed to an integral over closed loop by adding the circle at infinity to the integration path. The integrand has $K+1$-fold pole at $u = 0$.

Under these conditions one obtains

$$\begin{aligned}
I &= \frac{2\pi i}{1 - e^{-\pi\Delta}} \times R \times (R-1) .... \times (R - K - 1) \times (-1)^{\frac{N-K}{2} - K - 1} \ , \\
R &\equiv \frac{N-K}{2} + i\Delta \ .
\end{aligned} \tag{4.4.27}$$

This expression is non-vanishing for $\Delta \neq 0$. Thus it is not possible to satisfy orthogonality conditions without the un-physical $n = k, k_1 = 1/2$ constraint. The result is finite for $K > -1$ so that $k_1 > -1/2$ must be satisfied and if one allows only half-integers in the spectrum, one must have $k_1 \geq 0$, which is very natural if real conformal weights which are half integers are allowed.

4.4.7 How The Complex Eigenvalues Of The Radial Scaling Operator-Relate To Symplectic Conformal Weights?

Complexified Hamiltonians can be chosen to be eigenmodes of the radial scaling operator $r_M d/dr_M$, and the first guess was that the correct interpretation is as conformal weights. The problem is however that the eigenvalues are complex. Second problem is that general arguments are not enough to fix the spectrum of eigenvalues. There should be a direct connection to the dynamics defined by Kähler action and the Kähler-Dirac action defined by it.

The construction of WCW spinor structure in terms of second quantized induced spinor fields [K96] leads to the conclusion that the modes of induced spinor fields must be restricted at surfaces with 2-D CP_2 projection to guarantee vanishing W fields and well-defined em charge for them. In the generic case these surfaces are 2-D string world sheets (or possibly also partonic 2-surfaces) and in the non-generic case can be chosen to be such. The modes are labeled by generalized conformal weights assignable to complex or hypercomplex string coordinate. Conformal weights are expected to be integers from the experience with string models.

It is an open question whether these conformal weights are independent of the symplectic formal weights or not but on can consider also the possibility that they are dependent. Note hovewer that string coordinate is not reducible to the light-like radial coordinate in the generic case and one can imagine situations in which r_M is constant although string coordinate varies. Dependency would be achieved if the Hamiltonians are generalized eigen modes of $D = \gamma^x d/dx$, $x = log(r/r_0)$, satisfying $DH = \lambda\gamma^x H$ and thus of form $exp(\lambda x) = (r/r_0)^\lambda$ with the same spectrum of eigenvalues λ as associated with the Kähler-Dirac operator. That $log(r/r_0)$ naturally corresponds to the coordinate u assignable to the generalized eigen modes of Kähler-Dirac operator supports this interpretation.

The recent view is that the two conformal weights are independent. The conformal weights associated with the modes of Kähler-Dirac operator localized at string world sheets by the condition that the electromagnetic charge is well-defined for the modes (classical induced W field must vanish at string world sheets). The conformal weights of spinor modes would be integer valued as in string models. About super-symplectic conformal weights associated one cannot say this.

This revives the forgotten TGD inspired conjecture that the conformal weights associated with the generators (in the technical sense of the word) of the super-symplectic algebra are given by the negatives of the zeros of Riemann Zeta $h = -1/2 + iy_i$. Note that these conformal weights have negative real part having interpretation in terms of tachyonic ground state needed in p-adic mass calculations [K46]. The spectrum of conformal weights would be of form $h = n/2 + \sum_i n_i y_i$. This would conform with the association of Riemann Zeta to critical systems. From the identification of mass squared as conformal weight, the total conformal weights for the physical states should have vanishing imaginary part be therefore non-negative integers. This would give rise to what might be called conformal confinement.

4.5 Magnetic And Electric Representations Of WCW Hamiltonians

Symmetry considerations lead to the hypothesis that WCW Hamiltonians are apart from a factor depending on symplectic invariants equal to magnetic flux Hamiltonians. On the other hand, the hypothesis that Kähler function corresponds to a preferred extremal of Kähler action leads to the hypothesis that WCW Hamiltonians corresponds to classical charges associated with the Hamiltonians of the light cone boundary. These charges are very much like electric charges. The requirement that two approaches are equivalent leads to the hypothesis that magnetic and electric Hamiltonians are identical apart from a factor depending on isometry invariants. At the level of CP_2 corresponding duality corresponds to the self-duality of Kähler form stating that the magnetic and electric parts of Kähler form are identical.

4.5.1 Radial Symplectic Invariants

All $\delta M^4_+ \times CP_2$ symplectic transformations leave invariant the value of the radial coordinate r_M. Therefore the radial coordinate r_M of X^3 regarded as a function of $S^2 \times CP_2$ coordinates serves as height function. The number, type, ordering and values for the extrema for this height function in the interior and boundary components are isometry invariants. These invariants characterize not only the topology but also the size and shape of the 3-surface. The result implies that WCW metric indeed differentiates between 3-surfaces with the size of Planck length and with the size of galaxy. The characterization of these invariants reduces to Morse theory. The extrema correspond to topology changes for the two-dimensional (one-dimensional) $r_M = constant$ section of 3-surface (boundary of 3-surface). The height functions of sphere and torus serve as a good illustrations of the situation. A good example about non-topological extrema is provided by a sphere with two horns.

There are additional symplectic invariants. The "magnetic fluxes" associated with the δM^4_+ symplectic form

$$J_{S^2} = r_M^2 sin(\theta) d\theta \wedge d\phi$$

over any $X^2 \subset X^3$ are symplectic invariants. In particular, the integrals over $r_M = constant$ sections (assuming them to be 2-dimensional) are symplectic invariants. They give simply the solid angle $\Omega(r_M)$ spanned by $r_M = constant$ section and thus $r_M^2\Omega(r_M)$ characterizes transversal geometric size of the 3-surface. A convenient manner to discretize these invariants is to consider the Fourier components of these invariants in radial logarithmic plane wave basis discussed earlier:

$$\Omega(k) = \int_{r_{min}}^{r_{max}} (r_M/r_{max})^k \Omega(r_M)\frac{dr_M}{r_M} \ , \quad k = k_1 + ik_2 \ , \ perk_1 \geq 0 \ . \tag{4.5.1}$$

One must take into account that for each section in which the topology of $r_M = constant$ section remains constant one must associate invariants with separate components of the two-dimensional section. For a given value of r_M, r_M constant section contains several components (to visualize the situation consider torus as an example).

Also the quantities

$$\Omega^+(X^2) = \int_{X^2} |J| \equiv \int |\epsilon^{\alpha\beta} J_{\alpha\beta}|\sqrt{g_2}d^2x$$

are symplectic invariants and provide additional geometric information about 3-surface. These fluxes are non-vanishing also for closed surfaces and give information about the geometry of the boundary components of 3-surface (signed fluxes vanish for boundary components unless they enclose the tip of the light cone).

Since zero norm generators remain invariant under complexification, their contribution to the Kähler metric vanishes. It is not at all obvious whether WCW integration measure in these degrees of freedom exists at all. A localization in zero modes occurring in each quantum jump seems a more plausible and under suitable additional assumption it would have interpretation as a state function reduction. In string model similar situation is encountered; besides the functional integral determined by string action, one has integral over the moduli space.

If the effective 2-dimensionality implied by the strong form of general coordinate invariance discussed in the introduction is accepted, there is no need to integrate over the variable r_M and just the fluxes over the 2-surfaces X_i^2 identified as intersections of light like 3-D causal determinants with X^3 contain the data relevant for the construction of the WCW geometry. Also the symplectic invariants associated with these surfaces are enough.

4.5.2 Kähler Magnetic Invariants

The Kähler magnetic fluxes defined both the normal component of the Kähler magnetic field and by its absolute value

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \tag{4.5.2}$$

over suitably defined 2-surfaces are invariants under both Lorentz isometries and the symplectic transformations of CP_2 and can be calculated once X^3 is given.

For a closed surface $Q_m(X^2)$ vanishes unless the homology equivalence class of the surface is nontrivial in CP_2 degrees of freedom. In this case the flux is quantized. $Q_M^+(X^2)$ is non-vanishing for closed surfaces, too. Signed magnetic fluxes over non-closed surfaces depend on the boundary of X^2 only:

$$\begin{aligned} &\textstyle\int_{X^2} J = \int_{\delta X^2} A \ . \\ &J = dA \ . \end{aligned}$$

Un-signed fluxes can be written as sum of similar contributions over the boundaries of regions of X^2 in which the sign of J remains fixed.

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \tag{4.5.3}$$

There are also symplectic invariants, which are Lorentz covariants and defined as

$$\begin{aligned} Q_m(K,X^2) &= \int_{X^2} f_K J_{CP_2} \ , \\ Q_m^+(K,X^2) &= \int_{X^2} f_K|J_{CP_2}| \ , \\ f_{K\equiv(s,n,k)} &= e^{is\phi}\times\frac{\rho^{n-k}}{(1+\rho^2)^k}\times(\frac{r_M}{r_0})^k \end{aligned} \tag{4.5.4}$$

These symplectic invariants transform like an infinite-dimensional unitary representation of Lorentz group.

There must exist some minimal number of symplectically non-equivalent 2-surfaces of X^3, and the magnetic fluxes over the representatives these surfaces give thus good candidates for zero modes.

1. If effective 2-dimensionality is accepted, the surfaces X^2_i defined by the intersections of light like 3-D causal determinants X^3_l and X^3 provide a natural identification for these 2-surfaces.
2. Without effective 2-dimensionality the situation is more complex. Since symplectic transformations leave r_M invariant, a natural set of 2-surfaces X^2 appearing in the definition of fluxes are separate pieces for $r_M = constant$ sections of 3-surface. For a generic 3-surface, these surfaces are 2-dimensional and there is continuum of them so that discrete Fourier transforms of these invariants are needed. One must however notice that $r_M = constant$ surfaces could be be 3-dimensional in which case the notion of flux is not well-defined.

4.5.3 Isometry Invariants And Spin Glass Analogy

The presence of isometry invariants implies coset space decomposition $\cup_i G/H$. This means that quantum states are characterized, not only by the vacuum functional, which is just the exponential $exp(K)$ of Kähler function (Gaussian in lowest approximation) but also by a wave function in vacuum modes. Therefore the functional integral over the WCW decomposes into an integral over zero modes for approximately Gaussian functionals determined by $exp(K)$. The weights for the various vacuum mode contributions are given by the probability density associated with the zero modes. The integration over the zero modes is a highly problematic notion and it could be eliminated if a localization in the zero modes occurs in quantum jumps. The localization would correspond to a state function reduction and zero modes would be effectively classical variables correlated in one-one manner with the quantum numbers associated with the quantum fluctuating degrees of freedom.

For a given orbit K depends on zero modes and thus one has mathematical similarity with spin glass phase for which one has probability distribution for Hamiltonians appearing in the partition function $exp(-H/T)$. In fact, since TGD Universe is also critical, exact similarity requires that also the temperature is critical for various contributions to the average partition function of spin glass phase. The characterization of isometry invariants and zero modes of the Kähler metric provides a precise characterization for how TGD Universe is quantum analog of spin glass.

The spin glass analogy has been the basic starting point in the construction of p-adic field theory limit of TGD. The ultra-metric topology for the free energy minima of spin glass phase motivates the hypothesis that effective quantum average space-time possesses ultra-metric topology. This approach leads to excellent predictions for elementary particle masses and predicts even new branches of physics [K50, K88]. As a matter fact, an entire fractal hierarchy of copies of standard physics is predicted.

4.5.4 Magnetic Flux Representation Of The Symplectic Algebra

Accepting the strong form of general coordinate invariance implying effective two-dimensionality WCW Hamiltonians correspond to the fluxes associated with various 2-surfaces X^2_i defined by the intersections of light-like light-like 3-surfaces $X^3_{l,i}$ with X^3 at the boundaries of CD considered. Bearing in mind that zero energy ontology is the correct approach, one can restrict the consideration on fluxes at $\delta M^4_+ \times CP_2$ One must also remember that if the proposed symmetries hold true, it is in principle choose any partonic 2-surface in the conjectured slicing of the Minkowskian space-time sheet to partonic 2-surfaces parametrized by the points of stringy world sheets.vA physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K41] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

Generalized magnetic fluxes

Isometry invariants are just special case of the fluxes defining natural coordinate variables for WCW . Symplectic transformations of CP_2 act as $U(1)$ gauge transformations on the Kähler potential of CP_2 (similar conclusion holds at the level of $\delta M^4_+ \times CP_2$).

One can generalize these transformations to local symplectic transformations by allowing the Hamiltonians to be products of the CP_2 Hamiltonians with the real and imaginary parts of the functions $f_{m,n,k}$ (see Eq. 4.4.22) defining the Lorentz covariant function basis H_A, $A \equiv (a, m, n, k)$ at the light cone boundary: $H_A = H_a \times f(m, n, k)$, where a labels the Hamiltonians of CP_2.

One can associate to any Hamiltonian H^A of this kind both signed and unsigned magnetic flux via the following formulas:

$$\begin{aligned} Q_m(H_A|X^2) &= \int_{X^2} H_A J \ , \\ Q^+_m(H_A|X^2) &= \int_{X^2} H_A|J| \ . \end{aligned} \tag{4.5.5}$$

Here X^2 corresponds to any surface X^2_i resulting as intersection of X^3 with $X^3_{l,i}$. Both signed and unsigned magnetic fluxes and their superpositions

$$Q^{\alpha,\beta}_m(H_A|X^2) = \alpha Q_m(H_A|X^2) + \beta Q^+_m(H_A|X^2) \ , \ A \equiv (a, s, n, k) \tag{4.5.6}$$

provide representations of Hamiltonians. Note that symplectic invariants $Q^{\alpha,\beta}_m$ correspond to $H^A = 1$ and $H^A = f_{s,n,k}$. $H^A = 1$ can be regarded as a natural central term for the Poisson bracket algebra. Therefore, the isometry invariance of Kähler magnetic and electric gauge fluxes follows as a natural consequence.

The obvious question concerns about the correct values of the parameters α and β. One possibility is that the flux is an unsigned flux so that one has $\alpha = 0$. This option is favored by the construction of the WCW spinor structure involving the construction of the fermionic super charges anti-commuting to WCW Hamiltonians: super charges contain the square root of flux, which must be therefore unsigned. Second possibility is that magnetic fluxes are signed fluxes so that β vanishes.

One can define also the electric counterparts of the flux Hamiltonians by replacing J in the defining formulas with its dual $*J$

$$*J_{\alpha\beta} = \epsilon_{\alpha\beta}^{\ \ \gamma\delta} J_{\gamma\delta}.$$

For $H_A = 1$ these fluxes reduce to ordinary Kähler electric fluxes. These fluxes are however not symplectic covariants since the definition of the dual involves the induced metric, which is not symplectic invariant. The electric gauge fluxes for Hamiltonians in various representations of the color group ought to be important in the description of hadrons, not only as string like objects, but quite generally. These degrees of freedom would be identifiable as non-perturbative degrees of freedom involving genuinely classical Kähler field whereas quarks and gluons would correspond to the perturbative degrees of freedom, that is the interactions between CP_2 type extremals.

Poisson brackets

From the symplectic invariance of the radial component of Kähler magnetic field it follows that the Lie-derivative of the flux $Q_m^{\alpha,\beta}(H_A)$ with respect to the vector field $X(H_B)$ is given by

$$X(H_B)\cdot Q_m^{\alpha,\beta}(H_A) \quad = \quad Q_m^{\alpha,\beta}(\{H_B,H_A\}) \ . \tag{4.5.7}$$

The transformation properties of $Q_m^{\alpha,\beta}(H_A)$ are very nice if the basis for H_B transforms according to appropriate irreducible representation of color group and rotation group. This in turn implies that the fluxes $Q_m^{\alpha,\beta}(H_A)$ as functionals of 3-surface on given orbit provide a representation for the Hamiltonian as a functional of 3-surface. For a given surface X^3, the Poisson bracket for the two fluxes $Q_m^{\alpha,\beta}(H_A)$ and $Q_m^{\alpha,\beta}(H_B)$ can be defined as

$$\begin{aligned}\{Q_m^{\alpha,\beta}(H_A),Q_m^{\alpha,\beta}(H_B)\} &\equiv X(H_B)\cdot Q_m^{\alpha,\beta}(H_A)\\ &= Q_m^{\alpha,\beta}(\{H_A,H_B\}) = Q_m^{\alpha,\beta}(\{H_A,H_B\}) \ .\end{aligned} \tag{4.5.8}$$

The study of WCW gamma matrices identifiable as symplectic super charges demonstrates that the supercharges associated with the radial deformations vanish identically so that radial deformations correspond to zero norm degrees of freedom as one might indeed expect on physical grounds. The reason is that super generators involve the invariants $j^{ak}\gamma_k$ which vanish by $\gamma_{r_M}=0$.

The natural central extension associated with the symplectic group of CP_2 ($\{p,q\}=1!$) induces a central extension of this algebra. The central extension term resulting from $\{H_A,H_B\}$ when CP_2 Hamiltonians have $\{p,q\}=1$ equals to the symplectic invariant $Q_m^{\alpha,\beta}(f(m_a+m_b,n_a+n_b,k_a+k_b))$ on the right hand side. This extension is however anti-symmetric in symplectic degrees of freedom rather than in loop space degrees of freedom and therefore does not lead to the standard Kac Moody type algebra.

Quite generally, the Virasoro and Kac Moody algebras of string models are replaced in TGD context by much larger symmetry algebras. Kac Moody algebra corresponds to the deformations of light-like 3-surfaces respecting their light-likeness and leaving partonic 2-surfaces at δCD intact and are highly relevant to the elementary particle physics. This algebra allows a representation in terms of X_l^3 local Hamiltonians generating isometries of $\delta M_\pm^4\times CP_2$. Hamiltonian representation is essential for super-symmetrization since fermionic super charges anti-commute to Hamiltonians rather than vector fields: this is one of the deep differences between TGD and string models. Kac-Moody algebra does not contribute to WCW metric since by definition the generators vanish at partonic 2-surfaces. This is essential for the coset space property.

A completely new algebra is the CP_2 symplectic algebra localized with respect to the light cone boundary and relevant to the configuration space geometry. This extends to $S^2\times CP_2$ -or rather $\delta M_\pm^4\times CP_2$ symplectic algebra and this gives the strongest predictions concerning WCW metric. The local radial Virasoro localized with respect to $S^2\times CP_2$ acts in zero modes and has automatically vanishing norm with respect to WCW metric defined by super charges.

4.5.5 Symplectic Transformations Of $\Delta M_\pm^4\times CP_2$ As Isometries And Electric-Magnetic Duality

According to the construction of Kähler metric, symplectic transformations of $\delta M_\pm^4\times CP_2$ act as isometries whereas radial Virasoro algebra localized with respect to CP_2 has zero norm in the WCW metric.

Hamiltonians can be organized into light like unitary representations of $so(3,1)\times su(3)$ and the symmetry condition $Zg(X,Y)=0$ requires that the component of the metric is $so(3,1)\times su(3)$ invariant and this condition is satisfied if the component of metric between two different representations D_1 and D_2 of $so(3,1)\times su(3)$ is proportional to Glebch-Gordan coefficient $C_{D_1D_2,D_S}$ between $D_1\otimes D_2$ and singlet representation D_S. In particular, metric has components only between states having identical $so(3,1)\times su(3)$ quantum numbers.

Magnetic representation of WCW Hamiltonians means the action of the symplectic transformations of the light cone boundary as WCW isometries is an intrinsic property of the light cone boundary. If electric-magnetic duality holds true, the preferred extremal property only determines the conformal factor of the metric depending on zero modes. This is precisely as it should be if the group theoretical construction works. Hence it should be possible by a direct calculation check whether the metric defined by the magnetic flux Hamiltonians as half Poisson brackets in complex coordinates is invariant under isometries. Symplectic invariance of the metric means that matrix elements of the metric are left translates of the metric along geodesic lines starting from the origin of coordinates, which now naturally corresponds to the preferred extremal of Kähler action. Since metric derives from symplectic form this means that the matrix elements of symplectic form given by Poisson brackets of Hamiltonians must be left translates of their values at origin along geodesic line. The matrix elements in question are given by flux Hamiltonians and since symplectic transforms of flux Hamiltonian is flux Hamiltonian for the symplectic transform of Hamiltonian, it seems that the conditions are satisfied.

4.5.6 Quantum Counterparts Of The Symplectic Hamiltonians

The matrix elements of WCW Kähler metric can be expressed in terms of anti-commutators of WCW gamma matrices identified as super-symplectic super-charges, which might be called super-Hamiltonians. It is these operators which are the most relevant from the point of view of quantum TGD.

The generalization for the definition WCW super-Hamiltonians defining WCW gamma matrices is discussed in detail in [K73] feeds in the wisdom gained about preferred extremals of Kähler action and solutions of the Kähler-Dirac action: in particular, about their localization at string worlds sheets (right handed neutrino could be an exception). Second quantized Noether charges in turn define representation of WCW Hamiltonians as operators.

The basic formulas generalize as such: the only modification is that the super-Hamiltonian of $\delta M^4_\pm \times CP_2$ at given point of partonic 2-surface is replaced with the Noether super charge associated with the Hamiltonian obtained by integrating the 1-D super current over string emanating from partonic 2-surface. Right handed neutrino spinor is replaced with any mode of the Kähler-Dirac operator localized at string world sheet in the case of Kac-Moody sub-algebra of super-symplectic algebra corresponding to symplectic isometries at light-cone boundary and CP_2. The original proposal involved only the contractions with covariantly constant right- handed neutrino spinor mode but now one can allow contractions with all spinor modes - both quark like and leptonic ones. One obtains entire super-symplectic algebra and the direct sum of these algebras is used to construct physical states. This step is analogous to the replacement of point like particle with string.

The resulting super Hamiltonians define WCW gamma matrices. They are labelled by two conformal weights. The first one is the conformal weight associated with the light-like coordinate of $\delta M^4_\pm \times CP_2$. Second conformal weight is associated with the spinor mode and the coordinate along stringy curve and corresponds to the usual stringy conformal weight. The symplectic conformal weight can be more general - I have proposed its spectrum to be generated by the zeros of Riemann zeta. The total conformal weight of a physical state would be non-negative real integer meaning conformal confinement. Symplectic conformal symmetry can be assumed to be broken: an entire hierarchy of breakings is obtained corresponding to hierarchies of sub-algebra of the symplectic algebra isomorphic with it quantum criticalities, Planck constants, and dark matter. Breaking means that only the sub-algebra of super-symplectic algebra isomorphic to it corresponds vanishing elements of the WCW metric: in Hilbert space picture these gauge degrees of freedom correspond to zero norm states.

The presence of two conformal weights is in accordance with the idea that a generalization of conformal invariance to 4-D situation is in question. If Yangian extension of conformal symmetries is possible and would bring an additional integer n telling the degree of multi-locality of Yangian generators defined as the number of strings at which the generator acts (the original not proposal was as the number of partonic 2-surfaces). For super-symplectic algebra the degree of multi-locality equals to $n = 1$. Measurement resolution increases with n. This is also visible in the properties of space-time surfaces since string world sheets and possibly also partonic 2-surfaces and their light-like orbits provide the holographic data - kind of skeleton - determining space-time surface associated with them.

4.6 General Expressions For The Symplectic And Kähler Forms

One can derive general expressions for symplectic and Kähler forms as well as Kähler metric of WCW . The fact that these expressions involve only first variation of the Kähler action implies huge simplification of the basic formulas. Duality hypothesis leads to further simplifications of the formulas.

4.6.1 Closedness Requirement

The fluxes of Kähler magnetic and electric fields for the Hamiltonians of $\delta M^4_+ \times CP_2$ suggest a general representation for the components of the symplectic form of the WCW . The basic requirement is that Kähler form satisfies the defining condition

$$X \cdot J(Y,Z) + J([X,Y],Z) + J(X,[Y,Z]) = 0 \ , \tag{4.6.1}$$

where X, Y, Z are now vector fields associated with Hamiltonian functions defining WCW coordinates.

4.6.2 Matrix Elements Of The Symplectic Form As Poisson Brackets

Quite generally, the matrix element of $J(X(H_A), X(H_B))$ between vector fields $X(H_A))$ and $X(H_B))$ defined by the Hamiltonians H_A and H_B of $\delta M^4_+ \times CP_2$ isometries is expressible as Poisson bracket

$$J^{AB} = J(X(H_A), X(H_B)) = \{H_A, H_B\} \ . \tag{4.6.2}$$

J^{AB} denotes contravariant components of the symplectic form in coordinates given by a subset of Hamiltonians. The magnetic flux Hamiltonians $Q^{\alpha,\beta}_m(H_{A,k})$ of Eq. 4.5.5 provide an explicit representation for the Hamiltonians at the level of WCW so that the components of the symplectic form of the WCW are expressible as classical charges for the Poisson brackets of the Hamiltonians of the light cone boundary:

$$J(X(H_A), X(H_B)) = Q^{\alpha,\beta}_m(\{H_A, H_B\}) \ . \tag{4.6.3}$$

Recall that the superscript α, β refers the coefficients of J and $|J|$ in the superposition of these Kähler magnetic fluxes. Note that $Q^{\alpha,\beta}_m$ contains unspecified conformal factor depending on symplectic invariants characterizing Y^3 and is unspecified superposition of signed and unsigned magnetic fluxes.

This representation does not carry information about the tangent space of space-time surface at the partonic 2-surface, which motivates the proposal that also electric fluxes are present and proportional to magnetic fluxes with a factor K, which is symplectic invariant so that commutators of flux Hamiltonians come out correctly. This would give

$$Q^{\alpha,\beta}_m(H_A)_{em} = Q^{\alpha,\beta}_e(H_A) + Q^{\alpha,\beta}_m(H_A) = (1+K)Q^{\alpha,\beta}_m(H_A) \ . \tag{4.6.4}$$

Since Kähler form relates to the standard field tensor by a factor $e/\hbar$, flux Hamiltonians are dimensionless so that commutators do not involve $\hbar$. The commutators would come as

$$Q^{\alpha,\beta}_{em}(\{H_A, H_B\}) \rightarrow (1+K)Q^{\alpha,\beta}_m(\{H_A, H_B\}) \ . \tag{4.6.5}$$

The factor $1 + K$ plays the same role as Planck constant in the commutators.

WCW Hamiltonians vanish for the extrema of the Kähler function as variational derivatives of the Kähler action. Hence Hamiltonians are good candidates for the coordinates appearing as coordinates in the perturbative functional integral around extrema (with maxima giving dominating contribution). It is clear that WCW coordinates around a given extremum include only those Hamiltonians, which vanish at extremum (that is those Hamiltonians which span the tangent space of G/H) In Darboux coordinates the Poisson brackets reduce to the symplectic form

$$\begin{aligned} \{P^I, Q^J\} &= J^{IJ} = J_I \delta^{I,J} \ . \\ J_I &= 1 \ . \end{aligned} \tag{4.6.6}$$

It is not clear whether Darboux coordinates with $J_I = 1$ are possible in the recent case: probably the unit matrix on right hand side of the defining equation is replaced with a diagonal matrix depending on symplectic invariants so that one has $J_I \neq 1$. The integration measure is given by the symplectic volume element given by the determinant of the matrix defined by the Poisson brackets of the Hamiltonians appearing as coordinates. The value of the symplectic volume element is given by the matrix formed by the Poisson brackets of the Hamiltonians and reduces to the product

$$Vol = \prod_I J_I$$

in generalized Darboux coordinates.

Kähler potential (that is gauge potential associated with Kähler form) can be written in Darboux coordinates as

$$A = \sum_I J_I P_I dQ^I \ . \tag{4.6.7}$$

4.6.3 General Expressions For Kähler Form, Kähler Metric And Kähler Function

The expressions of Kähler form and Kähler metric in complex coordinates can obtained by transforming the contravariant form of the symplectic form from symplectic coordinates provided by Hamiltonians to complex coordinates:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} = \partial_{H^A} Z^i \partial_{H^B} \bar{Z}^j J^{AB} \ , \tag{4.6.8}$$

where J^{AB} is given by the classical Kahler charge for the light cone Hamiltonian $\{H^A, H^B\}$. Complex coordinates correspond to linear coordinates of the complexified Lie-algebra providing exponentiation of the isometry algebra via exponential mapping. What one must know is the precise relationship between allowed complex coordinates and Hamiltonian coordinates: this relationship is in principle calculable. In Darboux coordinates the expressions become even simpler:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} = \sum_I J(I)(\partial_{P^i} Z^i \partial_{Q^I} \bar{Z}^j - \partial_{Q^I} Z^i \partial_{P^I} \bar{Z}^j) \ . \tag{4.6.9}$$

Kähler function can be formally integrated from the relationship

$$\begin{aligned} A_{Z^i} &= i\partial_{Z^i} K \ , \\ A_{\bar{Z}^i} &= -i\partial_{Z^i} K \ . \end{aligned} \tag{4.6.10}$$

holding true in complex coordinates. Kähler function is obtained formally as integral

$$K = \int_0^Z (A_{Z^i} dZ^i - A_{\bar{Z}^i} d\bar{Z}^i) \ . \tag{4.6.11}$$

4.6.4 $Diff(X^3)$ Invariance And Degeneracy And Conformal Invariances Of The Symplectic Form

$J(X(H_A), X(H_B))$ defines symplectic form for the coset space G/H only if it is $Diff(X^3)$ degenerate. This means that the symplectic form $J(X(H_A), X(H_B))$ vanishes whenever Hamiltonian H_A or H_B is such that it generates diffeomorphism of the 3-surface X^3. If effective 2-dimensionality holds true, $J(X(H_A), X(H_B))$ vanishes if H_A or H_B generates two-dimensional diffeomorphism $d(H_A)$ at the surface X_i^2.

One can always write

$$J(X(H_A), X(H_B)) = X(H_A)Q(H_B|X_i^2) \ .$$

If H_A generates diffeomorphism, the action of $X(H_A)$ reduces to the action of the vector field X_A of some X_i^2-diffeomorphism. Since $Q(H_B|r_M)$ is manifestly invariant under the diffemorphisms of X^2, the result is vanishing:

$$X_A Q(H_B|X_i^2) = 0 \ ,$$

so that $Diff^2$ invariance is achieved.

The radial diffeomorphisms possibly generated by the radial Virasoro algebra do not produce trouble. The change of the flux integrand X under the infinitesimal transformation $r_M \rightarrow r_M + \epsilon r_M^n$ is given by $r_M^n dX/dr_M$. Replacing r_M with $r_M^{-n+1}/(-n+1)$ as variable, the integrand reduces to a total divergence dX/du the integral of which vanishes over the closed 2-surface X_i^2. Hence radial Virasoro generators having zero norm annihilate all matrix elements of the symplectic form. The induced metric of X_i^2 induces a unique conformal structure and since the conformal transformations of X_i^2 can be interpreted as a mere coordinate changes, they leave the flux integrals invariant.

4.6.5 Complexification And Explicit Form Of The Metric And Kähler Form

The identification of the Kähler form and Kähler metric in symplectic degrees of freedom follows trivially from the identification of the symplectic form and definition of complexification. The requirement that Hamiltonians are eigen states of angular momentum (and possibly Lorentz boost generator), isospin and hypercharge implies physically natural complexification. In order to fix the complexification completely one must introduce some convention fixing which states correspond to "positive" frequencies and which to "negative frequencies" and which to zero frequencies that is to decompose the generators of the symplectic algebra to three sets Can_+, Can_- and Can_0. One must distinguish between Can_0 and zero modes, which are not considered here at all. For instance, CP_2 Hamiltonians correspond to zero modes.

The natural complexification relies on the imaginary part of the radial conformal weight whereas the real part defines the $g = t + h$ decomposition naturally. The wave vector associated with the radial logarithmic plane wave corresponds to the angular momentum quantum number associated with a wave in S^1 in the case of Kac Moody algebra. One can imagine three options.

1. It is quite possible that the spectrum of k_2 does not contain $k_2 = 0$ at all so that the sector Can_0 could be empty. This complexification is physically very natural since it is manifestly invariant under $SU(3)$ and $SO(3)$ defining the preferred spherical coordinates. The choice of $SO(3)$ is unique if the classical four-momentum associated with the 3-surface is time like so that there are no problems with Lorentz invariance.

2. If $k_2 = 0$ is possible one could have

$$\begin{aligned}
Can_+ &= \{H^a_{m,n,k=k1_+ik_2}, k_2 > 0\} \ , \\
Can_- &= \{H^a_{m,n,k}, k_2 < 0\} \ , \\
Can_0 &= \{H^a_{m,n,k}, k_2 = 0\} \ .
\end{aligned} \tag{4.6.12}$$

3. If it is possible to $n_2 \neq 0$ for $k_2 = 0$, one could define the decomposition as

$$\begin{aligned} Can_+ &= \{H^a_{m,n,k}, k_2 > 0 \ \text{ or } \ k_2 = 0, n_2 > 0\} \ , \\ Can_- &= \{H^a_{m,n,k}, k_2 < 0 \ \text{ or} k_2 = 0, n_2 < 0\} \ , \\ Can_0 &= \{H^a_{m,n,k}, k_2 = n_2 = 0\} \ . \end{aligned} \tag{4.6.13}$$

In this case the complexification is unique and Lorentz invariance guaranteed if one can fix the $SO(2)$ subgroup uniquely. The quantization axis of angular momentum could be chosen to be the direction of the classical angular momentum associated with the 3-surface in its rest system.

The only thing needed to get Kähler form and Kähler metric is to write the half Poisson bracket defined by Eq. 4.6.15

$$\begin{aligned} J_f(X(H_A), X(H_B)) &= 2Im\,(iQ_f(\{H_A, H_B\}_{-+})) \ , \\ G_f(X(H_A), X(H_B)) &= 2Re\,(iQ_f(\{H_A, H_B\}_{-+})) \ . \end{aligned} \tag{4.6.14}$$

Symplectic form, and thus also Kähler form and Kähler metric, could contain a conformal factor depending on the isometry invariants characterizing the size and shape of the 3-surface. At this stage one cannot say much about the functional form of this factor.

4.6.6 Comparison Of CP_2 Kähler Geometry With Configuration Space Geometry

The explicit discussion of the role of $g = t + h$ decomposition of the tangent space of WCW provides deep insights to the metric of the symmetric space. There are indeed many questions to be answered. To what point of WCW (that is 3-surface) the proposed $g = t + h$ decomposition corresponds to? Can one derive the components of the metric and Kähler form from the Poisson brackets of complexified Hamiltonians? Can one characterize the point in question in terms of the properties of WCW Hamiltonians? Does the central extension of WCW reduce to the symplectic central extension of the symplectic algebra or can one consider also other options?

Cartan decomposition for CP_2

A good manner to gain understanding is to consider the CP_2 metric and Kähler form at the origin of complex coordinates for which the sub-algebra $h = u(2)$ defines the Cartan decomposition.

1. $g = t + h$ decomposition depends on the point of the symmetric space in general. In case of CP_2 $u(2)$ sub-algebra transforms as $g \circ u(2) \circ g^{-1}$ when the point s is replaced by gsg^{-1}. This is expected to hold true also in case of WCW (unless it is flat) so that the task is to identify the point of WCW at which the proposed decomposition holds true.
2. The Killing vector fields of h sub-algebra vanish at the origin of CP_2 in complex coordinates. The corresponding Hamiltonians need not vanish but their Poisson brackets must vanish. It is possible to add suitable constants to the Hamiltonians in order to guarantee that they vanish at origin.
3. It is convenient to introduce complex coordinates and decompose isometry generators to holomorphic components $J^a_+ = j^{ak}\partial_k$ and $j^a_- = j^{a\bar{k}}\partial_{\bar{k}}$. One can introduce what might be called half Poisson bracket and half inner product defined as

$$\begin{aligned} \{H^a, H^b\}_{-+} &\equiv \partial_{\bar{k}} H^a J^{\bar{k}l} \partial_l H^b \\ &= j^{ak} J_{k\bar{l}} j^{b\bar{l}} = -i(j^a_+, j^b_-) \ . \end{aligned} \tag{4.6.15}$$

One can express Poisson bracket of Hamiltonians and the inner product of the corresponding Killing vector fields in terms of real and imaginary parts of the half Poisson bracket:

$$\begin{aligned} \{H^a, H^b\} &= 2Im\left(i\{H^a, H^b\}_{-+}\right) \ , \\ (j^a, j^b) &= 2Re\left(i(j_+^a, j_-^b)\right) = 2Re\left(i\{H^a, H^b\}_{-+}\right) \ . \end{aligned} \quad (4.6.16)$$

What this means that Hamiltonians and their half brackets code all information about metric and Kähler form. Obviously this is of utmost importance in the case of the WCW metric whose symplectic structure and central extension are derived from those of CP_2.

Consider now the properties of the metric and Kähler form at the origin.

1. The relations satisfied by the half Poisson brackets can be written symbolically as

$$\begin{aligned} &\{h, h\}_{-+} = 0 \ , \\ &Re\left(i\{h, t\}_{-+}\right) = 0 \ , \quad Im\left(i\{h, t\}_{-+}\right) = 0 \ , \\ &Re\left(i\{t, t\}_{-+}\right) \neq 0 \ , \quad Im\left(i\{t, t\}_{-+}\right) \neq 0 \ . \end{aligned} \quad (4.6.17)$$

2. The first two conditions state that h vector fields have vanishing inner products at the origin. The first condition states also that the Hamiltonians for the commutator algebra $[h, h] = SU(2)$ vanish at origin whereas the Hamiltonian for $U(1)$ algebra corresponding to the color hyper charge need not vanish although it can be made vanishing. The third condition implies that the Hamiltonians of t vanish at origin.
3. The last two conditions state that the Kähler metric and form are non-vanishing between the elements of t. Since the Poisson brackets of t Hamiltonians are Hamiltonians of h, the only possibility is that $\{t, t\}$ Poisson brackets reduce to a non-vanishing $U(1)$ Hamiltonian at the origin or that the bracket at the origin is due to the symplectic central extension. The requirement that all Hamiltonians vanish at origin is very attractive aesthetically and forces to interpret $\{t, t\}$ brackets at origin as being due to a symplectic central extension. For instance, for S^2 the requirement that Hamiltonians vanish at origin would mean the replacement of the Hamiltonian $H = cos(\theta)$ representing a rotation around z-axis with $H_3 = cos(\theta) - 1$ so that the Poisson bracket of the generators H_1 and H_2 can be interpreted as a central extension term.
4. The conditions for the Hamiltonians of $u(2)$ sub-algebra state that their variations with respect to g vanish at origin. Thus $u(2)$ Hamiltonians have extremum value at origin.
5. Also the Kähler function of CP_2 has extremum at the origin. This suggests that in the case of the WCW the counterpart of the origin corresponds to the maximum of the Kähler function.

Cartan algebra decomposition at the level of WCW

The discussion of the properties of CP_2 Kähler metric at origin provides valuable guide lines in an attempt to understand what happens at the level of WCW . The use of the half bracket for WCW Hamiltonians in turn allows to calculate the matrix elements of the WCW metric and Kähler form explicitly in terms of the magnetic or electric flux Hamiltonians.

The earlier construction was rather tricky and formula-rich and not very convincing physically. Cartan decomposition had to be assigned with something and in lack of anything better it was assigned with Super Virasoro algebra, which indeed allows this kind of decompositions but without any strong physical justification.

It must be however emphasized that holography implying effective 2-dimensionality of 3-surfaces in some length scale resolution is absolutely essential for this construction since it allows to effectively reduce Kac-Moody generators associated with X_l^3 to $X^2 = X_l^3 \cap \delta M_\pm^4 \times CP_2$. In the similar manner super-symplectic generators can be dimensionally reduced to X^2. Number theoretical compactification forces the dimensional reduction and the known extremals are consistent with it [K10]. The construction of WCW spinor structure and metric in terms of the second quantized spinor fields [K96] relies to this picture as also the recent view about M-matrix [K19].

In this framework the coset space decomposition becomes trivial.

1. The algebra g is labeled by color quantum numbers of CP_2 Hamiltonians and by the label (m,n,k) labeling the function basis of the light cone boundary. Also a localization with respect to X^2 is needed. This is a new element as compared to the original view.
2. Super Kac-Moody algebra is labeled by color octet Hamiltonians and function basis of X^2. Since Lie-algebra action does not lead out of irreps, this means that Cartan algebra decomposition is satisfied.

4.6.7 Comparison With Loop Groups

It is useful to compare the recent approach to the geometrization of the loop groups consisting of maps from circle to Lie group G [A54], which served as the inspirer of the WCW geometry approach but later turned out to not apply as such in TGD framework.

In the case of loop groups the tangent space T corresponds to the local Lie-algebra $T(k,A) = exp(ik\phi)T_A$, where T_A generates the finite-dimensional Lie-algebra g and ϕ denotes the angle variable of circle; k is integer. The complexification of the tangent space corresponds to the decomposition

$$T = \{X(k>0,A)\} \oplus \{X(k<0,A)\} \oplus \{X(k=0,A)\} = T_+ \oplus T_- \oplus T_0$$

of the tangent space. Metric corresponds to the central extension of the loop algebra to Kac Moody algebra and the Kähler form is given by

$$J(X(k_1<0,A),X(k_2>0,B)) = k_2\delta(k_1+k_2)\delta(A,B) \ .$$

In present case the finite dimensional Lie algebra g is replaced with the Lie-algebra of the symplectic transformations of $\delta M^4_+ \times CP_2$ centrally extended using symplectic extension. The scalar function basis on circle is replaced with the function basis on an interval of length Δr_M with periodic boundary conditions; effectively one has circle also now.

The basic difference is that one can consider two kinds of central extensions now.

1. Central extension is most naturally induced by the natural central extension ($\{p,q\} = 1$) defined by Poisson bracket. This extension is anti-symmetric with respect to the generators of the symplectic group: in the case of the Kac Moody central extension it is symmetric with respect to the group G. The symplectic transformations of CP_2 might correspond to non-zero modes also because they are not exact symmetries of Kähler action. The situation is however rather delicate since $k=0$ light cone harmonic has a diverging norm due to the radial integration unless one poses both lower and upper radial cutoffs although the matrix elements would be still well defined for typical 3-surfaces. For Kac Moody group $U(1)$ transformations correspond to the zero modes. Light cone function algebra can be regarded as a local $U(1)$ algebra defining central extension in the case that only CP_2 symplectic transformations local with respect to δM^4_+ act as isometries: for Kac Moody algebra the central extension corresponds to an ordinary $U(1)$ algebra. In the case that entire light cone symplectic algebra defines the isometries the central extension reduces to a $U(1)$ central extension.

4.6.8 Symmetric Space Property Implies Ricci Flatness And Isometric Action Of Symplectic Transformations

The basic structure of symmetric spaces is summarized by the following structural equations

$$\begin{array}{l} g = h+t \ , \\ [h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ . \end{array} \tag{4.6.18}$$

In present case the equations imply that all commutators of the Lie-algebra generators of $Can(\neq 0)$ having non-vanishing integer valued radial quantum number n_2, possess zero norm. This condition is extremely strong and guarantees isometric action of $Can(\delta M^4_+ \times CP_2)$ as well as Ricci flatness of the WCW metric.

The requirement $[t,t] \subset h$ and $[h,t] \subset t$ are satisfied if the generators of the isometry algebra possess generalized parity P such that the generators in t have parity $P = -1$ and the generators belonging to h have parity $P = +1$. Conformal weight n must somehow define this parity. The first possibility to come into mind is that odd values of n correspond to $P = -1$ and even values to $P = 1$. Since n is additive in commutation, this would automatically imply $h \oplus t$ decomposition with the required properties. This assumption looks however somewhat artificial. TGD however forces a generalization of Super Algebras and N-S and Ramond type algebras can be combined to a larger algebra containing also Virasoro and Kac Moody generators labeled by half-odd integers. This suggests strongly that isometry generators are labeled by half integer conformal weight and that half-odd integer conformal weight corresponds to parity $P = -1$ whereas integer conformal weight corresponds to parity $P = 1$. Coset space would structure would state conformal invariance of the theory since super-symplectic generators with integer weight would correspond to zero modes.

Quite generally, the requirement that the metric is invariant under the flow generated by vector field X leads together with the covariant constancy of the metric to the Killing conditions

$$X \cdot g(Y,Z) \quad = \quad 0 = g([X,Y],Z) + g(Y,[X,Z]) \ . \tag{4.6.19}$$

If the commutators of the complexified generators in $Can(\neq 0)$ have zero norm then the two terms on the right hand side of Eq. (4.6.19) vanish separately. This is true if the conditions

$$Q_m^{\alpha,\beta}(\{H^A,\{H^B,H^C\}\}) \quad = \quad 0 \ , \tag{4.6.20}$$

are satisfied for all triplets of Hamiltonians in $Can_{\neq 0}$. These conditions follow automatically from the $[t,t] \subset h$ property and guarantee also Ricci flatness as will be found later.

It must be emphasized that for Kähler metric defined by purely magnetic fluxes, one cannot pose the conditions of Eq. (4.6.20) as consistency conditions on the initial values of the time derivatives of embedding space coordinates whereas in general case this is possible. If the consistency conditions are satisfied for a single surface on the orbit of symplectic group then they are satisfied on the entire orbit. Clearly, isometry and Ricci flatness requirements and the requirement of time reversal invariance might well force Kähler electric alternative.

4.7 Ricci Flatness And Divergence Cancelation

Divergence cancelation in WCW integration requires Ricci flatness and in this section the arguments in favor of Ricci flatness are discussed in detail.

4.7.1 Inner Product From Divergence Cancelation

Forgetting the delicacies related to the non-determinism of the Kähler action, the inner product is given by integrating the usual Fock space inner product defined at each point of WCW over the reduced WCW containing only the 3-surfaces Y^3 belonging to $\delta H = \delta M^4_+ \times CP_2$ ("light-cone boundary") using the exponent $exp(K)$ as a weight factor:

$$\begin{aligned} \langle \Psi_1 | \Psi_2 \rangle \quad &= \quad \int \overline{\Psi}_1(Y^3)\Psi_2(Y^3) exp(K)\sqrt{G} dY^3 \ , \\ \overline{\Psi_1}(Y^3)\Psi_2(Y^3) \quad &\equiv \quad \langle \Psi_1(Y^3)|\Psi_2(Y^3)\rangle_{Fock} \ . \end{aligned} \tag{4.7.1}$$

The degeneracy for the preferred extremals of Kähler action implies additional summation over the degenerate extremals associated with Y^3. The restriction of the integration on light cone boundary is Diff^4 invariant procedure and resolves in elegant manner the problems related to the integration over Diff^4 degrees of freedom. A variant of the inner product is obtained dropping the bosonic vacuum functional $exp(K)$ from the definition of the inner product and by assuming that it is included into the spinor fields themselves. Probably it is just a matter of taste how the necessary bosonic vacuum functional is included into the inner product: what is essential that the vacuum functional $exp(K)$ is somehow present in the inner product.

The unitarity of the inner product follows from the unitary of the Fock space inner product and from the unitarity of the standard L^2 inner product defined by WCW integration in the set of the L^2 integrable scalar functions. It could well occur that $Diff^4$ invariance implies the reduction of WCW integration to $C(\delta H)$.

Consider next the bosonic integration in more detail. The exponent of the Kähler function appears in the inner product also in the context of the finite dimensional group representations. For the representations of the non-compact groups (say $SL(2,R)$) in coset spaces (now $SL(2,R)/U(1)$ endowed with Kähler metric) the exponent of Kähler function is necessary in order to get square integrable representations [B26]. The scalar product for two complex valued representation functions is defined as

$$(f,g) \quad = \quad \int \overline{f} g exp(nK)\sqrt{g}dV \quad . \tag{4.7.2}$$

By unitarity, the exponent is an integer multiple of the Kähler function. In the present case only the possibility $n = 1$ is realized if one requires a complete cancelation of the determinants. In finite dimensional case this corresponds to the restriction to single unitary representation of the group in question.

The sign of the action appearing in the exponent is of decisive importance in order to make theory stable. The point is that the theory must be well defined at the limit of infinitely large system. Minimization of action is expected to imply that the action of infinitely large system is bound from above: the generation of electric Kähler fields gives negative contributions to the action. This implies that at the limit of the infinite system the average action per volume is non-positive. For systems having negative average density of action vacuum functional $exp(K)$ vanishes so that only configurations with vanishing average action per volume have significant probability. On the other hand, the choice $exp(-K)$ would make theory unstable: probability amplitude would be infinite for all configurations having negative average action per volume. In the fourth part of the book it will be shown that the requirement that average Kähler action per volume cancels has important cosmological consequences.

Consider now the divergence cancelation in the bosonic integration. One can develop the Kähler function as a Taylor series around maximum of Kähler function and use the contravariant Kähler metric as a propagator. Gaussian and metric determinants cancel each other for a unique vacuum functional. Ricci flatness guarantees that metric determinant is constant in complex coordinates so that one avoids divergences coming from it. The non-locality of the Kähler function as a functional of the 3-surface serves as an additional regulating mechanism: if $K(X^3)$ were a local functional of X^3 one would encounter divergences in the perturbative expansion.

The requirement that quantum jump corresponds to a quantum measurement in the sense of quantum field theories implies that quantum jump involves localization in zero modes. Localization in the zero modes implies automatically p-adic evolution since the decomposition of the WCW into sectors D_P labeled by the infinite primes P is determined by the corresponding decomposition in zero modes. Localization in zero modes would suggest that the calculation of the physical predictions does not involve integration over zero modes: this would dramatically simplify the calculational apparatus of the theory. Probably this simplification occurs at the level of practical calculations if U-matrix separates into a product of matrices associated with zero modes and fiber degrees of freedom.

One must also calculate the predictions for the ratios of the rates of quantum transitions to different values of zero modes and here one cannot actually avoid integrals over zero modes. To achieve this one is forced to define the transition probabilities for quantum jumps involving a localization in zero modes as

$$P(x,\alpha \to y,\beta) = \sum_{r,s} |S(r,\alpha \to s,\beta)|^2 |\Psi_r(x)|^2 |\Psi_s(y)|^2 \quad ,$$

where x and y correspond to the zero mode coordinates and r and s label a complete state functional basis in zero modes and $S(r,m \to s,n)$ involves integration over zero modes. In fact, only in this manner the notion of the localization in the zero modes makes mathematically sense at the level of S-matrix. In this case also unitarity conditions are well-defined. In zero modes state function basis can be freely constructed so that divergence difficulties could be avoided. An open question is whether this construction is indeed possible.

Some comments about the actual evaluation of the bosonic functional integral are in order.

1. Since WCW metric is degenerate and the bosonic propagator is essentially the contravariant metric, bosonic integration is expected to reduce to an integration over the zero modes. For instance, isometry invariants are variables of this kind. These modes are analogous to the parameters describing the conformal equivalence class of the orbit of the string in string models.
2. α_K is a natural small expansion parameter in WCW integration. It should be noticed that α_K, when defined by the criticality condition, could also depend on the coordinates parameterizing the zero modes.
3. Semiclassical approximation, which means the expansion of the functional integral as a sum over the extrema of the Kähler function, is a natural approach to the calculation of the bosonic integral. Symmetric space property suggests that for the given values of the zero modes there is only single extremum and corresponds to the maximum of the Kähler function. There are theorems (Duistermaat-Hecke theorem) stating that semiclassical approximation is exact for certain systems (for example for integrable systems [A55]). Symmetric space property suggests that Kähler function might possess the properties guaranteeing the exactness of the semiclassical approximation. This would mean that the calculation of the integral $\int exp(K)\sqrt{G}dY^3$ and even more complex integrals involving WCW spinor fields would be completely analogous to a Gaussian integration of free quantum field theory. This kind of reduction actually occurs in string models and is consistent with the criticality of the Kähler coupling constant suggesting that all loop integrals contributing to the renormalization of the Kähler action should vanish. Also the condition that WCW integrals are continuable to p-adic number fields requires this kind of reduction.

4.7.2 Why Ricci Flatness

It has been already found that the requirement of divergence cancelation poses extremely strong constraints on the metric of the WCW. The results obtained hitherto are the following.

1. If the vacuum functional is the exponent of Kähler function one gets rid of the divergences resulting from the Gaussian determinants and metric determinants: determinants cancel each other.
2. The non-locality of the Kähler action gives good hopes of obtaining divergence free perturbation theory.

The following arguments show that Ricci flatness of the metric is a highly desirable property.

1. Dirac operator should be a well defined operator. In particular its square should be well defined. The problem is that the square of Dirac operator contains curvature scalar, which need not be finite since it is obtained via two infinite-dimensional trace operations from the curvature tensor. In case of loop spaces [A54] the Kähler property implies that even Ricci tensor is only conditionally convergent. In fact, loop spaces with Kähler metric are Einstein spaces (Ricci tensor is proportional to metric) and Ricci scalar is infinite.

   In 3-dimensional case situation is even worse since the trace operation involves 3 summation indices instead of one! The conclusion is that Ricci tensor had better to vanish in vibrational degrees of freedom.
2. For Ricci flat metric the determinant of the metric is constant in geodesic complex coordinates as is seen from the expression for Ricci tensor [A61]

$$R_{k\bar{l}} = \partial_k\partial_{\bar{l}}ln(det(g)) \tag{4.7.3}$$

   in Kähler metric. This obviously simplifies considerably functional integration over WCW: one obtains just the standard perturbative field theory in the sense that metric determinant gives no contributions to the functional integration.

3. The constancy of the metric determinant results not only in calculational simplifications: it also eliminates divergences. This is seen by expanding the determinant as a functional Taylor series with respect to the coordinates of WCW. In local complex coordinates the first term in the expansion of the metric determinant is determined by Ricci tensor

$$\delta\sqrt{g} \propto R_{k\bar{l}} z^k \bar{z}^l \ . \tag{4.7.4}$$

In WCW integration using standard rules of Gaussian integration this term gives a contribution proportional to the contraction of the propagator with Ricci tensor. But since the propagator is just the contravariant metric one obtains Ricci scalar as result. So, in order to avoid divergences, Ricci scalar must be finite: this is certainly guaranteed if Ricci tensor vanishes.

4. The following group theoretic argument suggests that Ricci tensor either vanishes or is divergent. The holonomy group of the WCW is a subgroup of $U(n=\infty)$ ($D=2n$ is the dimension of the Kähler manifold) by Kähler property and Ricci flatness is guaranteed if the $U(1)$ factor is absent from the holonomy group. In fact Ricci tensor is proportional to the trace of the $U(1)$ generator and since this generator corresponds to an infinite dimensional unit matrix the trace diverges: therefore given element of the Ricci tensor is either infinite or vanishes. Therefore the vanishing of the Ricci tensor seems to be a mathematical necessity. This naïve argument doesn't hold true in the case of loop spaces, for which Kähler metric with finite non-vanishing Ricci tensor exists [A54] . Note however that also in this case the sum defining Ricci tensor is only conditionally convergent.

There are indeed good hopes that Ricci tensor vanishes. By the previous argument the vanishing of the Ricci tensor is equivalent with the absence of divergences in WCW integration. That divergences are absent is suggested by the non-locality of the Kähler function as a functional of 3-surface: the divergences of local field theories result from the locality of interaction vertices. Ricci flatness in vibrational degrees of freedom is not only necessary mathematically. It is also appealing physically: one can regard Ricci flat WCW as a vacuum solution of Einstein's equations $G^{\alpha\beta}=0$.

4.7.3 Ricci Flatness And Hyper Kähler Property

Ricci flatness property is guaranteed if WCW geometry is Hyper Kähler [A96, A44] (there exists 3 covariantly constant antisymmetric tensor fields, which can be regarded as representations of quaternionic imaginary units). Hyper Kähler property guarantees Ricci flatness because the contractions of the curvature tensor appearing in the components of the Ricci tensor transform to traces over Lie algebra generators, which are $SU(n)$ generators instead of $U(n)$ generators so that the traces vanish. In the case of the loop spaces left invariance implies that Ricci tensor in the vibrational degrees is a multiple of the metric tensor so that Ricci scalar has an infinite value. This is basically due to the fact that Kac-Moody algebra has $U(1)$ central extension.

Consider now the arguments in favor of Ricci flatness of the WCW.

1. The symplectic algebra of δM^4_+ takes effectively the role of the $U(1)$ extension of the loop algebra. More concretely, the $SO(2)$ group of the rotation group $SO(3)$ takes the role of $U(1)$ algebra. Since volume preserving transformations are in question, the traces of the symplectic generators vanish identically and in finite-dimensional this should be enough for Ricci flatness even if Hyper Kähler property is not achieved.

2. The comparison with CP_2 allows to link Ricci flatness with conformal invariance. The elements of the Ricci tensor are expressible in terms of traces of the generators of the holonomy group $U(2)$ at the origin of CP_2, and since $U(1)$ generator is non-vanishing at origin, the Ricci tensor is non-vanishing. In recent case the origin of CP_2 is replaced with the maximum of Kähler function and holonomy group corresponds to super-symplectic generators labelled by integer valued real parts k_1 of the conformal weights $k=k_1+i\rho$. If generators with $k_1=n$ vanish at the maximum of the Kähler function, the curvature scalar should vanish at the maximum and by the symmetric space property everywhere. These conditions correspond to Virasoro conditions in super string models.

A possible source of difficulties are the generators having $k_1 = 0$ and resulting as commutators of generators with opposite real parts of the conformal weights. It might be possible to assume that only the conformal weights $k = k_1 + i\rho$, $k_1 = 0, 1, ...$ are possible since it is the imaginary part of the conformal weight which defines the complexification in the recent case. This would mean that the commutators involve only positive values of k_1.

3. In the infinite-dimensional case the Ricci tensor involves also terms which are non-vanishing even when the holonomy algebra does not contain $U(1)$ factor. It will be found that symmetric space property guarantees Ricci flatness even in this case and the reason is essentially the vanishing of the generators having $k_1 = n$ at the maximum of Kähler function.

There are also arguments in favor of the Hyper Kähler property.

1. The dimensions of the embedding space and space-time are 8 and 4 respectively so that the dimension of WCW in vibrational modes is indeed multiple of four as required by Hyper Kähler property. Hyper Kähler property requires a quaternionic structure in the tangent space of WCW. Since any direction on the sphere S^2 defined by the linear combinations of quaternionic imaginary units with unit norm defines a particular complexification physically, Hyper Kähler property means the possibility to perform complexification in S^2-fold ways.
2. S^2-fold degeneracy is indeed associated with the definition of the complex structure of WCW. First of all, the direction of the quantization axis for the spherical harmonics or for the eigen states of Lorentz Cartan algebra at δM^4_+ can be chosen in S^2-fold ways. Quaternion conformal invariance means Hyper Kähler property almost by definition and the S^2-fold degeneracy for the complexification is obvious in this case.

If these naïve arguments survive a more critical inspection, the conclusion would be that the effective 2-dimensionality of light like 3-surfaces implying generalized conformal and symplectic symmetries would also imply Hyper Kähler property of WCW and make the theory well-defined mathematically. This obviously fixes the dimension of space-time surfaces as well as the dimension of Minkowski space factor of the embedding space.

In the sequel we shall show that Ricci flatness is guaranteed provided that the holonomy group of WCW is isomorphic to some subgroup of $SU(n = \infty)$ instead of $U(n = \infty)$ (n is the complex dimension of WCW) implied by the Kähler property of the metric. We also derive an expression for the Ricci tensor in terms of the structure constants of the isometry algebra and WCW metric. The expression for the Ricci tensor is formally identical with that obtained by Freed for loop spaces: the only difference is that the structure constants of the finite-dimensional group are replaced with the group $Can(\delta H)$. Also the arguments in favor of Hyper Kähler property are discussed in more detail.

4.7.4 The Conditions Guaranteeing Ricci Flatness

In the case of Kähler geometry Ricci flatness condition can be characterized purely Lie-algebraically: the holonomy group of the Riemann connection, which in general is subgroup of $U(n)$ for Kähler manifold of complex dimension n, must be subgroup of $SU(n)$ so that the Lie-algebra of this group consists of traceless matrices. This condition is easy to derive using complex coordinates. Ricci tensor is given by the following expression in complex vielbein basis

$$R^{A\bar{B}} = R^{A\bar{C}B}{}_{\bar{C}} \ , \tag{4.7.5}$$

where the latter summation is only over the antiholomorphic indices $\bar{C}$. Using the cyclic identities

$$\sum_{cycl\ \bar{C}B\bar{D}} R^{A\bar{C}B\bar{D}} = 0 \ , \tag{4.7.6}$$

the expression for Ricci tensor reduces to the form

$$R^{A\bar{B}} = R^{A\bar{B}C}{}_{C} \ , \tag{4.7.7}$$

where the summation is only over the holomorphic indices C. This expression can be regarded as a trace of the curvature tensor in the holonomy algebra of the Riemann connection. The trace is taken over holomorphic indices only: the traces over holomorphic and anti-holomorphic indices cancel each other by the antisymmetry of the curvature tensor. For Kähler manifold holonomy algebra is subalgebra of $U(n)$, when the complex dimension of manifold is n and Ricci tensor vanishes if and only if the holonomy Lie-algebra consists of traceless matrices, or equivalently: holonomy group is subgroup of $SU(n)$. This condition is expected to generalize also to the infinite-dimensional case.

We shall now show that if WCW metric is Kähler and possesses infinite-dimensional isometry algebra with the property that its generators form a complete basis for the tangent space (every tangent vector is expressible as a superposition of the isometry generators plus zero norm vector) it is possible to derive a representation for the Ricci tensor in terms of the structure constants of the isometry algebra and of the components of the metric and its inverse in the basis formed by the isometry generators and that Ricci tensor vanishes identically for the proposed complexification of the WCW provided the generators $\{H_{A,m\neq 0}, H_{B,n\neq 0}\}$ correspond to zero norm vector fields of WCW.

The general definition of the curvature tensor as an operator acting on vector fields reads

$$R(X,Y)Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z \ . \tag{4.7.8}$$

If the vector fields considered are isometry generators the covariant derivative operator is given by the expression

$$\begin{aligned} \nabla_X Y &= (Ad_X Y - Ad^*_X Y - Ad^*_Y X)/2 \ , \\ (Ad^*_X Y, Z) &= (Y, Ad_X Z) \ , \end{aligned} \tag{4.7.9}$$

where $Ad_X Y = [X,Y]$ and Ad^*_X denotes the adjoint of Ad_X with respect to WCW metric.

In the sequel we shall assume that the vector fields in question belong to the basis formed by the isometry generators. The matrix representation of Ad_X in terms of the structure constants $C_{X,Y:Z}$ of the isometry algebra is given by the expression

$$\begin{aligned} Ad^m_{Xn} &= C_{X,Y:Z}\hat{Y}_n Z^m \ , \\ [X,Y] &= C_{X,Y:Z}Z \ , \\ \hat{Y} &= g^{-1}(Y,V)V \ , \end{aligned} \tag{4.7.10}$$

where the summation takes place over the repeated indices and $\hat{Y}$ denotes the dual vector field of Y with respect to the WCW metric. From its definition one obtains for Ad^*_X the matrix representation

$$\begin{aligned} Ad^{*m}_{Xn} &= C_{X,Y:Z}\hat{Y}^m Z_n \ , \\ Ad^*_X Y &= C_{X,U:V}\, g(Y,U)g^{-1}(V,W)W = g(Y,U)g^{-1}([X,U],W)W \ , \end{aligned} \tag{4.7.11}$$

where the summation takes place over the repeated indices.

Using the representations of ∇_X in terms of Ad_X and its adjoint and the representations of Ad_X and Ad^*_X in terms of the structure constants and some obvious identities (such as $C_{[X,Y],Z:V} = C_{X,Y:U}C_{U,Z:V}$) one can by a straightforward but tedious calculation derive a more detailed expression for the curvature tensor and Ricci tensor. Straightforward calculation of the Ricci tensor has however turned to be very tedious even in the case of the diagonal metric and in the following we shall use a more convenient representation [A54] of the curvature tensor applying in case of the Kähler geometry.

The expression of the curvature tensor is given in terms of the so called Toeplitz operators T_X defined as linear operators in the "positive energy part" G_+ of the isometry algebra spanned by the $(1,0)$ parts of the isometry generators. In present case the positive and negative energy parts and cm part of the algebra can be defined just as in the case of loop spaces:

$$\begin{aligned} G_+ &= \{H^{Ak}|k>0\} \ , \\ G_- &= \{H^{Ak}|k<0\} \ , \\ G_0 &= \{H^{Ak}|k=0\} \ . \end{aligned} \tag{4.7.12}$$

Here H^{Ak} denote the Hamiltonians generating the symplectic transformations of δH. The positive energy generators with non-vanishing norm have positive radial scaling dimension: $k \geq 0$, which corresponds to the imaginary part of the scaling momentum $K = k_1 + i\rho$ associated with the factors $(r_M/r_0)^K$. A priori the spectrum of ρ is continuous but it is quite possible that the spectrum of ρ is discrete and $\rho = 0$ does not appear at all in the spectrum in the sense that the flux Hamiltonians associated with $\rho = 0$ elements vanish for the maximum of Kähler function which can be taken to be the point where the calculations are done.

T_X differs from Ad_X in that the negative energy part of $Ad_X Y = [X,Y]$ is dropped away:

$$\begin{aligned} T_X : G_+ &\rightarrow G_+ \ , \\ Y &\rightarrow [X,Y]_+ \ . \end{aligned} \tag{4.7.13}$$

Here "+" denotes the projection to "positive energy" part of the algebra. Using Toeplitz operators one can associate to various isometry generators linear operators $\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ acting on G_+:

$$\begin{aligned} \Phi(X_0) &= T_{X_0} \ , \ X_0 \varepsilon G_0 \ , \\ \Phi(X_-) &= T_{X_-} \ , \ X_- \varepsilon G_- \ , \\ \Phi(X_+) &= -T^*_{X_-} \ , \ X_+ \varepsilon G_+ \ . \end{aligned} \tag{4.7.14}$$

Here "*" denotes hermitian conjugate in the diagonalized metric: the explicit representation $\Phi(X_+)$ is given by the expression [A54]

$$\begin{aligned} \Phi(X_+) &= D^{-1} T_{X_-} D \ , \\ D X_+ &= d(X) X_- \ , \\ d(X) &= g(X_-, X_+) \ . \end{aligned} \tag{4.7.15}$$

Here $d(X)$ is just the diagonal element of metric assumed to be diagonal in the basis used. denotes the conformal factor associated with the metric.

The representations for the action of ,$\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ in terms of metric and structure constants of the isometry algebra are in the case of the diagonal metric given by the expressions

$$\begin{aligned} \Phi(X_0) Y_+ &= C_{X_0, Y_+ : U_+} U_+ \ , \\ \Phi(X_-) Y_+ &= C_{X_-, Y_+ : U_+} U_+ \ , \\ \Phi(X_+) Y_+ &= \frac{d(Y)}{d(U)} C_{X_-, Y_- : U_-} U_+ \ . \end{aligned} \tag{4.7.16}$$

The expression for the action of the curvature tensor in positive energy part G_+ of the isometry algebra in terms of the these operators is given as [A54] :

$$R(X,Y) Z_+ = \{[\Phi(X), \Phi(Y)] - \Phi([X,Y])\} Z_+ \ . \tag{4.7.17}$$

The calculation of the Ricci tensor is based on the observation that for Kähler manifolds Ricci tensor is a tensor of type $(1,1)$, and therefore it is possible to calculate Ricci tensor as the trace of the curvature tensor with respect to indices associated with G_+.

$$
Ricci(X_+, Y_-) \quad = \quad (\hat{Z}_+, R(X_+, Y_-)Z_+) \equiv Trace(R(X_+, Y_-)) \ , \tag{4.7.18}
$$

where the summation over Z_+ generators is performed.

Using the explicit representations of the operators Φ one obtains the following explicit expression for the Ricci tensor

$$
\begin{aligned}
Ricci(X_+, Y_-) &= Trace\{[D^{-1}T_{X_+}D, T_{Y_-}] - T_{[X_+,Y_-]_{|G_0+G_-}} \\
&- D^{-1}T_{[X_+,Y_-]_{|G_+}}D\} \ .
\end{aligned} \tag{4.7.19}
$$

This expression is identical to that encountered in case of loop spaces and the following arguments are repetition of those applying in the case of loop spaces.

The second term in the Ricci tensor is the only term present in the finite-dimensional case. This term vanishes if the Lie-algebra in question consists of traceless matrices. Since symplectic transformations are volume-preserving the traces of Lie-algebra generators vanish so that this term is absent. The last term gives a non-vanishing contribution to the trace for the same reason.

The first term is quadratic in structure constants and does not vanish in case of loop spaces. It can be written explicitly using the explicit representations of the various operators appearing in the formula:

$$
\begin{aligned}
Trace\{[D^{-1}T_{X_-}D, T_{Y_-}]\} &= \sum_{Z_+,U_+} [C_{X_-,U_-:Z_-}C_{Y_-,Z_+:U_+}\frac{d(U)}{d(Z)} \\
&- C_{X_-,Z_-:U_-}C_{Y_-,U_+:Z_+}\frac{d(Z)}{d(U)}] \ .
\end{aligned} \tag{4.7.20}
$$

Each term is antisymmetric under the exchange of U and Z and one might fail to conclude that the sum vanishes identically. This is not the case. By the diagonality of the metric with respect to radial quantum number, one has $m(X_-) = m(Y_-)$ for the non-vanishing elements of the Ricci tensor. Furthermore, one has $m(U) = m(Z) - m(Y)$, which eliminates summation over $m(U)$ in the first term and summation over $m(Z)$ in the second term. Note however, that summation over other labels related to symplectic algebra are present.

By performing the change $U \rightarrow Z$ in the second term one can combine the sums together and as a result one has finite sum

$$
\begin{aligned}
\sum_{0<m(Z)<m(X)} [C_{X_-,U_-:Z_-}C_{Y_-,Z_+:U_+}\frac{d(U)}{d(Z)} &= C \sum_{0<m(Z)<m(X)} \frac{m(X)}{m(Z)-m(X)} \ , \\
C &= \sum_{Z,U} C_{X,U:Z}C_{Y,Z:U}\frac{d_0(U)}{d_0(Z)} \ .
\end{aligned} \tag{4.7.21}
$$

Here the dependence of $d(X) = |m(X)|d_0(X)$ on $m(X)$ is factored out; $d_0(X)$ does not depend on k_X. The dependence on $m(X)$ in the resulting expression factorizes out, and one obtains just the purely group theoretic term C, which should vanish for the space to be Ricci flat.

The sum is quadratic in structure constants and can be visualized as a loop sum. It is instructive to write the sum in terms of the metric in the symplectic degrees of freedom to see the geometry behind the Ricci flatness:

$$
C \quad = \quad \sum_{Z,U} g([Y,Z],U)g^{-1}([X,U],Z) \ . \tag{4.7.22}
$$

Each term of this sum involves a commutator of two generators with a non-vanishing norm. Since tangent space complexification is inherited from the local coset space, the non-vanishing commutators in complexified basis are always between generators in $Can_{\neq 0}$; that is they do not not belong to rigid $su(2) \times su(3)$.

The condition guaranteeing Ricci flatness at the maximum of Kähler function and thus everywhere is simple. All elements of type $[X_{\neq 0}, Y_{\neq 0}]$ vanish or have vanishing norm. In case of CP_2 Kähler geometry this would correspond to the vanishing of the $U(2)$ generators at the origin of CP_2 (note that the holonomy group is $U(2)$ in case of CP_2). At least formally stronger condition is that the algebra generated by elements of this type, the commutator algebra associated with $Can_{\neq 0}$, consist of elements of zero norm. Already the (possibly) weaker condition implies that adjoint map $Ad_{X\neq 0}$ and its hermitian adjoint $Ad^*_{X_{\neq 0}}$ create zero norm states. Since isometry conditions involve also adjoint action the condition also implies that $Can_{\neq 0}$ acts as isometries. More concrete form for the condition is that all flux factors involving double Poisson bracket and three generators in $Can_{\neq 0}$ vanish:

$$Q_e(\{H_A, \{H_B, H_C\}\}) \quad = \quad 0 \ , \ \text{ for } H_A, H_B, H_C \text{ in } Can_{\neq 0} \ . \tag{4.7.23}$$

The vanishing of fluxes involving two Poisson brackets and three Hamiltonians guarantees isometry invariance and Ricci flatness and, as found in [K21] , is implied by the $[t,t] \subset h$ property of the Lie-algebra of coset space G/H having symmetric space structure.

The conclusion is that the mere existence of the proposed isometry group (guaranteed by the symmetric space property) implies the vanishing of the Ricci tensor and vacuum Einstein equations. The existence of the infinite parameter isometry group in turn follows basically from the condition guaranteeing the existence of the Riemann connection. Therefore vacuum Einstein equations seem to arise, not only as a consequence of a physically motivated variational principle but as a mathematical consistency condition in infinite dimensional Kähler geometry. The flux representation seems to provide elegant manner to formulate and solve these conditions and isometry invariance implies Ricci flatness.

4.7.5 Is WCW Metric Hyper Kähler?

The requirement that WCW integral integration is divergence free implies that WCW metric is Ricci flat. The so called Hyper-Kähler metrics [A96, A44] , [B48] are particularly nice representatives of Ricci flat metrics. In the following the basic properties of Hyper-Kähler metrics are briefly described and the problem whether Hyper Kähler property could realized in case of $M^4_+ \times CP_2$ is considered.

Hyper-Kähler property

Hyper-Kähler metric is a generalization of the Kähler metric. For Kähler metric metric tensor and Kähler form correspond to the complex numbers 1 and i and therefore define complex structure in the tangent space of the manifold. For Hyper Kähler metric tangent space allows three closed Kähler forms I, J, K, which with respect to the multiplication obey the algebra of quaternionic imaginary units and have square equal to - 1, which corresponds to the metric of Hyper Kähler space.

$$I^2 \quad = \quad J^2 = K^2 = -1 \quad IJ = -JI = K, \text{ etc. } . \tag{4.7.24}$$

To define Kähler structure one must choose one of the Kähler forms or any linear combination of I, J and K with unit norm. The group $SO(3)$ rotates different Kähler structures to each other playing thus the role of quaternion automorphisms. This group acts also as coordinate transformations in Hyper Kähler manifold but in general fails to act as isometries.

If K is chosen to define complex structure then K is tensor of type $(1,1)$ in complex coordinates, I and J being tensors of type $(2,0)+(0,2)$. The forms $I+iJ$ and $I-iJ$ are holomorphic and anti-holomorphic forms of type $(2,0)$ and $(0,2)$ respectively and defined standard step operators I_+ and I_- of $SU(2)$ algebra. The holonomy group of Hyper-Kähler metric is always $Sp(k)$, $k \leq dimM/4$, the group of $k \times k$ unitary matrices with quaternionic entries. This group is indeed subgroup of SU(2k), so that its generators are traceless and Hyper Kähler metric is therefore Ricci flat.

Hyper Kähler metrics have been encountered in the context of 3-dimensional super symmetric sigma models: a necessary prerequisite for obtaining $N = 4$ super-symmetric sigma model is that target space allows Hyper Kähler metric [B48, B12] . In particular, it has been found that Hyper Kähler property is decisive for the divergence cancelation.

Hyper-Kähler metrics arise also in monopole and instanton physics [A44] . The moduli spaces for monopoles have Hyper Kähler property. This suggests that Hyper Kähler property is characteristic for the configuration (or moduli) spaces of 4-dimensional Yang Mills types systems. Since YM action appears in the definition of WCW metric there are hopes that also in present case the metric possesses Hyper-Kähler property.

CP_2 allows what might be called almost Hyper-Kähler structure known as quaternionion structure. This means that the Weil tensor of CP_2 consists of three components in one-one correspondence with components of iso-spin and only one of them- the one corresponding to Kähler form- is covariantly constant. The physical interpretation is in terms of electroweak symmetry breaking selecting one isospin direction as a favored direction.

Does the "almost" Hyper-Kähler structure of CP_2 lift to a genuine Hyper-Kähler structure in WCW?

The Hyper-Kähler property of WCW metric does not seem to be in conflict with the general structure of TGD.

1. In string models the dimension of the "space-time" is two and Weyl invariance and complex structures play a decisive role in the theory. In present case the dimension of the space-time is four and one therefore might hope that quaternions play a similar role. Indeed, Weyl invariance implies YM action in dimension 4 and as already mentioned moduli spaces of instantons and monopoles enjoy the Hyper Kähler property.
2. Also the dimension of the embedding space is important. The dimension of Hyper Kähler manifold must be multiple of 4. The dimension of WCW is indeed infinite multiple of 8: each vibrational mode giving one "8".
3. The complexification of the WCW in symplectic degrees of freedom is inherited from $S^2 \times CP_2$ and CP_2 Kähler form defines the symplectic form of WCW. The point is that CP_2 Weyl tensor has 3 covariantly constant components, having as their square metric apart from sign. One of them is Kähler form, which is closed whereas the other two are non-closed forms and therefore fail to define Kähler structure. The group $SU(2)$ of electro-weak isospin rotations rotate these forms to each other. It would not be too surprising if one could identify WCW counterparts of these forms as representations of quaternionic units at the level of WCW. The failure of the Hyper Kähler property at the level of CP_2 geometry is due to the electro-weak symmetry breaking and physical intuition (in particular, p-adic mass calculations [K54]) suggests that electro-weak symmetry might not be broken at the level of WCW geometry).

A possible topological obstruction for the Hyper Kähler property is related to the cohomology of WCW: the three Kähler forms must be co-homologically trivial as is clear from the following argument. If any of 3 quaternionic 2-form is cohomologically nontrivial then by $SO(3)$ symmetry rotating Kähler forms to each other all must be co-homologically nontrivial. On the other hand, electro-weak isospin rotation leads to a linear combination of 3 Kähler forms and the flux associated with this form is in general not integer valued. The point is however that Kähler form forms only the $(1,1)$ part of the symplectic form and must be co-homologically trivial whereas the zero mode part is same for all complexifications and can be co-homologically nontrivial. The co-homological non-triviality of the zero mode part of the symplectic form is indeed a nice feature since it fixes the normalization of the Kähler function apart from a multiplicative integer. On the other hand the hypothesis that Kähler coupling strength is analogous to critical temperature provides a dynamical (and perhaps equivalent) manner to fix the normalization of the Kähler function.

Since the properties of the WCW metric are inherited from $M^4_+ \times CP_2$ then also the Hyper Kähler property should be understandable in terms of the embedding space geometry. In particular, the complex structure in CP_2 vibrational degrees of freedom is inherited from CP_2. Hyper Kähler property implies the existence of a continuum (sphere S^2) of complex structures: any linear superposition of 3 independent Kähler forms defines a respectable complex structure. Therefore also CP_2 should have this continuum of complex structures and this is certainly not the case.

Indeed, if we had instead of CP_2 Hyper Kähler manifold with 3 covariantly constant 2-forms then it would be easy to understand the Hyper Kähler structure of WCW. Given the Kähler structure of WCW would be obtained by replacing induced Kähler electric and magnetic fields in the definition of flux factors $Q(H_{A,m})$ with the appropriate component of the induced Weyl tensor. CP_2 indeed manages to be very nearly Hyper Kähler manifold!

How CP_2 fails to be Hyper Kähler manifold can be seen in the following manner. The Weyl tensor of CP_2 allows three independent components, which are self dual as 2-forms and rotated to each other by vielbein rotations.

$$\begin{aligned} W_{03} &= W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ W_{01} &= W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ W_{02} &= W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \end{aligned} \tag{4.7.25}$$

The component I_3 is just the Kähler form of CP_2. Remaining components are covariantly constant only with respect to spinor connection and not closed forms so that they cannot be interpreted as Maxwell fields. Their squares equal however apart from sign with the metric of CP_2, when appropriate normalization factor is used. If these forms were covariantly constant Kähler action defined by any linear superposition of these forms would indeed define Kähler structure in WCW and the group $SO(3)$ would rotate these forms to each other. The projections of the components of the Weyl tensor on 3-surface define 3 vector fields as their duals and only one of these vector fields (Kähler magnetic field) is divergenceless. One might regard these 3 vector fields as counter parts of quaternion units associated with the broken Hyper Kähler structure, that is quaternion structure. The interpretation in terms of electro-weak symmetry breaking is obvious.

One cannot exclude the possibility that the symplectic invariance of the induced Kähler electric field implies that the electric parts of the other two components of induced Weyl tensor are symplectic invariants. This is the minimum requirement. What is however obvious is that the magnetic parts cannot be closed forms for arbitrary 3-surfaces at light cone boundary. One counter example is enough and CP_2 type extremals seem to provide this counter example: the components of the induced Weyl tensor are just the same as they are for CP_2 and clearly not symplectically invariant.

Thus it seems that WCW could allow Hyper Kähler structure broken by electro-weak interactions but it cannot be inherited from CP_2. An open question is whether it allows genuine quaternionic structure. Good prospects for obtaining quaternionic structure are provided by the quaternionic counterpart QP_2 of CP_2, which is 8-dimensional and has coset space structure $QP_2 = Sp(3)/Sp(2) \times Sp(1)$. This choice does not seem to be consistent with the symmetries of the standard model. Note however that the over all symmetry group is obtained by replacing complex numbers with quaternions on the matrix representation of the standard model group.

Could different complexifications for M^4_+ and light like surfaces induce Hyper Kähler structure for WCW?

Quaternionic structure means also the existence of a family of complex structures parameterized by a sphere S^2. The complex structure of the WCW is inherited from the complex structure of some light like surface.

In the case of the light cone boundary δM^4_+ the complex structure corresponds to the choice of quantization axis of angular momentum for the sphere $r_M = constant$ so that the coordinates orthogonal to the quantization axis define a complex coordinate: the sphere S^2 parameterizes these choices. Thus there is a temptation to identify the choice of quantization axis with a particular imaginary unit and Hyper Kähler structure would directly relate to the properties rotation group. This would bring an additional item to the list of miraculous properties of light like surfaces of 4-dimensional space-times.

This might relate to the fact that WCW geometry is not determined by the symplectic algebra of CP_2 localized with respect to the light cone boundary as one might first expect but consists of $M^4_+ \times CP_2$ Hamiltonians so that infinitesimal symplectic transformation of CP_2 involves always also M^4_+-symplectic transformation. M^4_+ Hamiltonians are defined by a function basis generated as products of the Hamiltonians H_3 and $H_1 \pm iH_2$ generating rotations with respect to three orthogonal axes, and two of these Hamiltonians are complexified.

Also the light like 3-surfaces X_l^3 associated with quaternion conformal invariance are determined by some 2-surface X^2 and the choice of complex coordinates and if X^2 is sphere the choices are labelled by S^2. In this case, the presence of quaternion conformal structure would be almost obvious since it is possible to choose some complex coordinate in several ways and the choices are labelled by S^2. The choice of the complex coordinate in turn fixes 2-surface X^2 as a surface for which the remaining coordinates are constant. X^2 need not however be located at the elementary particle horizon unless one poses additional constraint. One might hope that different choices of X^2 resulting in this manner correspond to all possible different selections of the complex structure and that this choice could fix uniquely the conformal equivalence class of X^2 appearing as argument in elementary particle vacuum functionals. If X^2 has a more complex topology the identification is not so clear but since conformal algebra SL(2,C) containing algebra of rotation group is involved, one might argue that the choice of quantization axis also now involves S^2 degeneracy. If these arguments are correct one could conclude that Hyper Kähler structure is implicitly involved and guarantees Ricci flatness of the WCW metric.

Chapter 5

WCW Spinor Structure

5.1 Introduction

Quantum TGD should be reducible to the classical spinor geometry of the configuration space ("world of classical worlds" (WCW)). The possibility to express the components of WCW Kähler metric as anti-commutators of WCW gamma matrices becomes a practical tool if one assumes that WCW gamma matrices correspond to Noether super charges for super-symplectic algebra of WCW. The possibility to express the Kähler metric also in terms of Kähler function identified as Kähler for Euclidian space-time regions leads to a duality analogous to AdS/CFT duality.

5.1.1 Basic Principles

Physical states should correspond to the modes of the WCW spinor fields and the identification of the fermionic oscillator operators as super-symplectic charges is highly attractive. WCW spinor fields cannot, as one might naively expect, be carriers of a definite spin and unit fermion number. Concerning the construction of the WCW spinor structure there are some important clues.

Geometrization of fermionic statistics in terms of WCW spinor structure

The great vision has been that the second quantization of the induced spinor fields can be understood geometrically in terms of the WCW spinor structure in the sense that the anti-commutation relations for WCW gamma matrices require anti-commutation relations for the oscillator operators for free second quantized induced spinor fields.

1. One must identify the counterparts of second quantized fermion fields as objects closely related to the configuration space spinor structure. [B45] has as its basic field the anti-commuting field $\Gamma^k(x)$, whose Fourier components are analogous to the gamma matrices of the configuration space and which behaves like a spin 3/2 fermionic field rather than a vector field. This suggests that the are analogous to spin 3/2 fields and therefore expressible in terms of the fermionic oscillator operators so that their naturally derives from the anti-commutativity of the fermionic oscillator operators.

   As a consequence, WCW spinor fields can have arbitrary fermion number and there would be hopes of describing the whole physics in terms of WCW spinor field. Clearly, fermionic oscillator operators would act in degrees of freedom analogous to the spin degrees of freedom of the ordinary spinor and bosonic oscillator operators would act in degrees of freedom analogous to the "orbital" degrees of freedom of the ordinary spinor field.

2. The classical theory for the bosonic fields is an essential part of the WCW geometry. It would be very nice if the classical theory for the spinor fields would be contained in the definition of the WCW spinor structure somehow. The properties of the associated with the induced spinor structure are indeed very physical. The modified massless Dirac equation for the induced spinors predicts a separate conservation of baryon and lepton numbers. Contrary to the long held belief it seems that covariantly constant right handed neutrino does not generate . The differences between quarks and leptons result from the different couplings to

the CP_2 Kähler potential. In fact, these properties are shared by the solutions of massless Dirac equation of the embedding space.

3. Since TGD should have a close relationship to the ordinary quantum field theories it would be highly desirable that the second quantized free induced spinor field would somehow appear in the definition of the WCW geometry. This is indeed true if the complexified WCW gamma matrices are linearly related to the oscillator operators associated with the second quantized induced spinor field on the space-time surface and its boundaries. There is actually no deep reason forbidding the gamma matrices of the WCW to be spin half odd-integer objects whereas in the finite-dimensional case this is not possible in general. In fact, in the finite-dimensional case the equivalence of the spinorial and vectorial vielbeins forces the spinor and vector representations of the vielbein group $SO(D)$ to have same dimension and this is possible for $D = 8$-dimensional Euclidian space only. This coincidence might explain the success of 10-dimensional super string models for which the physical degrees of freedom effectively correspond to an 8-dimensional Euclidian space.

4. It took a long time to realize that the ordinary definition of the gamma matrix algebra in terms of the anti-commutators $\{\gamma_A, \gamma_B\} = 2g_{AB}$ must in TGD context be replaced with

$$\{\gamma_A^\dagger, \gamma_B\} = iJ_{AB} \quad .$$

where J_{AB} denotes the matrix elements of the Kähler form of the WCW. The presence of the Hermitian conjugation is necessary because WCW gamma matrices carry fermion number. This definition is numerically equivalent with the standard one in the complex coordinates. The realization of this delicacy is necessary in order to understand how the square of the WCW Dirac operator comes out correctly.

5. TGD as a generalized number theory vision leads to the understanding of how the second quantization of the induced spinor fields should be carried out and space-time conformal symmetries allow to explicitly solve the Dirac equation associated with the Kähler-Dirac action in the interior and at the 3-D light like causal determinants. An essentially new element is the notion of number theoretic braid forced by the fact that the Kähler-Dirac operator allows only finite number of generalized eigen modes so that the number of fermionic oscillator operators is finite. As a consequence, anti-commutation relations can be satisfied only for a finite set of points defined by the number theoretic braid, which is uniquely identifiable. The interpretation is in terms of finite measurement resolution. The finite Clifford algebra spanned by the fermionic oscillator operators is interpreted as the factor space $\mathcal{M}/\mathcal{N}$ of infinite hyper-finite factors of type II_1 defined by WCW Clifford algebra $\mathcal{N}$ and included Clifford algebra $\mathcal{M} \subset \mathcal{N}$ interpreted as the characterizer of the finite measurement resolution. Note that the finite number of eigenvalues guarantees that Dirac determinant identified as the exponent of Kähler function is finite. Finite number of eigenvalues is also essential for number theoretic universality.

Identification of WCW gamma matrices as super Hamiltonians and expression of WCW Kähler metric

The basic super-algebra corresponds to the fermionic oscillator operators and can be regarded as a generalization $\mathcal{N}$ super algebras by replacing $\mathcal{N}$ with the number of solutions of the Kähler-Dirac equation which can be infinite. This leads to QFT SUSY limit of TGD different in many respects crucially from standard SUSYs.

WCW gamma matrices are identified as super generators of super-symplectic and are expressible in terms of these oscillator operators. In the original proposal super-symplectic and super charges were assumed to be expressible as integrals over 2-dimensional partonic surfaces X^2 and interior degrees of freedom of X^4 can be regarded as zero modes representing classical variables in one-one correspondence with quantal degrees of freedom at X_l^3 as indeed required by quantum measurement theory.

It took quite long time to realize that it is possible to second quantize induced spinor fields by using just the standard canonical quantization. The only new element is the replacement of the ordinary gamma matrices with K-D gamma matrices identified as canonical momentum currents contracted with the embedding space gamma matrices. This allows to deduce super-generators of super-symplectic algebra as Noether supercharges assignable to the fermionic strings connecting partonic 2-surfaces. Their anti-commutators giving the matrix elements of WCW Kähler metric can be deduced explicitly. This is a decisive calculational advantage since the formal expression of the matrix elements in terms of second derivatives of Kähler function is not possible to calculate with the recent understanding. WCW gamma matrices provide also a natural identification for the counterparts of fermionic oscillator operators creating physical states.

One can also deduce the fermionic Hamiltonians as conserved Noether charges. The expressions for Hamiltonians generalized the earlier expressions as Hamiltonian fluxes in the sense that the embedding space Hamiltonian is replaced with the corresponding fermionic Noether charge. This replacement is analogous to a transition from field theory to string models requiring the replacement of points of partonic 2-surfaces with stringy curves connecting the points of two partonic 2-surfaces. One can consider also several strings emanating from a given partonic 2-surface. This leads to an extension of the super-symplectic algebra to a Yangian, whose generators are multi-local (multi-stringy) operators. This picture does not mean loss of effective 2-dimensionality implied by strong form of general coordinate invariance but allows genuine generalization of super-conformal invariance in 4-D context.

5.1.2 Kähler-Dirac Action

Supersymmetry fixes the interior part of Kähler-Dirac uniquely. The K-D gamma matrices are contractions of the canonical momentum currents of Kähler action with the embedding space gamma matrices and this gives field equations consistent with hermitian conjugation. The modes of K-D equation must be restricted to 2-D string world sheets with vanishing induced W boson fields in order that they have a well-defined em charge. It is not yet clear whether this restriction is part of variational principle or whether it is a property of spinor modes. For the latter option modes one can have 4-D modes if the space-time surface has CP_2 projection carrying vanishing W gauge potentials. Also covariantly constant right-handed neutrino defines this kind of mode.

The boundary terms of Kähler action and Kähler-Dirac action

A long standing question has been whether Kähler action could contain Chern-Simons term cancelling the Chern-Simons contribution of Kähler action at space-time interior at partonic orbit reducing to Chern-Simons terms so that only the contribution at space-like ends of space-time surface at the boundaries of causal diamond (CD) remains. This is however not necessary and super-symmetry would require Chern-Simons-Dirac term as boundary term in Dirac action. This however has unphysical implications since C-S-D Dirac operator acts on CP_2 coordinates only.

The intuitive expectation is that fermionic propagators assignable to string boundaries at light-like partonic orbits are needed in the construction of the scattering amplitudes. These boundaries can be locally space-like or light-like. One could add 1-D massles Dirac action with gamma matrices defined in the induced metric, which is by supersymmetry accompanied by the action defined by geodesic length, which however vanishes for light-like curves. Massless Dirac equation at the boundary of string world sheet fixes the boundary conditions for the spinor modes at the string world sheet. This option seems to be the most plausible at this moment.

Kähler-Dirac equation for induced spinor fields

It has become clear that Kähler-Dirac action with induced spinor fields localized at string world sheets carrying vanishing classical W fields, and the light-like boundaries of the string world sheets at light-like orbits of partonic 2-surfaces carrying massless Dirac operator for induced gamma matrices is the most natural looking option.

The light-like momentum associated with the boundary is a light-like curve of imbedding space and defines light-like 8-momentum, whose M^4 projection is in general time-like. This leads to an 8-D generalization of twistor formalism. The squares of the M^4 and CP_2 parts of the 8-momentum could be identified as mass squared for the embedding space spinor mode assignable to the ground state of super-symplectic representation. This would realize quantum classical correspondence for fermions. The four-momentum assignable to fermion line would have identification as gravitational four-momentum and that associated with the mode of embedding space spinor field as inertial four-momentum.

There are several approaches for solving the Kähler-Dirac (or Kähler-Dirac) equation.

1. The most promising approach assumes that the solutions are restricted on 2-D stringy world sheets and/or partonic 2-surfaces. This strange looking view is a rather natural consequence of both strong form of holography and of number theoretic vision, and also follows from the notion of finite measurement resolution having discretization at partonic 2-surfaces as a geometric correlate. Furthermore, the conditions stating that electric charge is well-defined for preferred extremals forces the localization of the modes to 2-D surfaces in the generic case. This also resolves the interpretational problems related to possibility of strong parity breaking effects since induce W fields and possibly also Z^0 field above weak scale, vanish at these surfaces.
2. One expects that stringy approach based on 4-D generalization of conformal invariance or its 2-D variant at 2-D preferred surfaces should also allow to understand the Kähler-Dirac equation. Conformal invariance indeed allows to write the solutions explicitly using formulas similar to encountered in string models. In accordance with the earlier conjecture, all modes of the Kähler-Dirac operator generate badly broken super-symmetries.
3. Well-definedness of em charge is not enough to localize spinor modes at string world sheets. Covariantly constant right-handed neutrino certainly defines solutions de-localized inside entire space-time sheet. This need not be the case if right-handed neutrino is not covarianty constant since the non-vanishing CP_2 part for the induced gamma matrices mixes it with left-handed neutrino. For massless extremals (at least) the CP_2 part however vanishes and right-handed neutrino allows also massless holomorphic modes de-localized at entire space-time surface and the de-localization inside Euclidian region defining the line of generalized Feynman diagram is a good candidate for the right-handed neutrino generating the least broken super-symmetry.This super-symmetry seems however to differ from the ordinary one in that ν_R is expected to behave like a passive spectator in the scattering. Also for the left-handed neutrino solutions localized inside string world sheet the condition that coupling to right-handed neutrino vanishes is guaranteed if gamma matrices are either purely Minkowskian or CP_2 like inside the world sheet.

Quantum criticality and K-D action

A detailed view about the physical role of quantum criticality results. Quantum criticality fixes the values of Kähler coupling strength as the analog of critical temperature. The recent formulation of quantum criticality states the existence of hierarchy of sub-algebras of super-symplectic algebras isomorphic with the original algebra. The conformal weights of given sub-algebra are n-multiples of those of the full algebra. n would also characterize the value of Planck constant $h_{eff} = n \times h$ assignable to various phases of dark matter. These sub-algebras correspond to a hierarchy of breakings of super-symplectic gauge symmetry to a sub-algebra. Accordingly the super-symplectic Noether charges of the sub-algebra annihilate physical states and the corresponding classical Noether charges vanish for Kähler action at the ends of space-time surfaces. This defines the notion of preferred extremal. These sub-algebras form an inclusion hierarchy defining a hierarchy of symmetry breakings. n would also characterize the value of Planck constant $h_{eff} = n \times h$ assignable to various phases of dark matter.

Quantum criticality implies that second variation of Kähler action vanishes for critical deformations defined by the sub-algebra and vanishing of the corresponding Noether charges and super-charges for physical stats. It is not quite clear whether the charges corresponding to broken super-symplectic symmetries are conserved. If this is the case, Kähler action is invariant under brokent symplectic transformations although the second variation is non-vanishing so these deformations contribute to Kähler metric and are thus quantum fluctuating dynamical degrees of freedom.

Quantum classical correspondence

Quantum classical correspondence (QCC) requires a coupling between quantum and classical and this coupling should also give rise to a generalization of quantum measurement theory. The big question mark is how to realize this coupling.

1. As already described, the massless Dirac equation for induced gamma matrices at the boundary of string world sheets gives as solutions for which local 8-momentum is light-like. The M^4 part of this momentum is in general time-like and can be identified as the 8-momentum of incoming fermion assignable to an embedding space spinor mode. The interpretation is as equivalence of gravitational and inertial masses.
2. QCC can be realized at the level of WCW Dirac operator and Kähler-Dirac operator contains only interior term. The vanishing of the normal component of fermion current replaces Chern-Simons Dirac operator at various boundary like surfaces. I have proposed that WCW spinor fields with given quantum charges in Cartan algebra are superpositions of space-time surfaces with same classical charges. A stronger form of QCC at the level of WCW would be that classical correlation functions for various geometric observables are identical with quantal correlation functions.

   QCC could be realized at the level of WCW by putting it in by hand. One can of course consider also the possibility that the equality of quantal and classical Cartan charges is realized by adding constraint terms realized using Lagrange multipliers at the space-like ends of space-time surface at the boundaries of CD. This procedure would be very much like the thermodynamical procedure used to fix the average energy or particle number of the the system with Lagrange multipliers identified as temperature or chemical potential. Since quantum TGD in zero energy ontology (ZEO) can be regarded as square root of thermodynamics, the procedure looks logically sound.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L22].

5.2 WCW Spinor Structure: General Definition

The basic problem in constructing WCW spinor structure is clearly the construction of the explicit representation for the gamma matrices of WCW . One should be able to identify the space, where these gamma matrices act as well as the counterparts of the "free" gamma matrices, in terms of which the gamma matrices would be representable using generalized vielbein coefficients.

5.2.1 Defining Relations For Gamma Matrices

The ordinary definition of the gamma matrix algebra is in terms of the anti-commutators

$$\{\gamma_A, \gamma_B\} = 2g_{AB} \quad .$$

This definition served implicitly also as a basic definition of the gamma matrix algebra in TGD context until the difficulties related to the understanding of WCW d'Alembertian defined in terms of the square of the Dirac operator forced to reconsider the definition. If WCW allows Kähler structure, the most general definition allows to replace the metric any covariantly constant Hermitian form. In particular, g_{AB} can be replaced with

$$\{\Gamma_A^\dagger, \Gamma_B\} = iJ_{AB} \quad , \tag{5.2.1}$$

where J_{AB} denotes the matrix element of the Kähler form of WCW . The reason is that gamma matrices carry fermion number and are non-hermitian in all coordinate systems. This definition is numerically equivalent with the standard one in the complex coordinates but in arbitrary coordinates situation is different since in general coordinates iJ_{kl} is a nontrivial positive square root of g_{kl}. The realization of this delicacy is necessary in order to understand how the square of WCW Dirac operator comes out correctly. Obviously, what one must do is the equivalent of replacing $D^2 = (\Gamma^k D_k)^2$ with $D\hat{D}$ with $\hat{D}$ defined as

$$\hat{D} = iJ^{kl}\Gamma_l^\dagger D_k \quad .$$

5.2.2 General Vielbein Representations

There are two ideas, which make the solution of the problem obvious.

1. Since the classical time development in bosonic degrees of freedom (induced gauge fields) is coded into the geometry of WCW it seems natural to expect that same applies in the case of the spinor structure. The time development of the induced spinor fields dictated by TGD counterpart of the massless Dirac action should be coded into the definition of the WCW spinor structure. This leads to the challenge of defining what classical spinor field means.

2. Since classical scalar field in WCW corresponds to second quantized boson fields of the embedding space same correspondence should apply in the case of the fermions, too. The spinor fields of WCW should correspond to second quantized fermion field of the embedding space and the space of the configuration space spinors should be more or less identical with the Fock space of the second quantized fermion field of embedding space or $X^4(X^3)$. Since classical spinor fields at space-time surface are obtained by restricting the spinor structure to the space-time surface, one might consider the possibility that life is really simple: the second quantized spinor field corresponds to the free spinor field of the embedding space satisfying the counterpart of the massless Dirac equation and more or less standard anti-commutation relations. Unfortunately life is not *so* simple as the construction of WCW spinor structure demonstrates: second quantization must be performed for induced spinor fields.

It is relatively simple to fill in the details once these basic ideas are accepted.

1. The only natural candidate for the second quantized spinor field is just the on X^4. Since this field is free field, one can indeed perform second quantization and construct fermionic oscillator operator algebra with unique anti-commutation relations. The space of WCW spinors can be identified as the associated with these oscillator operators. This space depends on 3-surface and strictly speaking one should speak of the Fock bundle having WCW as its base space.

2. The gamma matrices of WCW (or rather fermionic Kac Moody generators) are representable as super positions of the fermionic oscillator algebra generators:

$$\begin{aligned}
\Gamma_A^+ &= E_A^n a_n^\dagger \\
\Gamma_A^- &= \bar{E}_A^n a_n \\
iJ_{A\bar{B}} &= \sum_n E_A^n \bar{E}_B^n
\end{aligned} \tag{5.2.2}$$

   where E_A^n are the vielbein coefficients. Induced spinor fields can possess zero modes and there is no oscillator operators associated with these modes. Since oscillator operators are spin 1/2 objects, WCW gamma matrices are analogous to spin 3/2 spinor fields (in a very general sense). Therefore the generalized vielbein and WCW metric is analogous to the pair of spin 3/2 and spin 2 fields encountered in super gravitation! Notice that the contractions $j^{Ak}\Gamma_k$ of the complexified gamma matrices with the isometry generators are genuine spin 1/2 objects labeled by the quantum numbers labeling isometry generators. In particular, in CP_2 degrees of freedom these fermions are color octets.

3. A further great idea inspired by the symplectic and Kähler structures of WCW is that configuration gamma matrices are actually generators of super-symplectic symmetries. This simplifies enormously the construction allows to deduce explicit formulas for the gamma matrices.

5.2.3 Inner Product For WCW Spinor Fields

The conjugation operation for WCW spinor s corresponds to the standard $ket \rightarrow bra$ operation for the states of the Fock space:

$$\begin{aligned}
\Psi &\leftrightarrow |\Psi\rangle \\
\bar{\Psi} &\leftrightarrow \langle\Psi|
\end{aligned} \tag{5.2.3}$$

The inner product for WCW spinor s at a given point of WCW is just the standard Fock space inner product, which is unitary.

$$\bar{\Psi}_1(X^3)\Psi_2(X^3) = \langle\Psi_1|\Psi_2\rangle_{|X^3} \quad (5.2.4)$$

WCW inner product for two WCW spinor fields is obtained as the integral of the Fock space inner product over the whole WCW using the vacuum functional $exp(K)$ as a weight factor

$$\langle\Psi_1|\Psi_2\rangle = \int \langle\Psi_1|\Psi_2\rangle_{|X^3} exp(K)\sqrt{G}dX^3 \quad (5.2.5)$$

This inner product is obviously unitary. A modified form of the inner product is obtained by including the factor $exp(K/2)$ in the definition of the spinor field. In fact, the construction of the central extension for the isometry algebra leads automatically to the appearance of this factor in vacuum spinor field.

The inner product differs from the standard inner product for, say, Minkowski space spinors in that integration is over the entire WCW rather than over a time= constant slice of the WCW . Also the presence of the vacuum functional makes it different from the finite dimensional inner product. These are not un-physical features. The point is that (apart from classical non-determinism forcing to generalized the concept of 3-surface) Diff^4 invariance dictates the behavior of WCW spinor field completely: it is determined form its values at the moment of the big bang. Therefore there is no need to postulate any Dirac equation to determine the behavior and therefore no need to use the inner product derived from dynamics.

5.2.4 Holonomy Group Of The Vielbein Connection

Generalized vielbein allows huge gauge symmetry. An important constraint on physical observables is that they do not depend at all on the gauge chosen to represent the gamma matrices. This is indeed achieved using vielbein connection, which is now quadratic in fermionic oscillator operators. The holonomy group of the vielbein connection is the WCW counterpart of the electro-weak gauge group and its algebra is expected to have same general structure as the algebra of the WCW isometries. In particular, the generators of this algebra should be labeled by conformal weights like the elements of Kac Moody algebras. In present case however conformal weights are complex as the construction of WCW geometry demonstrates.

5.2.5 Realization Of WCW Gamma Matrices In Terms Of Super Symmetry Generators

In string models super symmetry generators behave effectively as gamma matrices and it is very tempting to assume that WCW gamma matrices can be regarded as generators of the symplectic algebra extended to super-symplectic Kac Moody type algebra. The experience with string models suggests also that radial Virasoro algebra extends to Super Virasoro algebra. There are good reasons to expect that WCW Dirac operator and its square give automatically a realization of this algebra. It this is indeed the case, then WCW spinor structure as well as Dirac equation reduces to mere group theory.

One can actually guess the general form of the super-symplectic algebra. The form is a direct generalization of the ordinary super Kac Moody algebra. The complexified super generators S_A are identifiable as WCW gamma matrices:

$$\Gamma_A = S_A \ . \quad (5.2.6)$$

The anti-commutators $\{\Gamma_A^\dagger, \Gamma_B\}_+ = i2J_{A,B}$ define a Hermitian matrix, which is proportional to the Kähler form of the configuration space rather than metric as usually. Only in complex coordinates the anti-commutators equal to the metric numerically. This is, apart from the multiplicative constant n, is expressible as the Poisson bracket of the WCW Hamiltonians H_A and H_B. Therefore one should be able to identify super generators $S_A(r_M)$ for each values of r_M as the counterparts of fluxes. The anti-commutators between the super generators S_A and their Hermitian conjugates should read as

$$\{S_A, S_B^\dagger\}_+ \quad = \quad iQ_m(H_{[A,B]}) \ . \tag{5.2.7}$$

and should be induced directly from the anti-commutation relations of free second quantized spinor fields of the embedding space restricted to the light cone boundary.

The commutation relations between s and super generators follow solely from the transformation properties of the super generators under symplectic transformations, which are same as for the Hamiltonians themselves

$$\{H_{Am}, S_{Bn}\}_- \quad = \quad S_{[Am,Bn]} \ \ , \tag{5.2.8}$$

and are of the same form as in the case of Super-Kac-Moody algebra.

The task is to derive an explicit representation for the super generators S_A in both cases. For obvious reason the spinor fields restricted to the 3-surfaces on the light cone boundary $\delta M^4_+ \times CP_2$ can be used. Leptonic/quark like oscillator operators are used to construct Ramond/NS type algebra.

What is then the strategy that one should follow?

1. WCW Hamiltonians correspond to either magnetic or electric flux Hamiltonians and the conjecture is that these representations are equivalent. It turns out that this electric-magnetic duality generalizes to the level of super charges. It also turns out that quark representation is the only possible option whereas leptonic super charges super-symmetrize the ordinary function algebra of the light cone boundary.
2. The simplest option would be that second quantized embedding space spinors could be used in the definition of super charges. This turns out to not work and one must second quantize the induced spinor fields.
3. The task is to identify a super-symmetric variational principle for the induced spinors: ordinary Dirac action does not work. It turns out that in the most plausible scenario the Kähler-Dirac action varied with respect to *both* embedding space coordinates and spinor fields is the fundamental action principle. The c-number parts of the conserved symplectic charges associated with this action give rise to bosonic conserved charges defining WCW Hamiltonians. The second quantization of the spinor fields reduces to the requirement that super charges and Hamiltonians generate super-symplectic algebra determining the anti-commutation relations for the induced spinor fields.

5.2.6 Central Extension As Symplectic Extension At WCW Level

The earlier attempts to understand the emergence of central extension of super-symplectic algebra were based on the notion of symplectic extension. This general view is not given up although it seems that this abstract approach is not very practical. Symplectic extension emerged originally in the attempts to construct formal expression for the WCW Dirac equation. The rather obvious idea was that the Dirac equation reduces to super Virasoro conditions with Super Virasoro generators involving the Dirac operator of the embedding space. The basic difficulty was the necessity to assign to the gamma matrices of the embedding space fermion number. In the recent formulation the Dirac operator of H does not appear in in the Super Virasoro conditions so that this problem disappears.

The proposal that Super Virasoro conditions should replaced with conditions stating that the commutator of super-symplectic and super Kac-Moody algebras annihilates physical states, looks rather feasible. One could call these conditions as WCW Dirac equation but at this moment I feel that this would be just play with words and mask the group theoretical content of these conditions. In any case, the formulas for the symplectic extension and action of isometry generators on WCW spinor deserve to be summarized.

Symplectic extension

The Abelian extension of the super-symplectic algebra is obtained by an extremely simple trick. Replace the ordinary derivatives appearing in the definition of, say spinorial isometry generator, by the covariant derivatives defined by a coupling to a multiple of the Kähler potential.

$$\begin{aligned} j^{Ak}\partial_k &\rightarrow j^{Ak}D_k \ , \\ D_k &= \partial_k + ikA_k/2 \ . \end{aligned} \tag{5.2.9}$$

where A_k denotes Kähler potential. The reality of the parameter k is dictated by the Hermiticity requirement and also by the requirement that Abelian extension reduces to the standard form in Cartan algebra. k is expected to be integer also by the requirement that covariant derivative corresponds to connection (quantization of magnetic charge).

The commutation relations for the centrally extended generators J^A read:

$$[J^A, J^B] = J^{[A,B]} + ikj^{Ak}J_{kl}j^{Bl} \equiv J^{[A,B]} + ikJ_{AB} \ . \tag{5.2.10}$$

Since Kähler form defines symplectic structure in WCW one can express Abelian extension term as a Poisson bracket of two Hamiltonians

$$J_{AB} \equiv j^{Ak}J_{kl}j^{Bl} = \{H^A, H^B\} \ . \tag{5.2.11}$$

Notice that Poisson bracket is well defined also when Kähler form is degenerate.

The extension indeed has acceptable properties:

1. Jacobi-identities reduce to the form

$$\sum_{cyclic} H^{[A,[B,C]]} = 0 \ , \tag{5.2.12}$$

and therefore to the Jacobi identities of the original Lie- algebra in Hamiltonian representation.

2. In the Cartan algebra Abelian extension reduces to a constant term since the Poisson bracket for two commuting generators must be a multiple of a unit matrix. This feature is clearly crucial for the non-triviality of the Abelian extension and is encountered already at the level of ordinary (q,p) Poisson algebra: although the differential operators ∂_p and ∂_q commute the Poisson bracket of the corresponding Hamiltonians p and q is nontrivial: $\{p,q\} = 1$. Therefore the extension term commutes with the generators of the Cartan subalgebra. Extension is also local $U(1)$ extension since Poisson algebra differs from the Lie-algebra of the vector fields in that it contains constant Hamiltonian ("1" in the commutator), which commutes with all other Hamiltonians and corresponds to a vanishing vector field.

3. For the generators not belonging to Cartan sub-algebra of CH isometries Abelian extension term is not annihilated by the generators of the original algebra and in this respect the extension differs from the standard central extension for the loop algebras. It must be however emphasized that for the super-symplectic algebra generators correspond to products of δM^4_+ and CP_2 Hamiltonians and this means that generators of say δM^4_+-local $SU(3)$ Cartan algebra are non-commuting and the commutator is completely analogous to central extension term since it is symmetric with respect to $SU(3)$ generators.

4. The proposed method yields a trivial extension in the case of Diff^4. The reason is the (four-dimensional!) Diff degeneracy of the Kähler form. Abelian extension term is given by the contraction of the Diff^4 generators with the Kähler potential

$$j^{Ak}J_{kl}j^{Bl} = 0 \ , \tag{5.2.13}$$

which vanishes identically by the Diff degeneracy of the Kähler form. Therefore neither 3- or 4-dimensional Diff invariance is not expected to cause any difficulties. Recall that 4-dimensional Diff degeneracy is what is needed to eliminate time like vibrational excitations from the spectrum of the theory. By the way, the fact that the loop space metric is not Diff degenerate makes understandable the emergence of Diff anomalies in string models [B45, B43] .

5. The extension is trivial also for the other zero norm generators of the tangent space algebra, in particular for the $k_2 = Im(k) = 0$ symplectic generators possible present so that these generators indeed act as genuine $U(1)$ transformations.

6. Concerning the solution of WCW Dirac equation the maximum of Kähler function is expected to be special, much like origin of Minkowski space and symmetric space property suggests that the construction of solutions reduces to this point. At this point the generators and Hamiltonians of the algebra h in the defining Cartan decomposition $g = h + t$ should vanish. h corresponds to integer values of $k_1 = Re(k)$ for Cartan algebra of super-symplectic algebra and integer valued conformal weights n for Super Kac-Moody algebra. The algebra reduces at the maximum to an exceptionally simple form since only central extension contributes to the metric and Kähler form. In the ideal case the elements of the metric and Kähler form could be even diagonal. The degeneracy of the metric might of course pose additional complications.

Super symplectic action on WCW spinor s

The generators of symplectic transformations are obtained in the spinor representation of the isometry group of WCW by the following formal construction. Take isometry generator in the spinor representation and add to the covariant derivative D_k defined by vielbein connection the coupling to the multiple of the Kähler potential: $D_k \rightarrow D_k + ikAk/2$.

$$\begin{aligned} J^A &= j^{Ak}D_k + D_l j_k \Sigma^{kl}/2 \ , \\ &\rightarrow \ \hat{J}^A = j^{Ak}(D_k + ikA_k/2) + D_l j_k^A \Sigma^{kl}/2 \ , \end{aligned} \tag{5.2.14}$$

This induces the required central term to the commutation relations. Introduce complex coordinates and define bosonic creation and annihilation operators as $(1,0)$ and $(0,1)$ parts of the modified isometry generators

$$\begin{aligned} B_A^\dagger &= J_+^A = j^{Ak}(D_k + ... \ , \\ B_A &= J_-^A = j^{A\bar{k}}(D_{\bar{k}} + ... \ . \end{aligned} \tag{5.2.15}$$

where "k" refers now to complex coordinates and "$\bar{k}$" to their conjugates.

Fermionic generators are obtained as the contractions of the complexified gamma matrices with the isometry generators

$$\begin{aligned} \Gamma_A^\dagger &= j^{Ak}\Gamma_k \ , \\ \Gamma_A &= j^{A\bar{k}}\Gamma_{\bar{k}} \ . \end{aligned} \tag{5.2.16}$$

Notice that the bosonic Cartan algebra generators obey standard oscillator algebra commutation relations and annihilate fermionic Cartan algebra generators. Hermiticity condition holds in the sense that creation type generators are hermitian conjugates of the annihilation operator type generators. There are two kinds of representations depending on whether one uses leptonic or quark like oscillator operators to construct the gammas. These will be assumed to correspond to Ramond and NS type generators with the radial plane waves being labeled by integer and half odd integer indices respectively.

The non-vanishing commutators between the Cartan algebra bosonic generators are given by the matrix elements of the Kähler form in the basis of formed by the isometry generators

$$[B_A^\dagger, B_B] = J(j^{A\dagger}, j^B) \equiv J_{\bar{A}B} \ . \tag{5.2.17}$$

and are isometry invariant quantities. The commutators between local $SU(3)$ generators not belonging to Cartan algebra are just those of the local gauge algebra with Abelian extension term added.

The anti-commutators between the fermionic generators are given by the elements of the metric (as opposed to Kähler form in the case of bosonic generators) in the basis formed by the isometry generators

$$\{\Gamma_A\dagger, \Gamma_B\} = 2g(j^{A\dagger}, j^B) \equiv 2g_{\bar{A}B} \ . \tag{5.2.18}$$

and are invariant under isometries. Numerically the commutators and anti-commutators differ only the presence of the imaginary unit and the scale factor R relating the metric and Kähler form to each other (the factor R is same for CP_2 metric and Kähler form).

The commutators between bosonic and fermionic generators are given by

$$[B_A, \Gamma_B] = \Gamma_{[A,B]} \ . \tag{5.2.19}$$

The presence of vielbein and rotation terms in the representation of the isometry generators is essential for obtaining these nice commutations relations. The commutators vanish identically for Cartan algebra generators. From the commutation relations it is clear that Super Kac Moody algebra structure is directly related to the Kähler structure of WCW : the anti-commutator of fermionic generators is proportional to the metric and the commutator of the bosonic generators is proportional to the Kähler form. It is this algebra, which should generate the solutions of the field equations of the theory.

The vielbein and rotational parts of the bosonic isometry generators are quadratic in the fermionic oscillator operators and this suggests the interpretation as the fermionic contribution to the isometry currents. This means that the action of the bosonic generators is essentially non-perturbative since it creates fermion anti-fermion pairs besides exciting bosonic degrees of freedom.

5.2.7 WCW Clifford Algebra As AHyper-Finite Factor Of Type II_1

The naïve expectation is that the trace of the unit matrix associated with the Clifford algebra spanned by WCW sigma matrices is infinite and thus defines an excellent candidate for a source of divergences in perturbation theory. This potential source of infinities remained un-noticed until it became clear that there is a connection with von Neumann algebras [A77]. In fact, for a separable Hilbert space defines a standard representation for so called [A65]. This guarantees that the trace of the unit matrix equals to unity and there is no danger about divergences.

Philosophical ideas behind von Neumann algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation * and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator A belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type II_1 [A65].

The definitions of adopted by von Neumann allow however more general algebras. Type I_n algebras correspond to finite-dimensional matrix algebras with finite traces whereas I_∞ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type III non-trivial traces are always infinite and the notion of trace becomes useless.

von Neumann, Dirac, and Feynman

The association of algebras of type I with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type II_1 as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac based on the notion of delta function, plus the emergence of Feynman graphs, the possibility to formulate the notion of delta function rigorously in terms of distributions, and the emergence of path integral approach meant that von Neumann approach was forgotten by particle physicists.

Algebras of type II_1 have emerged only much later in conformal and topological quantum field theories [A97, A58] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [A49, A70] relate closely to type II_1 factors. In topological quantum computation [B39] based on braid groups [A102] modular S-matrices they play an especially important role.

Clifford algebra of WCW as von Neumann algebra

The Clifford algebra of WCW provides a school example of a hyper-finite factor of type II_1, which means that fermionic sector does not produce divergence problems. Super-symmetry means that also "orbital" degrees of freedom corresponding to the deformations of 3-surface define similar factor. The general theory of hyper-finite factors of type II_1 is very rich and leads to rather detailed understanding of the general structure of S-matrix in TGD framework. For instance, there is a unitary evolution operator intrinsic to the von Neumann algebra defining in a natural manner single particle time evolution. Also a connection with 3-dimensional topological quantum field theories and knot theory, conformal field theories, braid groups, quantum groups, and quantum counterparts of quaternionic and octonionic division algebras emerges naturally. These aspects are discussed in detail in [K95].

5.3 Under What Conditions Electric Charge Is Conserved For The Kähler-Dirac Equation?

One might think that talking about the conservation of electric charge at 21st century is a waste of time. In TGD framework this is certainly not the case.

1. In quantum field theories there are two ways to define em charge: as electric flux over 2-D surface sufficiently far from the source region or in the case of spinor field quantum mechanically as combination of fermion number and vectorial isospin. The latter definition is quantum mechanically more appropriate.
2. There is however a problem. In standard approach to gauge theory Dirac equation in presence of charged classical gauge fields does not conserve electric charge as quantum number: electron

is transformed to neutrino and vice versa. Quantization solves the problem since the non-conservation can be interpreted in terms of emission of gauge bosons. In TGD framework this does not work since one does not have path integral quantization anymore. Preferred extremals carry classical gauge fields and the question whether em charge is conserved arises. Heuristic picture suggests that em charge must be conserved.

It seems that one should pose the well-definedness of spinorial em charge as an additional condition. Well-definedness of em charge is not the only problem. How to avoid large parity breaking effects due to classical Z^0 fields? How to avoid the problems due to the fact that color rotations induced vielbein rotation of weak fields? Does this require that classical weak fields vanish in the regions where the modes of induced spinor fields are non-vanishing?

This condition might be one of the conditions defining what it is to be a preferred extremal/solution of Kähler Dirac equation. It is not however trivial whether this kind of additional condition can be posed unless it follows automatically from the recent formulation for Kähler action and Kähler Dirac action. The common answer to these questions is restriction of the modes of induced spinor field to 2-D string world sheets (and possibly also partonic 2-surfaces) such that the induced weak fields vanish. This makes string/parton picture part of TGD. The vanishing of classical weak fields has also number theoretic interpretation: space-time surfaces would have quaternionic (hyper-complex) tangent space and the 2-surfaces carrying spinor fields complex (hyper-complex) tangent space.

5.3.1 Conservation Of EM Charge For Kähler Dirac Equation

What does the conservation of em charge imply in the case of the Kähler-Dirac equation? The obvious guess that the em charged part of the Kähler-Dirac operator must annihilate the solutions, turns out to be correct as the following argument demonstrates.

1. Em charge as coupling matrix can be defined as a linear combination $Q = aI+bI_3$, $I_3 = J_{kl}\Sigma^{kl}$, where I is unit matrix and I_3 vectorial isospin matrix, J_{kl} is the Kähler form of CP_2, Σ^{kl} denotes sigma matrices, and a and b are numerical constants different for quarks and leptons. Q is covariantly constant in $M^4 \times CP_2$ and its covariant derivatives at space-time surface are also well-defined and vanish.

2. The modes of the Kähler-Dirac equation should be eigen modes of Q. This is the case if the Kähler-Dirac operator D commutes with Q. The covariant constancy of Q can be used to derive the condition

$$\begin{aligned} [D,Q]\,\Psi &= D_1\Psi = 0 \ , \\ D &= \hat{\Gamma}^\mu D_\mu \ , \ \ D_1 = [D,Q] = \hat{\Gamma}_1^\mu D_\mu \ , \ \ \hat{\Gamma}_1^\mu = \left[\hat{\Gamma}^\mu, Q\right] \ . \end{aligned} \tag{5.3.1}$$

Covariant constancy of J is absolutely essential: without it the resulting conditions would not be so simple.

It is easy to find that also $[D_1,Q]\Psi = 0$ and its higher iterates $[D_n,Q]\Psi = 0$, $D_n = [D_{n-1},Q]$ must be true. The solutions of the Kähler-Dirac equation would have an additional symmetry.

3. The commutator $D_1 = [D,Q]$ reduces to a sum of terms involving the commutators of the vectorial isospin $I_3 = J_{kl}\Sigma^{kl}$ with the CP_2 part of the gamma matrices:

$$D_1 = [Q,D] = [I_3,\Gamma_r]\partial_\mu s^r T^{\alpha\mu} D_\alpha \ . \tag{5.3.2}$$

In standard complex coordinates in which $U(2)$ acts linearly the complexified gamma matrices can be chosen to be eigenstates of vectorial isospin. Only the charged flat space complexified gamma matrices Γ^A denoted by Γ^+ and Γ^- possessing charges +1 and -1 contribute to the right hand side. Therefore the additional Dirac equation $D_1\Psi = 0$ states

$$\begin{aligned} D_1\Psi &= [Q,D]\Psi = I_3(A) e_{Ar}\Gamma^A \partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi \\ &= (e_{+r}\Gamma^+ - e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha \Psi = 0 \ . \end{aligned} \tag{5.3.3}$$

The next condition is

$$D_2\Psi = [Q,D]\Psi = (e_{+r}\Gamma^+ + e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha \Psi = 0 \ . \quad (5.3.4)$$

Only the relative sign of the two terms has changed. The remaining conditions give nothing new.

4. These equations imply two separate equations for the two charged gamma matrices

$$\begin{aligned} D_+\Psi &= T_+^\alpha \Gamma^+ D_\alpha \Psi = 0 \ , \\ D_-\Psi &= T_-^\alpha \Gamma^- D_\alpha \Psi = 0 \ , \\ T_\pm^\alpha &= e_{\pm r}\partial_\mu s^r T^{\alpha\mu} \ . \end{aligned} \quad (5.3.5)$$

These conditions state what one might have expected: the charged part of the Kähler-Dirac operator annihilates separately the solutions. The reason is that the classical W fields are proportional to $e_{r\pm}$.

The above equations can be generalized to define a decomposition of the energy momentum tensor to charged and neutral components in terms of vierbein projections. The equations state that the analogs of the Kähler-Dirac equation defined by charged components of the energy momentum tensor are satisfied separately.

5. In complex coordinates one expects that the two equations are complex conjugates of each other for Euclidian signature. For the Minkowskian signature an analogous condition should hold true. The dynamics enters the game in an essential manner: whether the equations can be satisfied depends on the coefficients a and b in the expression $T = aG + bg$ implied by Einstein's equations in turn guaranteeing that the solution ansatz generalizing minimal surface solutions holds true [K10].

6. As a result one obtains three separate Dirac equations corresponding to the neutral part $D_0\Psi = 0$ and charged parts $D_\pm\Psi = 0$ of the Kähler-Dirac equation. By acting on the equations with these Dirac operators one obtains also that the commutators $[D_+, D_-]$, $[D_0, D_\pm]$ and also higher commutators obtained from these annihilate the induced spinor field model. Therefore entire -possibly- infinite-dimensional algebra would annihilate the induced spinor fields. In string model the counterpart of Dirac equation when quantized gives rise to Super-Virasoro conditions. This analogy would suggest that Kähler-Dirac equation gives rise to the analog of Super-Virasoro conditions in 4-D case. But what the higher conditions mean? Could they relate to the proposed generalization to Yangian algebra [A36] [B33, B24, B25]? Obviously these conditions resemble structurally Virasoro conditions $L_n|phys\rangle = 0$ and their supersymmetric generalizations, and might indeed correspond to a generalization of these conditions just as the field equations for preferred extremals could correspond to the Virasoro conditions if one takes seriously the analogy with the quantized string.

What could this additional symmetry mean from the point of view of the solutions of the Kähler-Dirac equation? The field equations for the preferred extremals of Kähler action reduce to purely algebraic conditions in the same manner as the field equations for the minimal surfaces in string model. Could this happen also for the Kähler-Dirac equation and could the condition on charged part of the Dirac operator help to achieve this?

This argument was very general and one can ask for simple ways to realize these conditions. Obviously the vanishing of classical W fields in the region where the spinor mode is non-vanishing defines this kind of condition.

5.3.2 About The Solutions Of Kähler Dirac Equation For Known Extremals

To gain perpective consider first Dirac equation in in H. Quite generally, one can construct the solutions of the ordinary Dirac equation in H from covariantly constant right-handed neutrino spinor playing the role of fermionic vacuum annihilated by the second half of complexified gamma matrices. Dirac equation reduces to Laplace equation for a scalar function and solution can be constructed from this "vacuum" by multiplying with the spherical harmonics of CP_2 and applying Dirac operator [K46]. Similar construction works quite generally thanks to the existence of covariantly constant right handed neutrino spinor. Spinor harmonics of CP_2 are only replaced with those of space-time surface possessing either hermitian structure or Hamilton-Jacobi structure (corresponding to Euclidian and Minkowskian signatures of the induced metric [K10, K96]). What is remarkable is that these solutions possess well-defined em charge although classical W boson fields are present.

This in sense that H d'Alembertian commutes with em charge matrix defined as a linear combination of unit matrix and the covariantly constant matrix $J^{kl}\Sigma_{kl}$ since the commutators of the covariant derivatives give constant Ricci scalar and $J^{kl}\Sigma_{kl}$ term to the d'Alembertian besides scalar d'Alembertian commuting with em charge. Dirac operator itself does not commute with em charge matrix since gamma matrices not commute with em charge matrix.

Consider now Kähler Dirac operator. The square of Kähler Dirac operator contains commutator of covariant derivatives which contains contraction $[\Gamma^\mu, \Gamma^\nu]\, F^{weak}_{\mu\nu}$ which is quadratic in sigma matrices of $M^4 \times CP_2$ and does not reduce to a constant term commuting which em charge matrix. Therefore additional condition is required even if one is satisfies with the commutativity of d'Alembertian with em charge. Stronger condition would be commutativity with the Kähler Dirac operator and this will be considered in the following.

To see what happens one must consider space-time regions with Minkowskian and Euclidian signature. What will be assumed is the existence of Hamilton-Jacobi structure [K10] meaning complex structure in Euclidian signature and hyper-complex plus complex structure in Minkowskian signature. The goal is to get insights about what the condition that spinor modes have a well-defined em charge eigenvalue requires. Or more concretely: is the localization at string world sheets guaranteeing well-defined value of em charge predicted by Kähler Dirac operator or must one introduce this condition separately? One can also ask whether this condition reduces to commutativity/co-commutativity in number theoretic vision.

1. CP_2 type vacuum extremals serve as a convenient test case for the Euclidian signature. In this case the Kähler-Dirac equation reduces to the massless ordinary Dirac equation in CP_2 allowing only covariantly constant right-handed neutrino as solution. Only part of CP_2 so that one give up the constraint that the solution is defined in the entire CP_2. In this case holomorphic solution ansatz obtained by assuming that solutions depend on the coordinates ξ^i, $i = 1, 2$ but not on their conjugates and that the gamma matrices $\Gamma^{\bar{i}}$, $i = 1, 2$, annihilate the solutions, works. The solutions ansatz and its conjugate are of exactly the same form as in case string models where one considers string world sheets instead of CP_2 region.

   The solutions are not restricted to 2-D string world sheets and it is not clear whether one can assign to them a well-defined em charge in any sense. Note that for massless Dirac equation in H one obtains all CP_2 harmonics as solutions, and it is possible to talk about em charge of the solution although solution itself is not restricted to 2-D surface of CP_2.

2. For massless extremals and a very wide class of solutions produced by Hamilton-Jacobi structure - perhaps all solutions representable locally as graphs for map $M^4 \to CP_2$ - canonical momentum densities are light-like and solutions are hyper-holomorphic in the coordinates associated with with string world sheet and annihilated by the conjugate gamma and arbitrary functions in transversal coordinates. This allows localization to string world sheets. The localization is now strictly dynamical and implied by the properties of Kähler Dirac operator.

3. For string like objects one obtains massless Dirac equation in $X^2 \times Y^2 \subset M^4 \times Y^2$, Y^2 a complex 2-surface in CP_2. Homologically trivial geodesic sphere corresponds to the simplest choice for Y^2. Modified Dirac operator reduces to a sum of massless Dirac operators associated with X^2 and Y^2. The most general solutions would have Y^2 mass. Holomorphic solutions reduces to product of hyper-holomorphic and holomorphic solutions and massless 2-D Dirac equation is satisfied in both factors.

   For instance, for S^2 a geodesic sphere and $X^2 = M^2$ one obtains M^2 massivation with mass squared spectrum given by Laplace operator for S^2. Conformal and hyper-conformal symmetries are lost, and one might argue that this is quite not what one wants. One must be however resist the temptation to make too hasty conclusions since the massivation of string like objects is expected to take place. The question is whether it takes place already at the level of fundamental spinor fields or only at the level of elementary particles constructed as many-fermion states of them as twistor Grassmann approach assuming massless M^4 propagators for the fundamental fermions strongly suggests [K86].

4. For vacuum extremals the Kähler Dirac operator vanishes identically so that it does not make sense to speak about solutions.

What can one conclude from these observations?

1. The localization of solutions to 2-D string world sheets follows from Kähler Dirac equation only for the Minkowskian regions representable as graphs of map $M^4 \rightarrow CP_2$ locally. For string like objects and deformations of CP_2 type vacuum extremals this is not expected to take place.
2. It is not clear whether one can speak about well-defined em charge for the holomorphic spinors annihilated by the conjugate gamma matrices or their conjugates. As noticed, for embedding space spinor harmonics this is however possible.
3. Strong form of conformal symmetry and the condition that em charge is well-defined for the nodes suggests that the localization at 2-D surfaces at which the charged parts of induced electroweak gauge fields vanish must be assumed as an additional condition. Number theoretic vision would suggest that these surfaces correspond to 2-D commutative or co-commutative surfaces. The string world sheets inside space-time surfaces would not emerge from theory but would be defined as basic geometric objects.

   This kind of condition would also allow duals of string worlds sheets as partonic 2-surfaces identified number theoretically as co-commutative surfaces. Commutativity and co-commutativity would become essential elemenents of the number theoretical vision.
4. The localization of solutions of the Kähler-Dirac action at string world sheets and partonic 2-surfaces as a constraint would mean induction procedure for Kähler-Dirac matrices from SX^4 to X^2 - that is projection. The resulting em neutral gamma matrices would correspond to tangent vectors of the string world sheet. The vanishing of the projections of charged parts of energy momentum currents would define these surfaces. The conditions would apply both in Minkowskian and Euclidian regions. An alternative interpretation would be number theoretical: these surface would be commutative or co-commutative.

5.3.3 Concrete Realization Of The Conditions Guaranteeing Well-Defined Em Charge

Well-definedness of the em charge is the fundamental condiiton on spinor modes. Physical intuition suggests that also classical Z^0 field should vanish - at least in scales longer than weak scale. Above the condition guaranteeing vanishing of em charge has been discussed at very general level. It has however turned out that one can understand situation by simply posing the simplest condition that one can imagine: the vanishing of classical W and possibly also Z^0 fields inducing mixing of different charge states.

1. Induced W fields mean that the modes of Kähler-Dirac equation do not in general have well-defined em charge. The problem disappears if the induced W gauge fields vanish. This does not yet guarantee that couplings to classical gauge fields are physical in long scales. Also classical Z^0 field should vanish so that the couplings would be purely vectorial. Vectoriality might be true in long enough scales only. If W and Z^0 fields vanish in all scales then electroweak forces are due to the exchanges of corresponding gauge bosons described as string like objects in TGD and represent non-trivial space-time geometry and topology at microscopic scale.
2. The conditions solve also another long-standing interpretational problem. Color rotations induce rotations in electroweak-holonomy group so that the vanishing of all induced weak fields also guarantees that color rotations do not spoil the property of spinor modes to be eigenstates of em charge.

One can study the conditions quite concretely by using the formulas for the components of spinor curvature [L2] (`http://tinyurl.com/z86o5qk`).

1. The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{lllllll} R_{01} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , & R_{23} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , \\ R_{02} & = & e^0\wedge e^2 - e^3\wedge e^1 \ , & R_{31} & = & -e^0\wedge e^2 + e^3\wedge e^1 \ , \\ R_{03} & = & 4e^0\wedge e^3 + 2e^1\wedge e^2 \ , & R_{12} & = & 2e^0\wedge e^3 + 4e^1\wedge e^2 \ . \end{array} \tag{5.3.6}$$

   $R_{01} = R_{23}$ and $R_{03} = -R_{31}$ combine to form purely left handed classical W boson fields and Z^0 field corresponds to $Z^0 = 2R_{03}$.

   Kähler form is given by

$$J \ = \ 2(e^0\wedge e^3 + e^1\wedge e^2) \ . \tag{5.3.7}$$

2. The vanishing of classical weak fields is guaranteed by the conditions

$$\begin{array}{r} e^0\wedge e^1 - e^2\wedge e^3 = 0 \ , \\ e^0\wedge e^2 - e^3\wedge e^1 \ , \\ 4e^0\wedge e^3 + 2e^1\wedge e^2 \ . \end{array} \tag{5.3.8}$$

3. There are many ways to satisfy these conditions. For instance, the condition $e^1 = a\times e^0$ and $e^2 = -a\times e^3$ with arbitrary a which can depend on position guarantees the vanishing of classical W fields. The CP_2 projection of the tangent space of the region carrying the spinor mode must be 2-D.

   Also classical Z^0 vanishes if $a^2 = 2$ holds true. This guarantees that the couplings of induced gauge potential are purely vectorial. One can consider other alternaties. For instance, one could require that only classical Z^0 field or induced Kähler form is non-vanishing and deduce similar condition.

4. The vanishing of the weak part of induced gauge field implies that the CP_2 projection of the region carrying spinor mode is 2-D. Therefore the condition that the modes of induced spinor field are restricted to 2-surfaces carrying no weak fields sheets guarantees well-definedness of em charge and vanishing of classical weak couplings. This condition does not imply string world sheets in the general case since the CP_2 projection of the space-time sheet can be 2-D.

How string world sheets could emerge?

1. Additional consistency condition to neutrality of string world sheets is that Kähler-Dirac gamma matrices have no components orthogonal to the 2-surface in question. Hence various fermionic would flow along string world sheet.
2. If the Kähler-Dirac gamma matrices at string world sheet are expressible in terms of two non-vanishing gamma matrices parallel to string world sheet and sheet and thus define an integrable distribution of tangent vectors, this is achieved. What is important that modified gamma matrices can indeed span lower than 4-D space and often do so as already described. Induced gamma matrices defined always 4-D space so that the restriction of the modes to string world sheets is not possible.
3. String models suggest that string world sheets are minimal surfaces of space-time surface or of embedding space but it might not be necessary to pose this condition separately.

In the proposed scenario string world sheets emerge rather than being postulated from beginning.

1. The vanishing conditions for induced weak fields allow also 4-D spinor modes if they are true for entire spatime surface. This is true if the space-time surface has 2-D projection. One can expect that the space-time surface has foliation by string world sheets and the general solution of K-D equation is continuous superposition of the 2-D modes in this case and discrete one in the generic case.

2. If the CP_2 projection of space-time surface is homologically non-trivial geodesic sphere S^2, the field equations reduce to those in $M^4 \times S^2$ since the second fundamental form for S^2 is vanishing. It is possible to have geodesic sphere for which induced gauge field has only em component?
3. If the CP_2 projection is complex manifold as it is for string like objects, the vanishing of weak fields might be also achieved.
4. Does the phase of cosmic strings assumed to dominate primordial cosmology correspond to this phase with no classical weak fields? During radiation dominated phase 4-D string like objects would transform to string world sheets.Kind of dimensional transmutation would occur.

Right-handed neutrino has exceptional role in K-D action.

1. Electroweak gauge potentials do not couple to ν_R at all. Therefore the vanishing of W fields is un-necessary if the induced gamma matrices do not mix right handed neutrino with left-handed one. This is guaranteed if M^4 and CP_2 parts of Kähler-Dirac operator annihilate separately right-handed neutrino spinor mode. Also ν_R modes can be interpreted as continuous superpositions of 2-D modes and this allows to define overlap integrals for them and induced spinor fields needed to define WCW gamma matrices and super-generators.
2. For covariantly constant right-handed neutrino mode defining a generator of super-symmetries is certainly a solution of K-D. Whether more general solutions of K-D exist remains to be checked out.

5.3.4 Connection With Number Theoretic Vision?

The interesting potential connection of the Hamilton-Jacobi vision to the number theoretic vision about field equations has been already mentioned.

1. The vision that associativity/co-associativity defines the dynamics of space-time surfaces boils down to $M^8 - H$ duality stating that space-time surfaces can be regarded as associative/co-associative surfaces either in M^8 or H [K82, K94]. Associativity reduces to hyper-quaternionicity implying that the tangent/normal space of space-time surface at each point contains preferred sub-space $M^2(x) \subset M^8$ and these sub-spaces forma an integrable distribution. An analogous condition is involved with the definition of Hamilton-Jacobi structure.
2. The octonionic representation of the tangent space of M^8 and H effectively replaces $SO(7,1)$ as tangent space group with its octonionic analog obtained by the replacement of sigma matrices with their octonionic counterparts defined by anti-commutators of gamma matrices. By non-associativity the resulting algebra is not ordinary Lie-algebra and exponentiates to a non-associative analog of Lie group. The original wrong belief was that the reduction takes place to the group G_2 of octonionic automorphisms acting as a subgroup of $SO(7)$. One can ask whether the conditions on the charged part of energy momentum tensor could relate to the reduction of $SO(7,1)$
3. What puts bells ringing is that the Kähler-Dirac equation for the octonionic representation of gamma matrices allows the conservation of electromagnetic charge in the proposed sense. The reason is that the left handed sigma matrices (W charges are left-handed) in the octonionic representation of gamma matrices vanish identically! What remains are vectorial=right-handed em and Z^0 charge which becomes proportional to em charge since its left-handed part vanishes. All spinor modes have a well-defined em charge in the octonionic sense defined by replacing embedding space spinor locally by its octonionic variant? Maybe this could explain why H spinor modes can have well-defined em charge contrary to the naïve expectations.
4. The non-associativity of the octonionic spinors is however a problem. Even non-commutativity poses problems - also at space-time level if one assumes quaternion-real analyticity for the spinor modes. Could one assume commutativity or co-commutativity for the induced spinor modes? This would mean restriction to associative or co-associative 2-surfaces and (hyper-)holomorphic depends on its (hyper-)complex coordinate. The outcome would be a localization to a hyper-commutative of commutative 2-surface, string world sheet or partonic 2-surface.

5. These conditions could also be interpreted by saying that for the Kähler Dirac operator the octonionic induced spinors assumed to be commutative/co-commutative are equivalent with ordinary induced spinors. The well-definedness of em charge for ordinary spinors would correspond to commutativity/co-commutativity for octonionic spinors. Even the Dirac equations based on induced and Kähler-Dirac gamma matrices could be equivalent since it is essentially holomorphy which matters.

To sum up, these considerations inspire to ask whether the associativity/co-associativity of the space-time surface is equivalent with the reduction of the field equations to stringy field equations stating that certain components of the induced metric in complex/Hamilton-Jacobi coordinates vanish in turn guaranteeing that field equations reduce to algebraic identifies following from the fact that energy momentum tensor and second fundamental form have no common components? Commutativity/co-commutativity would characterize fermionic dynamics and would have physical representation as possibility to have em charge eigenspinors. This should be the case if one requires that the two solution ansätze are equivalent.

5.4 Representation Of WCW Metric As Anti-Commutators Of Gamma Matrices Identified As Symplectic Super-Charges

WCW gamma matrices identified as symplectic super Noether charges suggest an elegant representation of WCW metric and Kähler form, which seems to be more practical than the representations in terms of Kähler function or representations guessed by symmetry arguments.

This representation is equivalent with the somewhat dubious representation obtained using symmetry arguments - that is by assuming that the half Poisson brackets of embedding space Hamiltonians defining Kähler form and metric can be lifted to the level of WCW, if the conformal gauge conditions hold true for the spinorial conformal algebra, which is the TGD counterpart of the standard Kac-Moody type algebra of the ordinary strings models. For symplectic algebra the hierarchy of breakings of super-conformal gauge symmetry is possible but not for the standard conformal algebras associated with spinor modes at string world sheets.

5.4.1 Expression For WCW Kähler Metric As Anticommutators As Symplectic Super Charges

During years I have considered several variants for the representation of symplectic Hamiltonians and WCW gamma matrices and each of these proposals have had some weakness. The key question has been whether the Noether currents assignable to WCW Hamiltonians should play any role in the construction or whether one can use only the generalization of flux Hamiltonians.

The original approach based on flux Hamiltonians did not use Noether currents.

1. Magnetic flux Hamiltonians do not refer to the space-time dynamics and imply genuine rather than only effective 2-dimensionality, which is more than one wants. If the sum of the magnetic and electric flux Hamiltonians and the weak form of self duality is assumed, effective 2-dimensionality might be achieved.

   The challenge is to identify the super-partners of the flux Hamiltonians and postulate correct anti-commutation relations for the induced spinor fields to achieve anti-commutation to flux Hamiltonians. It seems that this challenge leads to ad hoc constructions.

2. For the purposes of generalization it is useful to give the expression of flux Hamiltonian. Apart from normalization factors one would have

$$Q(H_A) = \int_{X^2} H_A J_{\mu\nu} dx^\mu \wedge dx^\nu \quad .$$

   Here A is a label for the Hamiltonian of $\delta M^4_\pm \times CP_2$ decomposing to product of $\delta M^4_\pm$ and CP_2 Hamiltonians with the first one decomposing to a product of function of the radial light-like coordinate r_M and Hamiltonian depending on S^2 coordinates. It is natural to assume that

Hamiltonians have well- defined $SO(3)$ and $SU(3)$ quantum numbers. This expressions serves as a natural starting point also in the new approach based on Noether charges.

The approach identifying the Hamiltonians as symplectic Noether charges is extremely natural from physics point of view but the fact that it leads to 3-D expressions involving the induced metric led to the conclusion that it cannot work. In hindsight this conclusion seems wrong: I had not yet realized how profound that basic formulas of physics really are. If the generalization of AdS/CFT duality works, Kähler action can be expressed as a sum of string area actions for string world sheets with string area in the effective metric given as the anti-commutator of the Kähler-Dirac gamma matrices for the string world sheet so that also now a reduction of dimension takes place. This is easy to understand if the classical Noether charges vanish for a sub-algebra of symplectic algebra for preferred extremals.

1. If all end points for strings are possible, the recipe for constructing super-conformal generators would be simple. The embedding space Hamiltonian H_A appearing in the expression of the flux Hamiltonian given above would be replaced by the corresponding symplectic quantum Noether charge $Q(H_A)$ associated with the string defined as 1-D integral along the string. By replacing Ψ or its conjugate with a mode of the induced spinor field labeled by electroweak quantum numbers and conformal weight nm one would obtain corresponding super-charged identifiable as WCW gamma matrices. The anti-commutators of the super-charges would give rise to the elements of WCW metric labelled by conformal weights n_1, n_2 not present in the naïve guess for the metric. If one assumes that the fermionic super-conformal symmetries act as gauge symmetries only $n_i = 0$ gives a non-vanishing matrix element.

   Clearly, one would have weaker form of effective 2-dimensionality in the sense that Hamiltonian would be functional of the string emanating from the partonic 2-surface. The quantum Hamiltonian would also carry information about the presence of other wormhole contacts- at least one- when wormhole throats carry Kähler magnetic monopole flux. If only discrete set for the end points for strings is possible one has discrete sum making possible easy p-adicization. It might happen that integrability conditions for the tangent spaces of string world sheets having vanishing W boson fields do not allow all possible strings.

2. The super charges obtained in this manner are not however entirely satisfactory. The problem is that they involve only single string emanating from the partonic 2-surface. The intuitive expectation is that there can be an arbitrarily large number of strings: as the number of strings is increased the resolution improves. Somehow the super-conformal algebra defined by Hamiltonians and super-Hamiltonians should generalize to allow tensor products of the strings providing more physical information about the 3-surface.

3. Here the idea of Yangian symmetry [K86] suggests itself strongly. The notion of Yangian emerges from twistor Grassmann approach and should have a natural place in TGD. In Yangian algebra one has besides product also co-product, which is in some sense "time-reversal" of the product. What is essential is that Yangian algebra is also multi-local.

   The Yangian extension of the super-conformal algebra would be multi-local with respect to the points of partonic surface (or multi-stringy) defining the end points of string. The basic formulas would be schematically

   $$O_1^A = f^A_{BC} T^B \otimes T^B \ ,$$

   where a summation of B, C occurs and f^A_{BC} are the structure constants of the algebra. The operation can be iterated and gives a hierarchy of n-local operators. In the recent case the operators are n-local symplectic super-charges with unit fermion number and symplectic Noether charges with a vanishing fermion number. It would be natural to assume that also the n-local gamma matrix like entities contribute via their anti-commutators to WCW metric and give multi-local information about the partonic 2-surface and 3-surface.

   The operation generating the algebra well-defined if one an assumes that the second quantization of induced spinor fields is carried out using the standard canonical quantization. One could even assume that the points involved belong to different partonic 2-surfaces belonging even at opposite boundaries of CD. The operation is also well-defined if one assumes that induced spinor fields at different space-time points at boundaries of CD always anti-commute. This could make sense at boundary of CD but lead to problems with embedding space-causality if assumed for the spinor modes at opposite boundaries of CD.

5.4.2 Handful Of Problems With A Common Resolution

Theory building could be compared to pattern recognition or to a solving a crossword puzzle. It is essential to make trials, even if one is aware that they are probably wrong. When stares long enough to the letters which do not quite fit, one suddenly realizes what one particular crossword must actually be and it is soon clear what those other crosswords are. In the following I describe an example in which this analogy is rather concrete.

I will first summarize the problems of ordinary Dirac action based on induced gamma matrices and propose Kähler-Dirac action as their solution.

Problems associated with the ordinary Dirac action

In the following the problems of the ordinary Dirac action are discussed and the notion of Kähler-Dirac action is introduced.

Minimal 2-surface represents a situation in which the representation of surface reduces to a complex-analytic map. This implies that induced metric is hermitian so that it has no diagonal components in complex coordinates $(z, \overline{z})$ and the second fundamental form has only diagonal components of type H^k_{zz}. This implies that minimal surface is in question since the trace of the second fundamental form vanishes. At first it seems that the same must happen also in the more general case with the consequence that the space-time surface is a minimal surface. Although many basic extremals of Kähler action are minimal surfaces, it seems difficult to believe that minimal surface property plus extremization of Kähler action could really boil down to the absolute minimization of Kähler action or some other general principle selecting preferred extremals as Bohr orbits [K21, K82].

This brings in mind a similar long-standing problem associated with the Dirac equation for the induced spinors. The problem is that right-handed neutrino generates super-symmetry only provided that space-time surface and its boundary are minimal surfaces. Although one could interpret this as a geometric symmetry breaking, there is a strong feeling that something goes wrong. Induced Dirac equation and super-symmetry fix the variational principle but this variational principle is not consistent with Kähler action.

One can also question the implicit assumption that Dirac equation for the induced spinors is consistent with the super-symmetry of the WCW geometry. Super-symmetry would obviously require that for vacuum extremals of Kähler action also induced spinor fields represent vacua. This is however not the case. This super-symmetry is however assumed in the construction of WCW geometry so that there is internal inconsistency.

Super-symmetry forces Kähler-Dirac equation

The above described three problems have a common solution. Nothing prevents from starting directly from the hypothesis of a super-symmetry generated by covariantly constant right-handed neutrino and finding a Dirac action which is consistent with this super-symmetry. Field equations can be written as

$$\begin{aligned} D_\alpha T^\alpha_k &= 0 \ , \\ T^\alpha_k &= \frac{\partial}{\partial h^k_{\ \alpha}} L_K \ . \end{aligned} \tag{5.4.1}$$

Here T^α_k is canonical momentum current of Kähler action. If super-symmetry is present one can assign to this current its super-symmetric counterpart

$$\begin{aligned} J^{\alpha k} &= \overline{\nu_R} \Gamma^k T^\alpha_l \Gamma^l \Psi \ , \\ D_\alpha J^{\alpha k} &= 0 \ . \end{aligned} \tag{5.4.2}$$

having a vanishing divergence. The isometry currents currents and super-currents are obtained by contracting T^{α_k} and J^{α_k} with the Killing vector fields of super-symmetries. Note also that the super current

$$J^{\alpha} = \overline{\nu_R} T^{\alpha}_l \Gamma^l \Psi \tag{5.4.3}$$

has a vanishing divergence.

By using the covariant constancy of the right-handed neutrino spinor, one finds that the divergence of the super current reduces to

$$D_{\alpha} J^{\alpha k} = \overline{\nu_R} \Gamma^k T^{\alpha}_l \Gamma^l D_{\alpha} \Psi \ . \tag{5.4.4}$$

The requirement that this current vanishes is guaranteed if one assumes that Kähler-Dirac equation

$$\begin{aligned} \hat{\Gamma}^{\alpha} D_{\alpha} \Psi &= 0 \ , \\ \hat{\Gamma}^{\alpha} &= T^{\alpha}_l \Gamma^l \ . \end{aligned} \tag{5.4.5}$$

This equation must be derivable from a Kähler-Dirac action. It indeed is. The action is given by

$$L = \overline{\Psi} \hat{\Gamma}^{\alpha} D_{\alpha} \Psi \ . \tag{5.4.6}$$

Thus the variational principle exists. For this variational principle induced gamma matrices are replaced with Kähler-Dirac gamma matrices and the requirement

$$D_{\mu} \hat{\Gamma}^{\mu} = 0 \tag{5.4.7}$$

guaranteeing that super-symmetry is identically satisfied if the bosonic field equations are satisfied. For the ordinary Dirac action this condition would lead to the minimal surface property. What sounds strange that the essentially hydrodynamical equations defined by Kähler action have fermionic counterpart: this is very far from intuitive expectations raised by ordinary Dirac equation and something which one might not guess without taking super-symmetry very seriously.

As a matter fact, any mode of Kähler-Dirac equation contracted with second quantized induced spinor field or its conjugate defines a conserved super charge. Also super-symplectic Noether charges and their super counterparts can be assigned to symplectic generators as Noether charges but they need not be conserved.

Second quantization of the K-D action

Second quantization of Kähler-Dirac action is crucial for the construction of the Kähler metric of world of classical worlds as anti-commutators of gamma matrices identified as super-symplectic Noether charges. To get a unique result, the anti-commutation relations must be fixed uniquely. This has turned out to be far from trivial.

1. *Canonical quantization works after all*

The canonical manner to second quantize fermions identifies spinorial canonical momentum densities and their conjugates as $\overline{\Pi} = \partial L_{K_D}/\partial_{\Psi} = \overline{\Psi}\Gamma^t$ and their conjugates. The vanishing of Γ^t at points, where the induced Kähler form J vanishes can cause problems since anti-commutation relations are not internally consistent anymore. This led me to give up the canonical quantization and to consider various alternatives consistent with the possibility that J vanishes. They were admittedly somewhat ad hoc. Correct (anti-)commutation relations for various fermionic Noether currents seem however to fix the anti-commutation relations to the standard ones. It seems that it is better to be conservative: the canonical method is heavily tested and turned out to work quite nicely.

The canonical manner to second quantize fermions identifies spinorial canonical momentum densities and their conjugates as $\overline{\Pi} = \partial L_{K_D}/\partial_\Psi = \overline{\Psi}\Gamma^t$ and their conjugates. The vanishing of Γ^t at points, where the induced Kähler form J vanishes can cause problems since anti-commutation relations are not internally consistent anymore. This led originally to give up the canonical quantization and to consider various alternatives consistent with the possibility that J vanishes. They were admittedly somewhat ad hoc. Correct commutation relations for various fermionic Noether currents seem however to fix the anti-commutation relations to the standard ones.

Consider first the 4-D situation without the localization to 2-D string world sheets. The canonical anti-commutation relations would state $\{\overline{\Pi}, \Psi\} = \delta^3(x,y)$ at the space-like boundaries of the string world sheet at either boundary of CD. At points where J and thus T^t vanishes, canonical momentum density vanishes identically and the equation seems to be inconsistent.

If fermions are localized at string world sheets assumed to always carry a non-vanishing J at their boundaries at the ends of space-time surfaces, the situation changes since Γ^t is non-vanishing. The localization to string world sheets, which are not vacua saves the situation. The problem is that the limit when string approaches vacuum could be very singular and discontinuous. In the case of elementary particle strings are associated with flux tubes carrying monopole fluxes so that the problem disappears.

It is better to formulate the anti-commutation relations for the modes of the induced spinor field. By starting from

$$\{\overline{\Pi}(x), \Psi(y)\} = \delta^1(x,y) \tag{5.4.8}$$

and contracting with $\Psi(x)$ and $\Pi(y)$ and integrating, one obtains using orthonormality of the modes of Ψ the result

$$\{b_m^\dagger, b_n\} = \gamma^0 \delta_{m,n} \tag{5.4.9}$$

holding for the nodes with non-vanishing norm. At the limit $J \to 0$ there are no modes with non-vanishing norm so that one avoids the conflict between the two sides of the equation.

The proposed anti-commutator would realize the idea that the fermions are massive. The following alternative starts from the assumption of 8-D light-likeness.

2. Does one obtain the analogy of SUSY algebra? In super Poincare algebra anti-commutators of super-generators give translation generator: anti-commutators are proportional to $p^k\sigma_k$. Could it be possible to have an anti-commutator proportional to the contraction of Dirac operator $p^k\sigma_k$ of 4-momentum with quaternionic sigma matrices having or 8-momentum with octonionic 8-matrices?

This would give good hopes that the GRT limit of TGD with many-sheeted space-time replaced with a slightly curved region of M^4 in long length scales has large $\mathcal{N}$ SUSY as an approximate symmetry: $\mathcal{N}$ would correspond to the maximal number of oscillator operators assignable to the partonic 2-surface. If conformal invariance is exact, it is just the number of fermion states for single generation in standard model.

1. The first promising sign is that the action principle indeed assigns a conserved light-like 8-momentum to each fermion line at partonic 2-surface. Therefore octonionic representation of sigma matrices makes sense and the generalization of standard twistorialization of four-momentum also. 8-momentum can be characterized by a pair of octonionic 2-spinors $(\lambda, \overline{\lambda})$ such that one has $\lambda\ \overline{\lambda}) = p^k\sigma_k$.
2. Since fermion line as string boundary is 1-D curve, the corresponding octonionic sub-spaces is just 1-D complex ray in octonion space and imaginary axes is defined by the associated imaginary octonion unit. Non-associativity and non-commutativity play no role and it is as if one had light like momentum in say z-direction.
3. One can select the ininitial values of spinor modes at the ends of fermion lines in such a way that they have well-defined spin and electroweak spin and one can also form linear

superpositions of the spin states. One can also assume that the 8-D algebraic variant of Dirac equation correlating M^4 and CP_2 spins is satisfied.

One can introduce oscillator operators $b^{\dagger}_{m,\alpha}$ and $b_{n,\alpha}$ with α denoting the spin. The motivation for why electroweak spin is not included as an index is due to the correlation between spin and electroweak spin. Dirac equation at fermion line implies a complete correlation between directions of spin and electroweak spin: if the directions are same for leptons (convention only), they are opposite for antileptons and for quarks since the product of them defines embedding space chirality which distinguishes between quarks and leptons. Instead of introducing electroweak isospin as an additional correlated index one can introduce 4 kinds of oscillator operators: leptonic and quark-like and fermionic and antifermionic.

4. For definiteness one can consider only fermions in leptonic sector. In hope of getting the analog of SUSY algebra one could modify the fermionic anti-commutation relations such that one has

$$\{b^{\dagger}_{m,\alpha}, b_{n,\beta}\} = \pm i\epsilon_{\alpha\beta}\delta_{m,n} \quad . \tag{5.4.10}$$

Here α is spin label and ϵ is the standard antisymmetric tensor assigned to twistors. The anti-commutator is clearly symmetric also now. The anti-commmutation relations with different signs $\pm$ at the right-hand side distinguish between quarks and leptons and also between fermions and anti-fermions. $\pm = 1$ could be the convention for fermions in lepton sector.

5. One wants combinations of oscillator operators for which one obtains anti-commutators having interpretation in terms of translation generators representing in terms of 8-momentum. The guess would be that the oscillator operators are given by

$$B^{\dagger}_n = b^{\dagger}_{m,\alpha}\lambda^{\alpha} \quad , \quad B_n = \overline{\lambda}^{\alpha} b_{m,\alpha} \quad . \tag{5.4.11}$$

The anti-commutator would in this case be given by

$$\begin{array}{l} \{B^{\dagger}_m, B_n\} = i\overline{\lambda}^{\alpha}\epsilon_{\alpha\beta}\lambda^{\beta}\delta_{m,n} \\ = Tr(p^k\sigma_k)\delta_{m,n} = 2p^0\delta_{m,n} \quad . \end{array} \tag{5.4.12}$$

The inner product is positive for positive value of energy p^0. This form of anti-commutator obviously breaks Lorentz invariance and this us due the number theoretic selection of preferred time direction as that for real octonion unit. Lorentz invariance is saved by the fact that there is a moduli space for the choices of the quaternion units parameterized by Lorentz boosts for CD.

The anti-commutator vanishes for covariantly constant antineutrino so that it does not generate sparticle states. Only fermions with non-vanishing four-momentum do so and the resulting algebra is very much like that associated with a unitary representation of super Poincare algebra.

6. The recipe gives one helicity state for lepton in given mode m (conformal weight). One has also antilepton with opposite helicity with $\pm = -1$ in the formula defining the anti-commutator. In the similar manner one obtains quarks and antiquarks.

7. Contrary to the hopes, one did not obtain the anti-commutator $p^k\sigma_k$ but $Tr(p^0\sigma_0)$. $2p^0$ is however analogous to the action of Dirac operator $p^k\sigma_k$ to a massless spinor mode with "wrong" helicity giving $2p^0\sigma^0$. Massless modes with wrong helicity are expected to appear in the fermionic propagator lines in TGD variant of twistor approach. Hence one might hope that the resulting algebra is consistent with SUSY limit.

The presence of 8-momentum at each fermion line would allow also to consider the introduction of anti-commutators of form $p^k(8)\sigma_k$ directly making $\mathcal{N} = 8$ SUSY at parton level manifest.

This expression restricts for time-like M^4 momenta always to quaternion and one obtains just the standard picture.

8. Only the fermionic states with vanishing conformal weight seem to be realized if the conformal symmetries associated with the spinor modes are realized as gauge symmetries. Super-generators would correspond to the fermions of single generation standard model: 4+4 =8 states altogether. Interestingly, $\mathcal{N} = 8$ correspond to the maximal SUSY for super-gravity. Right-handed neutrino would obviously generate the least broken SUSY. Also now mixing of M^4 helicities induces massivation and symmetry breaking so that even this SUSY is broken. One must however distinguish this SUSY from the super-symplectic conformal symmetry. The space in which SUSY would be realized would be partonic 2-surfaces and this distinguishes it from the usual SUSY. Also the conservation of fermion number and absence of Majorana spinors is an important distinction.

3. What about quantum deformations of the fermionic oscillator algebra?

Quantum deformation introducing braid statistics is of considerable interest. Quantum deformations are essentially 2-D phenomenon, and the experimental fact that it indeed occurs gives a further strong support for the localization of spinors at string world sheets. If the existence of anyonic phases is taken completely seriously, it supports the existence of the hierarchy of Planck constants and TGD view about dark matter. Note that the localization also at partonic 2-surfaces cannot be excluded yet.

I have wondered whether quantum deformation could relate to the hierarchy of Planck constants in the sense that $n = h_{eff}/h$ corresponds to the value of deformation parameter $q = exp(i2\pi/n)$.

A q-deformation of Clifford algebra of WCW gamma matrices is required. Clifford algebra is characterized in terms of anti-commutators replaced now by q-anticommutators. The natural identification of gamma matrices is as complexified gamma matrices. For q-deformation q-anti-commutators would define WCW Kähler metric. The commutators of the supergenerators should still give anti-symmetric sigma matrices. The q-anticommutation relations should be same in the entire sector of WCW considered and be induced from the q-anticommutation relations for the oscillator operators of induced spinor fields at string world sheets, and reflect the fact that permutation group has braid group as covering group in 2-D case so that braid statistics becomes possible.

In [A66] (`http://tinyurl.com/y9e6pg4d`) the q-deformations of Clifford algebras are discussed, and this discussion seems to apply in TGD framework.

1. It is assumed that a Lie-algebra g has action in the Clifford algebra. The q-deformations of Clifford algebra is required to be consistent with the q-deformation of the universal enveloping algebra Ug.
2. The simplest situation corresponds to group $su(2)$ so that Clifford algebra elements are labelled by spin $\pm 1/2$. In this case the q-anticommutor for creation operators for spin up states reduces to an anti-commutator giving q-deformation I_q of unit matrix but for the spin down states one has genuine q-anti-commutator containing besides I_q also number operator for spin up states at the right hand side.
3. The undeformed anti-commutation relations can be witten as

$$P^{+kl}_{ij} a_k a_l = 0 \ , \quad P^{+kl}_{ij} a^{\dagger}_k a^{\dagger}_l = 0 \ , \quad a^i a^{\dagger}_j + P^{ih}_{jk} a^{\dagger}_h a^k = \delta^i_j 1 \ . \tag{5.4.13}$$

   Here $P^{kl}_{ij} = \delta^i_l \delta^j_k$ is the permutator and $P^{+kl}_{ij} = (1+P)/2$ is projector. The q-deformation reduces to a replacement of the permutator and projector with q-permutator P_q and q-projector and P^+_q, which are both fixed by the quantum group.
4. Also the condition that deformed algebra has same Poincare series as the original one is posed. This says that the representation content is not changed that is the dimensions of summands in a representation as direct sum of graded sub-spaces are same for algebra and its q-deformation. If one has quantum group in a strict sense of the word (quasi-triangularity

(genuine braid group) rather that triangularity requiring that the square of the deformed permutator P_q is unit matrix, one can have two situations.

(a) $g = sl(N)$ (special linear group such as $SL(2, F)$, $F = R, C$) or $g = Sp(N = 2n)$ (symplectic group such as $Sp(2) = SL(2, R)$), which is subgroup of $sl(N)$. Creation (annihilation-) operators must form the N-dimensional defining representation of g.

(b) $g = sl(N)$ and one has direct sum of M N-dimensional defining representations of g. The M copies of representation are ordered so that they can be identified as strands of braid so that the deformation makes sense at the space-like ends of string world sheet naturally. q-projector is proportional to so called universal R-matrix.

5. It is also shown that q-deformed oscillator operators can be expressed as polynomials of the ordinary ones.

The following argument suggest that the g must correspond to the minimal choices $sl(2, R)$ (or $su(2)$) in TGD framework.

1. The q-Clifford algebra structure of WCW should be induced from that for the fermionic oscillator algebra. g cannot correspond to $su(2)_{spin} \times su(2)_{ew}$ since spin and weak isospin label fermionic oscillator operators beside conformal weights but must relate closely to this group. The physical reason is that the separate conservation of quark and lepton numbers and light-likeness in 8-D sense imply correlations between the components of the spinors and reduce g.

2. For a given H-chirality (quark/ lepton) 8-D light-likeness forced by massless Dirac equation at the light-like boundary of the string world sheet at parton orbit implies correlation between M^4 and CP_2 chiralities. Hence there are 4+4 spinor components corresponding to fermions and antifermions with physical (creation oeprators) and unphysical (annihilation operators) polarizations. This allows two creation operators with given H-chirality (quark or lepton) and fermion number. Same holds true for antifermions. By fermion number conservation one obtains a reduction to $SU(2)$ doublets and the quantum group would be $sl(2) = sp(2)$ for which “special linear” implies “symplectic”.

5.5 Quantum Criticality And Kähler-Dirac Action

The precise mathematical formulation of quantum criticality has remained one of the basic challenges of quantum TGD. The belief has been that the existence of conserved current for Kähler-Dirac equation are possible if Kähler action is critical for the 3-surface in question in the sense that the deformation in question corresponds to vanishing of second variation of Kähler action. The vanishing of the second variation states that the deformation of the Kähler-Dirac gamma matrix is divergence free just like the Kähler-Dirac gamma matrix itself and is therefore very natural.

2-D conformal invariance accompanies 2-D criticality and allows to satisfy these conditions for spinor modes localized at 2-D surfaces - string world sheets and possibly also partonic 2-surfaces. This localization is in the generic case forced by the conditions that em charge is well-defined for the spinor modes: this requires that classical W fields vanish and also the vanishing of classical Z^0 field is natural -at least above weak scale. Only 2 Kähler-Dirac gamma matrices can be non-vanishing and this is possible only for Kähler-Dirac action.

5.5.1 What Quantum Criticality Could Mean?

Quantum criticality is one of the basic guiding principles of Quantum TGD. What it means mathematically is however far from clear and one can imagine several meanings for it.

1. What is obvious is that quantum criticality implies quantization of Kähler coupling strength as a mathematical analog of critical temperature so that the theory becomes mathematically unique if only single critical temperature is possible. Physically this means the presence of long range fluctuations characteristic for criticality and perhaps assignable to the effective hierarchy of Planck constants having explanation in terms of effective covering spaces of the embedding space. This hierarchy follows from the vacuum degeneracy of Kähler action, which

in turn implies 4-D spin-glass degeneracy. It is easy to interpret the degeneracy in terms of criticality.

2. At more technical level one would expect criticality to correspond to deformations of a given preferred extremal defining a vanishing second variation of Kähler Kähler function or Kähler action.

   (a) For Kähler function this criticality is analogous to thermodynamical criticality. The Hessian matrix defined by the second derivatives of free energy or potential function becomes degenerate at criticality as function of control variables which now would be naturally zero modes not contribution to Kähler metric of WCW but appearing as parameters in it. The bevavior variables correspond to quantum fluctuating degrees of freedom and according to catastrophe theory a big change can in quantum fluctuating degrees of freedom at criticality for zero modes. This would be control of quantum state by varying classical variables. Cusp catastrophe is standard example of this. One can imagined also a situation in which the roles of zero modes and behavior variables change and big jump in the values of zero modes is induced by small variation in behavior variables. This would mean quantum control of classical variables.

   (b) Zero modes controlling quantum fluctuating variables in Kähler function would correspond to vanishing of also second derivatives of potential function at extremum in certain directions so that the matrix defined by second derivatives does not have maximum rank. Entire hierarchy of criticalities is expected and a good finite-dimensional model is provided by the catastrophe theory of Thom [A60]. Cusp catastrophe (see `http://tinyurl.com/yddpfdgo`) [A4] is the simplest catastrophe one can think of, and here the folds of cusp where discontinuous jump occurs correspond to criticality with respect to one control variable and the tip to criticality with respect to both control variables.

3. Quantum criticality makes sense also for Kähler action.

   (a) Now one considers space-time surface connecting which 3-surfaces at the boundaries of CD. The non-determinism of Kähler action allows the possibility of having several space-time sheets connecting the ends of space-time surface but the conditions that classical charges are same for them reduces this number so that it could be finite. Quantum criticality in this sense implies non-determinism analogous to that of critical systems since preferred extremals can co-incide and suffer this kind of bifurcation in the interior of CD. This quantum criticality can be assigned to the hierarchy of Planck constants and the integer n in $h_{eff} = n \times h$ [K29] corresponds to the number of degenerate space-time sheets with same Kähler action and conserved classical charges.

   (b) Also now one expects a hierarchy of criticalitiesandsince criticality and conformal invariance are closely related, a natural conjecture is that the fractal hierarchy of sub-algebras of conformal algebra isomorphic to conformal algebra itself and having conformal weights coming as multiples of n corresponds to the hierarchy of Planck constants. This hierarchy would define a hierarchy of symmetry breakings in the sense that only the sub-algebra would act as gauge symmetries.

   (c) The assignment of this hierarchy with super-symplectic algebra having conformal structure with respect to the light-like radial coordinate of light-cone boundary looks very attractive. An interesting question is what is the role of the super-conformal algebra associated with the isometries of light-cone boundary $R_+ \times S^2$ which are conformal transformations of sphere S^2 with a scaling of radial coordinate compensating the scaling induced by the conformal transformation. Does it act as dynamical or gauge symmetries?

4. I have discussed what criticality could mean for Kähler-Dirac action [K96].

   (a) I have conjectured that it leads to the existence of additional conserved currents defined by the variations which do not affect the value of Kähler action. These arguments are far from being mathematically rigorous and the recent view about the solutions of the Kähler-Dirac equation predicting that the spinor modes are restricted to 2-D string world sheets requires a modification of these arguments.

   (b) The basic challenge is to understand the mechanism making this kind of currents conserved: the same challenge is met already in the case of isometries since embedding space coordinates appear as parameters in Kähler-Dirac action. Kähler-Dirac equation is satisfied if the first variation of the canonical momentum densities contracted with the embedding space gamma matrices annihilates the spinor mode. Situation is analogous to massless Dirac equation: it does not imply the vanishing of four-momentum, only the vanishing of mass. One obtains conserved fermion current associated with deformations only if the deformation of the Kähler-Dirac gamma matrix is divergenceless just like the Kähler-Dirac gamma matrix itself. This conditions requires the vanishing of the second variation of Kähler action.

(c) It is far from obvious that these conditions can be satisfied. The localization of the spinor modes to string world sheets or partonic 2-surfaces guaranteeing in the generic case that em charge is well-defined for spinor modes implies holomorphy allowing to formulate current conservation for the deformations of the space-time surface for second quantized induced spinor field. The crux is that the deformation respects the holomorphy properties of the Kähler-Dirac gamma matrices at string world sheet and thus does not mix Γ^z with $\Gamma^{\overline{z}}$. The deformation of Γ^z has only z-component and also annihilates the holomorphic spinor.

This mechanism is possible only for Kähler-Dirac action since the Kähler-Dirac gamma matrices in directions orthogonal to the 2-surface must vanish and this is not possible for other actions. This also means that energy momentum tensor has rank 2 as a matrix. Cosmic string solutions are an exception since in this case CP_2 projection of space-time surface is 2-D and conditions guaranteing vanishing of classical W fields can be satisfied without the restriction to 2-surface.

The vacuum degeneracy of Kähler action strongly suggests that the number of critical deformations is always infinite and that these deformations define an infinite inclusion hierarchy of super-conformal algebras. This inclusion hierarchy would correspond to a fractal hierarchy of breakings of super-conformal symmetry generalizing the symmetry breaking hierarchies of gauge theories. These super-conformal inclusion hierarchies would realize the inclusion hierarchies for hyper-finite factors of type II_1.

5.5.2 Quantum Criticality And Fermionic Representation Of Conserved Charges Associated With Second Variations Of Kähler Action

It is rather obvious that TGD allows a huge generalizations of conformal symmetries. The development of the understanding of conservation laws has been however slow. Kähler-Dirac action provides excellent candidates for quantum counterparts of Noether charges. The problem is that the embedding space coordinates are in the role of classical external fields and induces spinor fields are second quantized so that it is not at all clear whether one obtains conserved charges.

What does the conservation of the fermionic Noether current require?

The obvious anser to the question of the title is that the conservation of the fermionic current requires the vanishing of the first variation of Kähler-Dirac action with respect to embedding space coordinates. This is certainly true but need not mean vanishing of the second variation of Kähler action as thought first. Hence fermionic conserved currents might be obtained for much more general variations than critical ones.

1. The Kähler-Dirac action assigns to a deformation of the space-time surface a conserved charge expressible as bilinears of fermionic oscillator operators only if the first variation of the Kähler-Dirac action under this deformation vanishes.

   The vanishing of the first variation for the Kähler-Dirac action is equivalent with the vanishing of the second variation for the Kähler action. This can be seen by the explicit calculation of the second variation of the Kähler-Dirac action and by performing partial integration for the terms containing derivatives of Ψ and $\overline{\Psi}$ to give a total divergence representing the difference of the charge at upper and lower boundaries of the causal diamond plus a four-dimensional integral of the divergence term defined as the integral of the quantity

$$\begin{aligned}
\Delta S_D &= \overline{\Psi}\Gamma^k D_\alpha J^\alpha_k \Psi \ , \\
J^\alpha_k &= \frac{\partial^2 L_K}{\partial h^k_\alpha \partial h^l_\beta}\delta h^k_\beta + \frac{\partial^2 L_K}{\partial h^k_\alpha \partial h^l}\delta h^l \ .
\end{aligned} \quad (5.5.1)$$

Here h^k_β denote partial derivative of the embedding space coordinates with respect to space-time coordinates. ΔS_D vanishes if this term vanishes:

$$D_\alpha J^\alpha_k = 0 \ .$$

The condition states the vanishing of the second variation of Kähler action. This can of course occur only for preferred deformations of X^4. One could consider the possibility that these deformations vanish at light-like 3-surfaces or at the boundaries of CD. Note that covariant divergence is in question so that J^α_k does not define conserved classical charge in the general case.

2. This condition is however un-necessarily strong. It is enough that that the deformation of Dirac operator anihilates the spinor mode, which can also change in the deformation. It must be possible to compensate the change of the covariant derivative in the deformation by a gauge transformation which requires that deformations act as gauge transformations on induce gauge potentials. This gives additional constraint and strongly suggests Kac-Moody type algebra for the deformations. Conformal transformations would satisfy this constraint and are suggested by quantum criticality.

3. It is essential that the Kähler-Dirac equation holds true so that the Kähler-Dirac action vanishes: this is needed to cancel the contribution to the second variation coming from the determinant of the induced metric. The condition that the Kähler-Dirac equation is satisfied for the deformed space-time surface requires that also Ψ suffers a transformation determined by the deformation. This gives

$$\delta\Psi = -\frac{1}{D} \times \Gamma^k J^\alpha_k \Psi \ . \quad (5.5.2)$$

Here $1/D$ is the inverse of the Kähler-Dirac operator defining the counterpart of the fermionic propagator.

4. The fermionic conserved currents associated with the deformations are obtained from the standard conserved fermion current

$$J^\alpha = \overline{\Psi}\Gamma^\alpha \Psi \ . \quad (5.5.3)$$

Note that this current is conserved only if the space-time surface is extremal of Kähler action: this is also needed to guarantee Hermiticity and same form for the Kähler-Dirac equation for Ψ and its conjugate as well as absence of mass term essential for super-conformal invariance. Note also that ordinary divergence rather only covariant divergence of the current vanishes.

The conserved currents are expressible as sums of three terms. The first term is obtained by replacing Kähler-Dirac gamma matrices with their increments in the deformation keeping Ψ and its conjugate constant. Second term is obtained by replacing Ψ with its increment $\delta\Psi$. The third term is obtained by performing same operation for $\delta\overline{\Psi}$.

$$J^\alpha = \overline{\Psi}\Gamma^k J^\alpha_k \Psi + \overline{\Psi}\hat{\Gamma}^\alpha \delta\Psi + \delta\overline{\Psi}\hat{\Gamma}^\alpha \Psi \ . \quad (5.5.4)$$

These currents provide a representation for the algebra defined by the conserved charges analogous to a fermionic representation of Kac-Moody algebra.

5. Also conserved super charges corresponding to super-conformal invariance are obtained. The first class of super currents are obtained by replacing Ψ or $\overline{\Psi}$ right handed neutrino spinor or its conjugate in the expression for the conserved fermion current and performing the above procedure giving two terms since nothing happens to the covariantly constant right handed-neutrino spinor. Second class of conserved currents is defined by the solutions of the Kähler-Dirac equation interpreted as c-number fields replacing Ψ or $\overline{\Psi}$ and the same procedure gives three terms appearing in the super current.

6. The existence of vanishing of second variations is analogous to criticality in systems defined by a potential function for which the rank of the matrix defined by second derivatives of the potential function vanishes at criticality. Quantum criticality becomes the prerequisite for the existence of quantum theory since fermionic anti-commutation relations in principle can be fixed from the condition that the algebra in question is equivalent with the algebra formed by the vector fields defining the deformations of the space-time surface defining second variations. Quantum criticality in this sense would also select preferred extremals of Kähler action as analogs of Bohr orbits and the spectrum of preferred extremals would be more or less equivalent with the expected existence of infinite-dimensional symmetry algebras.

It is far from obvious that the criticality conditions or even the weaker conditions guaranteing the existence of (say) isometry charges can be satisfied. It seems that the restriction of spinor modes to 2-D surfaces - string world sheets and possibly also partonic 2-surfaces - implied by the condition that em charge is well-define for them, is the manner to achieve this. The reason is that conformal invariance allows complexification of the Kähler-Dirac gamma matrices and allows to construct spinor modes as holomorphic modes and their conjugates. Holomorphy reduces K-D equation to algebraic condition that Γ^z annihilates the spinor mode. If this is true also the deformation of Γ^z then the existince of conserved current follows. It is essential that only two Kähler-Dirac gamma matrices are non-vanishing and this is possible only for Kähler-Dirac action.

About the general structure of the algebra of conserved charges

Some general comments about the structure of the algebra of conserved charges are in order.

1. Any Cartan algebra of the isometry group $P \times SU(3)$ (there are two types of them for P corresponding to linear and cylindrical Minkowski coordinates) defines critical deformations (one could require that the isometries respect the geometry of CD). The corresponding second order charges for Kähler action are conserved but vanish since the corresponding conjugate coordinates are cyclic for the Kähler metric and Kähler form so that the conserved current is proportional to the gradient of a Killing vector field which is constant in these coordinates.

2. Contrary to the original conclusion, the corresponding fermionic charges expressible as fermionic bilinears are first order in deformation and do not vanish! Four-momentum and color quantum numbers are defined for Kähler action as classical conserved quantities and for Kähler-Dirac action as quantal charges.

Critical manifold is infinite-dimensional for Kähler action

Some examples might help to understand what is involved.

1. The action defined by four-volume gives a first glimpse about what one can expect. In this case Kähler-Dirac gamma matrices reduce to the induced gamma matrices. Second variations satisfy d'Alembert type equation in the induced metric so that the analogs of massless fields are in question. Mass term is present only if some dimensions are compact. The vanishing of excitations at light-like boundaries is a natural boundary condition and might well imply that the solution spectrum could be empty. Hence it is quite possible that four-volume action leads to a trivial theory.

2. For the vacuum extremals of Kähler action the situation is different. There exists an infinite number of second variations and the classical non-determinism suggests that deformations vanishing at the light-like boundaries exist. For the canonical embedding of M^4 the equation for second variations is trivially satisfied. If the CP_2 projection of the vacuum extremal is

one-dimensional, the second variation contains a non-vanishing term and an equation analogous to massless d'Alembert equation for the increments of CP_2 coordinates is obtained. Also for the vacuum extremals of Kähler action with 2-D CP_2 projection all terms involving induced Kähler form vanish and the field equations reduce to d'Alembert type equations for CP_2 coordinates. A possible interpretation is as the classical analog of Higgs field. For the deformations of non-vacuum extremals this would suggest the presence of terms analogous to mass terms: these kind of terms indeed appear and are proportional to δs_k. M^4 degrees of freedom decouple completely and one obtains QFT type situation.

3. The physical expectation is that at least for the vacuum extremals the critical manifold is infinite-dimensional. The notion of finite measurement resolution suggests infinite hierarchies of inclusions of hyper-finite factors of type II_1 possibly having interpretation in terms of inclusions of the super conformal algebras defined by the critical deformations.

4. The properties of Kähler action give support for this expectation. The critical manifold is infinite-dimensional in the case of vacuum extremals. Canonical embedding of M^4 would correspond to maximal criticality analogous to that encountered at the tip of the cusp catastrophe. The natural guess would be that as one deforms the vacuum extremal the previously critical degrees of freedom are transformed to non-critical ones. The dimension of the critical manifold could remain infinite for all preferred extremals of the Kähler action. For instance, for cosmic string like objects any complex manifold of CP_2 defines cosmic string like objects so that there is a huge degeneracy is expected also now. For CP_2 type vacuum extremals M^4 projection is arbitrary light-like curve so that also now infinite degeneracy is expected for the deformations.

This leads to the conjecture that the critical deformations correspond to sub-algebras of super-conformal algebras with conformal weights coming as integer multiples of fixed integer m. One would have infinite hierarchy of breakings of conformal symmetry labelled by m. The super-conformal algebras would be effectively m-dimensional. Since all commutators with the critical sub-algebra would create zero energy states. In ordinary conformal field theory one have maximal criticality corresponding to $m = 1$.

Critical super-algebra and zero modes

The relationship of the critical super-algebra to WCW geometry is interesting.

1. The vanishing of the second variation plus the identification of Kähler function as a Kähler action for preferred extremals means that the critical variations are orthogonal to all deformations of the space-time surface with respect to the WCW metric.

   The original expectation was that critical deformations correspond to zero modes but this interpretation need not be correct since critical deformations can leave 3-surface invariant but affect corresponding preferred extremal: this would conform with the non-deterministic character of the dynamics which is indeed the basic signature of criticality. Rather, critical deformations are limiting cases of ordinary deformations acting in quantum fluctuating degrees of freedom.

   This conforms with the fact that WCW metric vanishes identically for canonically imbedded M^4 and that Kähler action has fourth order terms as first non-vanishing terms in perturbative expansion (for Kähler-Dirac the expansion is quadratic in deformation).

   Therefore the super-conformal algebra associated with the critical deformations has genuine physical content.

2. Since the action of X^4 local Hamiltonians of $\delta M^4_\times CP_2$ corresponds to the action in quantum fluctuating degrees of freedom, critical deformations cannot correspond to this kind of Hamiltonians.

3. The notion of finite measurement resolution suggests that the degrees of freedom which are below measurement resolution correspond to vanishing gauge charges. The sub-algebras of critical super-conformal algebra for which charges annihilate physical states could correspond to this kind of gauge algebras.

4. The conserved super charges associated with the vanishing second variations cannot give WCW metric as their anti-commutator. This would also lead to a conflict with the effective 2-dimensionality stating that WCW line-element is expressible as sum of contribution coming from partonic 2-surfaces as also with fermionic anti-commutation relations.

Connection with quantum criticality

The notion of quantum criticality of TGD Universe was originally inspired by the question how to make TGD unique if Kähler function for WCW is defined by the Kähler action for a preferred extremal assignable to a given 3-surface. Vacuum functional defined by the exponent of Kähler function is analogous to thermodynamical weight and the obviou idea with Kähler coupling strength taking the role of temperature. The obvious idea was that the value of Kähler coupling strength is analogous to critical temperature so that TGD would be more or less uniquely defined.

To understand the delicacies it is convenient to consider various variations of Kähler action first.

1. The variation can leave 3-surface invariant but modify space-time surface such that Kähler action remains invariant. In this case infinitesimal deformation reduces to a diffeomorphism at space-like 3-surface and perhaps also at light-like 3-surfaces. In this case the correspondence between X^3 and $X^4(X^3)$ would not be unique and one would have non-deterministic dynamics characteristic for critical systems. This criticality would correspond to criticality of Kähler action at X^3. Note that the original working hypothesis was that $X^4(X^3)$ is unique. The failure of the strict classical determinism implying spin glass type vacuum degeneracy indeed suggets that this is the case.
2. The variation could act on zero modes which do not affect Kähler metric which corresponds to (1, 1) part of Hessian in complex coordinates for WCW . Only the zero modes characterizing 3-surface appearing as parameters in the metric WCW would be affected and the result would be a generalization of conformal transformation. Kähler function would change but only due to the change in zero modes. These transformations do not seem to correspond to critical transformations since Kähler function changes.
3. The variation could act on 3-surface both in zero modes and dynamical degrees of freedom represented by complex coordinates. It would of course affect also the space-time surface. Criticality for Kähler function would mean that Kähler metric has zero modes at X^3 meaning that (1, 1) part of Hessian is degenerate. This could mean that in the vicinity of X^3 the Kähler form has non-definite signature: physically this is unacceptable since inner product in Hilbert space would not be positive definite.

Critical transformations might relate closely to the coset space decomposition of WCW to a union of coset spaces G/H labelled by zero modes.

1. The critical deformations leave 3-surface X^3 invariant as do also the transformations of H associated with X^3. If H affects $X^4(X^3)$ and corresponds to critical transformations then critical transformation would extend WCW to a bundle for which 3-surfaces would be base points and preferred extremals $X^4(X^3)$ would define the fiber. Gauge invariance with respect to H would generalize the assumption that $X^4(X^3)$ is unique.
2. Critical deformations could correspond to H or sub-group of H (which dependes on X^3). For other 3-surfaces than X^3 the action of H is non-trivial as the case of $CP_2 = SU(3)/U(2)$ makes easy to understand.
3. A possible identification of Lie-algebra of H is as a sub-algebra of Virasoro algebra associated with the symplectic transformations of $\delta M^4 \times CP_2$ and acting as diffeomorphisms for the light-like radial coordinate of δM^4_+. The sub-algebras of Virasoro algebra have conformal weights coming as integer multiplies of a given conformal weight m and form inclusion hierarchies suggesting a direct connection with finite measurement resolution realized in terms of inclusions of hyperfinite factors of type II_1. For $m > 1$ one would have breaking of maximal conformal symmetry. The action of these Virasoro algebra on symplectic algebra would make the corresponding sub-algebras gauge degrees of freedom so that the number of symplectic

generators generating non-gauge transformations would be finite. This result is not surprising since also for 2-D critical systems criticality corresponds to conformal invariance acting as local scalings.

The vanishing of the second variation for some deformations means that the system is critical, in the recent case quantum critical. Basic example of criticality is bifurcation diagram for cusp catastrophe. Quantum criticality realized as the vanishing of the second variation gives hopes about a more or less unique identification of preferred extremals and considered alternative identifications such as absolute minimization of Kähler action which is just the opposite of criticality.

One must be very cautious here: there are two criticalities: one for the extremals of Kähler action with respect to deformations of four-surface and second for the Kähler function itself with respect to deformations of 3-surface: these criticalities are not equivalent since in the latter case variation respects preferred extremal property unlike in the first case.

1. The criticality for preferred extremals would make 4-D criticality a property of all physical systems.
2. The criticality for Kähler function would be 3-D and might hold only for very special systems. In fact, the criticality means that some eigenvalues for the Hessian of Kähler function vanish and for nearby 3-surfaces some eigenvalues are negative. On the other hand the Kähler metric defined by (1, 1) part of Hessian in complex coordinates must be positive definite. Thus criticality might imply problems.

   This allows and suggests non-criticality of Kähler function coming from Kähler action for Euclidian space-time regions: this is mathematically the simplest situation since in this case there are no zero modes causing troubles in Gaussian approximation to functional integral. The Morse function coming from Kähler action in Minkowskian as imaginary contribution analogous to that appearing in path integral could be critical and allow non-definite signature in principle. In fact this is expected by the defining properties of Morse function.
3. The almost 2-dimensionality implied by strong form of holography suggests that the interior degrees of freedom of 3-surface can be regarded almost gauge degrees of freedom and that this relates directly to generalised conformal symmetries associated with symplectic isometries of WCW . These degrees of freedom are not critical in the sense inspired by G/H decomposition. The only plausible interaction seems to be that these degrees of freedom correspond to deformations in zero modes.

Both the super-symmetry of D_K and conservation Dirac Noether currents for Kähler-Dirac action have thus a connection with quantum criticality.

1. Finite-dimensional critical systems defined by a potential function $V(x^1, x^2, ..)$ are characterized by the matrix defined by the second derivatives of the potential function and the rank of system classifies the levels in the hierarchy of criticalities. Maximal criticality corresponds to the complete vanishing of this matrix. Thom's catastrophe theory classifies these hierarchies, when the numbers of behavior and control variables are small (smaller than 5). In the recent case the situation is infinite-dimensional and the criticality conditions give additional field equations as existence of vanishing second variations of Kähler action.
2. The vacuum degeneracy of Kähler action allows to expect that this kind infinite hierarchy of criticalities is realized. For a general vacuum extremal with at most 2-D CP_2 projection the matrix defined by the second variation vanishes because $J_{\alpha\beta} = 0$ vanishes and also the matrix $(J^{\alpha}_{\ k} + J_k^{\ \alpha})(J^{\beta}_{\ l} + J_l^{\ \beta})$ vanishes by the antisymmetry $J^{\alpha}_{\ k} = -J_k^{\ \alpha}$.

   The formulation of quantal version of Equivalence Principle (EP) in string picture demonstrates that the conservation of of fermionic Noether currents defining gravitational four-momentum and other Poincare quantum numbers requires that the deformation of the Kähler-Dirac equation obtained by replacing Kähler-Dirac gamma matrices with their deformations is also satisfied. Holomorphy can guarantee this. The original wrong conclusion was that this condition is equivalent with much stronger condition stating the vanishing of the second variation of Kähler action, which it is not. There is analogy for this: massless Dirac equation does not imply the vanishing of four-momentum.

3. Conserved bosonic and fermionic Noether charges would characterize quantum criticality. In particular, the isometries of the embedding space define conserved currents represented in terms of the fermionic oscillator operators if the second variations defined by the infinitesimal isometries vanish for the Kähler-Dirac action. For vacuum extremals the dimension of the critical manifold is infinite: maybe there is hierarchy of quantum criticalities for which this dimension decreases step by step but remains always infinite. This hierarchy could closely relate to the hierarchy of inclusions of hyper-finite factors of type II_1. Also the conserved charges associated with super-symplectic and Super Kac-Moody algebras would require infinite-dimensional critical manifold defined by the spectrum of second variations.

4. Phase transitions are characterized by the symmetries of the phases involved with the transitions, and it is natural to expect that dynamical symmetries characterize the hierarchy of quantum criticalities. The notion of finite quantum measurement resolution based on the hierarchy of Jones inclusions indeed suggests the existence of a hierarchy of dynamical gauge symmetries characterized by gauge groups in ADE hierarchy [K29] with degrees of freedom below the measurement resolution identified as gauge degrees of freedom.

5. Does this criticality have anything to do with the criticality against the phase transitions changing the value of Planck constant? If the geodesic sphere S^2_I for which induced Kähler form vanishes corresponds to the back of the CP_2 book (as one expects), this could be the case. The homologically non-trivial geodesic sphere S^12_{II} is as far as possible from vacuum extremals. If it corresponds to the back of CP_2 book, cosmic strings would be quantum critical with respect to phase transition changing Planck constant. They cannot however correspond to preferred extremals.

5.5.3 Preferred Extremal Property As Classical Correlate For Quantum Criticality, Holography, And Quantum Classical Correspondence

The Noether currents assignable to the Kähler-Dirac equation are conserved only if the first variation of the Kähler-Dirac operator D_K defined by Kähler action vanishes. This is equivalent with the vanishing of the second variation of Kähler action -at least for the variations corresponding to dynamical symmetries having interpretation as dynamical degrees of freedom which are below measurement resolution and therefore effectively gauge symmetries.

The vanishing of the second variation in interior of $X^4(X^3_l)$ is what corresponds exactly to quantum criticality so that the basic vision about quantum dynamics of quantum TGD would lead directly to a precise identification of the preferred extremals. Something which I should have noticed for more than decade ago! The question whether these extremals correspond to absolute minima remains however open.

The vanishing of second variations of preferred extremals -at least for deformations representing dynamical symmetries, suggests a generalization of catastrophe theory of Thom, where the rank of the matrix defined by the second derivatives of potential function defines a hierarchy of criticalities with the tip of bifurcation set of the catastrophe representing the complete vanishing of this matrix. In the recent case this theory would be generalized to infinite-dimensional context. There are three kind of variables now but quantum classical correspondence (holography) allows to reduce the types of variables to two.

1. The variations of $X^4(X^3_l)$ vanishing at the intersections of $X^4(X^3_l)$ with the light-like boundaries of causal diamonds CD would represent behavior variables. At least the vacuum extremals of Kähler action would represent extremals for which the second variation vanishes identically (the "tip" of the multi-furcation set).

2. The zero modes of Kähler function would define the control variables interpreted as classical degrees of freedom necessary in quantum measurement theory. By effective 2-dimensionality (or holography or quantum classical correspondence) meaning that the configuration space metric is determined by the data coming from partonic 2-surfaces X^2 at intersections of X^3_l with boundaries of CD, the interiors of 3-surfaces X^3 at the boundaries of CDs in rough sense correspond to zero modes so that there is indeed huge number of them. Also the variables characterizing 2-surface, which cannot be complexified and thus cannot contribute to the Kähler metric of WCW represent zero modes. Fixing the interior of the 3-surface would mean

fixing of control variables. Extremum property would fix the 4-surface and behavior variables if boundary conditions are fixed to sufficient degree.

3. The complex variables characterizing X^2 would represent third kind of variables identified as quantum fluctuating degrees of freedom contributing to the WCW metric. Quantum classical correspondence requires 1-1 correspondence between zero modes and these variables. This would be essentially holography stating that the 2-D "causal boundary" X^2 of $X^3(X^2)$ codes for the interior. Preferred extremal property identified as criticality condition would realize the holography by fixing the values of zero modes once X^2 is known and give rise to the holographic correspondence $X^2 \rightarrow X^3(X^2)$. The values of behavior variables determined by extremization would fix then the space-time surface $X^4(X^3_l)$ as a preferred extremal.

4. Clearly, the presence of zero modes would be absolutely essential element of the picture. Quantum criticality, quantum classical correspondence, holography, and preferred extremal property would all represent more or less the same thing. One must of course be very cautious since the boundary conditions at X^3_l involve normal derivative and might bring in delicacies forcing to modify the simplest heuristic picture.

5. There is a possible connection with the notion of self-organized criticality [B6] introduced to explain the behavior of systems like sand piles. Self-organization in these systems tends to lead "to the edge". The challenge is to understand how system ends up to a critical state, which by definition is unstable. Mechanisms for this have been discovered and based on phase transitions occurring in a wide range of parameters so that critical point extends to a critical manifold. In TGD Universe quantum criticality suggests a universal mechanism of this kind. The criticality for the preferred extremals of Kähler action would mean that classically all systems are critical in well-defined sense and the question is only about the degree of criticality. Evolution could be seen as a process leading gradually to increasingly critical systems. One must however distinguish between the criticality associated with the preferred extremals of Kähler action and the criticality caused by the spin glass like energy landscape like structure for the space of the maxima of Kähler function.

5.5.4 Quantum Criticality And Electroweak Symmetries

In the following quantum criticali and electroweak symmetries are discussed for Kähler-Dirac action.

What does one mean with quantum criticality?

Quantum criticality is one of the basic guiding principles of Quantum TGD. What it means mathematically is however far from clear and one can imagine several meanings for it.

1. What is obvious is that quantum criticality implies quantization of Kähler coupling strength as a mathematical analog of critical temperature so that the theory becomes mathematically unique if only single critical temperature is possible. Physically this means the presence of long range fluctuations characteristic for criticality and perhaps assignable to the effective hierarchy of Planck constants having explanation in terms of effective covering spaces of the embedding space. This hierarchy follows from the vacuum degeneracy of Kähler action, which in turn implies 4-D spin-glass degeneracy. It is easy to interpret the degeneracy in terms of criticality.

2. At more technical level one would expect criticality to corresponds to deformations of a given preferred extremal defining a vanishing second variation of Kähler Kähler function or Kähler action.

   (a) For Kähler function this criticality is analogous to thermodynamical criticality. The Hessian matrix defined by the second derivatives of free energy or potential function becomes degenerate at criticality as function of control variables which now would be naturally zero modes not contribution to Kähler metric of WCW but appearing as parameters in it. The bevavior variables correspond to quantum fluctuating degrees of freedom and according to catastrophe theory a big change can in quantum fluctuating degrees of freedom at criticality for zero modes. This would be control of quantum state by varying classical variables. Cusp catastrophe is standard example of this. One can imagined also a situation in which the roles of zero modes and behavior variables change and big jump in the values of zero modes is induced by small variation in behavior variables. This would mean quantum control of classical variables.

(b) Zero modes controlling quantum fluctuating variables in Kähler function would correspond to vanishing of also second derivatives of potential function at extremum in certain directions so that the matrix defined by second derivatives does not have maximum rank. Entire hierarchy of criticalities is expected and a good finite-dimensional model is provided by the catastrophe theory of Thom [A60]. Cusp catastrophe (see `http://tinyurl.com/yddpfdgo`) [A4] is the simplest catastrophe one can think of, and here the folds of cusp where discontinuous jump occurs correspond to criticality with respect to one control variable and the tip to criticality with respect to both control variables.

3. Quantum criticality makes sense also for Kähler action.

   (a) Now one considers space-time surface connecting which 3-surfaces at the boundaries of CD. The non-determinism of Kähler action allows the possibility of having several space-time sheets connecting the ends of space-time surface but the conditions that classical charges are same for them reduces this number so that it could be finite. Quantum criticality in this sense implies non-determinism analogous to that of critical systems since preferred extremals can co-incide and suffer this kind of bifurcation in the interior of CD. This quantum criticality can be assigned to the hierarchy of Planck constants and the integer n in $h_{eff} = n \times h$ [K29] corresponds to the number of degenerate space-time sheets with same Kähler action and conserved classical charges.

   (b) Also now one expects a hierarchy of criticalitiesandsince criticality and conformal invariance are closely related, a natural conjecture is that the fractal hierarchy of sub-algebras of conformal algebra isomorphic to conformal algebra itself and having conformal weights coming as multiples of n corresponds to the hierarchy of Planck constants. This hierarchy would define a hierarchy of symmetry breakings in the sense that only the sub-algebra would act as gauge symmetries.

   (c) The assignment of this hierarchy with super-symplectic algebra having conformal structure with respect to the light-like radial coordinate of light-cone boundary looks very attractive. An interesting question is what is the role of the super-conformal algebra associated with the isometries of light-cone boundary $R_+ \times S^2$ which are conformal transformations of sphere S^2 with a scaling of radial coordinate compensating the scaling induced by the conformal transformation. Does it act as dynamical or gauge symmetries?

4. I have discussed what criticality could mean for Kähler-Dirac action [K96].

   (a) I have conjectured that it leads to the existence of additional conserved currents defined by the variations which do not affect the value of Kähler action. These arguments are far from being mathematically rigorous and the recent view about the solutions of the Kähler-Dirac equation predicting that the spinor modes are restricted to 2-D string world sheets requires a modification of these arguments.

   (b) The basic challenge is to understand the mechanism making this kind of currents conserved: the same challenge is met already in the case of isometries since embedding space coordinates appear as parameters in Kähler-Dirac action. The existence of conserved currents does not actually require the vanishing of the second variation of Kähler action as claimed earlier. It is enough that the first variation of the canonical momentum densities contracted with the embedding space gamma matrices annihilates the spinor mode. Situation is analogous to massless Dirac equation: it does not imply the vanishing of four-momentum, only the vanishing of mass. Hence conserved currents are obtained also outside the quantum criticality.

   (c) It is far from obvious that these conditions can be satisfied. The localization of the spinor modes to string world sheets or partonic 2-surfaces guaranteeing in the generaic case that em charge is well-defined for spinor modes implies holomorphy allowing to formulate current conservation for currents associated with the deformations of the space-time surface for second quantized induced spinor field. The crux is that the deformation respects the holomorphy properties of the modified gamma matrices at string world sheet and thus does not mix Γ^z with $\Gamma^{\overline{z}}$. The deformation of Γ^z has only z-component and also annihilates the holomorphic spinor. This mechanism is possible only for Kähler-Dirac action since the Kähler-Dirac gamma matrices in directions orthogonal to the 2-surface must vanish and this is not possible for other actions. This also means that energy momentum tensor has rank 2 as matrix. Cosmic string solutions are an exception since in this case CP_2 projection of space-time surface is 2-D and conditions guaranteing vanishing of classical W fields can be satisfied.

In the following these arguments are formulated more precisely. The unexpected result is that critical deformations induce conformal scalings of the modified metric and electro-weak gauge transformations of the induced spinor connection at X^2. Therefore holomorphy brings in the Kac-Moody symmetries associated with isometries of H (gravitation and color gauge group) and quantum criticality those associated with the holonomies of H (electro-weak-gauge group) as additional symmetries.

The variation of modes of the induced spinor field in a variation of space-time surface respecting the preferred extremal property

Consider first the variation of the induced spinor field in a variation of space-time surface respecting the preferred extremal property. The deformation must be such that the deformed Kähler-Dirac operator D annihilates the modified mode. By writing explicitly the variation of the Kähler-Dirac action (the action vanishes by Kähler-Dirac equation) one obtains deformations and requiring its vanishing one obtains

$$\delta\Psi = D^{-1}(\delta D)\Psi \ . \tag{5.5.5}$$

D^{-1} is the inverse of the Kähler-Dirac operator defining the analog of Dirac propagator and δD defines vertex completely analogous to $\gamma^k\delta A_k$ in gauge theory context. The functional integral over preferred extremals can be carried out perturbatively by expressing δD in terms of δh^k and one obtains stringy perturbation theory around X^2 associated with the preferred extremal defining maximum of Kähler function in Euclidian region and extremum of Kähler action in Minkowskian region (stationary phase approximation).

What one obtains is stringy perturbation theory for calculating n-points functions for fermions at the ends of braid strands located at partonic 2-surfaces and representing intersections of string world sheets and partonic 2-surfaces at the light-like boundaries of CDs. δD- or more precisely, its partial derivatives with respect to functional integration variables - appear atthe vertices located anywhere in the interior of X^2 with outgoing fermions at braid ends. Bosonic propagators are replaced with correlation functions for δh^k. Fermionic propagator is defined by D^{-1}.

After 35 years or hard work this provides for the first time a reasonably explicit formula for the N-point functions of fermions. This is enough since by bosonic emergence these N-point functions define the basic building blocks of the scattering amplitudes. Note that bosonic emergence states that bosons corresponds to wormhole contacts with fermion and anti-fermion at the opposite wormhole throats.

What critical modes could mean for the induced spinor fields?

What critical modes could mean for the induced spinor fields at string world sheets and partonic 2-surfaces. The problematic part seems to be the variation of the Kähler-Dirac operator since it involves gradient. One cannot require that covariant derivative remains invariant since this would require that the components of the induced spinor connection remain invariant and this is quite too restrictive condition. Right handed neutrino solutions de-localized into entire X^2 are however an exception since they have no electro-weak gauge couplings and in this case the condition is obvious: Kähler-Dirac gamma matrices suffer a local scaling for critical deformations:

$$\delta\Gamma^\mu = \Lambda(x)\Gamma^\mu \ . \tag{5.5.6}$$

This guarantees that the Kähler-Dirac operator D is mapped to ΛD and still annihilates the modes of ν_R labelled by conformal weight, which thus remain unchanged.

What is the situation for the 2-D modes located at string world sheets? The condition is obvious. Ψ suffers an electro-weak gauge transformation as does also the induced spinor connection so that D_μ is not affected at all. Criticality condition states that the deformation of the space-time surfaces induces a conformal scaling of Γ^μ at X^2. It might be possible to continue this conformal scaling of the entire space-time sheet but this might be not necessary and this would mean that all critical deformations induced conformal transformations of the effective metric of the space-time surface defined by $\{\Gamma^\mu,\Gamma^\nu\} = 2G^{\mu\nu}$. Thus it seems that effective metric is indeed central concept (recall that if the conjectured quaternionic structure is associated with the effective metric, it might be possible to avoid problem related to the Minkowskian signature in an elegant manner).

In fact, one can consider even more general action of critical deformation: the modes of the induced spinor field would be mixed together in the infinitesimal deformation besides infinitesimal electroweak gauge transformation, which is same for all modes. This would extend electroweak gauge symmetry. Kähler-Dirac equation holds true also for these deformations. One might wonder whether the conjectured dynamically generated gauge symmetries assignable to finite measurement resolution could be generated in this manner.

The infinitesimal generator of a critical deformation J_M can be expressed as tensor product of matrix A_M acting in the space of zero modes and of a generator of infinitesimal electro-weak gauge transformation $T_M(x)$ acting in the same manner on all modes: $J_M = A_M \otimes T_M(x)$. A_M is a spatially constant matrix and $T_M(x)$ decomposes to a direct sum of left- and right-handed $SU(2)\times U(1)$ Lie-algebra generators. Left-handed Lie-algebra generator can be regarded as a quaternion and right handed as a complex number. One can speak of a direct sum of left-handed local quaternion $q_{M,L}$ and right-handed local complex number $c_{M,R}$. The commutator $[J_M,J_N]$ is given by $[J_M,J_N] = [A_M,A_N]\otimes\{T_M(x),T_N(x)\}+\{A_M,A_N\}\otimes[T_M(x),T_N(x)]$. One has $\{T_M(x),T_N(x)\} = \{q_{M,L}(x),q_{N,L}(x)\} \oplus \{c_{M,R}(x),c_{N,R}(x)\}$ and $[T_M(x),T_N(x)] = [q_{M,L}(x),q_{N,L}(x)]$. The commutators make sense also for more general gauge group but quaternion/complex number property might have some deeper role.

Thus the critical deformations would induce conformal scalings of the effective metric and dynamical electro-weak gauge transformations. Electro-weak gauge symmetry would be a dynamical symmetry restricted to string world sheets and partonic 2-surfaces rather than acting at the entire space-time surface. For 4-D de-localized right-handed neutrino modes the conformal scalings of the effective metric are analogous to the conformal transformations of M^4 for $\mathcal{N} = 4$ SYMs. Also ordinary conformal symmetries of M^4 could be present for string world sheets and could act as symmetries of generalized Feynman graphs since even virtual wormhole throats are massless. An interesting question is whether the conformal invariance associated with the effective metric is the analog of dual conformal invariance in $\mathcal{N} = 4$ theories.

Critical deformations of space-time surface are accompanied by conserved fermionic currents. By using standard Noetherian formulas one can write

$$J_i^\mu = \overline{\Psi}\Gamma^\mu\delta_i\Psi + \delta_i\overline{\Psi}\Gamma^\mu\Psi \ . \tag{5.5.7}$$

Here $\delta\Psi_i$ denotes derivative of the variation with respect to a group parameter labeled by i. Since $\delta\Psi_i$ reduces to an infinitesimal gauge transformation of Ψ induced by deformation, these currents are the analogs of gauge currents. The integrals of these currents along the braid strands at the ends of string world sheets define the analogs of gauge charges. The interpretation as Kac-Moody charges is also very attractive and I have proposed that the 2-D Hodge duals of gauge potentials could be identified as Kac-Moody currents. If so, the 2-D Hodge duals of J would define the quantum analogs of dynamical electro-weak gauge fields and Kac-Moody charge could be also seen as non-integral phase factor associated with the braid strand in Abelian approximation (the interpretation in terms of finite measurement resolution is discussed earlier).

One can also define super currents by replacing $\overline{\Psi}$ or Ψ by a particular mode of the induced spinor field as well as c-number valued currents by performing the replacement for both $\overline{\Psi}$ or Ψ. As expected, one obtains a super-conformal algebra with all modes of induced spinor fields acting as generators of super-symmetries restricted to 2-D surfaces. The number of the charges which do not annihilate physical states as also the effective number of fermionic modes could be finite and this would suggest that the integer $\mathcal{N}$ for the supersymmetry in question is finite. This would conform with the earlier proposal inspired by the notion of finite measurement resolution implying the replacement of the partonic 2-surfaces with collections of braid ends.

Note that Kac-Moody charges might be associated with "long" braid strands connecting different wormhole throats as well as short braid strands connecting opposite throats of wormhole contacts. Both kinds of charges would appear in the theory.

What is the interpretation of the critical deformations?

Critical deformations bring in an additional gauge symmetry. Certainly not all possible gauge transformations are induced by the deformations of preferred extremals and a good guess is that they correspond to holomorphic gauge group elements as in theories with Kac-Moody symmetry. What is the physical character of this dynamical gauge symmetry?

1. Do the gauge charges vanish? Do they annihilate the physical states? Do only their positive energy parts annihilate the states so that one has a situation characteristic for the representation of Kac-Moody algebras. Or could some of these charges be analogous to the gauge charges associated with the constant gauge transformations in gauge theories and be therefore non-vanishing in the absence of confinement. Now one has electro-weak gauge charges and these should be non-vanishing. Can one assign them to deformations with a vanishing conformal weight and the remaining deformations to those with non-vanishing conformal weight and acting like Kac-Moody generators on the physical states?

2. The simplest option is that the critical Kac-Moody charges/gauge charges with non-vanishing positive conformal weight annihilate the physical states. Critical degrees of freedom would not disappear but make their presence known via the states labelled by different gauge charges assignable to critical deformations with vanishing conformal weight. Note that constant gauge transformations can be said to break the gauge symmetry also in the ordinary gauge theories unless one has confinement.

3. The hierarchy of quantum criticalities suggests however entire hierarchy of electro-weak Kac-Moody algebras. Does this mean a hierarchy of electro-weak symmetries breakings in which the number of Kac-Moody generators not annihilating the physical states gradually increases as also modes with a higher value of positive conformal weight fail to annihilate the physical state?

   The only manner to have a hierarchy of algebras is by assuming that only the generators satisfying $n \ mod \ N = 0$ define the sub-Kac-Moody algebra annihilating the physical states so that the generators with $n \ mod \ N \neq 0$ would define the analogs of gauge charges. I have suggested for long time ago the relevance of kind of fractal hierarchy of Kac-Moody and Super-Virasoro algebras for TGD but failed to imagine any concrete realization.

   A stronger condition would be that the algebra reduces to a finite dimensional algebra in the sense that the actions of generators Q_n and Q_{n+kN} are identical. This would correspond to periodic boundary conditions in the space of conformal weights. The notion of finite measurement resolution suggests that the number of independent fermionic oscillator operators is proportional to the number of braid ends so that an effective reduction to a finite algebra is expected.

   Whatever the correct interpretation is, this would obviously refine the usual view about electro-weak symmetry breaking.

These arguments suggests the following overall view. The holomorphy of spinor modes gives rise to Kac-Moody algebra defined by isometries and includes besides Minkowskian generators associated with gravitation also SU(3) generators associated with color symmetries. Vanishing second variations in turn define electro-weak Kac-Moody type algebra.

Note that criticality suggests that one must perform functional integral over WCW by decomposing it to an integral over zero modes for which deformations of X^4 induce only an electro-weak gauge transformation of the induced spinor field and to an integral over moduli corresponding to the remaining degrees of freedom.

5.5.5 The Emergence Of Yangian Symmetry And Gauge Potentials As Duals Of Kac-Moody Currents

Yangian symmetry plays a key role in $\mathcal{N}=4$ super-symmetric gauge theories. What is special in Yangian symmetry is that the algebra contains also multi-local generators. In TGD framework multi-locality would naturally correspond to that with respect to partonic 2-surfaces and string world sheets and the proposal has been that the Super-Kac-Moody algebras assignable to string worlds sheets could generalize to Yangian.

Witten has written a beautiful exposition of Yangian algebras [B24]. Yangian is generated by two kinds of generators J^A and Q^A by a repeated formation of commutators. The number of commutations tells the integer characterizing the multi-locality and provides the Yangian algebra with grading by natural numbers. Witten describes a 2-dimensional QFT like situation in which one has 2-D situation and Kac-Moody currents assignable to real axis define the Kac-Moody charges as integrals in the usual manner. It is also assumed that the gauge potentials defined by the 1-form associated with the Kac-Moody current define a flat connection:

$$\partial_\mu j^A_\nu - \partial_\nu j^A_\nu + [j^A_\mu, j^A_\nu] = 0 \ . \tag{5.5.8}$$

This condition guarantees that the generators of Yangian are conserved charges. One can however consider alternative ways to obtain the conservation.

1. The generators of first kind - call them J^A - are just the conserved Kac-Moody charges. The formula is given by

$$J_A = \int_{-\infty}^{\infty} dx j^{A0}(x,t) \ . \tag{5.5.9}$$

2. The generators of second kind contain bi-local part. They are convolutions of generators of first kind associated with different points of string described as real axis. In the basic formula one has integration over the point of real axis.

$$Q^A = f^A_{BC}\int_{-\infty}^{\infty} dx \int_x^{\infty} dy j^{B0}(x,t) j^{C0}(y,t) - 2\int_{-\infty}^{\infty} j^A_x dx \ . \tag{5.5.10}$$

   These charges are indeed conserved if the curvature form is vanishing as a little calculation shows.

How to generalize this to the recent context?

1. The Kac-Moody charges would be associated with the braid strands connecting two partonic 2-surfaces - Strands would be located either at the space-like 3-surfaces at the ends of the space-time surface or at light-like 3-surfaces connecting the ends. Kähler-Dirac equation would define Super-Kac-Moody charges as standard Noether charges. Super charges would be obtained by replacing the second quantized spinor field or its conjugate in the fermionic bilinear by particular mode of the spinor field. By replacing both spinor field and its conjugate by its mode one would obtain a conserved c-number charge corresponding to an anti-commutator of two fermionic super-charges. The convolution involving double integral is however not number theoretically attractive whereas single 1-D integrals might make sense.

2. An encouraging observation is that the Hodge dual of the Kac-Moody current defines the analog of gauge potential and exponents of the conserved Kac-Moody charges could be identified as analogs for the non-integrable phase factors for the components of this gauge potential. This identification is precise only in the approximation that generators commute since only in this case the ordered integral $P(exp(i\int Adx))$ reduces to $P(exp(i\int Adx))$.Partonic 2-surfaces connected by braid strand would be analogous to nearby points of space-time in its discretization implying that Abelian approximation works. This conforms with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

This would make possible a direct identification of Kac-Moody symmetries in terms of gauge symmetries. For isometries one would obtain color gauge potentials and the analogs of gauge potentials for graviton field (in TGD framework the contraction with M^4 vierbein would transform tensor field to 4 vector fields). For Kac-Moody generators corresponding to holonomies one would obtain electroweak gauge potentials. Note that super-charges would give rise to a collection of spartners of gauge potentials automatically. One would obtain a badly broken SUSY with very large value of $\mathcal{N}$ defined by the number of spinor modes as indeed speculated earlier [?].

3. The condition that the gauge field defined by 1-forms associated with the Kac-Moody currents are trivial looks unphysical since it would give rise to the analog of topological QFT with gauge potentials defined by the Kac-Moody charges. For the duals of Kac-Moody currents defining gauge potentials only covariant divergence vanishes implying that curvature form is

$$F_{\alpha\beta} = \epsilon_{\alpha\beta}[j_\mu, j^\mu] \ , \tag{5.5.11}$$

so that the situation does not reduce to topological QFT unless the induced metric is diagonal. This is not the case in general for string world sheets.

4. It seems however that there is no need to assume that j_μ defines a flat connection. Witten mentions that although the discretization in the definition of J^A does not seem to be possible, it makes sense for Q^A in the case of $G = SU(N)$ for any representation of G. For general G and its general representation there exists no satisfactory definition of Q. For certain representations, such as the fundamental representation of $SU(N)$, the definition of Q^A is especially simple. One just takes the bi-local part of the previous formula:

$$Q^A = f^A_{BC} \sum_{i<j} J^B_i J^C_j \ . \tag{5.5.12}$$

What is remarkable that in this formula the summation need not refer to a discretized point of braid but to braid strands ordered by the label i by requiring that they form a connected polygon. Therefore the definition of J^A could be just as above.

5. This brings strongly in mind the interpretation in terms of twistor diagrams. Yangian would be identified as the algebra generated by the logarithms of non-integrable phase factors in Abelian approximation assigned with pairs of partonic 2-surfaces defined in terms of Kac-Moody currents assigned with the Kähler-Dirac action. Partonic 2-surfaces connected by braid strand would be analogous to nearby points of space-time in its discretization. This would fit nicely with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

The resulting algebra satisfies the basic commutation relations

$$\left[J^A, J^B\right] = f^{AB}_C J^C \ , \ \left[J^A, Q^B\right] = f^{AB}_C Q^C \ . \tag{5.5.13}$$

plus the rather complex Serre relations described in [B24].

5.6 Kähler-Dirac Equation And Super-Symmetries

The previous considerations concerning super-conformal symmetries and space-time SUSY have been based on general arguments. The new vision about preferred extremals and Kähler-Dirac equation however leads to a rather detailed understanding of super-conformal symmetries at the level of field equations and is bound to modify the existing vision about super-conformal symmetries.

Whether TGD predicts some variant of space-time SUSY or not has been a long-standing issue: the reason is that TGD does not allow Majorana spinors since fermion number conservation is exact. The more precise formulation of field equations made possible by the realization that spinor modes are localized at string world sheets allows to conclude that the analog of broken $\mathcal{N}=8$ SUSY is predicted at parton level and that right-handed neutrino generates the minimally broken $\mathcal{N}=2$ sub-SUSY.

One important outcome of criticality is the identification of gauge potentials as duals of Kac-Moody currents at the boundaries of string world sheets: quantum gauge potentials are defined only where they are needed that is string curves defining the non-integrable phase factors. This gives also rise to the realization of the conjectured Yangian in terms of the Kac-Moody charges and commutators in accordance with the earlier conjecture.

5.6.1 Super-Conformal Symmetries

It is good to summarize first the basic ideas about Super-Virasoro representations. TGD allows two kinds of super-conformal symmetries.

1. The first super-conformal symmetry is associated with $\delta M^4_\pm \times CP_2$ and corresponds to symplectic symmetries of $\delta M^4_\pm \times CP_2$. The reason for extension of conformal symmetries is metric 2-dimensionality of the light-like boundary $\delta M^4_\pm$ defining upper/lower boundary of causal diamond (CD). This super-conformal symmetry is something new and corresponds to replacing finite-dimensional Lie-group G for Kac-Moody symmetry with infinite-dimensional symplectic group. The light-like radial coordinate of $\delta M^4_\pm$ takes the role of the real part of complex coordinate z for ordinary conformal symmetry. Together with complex coordinate of S^2 it defines 3-D restriction of Hamilton-Jacobi variant of 4-D super-conformal symmetries. One can continue the conformal symmetries from light-cone boundary to CD by forming a slicing by parallel copies of $\delta M^4_\pm$. There are two possible slicings corresponding to the choices δM^4_+ and δM^4_- assignable to the upper and lower boundaries of CD. These two choices correspond to two arrows of geometric time for the basis of zero energy states in ZEO.
2. Super-symplectic degrees of freedom determine the electroweak and color quantum numbers of elementary particles. Bosonic emergence implies that ground states assignable to partonic 2-surfaces correspond to partial waves in $\delta M^4_\pm$ and one obtains color partial waves in particular. These partial waves correspond to the solutions for the Dirac equation in embedding space and the correlation between color and electroweak quantum numbers is not quite correct. Super-Kac-Moody generators give the compensating color for massless states obtained from tachyonic ground states guaranteeing that standard correlation is obtained. Super-symplectic degrees are therefore directly visible in particle spectrum. One can say that at the point-like limit the WCW spinors reduce to tensor products of embedding space spinors assignable to the center of mass degrees of freedom for the partonic 2-surfaces defining wormhole throats.

   I have proposed a physical interpretation of super-symplectic vibrational degrees of freedom in terms of degrees of freedom assignable to non-perturbative QCD. These degrees of freedom would be responsible for most of the baryon masses but their theoretical understanding is lacking in QCD framework.
3. The second super-conformal symmetry is assigned light-like 3-surfaces and to the isometries and holonomies of the embedding space and is analogous to the super-Kac-Moody symmetry of string models. Kac-Moody symmetries could be assigned to the light-like deformations of light-like 3-surfaces. Isometries give tensor factor $E^2 \times SU(3)$ and holonomies factor $SU(2)_L \times U(1)$. Altogether one has 5 tensor factors to super-conformal algebra. That the number is just five is essential for the success p-adic mass calculations [K54, K46].

   The construction of solutions of the Kähler-Dirac equation suggests strongly that the fermionic representation of the Super-Kac-Moody algebra can be assigned as conserved charges associated with the space-like braid strands at both the 3-D space-like ends of space-time surfaces and with the light-like (or space-like with a small deformation) associated with the light-like 3-surfaces. The extension to Yangian algebra involving higher multi-linears of super-Kac Moody generators is also highly suggestive. These charges would be non-local and assignable to several wormhole contacts simultaneously. The ends of braids would correspond points of partonic 2-surfaces defining a discretization of the partonic 2-surface having interpretation in terms of finite measurement resolution.

   These symmetries would correspond to electroweak and strong gauge fields and to gravitation. The duals of the currents giving rise to Kac-Moody charges would define the counterparts of gauge potentials and the conserved Kac-Moody charges would define the counterparts of non-integrable phase factors in gauge theories. The higher Yangian charges would define generalization of non-integrable phase factors. This would suggest a rather direct connection with the twistorial program for calculating the scattering amplitudes implies also by zero energy ontology.

Quantization recipes have worked in the case of super-string models and one can ask whether the application of quantization to the coefficients of powers of complex coordinates or Hamilton-Jacobi coordinates could lead to the understanding of the 4-D variants of the conformal symmetries and give detailed information about the representations of the Kac-Moody algebra too.

5.6.2 WCW Geometry And Super-Conformal Symmetries

The vision about the geometry of WCW has been roughly the following and the recent steps of progress induce to it only small modifications if any.

1. Kähler geometry is forced by the condition that hermitian conjugation allows geometrization. Kähler function is given by the Kähler action coming from space-time regions with Euclidian signature of the induced metric identifiable as lines of generalized Feynman diagrams. Minkowskian regions give imaginary contribution identifiable as the analog of Morse function and implying interference effects and stationary phase approximation. The vision about quantum TGD as almost topological QFT inspires the proposal that Kähler action reduces to 3-D terms reducing to Chern-Simons terms by the weak form of electric-magnetic duality. The recent proposal for preferred extremals is consistent with this property realizing also holography implied by general coordinate invariance. Strong form of general coordinate invariance implying effective 2-dimensionality in turn suggests that Kähler action is expressible string world sheets and possibly also areas of partonic 2-surfaces.
2. The complexified gamma matrices of WCW come as hermitian conjugate pairs and anti-commute to the Kähler metric of WCW . Also bosonic generators of symplectic transformations of $\delta M^4_\pm \times CP_2$ a assumed to act as isometries of WCW geometry can be complexified and appear as similar pairs. The action of isometry generators co-incides with that of symplectic generators at partonic 2-surfaces and string world sheets but elsewhere inside the space-time surface it is expected to be deformed from the symplectic action. The super-conformal transformations of $\delta M^4_\pm \times CP_2$ acting on the light-like radial coordinate of $\delta M^4_\pm$ act as gauge symmetries of the geometry meaning that the corresponding WCW vector fields have zero norm.
3. WCW geometry has also zero modes which by definition do not contribute to WCW metric expect possibly by the dependence of the elements of WCW metric on zero modes through a conformal factor. In particular, induced CP_2 Kähler form and its analog for sphere $r_M = constant$ of light cone boundary are symplectic invariants, and one can define an infinite number of zero modes as invariants defined by Kähler fluxes over partonic 2-surfaces and string world sheets. This requires however the slicing of CD parallel copies of δM^4_+ or δM^4_-. The physical interpretation of these non-quantum fluctuating degrees of freedom is as classical variables necessary for the interpretation of quantum measurement theory. Classical variable would metaphorically correspond the position of the pointer of the measurement instrument.
4. The construction receives a strong philosophical inspiration from the geometry of loop spaces. Loop spaces allow a unique Kähler geometry with maximal isometry group identifiable as Kac-Moody group. The reason is that otherwise Riemann connection does not exist. The only problem is that curvature scalar diverges since the Riemann tensor is by constant curvature property proportional to the metric. In 3-D case one would have union of constant curvature spaces labelled by zero modes and the situation is expected to be even more restrictive. The conjecture indeed is that WCW geometry exists only for $H = M^4 \times CP_2$: infinite-D Kähler geometric existence and therefore physics would be unique. One can also hope that Ricci scalar is finite and therefore zero by the constant curvature property so that Einstein's equations are satisfied.
5. The matrix elements of WCW Kähler metric are given in terms of the anti-commutators of the fermionic Noether super-charges associated with symplectic isometry currents. A given mode

of induced spinor field characterized by embedding space chirality (quark or lepton), by spin and weak spin plus conformal weight n. If the super-conformal transformations for string modes act gauge transformations only the spinor modes with vanishing conformal weight correspond to non-zero modes of the WCW metric and the situation reduces essentially to the analog of $\mathcal{N}=8$ SUSY.

The WCW Hamiltonians generating symplectic isometries correspond to the Hamiltonians spanning the symplectic group of $\delta M^4_\pm \times CP_2$. One can say that the space of quantum fluctuating degrees of freedom is this symplectic group of $\delta M^4_\pm \times CP_2$ or its subgroup or coset space: this must have very deep implications for the structure of the quantum TGD.

An interesting possibility is that the radial conformal weights of the symplectic algebra are linear combinations of the zeros of Riemann Zeta with integer coefficients. Also this option allows to realize the hierarchy of super-symplectic conformal symmetry breakings in terms of sub-algebras isomorphic to the entire super-symplectic algebra. WCW would have fractal structure corresponding to a hierarchy of quantum criticalities.

6. The localization of the induced spinors to string world sheets means that the super-symplectic Noether charges are associated with strings connecting partonic 2-surfaces. The physically obvious fact that given partonic surface can be accompanied by an arbitrary number of strings, forces a generalization of the super-symplectic algebra to a Yangian containing infinite number of n-local variants of various super-symplectic Noether charges. For instance, four -momentum is accompanied by multi-stringy variants involving four-momentum P^A_0 and angular momentum generators. At the first level of the hierarchy one has $P^A_1 = f^A_{BC} P^B_0 \otimes J^C$. This hierarchy might play crucial role in understanding of the four-momenta of bound states.

7. Zero energy ontology brings in additional delicacies. Basic objects are now unions of partonic 2-surfaces at the ends of CD. One can generalize the expressions for the isometry generators in a straightforward manner by requiring that given isometry restricts to a symplectic transformation at partonic 2-surfaces and string world sheets.

8. One could criticize the effective metric 2-dimensionality forced by the general consistency arguments as something non-physical. The WCW Hamiltonians are expressed using only the data at partonic 2-surfaces and string string world sheets: this includes also 4-D tangent space data via the weak form of electric-magnetic duality so that one has only effective 2-dimensionality. Obviously WCW geometry must huge large gauge symmetries besides zero modes. The hierarchy of super-symplectic symmetries indeed represent gauge symmetries of this kind.

   Effective 2-dimensionality realizing strong form of holography in turn is induced by the strong form of general coordinate invariance. Light-like 3-surfaces at which the signature of the induced metric changes must be equivalent with the 3-D space-like ends of space-time surfaces at the light-boundaries of space-time surfaces as far as WCW geometry is considered. This requires that the data from their 2-D intersections defining partonic 2-surfaces should dictate the WCW geometry. Note however that Super-Kac-Moody charges giving information about the interiors of 3-surfaces appear in the construction of the physical states.

5.6.3 The Relationship Between Inertial Gravitational Masses

The relationship between inertial and gravitational masses and Equivalence Principle have been on of the longstanding problems in TGD. Not surprisingly, the realization how GRT space-time relates to the many-sheeted space-time of TGD finally allowed to solve the problem.

ZEO and non-conservation of Poincare charges in Poincare invariant theory of gravitation

In positive energy ontology the Poincare invarance of TGD is in sharpt contrast with the fact that GRT based cosmology predicts non-conservation of Poincare charges (as a matter fact, the definition of Poincare charges is very questionable for general solutions of field equations).

In zero energy ontology (ZEO) all conserved (that is Noether-) charges of the Universe vanish identically and their densities should vanish in scales below the scale defining the scale for observations and assignable to causal diamond (CD). This observation allows to imagine a ways out of what seems to be a conflict of Poincare invariance with cosmological facts.

ZEO would explain the local non-conservation of average energies and other conserved quantum numbers in terms of the contributions of sub-CDs analogous to quantum fluctuations. Classical gravitation should have a thermodynamical description if this interpretation is correct. The average values of the quantum numbers assignable to a space-time sheet would depend on the size of CD and possibly also its location in M^4. If the temporal distance between the tips of CD is interpreted as a quantized variant of cosmic time, the non-conservation of energy-momentum defined in this manner follows. One can say that conservation laws hold only true in given scale defined by the largest CD involved.

Equivalence Principle at quantum level

The interpretation of EP at quantum level has developed slowly and the recent view is that it reduces to quantum classical correspondence meaning that the classical charges of Kähler action can be identified with eigen values of quantal charges associated with Kähler-Dirac action.

1. At quantum level I have proposed coset representations for the pair of super-symplectic algebras assignable to the light-like boundaries of CD and the Super Kac-Moody algebra assignable to the light-like 3-surfaces defining the orbits of partonic 2-surfaces as realization of EP. For coset representation the differences of super-conformal generators would annihilate the physical states so that one can argue that the corresponding four-momenta are identical. One could even say that one obtains coset representation for the "vibrational" parts of the super-conformal algebras in question. It is now clear that this idea does not work. Note however that coset representations occur naturally for the subalgebras of symplectic algebra and Super Kac-Moody algebra and are naturally induced by finite measurement resolution.
2. The most recent view (2014) about understanding how EP emerges in TGD is described in [K90] and relies heavily on superconformal invariance and a detailed realisation of ZEO at quantum level. In this approach EP corresponds to quantum classical correspondence (QCC): four-momentum identified as classical conserved Noether charge for space-time sheets associated with Käbler action is identical with quantal four-momentum assignable to the representations of super-symplectic and super Kac-Moody algebras as in string models and having a realisation in ZEO in terms of wave functions in the space of causal diamonds (CDs).
3. The latest realization is that the eigenvalues of quantal four-momentum can be identified as eigenvalues of the four-momentum operator assignable to the Kähler-Dirac equation. This realisation seems to be consistent with the p-adic mass calculations requiring that the super-conformal algebra acts in the tensor product of 5 tensor factors.

Equivalence Principle at classical level

How Einstein's equations and General Relativity in long length scales emerges from TGD has been a long-standing interpretational problem of TGD.

The first proposal making sense even when one does not assume ZEO is that vacuum extremals are only approximate representations of the physical situation and that small fluctuations around them give rise to an inertial four-momentum identifiable as gravitational four-momentum identifiable in terms of Einstein tensor. EP would hold true in the sense that the average gravitational four-momentum would be determined by the Einstein tensor assignable to the vacuum extremal. This interpretation does not however take into account the many-sheeted character of TGD spacetime and is therefore questionable.

The resolution of the problem came from the realization that GRT is only an effective theory obtained by endowing M^4 with effective metric.

1. The replacement of superposition of fields with superposition of their effects means replacing superposition of fields with the set-theoretic union of space-time surfaces. Particle experiences sum of the effects caused by the classical fields at the space-time sheets (see **Fig.** `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg` or **Fig. ??** in the appendix of this book).

2. This is true also for the classical gravitational field defined by the deviation from flat Minkowski metric instandard M^4 coordinates for the space-time sheets. One can define effective metric as sum of M^4 metric and deviations. This effective metric would correspond to that of General Relativity. This resolves long standing issues relating to the interpretation of TGD.

3. Einstein's equations could hold true for the effective metric. They are motivated by the underlying Poincare invariance which cannot be realized as global conservation laws for the effective metric. The conjecture vanishing of divergence of Kähler energy momentum tensor can be seen as the microscopic justification for the claim that Einstein's equations hold true for the effective space-time.

4. The breaking of Poincare invariance could have interpretation as effective breaking in zero energy ontology (ZEO), in which various conserved charges are length dependent and defined separately for each causal diamond (CD).

One can of course consider the possibility that Einstein's equations generalize for preferred extremals of Kähbler action. This would actually represent at space-time level the notion of QCC rather than realise QCC interpreted as EP. The condition that the energy momentum tensor for Kähler action has vanishing covariant divergence would be satisfied in GRT if Einstein's equations with cosmological term hold true. This is the case also now but one can consider also more general solutions in which one has two cosmological constants which are not genuine constants anymore: this approach is not promising.

An interesting question is whether inertial-gravitational duality generalizes to the case of color gauge charges so that color gauge fluxes would correspond to "gravitational" color charges and the charges defined by the conserved currents associated with color isometries would define "inertial" color charges. Since the induced color fields are proportional to color Hamiltonians multiplied by Kähler form they vanish identically for vacuum extremals in accordance with "gravitational" color confinement.

Constraints from p-adic mass calculations and ZEO

A further important physical input comes from p-adic thermodynamics forming a core element of p-adic mass calculations.

1. The first thing that one can get worried about relates to the extension of conformal symmetries. If the conformal symmetries generalize to $D = 4$, how can one take seriously the results of p-adic mass calculations based on 2-D conformal invariance? There is no reason to worry. The reduction of the conformal invariance to 2-D one for the preferred extremals takes care of this problem. This however requires that the fermionic contributions assignable to string world sheets and/or partonic 2-surfaces - Super- Kac-Moody contributions - should dictate the elementary particle masses. For hadrons also symplectic contributions should be present. This is a valuable hint in attempts to identify the mathematical structure in more detail.

2. ZEO suggests that all particles, even virtual ones correspond to massless wormhole throats carrying fermions. As a consequence, twistor approach would work and the kinematical constraints to vertices would allow the cancellation of divergences. This would suggest that the p-adic thermal expectation value is for the longitudinal M^2 momentum squared (the definition of CD selects $M^1 \subset M^2 \subset M^4$ as also does number theoretic vision). Also propagator would be determined by M^2 momentum. Lorentz invariance would be obtained by integration of the moduli for CD including also Lorentz boosts of CD.

3. In the original approach one allows states with arbitrary large values of L_0 as physical states. Usually one would require that L_0 annihilates the states. In the calculations however mass squared was assumed to be proportional L_0 apart from vacuum contribution. This is a questionable assumption. ZEO suggests that total mass squared vanishes and that one can decompose mass squared to a sum of longitudinal and transversal parts. If one can do the same decomposition to longitudinal and transverse parts also for the Super Virasoro algebra then one can calculate longitudinal mass squared as a p-adic thermal expectation in the transversal super-Virasoro algebra and only states with $L_0 = 0$ would contribute and one would have conformal invariance in the standard sense.

4. In the original approach the assumption motivated by Lorentz invariance has been that mass squared is replaced with conformal weight in thermodynamics, and that one first calculates the thermal average of the conformal weight and then equates it with mass squared. This assumption is somewhat ad hoc. ZEO however suggests an alternative interpretation in which one has zero energy states for which longitudinal mass squared of positive energy state derive from p-adic thermodynamics. Thermodynamics - or rather, its square root - would become part of quantum theory in ZEO. M-matrix is indeed product of hermitian square root of density matrix multiplied by unitary S-matrix and defines the entanglement coefficients between positive and negative energy parts of zero energy state.
5. The crucial constraint is that the number of super-conformal tensor factors is $N = 5$: this suggests that thermodynamics applied in Super-Kac-Moody degrees of freedom assignable to string world sheets is enough, when one is interested in the masses of fermions and gauge bosons. Super-symplectic degrees of freedom can also contribute and determine the dominant contribution to baryon masses. Should also this contribution obey p-adic thermodynamics in the case when it is present? Or does the very fact that this contribution need not be present mean that it is not thermal? The symplectic contribution should correspond to hadronic p-adic length prime rather the one assignable to (say) u quark. Hadronic p-adic mass squared and partonic p-adic mass squared cannot be summed since primes are different. If one accepts the basic rules [K55], longitudinal energy and momentum are additive as indeed assumed in perturbative QCD.
6. Calculations work if the vacuum expectation value of the mass squared must be assumed to be tachyonic. There are two options depending on whether one whether p-adic thermodynamics gives total mass squared or longitudinal mass squared.
   (a) One could argue that the total mass squared has naturally tachyonic ground state expectation since for massless extremals longitudinal momentum is light-like and transversal momentum squared is necessary present and non-vanishing by the localization to topological light ray of finite thickness of order p-adic length scale. Transversal degrees of freedom would be modeled with a particle in a box.
   (b) If longitudinal mass squared is what is calculated, the condition would require that transversal momentum squared is negative so that instead of plane wave like behavior exponential damping would be required. This would conform with the localization in transversal degrees of freedom.

5.6.4 Realization Of Space-Time SUSY In TGD

The generators of super-conformal algebras are obtained by taking fermionic currents for second quantized fermions and replacing either fermion field or its conjugate with its particular mode. The resulting super currents are conserved and define super charges. By replacing both fermion and its conjugate with modes one obtains c-number valued currents. In this manner one also obtains the analogs of super-Poincare generators labelled by the conformal weight and other spin quantum numbers as Noether charges so that space-time SUSY is suggestive.

The super-conformal invariance in spinor modes is expected to be gauge symmetry so that only the generators with vanishing string world sheet conformal weight create physical states. This would leave only the conformal quantum numbers characterizing super-symplectic generators (radial conformal weight included) under consideration and the hierarchy of its sub-algebras acting as gauge symmetries giving rise to a hierarchy of criticalities having interpretation in terms of dark matter.

As found in the earlier section, the proposed anti-commutation relations for fermionic oscillator operators at the ends of string world sheets can be formulated so that they are analogous to those for Super Poincare algebra. The reason is that field equations assign a conserved 8-momentum to the light-like geodesic line defining the boundary of string at the orbit of partonic 2-surface. Octonionic representation of sigma matrices making possible generalization of twistor formalism to 8-D context is also essential. As a matter, the final justification for the analog of space-time came from the generalization of twistor approach to 8-D context.

By counting the number of spin and weak isospin components of embedding space spinors satisfying massless algebraic Dirac equation one finds that broken $\mathcal{N} = 8$ SUSY is the expected space-time SUSY. $\mathcal{N} = 2$ SUSY assignable to right-handed neutrino is the least broken sub-SUSY and one is forced to consider the possibility that spartners correspond to dark matter with $h_{eff} = n \times h$ and therefore remaining undetected in recent particle physics experiments.

Super-space viz. Grassmann algebra valued fields

Standard SUSY induces super-space extending space-time by adding anti-commuting coordinates as a formal tool. Many mathematicians are not enthusiastic about this approach because of the purely formal nature of anti-commuting coordinates. Also I regard them as a non-sense geometrically and there is actually no need to introduce them as the following little argument shows.

Grassmann parameters (anti-commuting theta parameters) are generators of Grassmann algebra and the natural object replacing super-space is this Grassmann algebra with coefficients of Grassmann algebra basis appearing as ordinary real or complex coordinates. This is just an ordinary space with additional algebraic structure: the mysterious anti-commuting coordinates are not needed. To me this notion is one of the conceptual monsters created by the over-pragmatic thinking of theoreticians.

This allows allows to replace field space with super field space, which is completely well-defined object mathematically, and leave space-time untouched. Linear field space is simply replaced with its Grassmann algebra. For non-linear field space this replacement does not work. This allows to formulate the notion of linear super-field just in the same manner as it is done usually.

The generators of super-symmetries in super-space formulation reduce to super translations, which anti-commute to translations. The super generators Q_α and $\overline{Q}_{\dot{\beta}}$ of super Poincare algebra are Weyl spinors commuting with momenta and anti-commuting to momenta:

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\ \dot{beta}} P_\mu \ . \tag{5.6.1}$$

One particular representation of super generators acting on super fields is given by

$$\begin{aligned} D_\alpha &= i\frac{\partial}{\partial_{\theta_\alpha}} \ , \\ D_{\dot{\alpha}} &= i\frac{\partial}{\partial_{\overline{\theta}\ \dot{alpha}}} + \theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu \end{aligned} \tag{5.6.2}$$

Here the index raising for 2-spinors is carried out using antisymmetric 2-tensor $\epsilon^{\alpha\beta}$. Super-space interpretation is not necessary since one can interpret this action as an action on Grassmann algebra valued field mixing components with different fermion numbers.

Chiral superfields are defined as fields annihilated by $D_{\dot{\alpha}}$. Chiral fields are of form $\Psi(x^\mu + i\overline{\theta}\sigma^\mu\theta, \theta)$. The dependence on $\overline{\theta}_{\dot{\alpha}}$ comes only from its presence in the translated Minkowski coordinate annihilated by $D_{\dot{\alpha}}$. Super-space enthusiast would say that by a translation of M^4 coordinates chiral fields reduce to fields, which depend on θ only.

The space of fermionic Fock states at partonic 2-surface as TGD counterpart of chiral super field

As already noticed, another manner to realize SUSY in terms of representations the super algebra of conserved super-charges. In TGD framework these super charges are naturally associated with the modified Dirac equation, and anti-commuting coordinates and super-fields do not appear anywhere. One can however ask whether one could identify a mathematical structure replacing the notion of chiral super field.

In [?] it was proposed that generalized chiral super-fields could effectively replace induced spinor fields and that second quantized fermionic oscillator operators define the analog of SUSY algebra. One would have $\mathcal{N} = \infty$ if all the conformal excitations of the induced spinor field restricted on 2-surface are present. For right-handed neutrino the modes are labeled by two integers and de-localized to the interior of Euclidian or Minkowskian regions of space-time sheet.

The obvious guess is that chiral super-field generalizes to the field having as its components many-fermions states at partonic 2-surfaces with theta parameters and their conjugates in one-one correspondence with fermionic creation operators and their hermitian conjugates.

1. Fermionic creation operators - in classical theory corresponding anti-commuting Grassmann parameters - replace theta parameters. Theta parameters and their conjugates are not in one-one correspondence with spinor components but with the fermionic creation operators and their hermitian conjugates. One can say that the super-field in question is defined in the "world of classical worlds" (WCW) rather than in space-time. Fermionic Fock state at the partonic 2-surface is the value of the chiral super field at particular point of WCW .
2. The matrix defined by the $\sigma^\mu\partial_\mu$ is replaced with a matrix defined by the Kähler-Dirac operator D between spinor modes acting in the solution space of the Kähler-Dirac equation. Since Kähler-Dirac operator annihilates the modes of the induced spinor field, super covariant derivatives reduce to ordinary derivatives with respect the theta parameters labeling the modes. Hence the chiral super field is a field that depends on θ_m or conjugates $\overline{\theta}_m$ only. In second quantization the modes of the chiral super-field are many-fermion states assigned to partonic 2-surfaces and string world sheets. Note that this is the only possibility since the notion of super-coordinate does not make sense now.
3. It would seem that the notion of super-field does not bring anything new. This is not the case. First of all, the spinor fields are restricted to 2-surfaces. Second point is that one cannot assign to the fermions of the many-fermion states separate non-parallel or even parallel four-momenta. The many-fermion state behaves like elementary particle. This has non-trivial implications for propagators and a simple argument [**?**] leads to the proposal that propagator for N-fermion partonic state is proportional to $1/p^N$. This would mean that only the states with fermion number equal to 1 or 2 behave like ordinary elementary particles.

5.6.5 Comparison Of TGD And Stringy Views About Super-Conformal Symmetries

The best manner to represent TGD based view about conformal symmetries is by comparison with the conformal symmetries of super string models.

Basic differences between the realization of super conformal symmetries in TGD and in super-string models

The realization super conformal symmetries in TGD framework differs from that in string models in several fundamental aspects.

1. In TGD framework super-symmetry generators acting as configuration space gamma matrices carry either lepton or quark number. Majorana condition required by the hermiticity of super generators which is crucial for super string models would be in conflict with the conservation of baryon and lepton numbers and is avoided. This is made possible by the realization of bosonic generators represented as Hamiltonians of X^2-local symplectic transformations rather than vector fields generating them [K21]. This kind of representation applies also in Kac-Moody sector since the local transversal isometries localized in X^3_l and respecting light-likeness condition can be regarded as X^2 local symplectic transformations, whose Hamiltonians generate also isometries. Localization is not complete: the functions of X^2 coordinates multiplying symplectic and Kac-Moody generators are functions of the symplectic invariant $J = \epsilon^{\mu\nu}J_{\mu\nu}$ so that effective one-dimensionality results but in different sense than in conformal field theories. This realization of super symmetries is what distinguishes between TGD and super string models and leads to a totally different physical interpretation of super-conformal symmetries. The fermionic representations of super-symplectic and super Kac-Moody generators can be identified as Noether charges in standard manner.
2. A long-standing problem of quantum TGD was that stringy propagator $1/G$ does not make sense if G carries fermion number. The progress in the understanding of second quantization of the modified Dirac operator made it however possible to identify the counterpart of G as a c-number valued operator and interpret it as different representation of G [K19].
3. The notion of super-space is not needed at all since Hamiltonians rather than vector fields represent bosonic generators, no super-variant of geometry is needed. The distinction between Ramond and N-S representations important for $N = 1$ super-conformal symmetry and

allowing only ground state weight 0 an 1/2 disappears. Indeed, for $N = 2$ super-conformal symmetry it is already possible to generate spectral flow transforming these Ramond and N-S representations to each other (G_n is not Hermitian anymore).

4. If Kähler action defines the Kähler-Dirac operator, the number of spinor modes could be finite. One must be here somewhat cautious since bound state in the Coulomb potential associated with electric part of induced electro-weak gauge field might give rise to an infinite number of bound states which eigenvalues converging to a fixed eigenvalue (as in the case of hydrogen atom). Finite number of generalized eigenmodes means that the representations of super-conformal algebras reduces to finite-dimensional ones in TGD framework. Also the notion of number theoretic braid indeed implies this. The physical interpretation would be in terms of finite measurement resolution. If Kähler action is complexified to include imaginary part defined by CP breaking instanton term, the number of stringy mass square eigenvalues assignable to the spinor modes becomes infinite since conformal excitations are possible. This means breakdown of exact holography and effective 2-dimensionality of 3-surfaces. It seems that the inclusion of instanton term is necessary for several reasons. The notion of finite measurement resolution forces conformal cutoff also now. There are arguments suggesting that only the modes with vanishing conformal weight contribute to the Dirac determinant defining vacuum functional identified as exponent of Kähler function in turn identified as Kähler action for its preferred extremal.

5. What makes spinor field mode a generator of gauge super-symmetry is that is c-number and not an eigenmode of $D_K(X^2)$ and thus represents non-dynamical degrees of freedom. If the number of eigen modes of $D_K(X^2)$ is indeed finite means that most of spinor field modes represent super gauge degrees of freedom.

The super generators G are not Hermitian in TGD!

The already noticed important difference between TGD based and the usual Super Virasoro representations is that the Super Virasoro generator G cannot Hermitian in TGD. The reason is that WCW gamma matrices possess a well defined fermion number. The hermiticity of the WCW gamma matrices Γ and of the Super Virasoro current G could be achieved by posing Majorana conditions on the second quantized H-spinors. Majorana conditions can be however realized only for space-time dimension $D \ mod\ 8 = 2$ so that super string type approach does not work in TGD context. This kind of conditions would also lead to the non-conservation of baryon and lepton numbers.

An analogous situation is encountered in super-symmetric quantum mechanics, where the general situation corresponds to super symmetric operators S, $S^\dagger$, whose anti-commutator is Hamiltonian: $\{S, S^\dagger\} = H$. One can define a simpler system by considering a Hermitian operator $S_0 = S + S^\dagger$ satisfying $S_0^2 = H$: this relation is completely analogous to the ordinary Super Virasoro relation $GG = L$. On basis of this observation it is clear that one should replace ordinary Super Virasoro structure $GG = L$ with $GG^\dagger = L$ in TGD context.

It took a long time to realize the trivial fact that $N = 2$ super-symmetry is the standard physics counterpart for TGD super symmetry. $N = 2$ super-symmetry indeed involves the doubling of super generators and super generators carry $U(1)$ charge having an interpretation as fermion number in recent context. The so called short representations of $N = 2$ super-symmetry algebra can be regarded as representations of $N = 1$ super-symmetry algebra.

WCW gamma matrix Γ_n, $n > 0$ corresponds to an operator creating fermion whereas Γ_n, $n < 0$ annihilates anti-fermion. For the Hermitian conjugate $\Gamma_n^\dagger$ the roles of fermion and anti-fermion are interchanged. Only the anti-commutators of gamma matrices and their Hermitian conjugates are non-vanishing. The dynamical Kac Moody type generators are Hermitian and are constructed as bilinears of the gamma matrices and their Hermitian conjugates and, just like conserved currents of the ordinary quantum theory, contain parts proportional to $a^\dagger a$, $b^\dagger b$, $a^\dagger b^\dagger$ and ab (a and b refer to fermionic and anti-fermionic oscillator operators). The commutators between Kac Moody generators and Kac Moody generators and gamma matrices remain as such.

For a given value of m G_n, $n > 0$ creates fermions whereas G_n, $n < 0$ annihilates anti-fermions. Analogous result holds for $G_n^\dagger$. Virasoro generators remain Hermitian and decompose just like Kac Moody generators do. Thus the usual anti-commutation relations for the super

Virasoro generators must be replaced with anti-commutations between G_m and $G_n^\dagger$ and one has

$$\begin{aligned} \{G_m, G_n^\dagger\} &= 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m,-n} \ , \\ \{G_m, G_n\} &= 0 \ , \\ \{G_m^\dagger, G_n^\dagger\} &= 0 \ . \end{aligned} \tag{5.6.3}$$

The commutators of type $[L_m, L_n]$ are not changed. Same applies to the purely kinematical commutators between L_n and $G_m/G_m^\dagger$.

The Super Virasoro conditions satisfied by the physical states are as before in case of L_n whereas the conditions for G_n are doubled to those of G_n, $n < 0$ and $G_n^\dagger$, $n > 0$.

What could be the counterparts of stringy conformal fields in TGD framework?

The experience with string models would suggest the conformal symmetries associated with the complex coordinates of X^2 as a candidate for conformal super-symmetries. One can imagine two counterparts of the stringy coordinate z in TGD framework.

1. Super-symplectic and super Kac-Moody symmetries are local with respect to X^2 in the sense that the coefficients of generators depend on the invariant $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ rather than being completely free [K21]. Thus the real variable J replaces complex (or hyper-complex) stringy coordinate and effective 1-dimensionality holds true also now but in different sense than for conformal field theories.

2. The slicing of X^4 by string world sheets Y^2 and partonic 2-surfaces X^2 implied by number theoretical compactification implies string-parton duality and involves the super conformal fermionic gauge symmetries associated with the coordinates u and w in the dual dimensional reductions to stringy and partonic dynamics. These coordinates define the natural analogs of stringy coordinate. The effective reduction of X_l^3 to braid by finite measurement resolution implies the effective reduction of $X^4(X^3)$ to string world sheet. This implies quite strong resemblance with string model. The realization that spinor modes with well- define em charge must be localized at string world sheets makes the connection with strings even more explicit [K96].

   One can understand how Equivalence Principle emerges in TGD framework at space-time level when many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig.** 9 in the appendix of this book) is replaced with effective space-time lumping together the space-time sheets to M^4 endowed with effective metric. The quantum counterpart EP has most feasible interpretation in terms of Quantum Classical Correspondence (QCC): the conserved Kähler four-momentum equals to an eigenvalue of conserved Kähler-Dirac four-momentum acting as operator.

3. The conformal fields of string model would reside at X^2 or Y^2 depending on which description one uses and complex (hyper-complex) string coordinate would be identified accordingly. Y^2 could be fixed as a union of stringy world sheets having the strands of number theoretic braids as its ends. The proposed definition of braids is unique and characterizes finite measurement resolution at space-time level. X^2 could be fixed uniquely as the intersection of X_l^3 (the light-like 3-surface at which induced metric of space-time surface changes its signature) with $\delta M_\pm^4 \times CP_2$. Clearly, wormhole throats X_l^3 would take the role of branes and would be connected by string world sheets defined by number theoretic braids.

4. An alternative identification for TGD parts of conformal fields is inspired by $M^8 - H$ duality. Conformal fields would be fields in WCW . The counterpart of z coordinate could be the hyper-octonionic M^8 coordinate m appearing as argument in the Laurent series of WCW Clifford algebra elements. m would characterize the position of the tip of CD and the fractal hierarchy of CDs within CDs would give a hierarchy of Clifford algebras and thus inclusions of hyper-finite factors of type II_1. Reduction to hyper-quaternionic field -that is field in M^4 center of mass degrees of freedom- would be needed to obtained associativity. The arguments m at various level might correspond to arguments of N-point function in quantum field theory.

5.7 Still about induced spinor fields and TGD counterpart for Higgs

The understanding of the modified Dirac equation and of the possible classical counterpart of Higgs field in TGD framework is not completely satisfactory. The emergence of twistor lift of Kähler action [K31] [L33] inspired a fresh approach to the problem and it turned out that a very nice understanding of the situation emerges.

More precise formulation of the Dirac equation for the induced spinor fields is the first challenge. The well-definedness of em charge has turned out to be very powerful guideline in the understanding of the details of fermionic dynamics. Although induced spinor fields have also a part assignable space-time interior, the spinor modes at string world sheets determine the fermionic dynamics in accordance with strong form of holography (SH).

The well-definedness of em charged is guaranteed if induced spinors are associated with 2-D string world sheets with vanishing classical W boson fields. It turned out that an alternative manner to satisfy the condition is to assume that induced spinors at the boundaries of string world sheets are neutrino-like and that these string world sheets carry only classical W fields. Dirac action contains 4-D interior term and 2-D term assignable to string world sheets. Strong form of holography (SH) allows to interpret 4-D spinor modes as continuations of those assignable to string world sheets so that spinors at 2-D string world sheets determine quantum dynamics.

Twistor lift combined with this picture allows to formulate the Dirac action in more detail. Well-definedness of em charge implies that charged particles are associated with string world sheets assignable to the magnetic flux tubes assignable to homologically non-trivial geodesic sphere and neutrinos with those associated with homologically trivial geodesic sphere. This explains why neutrinos are so light and why dark energy density corresponds to neutrino mass scale, and provides also a new insight about color confinement.

A further important result is that the formalism works only for embedding space dimension $D=8$. This is due the fact that the number of vector components is the same as the number of spinor components of fixed chirality for $D=8$ and corresponds directly to the octonionic triality.

p-Adic thermodynamics predicts elementary particle masses in excellent accuracy without Higgs vacuum expectation: the problem is to understand fermionic Higgs couplings. The observation that CP_2 part of the modified gamma matrices gives rise to a term mixing M^4 chiralities contain derivative allows to understand the mass-proportionality of the Higgs-fermion couplings at QFT limit.

5.7.1 More precise view about modified Dirac equation

Consistency conditions demand that modified Dirac equation with modified gamma matrices Γ^α defined as contractions $\Gamma^\alpha = T^{\alpha k}\gamma_k$ of canonical momentum currents $T^{\alpha k}$ associated with the bosonic action with embedding space gamma matrices γ_k [K96, K73]. The Dirac operator is not hermitian in the sense that the conjugation for the Dirac equation for Ψ does not give Dirac equation for $\overline{\Psi}$ unless the modified gamma matrices have vanishing covariant divergence as vector at space-time surface. This says that classical field equations are satisfied. This consistency condition holds true also for spinor modes possibly localized at string world sheets to which one can perhaps assign area action plus topological action defined by Kähler magnetic flux. The interpretation is in terms of super-conformal invariance.

The challenge is to formulate this picture more precisely and here I have not achieved a satisfactory formulation. The question has been whether interior spinor field Ψ are present at all, whether only Ψ is present and somehow becomes singular at string world sheets, or whether both stringy spinors Ψ_s and interior spinors Ψ are present. Both Ψ and Ψ_s could be present and Ψ_s could serve as source for interior spinors with the same H-chirality.

The strong form of holography (SH) suggests that interior spinor modes Ψ_n are obtained as continuations of the stringy spinor modes $\Psi_{s,n}$ and one has $\Psi = \Psi_s$ at string world sheets. Dirac action would thus have a term localized at strong world sheets and bosonic action would contain similar term by the requirement of super-conformal symmetry. Can one realize this intuition?

1. Suppose that Dirac action has interior and stringy parts. For the twistor lift of TGD [L33] the interior part with gamma matrices given by the modified gamma matrices associated

with the sum of Kähler action and volume action proportional to cosmological constant Λ. The variation with respect to the interior spinor field Ψ gives modified Dirac equation in the interior with source term from the string world sheet. The H-chiralites of Ψ and Psi_s would be same. Quark like and leptonic H-chiralities have different couplings to Kähler gauge potential and mathematical consistency strongly encourages this.

What is important is that the string world sheet part, which is bilinear in interior and string world sheet spinor fields Ψ and Ψ_s and otherwise has the same form as Dirac action. The natural assumption is that the stringy Dirac action corresponds to the modified gamma matrices assignable to area action.

2. String world sheet must be minimal surface: otherwise hermiticity is lost. This can be achieved either by adding to the Kähler action string world sheet area term. Whatever the correct option is, quantum criticality should determine the value of string tension. The first string model inspired guess is that the string tension is proportional to gravitational constant $1/G = 1/l_P^2$ defining the radius fo M^4 twistor sphere or to $1/R^2$, R CP_2 radius. This would however allow only strings not much longer than l_P or R. A more natural estimate is that string tension is proportional to the cosmological constant Λ and depends on p-adic length scale as $1/p$ so that the tension becomes small in long length scales. Since Λ coupling contant type parameter, this estimate looks rather reasonable.

3. The variation of stringy Dirac action with action density

$$L = [\overline{\Psi}_s D_s^{\rightarrow}\Psi - \overline{\Psi}_s D_s^{\leftarrow}\Psi]\sqrt{g_2} + h.c. \quad (5.7.1)$$

with respect to stringy spinor field Ψ_s gives for Ψ Dirac equation $D_s\Psi = 0$ if there are no Lagrange multiplier terms (see below). The variation in interior gives $D\Psi = S = D_s\Psi_s$, where the source term S is located at string world sheets. Ψ satisfies at string world sheet the analog of 2-D massless Dirac equation associated with the induced metric. This is just what stringy picture suggests.

The stringy source term for D equals to $D_s\Psi_s$ localized at string world sheets: the construction of solutions would require the construction of propagator for D, and this does not look an attractive idea. For $D_s\Psi_s = 0$ the source term vanishes. Holomorphy for Ψ_s indeed implies $D_s\Psi = 0$.

4. $\Psi_s = \Psi$ would realize SH as a continuation of Ψ_s from string world sheet to Ψ in the interior. Could one introduce Lagrange multiplier term

$$L_1 = \overline{\Lambda}(\Psi - \Psi_s) + h.c.$$

to realize $\Psi_s = \Psi$? Lagrange multiplier spinor field Λ would serve a source in the Dirac equation for $\Psi = \Psi_s$ and Ψ should be constructed at string world sheet in terms of stringy fermionic propagator with Λ as source. The solution for Ψ_s would require the construction of 2-D stringy propagator for Ψ_s but in principle this is not a problem since the modes can be solved by holomorphy in hypercomplex stringy coordinate. The problem of this option is that the H-chiralities of Λ and Ψ would be opposite and the coupling of opposite H-chiralities is not in spirit with H-chirality conservation.

A possible cure is to replace the Lagrange multiplier term with

$$L_1 = \overline{\Lambda}^k\gamma_k(\Psi - \Psi_s) + h.c. \ . \quad (5.7.2)$$

The variation with respect to the spin 3/2 field Λ^k would give 8 conditions - just the number of spinor components for given H-chirality - forcing $\Psi = \Psi_s$! $D = 8$ would be in crucial role! In other embedding space dimensions the number of conditions would be too high or too low. One would however obtain

$$D_s\Psi = D_s\Psi_s = \Lambda^k\gamma_k \ . \quad (5.7.3)$$

One could of course solve Ψ at string world sheet from $\Lambda^k\gamma_k$ by constructing the 2-D propagator associated with D_s. Conformal symmetry for the modes however implies $D_s\Psi = 0$ so that one has actually $\Lambda^k = 0$ and Λ^k remains mere formal tool to realize the constraint $\Psi = \Psi_s$ in mathematically rigorous manner for embedding space dimension $D = 8$. This is a new very powerful argument in favor of TGD.

5. At the string world sheets Ψ would be annihilated both by D and D_s. The simplest possibility is that the actions of D and D_s are proportional to each other at string world sheets. This poses conditions on string world sheets, which might force the CP_2 projection of string world sheet to belong to a geodesic sphere or circle of CP_2. The idea that string world sheets and also 3-D surfaces with special role in TGD could correspond to singular manifolds at which trigonometric functions representing CP_2 coordinates tend to go outside their allowed value range supports this picture. This will be discussed below.
   (a) For the geodesic sphere of type II induced Kähler form vanishes so that the action of 4-D Dirac massless operator would be determined by the volume term (cosmological constant). Could the action of D reduce to that of D_s at string world sheets? Does this require a reduction of the metric to an orthogonal direct sum from string world sheet tangent space and normal space and that also normal part of D annihilates the spinors at the string world sheet? The modes of Ψ at string world sheets are locally constant with respect to normal coordinates.
   (b) For the geodesic sphere of type I induced Kähler form is non-vanishing and brings an additional term to D coming from CP_2 degrees of freedom. This might lead to trouble since the gamma matrix structures of D and D_s would be different. One could however add to string world sheet bosonic action a topological term as Kähler magnetic flux. Although its contribution to the field equations is trivial, the contribution to the modified gamma matrices is non-vanishing and equal to the contraction $J^{\alpha k}\gamma_k$ of half projection of the Kähler form with CP_2 gamma matrices. The presence of this term could allow the reduction of $D\Psi_s = 0$ and $D_s\Psi_s = 0$ to each other also in this case.

5.7.2 A more detailed view about string world sheets

In TGD framework gauge fields are induced and what typically occurs for the space-time surfaces is that they tend to "go out" from CP_2. Could various lower-D surfaces of space-time surface correspond to sub-manifolds of space-time surface?

1. To get a concrete idea about the situation it is best to look what happens in the case of sphere $S^2 = CP_1$. In the case of sphere S^2 the Kähler form vanishes at South and North poles. Here the dimension is reduced by 2 since all values of ϕ correspond to the same point. $sin(\Theta)$ equals to 1 at equator - geodesic circle - and here Kähler form is non-vanishing. Here dimension is reduced by 1 unit. This picture conforms with the expectations in the case of CP_2 These two situations correspond to 1-D and 2-D geodesic sub-manifolds.
2. CP_2 coordinates can be represented as cosines or sines of angles and the modules of cosine or sine tends to become larger than 1 (see `http://tinyurl.com/z3coqau`). In Eguchi-Hanson coordinates (r, Θ, Φ, Ψ) the coordinates r and Θ give rise to this kind of trigonometric coordinates. For the two cyclic angle coordinates (Φ, Ψ) one does not encounter this problem.
3. In the case of CP_2 only geodesic sub-manifolds with dimensions $D = 0, 1, 2$ are possible. 1-D geodesic submanifolds carry vanishing induce spinor curvature. The impossibility of 3-D geodesic sub-manifolds would suggest that 3-D surfaces are not important. CP_2 has two geodesic spheres: S^2_I is homologically non-trivial and S^2_{II} homologically trivial (see `http://tinyurl.com/z3coqau`).
   (a) Let us consider S^2_I first. CP_2 has 3 poles, which obviously relates to SU(3), and in Eguchi Hanson coordinates (r, θ, Φ, Ψ) the surface $r = \infty$ is one of them and corresponds - not to a 3-sphere - but homologically non-trivial geodesic 2- sphere, which is complex sub-manifold and orbits of $SU(2) \times U(1)$ subgroup. Various values of the coordinate Ψ correspond to same point as those of Φ at the poles of S^2. The Kähler form J and classical Z^0 and γ fields are non-vanishing whereas W gauge fields vanish leaving only induced γ

and Z^0 field as one learns by studying the detailed expressions for the curvature of spinor curvature and vierbein of CP_2.
String world sheet could have thus projection to S^2_I but both γ and Z^0 would be vanishing except perhaps at the boundaries of string world sheet, where Z^0 would naturally vanish in the picture provided by standard model. One can criticize the presence of Z^0 field since it would give a parity breaking term to the modified Dirac operator. SH would suggest that the reduction to electromagnetism at string boundaries might make sense as counterpart for standard model picture. Note that the original vision was that besides induced Kähler form and em field also Z^0 field could vanish at string world sheets.

(b) The homologically trivial geodesic sphere S^2_{II} is the orbit of $SO(3)$ subgroup and not a complex manifold. By looking the standard example about S^2_I, one finds that the both J, Z_0, and γ vanish and only the W components of spinor connection are non-vanishing. In this case the notion of em charge would not be well-defined for S^2_{II} without additional conditions. Partonic 2-surfaces, their light-like orbits, and boundaries of string world sheets could do so since string world sheets have 1-D intersection with with the orbits. This picture would make sense for the minimal surfaces replacing vacuum extremals in the case of twistor lift of TGD.
Since em fields are not present, the presence of classical W fields need not cause problems. The absence of classical em fields however suggests that the modes of induced spinor fields at boundaries of string worlds sheets must be em neutral and represent therefore neutrinos. The safest but probably too strong option would be right-handed neutrino having no coupling spinor connection but coupling to the CP_2 gamma matrices transforming it to left handed neutrino. Recall that ν_R represents a candidate for super-symmetry.
Neither charged leptons nor quarks would be allowed at string boundaries and classical W gauge potentials should vanish at the boundaries if also left-handed neutrinos are allowed: this can be achieved in suitable gauge. Quarks and charged leptons could reside only at string world sheets assignable to monopole flux tubes. This could relate to color confinement and also to the widely different mass scales of neutrinos and other fermions as will be found.

To sum up, the new result is that the distinction between neutrinos and other fermions could be understood in terms of the condition that em charge is well-defined. What looked originally a problem of TGD turns out to be a powerful predictive tool.

5.7.3 Classical Higgs field again

A motivation for returning back to Higgs field comes from the twistor lift of Kähler action.

1. The twistor lift of TGD [K31] [L33] brings in cosmological constant as the coefficient of volume term resulting in dimensional reduction of 6-D Kähler action for twistor space of space-time surface realized as surface in the product of twistor space of M^4 and CP_2. The radius of the sphere of M^4 twistor bundle corresponds to Planck length. Volume term is extremely small but removes the huge vacuum degeneracy of Kähler action. Vacuum extremals are replaced by 4-D minimal surfaces and modified Dirac equation is just the analog of massless Dirac equation in complete analogy with string models.
2. The well-definedness and conservation of fermionic em charges and SH demand the localization of fermions to string world sheets. The earlier picture assumed only em fields at string world sheets. More precise picture allows also W fields.
3. The first guess is that string world sheets are minimal surfaces and this is supported by the previous considerations demanding also string area term and Kähler magnetic flux tube. Here gravitational constant assignable to M^4 twistor space would be the first guess for the string tension.

What one can say about the possible existence of classical Higgs field?

1. TGD predicts both Higgs type particles and gauge bosons as bound states of fermions and antifermions and they differ only in that their polarization are in M^4 resp. CP_2 tangent space. p-adic thermodynamics [K46] gives excellent predictions for elementary particle masses

in TGD framework. Higgs vacuum expectation is not needed to predict fermion or boson masses. Standard model gives only a parametrization of these masses by assuming that Higgs couplings to fermions are proportional to their masses, it does not predict them.

The experimental fact is however that the couplings of Higgs are proportional to fermion masses and TGD should be able to predict this and there is a general argument for the proportonality, which however should be deduced from basic TGD. Can one achieve this?

2. Can one imagine any candidate for the classical Higgs field? There is no covariantly constant vector field in CP_2, whose space-time projection could define a candidate for classical Higgs field. This led years ago before the model for how bosons emerge from fermions to the wrong conclusion that TGD does not predict Higgs.

   The first guess for the possibly existing classical counterpart of Higgs field would be as CP_2 part for the divergence of the space-time vector defined modified gamma matrices expressible in terms of canonical momentum currents having natural interpretation as a generalization of force for point like objects to that for extended objects. Higgs field in this sense would however vanish by above consistency conditions and would not couple to spinors at all.

   Classical Higgs field should have only CP_2 part being CP_2 vector. What would be also troublesome that this proposale for classical Higgs field would involve second derivatives of embedding space coordinates. Hence it seems that there is no hope about geometrization of classical Higgs fields.

3. The contribution of the induced Kähler form gives to the modified gamma matrices a term expressible solely in terms of CP_2 gamma matrices. This term appears in modified Dirac equation and mixes M^4 chiralities - a signal for the massivation. This term is analogous to Higgs term expect that it contains covariant derivative.

   The question that I have not posed hitherto is whether this term could at QFT limit of TGD give rise to vacuum expectation of Higgs. The crucial observation is that the presence of derivative, which in quantum theory corresponds roughly to mass proportionality of chirality mixing coupling at QFT limit. This could explain why the coupling of Higgs field to fermions is proportional to the mass of the fermion at QFT limit!

4. For S^2_{II} type string world sheets assignable to neutrinos the contribution to the chirality mixing coupling should be of order of neutrino mass. The coefficient $1/L^4$ of the volume term defining cosmological constant [L33] separates out as over all factor in massless Dirac equation and the parameter characterizing the mass scale causing the mixing is of order $m = \omega_1\omega_2 R$. Here ω_1 characterizes the scale of gradient for CP_2 coordinates. The simplest minimal surface is that for which CP_2 projection is geodesic line with $\Phi = \omega_1 t$. ω_2 characterizes the scale of the gradient of spinor mode.

   Assuming $\omega_1 = \omega_2 \equiv \omega$ the scale m is of order neutrino mass $m_\nu \simeq .1$ eV from the condition $m \sim \omega^2 R \sim m_\nu$. This gives the estimate $\omega \sim \sqrt{m_{CP_2} m_\nu} \sim 10^2 m_p$ from $m_{CP_2} \sim 10^{-4} m_P$, which is weak mass scale and therefore perfectly sensible. The reduction $\Delta c/c$ of the light velocity from maximal signal velocity due the replacement $g_{tt} = 1 - R^2\omega^2$ is $\Delta c/c \sim 10^{-34}$ and thus completely negligible. This estimate does not make sense for charged fermions, which correspond to S^2_I type string world sheets.

   A possible problem is that if the value of the cosmological constant Λ evolves as $1/p$ as function of the length mass scale the mass scale of neutrinos should increase in short scales. This looks strange unless the mass scale remains below the cosmic temperature so that neutrinos would be always effectively massless.

5. For S^2_I type string world sheets assignable to charged fermions Kähler action dominates and the mass scales are expected to be higher than for neutrinos. For S^2_I type strings the modified gamma matrices contain also Kähler term and a rough estimate is that the ratio of two contributions is the ratio of the energy density of Kähler action to vacuum energy density. As Kähler energy density exceeds the value corresponding to vacuum energy density $1/L^4$, $L \sim 40$ μm, Kähler action density begins to dominate over dark energy density.

To sum up, this picture suggest that the large difference between the mass scales of neutrinos and em charged fermions is due to the fact that neutrinos are associated with string world sheet of type II and em charged fermions with string world sheets of type I. Both strings world sheets would be accompanied by flux tubes but for charged particles the flux tubes would carry Kähler magnetic flux. Cosmological constant forced by twistor lift would make neutrinos massive and allow to understand neutrino mass scale.

Chapter 6

Recent View about Kähler Geometry and Spin Structure of WCW

6.1 Introduction

The construction of Kähler geometry of WCW ("world of classical worlds") is fundamental to TGD program. I ended up with the idea about physics as WCW geometry around 1985 and made a breakthrough around 1990, when I realized that Kähler function for WCW could correspond to Kähler action for its preferred extremals defining the analogs of Bohr orbits so that classical theory with Bohr rules would become an exact part of quantum theory and path integral would be replaced with genuine integral over WCW. The motivating construction was that for loop spaces leading to a unique Kähler geometry [A54]. The geometry for the space of 3-D objects is even more complex than that for loops and the vision still is that the geometry of WCW is unique from the mere existence of Riemann connection.

The basic idea is that WCW is union of symmetric spaces G/H labelled by zero modes which do not contribute to the WCW metric. There have been many open questions and it seems the details of the ealier approach [?]ust be modified at the level of detailed identifications and interpretations.

1. A longstanding question has been whether one could assign Equivalence Principle (EP) with the coset representation formed by the super-Virasoro representation assigned to G and H in such a way that the four-momenta associated with the representations and identified as inertial and gravitational four-momenta would be identical. This does not seem to be the case. The recent view will be that EP reduces to the view that the classical four-momentum associated with Kähler action is equivalent with that assignable to Kähler-Dirac action supersymmetrically related to Kähler action: quantum classical correspondence (QCC) would be in question. Also strong form of general coordinate invariance implying strong form of holography in turn implying that the super-symplectic representations assignable to space-like and light-like 3-surfaces are equivalent could imply EP with gravitational and inertial four-momenta assigned to these two representations.

   At classical level EP follows from the interpretation of GRT space-time as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrices of space-time sheets from Minkowski metric. Poincare invariance suggests strongly classical EP for the GRT limit in long length scales at least.

2. The detailed identification of groups G and H and corresponding algebras has been a long-standing problem. Symplectic algebra associated with$\delta M^4_\pm \times CP2$ ($\delta M^4_\pm$ is light-cone boundary - or more precisely, with the boundary of causal diamond (CD) defined as Cartesian product of CP_2 with intersection of future and past direct light cones of M^4 has Kac-Moody

type structure with light-like radial coordinate replacing complex coordinate z. Virasoro algebra would correspond to radial diffeomorphisms. I have also introduced Kac-Moody algebra assigned to the isometries and localized with respect to internal coordinates of the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian and which serve as natural correlates for elementary particles (in very general sense!). This kind of localization by force could be however argued to be rather ad hoc as opposed to the inherent localization of the symplectic algebra containing the symplectic algebra of isometries as sub-algebra. It turns out that one obtains direct sum of representations of symplectic algebra and Kac-Moody algebra of isometries naturally as required by the success of p-adic mass calculations.

3. The dynamics of Kähler action is not visible in the earlier construction. The construction also expressed WCW Hamiltonians as 2-D integrals over partonic 2-surfaces. Although strong form of general coordinate invariance (GCI) implies strong form of holography meaning that partonic 2-surfaces and their 4-D tangent space data should code for quantum physics, this kind of outcome seems too strong. The progress in the understanding of the solutions of Kähler-Dirac equation led however to the conclusion that spinor modes other than right-handed neutrino are localized at string world sheets with strings connecting different partonic 2-surfaces. This leads to a modification of earlier construction in which WCW super-Hamiltonians are essentially integrals with integrand identified as a Noether super current for the deformations in G Each spinor mode gives rise to super current and the modes of right-handed neutrino and other fermions differ in an essential ways. Right-handed neutrino would correspond to symplectic algebra and other modes to the Kac-Moody algebra and one obtains the crucial 5 tensor factors of Super Virasoro required by p-adic mass calculations.

   The matrix elements of WCW metric between Killing vectors are expressible as anti-commutators of super-Hamiltonians identifiable as contractions of WCW gamma matrices with these vectors and give Poisson brackets of corresponding Hamiltonians. The anti-commutation relates of induced spinor fields are dictated by this condition. Everything is 3-dimensional although one expects that symplectic transformations localized within interior of X^3 act as gauge symmetries so that in this sense effective 2-dimensionality is achieved. The components of WCW metric are labelled by standard model quantum numbers so that the connection with physics is extremely intimate.

4. An open question in the earlier visions was whether finite measurement resolution is realized as discretization at the level of fundamental dynamics. This would mean that only certain string world sheets from the slicing by string world sheets and partonic 2-surfaces are possible. The requirement that anti-commutations are consistent suggests that string world sheets correspond to surfaces for which Kähler magnetic field is constant along string in well defined sense ($J_{\mu\nu}\epsilon^{\mu\nu}g^{1/2}$ remains constant along string). It however turns that by a suitable choice of coordinates of 3-surface one can guarantee that this quantity is constant so that no additional constraint results.

5. Quantum criticality is one of the basic notions of quantum TGD and its relationship to coset construction has remained unclear. In this chapter the concrete realization of criticality in terms of symmetry breaking hierarchy of Super Virasoro algebra acting on symplectic and Kac-Moody algebras. Also a connection with finite measurement resolution - second key notion of TGD - emerges naturally.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L22].

6.2 WCW As A Union Of Homogenous Or Symmetric Spaces

The physical interpretation and detailed mathematical understanding of super-conformal symmetries has developed rather slowly and has involved several side tracks. In the following I try to summarize the basic picture with minimal amount of formulas with the understanding that the statement "Noether charge associated with geometrically realized Kac-Moody symmetry" is enough for the reader to write down the needed formula explicitly. Formula oriented reader might deny the value of the approach giving weight to principles. My personal experience is that piles of formulas too often hide the lack of real understanding.

In the following the vision about WCW as union of coset spaces is discussed in more detail.

6.2.1 Basic Vision

The basic view about coset space construction for WCW has not changed.

1. The idea about WCW as a union of coset spaces G/H labelled by zero modes is extremely attractive. The structure of homogenous space [A14] (`http://tinyurl.com/y7u2t8jo`) means at Lie algebra level the decomposition $g = h \oplus t$ to sub-Lie-algebra h and its complement t such that $[h, t] \subset t$ holds true. Homogeneous spaces have G as its isometries. For symmetric space the additional condition $[t, t] \subset h$ holds true and implies the existence of involution changing at the Lie algebra level the sign of elements of t and leaving the elements of h invariant. The assumption about the structure of symmetric space [A31] (`http://tinyurl.com/ycouv7uh`) implying covariantly constant curvature tensor is attractive in infinite-dimensional case since it gives hopes about calculability.

   An important source of intuition is the analogy with the construction of CP_2, which is symmetric space A particular choice of h corresponds to Lie-algebra elements realized as Killing vector fields which vanish at particular point of WCW and thus leave 3-surface invariant. A preferred choice for this point is as maximum or minimum of Kähler function. For this 3-surface the Hamiltonians of h should be stationary. If symmetric space property holds true then commutators of $[t, t]$ also vanish at the minimum/maximum. Note that Euclidian signature for the metric of WCW requires that Kähler function can have only maximum or minimum for given zero modes.

2. The basic objection against TGD is that one cannot use the powerful canonical quantization using the phase space associated with configuration space - now WCW . The reason is the extreme non-linearity of the Kähler action and its huge vacuum degeneracy, which do not allow the construction of Hamiltonian formalism. Symplectic and Kähler structure must be realized at the level of WCW . In particular, Hamiltonians must be represented in completely new manner. The key idea is to construct WCW Hamiltonians as anti-commutators of super-Hamiltonians defining the contractions of WCW gamma matrices with corresponding Killing vector fields and therefore defining the matrix elements of WCW metric in the tangent vector basis defined by Killing vector fields. Super-symmetry therefre gives hopes about constructing quantum theory in which only induced spinor fields are second quantized and embedding space coordinates are treated purely classically.

3. It is important to understand the difference between symmetries and isometries assigned to the Kähler function. Symmetries of Kähler function do not affect it. The symmetries of Kähler action are also symmetries of Kähler action because Kähler function is Kähler action for a preferred extremal (here there have been a lot of confusion). Isometries leave invariant only the quadratic form defined by Kähler metric $g_{M\overline{N}} = \partial_M \partial_{\overline{L}} K$ but not Kähler function in general. For G/H decomposition G represents isometries and H both isometries and symmetries of Kähler function.

   CP_2 is familiar example: $SU(3)$ represents isometries and $U(2)$ leaves also Kähler function invariant since it depends on the $U(2)$ invariant radial coordinate r of CP_2. The origin $r = 0$ is left invariant by $U(2)$ but for $r > 0$ $U(2)$ performs a rotation at $r = constant$ 3-sphere. This simple picture helps to understand what happens at the level of WCW .

   How to then distinguish between symmetries and isometries? A natural guess is that one obtains also for the isometries Noether charges but the vanishing of boundary terms at spatial infinity crucial in the argument leading to Noether theorem as $\Delta S = \Delta Q = 0$ does not hold true anymore and one obtains charges which are not conserved anymore. The symmetry breaking contributions would now come from effective boundaries defined by wormhole throats at which the induce metric changes its signature from Minkowskian to Euclidian. A more delicate situation is in which first order contribution to ΔS vanishes and therefore also ΔQ and the contribution to ΔS comes from second variation allowing also to define Noether charge which is not conserved.

4. The simple picture about CP_2 as symmetric space helps to understand the general vision if one assumes that WCW has the structure of symmetric space. The decomposition $g = h + t$ corresponds to decomposition of symplectic deformations to those which vanish at 3-surface (h) and those which do not (t).

   For the symmetric space option, the Poisson brackets for super generators associated with t give Hamiltonians of h identifiable as the matrix elements of WCW metric. They would not vanish although they are stationary at 3-surface meaning that Riemann connection vanishes at 3-surface. The Hamiltonians which vanish at 3-surface X^3 would correspond to t and the Hamiltonians for which Killing vectors vanish and which therefore are stationary at X^3 would correspond to h. Outside X^3 the situation would of course be different. The metric would be obtained by parallel translating the metric from the preferred point of WCW to elsewhere and symplectic transformations would make this parallel translation.

   For the homogenous space option the Poisson brackets for super generators of t would still give Hamiltonians identifiable as matrix elements of WCW metric but now they would be necessary those of h. In particular, the Hamiltonians of t do not in general vanish at X^3.

6.2.2 Equivalence Principle And WCW

6.2.3 Equivakence Principle At Quantum And Classical Level

Quite recently I returned to an old question concerning the meaning of Equivalence Principle (EP) in TGD framework.

Heretic would of course ask whether the question about whether EP is true or not is a pseudo problem due to uncritical assumption there really are two different four-momenta which must be identified. If even the identification of these two different momenta is difficult, the pondering of this kind of problem might be waste of time.

At operational level EP means that the scattering amplitudes mediated by graviton exchange are proportional to the product of four-momenta of particles and that the proportionality constant does not depend on any other parameters characterizing particle (except spin). The are excellent reasons to expect that the stringy picture for interactions predicts this.

1. The old idea is that EP reduces to the coset construction for Super Virasoro algebra using the algebras associated with G and H. The four-momenta assignable to these algebras would be identical from the condition that the differences of the generators annihilate physical states and identifiable as inertial and gravitational momenta. The objection is that for the preferred 3-surface H by definition acts trivially so that time-like translations leading out from the boundary of CD cannot be contained by H unlike G. Hence four-momentum is not associated with the Super-Virasoro representations assignable to H and the idea about assigning EP to coset representations does not look promising.

2. Another possibility is that EP corresponds to quantum classical correspondence (QCC) stating that the classical momentum assignable to Kähler action is identical with gravitational momentum assignable to Super Virasoro representations. This forced to reconsider the questions about the precise identification of the Kac-Moody algebra and about how to obtain the magic five tensor factors required by p-adic mass calculations [K90].

   A more precise formulation for EP as QCC comes from the observation that one indeed obtains two four-momenta in TGD approach. The classical four-momentum assignable to the Kähler action and that assignable to the Kähler-Dirac action. This four-momentum is an operator and QCC would state that given eigenvalue of this operator must be equal to the value of classical four-momentum for the space-time surfaces assignable to the zero energy state in question. In this form EP would be highly non-trivial. It would be justified by the Abelian character of four-momentum so that all momentum components are well-defined also quantum mechanically. One can also consider the splitting of four-momentum to longitudinal and transversal parts as done in the parton model for hadrons: this kind of splitting would be very natural at the boundary of CD. The objection is that this correspondence is nothing more than QCC.

3. A further possibility is that duality of light-like 3-surfaces and space-like 3-surfaces holds true. This is the case if the action of symplectic algebra can be defined at light-like 3-surfaces or even

for the entire space-time surfaces. This could be achieved by parallel translation of light-cone boundary providing slicing of CD. The four-momenta associated with the two representations of super-symplectic algebra would be naturally identical and the interpretation would be in terms of EP.

One should also understand how General Relativity and EP emerge at classical level. The understanding comes from the realization that GRT is only an effective theory obtained by endowing M^4 with effective metric.

1. The replacement of superposition of fields with superposition of their effects means replacing superposition of fields with the set-theoretic union of space-time surfaces. Particle experiences sum of the effects caused by the classical fields at the space-time sheets.
2. This is true also for the classical gravitational field defined by the deviation from flat Minkowski metric instandard M^4 coordinates for the space-time sheets. One can define effective metric as sum of M^4 metric and deviations. This effective metric would correspond to that of General Relativity. This resolves long standing issues relating to the interpretation of TGD.
3. Einstein's equations could hold true for the effective metric. They are motivated by the underlying Poincare invariance which cannot be realized as global conservation laws for the effective metric. The conjecture vanishing of divergence of Kähler energy momentum tensor can be seen as the microscopic justification for the claim that Einstein's equations hold true for the effective space-time.
4. The breaking of Poincare invariance could have interpretation as effective breaking in zero energy ontology (ZEO), in which various conserved charges are length dependent and defined separately for each causal diamond (CD).

One can of course consider the possibility that Einstein's equations generalize for preferred extremals of Kähbler action. This would actually represent at space-time level the notion of QCC rather than realise QCC interpreted as EP. The condition that the energy momentum tensor for Kähler action has vanishing covariant divergence would be satisfied in GRT if Einstein's equations with cosmological term hold true. This is the case also now but one can consider also more general solutions in which one has two cosmological constants which are not genuine constants anymore: this idea is however not promising.

An interesting question is whether inertial-gravitational duality generalizes to the case of color gauge charges so that color gauge fluxes would correspond to "gravitational" color charges and the charges defined by the conserved currents associated with color isometries would define "inertial" color charges. Since the induced color fields are proportional to color Hamiltonians multiplied by Kähler form they vanish identically for vacuum extremals in accordance with "gravitational" color confinement.

6.2.4 Criticism Of The Earlier Construction

The earlier detailed realization of super-Hamiltonians and Hamiltonians can be criticized.

1. Even after these more than twenty years it looks strange that the Hamiltonians should reduce to flux integrals over partonic 2-surfaces. The interpretation has been in terms of effective 2-dimensionality suggested strongly by strong form of general coordinate invariance stating that the descriptions based on light-like orbits of partonic 2-surfaces and space-like three surfaces at the ends of causal diamonds are dual so that only partonic 2-surfaces and 4-D tangent space data at them would matter. Strong form of holography implies effective 2-dimensionality but this should correspond gauge character for the action of symplectic generators in the interior the space-like 3-surfaces at the ends of CDs, which is something much milder.

   One expects that the strings connecting partonic 2-surfaces could bring something new to the earlier simplistic picture. The guess is that embedding space Hamiltonian assignable to a point of partonic 2-surface should be replaced with that defined as integral over string attached to the point. Therefore the earlier picture would suffer no modification at the level of general formulas.

2. The fact that the dynamics of Kähler action and Kähler-Dirac action are not directly involved with the earlier construction raises suspicions. I have proposed that Kähler function could allow identification as Dirac determinant [K96] but one would expect more intimate connection. Here the natural question is whether super-Hamiltonians for the Kähler-Dirac action could correspond to Kähler charges constructible using Noether's theorem for corresponding deformations of the space-time surface and would also be identifiable as WCW gamma matrices.

6.2.5 Is WCW Homogenous Or Symmetric Space?

A key question is whether WCW can be symmetric space [A31] (`http://tinyurl.com/y8ojglkb`) or whether only homogenous structure is needed. The lack of covariant constancy of curvature tensor might produce problems in infinite-dimensional context.

The algebraic conditions for symmetric space are $g = h + t$, $[h,t] \subset t$, $[t,t] \subset h$. The latter condition is the difficult one.

1. δCD Hamiltonians should induce diffeomorphisms of X^3 indeed leaving it invariant. The symplectic vector fields would be parallel to X^3. A stronger condition is that they induce symplectic transformations for which all points of X^3 remain invariant. Now symplectic vector fields vanish at preferred 3-surface (note that the symplectic transformations are r_M local symplectic transformations of $S^2 \times CP_2$).

2. For Kac-Moody algebra inclusion $H \subset G$ for the finite-dimensional Lie-algebra induces the structure of symmetric space. If entire algebra is involved this does not look physically very attractive idea unless one believes on symmetry breaking for both $SU(3)$, $U(2)_{ew}$, and $SO(3)$ and E_2 (here complex conjugation corresponds to the involution). If one assumes only Kac-Moody algebra as critical symmetries, the number of tensor factors is 4 instead of five, and it is not clear whether one can obtain consistency with p-adic mass calculations.

   Examples of 3-surfaces remaining invariant under $U(2)$ are 3-spheres of CP_2. They could correspond to intersections of deformations of CP_2 type vacuum extremals with the boundary of CD. Also geodesic spheres S^2 of CP_2 are invariant under $U(2)$ subgroup and would relate naturally to cosmic strings. The corresponding 3-surface would be $L \times S^2$, where L is a piece of light-like radial geodesic.

3. In the case of symplectic algebra one can construct the embedding space Hamiltonians inducing WCW Hamiltonians as products of elements of the isometry algebra of $S^2 \times CP_2$ for with parity under involution is well-defined. This would give a decomposition of the symplectic algebra satisfying the symmetric space property at the level embedding space. This decomposition does not however look natural at the level of WCW since the only single point of CP_2 and light-like geodesic of δM^4_+ can be fixed by $SO(2) \times U(2)$ so that the 3-surfaces would reduce to pieces of light rays.

4. A more promising involution is the inversion $r_M \to 1/r_M$ of the radial coordinate mapping the radial conformal weights to their negatives. This corresponds to the inversion in Super Virasoro algebra. t would correspond to functions which are odd functions of $u \equiv log(r_M/r_0)$ and h to even function of u. Stationary 3-surfaces would correspond to $u = 1$ surfaces for which $log(u) = 0$ holds true. This would assign criticality with Virasoro algebra as one expects on general grounds.

   $r_M = constant$ surface would most naturally correspond to a maximum of Kähler function which could indeed be highly symmetric. The elements with even u-parity should define Hamiltonians, which are stationary at the maximum of Kähler function. For other 3-surfaces obtained by $/r_M$-local) symplectic transformations the situation is different: now H is replaced with its symplectic conjugate hHg^{-1} of H is acceptable and if the conjecture is true one would obtained 3-surfaces assignable to perturbation theory around given maximum as symplectic conjugates of the maximum. The condition that H leaves X^3 invariant in poin-twise manner is certainly too strong and imply that the 3-surface has single point as CP_2 projection.

5. One can also consider the possibility that critical deformations correspond to h and non-critical ones to t for the preferred 3-surface. Criticality for given h would hold only for a

preferred 3-surface so that this picture would be very similar that above. Symplectic conjugates of h would define criticality for other 3-surfaces. WCW would decompose to a union corresponding to different criticalities perhaps assignable to the hierarchy of sub-algebras of conformal algebra labelled by integer whose multiples give the allowed conformal weights. Hierarchy of breakings of conformal symmetries would characterize this hierarchy of sectors of WCW .

For sub-algebras of the conformal algebras (Kac-Moody and symplectic algebra) the condition $[t,t] \subset h$ cannot hold true so that one would obtain only the structure of homogenous space.

6.2.6 Symplectic And Kac-Moody Algebras As Basic Building Bricks

6.3 Updated View About Kähler Geometry Of WCW

During last years the understanding of the mathematical aspects of TGD and of its connection with the experimental world has developed rapidly.

TGD differs in several respects from quantum field theories and string models. The basic mathematical difference is that the mathematically poorly defined notion of path integral is replaced with the mathematically well-defined notion of functional integral defined by the Kähler function defining Kähler metric for WCW ("world of classical worlds"). Apart from quantum jump, quantum TGD is essentially theory of classical WCW spinor fields with WCW spinors represented as fermionic Fock states. One can say that Einstein's geometrization of physics program is generalized to the level of quantum theory.

It has been clear from the beginning that the gigantic super-conformal symmetries generalizing ordinary super-conformal symmetries are crucial for the existence of WCW Kähler metric. The detailed identification of Kähler function and WCW Kähler metric has however turned out to be a difficult problem. It is now clear that WCW geometry can be understood in terms of the analog of AdS/CFT duality between fermionic and space-time degrees of freedom (or between Minkowskian and Euclidian space-time regions) allowing to express Kähler metric either in terms of Kähler function or in terms of anti-commutators of WCW gamma matrices identifiable as super-conformal Noether super-charges for the symplectic algebra assignable to $\delta M^4_\pm \times CP_2$. The string model type description of gravitation emerges and also the TGD based view about dark matter becomes more precise. String tension is however dynamical rather than pregiven and the hierarchy of Planck constants is necessary in order to understand the formation of gravitationally bound states. Also the proposal that sparticles correspond to dark matter becomes much stronger: sparticles actually are dark variants of particles.

A crucial element of the construction is the assumption that super-symplectic and other super-conformal symmetries having the same structure as 2-D super-conformal groups can be seen a broken gauge symmetries such that sub-algebra with conformal weights coming as n-ples of those for full algebra act as gauge symmetries. In particular, the Noether charges of this algebra vanish for preferred extremals- this would realize the strong form of holography implied by strong form of General Coordinate Invariance. This gives rise to an infinite number of hierarchies of conformal gauge symmetry breakings with levels labelled by integers $n(i)$ such that $n(i)$ divides $n(i+1)$ interpreted as hierarchies of dark matter with levels labelled by the value of Planck constant $h_{eff} = n \times h$. These hierarchies define also hierarchies of quantum criticalities, and are proposed to give rise to inclusion hierarchies of hyperfinite factors of II_1 having interpretation in terms of finite cognitive resolution with inclusions being characterized by the integers $n(+1)/n(i)$.

These hierarchies are fundamental for the understanding of living matter. Living matter is fighting in order to stay at criticality and uses metabolic energy and homeostasis to achieve this. In the biological death of the system (self) a phase transition increasing h_{eff} finally takes place. The sub-selves of self experienced by self as mental images however die and are reborn at opposite boundary of the corresponding causal diamond (CD) and they genuinely evolve so that self can be said to become wiser even without dying! The purpose of this fighting against criticality would thus allow a possibility for sub-selves to evolve via subsequent re-incarnations. One interesting prediction is the possibility of time reversed mental images. The challenge is to understand what they do mean at the level of conscious experience.

6.3.1 Kähler Function, Kähler Action, And Connection With String Models

The definition of Kähler function in terms of Kähler action is possible because space-time regions can have also Euclidian signature of induced metric. Euclidian regions with 4-D CP_2 projection - wormhole contacts - are identified as lines of generalized Feynman diagrams - space-time correlates for basic building bricks of elementary particles. Kähler action from Minkowskian regions is imaginary and gives to the functional integrand a phase factor crucial for quantum field theoretic interpretation. The basic challenges are the precise specification of Kähler function of "world of classical worlds" (WCW) and Kähler metric.

There are two approaches concerning the definition of Kähler metric: the conjecture analogous to AdS/CFT duality is that these approaches are mathematically equivalent.

1. The Kähler function defining Kähler metric can be identified as Kähler action for space-time regions with Euclidian signature for a preferred extremal containing 3-surface as the ends of the space-time surfaces inside causal diamond (CD). Minkowskian space-time regions give to Kähler action an imaginary contribution interpreted as the counterpart of quantum field theoretic action. The exponent of Kähler function gives rise to a mathematically well-defined functional integral in WCW . WCW metric is dictated by the Euclidian regions of space-time with 4-D CP_2 projection.

   The basic question concerns the attribute "preferred". Physically the preferred extremal is analogous to Bohr orbit. What is the mathematical meaning of preferred extremal of Kähler action? The latest step of progress is the realization that the vanishing of generalized conformal charges for the ends of the space-time surface fixes the preferred extremals to high extent and is nothing but classical counterpart for generalized Virasoro and Kac-Moody conditions.

2. Fermions are also needed. The well-definedness of electromagnetic charge led to the hypothesis that spinors are restricted at string world sheets. One could also consider associativity as basic contraint to fermionic dynamics combined with the requirement that octonionic representation for gamma matrices is equivalent with the ordinary one. The conjecture is that this leads to the same outcome. This point is highly non-trivial and will be discussed below separately.

3. Second manner to define Kähler metric is as anticommutators of WCW gamma matrices identified as super-symplectic Noether charges for the Dirac action for induced spinors with string tension proportional to the inverse of Newton's constant. These charges are associated with the 1-D space-like ends of string world sheets connecting the wormhole throats. WCW metric contains contributions from the spinor modes associated with various string world sheets connecting the partonic 2-surfaces associated with the 3-surface.

   It is clear that the information carried by WCW metric about 3-surface is rather limited and that the larger the number of string world sheets, the larger the information. This conforms with strong form of holography and the notion of measurement resolution as a property of quantums state. Duality clearly means that Kähler function is determined either by space-time dynamics inside Euclidian wormhole contacts or by the dynamics of fermionic strings in Minkowskian regions outside wormhole contacts. This duality brings strongly in mind AdS/CFT duality. One could also speak about fermionic emergence since Kähler function is dictated by the Kähler metric part from a real part of gradient of holomorphic function: a possible identification of the exponent of Kähler function is as Dirac determinant.

6.3.2 Symmetries of WCW

Towards the end of year 2023 a dramatic progress in the understanding of WCW geometry took place and the following piece of text summarizes the findings. It turned that the original intuitive picture was surprisingly near to what now looks the correct view.

The situation before 2023

WCW geometry exists only if it has maximal isometries. I have proposed that WCW could be regarded as a union of generalized symmetric spaces labelled by zero modes which do not contribute to the metric. The induced Kähler field is invariant under symplectic transformations of CP_2 and would therefore define zero mode degrees of freedom if one assumes that WCW metric has symplectic transformations as isometries. In particular, Kähler magnetic fluxes would define zero modes and are quantized closed 2-surfaces. The induced metric appearing in Kähler action is however not zero mode degree of freedom. If the action contains volume term, the assumption about union of symmetric spaces is not well-motivated.

Symplectic transformations are not the only candidates for the isometries of WCW. The basic picture about what these maximal isometries could be, is partially inspired by string models.

1. A weaker proposal is that the symplectomorphisms of H define only symplectomorphisms of WCW. Extended conformal symmetries define also a candidate for isometry group. Remarkably, light-like boundary has an infinite-dimensional group of isometries which are in 1-1 correspondence with conformal symmetries of $S^2 \subset S^2 \times R_+ = \delta M^4_+$.
2. Extended Kac Moody symmetries induced by isometries of δM^4_+ are also natural candidates for isometries. The motivation for the proposal comes from physical intuition deriving from string models. Note they do not include Poincare symmetries, which act naturally as isometries in the moduli space of causal diamonds (CDs) forming the "spine" of WCW.
3. The light-like orbits of partonic 2-surfaces might allow separate symmetry algebras. One must however notice that there is exchange of charges between interior degrees of freedom and partonic 2-surfaces. The essential point is that one can assign to these surface conserved charges when the dual light-like coordinate defines time coordinate. This picture also assumes a slicing of space-time surface by by the partonic orbits for which partonic orbits associated with wormrhole throats and boundaries of the space-time surface would be special. This slicing would correspond to Hamilton-Jacobi structure.
4. Fractal hierarchy of symmetry algebras with conformal weights, which are non-negative integer multiples of fundamental conformal weights, is essential and distinguishes TGD from string models. Gauge conditions are true only the isomorphic subalgebra and its commutator with the entire algebra and the maximal gauge symmetry to a dynamical symmetry with generators having conformal weights below maximal value. This view also conforms with p-adic mass calculations.
5. The realization of the symmetries for 3-surfaces at the boundaries of CD and for light-like orbits of partonic 2-surfaces is known. The problem is how to extend the symmetries to the interior of the space-time surface. It is natural to expect that the symmetries at partonic orbits and light-cone boundary extend to the same symmetries.

Realization Of Super-Conformal Symmetries

The detailed realization of various super-conformal symmetries has been also a long standing problem.

1. Super-conformal symmetry requires that Dirac action for string world sheets is accompanied by string world sheet area as part of bosonic action. String world sheets are implied and can be present only in Minkowskian regions if one demands that octonionic and ordinary representations of induced spinor structure are equivalent (this requires vanishing of induced spinor curvature to achieve associativity in turn implying that CP_2 projection is 1-D). Note that 1-dimensionality of CP_2 projection is symplectically invariant property. Kähler action is not invariant under symplectic transformations. This is necessary for having non-trivial Kähler metric. Whether WCW really possesses super-symplectic isometries remains an open problem.
2. Super-conformal symmetry also demands that Kähler action is accompanied by what I call Kähler-Dirac action with gamma matrices defined by the contractions of the canonical momentum currents with embedding space-gamma matrices. Both the well-definedness of em charge and equivalence of octonionic spinor dynamics with ordinary one require the restriction of spinor modes to string world sheets with light-like boundaries at wormhole throats. K-D action with the localization of induced spinors at string world sheets is certainly the minimal option to consider.

3. Strong form of holography suggested by strong form of general coordinate invariance strongly suggests that super-conformal symmetry is broken gauge invariance in the sense that the clasical super-conformal charges for a sub-algebra of the symplectic algebra with conformal weights vanishing modulo some integer n vanish. The proposal is that n corresponds to the effective Planck constant as $h_{eff}/h = n$. The standard conformal symmetries for spinors modes at string world sheets is always unbroken gauge symmetry.

The conserved charges associated with holomorphies

Generalized holomorphy not only solves explicitly the equations of motion [L111] but, as found quite recently, also gives corresponding conserved Noether currents and charges.

1. Generalized holomorphy algebra generalizes the Super-Virasoro algebra and the Super-Kac-Moody algebra related to the conformal invariance of the string model. The corresponding Noether charges are conserved. Modified Dirac action allows to construct the supercharges having interpretation as WCW gamma matrices. This suggests an answer to a longstanding question related to the isometries of the "world of the classical worlds" (WCW).
2. Either the generalized holomorphies or the symplectic symmetries of $H = M^4 \times CP_2$ or both together define WCW isometries and corresponding super algebra. It would seem that symplectic symmetries induced from H are not necessarily needed and might correspond to symplectic symmetries of WCW. One would obtain a close similarity with the string model, except that one has half-algebra for which conformal weights are proportional to non-negative integers and gauge conditions only apply to an isomorphic subalgebra. These are labeled by positive integers and one obtains a hierarchy.
3. By their light-likeness, the light cone boundary and orbits of partonic 2-surfaces allow an infinite-dimensional isometry group. This is possible only in dimension four. Its transformations are generalized conformal transformations of 2-sphere (partonic 2-surface) depending on light-like radial coordinate such that the radial scaling compensates for the usual conformal scaling of the metric. The WCW isometries would thus correspond to the isometries of the parton orbit and of the boundary of the light cone! These two representations could provide alternative representations for the charges if the strong form of holography holds true and would realize a strong form of holography. Perhaps these realizations deserve to be called inertial and gravitational charges.

   Can these transformations leave the action invariant? For the light-cone boundary, this looks obvious if the light-cone is sliced by a surface parallel to the light-cone boundary. Note however that the tip of this surface might produce problems. A slicing defined by the Hamilton-Jacobi structure would be naturally associated with partonic orbits.
4. What about Poincare symmetries? They would act on the center of mass coordinates of causal diamonds (CDs) as found already earlier [L118]. CDs form the "spine" of WCW, which can be regarded as fiber space with fiber for a given CD containing as a fiber the space-time surfaces inside it.

The super-symmetric counterparts of holomorphic charges for the modified Dirac action and bilinear in fermionic oscillator operators associated with the second quantization of free spinor fields in H, define gamma matrices of WCW. Their anticommutators define the Kähler metric of WCW. There is no need to calculate either the action defining the classical Kähler action defining the Kähler function or its derivatives with respect to WCW complex coordinates and their conjugates. What is important is that this makes it possible to speak about WCW metric also for number theoretical discretization of WCW with space-time surfaces replaced with their number theoretic discretizations.

Could generalized holomorphy allow to sharpen the existing views?

This picture is rather speculative, allows several variants, and is not proven. There is now however a rather convincing ansatz for the general form of preferred extremals. This proposal relies on the realization of holography as generalized 4-D holomorphy. Could it help to make the picture more precise?

1. Explicit solution of field equations in terms of the generalized holomorphy is now known. The solution ansatz is independent of action as long it is general coordinate invariance depending only on the induced geometric structures. Space-time surfaces would be minimal surfaces apart from lower-dimensional singular surfaces at which the field equations involve the entire action. Only the singularities, classical charges and positions of topological interaction vertices depend on the choice of the action [L111]. Kähler action plus volume term is the choice of action forced by twistor lift making the choice of H unique.

2. The universality has a very intriguing implication. One can assign to any action of this kind conserved Noether currents and their fermionic counterparts (also super counterparts). One would have a huge algebra of conserved currents characterizing the space-time geometry. The corresponding charges can be made conserved by suitably modifying the form of holomorphic functions of the ansatz and therefore the time derivatives $\partial_t h^k$ at the 3-D end of space-time surface at the boundary CD. This need not be the case for all deformations of partonic orbits. In any case, the 3-D holographic data seem to be dual as the strong form of holography suggests. The discussion of the symplectic symmetries leads to the conclusion that they give rise to conserved charges at the partonic 3-surfaces obeying Chern-Simons-Kähler dynamics, which is non-deterministic.

3. Hamilton-Jacobi structures emerge naturally as generalized conformal structures of space-time surfaces and M^4 [L114]. This inspires a proposal for a generalization of modular invariance and of moduli spaces as subspaces of Teichmüller spaces.

4. One can assign to holomorphy conserved Noether charges. The conservation reduces to the algebraic conditions satisfied for the same reason as field equations, i.e. the conservation conditions involving contractions of complex tensors of type (1,1) with tensors of type (2,0) and (0,2). The charges have the same form as Noether charges but it is not completely clear whether the action remains invariant under these transformations. This point is non-trivial since Noether theorem says that invariance of the action implies the existence of conserved charges but not vice versa. Could TGD represent a situation in which the equivalence between symmetries of action and conservation laws fails?

   Also string models have conformal symmetries but in this case 2-D area form suffers conformal scaling. Also the fact that holomorphic ansatz is satisfied for such a large class of actions apart from singularities suggests that the action is not invariant.

5. The action should define Kähler function for WCW identified as the space of Bohr orbits. WCW Kähler metric is defined in terms of the second derivatives of the Kähler action of type (1,1) with respect to complex coordinates of WCW. Does the invariance of the action under holomorphies imply a trivial Kähler metric and constant Kähler function?

   Here one must be very cautious since by holography the variations of the space-time surface are induced by those of 3-surface defining holographic data so that the entire space-time surface is modified and the action can change. The presence of singularities, analogous to poles and cuts of an analytic function and representing particles, suggests that the action represents the interactions of particles and must change. Therefore the action might not be invariant under holomorphies. The parameters characterizing the singularities should affect the value of the action just as the positions of these singularities in 2-D electrostatistics affect the Coulomb energy.

   Generalized conformal charges and supercharges define a generalization of Super Virasoro algebra of string models. Also Kac-Moody algebra assignable to the isometries of $\delta M^4_+ \times CP_2$ and light H generalizes trivially.

6. An absolutely essential point is that generalized holomorphisms are *not* symmetries of Kähler function since otherwise Kähler metric involving second derivatives of type (1,1) with respect to complex coordinates of WCW is non-trivial if defined by these symmetry generators as differential operators. If Kähler function is equal to Kähler action, as it seems, Kähler action cannot be invariant under generalized holomorphies.

   Noether's theorem states that the invariance of the action under a symmetry implies the conservation of corresponding charge but does *not* claim that the existence of conserved Noether currents implies invariance of the action. Since Noether currents are conserved now, one would have a concrete example about the situation in which the inverse of Noether's theorem does not hold true. In a string model based on area action, conformal transformations of complex string coordinates give rise to conserved Noether currents as one easily checks. The area element defined by the induced metric suffers a conformal scaling so that the action is not invariant in this case.

Challenging the existing picture of WCW geometry

These findings make it possible to challenge and perhaps sharpen the existing speculations concerning the metric and isometries of WCW.

I have considered the possibility that also the symplectomorphisms of $\delta M^4 + \times CP_2$ could define WCW isometries. This actually the original proposal. One can imagine two options.

1. The continuation of symplectic transformations to transformations of the space-time surface from the boundary of light-cone or from the orbits partonic 2-surfaces should give rise to conserved Noether currents but it is not at all obvious whether this is the case.
2. One can assign conserved charges to the time evolution of the 3-D boundary data defining the holographic data: the time coordinate for the evolution would correspond to the light-like coordinate of light-cone boundary or partonic orbit. This option I have not considered hitherto. It turns out that this option works!

The conclusion would be that generalized holomorphies give rise to conserved charges for 4-D time evolution and symplectic transformations give rise to conserved charged for 3-D time evolution associated with the holographic data.

About extremals of Chern-Simons-Kähler action

Let us look first the general nature of the solutions to the extremization of Chern-Simons-Kähler action.

1. The light-likeness of the partonic orbits requires Chern-Simons action, which is equivalent to the topological action $J \wedge J$, which is total divergence and is a symplectic in variant. The field equations at the boundary cannot involve induced metric so that only induced symplectic structure remains. The 3-D holographic data at partonic orbits would extremize Cherns-Simons-Kähler action. Note that at the ends of the space-time surface about boundaries of CD one cannot pose any dynamics.
2. If the induced Kähler form has only the CP_2 part, the variation of Chern-Simons-Kähler form would give equations satisfied if the CP_2 projection is at most 2-dimensional and Chern-Simons action would vanish and imply that instanton number vanishes.
3. If the action is the sum of M^4 and CP_2 parts, the field equations in M^4 and CP_2 degrees of freedom would give the same result. If the induced Kähler form is identified as the sum of the M^4 and CP_2 parts, the equations also allow solutions for which the induced M^4 and CP_2 Kähler forms sum up to zero. This phase would involve a map identifying M^4 and CP_2 projections and force induce Kähler forms to be identical. This would force magnetic charge in M^4 and the question is whether the line connecting the tips of the CD makes non-trivial homology possible. The homology charges and the 2-D ends of the partonic orbit cancel each other so that partonic surfaces can have monopole charge.

   The conditions at the partonic orbits do not pose conditions on the interior and should allow generalized holomorphy. The following considerations show that besides homology charges as Kähler magnetic fluxes also Hamiltonian fluxes are conserved in Chern-Simons-Kähler dynamics.

Can one assign conserved charges with symplectic transformations or partonic orbits and 3-surfaces at light-cone boundary?

The geometric picture is that symplectic symmetries are Hamiltonian flows along the light-like partonic orbits generated by the projection A_t of the Kähler gauge potential in the direction of the light-like time coordinate. The physical picture is that the partonic 2-surface is a Kähler charged particle that couples to the Hamilton $H = A_t$. The Hamiltonians H_A are conserved in this time evolution and give rise to conserved Noether currents. The corresponding conserved charge is integral over the 2-surface defined by the area form defined by the induced Kähler form.

Let's examine the change of the Chern-Simons-Kähler action in a deformation that corresponds, for example, to the CP_2 symplectic transformation generated by Hamilton H_A. M^4 symplectic transformations can be treated in the same way:here however M^4 Kähler form would be involved, assumed to accompany Hamilton-Jacobi structure as a dynamically generated structure.

1. Instanton density for the induced Kähler form reduces to a total divergence and gives Chern-Simons-Kähler action, which is TGD analog of topological action. This action should change in infinitesimal symplectic transformations by a total divergence, which should vanish for extremals and give rise to a conserved current. The integral of the divergence gives a vanishing charge difference between the ends of the partonic orbit. If the symplectic transformations define symmetries, it should be possible to assign to each Hamiltonian H_A a conserved charge. The corresponding quantal charge would be associated with the modified Dirac action.

2. The conserved charge would be an integral over X^2. The surface element is not given by the metric but by the symplectic structure, so that it is preserved in symplectic transformations. The 2-surface of the time evolution should correspond to the Hamiltonian time transformation generated by the projection $A_\alpha = A_k \partial_\alpha s^k$ of the Kähler gauge potential A_k to the direction of light-like time coordinate $x^\alpha \equiv t$.

3. The effect of the generator $j_A^k = J^{kl}\partial_l H_A$ on the Kähler potential A_l is given by $j_A^k \partial_k A_l$. This can be written as $\partial_k A_l = J_{kl} + \partial_l A_k$. The first term gives the desired total divergence $\partial_\alpha(\epsilon^{\alpha\beta\gamma} J_{\beta\gamma} H_A)$.

   The second term is proportional to the term $\partial_\alpha H_A - \{A_\alpha, H\}$. Suppose that the induced Kähler form is transversal to the light-like time coordinate t, i.e. the induced Kähler form does not have components of form $J_{t\mu}$. In this kind of situation the only possible choice for α corresponds to the time coordinate t. In this situation one can perform the replacement $\partial_\alpha H_A - \{A_\alpha, H\} \rightarrow dH_A/dt - \{A_t, H\}$ This corresponds to a Hamiltonian time evolution generated by the projection A_t acting as a Hamiltonian. If this is really a Hamiltonian time evolution, one has $dH_A/dt - \{A, H\} = 0$. Because the Poisson bracket represents a commutator, the Hamiltonian time evolution equation is analogous to the vanishing of a covariant derivative of H_A along light-like curves: $\partial_t H_A + [A, H_A] = 0$. The physical interpretation is that the partonic surface develops like a particle with a Kähler charge. As a consequence the change of the action reduces to a total divergence.

   An explicit expression for the conserved current $J_A^\alpha = H_A \epsilon^{\alpha\beta\gamma} J_{\beta\gamma}$ can be derived from the vanishing of the total divergence. Symplectic transformations on X^2 generate an infinite-dimensional symplectic algebra. The charge is given by the Hamiltonian flux $Q_A = \int H_A J_{\beta\gamma} dx^\alpha \wedge dx^\beta$.

4. If the projection of the partonic path CP_2 or M^4 is 2-D, then the light-like geodesic line corresponds to the path of the parton surface. If A_l can be chosen parallel to the surface, its projection in the direction of time disappears and one has $A_t = 0$. In the more general case, X^2 could, for example, rotate in CP_2. In this case A_t is nonvanishing. If J is transversal (no Kähler electric field), charge conservation is obtained.

Do the above observations apply at the boundary of the light-cone?

1. Now the 3-surface is space-like and Chern-Simons-Kähler action makes sense. It is not necessary but emerges from the "instanton density" for the Kähler form. The symplectic transformations of $\delta M^4_+ \times CP_2$ are the symmetries. The most time evolution associated with the radial light-like coordinate would be from the tip of the light-cone boundary to the boundary of CD. Conserved charges as homological invariants defining symplectic algebra would be associated with the 2-D slices of 3-surfaces. For closed 3-surfaces the total charges from the sheets of 3-space as covering of δM^4_+ must sum up to zero.

2. Interestingly, the original proposal [K21] for the isometries of WCW was that the Hamiltonian fluxes assignable to M^4 and CP_2 degrees of freedom at light-like boundary act define the charges associated with the WCW isometries as symplectic transformations so that a strong

form of holography would have been be realized and space-time surface would have been effectively 2-dimensional. The recent view is that these symmetries pose conditions only on the 3-D holographic data. The holographic charges would correspond to additional isometries of WCW and would be well-defined for the 3-surfaces at the light-cone boundary.

To sum up, one can imagine many options but the following picture is perhaps the simplest one and is supported by physical intuition and mathematical facts. The isometry algebra of $\delta M^4_+ \times CP_2$ consists of generalized conformal and KM algebras at 3-surfaces in $\delta M^4_+ \times CP_2$ and symplectic algebras at the light cone boundary and 3-D light-like partonic orbits. The latter symmetries give constraints on the 3-D holographic data. It is still unclear whether one can assign generalized conformal and Kac-Moody charges to Chern-Simons-Kähler action. The isomorphic subalgebras labelled by a positive integer and their commutators with the entire algebra would annihilate the physical states. The isomorphic subalgebras labelled by a positive integer and their commutators with the entire algebra would annihilate the physical states. These two representations would generalize the notions of inertial and gravitational mass and their equivalence would generalize the Equivalence Principle.

Objection against the idea about theoretician friendly Mother Nature

One of the key ideas behind the TGD view of dark matter is that Nature is theoretician friendly [L112]. When the coupling strength proportional to $\hbar_{eff}$ becomes so large that perturbation series ceases to converge, a phase transition increasing the value of h_{eff} takes place so that the perturbation series converges.

One can however argue that this argument is quantum field-theoretic and does not apply in TGD since holography changes the very concept of perturbation theory. There is no path integral to worry about. Path integral is indeed such a fundamental concept that one expects it to have some approximate counterpart also in the TGD Universe. Bohr orbits are not completely deterministic: could the sum over the Bohr orbits however translate to an approximate description as a path integral at the QFT limit? The dynamics of light-like partonic orbits is indeed non-deterministic and could give rise to an analog of path integral as a finite sum.

1. The dynamics implied by Chern-Simons-Kähler action assignable to the partonic 3-surface with light-one coordinate in the role of time, is very topological in that the partonic orbits is light-like 3-surface and has 2-D CP_2 and M^4 projections unless the induced M^4 and CP_2 Kähler forms sum up to zero. The light-likeness of the projection is a very loose condition and and the sum over partonic orbits as possible representation of holographic data analogous to initial values (light-likeness!) is therefore analogous to the sum over all paths appearing as a representation of Schrödinger equation in wave mechanics.

   One would have an analog of 1-D QFT. This means that the infinities of quantum field theories are absent but for a large enough coupling strength $g^2/4\pi\hbar$ the perturbation series fails to converge. The increase of h_{eff} would resolve the problem. For instance, Dirac equation in atomic physics makes unphysical predictions when the value of nuclear charge is larger than $Z \sim 137$.

2. I have also considered a discrete variant of this picture motivated by the fact that the presence of the volume term in the action implies that the M^4 projection of the CP_2 type extremal is a light-like geodesic line. The light-like orbits would consist of pieces of light-like geodesics implying that the average velocity would be smaller than c: this could be seen as a correlate for massivation.

   The points at which the direction of segment changes would correspond to points at which energy and momentum transfer between the partonic orbit and environment takes place. This kind of quantum number transfer might occur at least for the fermionic lines as boundaries of string world sheets. They could be described quantum mechanically as interactions with classical fields in the same way as the creation of fermion pairs as a fundamental vertex [L111]. The same universal 2-vertex would be in question.

   At these points the minimal surface property would fail and the trace of the second fundamental form would not vanish but would have a delta function-like singularity. The CP_2 part of the second fundamental form has quantum numbers of Higgs so that there would be an analogy with the standard description of massivation by the Higgs mechanism. Higgs would be only where the vertices are.

3. What is intriguing, that the light-likeness of the projection of the CP_2 type extremals in M^4 leads to Virasoro conditions assignable to M^4 coordinates and this eventually led to the idea of conformal symmetries as isometries as WCW. In the case of the partonic orbits, the light-like curve would be in $M^4 \times CP_2$ but it would not be surprising if the generalization of the Virasoro conditions would emerge also now.

   One can write M^4 and CP_2 coordinates for the light-like curve as Fourier expansion in powers of $exp(it)$, where t is the light-like coordinate. This gives $h^k = \sum h^k_n exp(int)$. If the CP_2 projection of the orbits of the partonic 2-surface is geodesic circle, CP_2 metric s_{kl} is constant, the light-likeness condition $h_{kl}\partial_t h^k \partial_t h^l = 0$ gives $Re[h_{kl} \sum_m h^k_{n-m}\overline{h}^l_m] = 0$. This does not give Virasoro conditions.

   The condition $d/dt(h_{kl}\partial_t h^k \partial_t h^l = 0) = 0$ however gives the standard Virasoro condition in quantization condition stating that the operator counterparts of quantities $L_n = Re[h_{kl} \sum_m (n-m)h^k_{n-m}\overline{h}^l_m]$ annihilate the physical states. What is interesting is that the latter condition also allows time-like (and even space-like) geodesics.

   Could massivation mean a failure of light-likeness? For piecewise light-like geodesics the light-likeness condition would be true only inside the segments. By taking Fourier transform one expects to obtain Virasoro conditions with a cutoff analogous to the momentum cutoff in condensed matter physics for crystals.

4. In TGD the Virasoro, Kac-Moody algebras and symplectic algebras are replaced by half-algebras and the gauge conditions are satisfied for conformal weights which are n-multiples of fundamentals with with n larger than some minimal value. This would dramatically reduce the effects of the non-determinism and could make the sum over all paths allowed by the light-likeness manifestly finite and reduce it to a sum with a finite number of terms. This cutoff in degrees of freedom would correspond to a genuinely physical cutoff due to the finite measurement resolution coded to the number theoretical anatomy of the space-time surfaces. This cutoff is analogous to momentum cutoff and could at the space-time picture correspond to finite minimum length for the light-like segments of the orbit of the partoic 2-surface.

Boundary conditions at partonic orbits and holography

TGD reduces coupling constant evolution to a number theoretical evolution of the coupling parameters of the action identified as Kähler function for WCW. An interesting question is how the 3-D holographic data at the partonic orbits relates to the corresponding 3-D data at the ends of space-time surfaces at the boundary of CD, and how it relates to coupling constant evolution.

1. The twistor lift of TGD strongly favours 6-D Kähler action, which dimensionally reduces to Kähler action plus volume term plus topological $\int J \wedge J$ term reducing to Chern Simons-Kähler action. The coefficients of these terms are proposed to be expressible in terms of number theoretical invariants characterizing the algebraic extensions of rationals and polynomials determining the space-time surfaces by $M^8 - H$ duality.

   Number theoretical coupling constant evolution would be discrete. Each extension of rationals would give rise to its own coupling parameters involving also the ramified primes characterizing the polynomials involved and identified as p-adic length scales.

2. The time evolution of the partonic orbit would be non-deterministic but subject to the light-likeness constraint and boundary conditions guaranteeing conservation laws. The natural expectation is that the boundary/interface conditions for a given action cannot be satisfied for all partonic orbits (and other singularities). The deformation of the partonic orbit requiring that boundary conditions are satisfied, does not affect X^3 but the time derivatives $\partial_t h^k$ at X^3 are affected since the form of the holomorphic functions defining the space-time surface would change. The interpretation would be in terms of duality of the holographic data associated with the partonic orbits *resp.* X^3.

   There can of course exist deformations, which require the change of the coupling parameters of the action to satisfy the boundary conditions. One can consider an analog of renormalization group equations in which the deformation corresponds to a modification of the coupling parameters of the action, most plausibly determined by the twistor lift. Coupling parameters would label different regions of WCW and the space-time surfaces possible for two different sets of coupling parameters would define interfaces between these regions.

In order to build a more detailed view one must fix the details related to the action whose value defines the WCW Kähler function.

1. If Kähler action is identified as Kähler action, the identification is unique. There is however the possibility that the imaginary exponent of the instanton term or the contribution from the Euclidean region is not included in the definition of Kähler function. For instance instanton term could be interpreted as a phase of quantum state and would not contribute.

2. Both Minkowskian and Euclidean regions are involved and the Euclidean signature poses problems. The definition of the determinant as $\sqrt{-g_4}$ is natural in Minkowskian regions but gives an imaginary contribution in Euclidean regions. $\sqrt{|g_4|}$ is real in both regions. $i\sqrt{g_4}$ is real in Minkowskian regions but imaginary in the Euclidean regions.

   There is also a problem related to the instanton term, which does not depend on the metric determinant at all. In QFT context the instanton term is imaginary and this is important for instance in QCD in the definition of CP breaking vacuum functional. Should one include only the 4-D or possibly only Minkowskian contribution to the Kähler function imaginary coefficient for the instanton/Euclidian term would be possible?

3. Boundary conditions guaranteeing the conservation laws at the partonic orbits must be satisfied. Consider the $\sqrt{|g_4|}$ case. Charge transfer between Euclidean and Minkowskian regions. If the C-S-K term is real, also the charge transfer between partonic orbit and 4-D regions is possible. The boundary conditions at the partonic orbit fix it to a high degree and also affect the time derivatives $\partial_t h^k$ at X^3. This option looks physically rather attractive because classical conserved charges would be real.

   If the C-S-K term is imaginary it behaves like a free particle since charge exchange with Minkowskian and Euclidean regions is not possible. A possible interpretation of the possible M^4 contribution to momentum could be in terms of decay width. The symplectic charges do not however involve momentum. The imaginary contribution to momentum could therefore come only from the Euclidean region.

4. If the Euclidean contribution is imaginary, it seems that it cannot be included in the Kähler function. Since in M^8 picture the momenta of virtual fermions are in general complex, one could consider the possibility that Euclidean contribution to the momentum is imaginary and allows an interpretation as a decay width.

The TGD counterparts of the gauge conditions of string models

The string model picture forces to ask whether the symplectic algebras and the generalized conformal and Kac-Moody algebras could act as gauge symmetries.

1. In string model picture conformal invariance would suggest that the generators of the generalized conformal and KM symmetries act as gauge transformations annihilate the physical states. In the TGD framework, this does not however make sense physically. This also suggests that the components of the metric defined by supergenerators of generalized conformal and Kac Moody transformations vanish. If so, the symplectomorphisms $\delta M^4_+ \times CP_2$ localized with respect to the light-like radial coordinate acting as isometries would be needed. The half-algebras of both symplectic and conformal generators are labelled by a non-negative integer defining an analog of conformal weight so there is a fractal hierarchy of isomorphic subalgebras in both cases.

2. TGD forces to ask whether only subalgebras of both conformal and Kac-Moody half algebras, isomorphic to the full algebras, act as gauge algebras. This applies also to the symplectic case. Here it is essential that only the half algebra with non-negative multiples of the fundamental conformal weights is allowed. For the subalgebra annihilating the states the conformal weights would be fixed integer multiples of those for the full algebra. The gauge property would be true for all algebras involved. The remaining symmetries would

be genuine dynamical symmetries of the reduced WCW and this would reflect the number theoretically realized finite measurement resolution. The reduction of degrees of freedom would also be analogous to the basic property of hyperfinite factors assumed to play a key role in thee definition of finite measurement resolution.

3. For strong holography, the orbits of partonic 2-surfaces and boundaries of the spacetime surface at δM^4_+ would be dual in the information theoretic sense. Either would be enough to determine the space-time surface.

6.3.3 Interior Dynamics For Fermions, The Role Of Vacuum Extremals, And Dark Matter

The key role of CP_2-type and M^4-type vacuum extremals has been rather obvious from the beginning but the detailed understanding has been lacking. Both kinds of extremals are invariant under symplectic transformations of $\delta M^4 \times CP_2$, which inspires the idea that they give rise to isometries of WCW . The deformations CP_2-type extremals correspond to lines of generalized Feynman diagrams. M^4 type vacuum extremals in turn are excellent candidates for the building bricks of many-sheeted space-time giving rise to GRT space-time as approximation. For M^4 type vacuum extremals CP_2 projection is (at most 2-D) Lagrangian manifold so that the induced Kähler form vanishes and the action is fourth-order in small deformations. This implies the breakdown of the path integral approach and of canonical quantization, which led to the notion of WCW .

If the action in Minkowskian regions contains also string area, the situation changes dramatically since strings dominate the dynamics in excellent approximation and string theory should give an excellent description of the situation: this of course conforms with the dominance of gravitation.

String tension would be proportional to $1/\hbar G$ and this raises a grave classical counter argument. In string model massless particles are regarded as strings, which have contracted to a point in excellent approximation and cannot have length longer than Planck length. How this can be consistent with the formation of gravitationally bound states is however not understood since the required non-perturbative formulation of string model required by the large valued of the coupling parameter GMm is not known.

In TGD framework strings would connect even objects with macroscopic distance and would obviously serve as correlates for the formation of bound states in quantum level description. The classical energy of string connecting say the two wormhole contacts defining elementary particle is gigantic for the ordinary value of $\hbar$ so that something goes wrong.

I have however proposed [K76, K62, K63] that gravitons - at least those mediating interaction between dark matter have large value of Planck constant. I talk about gravitational Planck constant and one has $\hbar_{eff} = \hbar_{gr} = GMm/v_0$, where $v_0/c < 1$ (v_0 has dimensions of velocity). This makes possible perturbative approach to quantum gravity in the case of bound states having mass larger than Planck mass so that the parameter GMm analogous to coupling constant is very large. The velocity parameter v_0/c becomes the dimensionless coupling parameter. This reduces the string tension so that for string world sheets connecting macroscopic objects one would have $T \propto v_0/G^2Mm$. For $v_0 = GMm/\hbar$, which remains below unity for Mm/m^2_{Pl} one would have $h_{gr}/h = 1$. Hence action remains small and its imaginary exponent does not fluctuate wildly to make the bound state forming part of gravitational interaction short ranged. This is expected to hold true for ordinary matter in elementary particle scales. The objects with size scale of large neutron (100 μm in the density of water) - probably not an accident - would have mass above Planck mass so that dark gravitons and also life would emerge as massive enough gravitational bound states are formed. $h_{gr} = h_{eff}$ hypothesis is indeed central in TGD based view about living matter.

If one assumes that for non-standard values of Planck constant only n-multiples of super-conformal algebra in interior annihilate the physical states, interior conformal gauge degrees of freedom become partly dynamical. The identification of dark matter as macroscopic quantum phases labeled by $h_{eff}/h = n$ conforms with this.

The emergence of dark matter corresponds to the emergence of interior dynamics via breaking of super-conformal symmetry. The induced spinor fields in the interior of flux tubes obeying Kähler Dirac action should be highly relevant for the understanding of dark matter. The assumption that dark particles have essentially same masses as ordinary particles suggests that dark fermions correspond to induced spinor fields at both string world sheets and in the space-time interior: the spinor fields in the interior would be responsible for the long range correlations characterizing $h_{eff}/h = n$. Magnetic flux tubes carrying dark matter are key entities in TGD inspired quantum biology. Massless extremals represent second class of M^4 type non-vacuum extremals.

This view forces once again to ask whether space-time SUSY is present in TGD and how it is realized. With a motivation coming from the observation that the mass scales of particles and sparticles most naturally have the same p-adic mass scale as particles in TGD Universe I have proposed that sparticles might be dark in TGD sense. The above argument leads to ask whether the dark variants of particles correspond to states in which one has ordinary fermion at string world sheet and 4-D fermion in the space-time interior so that dark matter in TGD sense would almost by definition correspond to sparticles!

6.3.4 Classical Number Fields And Associativity And Commutativity As Fundamental Law Of Physics

The dimensions of classical number fields appear as dimensions of basic objects in quantum TGD. Embedding space has dimension 8, space-time has dimension 4, light-like 3-surfaces are orbits of 2-D partonic surfaces. If conformal QFT applies to 2-surfaces (this is questionable), one-dimensional structures would be the basic objects. The lowest level would correspond to discrete sets of points identifiable as intersections of real and p-adic space-time sheets. This suggests that besides p-adic number fields also classical number fields (reals, complex numbers, quaternions, octonions [A82]) are involved [K82] and the notion of geometry generalizes considerably. In the recent view about quantum TGD the dimensional hierarchy defined by classical number field indeed plays a key role. $H = M^4 \times CP_2$ has a number theoretic interpretation and standard model symmetries can be understood number theoretically as symmetries of hyper-quaternionic planes of hyper-octonionic space.

The associativity condition $A(BC) = (AB)C$ suggests itself as a fundamental physical law of both classical and quantum physics. Commutativity can be considered as an additional condition. In conformal field theories associativity condition indeed fixes the n-point functions of the theory. At the level of classical TGD space-time surfaces could be identified as maximal associative (hyper-quaternionic) sub-manifolds of the embedding space whose points contain a preferred hyper-complex plane M^2 in their tangent space and the hierarchy finite fields-rationals-reals-complex numbers-quaternions-octonions could have direct quantum physical counterpart [K82]. This leads to the notion of number theoretic compactification analogous to the dualities of M-theory: one can interpret space-time surfaces either as hyper-quaternionic 4-surfaces of M^8 or as 4-surfaces in $M^4 \times CP_2$. As a matter fact, commutativity in number theoretic sense is a further natural condition and leads to the notion of number theoretic braid naturally as also to direct connection with super string models.

At the level of Kähler-Dirac action the identification of space-time surface as a hyper-quaternionic sub-manifold of H means that the modified gamma matrices of the space-time surface defined in terms of canonical momentum currents of Kähler action using octonionic representation for the gamma matrices of H span a hyper-quaternionic sub-space of hyper-octonions at each point of space-time surface (hyper-octonions are the subspace of complexified octonions for which imaginary units are octonionic imaginary units multiplied by commutating imaginary unit). Hyper-octonionic representation leads to a proposal for how to extend twistor program to TGD framework [K96, K86].

How to achieve associativity in the fermionic sector?

In the fermionic sector an additional complication emerges. The associativity of the tangent- or normal space of the space-time surface need not be enough to guarantee the associativity at the level of Kähler-Dirac or Dirac equation. The reason is the presence of spinor connection. A possible cure could be the vanishing of the components of spinor connection for two conjugates of quaternionic coordinates combined with holomorphy of the modes.

1. The induced spinor connection involves sigma matrices in CP_2 degrees of freedom, which for the octonionic representation of gamma matrices are proportional to octonion units in

Minkowski degrees of freedom. This corresponds to a reduction of tangent space group $SO(1,7)$ to G_2. Therefore octonionic Dirac equation identifying Dirac spinors as complexified octonions can lead to non-associativity even when space-time surface is associative or co-associative.

2. The simplest manner to overcome these problems is to assume that spinors are localized at 2-D string world sheets with 1-D CP_2 projection and thus possible only in Minkowskian regions. Induced gauge fields would vanish. String world sheets would be minimal surfaces in $M^4 \times D^1 \subset M^4 \times CP_2$ and the theory would simplify enormously. String area would give rise to an additional term in the action assigned to the Minkowskian space-time regions and for vacuum extremals one would have only strings in the first approximation, which conforms with the success of string models and with the intuitive view that vacuum extremals of Kähler action are basic building bricks of many-sheeted space-time. Note that string world sheets would be also symplectic covariants.

   Without further conditions gauge potentials would be non-vanishing but one can hope that one can gauge transform them away in associative manner. If not, one can also consider the possibility that CP_2 projection is geodesic circle S^1: symplectic invariance is considerably reduces for this option since symplectic transformations must reduce to rotations in S^1.

3. The fist heavy objection is that action would contain Newton's constant G as a fundamental dynamical parameter: this is a standard recipe for building a non-renormalizable theory. The very idea of TGD indeed is that there is only single dimensionless parameter analogous to critical temperature. One can of coure argue that the dimensionless parameter is $\hbar G/R^2$, R CP_2 "radius".

   Second heavy objection is that the Euclidian variant of string action exponentially damps out all string world sheets with area larger than $\hbar G$. Note also that the classical energy of Minkowskian string would be gigantic unless the length of string is of order Planck length. For Minkowskian signature the exponent is oscillatory and one can argue that wild oscillations have the same effect.

   The hierarchy of Planck constants would allow the replacement $\hbar \to \hbar_{eff}$ but this is not enough. The area of typical string world sheet would scale as h_{eff} and the size of CD and gravitational Compton lengths of gravitationally bound objects would scale as $\sqrt{h_{eff}}$ rather than $\hbar_{eff} = GMm/v_0$, which one wants. The only way out of problem is to assume $T \propto (\hbar/h_{eff})^2 \times (1/h_{bar}G)$. This is however un-natural for genuine area action. Hence it seems that the visit of the basic assumption of superstring theory to TGD remains very short.

Is super-symmetrized Kähler-Dirac action enough?

Could one do without string area in the action and use only K-D action, which is in any case forced by the super-conformal symmetry? This option I have indeed considered hitherto. K-D Dirac equation indeed tends to reduce to a lower-dimensional one: for massless extremals the K-D operator is effectively 1-dimensional. For cosmic strings this reduction does not however take place. In any case, this leads to ask whether in some cases the solutions of Kähler-Dirac equation are localized at lower-dimensional surfaces of space-time surface.

1. The proposal has indeed been that string world sheets carry vanishing W and possibly even Z fields: in this manner the electromagnetic charge of spinor mode could be well-defined. The vanishing conditions force in the generic case 2-dimensionality.

   Besides this the canonical momentum currents for Kähler action defining 4 embedding space vector fields must define an integrable distribution of two planes to give string world sheet. The four canonical momentum currents $\Pi_k\alpha = \partial L_K/\partial_{\partial_\alpha h^k}$ identified as embedding 1-forms can have only two linearly independent components parallel to the string world sheet. Also the Frobenius conditions stating that the two 1-forms are proportional to gradients of two embedding space coordinates Φ_i defining also coordinates at string world sheet, must be satisfied. These conditions are rather strong and are expected to select some discrete set of string world sheets.

2. To construct preferred extremal one should fix the partonic 2-surfaces, their light-like orbits defining boundaries of Euclidian and Minkowskian space-time regions, and string world sheets.

At string world sheets the boundary condition would be that the normal components of canonical momentum currents for Kähler action vanish. This picture brings in mind strong form of holography and this suggests that might make sense and also solution of Einstein equations with point like sources.

3. The localization of spinor modes at 2-D surfaces would would follow from the well-definedness of em charge and one could have situation is which the localization does not occur. For instance, covariantly constant right-handed neutrinos spinor modes at cosmic strings are completely de-localized and one can wonder whether one could give up the localization inside wormhole contacts.

4. String tension is dynamical and physical intuition suggests that induced metric at string world sheet is replaced by the anti-commutator of the K-D gamma matrices and by conformal invariance only the conformal equivalence class of this metric would matter and it could be even equivalent with the induced metric. A possible interpretation is that the energy density of Kähler action has a singularity localized at the string world sheet.

   Another interpretation that I proposed for years ago but gave up is that in spirit with the TGD analog of AdS/CFT duality the Noether charges for Kähler action can be reduced to integrals over string world sheet having interpretation as area in effective metric. In the case of magnetic flux tubes carrying monopole fluxes and containing a string connecting partonic 2-surfaces at its ends this interpretation would be very natural, and string tension would characterize the density of Kähler magnetic energy. String model with dynamical string tension would certainly be a good approximation and string tension would depend on scale of CD.

5. There is also an objection. For M^4 type vacuum extremals one would not obtain any non-vacuum string world sheets carrying fermions but the successes of string model strongly suggest that string world sheets are there. String world sheets would represent a deformation of the vacuum extremal and far from string world sheets one would have vacuum extremal in an excellent approximation. Situation would be analogous to that in general relativity with point particles.

6. The hierarchy of conformal symmetry breakings for K-D action should make string tension proportional to $1/h_{eff}^2$ with $h_{eff} = h_{gr}$ giving correct gravitational Compton length $\Lambda_{gr} = GM/v_0$ defining the minimal size of CD associated with the system. Why the effective string tension of string world sheet should behave like $(\hbar/\hbar_{eff})^2$?

   The first point to notice is that the effective metric $G^{\alpha\beta}$ defined as $h^{kl}\Pi_k^\alpha\Pi_l^\beta$, where the canonical momentum current $\Pi_k\alpha = \partial L_K/\partial_{\partial_\alpha h^k}$ has dimension $1/L^2$ as required. Kähler action density must be dimensionless and since the induced Kähler form is dimensionless the canonical momentum currents are proportional to $1/\alpha_K$.

   Should one assume that α_K is fundamental coupling strength fixed by quantum criticality to $\alpha_K = 1/137$? Or should one regard g_K^2 as fundamental parameter so that one would have $1/\alpha_K = \hbar_{eff}/4\pi g_K^2$ having spectrum coming as integer multiples (recall the analogy with inverse of critical temperature)?

   The latter option is the in spirit with the original idea stating that the increase of h_{eff} reduces the values of the gauge coupling strengths proportional to α_K so that perturbation series converges (Universe is theoretician friendly). The non-perturbative states would be critical states. The non-determinism of Kähler action implying that the 3-surfaces at the boundaries of CD can be connected by large number of space-time sheets forming n conformal equivalence classes. The latter option would give $G^{\alpha\beta} \propto h_{eff}^2$ and $det(G) \propto 1/h_{eff}^2$ as required.

7. It must be emphasized that the string tension has interpretation in terms of gravitational coupling on only at the GRT limit of TGD involving the replacement of many-sheeted space-time with single sheeted one. It can have also interpretation as hadronic string tension or effective string tension associated with magnetic flux tubes and telling the density of Kähler magnetic energy per unit length.

   Superstring models would describe only the perturbative Planck scale dynamics for emission and absorption of $h_{eff}/h = 1$ on mass shell gravitons whereas the quantum description of bound states would require $h_{eff}/n > 1$ when the masses. Also the effective gravitational constant associated with the strings would differ from G.

The natural condition is that the size scale of string world sheet associated with the flux tube mediating gravitational binding is $G(M+m)/v_0$, By expressing string tension in the form $1/T = n^2\hbar G_1$, $n = h_{eff}/h$, this condition gives $\hbar G_1 = \hbar^2/M_{red}^2$, $M_{red} = Mm/(M+m)$. The effective Planck length defined by the effective Newton's constant G_1 analogous to that appearing in string tension is just the Compton length associated with the reduced mass of the system and string tension equals to $T = [v_0/G(M+m)]^2$ apart from a numerical constant ($2G(M+m)$ is Schwartschild radius for the entire system). Hence the macroscopic stringy description of gravitation in terms of string differs dramatically from the perturbative one. Note that one can also understand why in the Bohr orbit model of Nottale [E1] for the planetary system and in its TGD version [K76] v_0 must be by a factor 1/5 smaller for outer planets rather than inner planets.

Are 4-D spinor modes consistent with associativity?

The condition that octonionic spinors are equivalent with ordinary spinors looks rather natural but in the case of Kähler-Dirac action the non-associativity could leak in. One could of course give up the condition that octonionic and ordinary K-D equation are equivalent in 4-D case. If so, one could see K-D action as related to non-commutative and maybe even non-associative fermion dynamics. Suppose that one does not.

1. K-D action vanishes by K-D equation. Could this save from non-associativity? If the spinors are localized to string world sheets, one obtains just the standard stringy construction of conformal modes of spinor field. The induce spinor connection would have only the holomorphic component A_z. Spinor mode would depend only on z but K-D gamma matrix Γ^z would annihilate the spinor mode so that K-D equation would be satisfied. There are good hopes that the octonionic variant of K-D equation is equivalent with that based on ordinary gamma matrices since quaternionic coordinated reduces to complex coordinate, octonionic quaternionic gamma matrices reduce to complex gamma matrices, sigma matrices are effectively absent by holomorphy.
2. One can consider also 4-D situation (maybe inside wormhole contacts). Could some form of quaternion holomorphy [A101] [K86] allow to realize the K-D equation just as in the case of super string models by replacing complex coordinate and its conjugate with quaternion and its 3 conjugates. Only two quaternion conjugates would appear in the spinor mode and the corresponding quaternionic gamma matrices would annihilate the spinor mode. It is essential that in a suitable gauge the spinor connection has non-vanishing components only for two quaternion conjugate coordinates. As a special case one would have a situation in which only one quaternion coordinate appears in the solution. Depending on the character of quaternionion holomorphy the modes would be labelled by one or two integers identifiable as conformal weights.

   Even if these octonionic 4-D modes exists (as one expects in the case of cosmic strings), it is far from clear whether the description in terms of them is equivalent with the description using K-D equation based ordinary gamma matrices. The algebraic structure however raises hopes about this. The quaternion coordinate can be represented as sum of two complex coordinates as $q = z_1 + Jz_2$ and the dependence on two quaternion conjugates corresponds to the dependence on two complex coordinates z_1, z_2. The condition that two quaternion complexified gammas annihilate the spinors is equivalent with the corresponding condition for Dirac equation formulated using 2 complex coordinates. This for wormhole contacts. The possible generalization of this condition to Minkowskian regions would be in terms Hamilton-Jacobi structure.

   Note that for cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ the associativity condition for S^2 sigma matrix and without assuming localization demands that the commutator of Y^2 imaginary units is proportional to the imaginary unit assignable to X^2 which however depends on point of X^2. This condition seems to imply correlation between Y^2 and S^2 which does not look physical.

To summarize, the minimal and mathematically most optimistic conclusion is that Kähler-Dirac action is indeed enough to understand gravitational binding without giving up the associativity of the fermionic dynamics. Conformal spinor dynamics would be associative if the spinor modes are localized at string world sheets with vanishing W (and maybe also Z) fields guaranteeing well-definedness of em charge and carrying canonical momentum currents parallel to them. It is not quite clear whether string world sheets are present also inside wormhole contacts: for CP_2 type vacuum extremals the Dirac equation would give only right-handed neutrino as a solution (could they give rise to $N = 2$ SUSY?).

The construction of preferred extremals would realize strong form of holography. By conformal symmetry the effective metric at string world sheet could be conformally equivalent with the induced metric at string world sheets.

Dynamical string tension would be proportional to $\hbar/h_{eff}^2$ due to the proportionality $\alpha_K \propto 1/h_{eff}$ and predict correctly the size scales of gravitationally bound states for $\hbar_{gr} = \hbar_{eff} = GMm/v_0$. Gravitational constant would be a prediction of the theory and be expressible in terms of α_K and R^2 and $\hbar_{eff}$ ($G \propto R^2/g_K^2$).

In fact, all bound states - elementary particles as pairs of wormhole contacts, hadronic strings, nuclei [K52], molecules, etc. - are described in the same manner quantum mechanically. This is of course nothing new since magnetic flux tubes associated with the strings provide a universal model for interactions in TGD Universe. This also conforms with the TGD counterpart of AdS/CFT duality.

6.4 About some unclear issues of TGD

TGD has been in the middle of palace revolution during last two years and it is almost impossible to keep the chapters of the books updated. Adelic vision and twistor lift of TGD are the newest developments and there are still many details to be understood and errors to be corrected. The description of fermions in TGD framework has contained some unclear issues. Hence the motivation for the following brief comments.

There questions about the adelic vision about symmetries. Do the cognitive representations implying number theoretic disretization of the space-time surface lead to the breaking of the basic symmetries and are preferred embedding space coordinates actually necessary?

In the fermionic sector there are many questions deserving clarification. How quantum classical correspondence (QCC) is realized for fermions? How is SH realized for fermions and how does it lead to the reduction of dimension $D = 4$ to $D = 2$ (apart from number theoretical discretization)? Can scattering amplitudes be really formulated by using only the data at the boundaries of string sheets and what does this mean from the point of view of the modified Dirac equation? Are the spinors at light-like boundaries limiting values or sources? A long-standing issue concerns the fermionic anti-commutation relations: what motivated this article was the solution of this problem. There is also the general problem about whether statistical entanglement is "real".

6.4.1 Adelic vision and symmetries

In the adelic TGD SH is weakened: also the points of the space-time surface having embedding space coordinates in an extension of rationals (cognitive representation) are needed so that data are not precisely 2-D. I have believed hitherto that one must use preferred coordinates for the embedding space H - a subset of these coordinates would define space-time coordinates. These coordinates are determined apart from isometries. Does the number theoretic discretization imply loss of general coordinate invariance and also other symmetries?

The reduction of symmetry groups to their subgroups (not only algebraic since powers of e define finite-dimensional extension of p-adic numbers since e^p is ordinary p-adic number) is genuine loss of symmetry and reflects finite cognitive resolution. The physics itself has the symmetries of real physics.

The assumption about preferred embedding space coordinates is actually not necessary. Different choices of H-coordinates means only different and non-equivalent cognitive representations. Spherical and linear coordinates in finite accuracy do not provide equivalent representations.

6.4.2 Quantum-classical correspondence for fermions

Quantum-classical correspondence (QCC) for fermions is rather well-understood but deserves to be mentioned also here.

QCC for fermions means that the space-time surface as preferred extremal should depend on fermionic quantum numbers. This is indeed the case if one requires QCC in the sense that the fermionic representations of Noether charges in the Cartan algebras of symmetry algebras are equal to those to the classical Noether charges for preferred extremals.

Second aspect of QCC becomes visible in the representation of fermionic states as point like particles moving along the light-like curves at the light-like orbits of the partonic 2-surfaces (curve at the orbit can be locally only light-like or space-like). The number of fermions and antifermions dictates the number of string world sheets carrying the data needed to fix the preferred extremal by SH. The complexity of the space-time surface increases as the number of fermions increases.

6.4.3 Strong form of holography for fermions

It seems that scattering amplitudes can be formulated by assigning fermions with the boundaries of strings defining the lines of twistor diagrams [K31, L50]. This information theoretic dimensional reduction from $D = 4$ to $D = 2$ for the scattering amplitudes can be partially understood in terms of strong form of holography (SH): one can construct the theory by using the data at string worlds sheets and/or partonic 2-surfaces at the ends of the space-time surface at the opposite boundaries of causal diamond (CD).

4-D modified Dirac action would appear at fundamental level as supersymmetry demands but would be reduced for preferred extremals to its 2-D stringy variant serving as effective action. Also the value of the 4-D action determining the space-time dynamics would reduce to effective stringy action containing area term, 2-D Kähler action, and topological Kähler magnetic flux term. This reduction would be due to the huge gauge symmetries of preferred extremals. Sub-algebra of super-symplectic algebra with conformal weigths coming as n-multiples of those for the entire algebra and the commutators of this algebra with the entire algebra would annihilate the physical states, and thecorresponding classical Noether charges would vanish.

One still has the question why not the data at the entire string world sheets is not needed to construct scattering amplitudes. Scattering amplitudes of course need not code for the entire physics. QCC is indeed motivated by the fact that quantum experiments are always interpreted in terms of classical physics, which in TGD framework reduces to that for space-time surface.

6.4.4 The relationship between spinors in space-time interior and at boundaries between Euclidian and Minkoskian regions

Space-time surface decomposes to interiors of Minkowskian and Euclidian regions. At light-like 3-surfaces at which the four-metric changes, the 4-metric is degenerate. These metrically singular 3-surfaces - partonic orbits- carry the boundaries of string world sheets identified as carriers of fermionic quantum numbers. The boundaries define fermion lines in the twistor lift of TGD [K31, L50]. The relationship between fermions at the partonic orbits and interior of the space-time surface has however remained somewhat enigmatic.

So: What is the precise relationship between induced spinors Ψ_B at light-like partonic 3-surfaces and Ψ_I in the interior of Minkowskian and Euclidian regions? Same question can be made for the spinors Ψ_B at the boundaries of string world sheets and Ψ_I in interior of the string world sheets. There are two options to consider:

- Option I: Ψ_B is the limiting value of Ψ_I .
- Option II: Ψ_B serves as a source of Ψ_I .

For the Option I it is difficult to understand the preferred role of Ψ_B.

I have considered Option II already years ago but have not been able to decide.

1. That scattering amplitudes could be formulated only in terms of sources only, would fit nicely with SH, twistorial amplitude construction, and also with the idea that scattering amplitudes in gauge theories can be formulated in terms of sources of boson fields assignable to vertices and propagators. Now the sources would become fermionic.

2. One can take gauge theory as a guideline. One adds to free Dirac equation source term $\gamma^k A_k \Psi$. Therefore the natural boundary term in the action would be of the form (forgetting overall scale factor)

$$S_B = \overline{\Psi}_I \Gamma^\alpha (C-S) A_\alpha \Psi_B + c.c \quad .$$

Here the modified gamma matrix is $\Gamma^\alpha(C-S)$ (contravariant form is natural for light-like 3-surfaces) is most naturally defined by the boundary part of the action - naturally Chern-Simons term for Kähler action. A denotes the Kähler gauge potential.

3. The variation with respect to Ψ_B gives

$$G^\alpha (C-S) A_\alpha \Psi_I = 0$$

at the boundary so that the C-S term and interaction term vanish. This does not however imply vanishing of the source term! This condition can be seen as a boundary condition.

The same argument applies also to string world sheets.

6.4.5 About second quantization of the induced spinor fields

The anti-commutation relations for the induced spinors have been a long-standing issue and during years I have considered several options. The solution of the problem looks however stupifuingly simple. The conserved fermion currents are accompanied by super-currents obtained by replacing Ψ with a mode of the induced spinor field to get $\overline{u}_n \Gamma^\alpha \Psi$ or $\overline{\Psi} \Gamma^\alpha u_n$ with the conjugate of the mode. One obtains infinite number of conserved super currents. One can also replace both Ψ and $\overline{\Psi}$ in this manner to get purely bosonic conserved currents $\overline{u}_m \Gamma^\alpha u_n$ to which one can assign a conserved bosonic charges Q_{mn}.

I noticed this years ago but did not realize that these bosonic charges define naturally anti-commutators of fermionic creation and annihilation operators! The ordinary anti-commutators of quantum field theory follow as a special case! By a suitable unitary transformation of the spinor basis one can diagonalize the hermitian matrix defined by Q_{mn} and by performing suitable scalings one can transform anti-commutation relations to the standard form. An interesting question is whether the diagonalization is needed, and whether the deviation of the diagonal elements from unity could have some meaning and possibly relate to the hierarchy $h_{eff} = n \times h$ of Planck constants - probably not.

6.4.6 Is statistical entanglement "real" entanglement?

The question about the "reality" of statistical entanglement has bothered me for years. This entanglement is maximal and it cannot be reduced by measurement so that one can argue that it is not "real". Quite recently I learned that there has been a longstanding debate about the statistical entanglement and that the issue still remains unresolved.

The idea that all electrons of the Universe are maximally entangled looks crazy. TGD provides several variants for solutions of this problem. It could be that only the fermionic oscillator operators at partonic 2-surfaces associated with the space-time surface (or its connected component) inside given CD anti-commute and the fermions are thus indistinguishable. The extremist option is that the fermionic oscillator operators belonging to a network of partonic 2-surfaces connected by string world sheets anti-commute: only the oscillator operators assignable to the same scattering diagram would anti-commute.

What about QCC in the case of entanglement. ER-EPR correspondence introduced by Maldacena and Susskind for 4 years ago proposes that blackholes (maybe even elementary particles) are connected by wormholes. In TGD the analogous statement emerged for more than decade ago - magnetic flux tubes take the role of wormholes in TGD. Magnetic flux tubes were assumed to be accompanied by string world sheets. I did not consider the question whether string world sheets are always accompanied by flux tubes.

What could be the criterion for entanglement to be "real"? "Reality" of entanglement demands some space-time correlate. Could the presence of the flux tubes make the entanglement "real"? If statistical entanglement is accompanied by string connections without magnetic flux tubes, it would not be "real": only the presence of flux tubes would make it "real". Or is the presence of strings enough to make the statistical entanglement "real". In both cases the fermions associated with disjoint space-time surfaces or with disjoint CDs would not be indistinguishable. This looks rather sensible.

The space-time correlate for the reduction of entanglement would be the splitting of a flux tube and fermionic strings inside it. The fermionic strings associated with flux tubes carrying monopole flux are closed and the return flux comes back along parallel space-time sheet. Also fermionic string has similar structure. Reconnection of this flux tube with shape of very long flattened square splitting it to two pieces would be the correlate for the state function reduction reducing the entanglement with other fermions and would indeed decouple the fermion from the network.

6.5 About The Notion Of Four-Momentum In TGD Framework

The starting point of TGD was the energy problem of General Relativity [K90]. The solution of the problem was proposed in terms of sub-manifold gravity and based on the lifting of the isometries of space-time surface to those of $M^4 \times CP_2$ in which space-times are realized as 4-surfaces so that Poincare transformations act on space-time surface as an 4-D analog of rigid body rather than moving points at space-time surface. It however turned out that the situation is not at all so simple.

There are several conceptual hurdles and I have considered several solutions for them. The basic source of problems has been Equivalence Principle (EP): what does EP mean in TGD framework [K90] ? A related problem has been the interpretation of gravitational and inertial masses, or more generally the corresponding 4-momenta. In General Relativity based cosmology gravitational mass is not conserved and this seems to be in conflict with the conservation of Noether charges. The resolution is in terms of zero energy ontology (ZEO), which however forces to modify slightly the original view about the action of Poincare transformations.

A further problem has been quantum classical correspondence (QCC): are quantal four-momenta associated with super conformal representations and classical four-momenta associated as Noether charges with Kähler action for preferred extremals identical? Could inertial-gravitational duality - that is EP - be actually equivalent with QCC? Or are EP and QCC independent dualities. A powerful experimental input comes p-adic mass calculations [K54] giving excellent predictions provided the number of tensor factors of super-Virasoro representations is five, and this input together with Occam's razor strongly favors QCC=EP identification.

There is also the question about classical realization of EP and more generally, TGD-GRT correspondence.

Twistor Grassmannian approach has meant a technical revolution in quantum field theory (for attempts to understand and generalize the approach in TGD framework see [K86]. This approach seems to be extremely well suited to TGD and I have considered a generalization of this approach from $\mathcal{N} = 4$ SUSY to TGD framework by replacing point like particles with string world sheets in TGD sense and super-conformal algebra with its TGD version: the fundamental objects are now massless fermions which can be regarded as on mass shell particles also in internal lines (but with unphysical helicity). The approach solves old problems related to the realization of stringy amplitudes in TGD framework, and avoids some problems of twistorial QFT (IR divergences and the problems due to non-planar diagrams). The Yangian [A36] [B33, B24, B25] variant of 4-D conformal symmetry is crucial for the approach in $\mathcal{N} = 4$ SUSY, and implies the recently introduced notion of amplituhedron [B15]. A Yangian generalization of various super-conformal algebras seems more or less a "must" in TGD framework. As a consequence, four-momentum is expected to have characteristic multilocal contributions identifiable as multipart on contributions now and possibly relevant for the understanding of bound states such as hadrons.

6.5.1 Scale Dependent Notion Of Four-Momentum In Zero Energy Ontology

Quite generally, General Relativity does not allow to identify four-momentum as Noether charges but in GRT based cosmology one can speak of non-conserved mass [K77], which seems to be in conflict with the conservation of four-momentum in TGD framework. The solution of the problem comes in terms of zero energy ontology (ZEO) [K4, K93], which transforms four-momentum to a scale dependent notion: to each causal diamond (CD) one can assign four-momentum assigned with say positive energy part of the quantum state defined as a quantum superposition of 4-surfaces inside CD.

ZEO is necessary also for the fusion of real and various p-adic physics to single coherent whole. ZEO also allows maximal "free will" in quantum jump since every zero energy state can be created from vacuum and at the same time allows consistency with the conservation laws. ZEO has rather dramatic implications: in particular the arrow of thermodynamical time is predicted to vary so that second law must be generalized. This has especially important implications in living matter, where this kind of variation is observed.

More precisely, this superposition corresponds to a spinor field in the "world of classical worlds" (WCW) [K93]: its components - WCW spinors - correspond to elements of fermionic Fock basis for a given 4-surface - or by holography implied by general coordinate invariance (GCI) - for 3-surface having components at both ends of CD. Strong form of GGI implies strong form of holography (SH) so that partonic 2-surfaces at the ends of space-time surface plus their 4-D tangent space data are enough to fix the quantum state. The classical dynamics in the interior is necessary for the translation of the outcomes of quantum measurements to the language of physics based on classical fields, which in turn is reduced to sub-manifold geometry in the extension of the geometrization program of physics provided by TGD.

Holography is very much reminiscent of QCC suggesting trinity: GCI-holography-QCC. Strong form of holography has strongly stringy flavor: string world sheets connecting the wormhole throats appearing as basic building bricks of particles emerge from the dynamics of induced spinor fields if one requires that the fermionic mode carries well-defined electromagnetic charge [K96].

6.5.2 Are The Classical And Quantal Four-Momenta Identical?

One key question concerns the classical and quantum counterparts of four-momentum. In TGD framework classical theory is an exact part of quantum theory. Classical four-momentum corresponds to Noether charge for preferred extremals of Kähler action. Quantal four-momentum in turn is assigned with the quantum superposition of space-time sheets assigned with CD - actually WCW spinor field analogous to ordinary spinor field carrying fermionic degrees of freedom as analogs of spin. Quantal four-momentum emerges just as it does in super string models - that is as a parameter associated with the representations of super-conformal algebras. The precise action of translations in the representation remains poorly specified. Note that quantal four-momentum does not emerge as Noether charge: at at least it is not at all obvious that this could be the case.

Are these classical and quantal four-momenta identical as QCC would suggest? If so, the Noether four-momentum should be same for all space-time surfaces in the superposition. QCC suggests that also the classical correlation functions for various general coordinate invariant local quantities are same as corresponding quantal correlation functions and thus same for all 4-surfaces in quantum superposition - this at least in the measurement resolution used. This would be an extremely powerful constraint on the quantum states and to a high extend could determined the U-, M-, and S-matrices.

QCC seems to be more or less equivalent with SH stating that in some respects the descriptions based on classical physics defined by Kähler action in the interior of space-time surface and the quantal description in terms of quantum states assignable to the intersections of space-like 3-surfaces at the boundaries of CD and light-like 3-surfaces at which the signature of induced metric changes. SH means effective 2-dimensionality since the four-dimensional tangent space data at partonic 2-surfaces matters. SH could be interpreted as Kac-Mody and symplectic symmetries meaning that apart from central extension they act almost like gauge symmetries in the interiors of space-like 3-surfaces at the ends of CD and in the interiors of light-like 3-surfaces representing orbits of partonic 2-surfaces. Gauge conditions are replaced with Super Virasoro conditions. The word "almost" is of course extremely important.

6.5.3 What Equivalence Principle (EP) Means In Quantum TGD?

EP states the equivalence of gravitational and inertial masses in Newtonian theory. A possible generalization would be equivalence of gravitational and inertial four-momenta. In GRT this correspondence cannot be realized in mathematically rigorous manner since these notions are poorly defined and EP reduces to a purely local statement in terms of Einstein's equations.

What about TGD? What could EP mean in TGD framework?

1. Is EP realized at both quantum and space-time level? This option requires the identification of inertial and gravitational four-momenta at both quantum and classical level. It is now clear that at classical level EP follows from very simple assumption that GRT space-time is obtained by lumping together the space-time sheets of the many-sheeted space-time and by the identification the effective metric as sum of M^4 metric and deviations of the induced metrics of space-time sheets from M^2 metric: the deviations indeed define the gravitational field defined by multiply topologically condensed test particle. Similar description applies to gauge fields. EP as expressed by Einstein's equations would follow from Poincare invariance at microscopic level defined by TGD space-time. The effective fields have as sources the energy momentum tensor and YM currents defined by topological inhomogenities smaller than the resolution scale.

2. QCC would require the identification of quantal and classical counterparts of both gravitational and inertial four-momenta. This would give three independent equivalences, say $P_{I,class} = P_{I,quant}$, $P_{gr,class} = P_{gr,quant}$, $P_{gr,class} = P_{I,quant}$, which imply the remaining ones.

   Consider the condition $P_{gr,class} = P_{I,class}$. At classical level the condition that the standard energy momentum tensor associated with Kähler action has a vanishing divergence is guaranteed if Einstein's equations with cosmological term are satisfied. If preferred extremals satisfy this condition they are constant curvature spaces for non-vanishing cosmological constant. It must be emphasized that field equations are extremely non-linear and one must also consider preferred extremals (which could be identified in terms of space-time regions having so called Hamilton-Jacobi structure): hence these proposals are guesses motivated by what is known about exact solutions of field equations.

   Consider next $P_{gr,class} = P_{I,class}$. At quantum level I have proposed coset representations for the pair of super conformal algebras g and $h \subset g$ which correspond to the coset space decomposition of a given sector of WCW with constant values of zero modes. The coset construction would state that the differences of super-Virasoro generators associated with g resp. h annhilate physical states.

   The identification of the algebras g and h is not straightforward. The algebra g could be formed by the direct sum of super-symplectic and super Kac-Moody algebras and its sub-algebra h for which the generators vanish at partonic 2-surface considered. This would correspond to the idea about WCW as a coset space G/H of corresponding groups (consider as a model $CP_2 = SU(3)/U(2)$ with $U(2)$ leaving preferred point invariant). The sub-algebra h in question includes or equals to the algebra of Kac-Moody generators vanishing at the partonic 2-surface. A natural choice for the preferred WCW point would be as maximum of Kähler function in Euclidian regions: positive definiteness of Kähler function allows only single maximum for fixed values of zero modes). Coset construction states that differences of super Virasoro generators associated with g and h annihilate physical states. This implies that corresponding four-momenta are identical that is Equivalence Principle.

3. Does EP at quantum level reduce to one aspect of QCC? This would require that classical Noether four-momentum identified as inertial momentum equals to the quantal four-momentum assignable to the states of super-conformal representations and identifiable as gravitational four-momentum. There would be only one independent condition: $P_{class} \equiv P_{I,class} = P_{gr,quant} \equiv P_{quant}$.

   Holography realized as AdS/CFT correspondence states the equivalence of descriptions in terms of gravitation realized in terms of strings in 10-D space-time and gauge fields at the boundary of AdS. What is disturbing is that this picture is not completely equivalent with the proposed one. In this case the super-conformal algebra would be direct sum of super-symplectic and super Kac-Moody parts.

Which of the options looks more plausible? The success of p-adic mass calculations [K54] have motivated the use of them as a guideline in attempts to understand TGD. The basic outcome was that elementary particle spectrum can be understood if Super Virasoro algebra has five tensor factors. Can one decide the fate of the two approaches to EP using this number as an input?

This is not the case. For both options the number of tensor factors is five as required. Four tensor factors come from Super Kac-Moody and correspond to translational Kac-Moody type degrees of freedom in M^4, to color degrees of freedom and to electroweak degrees of freedom ($SU(2) \times U(1)$). One tensor factor comes from the symplectic degrees of freedom in $\Delta CD \times CP_2$ (note that Hamiltonians include also products of δCD and CP_2 Hamiltonians so that one does not have direct sum!).

The reduction of EP to the coset structure of WCW sectors is extremely beautiful property. But also the reduction of EP to QCC looks very nice and deep. It is of course possible that the two realizations of EP are equivalent and the natural conjecture is that this is the case.

For QCC option the GRT inspired interpretation of Equivalence Principle at space-time level remains to be understood. Is it needed at all? The condition that the energy momentum tensor of Kähler action has a vanishing divergence leads in General Relativity to Einstein equations with cosmological term. In TGD framework preferred extremals satisfying the analogs of Einstein's equations with several cosmological constant like parameters can be considered.

Should one give up this idea, which indeed might be wrong? Could the divergence of of energy momentum tensor vanish only asymptotically as was the original proposal? Or should one try to generalize the interpretation? QCC states that quantum physics has classical correlate at space-time level and implies EP. Could also quantum classical correspondence itself have a correlate at space-time level. If so, space-time surface would able to represent abstractions as statements about statements about.... as the many-sheeted structure and the vision about TGD physics as analog of Turing machine able to mimic any other Turing machine suggest.

6.5.4 TGD-GRT Correspondence And Equivalence Principle

One should also understand how General Relativity and EP emerge at classical level. The understanding comes from the realization that GRT is only an effective theory obtained by endowing M^4 with effective metric.

1. The replacement of superposition of fields with superposition of their effects means replacing superposition of fields with the set-theoretic union of space-time surfaces. Particle experiences sum of the effects caused by the classical fields at the space-time sheets (see **Fig.** `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg` or **Fig. ??** in the appendix of this book).
2. This is true also for the classical gravitational field defined by the deviation from flat Minkowski metric instandard M^4 coordinates for the space-time sheets. One can define effective metric as sum of M^4 metric and deviations. This effective metric would correspond to that of General Relativity. This resolves long standing issues relating to the interpretation of TGD.
3. Einstein's equations could hold true for the effective metric. They are motivated by the underlying Poincare invariance which cannot be realized as global conservation laws for the effective metric. The conjecture vanishing of divergence of Kähler energy momentum tensor can be seen as the microscopic justification for the claim that Einstein's equations hold true for the effective space-time.
4. The breaking of Poincare invariance could have interpretation as effective breaking in zero energy ontology (ZEO), in which various conserved charges are length dependent and defined separately for each causal diamond (CD).

One can of course consider the possibility that Einstein's equations generalize for preferred extremals of Kähbler action. This would actually represent at space-time level the notion of QCC rather than realise QCC interpreted as EP. The condition that the energy momentum tensor for Kähler action has vanishing covariant divergence would be satisfied in GRT if Einstein's equations with cosmological term hold true. This is the case also now but one can consider also more general solutions in which one has two cosmological constants which are not genuine constants anymore. It has turned out that this line of approach is too adhoc to be taken seriously.

6.5.5 How Translations Are Represented At The Level Of WCW ?

The four-momentum components appearing in the formulas of super conformal generators correspond to infinitesimal translations. In TGD framework one must be able to identify these infinitesimal translations precisely. As a matter of fact, finite measurement resolution implies that it is probably too much to assume infinitesimal translations. Rather, finite exponentials of translation generators are involved and translations are discretized. This does not have practical signficance since for optimal resolution the discretization step is about CP_2 length scale.

Where and how do these translations act at the level of WCW ? ZEO provides a possible answer to this question.

Discrete Lorentz transformations and time translations act in the space of CDs: inertial four-momentum

Quantum state corresponds also to wave function in moduli space of CDs. The moduli space is obtained from given CD by making all boosts for its non-fixed boundary: boosts correspond to a discrete subgroup of Lorentz group and define a lattice-like structure at the hyperboloid for which proper time distance from the second tip of CD is fixed to $T_n = n \times T(CP_2)$. The quantization of cosmic redshift for which there is evidence, could relate to this lattice generalizing ordinary 3-D lattices from Euclidian to hyperbolic space by replacing translations with boosts (velocities).

The additional degree of freedom comes from the fact that the integer $n > 0$ obtains all positive values. One has wave functions in the moduli space defined as a pile of these lattices defined at the hyperboloid with constant value of $T(CP_2)$: one can say that the points of this pile of lattices correspond to Lorentz boosts and scalings of CDs defining sub- WCW : s.

The interpretation in terms of group which is product of the group of shifts $T_n(CP_2) \rightarrow T_{n+m}(CP_2)$ and discrete Lorentz boosts is natural. This group has same Cartesian product structure as Galilean group of Newtonian mechanics. This would give a discrete rest energy and by Lorentz boosts discrete set of four-momenta giving a contribution to the four-momentum appearing in the super-conformal representation.

What is important that each state function reduction would mean localisation of either boundary of CD (that is its tip). This localization is analogous to the localization of particle in position measurement in E^3 but now discrete Lorentz boosts and discrete translations $T_n - - > T_{n+m}$ replace translations. Since the second end of CD is necessary del-ocalized in moduli space, one has kind of flip-flop: localization at second end implies de-localization at the second end. Could the localization of the second end (tip) of CD in moduli space correspond to our experience that momentum and position can be measured simultaneously? This apparent classicality would be an illusion made possible by ZEO.

The flip-flop character of state function reduction process implies also the alternation of the direction of the thermodynamical time: the asymmetry between the two ends of CDs would induce the quantum arrow of time. This picture also allows to understand what the experience growth of geometric time means in terms of CDs.

The action of translations at space-time sheets

The action of embedding space translations on space-time surfaces possibly becoming trivial at partonic 2-surfaces or reducing to action at δCD induces action on space-time sheet which becomes ordinary translation far enough from end end of space-time surface. The four-momentum in question is very naturally that associated with Kähler action and would therefore correspond to inertial momentum for $P_{I,class} = P_{quant,gr}$ option. Indeed, one cannot assign quantal four-momentum to Kähler action as an operator since canonical quantization badly fails. In finite measurement infinitesimal translations are replaced with their exponentials for $P_{I,class} = P_{quant,gr}$ option.

What looks like a problem is that ordinary translations in the general case lead out from given CD near its boundaries. In the interior one expects that the translation acts like ordinary translation. The Lie-algebra structure of Poincare algebra including sums of translation generators with positive coefficient for time translation is preserved if only time-like superpositions if generators are allowed also the commutators of time-like translation generators with boost generators give time like translations. This defines a Lie-algebraic formulation for the arrow of geometric time. The action of time translation on preferred extremal would be ordinary translation plus continuation of the translated preferred extremal backwards in time to the boundary of CD. The transversal space-like translations could be made Kac-Moody algebra by multiplying them with functions which vanish at δCD.

A possible interpretation would be that $P_{quant,gr}$ corresponds to the momentum assignable to the moduli degrees of freedom and $P_{cl,I}$ to that assignable to the time like translations. $P_{quant,gr} = P_{cl,I}$ would code for QCC. Geometrically quantum classical correspondence would state that time-like translation shift both the interior of space-time surface and second boundary of CD to the geometric future/past while keeping the second boundary of space-time surface and CD fixed.

6.5.6 Yangian And Four-Momentum

Yangian symmetry implies the marvellous results of twistor Grassmannian approach to $\mathcal{N} = 4$ SUSY culminating in the notion of amplituhedron which promises to give a nice projective geometry interpretation for the scattering amplitudes [B15]. Yangian symmetry is a multilocal generalization of ordinary symmetry based on the notion of co-product and implies that Lie algebra generates receive also multilocal contributions. I have discussed these topics from slightly different point of view in [K86], where also references to the work of pioneers can be found.

Yangian symmetry

The notion equivalent to that of Yangian was originally introduced by Faddeev and his group in the study of integrable systems. Yangians are Hopf algebras which can be assigned with Lie algebras as the deformations of their universal enveloping algebras. The elegant but rather cryptic looking definition is in terms of the modification of the relations for generating elements [K86]. Besides ordinary product in the enveloping algebra there is co-product Δ which maps the elements of the enveloping algebra to its tensor product with itself. One can visualize product and co-product is in terms of particle reactions. Particle annihilation is analogous to annihilation of two particle so single one and co-product is analogous to the decay of particle to two. Δ allows to construct higher generators of the algebra.

Lie-algebra can mean here ordinary finite-dimensional simple Lie algebra, Kac-Moody algebra or Virasoro algebra. In the case of SUSY it means conformal algebra of M^4- or rather its super counterpart. Witten, Nappi and Dolan have described the notion of Yangian for super-conformal algebra in very elegant andconcrete manner in the article *Yangian Symmetry in D=4 superconformal Yang-Mills theory* [B24]. Also Yangians for gauge groups are discussed.

In the general case Yangian resembles Kac-Moody algebra with discrete index n replaced with a continuous one. Discrete index poses conditions on the Lie group and its representation (adjoint representation in the case of $\mathcal{N} = 4$ SUSY). One of the conditions conditions is that the tensor product $R \otimes R^*$ for representations involved contains adjoint representation only once. This condition is non-trivial. For $SU(n)$ these conditions are satisfied for any representation. In the case of $SU(2)$ the basic branching rule for the tensor product of representations implies that the condition is satisfied for the product of any representations.

Yangian algebra with a discrete basis is in many respects analogous to Kac-Moody algebra. Now however the generators are labelled by non-negative integers labeling the light-like incoming and outgoing momenta of scattering amplitude whereas in in the case of Kac-Moody algebra also negative values are allowed. Note that only the generators with non-negative conformal weight appear in the construction of states of Kac-Moody and Virasoro representations so that the extension to Yangian makes sense.

The generating elements are labelled by the generators of ordinary conformal transformations acting in M^4 and their duals acting in momentum space. These two sets of elements can be labelled by conformal weights $n = 0$ and $n = 1$ and and their mutual commutation relations are same as for Kac-Moody algebra. The commutators of $n = 1$ generators with themselves are however something different for a non-vanishing deformation parameter h. Serre's relations characterize the difference and involve the deformation parameter h. Under repeated commutations the generating elements generate infinite-dimensional symmetric algebra, the Yangian. For $h = 0$ one obtains just one half of the Virasoro algebra or Kac-Moody algebra. The generators with $n > 0$ are $n + 1$-local in the sense that they involve $n + 1$-forms of local generators assignable to the ordered set of incoming particles of the scattering amplitude. This non-locality generalizes the notion of local symmetry and is claimed to be powerful enough to fix the scattering amplitudes completely.

How to generalize Yangian symmetry in TGD framework?

As far as concrete calculations are considered, it is not much to say. It is however possible to keep discussion at general level and still say something interesting (as I hope!). The key question is whether it could be possible to generalize the proposed Yangian symmetry and geometric picture behind it to TGD framework.

1. The first thing to notice is that the Yangian symmetry of $\mathcal{N} = 4$ SUSY in question is quite too limited since it allows only single representation of the gauge group and requires massless particles. One must allow all representations and massive particles so that the representation of symmetry algebra must involve states with different masses, in principle arbitrary spin and arbitrary internal quantum numbers. The candidates are obvious: Kac-Moody algebras [A19] and Virasoro algebras [A30] and their super counterparts. Yangians indeed exist for arbitrary super Lie algebras. In TGD framework conformal algebra of Minkowski space reduces to Poincare algebra and its extension to Kac-Moody allows to have also massive states.
2. The formal generalization looks surprisingly straightforward at the formal level. In zero energy ontology one replaces point like particles with partonic two-surfaces appearing at the ends of light-like orbits of wormhole throats located to the future and past light-like boundaries of causal diamond ($CD \times CP_2$ or briefly CD). Here CD is defined as the intersection of future and past directed light-cones. The polygon with light-like momenta is naturally replaced with a polygon with more general momenta in zero energy ontology and having partonic surfaces as its vertices. Non-point-likeness forces to replace the finite-dimensional super Lie-algebra with infinite-dimensional Kac-Moody algebras and corresponding super-Virasoro algebras assignable to partonic 2-surfaces.
3. This description replaces disjoint holomorphic surfaces in twistor space with partonic 2-surfaces at the boundaries of $CD \times CP_2$ so that there seems to be a close analogy with Cachazo-Svrcek-Witten picture. These surfaces are connected by either light-like orbits of partonic 2-surface or space-like 3-surfaces at the ends of CD so that one indeed obtains the analog of polygon.

What does this then mean concretely (if this word can be used in this kind of context)?

1. At least it means that ordinary Super Kac-Moody and Super Virasoro algebras associated with isometries of $M^4 \times CP_2$ annihilating the scattering amplitudes must be extended to a co-algebras with a non-trivial deformation parameter. Kac-Moody group is thus the product of Poincare and color groups. This algebra acts as deformations of the light-like 3-surfaces representing the light-like orbits of particles which are extremals of Chern-Simon action with the constraint that weak form of electric-magnetic duality holds true. I know so little about the mathematical side that I cannot tell whether the condition that the product of the representations of Super-Kac-Moody and Super-Virasoro algebras contains adjoint representation only once, holds true in this case. In any case, it would allow all representations of finite-dimensional Lie group in vertices whereas $\mathcal{N} = 4$ SUSY would allow only the adjoint.
2. Besides this ordinary kind of Kac-Moody algebra there is the analog of Super-Kac-Moody algebra associated with the light-cone boundary which is metrically 3-dimensional. The finite-dimensional Lie group is in this case replaced with infinite-dimensional group of symplectomorphisms of $\delta M^4_{+/-}$ made local with respect to the internal coordinates of the partonic 2-surface. This picture also justifies p-adic thermodynamics applied to either symplectic or isometry Super-Virasoro and giving thermal contribution to the vacuum conformal and thus to mass squared.
3. The construction of TGD leads also to other super-conformal algebras and the natural guess is that the Yangians of all these algebras annihilate the scattering amplitudes.

4. Obviously, already the starting point symmetries look formidable but they still act on single partonic surface only. The discrete Yangian associated with this algebra associated with the closed polygon defined by the incoming momenta and the negatives of the outgoing momenta acts in multi-local manner on scattering amplitudes. It might make sense to speak about polygons defined also by other conserved quantum numbers so that one would have generalized light-like curves in the sense that state are massless in 8-D sense.

Could Yangian symmetry provide a new view about conserved quantum numbers?

The Yangian algebra has some properties which suggest a new kind of description for bound states. The Cartan algebra generators of $n = 0$ and $n = 1$ levels of Yangian algebra commute. Since the co-product Δ maps $n = 0$ generators to $n = 1$ generators and these in turn to generators with high value of n, it seems that they commute also with $n \geq 1$ generators. This applies to four-momentum, color isospin and color hyper charge, and also to the Virasoro generator L_0 acting on Kac-Moody algebra of isometries and defining mass squared operator.

Could one identify total four momentum and Cartan algebra quantum numbers as sum of contributions from various levels? If so, the four momentum and mass squared would involve besides the local term assignable to wormhole throats also n-local contributions. The interpretation in terms of n-parton bound states would be extremely attractive. n-local contribution would involve interaction energy. For instance, string like object would correspond to $n = 1$ level and give $n = 2$-local contribution to the momentum. For baryonic valence quarks one would have 3-local contribution corresponding to $n = 2$ level. The Yangian view about quantum numbers could give a rigorous formulation for the idea that massive particles are bound states of massless particles.

6.6 Generalization Of Ads/CFT Duality To TGD Framework

AdS/CFT duality has provided a powerful approach in the attempts to understand the non-perturbative aspects of super-string theories. The duality states that conformal field theory in n-dimensional Minkowski space M^n identifiable as a boundary of $n+1$-dimensional space AdS_{n+1} is dual to a string theory in $AdS_{n+1} \times S^{9-n}$.

As a mathematical discovery the duality is extremely interesting but it seems that it need not have much to do with physics. From TGD point of view the reason is obvious: the notion of conformal invariance is quite too limited. In TGD framework conformal invariance is extended to a super-symplectic symmetry in $\delta M^4_\pm \times CP_2$, whose Lie-algebra has the structure of conformal algebra. Also ordinary super-conformal symmetries associated with string world sheets are present as well as generalization of 2-D conformal symmetries to their analogs at light-cone boundary and light-like orbits of partonic 2-surfaces. In this framework AdS/CFT duality is expected to be modified and this seems to be the case.

The matrix elements of Kähler metric of WCW can be expressed in two ways. As contractions of the derivatives $\partial_K \partial_{\overline{L}} K$ of the Kähler function of WCW with isometry generators or as anticommutators of WCW gamma matrices identified as supersymplectic Noether super charges assignable to fermioni strings connecting partonic 2-surfaces. Kähler function is identified as Kähler action for the Euclidian space-time regions with 4-D CP_2 projection. Kähler action defines the Kähler-Dirac gamma matrices appearing in K-D action as contractions of canonical momentum currents with embedding space gamma matrices. Bosonic and fermionic degrees of freedom are therefore dual in a well-defined sense.

This observation suggests various generalizations. There is super-symmetry between Kähler action and Kähler-Dirac action. The problem is that induced spinor fields are localized at 2-D string world sheets. Strong form of holography implying effective 2-dimensionality suggests the solution to the paradox. The paradox disappears if the Kähler action is expressible as string area for the effective metric defined by the anti-commutators of K-D gamma matrices at string world sheet. This expression allows to understand how strings connecting partonic 2-surfaces give rise to the formation of gravitationally bound states. Bound states of macroscopic size are however possible only if one allows hierarchy of Planck constants. This representation of Kähler action can be seen as one aspect of the analog of AdS/CFT duality in TGD framework.

One can imagine also other realizations. For instance, Dirac determinant for the spinors associated with string world sheets should reduce to the exponent of Kähler action.

6.6.1 Does The Exponent Of Chern-Simons Action Reduce To The Exponent Of The Area Of Minimal Surfaces?

As I scanned of hep-th I found an interesting article (see `http://tinyurl.com/ycpkrg4f`) by Giordano, Peschanski, and Seki [B41] based on AdS/CFT correspondence. What is studied is the high energy behavior of the gluon-gluon and quark-quark scattering amplitudes of $\mathcal{N}=4$ SUSY.

1. The proposal made earlier by Aldaya and Maldacena (see `http://tinyurl.com/ybnk6kbs`) [B11] is that gluon-gluon scattering amplitudes are proportional to the imaginary exponent of the area of a minimal surface in AdS_5 whose boundary is identified as *momentum space*. The boundary of the minimal surface would be polygon with light-like edges: this polygon and its dual are familiar from twistor approach.
2. Giordano, Peschanski, and Seki claim that quark-quark scattering amplitude for heavy quarks corresponds to the exponent of the area for a minimal surface in the *Euclidian* version of AdS_5 which is hyperbolic space (space with a constant negative curvature): it is interpreted as a counterpart of WCW rather than momentum space and amplitudes are obtained by analytic continuation. For instance, a universal Regge behavior is obtained. For general amplitudes the exponent of the area alone is not enough since it does not depend on gluon quantum numbers and vertex operators at the edges of the boundary polygon are needed.

In the following my intention is to consider the formulation of this conjecture in quantum TGD framework. I hasten to inform that I am not a specialist in AdS/CFT and can make only general comments inspired by analogies with TGD and the generalization of AdS/CFT duality to TGD framework based on the localization of induced spinors at string world sheets, supersymmetry between bosonic and fermionic degrees of freedom at the level of WCW , and the notion of effective metric at string world sheets.

6.6.2 Does Kähler Action Reduce To The Sum Of Areas Of Minimal Surfaces In Effective Metric?

Minimal surface conjectures are highly interesting from TGD point of view. The weak form of electric magnetic duality implies the reduction of Kähler action to 3-D Chern-Simons terms. Effective 2-dimensionality implied by the strong form of General Coordinate Invariance suggests a further reduction of Chern-Simons terms to 2-D terms and the areas of string world sheet and of partonic 2-surface are the only non-topological options that one can imagine. Skeptic could of course argue that the exponent of the minimal surface area results as a characterizer of the quantum state rather than vacuum functional. In the following I end up with the proposal that the Kähler action should reduce to the sum of string world sheet areas in the effective metric defines by the anticommutators of Kähler-Dirac gamma matrices at string world sheets.

Let us look this conjecture in more detail.

1. In zero energy ontology twistor approach is very natural since all physical states are bound states of massless particles. Also virtual particles are composites of massless states. The possibility to have both signs of energy makes possible space-like momenta for wormhole contacts. Mass shell conditions at internal lines imply extremely strong constraints on the virtual momenta and both UV and IR finiteness are expected to hold true.
2. The weak form of electric magnetic duality [K96] implies that the exponent of Kähler action reduces to the exponent of Chern-Simons term for 3-D space-like surfaces at the ends of space-time surface inside CD and for light-like 3-surfaces. The coefficient of this term is complex since the contribution of Minkowskian regions of the space-time surface is imaginary ($\sqrt{g_4}$ is imaginary) and that of Euclidian regions (generalized Feynman diagrams) real. The Chern-Simons term from Minkowskian regions is like Morse function and that from Euclidian regions defines Kähler function and stationary phase approximation makes sense. The two contributions are different since the space-like 3-surfaces contributing to Kähler function and Morse function are different.

3. Electric magnetic duality [K96] leads also to the conclusion that wormhole throats carrying elementary particle quantum numbers are Kähler magnetic monopoles. This forces to identify elementary particles as string like objects with ends having opposite monopole charges. Also more complex configurations are possible.

   It is not quite clear what the scale of the stringyness is. The natural first guess inspired by quantum classical correspondence is that it corresponds to the p-adic length scale of the particle characterizing its Compton length. Second possibility is that it corresponds to electroweak scale. For leptons stringyness in Compton length scale might not have any fatal implications since the second end of string contains only neutrinos neutralizing the weak isospin of the state. This kind of monopole pairs could appear even in condensed matter scales: in particular if the proposed hierarchy of Planck constants [K29] is realized.

4. Strong form of General Coordinate Invariance requires effective 2-dimensionality. In given UV and IR resolutions either partonic 2-surfaces or string world sheets form a finite hierarchy of CDs inside CDs with given CD characterized by a discrete scale coming as an integer multiple of a fundamental scale (essentially CP_2 size). The string world sheets have boundaries consisting of either light-like curves in induced metric at light-like wormhole throats and space-like curves at the ends of CD whose M^4 projections are light-like. These braids intersect partonic 2-surfaces at discrete points carrying fermionic quantum numbers.

   This implies a rather concrete analogy with $AdS_5 \times S_5$ duality, which describes gluons as open strings. In zero energy ontology (ZEO) string world sheets are indeed a fundamental notion and the natural conjecture is that these surfaces are minimal surfaces whose area by quantum classical correspondence depends on the quantum numbers of the external particles. String tension in turn should depend on gauge couplings -perhaps only Kähler coupling strength- and geometric parameters like the size scale of CD and the p-adic length scale of the particle.

5. One can of course ask whether the metric defining the string area is induced metric or possibly the metric defined by the anti-commutators of Kähler-Dirac gamma matrices. The recent view does not actually leave any other alternative. The analog of AdS/CFT duality together with supersymmetry demands that Kähler action is proportional to the sum of the areas of string world sheets in this effective metric. Whether the vanishing of induced W fields (and possibly also Z^0) making possible well-defined em charge for the spinor nodes is realized by the condition that the string world sheet is a miniml surface in the effective metric remains an open question.

   The assumption that ordinary minimal surfaces are in question is not consistent with the TGD view about the formation of gravitational bound states and if string tension is $1/\hbar G$ as in string models, only bound states with size of order Planck length are possible. This strongly favors effective metric giving string tension proportional to $1/h_{eff}^2$. How $1/h_{eff}^2$ proportionality might be understood is discussed in [?] in terms electric-magnetic duality.

6. One can of course still consider also the option that ordinary minimal surfaces are in question. Are the minimal surfaces in question minimal surfaces of the embedding space $M^4 \times CP_2$ or of the space-time surface X^4? All possible 2-surfaces at the boundary of CD must be allowed so that they cannot correspond to minimal surfaces in $M^4 \times CP_2$ unless one assumes that they emerge in stationary phase approximation only. The boundary conditions at the ends of CD could however be such that *any* partonic 2-surface correspond to a minimal surfaces in X^4. Same applies to string world sheets. One might even hope that these conditions combined with the weak form of electric magnetic duality fixes completely the boundary conditions at wormhole throats and space-like ends of space-time surface.

   The trace of the second fundamental form orthogonal to the string world sheet/partonic 2-surface as sub-manifold of space-time surface would vanish: this is nothing but a generalization of the geodesic motion obtained by replacing word line with a 2-D surface. It does not imply the vanishing of the trace of the second fundamental form in $M^4 \times CP_2$ having interpretation as a generalization of particle acceleration [K90]. Effective 2-dimensionality would be realized if Chern-Simons terms reduce to a sum of the areas of these minimal surfaces.

These arguments suggest that scattering amplitudes are proportional to the product of exponents of 2-dimensional actions which can be either imaginary or real. Imaginary exponent would be proportional to the total area of string world sheets and the imaginary unit would come naturally from $\sqrt{g_2}$, where g_2 is effective metric most naturally. Teal exponent proportional to the total area of partonic 2-surfaces. The coefficient of these areas would not in general be same.

The reduction of the Kähler action from Minkowskian regions to Chern-Simons terms means that Chern-Simons terms reduce to actions assignable to string world sheets. The equality of the Minkowskian and Euclidian Chern-Simons terms is suggestive but not necessarily true since there could be also other Chern-Simons contributions than those assignable to wormhole throats and the ends of space-time. The equality would imply that the total area of string world sheets equals to the total area of partonic 2-surfaces suggesting strongly a duality meaning that either Euclidian or Minkowskian regions carry the needed information.

6.6.3 Surface Area As Geometric Representation Of Entanglement Entropy?

I encountered a link to a talk by James Sully and having the title "Geometry of Compression" (see `http://tinyurl.com/ycuu8xcr`). I must admit that I understood very little about the talk. My not so educated guess is however that information is compressed: UV or IR cutoff eliminating entanglement in short length scales and describing its presence in terms of density matrix - that is thermodynamically - is another manner to say it. The TGD inspired proposal for the interpretation of the inclusions of hyper-finite factors of type II_1 (HFFs) [K95] is in spirit with this.

The space-time counterpart for the compression would be in TGD framework discretization. Discretizations using rational points (or points in algebraic extensions of rationals) make sense also p-adically and thus satisfy number theoretic universality. Discretization would be defined in terms of intersection (rational or in algebraic extension of rationals) of real and p-adic surfaces. At the level of "world of classical worlds" the discretization would correspond to - say - surfaces defined in terms of polynomials, whose coefficients are rational or in some algebraic extension of rationals. Pinary UV and IR cutoffs are involved too. The notion of p-adic manifold allows to interpret the p-adic variants of space-time surfaces as cognitive representations of real space-time surfaces.

Finite measurement resolution does not allow state function reduction reducing entanglement totally. In TGD framework also negentropic entanglement stable under Negentropy Maximization Principle (NMP) is possible [K49]. For HFFs the projection into single ray of Hilbert space is indeed impossible: the reduction takes always to infinite-D sub-space.

The visit to the URL was however not in vain. There was a link to an article (see `http://tinyurl.com/y9h3qtr8`) [B58] discussing the geometrization of entanglement entropy inspired by the AdS/CFT hypothesis.

Quantum classical correspondence is basic guiding principle of TGD and suggests that entanglement entropy should indeed have space-time correlate, which would be the analog of Hawking-Bekenstein entropy.

Generalization of AdS/CFT to TGD context

AdS/CFT generalizes to TGD context in non-trivial manner. There are two alternative interpretations, which both could make sense. These interpretations are not mutually exclusive. The first interpretation makes sense at the level of "world of classical worlds" (WCW) with symplectic algebra and extended conformal algebra associated with $\delta M^4_\pm$ replacing ordinary conformal and Kac-Moody algebras. Second interpretation at the level of space-time surface with the extended conformal algebras of the light-likes orbits of partonic 2-surfaces replacing the conformal algebra of boundary of AdS^n.

1. First interpretation

For the first interpretation 2-D conformal invariance is generalised to 4-D conformal invariance relying crucially on the 4-dimensionality of space-time surfaces and Minkowski space.

1. One has an extension of the conformal invariance provided by the symplectic transformations of $\delta CD \times CP_2$ for which Lie algebra has the structure of conformal algebra with radial light-like coordinate of δM^4_+ replacing complex coordinate z.

2. One could see the counterpart of AdS_n as embedding space $H = M^4 \times CP_2$ completely unique by twistorial considerations and from the condition that standard model symmetries are

obtained and its causal diamonds defined as sub-sets $CD \times CP_2$, where CD is an intersection of future and past directed light-cones. I will use the shorthand CD for $CD \times CP_2$. Strings in $AdS_5 \times S^5$ are replaced with space-time surfaces inside 8-D CD.

3. For this interpretation 8-D CD replaces the 10-D space-time $AdS_5 \times S^5$. 7-D light-like boundaries of CD correspond to the boundary of say AdS_5, which is 4-D Minkowski space so that zero energy ontology (ZEO) allows rather natural formulation of the generalization of AdS/CFT correspondence since the positive and negative energy parts of zero energy states are localized at the boundaries of CD.

Second interpretation

For the second interpretation relies on the observation that string world sheets as carriers of induced spinor fields emerge in TGD framework from the condition that electromagnetic charge is well-defined for the modes of induced spinor field.

1. One could see the 4-D space-time surfaces X^4 as counterparts of AdS_4. The boundary of AdS_4 is replaced in this picture with 3-surfaces at the ends of space-time surface at opposite boundaries of CD and by strong form of holography the union of partonic 2-surfaces defining the intersections of the 3-D boundaries between Euclidian and Minkowskian regions of space-time surface with the boundaries of CD. Strong form of holography in TGD is very much like ordinary holography.

2. Note that one has a dimensional hierarchy: the ends of the boundaries of string world sheets at boundaries of CD as point-like partices, boundaries as fermion number carrying lines, string world sheets, light-like orbits of partonic 2-surfaces, 4-surfaces, embedding space $M^4 \times CP_2$. Clearly the situation is more complex than for AdS/CFT correspondence.

3. One can restrict the consideration to 3-D sub-manifolds X^3 at either boundary of causal diamond (CD): the ends of space-time surface. In fact, the position of the other boundary is not well-defined since one has superposition of CDs with only one boundary fixed to be piece of light-cone boundary. The delocalization of the other boundary is essential for the understanding of the arrow of time. The state function reductions at fixed boundary leave positive energy part (say) of the zero energy state at that boundary invariant (in positive energy ontology entire state would remain unchanged) but affect the states associated with opposite boundaries forming a superposition which also changes between reduction: this is analog for unitary time evolution. The average for the distance between tips of CDs in the superposition increases and gives rise to the flow of time.

4. One wants an expression for the entanglement entropy between X^3 and its partner. Bekenstein area law allows to guess the general expression for the entanglement entropy: for the proposal discussed in the article the entropy would be the area of the boundary of X^3 divided by gravitational constant: $S = A/4G$. In TGD framework gravitational constant might be replaced by the square of CP_2 radius apart from numerical constant. How gravitational constant emerges in TGD framework is not completely understood although one can deduce for it an estimate using dimensional analyses. In any case, gravitational constant is a parameter which characterizes GRT limit of TGD in which many-sheeted space-time is in long scales replaced with a piece of Minkowski space such that the classical gravitational fields and gauge potentials for sheets are summed. The physics behind this relies on the generalization of linear superposition of fields: the effects of different space-time sheets particle touching them sum up rather than fields.

5. The counterpart for the boundary of X^3 appearing in the proposal for the geometrization of the entanglement entropy naturally corresponds to partonic 2-surface or a collection of them if strong form of holography holds true.

   There is however also another candidate to be considered! Partonic 2-surfaces are basic objects, and one expects that the entanglement between fundamental fermions associated with distinct partonic 2-surfaces has string world sheets as space-time correlates. Could the area of the string world sheet in the effective metric defined by the anti-commutators of K-D gamma matrices at string world sheet provide a measure for entanglement entropy? If this conjecture is correct: the entanglement entropy would be proportional to Kähler action. Also negative values are possible for Kähler action in Minkowskian regions but in TGD framework number theoretic entanglement entropy having also negative values emerges naturally.

Which of these guesses is correct, if any? Or are they equivalent?

With what kind of systems 3-surfaces can entangle?

With what system X^3 is entangled/can entangle? There are several options to consider and they could correspond to the two TGD variants for the AdS/CFT correspondence.

1. X^3 could correspond to a wormhole contact with Euclidian signature of induced metric. The entanglement would be between it and the exterior region with Minkowskian signature of the induced metric.

2. X^3 could correspond to single sheet of space-time surface connected by wormhole contacts to a larger space-time sheet defining its environment. More precisely, X^3 and its complement would be obtained by throwing away the wormhole contacts with Euclidian signature of induce metric. Entanglement would be between these regions. In the generalization of the formula

$$S = \frac{A}{4\hbar G}$$

area A would be replaced by the total area of partonic 2-surfaces and G perhaps with CP_2 length scale squared.

3. In ZEO the entanglement could also correspond to time-like entanglement between the 3-D ends of the space-time surface at opposite light-like boundaries of CD. M-matrix, which can be seen as the analog of thermal S-matrix, decomposes to a product of hermitian square root of density matrix and unitary S-matrix and this hermitian matrix could also define p-adic thermodynamics. Note that in ZEO quantum theory can be regarded as square root of thermodynamics.

Minimal surface property is not favored in TGD framework

Minimal surface property for the 3-surfaces X^3 at the ends of space-time surface looks at first glance strange but a proper generalization of this condition makes sense if one assumes strong form of holography. Strong form of holography realizes General Coordinate Invariance (GCI) in strong sense meaning that light-like parton orbits and space-like 3-surfaces at the ends of space-time surfaces are equivalent physically. As a consequence, partonic 2-surfaces and their 4-D tangent space data must code for the quantum dynamics.

The mathematical realization is in terms of conformal symmetries accompanying the symplectic symmetries of $\delta M^4_\pm \times CP_2$ and conformal transformations of the light-like partonic orbit [K96]. The generalizations of ordinary conformal algebras correspond to conformal algebra, Kac-Moody algebra at the light-like parton orbits and to symplectic transformations $\delta M^4 \times CP_2$ acting as isometries of WCW and having conformal structure with respect to the light-like radial coordinate plus conformal transformations of δM^4_+, which is metrically 2-dimensional and allows extended conformal symmetries.

1. If the conformal realization of the strong form of holography works, conformal transformations act at quantum level as gauge symmetries in the sense that generators with no-vanishing conformal weight are zero or generate zero norm states. Conformal degeneracy can be eliminated by fixing the gauge somehow. Classical conformal gauge conditions analogous to Virasoro and Kac-Moody conditions satisfied by the 3-surfaces at the ends of CD are natural in this respect. Similar conditions would hold true for the light-like partonic orbits at which the signature of the induced metric changes.

2. What is also completely new is the hierarchy of conformal symmetry breakings associated with the hierarchy of Planck constants $h_{eff}/h = n$ [K29]. The deformations of the 3-surfaces which correspond to non-vanishing conformal weight in algebra or any sub-algebra with conformal weights vanishing modulo n give rise to vanishing classical charges and thus do not affect the value of the Kähler action [K96].

   The inclusion hierarchies of conformal sub-algebras are assumed to correspond to those for hyper-finite factors. There is obviously a precise analogy with quantal conformal invariance conditions for Virasoro algebra and Kac-Moody algebra. There is also hierarchy of inclusions which corresponds to hierarchy of measurement resolutions. An attractive interpretation is that singular conformal transformations relate to each other the states for broken conformal symmetry. Infinitesimal transformations for symmetry broken phase would carry fractional conformal weights coming as multiples of $1/n$.

3. Conformal gauge conditions need not reduce to minimal surface conditions holding true for all variations.

4. Note that Kähler action reduces to Chern-Simons term at the ends of CD if weak form of electric magnetic duality holds true. The conformal charges at the ends of CD cannot however reduce to Chern-Simons charges by this condition since only the charges associated with CP_2 degrees of freedom would be non-trivial.

The way out of the problem is provided by the generalization of AdS/CFT conjecture. String area is estimated in the effective metric provided by the anti-commutator of K-D gamma matrices at string world sheet.

6.6.4 Related Ideas

p-Adic mass calculations led to the introduction of the p-adic variant of Bekenstein-Hawkin law in which Planck length is replaced by p-adic length scale. This generalization is in spirit with the idea that string world sheet area is estimated in effective rather than induced metric.

p-Adic variant of Bekenstein-Hawking law

When the 3-surface corresponds to elementary particle, a direct connection with p-adic thermodynamics suggests itself and allows to answer the questions above. p-Adic thermodynamics could be interpreted as a description of the entanglement with environment. In ZEO the entanglement could also correspond to time-like entanglement between the 3-D ends of the space-time surface at opposite light-like boundaries of CD. M-matrix, which can be seen as the analog of thermal S-matrix, decomposes to a product of hermitian square root of density matrix and unitary S-matrix and this hermitian matrix could also define p-adic thermodynamics.

1. p-Adic thermodynamics [K54] would not be for energy but for mass squared (or scaling generator L_0) would describe the entanglement of the particle with environment defined by the larger space-time sheet. Conformal weights would comes as positive powers of integers (p_0^L would replace $exp(-H/T)$ to guarantee the number theoretical existence and convergence of the Boltzmann weight: note that conformal invariance that is integer spectrum of L_0 is also essential).

2. The interactions with environment would excite very massive CP_2 mass scale excitations (mass scale is about 10^{-4} times Planck mass) of the particle and give it thermal mass squared identifiable as the observed mass squared. The Boltzmann weights would be extremely small having p-adic norm about $1/p^n$, p the p-adic prime: $M_{127} = 2^{127} - 1$ for electron.

3. One of the first ideas inspired by p-adic vision was that p-adic entropy could be seen as a p-adic counterpart of Bekenstein-Hawking entropy [K57]. $S = (R^2/\hbar^2) \times M^2$ holds true identically apart from numerical constant. Note that one could interpret $R^2M/\hbar$ as the counterpart of Schwartschild radius. Note that this radius is proportional to $1/\sqrt{p}$ so that the area A would correspond to the area defined by Compton length. This is in accordance with the third option.

What is the space-time correlate for negentropic entanglement?

The new element brought in by TGD framework is that number theoretic entanglement entropy is negative for negentropic entanglement assignable to unitary entanglement (in the sense that entanglement matrix is proportional to a unitary matrix) and NMP states that this negentropy increases [K49]. Since entropy is essentially number of energy degenerate states, a good guess is that the number $n = h_{eff}/h$ of space-time sheets associated with h_{eff} defines the negentropy. An attractive space-time correlate for the negentropic entanglement is braiding. Braiding defines unitary S-matrix between the states at the ends of braid and this entanglement is negentropic. This entanglement gives also rise to topological quantum computation.

6.6.5 The Importance Of Being Light-Like

The singular geometric objects associated with the space-time surface have become increasingly important in TGD framework. In particular, the recent progress has made clear that these objects might be crucial for the understanding of quantum TGD. The singular objects are associated not only with the induced metric but also with the effective metric defined by the anti-commutators of the Kähler-Dirac gamma matrices appearing in the Kähler-Dirac equation and determined by the Kähler action.

The singular objects associated with the induced metric

Consider first the singular objects associated with the induced metric.

1. At light-like 3-surfaces defined by wormhole throats the signature of the induced metric changes from Euclidian to Minkowskian so that 4-metric is degenerate. These surfaces are carriers of elementary particle quantum numbers and the 4-D induced metric degenerates locally to 3-D one at these surfaces.

2. Braid strands at partonic orbits - fermion lines identified as boundaries of string world sheets in the more recent terminology - are most naturally light-like curves: this correspond to the boundary condition for open strings. One can assign fermion number to the braid strands. Braid strands allow an identification as curves along which the Euclidian signature of the string world sheet in Euclidian region transforms to Minkowskian one. Number theoretic interpretation would be as a transformation of complex regions to hyper-complex regions meaning that imaginary unit i satisfying $i^2 = -1$ becomes hyper-complex unit e satisfying $e^2 = 1$. The complex coordinates $(z, \overline{z})$ become hyper-complex coordinates $(u = t + ex, v = t - ex)$ giving the standard light-like coordinates when one puts $e = 1$.

The singular objects associated with the effective metric

There are also singular objects assignable to the effective metric. According to the simple arguments already developed, string world sheets and possibly also partonic 2-surfaces are singular objects with respect to the effective metric defined by the anti-commutators of the Kähler-Dirac gamma matrices rather than induced gamma matrices. Therefore the effective metric might be more than a mere formal structure. The following is of course mere speculation and should be taken as such.

1. For instance, quaternionicity of the space-time surface *might* allow an elegant formulation in terms of the effective metric avoiding the problems due to the Minkowski signature. This is achieved if the effective metric has Euclidian signature $\epsilon \times (1, 1, 1, 1)$, $\epsilon = \pm 1$ or a complex counterpart of the Minkowskian signature $\epsilon(1, 1, -1, -1)$.

2. String word sheets and perhaps also partonic 2-surfaces might be be understood as singularities of the effective metric. What happens that the effective metric with Euclidian signature $\epsilon \times (1, 1, 1, 1)$ transforms to the signature $\epsilon(1, 1, -1, -1)$ (say) at string world sheet so that one would have the degenerate signature $\epsilon \times (1, 1, 0, 0)$ at the string world sheet.

   What is amazing is that this works also number theoretically. It came as a total surprise to me that the notion of hyper-quaternions as a closed algebraic structure indeed exists. The hyper-quaternionic units would be given by $(1, I, iJ, iK)$, where i is a commuting imaginary unit satisfying $i^2 = -1$. Hyper-quaternionic numbers defined as combinations of these units with real coefficients do form a closed algebraic structure which however fails to be a number field just like hyper-complex numbers do. Note that the hyper-quaternions obtained with real coefficients from the basis $(1, iI, iJ, iK)$ fail to form an algebra since the product is not hyper-quaternion in this sense but belongs to the algebra of complexified quaternions. The same problem is encountered in the case of hyper-octonions defined in this manner. This has been a stone in my shoe since I feel strong disrelish towards Wick rotation as a trick for moving between different signatures.

3. Could also partonic 2-surfaces correspond to this kind of singular 2-surfaces? In principle, 2-D surfaces of 4-D space intersect at discrete points just as string world sheets and partonic 2-surfaces do so that this might make sense. By complex structure the situation is algebraically equivalent to the analog of plane with non-flat metric allowing all possible signatures (ϵ_1, ϵ_2) in various regions. At light-like curve either ϵ_1 or ϵ_2 changes sign and light-like curves for these two kinds of changes can intersect as one can easily verify by drawing what happens. At the intersection point the metric is completely degenerate and simply vanishes.

4. Replacing real 2-dimensionality with complex 2-dimensionality, one obtains by the universality of algebraic dimension the same result for partonic 2-surfaces and string world sheets. The braid ends at partonic 2-surfaces representing the intersection points of 2-surfaces of this kind would have completely degenerate effective metric so that the Kähler-Dirac gamma matrices would vanish implying that energy momentum tensor vanishes as does also the induced Kähler field.

5. The effective metric suffers a local conformal scaling in the critical deformations identified in the proposed manner. Since ordinary conformal group acts on Minkowski space and leaves the boundary of light-cone invariant, one has two conformal groups. It is not however clear whether the M^4 conformal transformations can act as symmetries in TGD, where the presence of the induced metric in Kähler action breaks M^4 conformal symmetry. As found, also in TGD framework the Kac-Moody currents assigned to the braid strands generate Yangian: this is expected to be true also for the Kac-Moody counterparts of the conformal algebra associated with quantum criticality. On the other hand, in twistor program one encounters also two conformal groups and the space in which the second conformal group acts remains somewhat mysterious object. The Lie algebras for the two conformal groups generate the conformal Yangian and the integrands of the scattering amplitudes are Yangian invariants. Twistor approach should apply in TGD if zero energy ontology is right. Does this mean a deep connection?

   What is also intriguing that twistor approach in principle works in strict mathematical sense only at signatures $\epsilon \times (1, 1, -1 - 1)$ and the scattering amplitudes in Minkowski signature are obtained by analytic continuation. Could the effective metric give rise to the desired signature? Note that the notion of massless particle does not make sense in the signature $\epsilon \times (1, 1, 1, 1)$.

These arguments provide genuine a support for the notion of quaternionicity and suggest a connection with the twistor approach.

6.7 Could One Define Dynamical Homotopy Groups In WCW?

Agostino Prastaro - working as professor at the University of Rome - has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action in TGD [A41, A42]. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis [K94].

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could this mean that the natural topology in the parameter space of Noether charges zero modes of WCW metric) is p-adic? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the six lowest dynamical homotopy groups of WCW would be non-trivial. The finite number of these groups dictate by the dimension of embedding space suggests also an interpretation as analogs of homology groups.

In the following the notion of cobordism is briefly discussed and the idea of Prastaro about assigning cobordism with partial differential equations is discussed.

6.7.1 About Cobordism As A Concept

To get some background consider first the notion of cobordism (`http://tinyurl.com/y7wdhtmv`).

1. Thom's cobordism theory [A92] is inspired by the question "When an n-manifold can be represented as a boundary of $n+1$-manifold". One can also pose additional conditions such as continuity, smoothness, orientability, one can add bundles structures and require that they are induced to n-manifold from that of $n+1$-manifold. One can also consider sub-manifolds of some higher-dimensional manifold.

   One can also fix n-manifold M and ask "What is the set of n-manifolds N with the property that there exists $n+1$-manifold W having union of $M \cup N$ as its boundary". One can also allow M to have boundary and pose the same question by allowing also the boundary of connecting $n+1$-manifold W contain also the orbits of boundaries of M and N.

   The cobordism class of M can be defined as the set of manifolds N cobordant with M - that is connectable in this manner. They have same cobordism class since cobordism is equivalence relation. The classes form also a group with respect to disjoint union. Cobordism is much rougher equivalence relation than diffeomorphy or homeomorphy since topology changes are possible. For instance, every 3-D closed un-oriented manifold is a boundary of a 4-manifold! Same is true for orientable cobordisms. Cobordism defines a category: objects are (say closed) manifolds and morphisms are cobordisms.

2. The basic result of Morse, Thom, and Milnor is that cobordism as topology changes can be engineered from elementary cobordisms. One take manifold $M \times I$ and imbeds to its other n-dimensional end the manifold $S^p \times D^q$, $n = p + q$, removes its interior and glues back $D^{p+1} \times S^{q-1}$ along its boundary to the boundary of the resulting hole. This gives n-manifold with different topology, call it N. The outcome is a cobordism connecting M and N unless there are some obstructions.

   There is a connection with Morse theory (`http://tinyurl.com/ych4chg9`) in which cobordism can be seen as a mapping of W to a unit interval such that the inverse images define a slicing of W and the inverse images at ends correspond to M and N.

3. One can generalize the abstract cobordism to that for n-sub-manifolds of a given embedding space. This generalization is natural in TGD framework. This might give less trivial results since not all connecting manifolds are imbeddable into a given embedding space. If connecting 4-manifolds connecting 3-manifolds with Euclidian signature (of induced metric) are assumed to have a Minkowskian signature, one obtains additional conditions, which might be too strong (the classical result of Geroch [A93] implies that non-trivial cobordism implies closed time loops - impossible in TGD).

   From TGD point of view this is too strong a condition and in TGD framework space-time surfaces with both Euclidian and Minkowskian signature of the induced metric are allowed. Also cobordisms singular as 4-surfaces are analogous to 3-vertices of Feynman diagrams are allowed.

6.7.2 Prastaro's Generalization Of Cobordism Concept To The Level Of Partial Differential Equations

I am not enough mathematician in technical sense of the word to develop overall view about what Prastaro has done and I have caught only the basic idea. I have tried to understand the articles

[A41, A42] with title “Geometry of PDE’s. I/II: Variational PDE’s and integral bordism groups” (`http://tinyurl.com/yb9wey8c` and `http://tinyurl.com/y9x55qmk`), which seem to correspond to my needs. The key idea is to generalize the cobordism concept also to partial differential equations with cobordism replaced with the time evolution defined by partial differential equation. In particular, to geometric variational principles defining as their extremals the counterparts of cobordisms.

Quite generally, and especially so in the case of the conservation of Noether charges give rise to strong selection rules since two n-surfaces with different classical charges cannot be connected by extremals of the variational principle. Note however that the values of the conserved charges depend on the normal derivatives of the embedding space coordinates at the n-dimensional ends of cobordism. If one poses additional conditions fixing these normal derivatives, the selection rules become even stronger. In TGD framework Bohr orbit property central for the notion of WCW geometry and holography allows to hope that conserved charges depend on 3-surfaces only.

What is so beautiful in this approach that it promises to generalize the notion of cobordism and perhaps also the notions of homotopy/homology groups so that they would apply to partial differential equations quite generally, and especially so in the case of geometric variational principles giving rise to $n+1$-surfaces connecting n-surfaces characterizing the initial and final states classically. TGD with $n = 3$ seems to be an ideal applications for these ideas.

Prastaro also proposes a generalization of cobordism theory to super-manifolds and quantum super-manifolds. The generalization in the case of quantum theory utilizing path integral does not not pose conditions on classical connecting field configurations. In TGD framework these generalizations are not needed since fermion number is geometrized in terms of embedding space gamma matrices and super(-symplectic) symmetry is realized differently.

6.7.3 Why Prastaro’s Idea Resonates So Strongly With TGD

Before continuing I want to make clear why Prastaro’s idea resonates so strongly with TGD.

1. One of the first ideas as I started to develop TGD was that there might be selection rules analogous to those of quantum theory telling which 3-surfaces can be connected by a space-time surface. At that time I still believed in path integral formalism assuming that two 3-surfaces at different time slices with different values of Minkowski time can be connected by any space-time surface for which embedding space coordinates have first derivatives.

   I soon learned about Thom’s theory but was greatly disappointed since no selection rules were involved in the category of abstract 3-manifolds. I thought that possible selection rules should result from the imbeddability of the connecting four-manifold to $H = M^4 \times CP_2$ but my gut feeling was that these rules are more or less trivial since so many connecting 4-manifolds exist and some of them are very probably imbeddable.

   One possible source of selection rules could have been the condition that the induced metric has Minkowskian signature - one could justify it in terms of classical causality. This restricts strongly topology change in general relativity (`http://tinyurl.com/y6vuopgj`). Geroch’s classical result [A93] states that non-trivial smooth Lorentz cobordism between compact 3-surfaces implies the existence of closed time loop - not possible in TGD framework. Second non-encouraging result is that scalar field propagating in trouser topology leads to an occurrence of infinite energy burst (`http://tinyurl.com/ybbuwyfj`).

   In the recent formulation of TGD however also Euclidian signature of the induced metric is allowed. For space-time counterparts of 3-particle vertices three space-time surfaces are glued along their smooth 3-D ends whereas space-time surface fails to be everywhere smooth manifold. This picture fits nicely with the idea that one can engineer space-time surfaces by gluing them together along their ends.

2. At that time (before 1980) the discovery of the geometry of the “World of Classical Worlds” (WCW) as a possible solution to the failures of canonical quantization and path integral formalism was still at distance of ten years in future. Around 1985 I discovered the notion of WCW. I made some unsuccessful trials to construct its geometry, and around 1990 finally realized that 4-D general coordinate invariance is needed although basic objects are 3-D surfaces.

This is realized if classical physics is an exact part of quantum theory - not only something resulting in a stationary phase approximation. Classical variational principle should assign to a 3-surface a physically unique space-time surface - the analog of Bohr orbit - and the action for this surface would define Kähler function defining the Kähler geometry of WCW using standard formula.

This led to a notion of preferred extremal: absolute minimum of Kähler action was the first guess and might indeed make sense in the space-time regions with Euclidian signature of induced metric but not in Minkowskian regions, which give to the vacuum functional and exponential of Minkowskian Kähler action multiplied by imaginary unit coming from $\sqrt{g}$ - just as in quantum field theories. Euclidian regions give the analog of the free energy exponential of thermodynamics and transform path integral to mathematically well-defined functional integral.

3. After having discovered the notion of preferred extremal, I should have also realized that an interesting generalization of cobordism theory might make sense after all, and could even give rise to the classical counterparts of the selection rules! For instance, conservation of isometry charges defines equivalence classes of 3-surfaces endowed with tangent space data. Bohr orbit property could fix the tangent space data (normal derivatives of embedding space coordinates) so that conserved classical charges would characterize 3-surfaces alone and thus cobordism equivalence classes and become analogous to topological invariants. This would be in spirit with the attribute "Topological" in TGD!

6.7.4 What Preferred Extremals Are?

The topology of WCW has remained mystery hitherto - partly due to my very limited technical skills and partly by the lack of any real physical idea. The fact, that p-adic topology seems to be natural at least as an effective topology for the maxima of Kähler function of WCW gave a hint but this was not enough.

I hope that the above summary has made clear why the idea about dynamical cobordism and even dynamical homotopy theory is so attractive in TGD framework. One could even hope that dynamics determines not only Kähler geometry but also the topology of WCW to some extend at least! To get some idea what might be involved one must however first tell about the recent situation concerning the notion of preferred extremal.

1. The recent formulation for the notion of preferred extremal relies on strong form of General Coordinate Invariance (SGCI). SGCI states that two kinds of 3-surfaces can identified as fundamental objects. Either the light-light 3-D orbits of partonic 2-surfaces defining boundaries between Minkowskian and Euclidian space-time regions or the space-like 3-D ends of space-time surfaces at boundaries of CD. Since both choices are equally good, partonic 2-surfaces and their tangent space-data at the ends of space-time should be the most economic choice.

   This eventually led to the realization that partonic 2-surfaces and string world sheets should be enough for the formulation of quantum TGD. Classical fields in the interior of space-time surface would be needed only in quantum measurement theory, which demands classical physics in order to interpret the experiments.

2. The outcome is strong form of holography (SH) stating that quantum physics should be coded by string world sheets and partonic 2-surfaces inside given causal diamond (CD). SH is very much analogous to the AdS/CFT correspondence but is much simpler: the simplicity is made possible by much larger group of conformal symmetries.

   If these 2-surfaces satisfy some consistency conditions one can continue them to 4-D space-time surface inside CD such that string world sheets are surfaces inside them satisfying the condition that charged (possibly all) weak gauge potentials identified as components of the induced spinor connection vanish at the string world sheets and also that energy momentum currents flow along these surfaces. String world sheets carry second quantized free induced spinor fields and fermionic oscillator operator basis is used to construct WCW gamma matrices.

3. The 3-surfaces at the ends of WCW must satisfy strong conditions to guarantee effective 2-dimensionality. Quantum criticality suggests the identification of these conditions. All Noether charges assignable to a sub-algebra of super-symplectic algebra isomorphic to it and having conformal weights which are n-multiples of those of entire algebra vanish/annihilate quantum states. One has infinite fractal hierarchy of broken super-conformal symmetries with the property that the sub-algebra is isomorphic with the entire algebra. This like a ball at the top of ball at the top of ....

   The speculative vision is that super-symplectic subalgebra with weights coming as n-ples of those for the entire algebra acts as an analog of conformal gauge symmetries on light-like orbits of partonic 2-surfaces, and gives rise to a pure gauge degeneracy whereas other elements of super-symplectic algebra act as dynamical symmetries. The hierarchy of quantum criticalities defines hierarchies of symmetry breakings characterized by hierarchies of sub-algebras for which one n_{i+1} is divisible by n_i. The proposal is that conformal gauge invariance means that the analogs of Bohr orbits are determined only apart from conformal gauge transformations forming to n_i conformal equivalence classes so that effectively one has n_i discrete degrees of freedom assignable to light-like partonic orbits.

4. In this framework manifolds M and N would correspond the 3-surfaces at the boundaries of CD and containing a collection strings carrying induced spinor fields. The connecting 4-surface W would contain string world sheets and the light-like orbits of partonic 2-surfaces as simultaneous boundaries for Minkowskian and Euclidian regions.

Propagator line has several meanings depending on whether one considers particles as strings, as single fermion states localizable at the ends of strings, or as Euclidian space-time regions or their light-like boundaries with singular induced metric having vanishing determinant. Vertices appear as generalizations of the stringy vertices and as generalization of the vertices of Feynman diagrams in which the incoming 4-surfaces meet along their ends.

1. Propagator line has several meanings depending on whether one considers particles as strings, as single fermion states localizable at the ends of strings, or as Euclidian space-time regions or their light-like boundaries with degenerate induced metric with vanishing determinant. Vertices appear as generalizations of the stringy vertices and as generalization of the vertices of Feynman diagrams in which the incoming 4-surfaces meet along their ends.

   (a) The lines of generalized Feynman graphs defined in topological sense are identified as slightly deformed pieces of CP_2 defining wormhole contacts connecting two Minkowskian regions and having wormhole throats identified as light-like parton orbits as boundaries. Since there is a magnetic monopole flux through the wormhole contacts they must appear as pairs (also larger number is possible) in order that magnetic field lines can close. Elementary particles correspond to pairs of wormhole contacts. At both space-time sheets the throats are connected by magnetic flux tubes carrying monopole flux so that a closed flux tube results having a shape of an extremely flattened square and having wormhole contacts at its ends. It is a matter of taste, whether to call the light-like wormhole throats or their interiors as lines of the generalized Feynman/twistor diagrams.

   The light-like orbits of partonic 2-surfaces bring strongly in mind the light-like 3-surfaces along which radiation fields can be restricted - kind of shockwaves at which the signature of the induced space-time metric changes its signature.

   (b) String world sheets as orbits of strings are also in an essential role and could be seen as particle like objets. String world sheets could as kind of singular solutions of field equations analogous to characteristics of hyperbolic differential equations. The isometry currents of Kähler action flow along string world sheets and field equations restricted to them are satisfied. As if one would have 2-dimensional solution. $\sqrt{g_4}$ would of course vanishes for genuinely 2-D solution but this one can argue that this is not a problem since $\sqrt{g_4}$ can be eliminated from field equations. String world sheets could serve as 2-D a analoga for a solution of hyperbolic field equations defining expanding wave front localized at 3-D light-like surface.

(c) Propagation in the third sense of word is assignable to the ends of string world sheets at the light-like orbits of partonic 2-surfaces and possibly carrying fermion number. One could say that in TGD one has both fundamental fermions serving as building bricks of elementary particles and strings characterizing interactions between particles. Fermion lines are massless in 8-D sense. By strong form of holography this quantum description has 4-D description space-time description as a classical dual.

2. The topological description of interaction vertices brings in the most important deviation from the standard picture behind cobordism: space-time surfaces are not smooth in TGD framework. One allows topological analogs of 3-vertices of Feynman diagrams realized by connecting three 4-surfaces along their smooth 3-D ends. 3-vertex is also an analog (actually much more!) for the replication in biology. This vertex is *not* the analog of stringy trouser vertex for which space-time surface is continuous whereas 3-surface at the vertex is singular (also trouser vertex could appear in TGD).

   The analog of trouser vertex for string world sheets means splitting of string and fermionic field modes decompose into superposition of modes propagating along the two branches. For instance, the propagation of photon along two paths could correspond to its geometric decay at trouser vertex not identifiable as "decay" to two separate particles.

   For the analog of 3-vertex of Feynman diagram the 3-surface at the vertex is non-singular but space-time surface is singular. The gluing along ends corresponds to genuine 3-particle vertex.

   The view about solution of PDEs generalizes dramatically but the general idea about cobordism might make sense also in the generalized context.

6.7.5 Could Dynamical Homotopy/Homology Groups Characterize WCW Topology?

The challenge is to at least formulate (with my technical background one cannot dream of much more) the analog of cobordism theory in this framework. One can actually hope even the analog of homotopy/homology theory.

1. To a given 3-surface one can assign its cobordism class as the set of 3-surfaces at the opposite boundary of CD connected by a preferred extremal. The 3-surfaces in the same cobordism class are characterized by same conserved classical Noether charges, which become analogs of topological invariants.

   One can also consider generalization of cobordisms as analogs to homotopies by allowing return from the opposite boundary of CD. This would give rise to first homotopy groupoid. One can even go back and forth several times. These dynamical cobordisms allow to divide 3-surfaces at given boundary of CD in equivalence classes characterized among other things by same values of conserved charges. One can also return to the original 3-surface. This could give rise to the analog of the first homotopy group Π_1.

2. If one takes the homotopy interpretation literally one must conclude that the 3-surfaces with different conserved Noether charges cannot be connected by any path in WCW - they belong to disjoint components of the WCW! The zeroth dynamical homotopy group Π_0 of WCW would be non-trivial and its elements would be labelled by the conserved Noether charges defining topological invariants!

   The values of the classical Noether charges would label disjoint components of WCW. The topology for the space of these parameters would be totally disconnected - no two points cannot be connected by a continuous path. p-Adic topologies are indeed totally disconnected. Could it be that p-adic topology is natural for the conserved classical Noether charges and the sectors of WCW are characetrized by p-adic number fields and their algebraic extensions?

   Long time ago I noticed that the 4-D spin glass degeneracy induced by the huge vacuum degeneracy of Kähler action implies analogy between the space of maxima of Kähler function and the energy landscape of spin glass systems [K57]. Ultrametricity (`http://tinyurl.com/y6vswdoh`) is the basic property of the topology of the spin glass energy landscape. p-Adic topology is ultrametric and the proposal was that the effective topology for the space of maxima could be p-adic.

3. Isometry charges are the most important Noether charges. These Noether charges are very probably not the only conserved charges. Also the generators in the complement of the gauge sub-algebra of symplectic algebra acting as gauge conformal symmetries could be conserved. All these conserved Noether charges would define a parameter space with a natural p-adic topology.

   Since integration is problematic p-adically, one can ask whether only discrete quantum superpositions of 3-surfaces with different classical charges are allowed or whether one should even assume fixed values for the total classical Noether charges appearing in the scattering amplitudes.

   I have proposed this kind of approach for the zero modes of WCW geometry not contributing to the Kähler metric except as parameters. The integration for zero modes is also problematic because there is no metric, which would define the integration measure. Since classical charges do not correspond to quantum fluctuating degrees of freedom they should correspond to zero modes. Hence these arguments are equivalent.

The above argument led to the identification of the analogs of the homotopy group Π_0 and led to the idea about homotopy groupoid/group Π_1. The elements of Π_1 would correspond to space-time surfaces, which run arbitrary number of times fourth and back and return to the initial 3-surface at the boundary of CD. If the two preferred extremals connecting same pair of 3-surfaces can be deformed to each other, one can say that they are equivalent as dynamical homotopies (or cobordisms). What could be the allowed deformations? Are they cobordisms of cobordisms? What this could mean? Could they define the analog of homotopy groupoid Π_2 as foliations of preferred extremals connecting the same 3-surfaces?

1. The number theoretic vision about generalized Feynman diagrams suggests a possible approach. Number theoretic ideas combined with the generalization of twistor approach [K94, K86] led to the vision that generalized Feynman graphs can be identified as sequences or webs of algebraic operations in the co-algebra defined by the Yangian assignable to super-symplectic algebra [A36] [B33, B24, B25] and acting as symmetries of TGD. Generalized Feynman graphs would represent algebraic computations. Computations can be done in very many different ways and each of them corresponds to a generalized Feynman diagram. These computations transform give same final collection of "numbers" when the initial collection of "numbers" is given. Does this mean that the corresponding scattering amplitudes must be identical?

   If so, a huge generalization of the duality symmetry of the hadronic string models would suggest itself. All computations can be reduced to minimal computations. Accordingly, generalized Feynman diagrams can be reduced to trees by eliminating loops by moving the ends of the loops to same point and snipping the resulting tadpole out! The snipped of tadpole would give a mere multiplicative factor to the amplitude contributing nothing to the scattering rate - just like vacuum bubbles contribute nothing in the case of ordinary Feynman diagrams.

2. How this symmetry could be realized? Could one just assume that only the minimal generalized Feynman diagrams contribute? - not a very attractive option. Or could one hope that only tree diagrams are allowed by the classical dynamics: this was roughly the original vision? The huge vacuum degeneracy of Kähler action implying non-determinism does not encourage this option. The most attractive and most predictive realization conforming with the idea about generalized Feynman diagrammatics as arithmetics would be that all the diagrams differing by these moves give the same result. An analogous symmetry has been discovered for twistor diagrams.

3. Suppose one takes seriously the snipping of a tadpole away from diagram as a move, which does not affect the scattering amplitude. Could this move correspond to an allowed elementary cobordism of preferred extremal? If so, scattering amplitudes would have purely

topological meaning as representations of the elements of cobordism classes! TGD would indeed be what it was proposed to be but in much deeper sense than I thought originally. This could also conform with the interpretation of classical charges as topological invariants, realize adelic physics at the level of WCW, and conform with the idea about TGD as almost topological QFT and perhaps generalizing it to topological QFT in generalized sense.

4. One can imagine several interpretations for the snipping operation at space-time level. TGD allows a huge classical vacuum degeneracy: all space-time surfaces having Lagrangian manifold of CP_2 as their CP_2 projection are vacuum extremals of Kähler action. Also all CP_2 type extremals having 1-D light-like curve as M^4 projection are vacuum extremals but have non-vanishing Kähler action. This would not matter if one does not have superpositions since multiplicative factors are eliminated in scattering amplitudes. Could the tadpoles correspond to CP_2 type vacuum extremals at space-time level?

   There is also an alternative interpretation. In ZEO causal diamonds (CD) form a hierarchy and one can imagine that the sub-CDs of given CD correspond to quantum fluctuations. Could tadpoles be assigned to sub-CDs of CD be considered+

5. In this manner one could perhaps define elements of homotopy groupoid Π_2 as foliations preferred extremals with same ends - these would be 5-D surfaces. If one has two such 5-D foliations with the same 4-D ends, one can form the reverse of the other and form a closed surface. This would be analogous to a map of S^2 to WCW. If the two 5-D foliations cannot be transformed to each other, one would have something, which might be regarded as a non-trivial element of dynamical homotopy group Π_2.

One can ask whether one could define also the analogs of higher homology or homotopy grouppoids and groupoids Π_3 up to Π_5 - the upper bound $n = 5 = 8 - 3$ comes from the fact that foliations of foliations.. can have maximum dimension $D = 8$ and from the dimension of $D = 3$ of basic objects.

1. One could form a foliation of the foliations of preferred extremals as the element of the homotopy groupoid Π_3. Could allowed moves reduce to the snipping operation for generalized Feynman diagrams but performed along direction characterized by a new foliation parameter.

2. The topology of the zero mode sector of WCW parameterized by fixed values of conserved Noether charges as element of Π_0 could be characterized by dynamical homotopy groups Π_n, $n = 1, ..., 5$ - at least partially. These degrees of freedom could correspond to quantum fluctuating degrees of freedom. The Kähler structure of WCW and finite-D analogy suggests that all odd dynamical homotopy groups vanish so that Π_0, Π_2 and Π_4 would be the only non-trivial dynamical homotopy groups. The vanishing of Π_1 would imply that there is only single minimal generalized Feynman diagram contributing to the scattering amplitude. This also true if Feynman diagrams correspond to arithmetic operations.

3. Whether one should call these groups homotopy groups or homology groups is not obvious. The construction means that the foliations of foliations of ... can be seen as images of spheres suggesting “homotopy”. The number of these groups is determined by the dimension of embedding space, which suggests “homology”.

4. Clearly, the surfaces defining the dynamical homotopy groups/groupoids would be analogs of branes of M-theory but would be obtained constructing paths of paths of paths... by starting from preferred extremals. The construction of so called n-groups (`http://tinyurl.com/yckcjcln`) brings strongly in mind this construction.

6.7.6 Appendix: About Field Equations Of TGD In Jet Bundle Formulation

Prastaro utilizes jet bundle (`http://tinyurl.com/yb2575bm`) formulation of partial differential equations (PDEs). This notion allows a very terse formulation of general PDEs as compared to the old-fashioned but much more concrete formulation that I have used. The formulation is rather formula rich and reader might lose easily his/her patience since one must do hard work to learn which formulas follow trivially from the basic definitions.

I will describe this formulation in TGD framework briefly but without explicit field equations, which can be found at [K10]. To my view a representation by using a concrete example is always more reader friendly than the general formulas derived in some reference. I explain my view about the general ideas behind jet bundle formulation with minimal number amount of formulas. The reader can find explicit formulas from the Wikipedia link above.

The basic goal is to have a geometric description of PDE. In TGD framework the geometric picture is of course present from beginning: field patterns as 4-surfaces in field space - somewhat formal geometric objects - are replaced with genuine 4-surfaces in $M^4 \times CP_2$.

Field equations as conservation laws, Frobenius integrability conditions, and a connection with quaternion analyticity

The following represents qualitative picture of field equations of TGD trying to emphasize the physical aspects. Also the possibility that Frobenius integrability conditions are satisfied and correspond to quaternion analyticity is discussed.

1. Kähler action is Maxwell action for induced Kähler form and metric expressible in terms of embedding space coordinates and their gradients. Field equations reduce to those for embedding space coordinates defining the primary dynamical variables. By GCI only four of them are independent dynamical variables analogous to classical fields.

2. The solution of field equations can be interpreted as a section in fiber bundle. In TGD the fiber bundle is just the Cartesian product $X^4 \times CD \times CP_2$ of space-time surface X^4 and causal diamond $CD \times CP_2$. CD is the intersection of future and past directed light-cones having two light-like boundaries, which are cone-like pieces of light-boundary $\delta M^4_\pm \times CP_2$. Space-time surface serves as base space and $CD \times CP_2$ as fiber. Bundle projection Π is the projection to the factor X^4. Section corresponds to the map $x \rightarrow h^k(x)$ giving embedding space coordinates as functions of space-time coordinates. Bundle structure is now trivial and rather formal.

   By GCI one could also take suitably chosen 4 coordinates of $CD \times CP_2$ as space-time coordinates, and identify $CD \times CP_2$ as the fiber bundle. The choice of the base space depends on the character of space-time surface. For instance CD, CP_2 or $M^2 \times S^2$ (S^2 a geodesic sphere of CP_2), could define the base space. The bundle projection would be projection from $CD \times CP_2$ to the base space. Now the fiber bundle structure can be non-trivial and make sense only in some space-time region with same base space.

3. The field equations derived from Kähler action must be satisfied. Even more: one must have a *preferred* extremal of Kähler action. One poses boundary conditions at the 3-D ends of space-time surfaces and at the light-like boundaries of $CD \times CP_2$.

   One can fix the values of conserved Noether charges at the ends of CD (total charges are same) and require that the Noether charges associated with a sub-algebra of super-symplectic algebra isomorphic to it and having conformal weights coming as n-ples of those for the entire algebra, vanish. This would realize the effective 2-dimensionality required by SH. One must pose boundary conditions also at the light-like partonic orbits. So called weak form of electric-magnetic duality is at least part of these boundary conditions.

   It seems that one must restrict the conformal weights of the entire algebra to be non-negative $r \geq 0$ and those of subalgebra to be positive: $mn > 0$. The condition that also the commutators of sub-algebra generators with those of the entire algebra give rise to vanishing Noether charges implies that all algebra generators with conformal weight $m \geq n$ vanish so the dynamical algebra becomes effectively finite-dimensional. This condition generalizes to the action of super-symplectic algebra generators to physical states.

   M^4 time coordinate cannot have vanishing time derivative dm^0/dt so that four-momentum is non-vanishing for non-vacuum extremals. For CP_2 coordinates time derivatives ds^k/dt can vanish and for space-like Minkowski coordinates dm^i/dt can be assumed to be non-vanishing if M^4 projection is 4-dimensional. For CP_2 coordinates $ds^k/dt = 0$ implies the vanishing of electric parts of induced gauge fields. The non-vacuum extremals with the largest conformal gauge symmetry (very small n) would correspond to cosmic string solutions for which induced gauge fields have only magnetic parts. As n increases, also electric parts are generated. Situation becomes increasingly dynamical as conformal gauge symmetry is reduced and dynamical conformal symmetry increases.

4. The field equations involve besides embedding space coordinates h^k also their partial derivatives up to second order. Induced Kähler form and metric involve first partial derivatives $\partial_\alpha h^k$ and second fundamental form appearing in field equations involves second order partial derivatives $\partial_\alpha \partial_\beta h^k$.

   Field equations are hydrodynamical, in other worlds represent conservation laws for the Noether currents associated with the isometries of $M^4 \times CP_2$. By GCI there are only 4 independent dynamical variables so that the conservation of $m \leq 4$ isometry currents is enough if chosen to be independent. The dimension m of the tangent space spanned by the conserved currents can be smaller than 4. For vacuum extremals one has $m = 0$ and for massless extremals (MEs) $m = 1$! The conservation of these currents can be also interpreted as an existence of $m \leq 4$ closed 3-forms defined by the duals of these currents.

5. The hydrodynamical picture suggests that in some situations it might be possible to assign to the conserved currents flow lines of currents even globally. They would define $m \leq 4$ global coordinates for some subset of conserved currents (4+8 for four-momentum and color quantum numbers). Without additional conditions the individual flow lines are well-defined but do not organize to a coherent hydrodynamic flow but are more like orbits of randomly moving gas particles. To achieve global flow the flow lines must satisfy the condition $d\phi^A/dx^\mu = k^A_B J^B_\mu$ or $d\phi^A = k^A_B J^B$ so that one can special of 3-D family of flow lines parallel to $k^A_B J^B$ at each point - I have considered this kind of possibly in [K10] at detail but the treatment is not so general as in the recent case.

   Frobenius integrability conditions (`http://tinyurl.com/yc6apam2`) follow from the condition $d^2\phi^A = 0 = dk^A_B \wedge J^B + k^A_B dJ^B = 0$ and implies that dJ^B is in the ideal of exterior algebra generated by the J^A appearing in $k^A_B J^B$. If Frobenius conditions are satisfied, the field equations can define coordinates for which the coordinate lines are along the basis elements for a sub-space of at most 4-D space defined by conserved currents. Of course, the possibility that for preferred extremals there exists $m \leq 4$ conserved currents satisfying integrability conditions is only a conjecture.

   It is quite possible to have $m < 4$. For instance for vacuum extremals the currents vanish identically For MEs various currents are parallel and light-like so that only single light-like coordinate can be defined globally as flow lines. For cosmic strings (cartesian products of minimal surfaces X^2 in M^4 and geodesic spheres S^2 in CP_2 4 independent currents exist). This is expected to be true also for the deformations of cosmic strings defining magnetic flux tubes.

6. Cauchy-Riemann conditions in 2-D situation represent a special case of Frobenius conditions. Now the gradients of real and imaginary parts of complex function $w = w(z) = u + iv$ define two conserved currents by Laplace equations. In TGD isometry currents would be gradients apart from scalar function multipliers and one would have generalization of C-R conditions. In citeallbprefextremals,twistorstory I have considered the possibility that the generalization of Cauchy-Riemann-Fuerter conditions [A101, A86] (`http://tinyurl.com/yb8l34b5`) could define quaternion analyticity - having many non-equivalent variants - as a defining property of preferred extremals. The integrability conditions for the isometry currents would be the natural physical formulation of CRF conditions. Different variants of CRF conditions would correspond to varying number of independent conserved isometry currents.

7. The problem caused by GCI is that there is infinite number of coordinate choices. How to pick a physically preferred coordinate system? One possible manner to do this is to use coordinates for the projection of space-time surface to some preferred sub-space of embedding - geodesic manifold is an excellent choice. Only $M^1 \times X^3$ geodesic manifolds are not possible but these correspond to vacuum extremals.

One could also consider a philosophical principle behind integrability. The variational principle itself could give rise to at least some preferred space-time coordinates in the same manner as TGD based quantum physics would realize finite measurement resolution in terms of inclusions of HFFs in terms of hierarchy of quantum criticalities and fermionic strings connecting partonic 2-surfaces. Frobenius integrability of the isometry currents would define some preferred coordinates. Their number need not be the maximal four however.

For instance, for massless extremals only light-like coordinate corresponding to the light-like momentum is obtained. To this one can however assign another local light-like coordinate uniquely to obtain integrable distribution of planes M^2. The solution ansatz however defines directly an integrable choice of two pairs of coordinates at embedding space level usable also as space-time coordinates - light-like local direction defining local plane M^2 and polarization direction defining a local plane E^2. These choices define integrable distributions of orthogonal planes and local hypercomplex and complex coordinates. Pair of analogs of C-R equations is the outcome. I have called these coordinates Hamilton-Jacobi coordinates for M^4.

8. This picture allows to consider a generalization of the notion of solution of field equation to that of integral manifold (`http://tinyurl.com/yajn7cuz`. If the number of independent isometry currents is smaller than 4 (possibly locally) and the integrability conditions hold true, lower-dimensional sub-manifolds of space-time surface define integral manifolds as kind of lower-dimensional effective solutions. Genuinely lower-dimensional solutions would of course have vanishing $\sqrt{g_4}$ and vanishing Kähler action.

   String world sheets can be regarded as 2-D integral surfaces. Charged (possibly all) weak boson gauge fields vanish at them since otherwise the electromagnetic charge for spinors would not be well-defined. These conditions force string world sheets to be 2-D in the generic case. In special case 4-D space-time region as a whole can satisfy these conditions. Well-definedness of Kähler-Dirac equation [K96, K73] demands that the isometry currents of Kähler action flow along these string world sheets so that one has integral manifold. The integrability conditions would allow $2 < m \leq n$ integrable flows outside the string world sheets, and at string world sheets one or two isometry currents would vanish so that the flows would give rise 2-D independent sub-flow.

9. The method of characteristics (`http://tinyurl.com/y9dcdayt`) is used to solve hyperbolic partial differential equations by reducing them to ordinary differential equations. The (say 4-D) surface representing the solution in the field space has a foliation using 1-D characteristics. The method is especially simple for linear equations but can work also in the non-linear case. For instance, the expansion of wave front can be described in terms of characteristics representing light rays. It can happen that two characteristics intersect and a singularity results. This gives rise to physical phenomena like caustics and shock waves.

   In TGD framework the flow lines for a given isometry current in the case of an integrable flow would be analogous to characteristics, and one could also have purely geometric counterparts of shockwaves and caustics. The light-like orbits of partonic 2-surface at which the signature of the induced metric changes from Minkowskian to Euclidian might be seen as an example about the analog of wave front in induced geometry. These surfaces serve as carriers of fermion lines in generalized Feynman diagrams. Could one see the particle vertices at which the 4-D space-time surfaces intersect along their ends as analogs of intersections of characteristics - kind of caustics? At these 3-surfaces the isometry currents should be continuous although the space-time surface has "edge".

10. The analogy with ordinary analyticity suggests that it might be possible to interpret string world sheets and partonic 2-surfaces appearing in strong form of holography (SH) as co-dimension 2 surfaces analogous to poles of analytic function in complex plane. Light-like 3-surfaces might be seen as analogs of cuts. The coding of analytic function by its singularities could be seen as analog of SH.

Jet bundle formalism

Jet bundle formalism (`http://tinyurl.com/yb2575bm`) is a modern manner to formulate PDEs in a coordinate independent manner emphasizing the local algebraic character of field equations. In

TGD framework GCI of course guarantees this automatically. Beside this integrability conditions formulated in terms of Cartan's contact forms are needed.

1. The basic idea is to take the partial derivatives of embedding space coordinates as functions of space-time coordinates as independent variables. This increases the number of independent variables. Their number depends on the degree of the jet defined and for partial differential equation of order r, for n dependent variables, and for N independent variables the number of new degrees of freedom is determined by r, n, and N - just by counting the total number of various partial derivatives from $k = 0$ to r. For $r = 1$ (first order PDE) it is $N \times (1 + n)$.

2. Jet at given space-time point is defined as a Taylor polynomial of the embedding space coordinates as functions of space-time coordinates and is characterized by the partial derivatives at various points treated as independent coordinates analogous to embedding space coordinate. Jet degree r is characterized by the degree of the Taylor polynomial. One can sum and multiply jets just like Taylor polynomials. Jet bundle assigns to the fiber bundle associated with the solutions of PDE corresponding jet bundle with fiber at each point consisting of jets for the independent variables ($CD \times CP_2$ coordinates) as functions of the dependent variables (space-time coordinates).

3. The field equations from the variation of Kähler action are second order partial differential equations and in terms of jet coefficients they reduce to local algebraic equations plus integrability conditions. Since TGD is very non-linear one obtains polynomial equations at each point - one for each embedding space coordinate. Their number reduces to four by GCI. The minimum degree of jet bundle is $r = 2$ if one wants algebraic equations since field equations are second order PDEs.

4. The local algebraic conditions are not enough. One must have also conditions stating that the new independent variables associated with partial derivatives of various order reduces to appropriate multiple partial derivatives of embedding space coordinates. These conditions can be formulated in terms of Cartan's contact forms, whose vanishing states these conditions. For instance, if dh^k is replaced by independent variable u^k, the condition $dh^k - u^k = 0$ is true for the solution surfaces.

5. In TGD framework there are good motivations to break the non-orthodoxy and use 1-jets so that algebraic equations replaced by first order PDEs plus conditions requiring vanishing of contact forms. These equations state the conservation of isometry currents implying that the 3-forms defined by the duals of isometry currents are closed. As found, this formulation reveals in TGD framework the hydrodynamic picture and suggests conditions making the system integrable in Frobenius sense.

6.8 Twistor lift of TGD and WCW geometry

In the following a view about WCW geometry forced by twistor lift of TGD [K86, L33, L50, L57] is summarized. Twistor lift brings to the action a volume term but without breaking conformal invariance and without introducing cosmological constant as a fundamental dimensional dynamical coupling. The proposed construction of the gamma matrices of WCW giving rise to Kähler metric as anti-commutators is now in terms of the Noether super charges associated with the super-symplectic algebra. This I dare to regard as a very important step of progress.

6.8.1 Possible weak points of the earlier vision

To make progress it is wise to try to identify the possible weak points of the earlier vision.

1. The huge vacuum degeneracy of Kähler action [K39] defining the Kähler function of WCW Kähler metric is analogous to gauge degeneracy of Maxwell action and coded by symplectic transformations of CP_2. It implies that the degeneracy of the metric increases as one approaches vacuum extremals and is maximal for the space-time surfaces representing canonical embeddings of Minkowski space: Kähler action vanishes up to fourth order in deformation.

The original interpretation was in terms of 4-D spin glass degeneracy assumed to be induced by quantum degeneracy.

One could however argue that classical non-determinism of Kähler action is not acceptable and that a small term removing the vacuum degeneracy is needed to make the situation mathematically acceptable. There is an obvious candidate: a volume term having an interpretation in terms of cosmological constant. This term however seems to mean the presence of length scale as a fundamental constant and is in conflict with the basic lesson learned from gauge theories teaching that only dimensionless couplings can be allowed.

2. The construction of WCW Kähler metric relies on the hypothesis that the basic result from the construction of loop space geometries [A54] generalizes: the Kähler metric should be essentially unique from the condition that the isometry group is maximal - this guarantees the existence of Riemann connection. For $D = 3$ this condition is expected to be even stronger than for $D = 1$.

   The hypothesis is that in zero energy ontology (ZEO) the symplectic group acting at the light-like boundaries of causal diamond (CD) (one has CD= $cd \times CP_2$, where cd is the intersection of future and past directed light-cones) acts as the isometries of the Kähler metric.

   It would be enough to identify complexified WCW gamma matrices and define WCW metric in terms of their anti-commutators. The natural proposal is that gamma matrices are expressible as linear combinations of fermionic oscillator operators for second quantized induced spinor fields at space-time surface. One could even ask whether fermionic super charges and conserved fermionic Noether charges are involved with the construction.

   The explicit construction of gamma matrices [K96, K73] has however been based on somewhat ad hoc formulas, and what I call effective 2-dimensionality argued to follow from quantum criticality is somewhat questionable as exact notion.

6.8.2 Twistor lift of TGD and ZEO

Twistor lift of TGD and ZEO meant a revolution in the view about WCW geometry and spinor structure.

1. The basic idea is to replace 4-D Kähler action with dimensionally reduced 6-D Kähler for the analog of twistor space of space-time surface. The induction procedure for the spinors would be generalized so that it applies to twistor structure [L55]. The twistor structure of the embedding space is identified as the product of twistor spaces $M^4 \times S^2$ of M^4 and $SU(3)/U(1) \times U(1)$ of CP_2. In momentum degrees of freedom the twistor space of M^4 would be the usual CP_3.

   Remarkably, M^4 and CP_2 are the only spaces allowing twistor space with Kähler structure [A74]. In the case of M^4 the Kähler structure is a generalization of that for E^4. TGD would be unique from the existence of twistor lift. This predicts CP breaking at fundamental level possibly responsible for CP breaking and matter-antimatter asymmetry.

2. One would still have Kähler coupling strength α_K as the only single dimensionless coupling strength, whose spectrum is dictated by quantum criticality meaning that it is analogous to critical temperature. All coupling constant like parameters would be determined by quantum criticality. Cosmological constant would not be fundamental constant and this makes itself visible also in the concrete expressions for conserved Noether currents. The breaking of the scale invariance removing vacuum degeneracy of 4-D Kähler action would be analogous to spontaneous symmetry breaking and would remove vacuum degeneracy and classical non-determinism.

   The volume term would emerge from dimensional reduction required to give for the 6-surface the structure of S^2 bundle having space-time surfaces as base space. Cosmological constant would be determined by dynamics and depend on p-adic length scale depending in the average on length scale of space-time sheet proportional to the cosmic time sense like $1/a^2$, a cosmic time. This would solve the problem of large cosmological constant and predict extremely small cosmological constant in cosmic scales in the recent cosmology. This suggests that in long length scales one still has spin glass degeneracy realized in terms of many-sheeted space-time.

3. In ZEO 3-surface correspond to a union of 3-surfaces at the ends of space-time surfaces at boundaries of CD. There are many characterizations of quantum criticality.

    (a) Preferred extremal property and quantum criticality would mean that one has simultaneously an extremal of both 4-D Kähler action and volume term except at singular 2-surfaces identified as string world sheets and their boundaries. In accordance with the universality of quantum critical dynamics, one would have outside singularities local dynamics without dependence on Kähler coupling strength. The interpretation would be as geometric generalization of massless fields also characterizing criticality.

    (b) Another characterization of preferred extremal is as a space-time surfaces using sub-algebra S_m of symplectic algebra S for which generators have conformal weights coming as m-tuples of those for the full symplectic algebra. Both S_m and $[S, S_m]$ would have vanishing Noether charges. For the induced spinor fields analogous condition would hold true. Effectively the infinite number of radial conformal weights of the symplectic algebra associated with the light-like radial coordinate of $\delta M^4_\pm$ would reduce to a finite number.

    (c) A further characterization would be in terms of $M^8 - H$ duality [L42]. Preferred extremals in H would be images of of space-time surfaces in M^8 under $M^8 - H$ duality. The latter would correspond to roots of octonionic polynomials with coefficients in an extension of rationals. Therefore space-time surfaces in H satisfying field equations plus preferred extremal conditions would correspond to surfaces described by algebraic equations in M^8. Algebraic dynamics would be dual to differential dynamics.

    (d) In adelic physics [L48, L47] the hierarchy of Planck constants $h_{eff}/h_0 = n$ with n having an interpretation as dimensions of Galois group of extension of rationals would define further correlate of quantum criticality. The scaled up Compton lengths proportional to h_{eff} would characterize the long range fluctuations associated with quantum criticality.

6.8.3 The revised view about WCW metric and spinor structure

In this framework one can take a fresh approach to the construction of the spinor structure and Kähler metric of WCW. The basic vision is rather conservative. Rather than inducing ad hoc formulas for WCW gamma matrices one tries to identify Noether the elements super-algebra as Noether charges containing also the gamma matrices as Noether super charges.

1. The simplest guess is that the algebra generated by fermionic Noether charges Q^A for symplectic transformations $h^k \to h^k + \epsilon j^{Ak}$ assumed to induce isometries of WCW and Noether supercharges Q_n and their conjugates for the shifts $\Psi \to \Psi + \epsilon u_n$, where u_n is a solution of the modified Dirac equation, and ϵ is Grassmann number are enough to generate algebra containing the gamma matrix algebra.

2. The commutators $\Gamma^A_n = [Q^A, Q_n]$ are super-charges labelled by (A, n). One would like to identify them as gamma matrices of WCW. The problem is that they are labelled by (A, n) whereas isometry generators are labelled by A only just as symplectic Noether charges. Do all supercharges Γ^A_n except Γ^A_0 corresponding to $u_0 = constant$ annihilate the physical states so that one would have 1-1 correspondence? This would be analogous to what happens quite generally in super-conformal algebras.

3. The anti-commutators of Γ^A_0 would give the components of the Kähler metric. The allowance of singular surfaces having 2-D string world sheets as singularities would give to the metric also stringy component besides 3-D component and possible 0-D components at the ends of string. Metric 2-D property would not be exact as assumed originally.

This construction can be blamed for the lack of explicitness. The general tendency in the development of TGD has been replacement of explicit but somewhat ad hoc formulas with principles. Maybe this reflects to my own ageing and increasing laziness but my own view is that principles are what matter and get abstracted only very slowly. The less formulas, the better!

6.9 Does 4-D action generate lower-dimensional terms dynamically?

The original proposal was that the action defining the preferred extremals is 4-D Kähler action. Later it became obvious that there must be also 2-D string world sheet term present and probably also 1-D term associated with string boundaries at partonic 2-surfaces. The question has been whether these lower-D terms in the action are primary of generated dynamically. By super-conformal symmetry the same question applies to the fermionic part of the action. The recent formulation based on the twistor lift of TGD contains also volume term but the question remains the same.

During years several motivations for the proposal that preferred extremals of action principle including also volume term for twistor lift of Kähler action are minimal surfaces which are singular at 2-D string world sheets and perhaps also at their boundaries.

In particular, quantum criticality would be realized as a minimal surface property realized by holomorphy in suitably generalized sense [L61, L55]. The reason is that the holomorphic solutions of minimal surface equations involve no coupling parameters as the universality of the dynamics at quantum criticality demands.

Minimal surface equation would be true apart from possible singular surfaces having dimension $D = 2, 1, 0$. $D = 2$ corresponds to string world sheets and partonic 2-surfaces. If there are 0-D singularities they would be associated with the ends of orbits of partonic 2-surfaces at boundaries of causal diamond (CD). Minimal surfaces are solutions of non-linear variant of massless d'Alembertian having as effective sources the singular surfaces at which d'Alembertian equation fails. The analogy with gauge theories is highly suggestive: singular surfaces would act as sources of massless field.

Strings world sheets seem to be necessary. The basic question is whether the singular surfaces are postulated from the beginning and there is action associated with them or whether they emerge dynamical from 4-D action. One can consider two extreme options.

Option I: There is an explicit assignment of action to the singular surfaces from the beginning. A transfer of Noether charges between space-time interior and string world sheets is possible. This kind of transfer process can take place also between string world sheets and their light-like boundaries and happens if the normal derivatives of embedding space coordinates are discontinuous at the singular surface.

Option II: No separate action is assigned with the singular surfaces. There could be a transfer of Noether charges between 4-D Kähler and volume degrees of freedom at the singular surfaces causing the failure of minimal surface property in 4-D sense. But could singular surfaces carry Noether currents as 2-D delta function like densities?

This is possible if the discontinuity of the normal derivatives generates a 2-D singular term to the action. Conservation laws require that at string world sheets energy momentum tensor should degenerate to a 2-D tensor parallel to and concentrated at string world sheet. Only 4-D action would be needed - this was actually the original proposal. Strings and particles would be essentially edges of space-time - this is not possible in GRT. Same could happen also at its boundaries giving rise to point like particles. Super-conformal symmetry would make this possible also in the fermionic sector.

For both options the singular surfaces would provide a concrete topological picture about the scattering process at the level of single space-time surface and telling what happens to the initial state. The question is whether Option I actually reduces to Option II. If the 2-D term is generated to 4-D action dynamically, there is no need to postulate primary 2-D action.

6.9.1 Can Option II generate separate 2-D action dynamically?

The following argument shows that Option II with 4-D primary action can generate dynamically 2-D term into the action so that no primary action need to be assigned with string world sheets.

Dimensional hierarchy of surfaces and strong form of holography

String world sheets having light-like boundaries at the light-like orbits of partonic 2-surfaces are certainly needed to realize strong form of holography [K96]. Partonic 2-surfaces emerge automatically as the ends of the orbits of wormhole contacts.

1. There could (but neet not) be a separate terms in the primary action corresponding to string world sheets and their boundaries. This hierarchy bringing in mind branes would correspond to the hierarchy of classical number fields formed by reals, complex numbers, quaternions (space-time surface), and octonions (embedding space in M^8-side of M^8 duality). The tangent - or normal spaces of these surfaces would inherit real, complex, and quaternionic structures as induced structure. The number theoretic interpretation would allow to see these surfaces as images of those surfaces in M^8 mapped to H by $M^8 - H$ duality. Therefore it would be natural to assign action to these surfaces.

2. This makes in principle possible the transfer of classical and quantum charges between space-time interior and string world sheets and between from string world sheets to their light-like boundaries. TGD variant of twistor Grassmannian approach [L50, L57] relies on the assumption that the boundaries of string world sheets at partonic orbits carry quantum numbers. Quantum criticality realized in terms of minimal surface property realized holomorphically is central for TGD and one can ask whether it could play a role in the definition of S-matrix and identification of particles as geometric objects.

3. For preferred extremals string world sheets (partonic 2-surfaces) would be complex (co-complex) manifolds in octonionic sense. Minimal surface equations would hold true outside string world sheets. Conservation of various charges would require that the divergences of canonical momentum currents at string world sheet would be equal to the discontinuities of the normal components of the canonical momentum currents in interior. These discontinuities would correspond to discontinuities of normal derivatives of embedding space coordinates and are acceptable. Similar conditions would hold true at the light-like boundaries of string world sheets at light-like boundaries of parton orbits. String world sheets would not be minimal surfaces and minimal surface property for space-time surface would fail at these surfaces.

   Quantum criticality for string world sheets would also correspond to minimal surface property. If this is realized in terms of holomorphy, the field equations for Kähler and volume parts at string world sheets would be satisfied separately and the discontinuities of normal components for the canonical momentum currents in the interior would vanish at string world sheets.

4. The idea about asymptotic states as free particles would suggest that normal components of canonical momentum currents are continuous near the boundaries of CD as boundary conditions at least. The same must be true at the light-like boundaries of string world sheets. Minimal surface property would reduce to the property of being light-like geodesics at light-like partonic 2-surface. If this is not assumed, the orbit is space-like. Even if these conditions are realized, one can imagine the possibility that at string world sheets 4-D minimal surface equation fails and there is transfer of charges between Kähler and volume degrees of freedom (Option II) and therefore breaking of quantum criticality.

   If the exchange of Noether charges vanishes everywhere at string world sheets and boundaries, one could argue that they represent independent degrees of freedom and that TGD reduces to string model. The proposed equation for coupling constant evolution however contains a coefficients depending on the total action so that this would not be the case.

5. Assigning action to the lower-D objects requires additional coupling parameters. One should be able to express these parameters in terms of the parameters appearing in 4-D action (α_K and cosmological constant). For string sheets the action containing cosmological term is 4-D and Kähler action for $X^2 \times S^2$, where S^2 is non-dynamical twistor sphere is a good guess. Kähler action gets contributions from X^2 and S^2. If the 2-D action is generated dynamically as a singular term of 4-D action its coupling parameters are those of 4-D action.

6. There is a temptation to interpret this picture as a realization of strong form of holography (SH) in the sense that one can deduce the space-time surfaces by using data at string world sheets and partonic 2-surfaces and their light-like orbits. The vanishing of normal components of canonical momentum currents would fix the boundary conditions.

If double holography $D = 4 \rightarrow D = 2 \rightarrow D = 1$ were satisfied it might be even possible to reduce the construction of S-matrix to the proposed variant of twistor Grassmann approach. This need not be the case: p-adic mass calculations rely on p-adic thermodynamics for the excitions of massless particles having CP_2 mass scale and it would seem that the double holography can makes sense for massless states only.

In M^8-picture [L42] the information about space-time surface is coded by a polynomial defined at real line having coefficients in an extension of rationals. This real line for octonions corresponds to the time axis in the rest system rather than light-like orbit as light-like boundary of string world sheet.

Stringy quantum criticality?

The original intuition [L61] was that there are canonical momentum currents between Kähler and volume degrees of freedom at singular surfaces but no transfer of canonical momenta between interior and string world sheets nor string world sheets and their boundaries. Also string world sheets would be minimal surfaces as also the intuition from string models suggests. Could also the stringy quantum criticality be realized?

1. Some embedding space coordinates h^k must have discontinuous partial derivatives in directions normal to the string world sheet so that 3-surface has 1-D edge along fermionic string connecting light-like curves at partonic 2-surfaces in both Minkowskian and Euclidian regions. A closed highly flattened rectangle with long and short edges would be associated with closed monopole flux tube in the case of wormhole contact pairs assigned with elementary particles. 3-surfaces would be "edgy" entities and space-time surfaces would have 2-D and 1-D edges. In condensed matter physics these edges would be regarded as defects.

2. Quantum criticality demands that the dynamics of string world sheets and of interior effectively decouple. Same must take place for the dynamics of string world sheets and their boundaries. Decoupling allows also string world sheets to be minimal surfaces as analogs of complex surfaces whereas string world sheet boundaries would be light-like (their deformations are always space-like) so that one obtains both particles and string like objects.

3. By field equations the sums for the divergences of stringy canonical momentum currents and the corresponding singular parts of these currents in the interior must vanish. By quantum criticality in interior the divergencespf Kähler and volume terms vanish separately. Same must happen for the sums in case of string world sheets and their boundaries. The discontinuity of normal derivatives implies that the contribution from the normal directions to the divergence reduces to the sum of discontinuities in two normal directions multiplied by 2-D delta function. Thid contribution is in the general case equal to the divergence of corresponding stringy canonical momentum current but must vanish if one has quantum criticality also at string world sheets and their boundaries.

   The separate continuity of Kähler and volume parts of canonical momentum currents would guarantee this but very probably implies the continuity of the induced metric and Kähler form and therefore of normal derivatives so that there would be no singularity. However, the condition that total canonical momentum currents are continuous makes sense, and indeed implies a transfer of various conserved charges between Kähler action and volume degrees of freedom at string world sheets and their boundaries in normal directions as was conjectured in [L61].

4. What about the situation in fermionic degrees of freedom? The action for string world sheet X^2 would be essentially of Kähler action for $X^2 \times S^2$, where S^2 is twistor sphere. Since the modified gamma matrices appearing in the modified Dirac equation are determined in terms of canonical momentum densities assignable to the modified Dirac action, there could be similar transfer of charges involved with the fermionic sector and the divergences of Noether charges and super-charges assignable to the volume action are non-vanishing at the singular surfaces. The above mechanism would force decoupling between interior spinors and string world sheets spinors also for the modified Dirac equation since modified gamma matrices are determined by the bosonic action.

Remark: There is a delicacy involved with the definition of modified gamma matrices, which for volume term are proportional to the induced gamma matrices (projections of the embedding space gamma matrices to space-time surface). Modified gamma matrices are proportional to the contractions $T^{\alpha}_k\Gamma^k$ of canonical momentum densities $T^{\alpha_k} = \partial L/\partial(\partial_\alpha h^k)$ with embedding space gamma matrices Γ^k. To get dimension correctly in the case of volume action one must divide away the factor $\Lambda/8\pi G$. Therefore fermionic super-symplectic currents do not involve this factor as required.

It remains an open question whether the string quantum criticality is realized everywhere or only near the ends of string world sheets near boundaries of CD.

String world sheet singularities as infinitely sharp edges and dynamical generation of string world sheet action

The condition that the singularities are 2-D string world sheets forces 1-D edges of 3-surfaces to be infinitely sharp.

Consider an edge at 3-surface. The divergence from the discontinuity contains contributions from two normal coordinates proportional to a delta function for the normal coordinate and coming from the discontinuity. The discontinuity must be however localized to the string rather than 2-surface. There must be present also a delta function for the second normal coordinate. Hence the value of also discontinuity must be infinite. One would have infinitely sharp edge. A concrete example is provided by function $y = |x|^\alpha$ $\alpha < 1$. This kind of situation is encountered in Thom's catastrope theory for the projection of the catastrophe: in this case one has $\alpha = 1/2$. This argument generalizes to 3-D case but visualization is possible only as a motion of infinitely sharp edge of 3-surface.

Kähler form and metric are second degree monomials of partial derivatives so that an attractive assumption is that $g_{\alpha\beta}$, $J_{\alpha\beta}$ and therefore also the components of volume and Kähler energy momentum tensor are continuous. This would allow $\partial_{n_i} h^k$ to become infinite and change sign at the discontinuity as the idea about infinitely sharp edge suggests. This would reduce the continuity conditions for canonical momentum currents to rather simple form

$$T^{n_i n_j} \Delta \partial_{n_j} h^k = 0 \ . \tag{6.9.1}$$

which in turn would give

$$T^{n_i n_j} = 0 \tag{6.9.2}$$

stating that canonical momentum is conserved but transferred between Kähler and volume degrees of freedom. One would have a condition for a continuous quantity conforming with the intuitive view about boundary conditions due to conservation laws. The condition would state that energy momentum tensor reduces to that for string world sheet at the singularity so that the system becomes effectively 2-D. I have already earlier proposed this condition.

The reduction of 4-D locally to effectively 2-D system raises the question whether any separate action is needed for string world sheets (and their boundaries)? The generated 2-D action would be similar to the proposed 2-D action. By super-conformal symmetry similar generation of 2-D action would take place also in the fermionic degrees of freedom. I have proposed also this option already earlier. This would mean that Option II is enough.

The following gives a more explicit analysis of the singularities. The vanishing on the discontinuity for the sum of normal derivative gives terms with varying degree of divergence. Denote by n_i resp. t_i the coordinate indices in the normal resp. tangent space. Suppose that some derivative $\partial_{n_i} h^k$ become infinite at string. One can introduce degree n_D of divergence for a quantity appearing as part of canonical momentum current as the degree of the highest monomial consisting of the diverging derivatives $\partial_{n_i} h^k$ appearing in quantity in question. For the leading term in continuity conditions for canonical momentum currents of total action one should have $n_D = 2$ to give the required 2-D delta function singularity.

- $\partial_{n_i} h^k$ has $n_D \leq 1$. If it is also discontinuous - say changes sign - one has $n_D = 2$ for $\Delta \partial_{n_i} h^k$ in direction n_i.

- One has $n_D(g_{t_i t_j}) = 0$, $n_D(g_{t_i n_j}) = 1$, $n_D(g_{n_i n_i}) = 2$ and $n_D(g_{n_i n_j}) = 1$ or 2 for $i \neq j$. One has $n_D(g) = 4$ ($g = det(g_{\alpha\beta})$). For contravariant metric one gas $n_D(g^{t_i t_j}) = 0$ and $n_D(g^{n_i j}) = n_D(g^{n_i n_j}) = -2$ as is easy to see from the formula for $g^{\alpha\beta}$ in terms of cofactors.

- Both Kähler and volume terms in canonical momentum current are proportional to $\sqrt{g}$ with $n_D(\sqrt{g}) = 2$ having leading term proportional to 2-determinant $\sqrt{det(g_{n_i n_j})}$. In Kähler action the leading term comes from tangent space part J_{ij} and has $n_D = -1$ coming from the partial derivative. The remaining parts involving $J_{t_i n_j}$ or $J_{n_i n_j}$ have $n_D < 0$.

- Consider the behavior of the contribution of volume term to the canonical momentum currents. For $g^{n_i t_j} \partial_{t_j} h^k \sqrt{g}$ one has $n_D = 0$ so that this term is finite. For $g^{n_i n_j} \partial_{n_j} h^k \sqrt{g}$ one has $n_D \leq 1$ and this term can be infinite as also its discontinuity coming solely from the change of sign for $\partial_{n_j} h^k$. If $\partial_{n_j} h^k$ is infinite and changes sign, one can have $n_D = 2$ as required by 2-D delta function singularity.

  The continuity condition for the canonical momentum current would state the vanishing of $n_D = 2$ discontinuity but would not imply separate vanishing of discontinuity for Kähler and volume parts of canonical momentum currents - this in accordance with the idea about canonical momentum transfer. If the sign of partial derivative only changes the coefficient of the partial derivative must vanish so that the condition reduces to the condition $T^{n_i n_j} = 0$ already given for the components of the total energy momentum tensor, which would be continuous by the above assumption.

6.9.2 Kähler calibrations: an idea before its time?

While updating book introductions I was surprised to find that I had talked about so called calibrations of sub-manifolds as something potentially important for TGD and later forgotten the whole idea! A closer examination however demonstrated that I had ended up with the analog of this notion completely independently later as the idea that preferred extremals are minimal surfaces apart form 2-D singular surfaces, where there would be exchange of Noether charges between Kähler and volume degrees of freedom.

1. The original idea that I forgot too soon was that the notion of calibration (see `http://tinyurl.com/y3lyead3`) generalizes and could be relevant for TGD. A calibration in Riemann manifold M means the existence of a k-form ϕ in M such that for any orientable k-D sub-manifold the integral of ϕ over M equals to its k-volume in the induced metric. One can say that metric k-volume reduces to homological k-volume.

   Calibrated k-manifolds are minimal surfaces in their homology class, in other words their volume is minimal. Kähler calibration is induced by the k^{th} power of Kähler form and defines calibrated sub-manifold of real dimension $2k$. Calibrated sub-manifolds are in this case precisely the complex sub-manifolds. In the case of CP_2 they would be complex curves (2-surfaces) as has become clear.

2. By the Minkowskian signature of M^4 metric, the generalization of calibrated sub-manifold so that it would apply in $M^4 \times CP_2$ is non-trivial. Twistor lift of TGD however forces to introduce the generalization of Kähler form in M^4 (responsible for CP breaking and matter antimatter asymmetry) and calibrated manifolds in this case would be naturally analogs of string world sheets and partonic 2-surfaces as minimal surfaces. Cosmic strings are Cartesian products of string world sheets and complex curves of CP_2. Calibrated manifolds, which do not reduce to Cartesian products of string world sheets and complex surfaces of CP_2 should also exist and are minimal surfaces.

   One can also have 2-D calibrated surfaces and they could correspond to string world sheets and partonic 2-surfaces which also play key role in TGD. Even discrete points assignable to partonic 2-surfaces and representing fundamental fermions play a key role and would trivially correspond to calibrated surfaces.

3. Much later I ended up with the identification of preferred extremals as minimal surfaces by totally different route without realizing the possible connection with the generalized calibrations. Twistor lift and the notion of quantum criticality led to the proposal that preferred extremals for the twistor lift of Kähler action containing also volume term are minimal surfaces. Preferred extremals would be separately minimal surfaces and extrema of Kähler action and generalization of complex structure to what I called Hamilton-Jacobi structure would be an essential element. Quantum criticality outside singular surfaces would be realized as decoupling of the two parts of the action. May be all preferred extremals be regarded as calibrated in generalized sense.

   If so, the dynamics of preferred extremals would define a homology theory in the sense that each homology class would contain single preferred extremal. TGD would define a generalized topological quantum field theory with conserved Noether charges (in particular rest energy) serving as generalized topological invariants having extremum in the set of topologically equivalent 3-surfaces.

   It is interesting to recall that the original proposal for the preferred extremals as absolute minima of Kähler action has transformed during years to a proposal that they are absolute minima of volume action within given homology class and having fixed ends at the boundaries of CD.

4. The experience with CP_2 would suggest that the Kähler structure of M^4 defining the counterpart of form ϕ is unique. There is however infinite number of different closed self-dual Kähler forms of M^4 defining what I have called Hamilton-Jacobi structures. These forms can have subgroups of Poincare group as symmetries. For instance, magnetic flux tubes correspond to given cylindrically symmetry Kähler form. The problem disappears as one realizes that Kähler structures characterize families of preferred extremals rather than M^4 itself.

If the notion of calibration indeed generalizes, one ends up with the same outcome - preferred extremals as minimal surfaces with 2-D string world sheets and partonic 2-surfaces as singularities - from many different directions.

1. Quantum criticality requires that dynamics does not depend on coupling parameters so that extremals must be separately extremals of both volume term and Kähler action and therefore minimal surfaces for which these degrees of freedom decouple except at singular 2-surfaces, where the necessary transfer of Noether charges between two degrees of freedom takes place at these. One ends up with string picture but strings alone are of course not enough. For instance, the dynamical string tension is determined by the dynamics for the twistor lift.

2. Almost topological QFT picture implies the same outcome: topological QFT property fails only at the string world sheets.

3. Discrete coupling constant evolution, vanishing of loop corrections, and number theoretical condition that scattering amplitudes make sense also in p-adic number fields, requires a representation of scattering amplitudes as sum over resonances realized in terms of string world sheets.

4. In the standard QFT picture about scattering incoming states are solutions of free massless field equations and interaction regions the fields have currents as sources. This picture is realized by the twistor lift of TGD in which the volume action corresponds to geodesic length and Kähler action to Maxwell action and coupling corresponds to a transfer of Noether charges between volume and Kähler degrees of freedom. Massless modes are represented by minimal surfaces arriving inside causal diamond (CD) and minimal surface property fails in the scattering region consisting of string world sheets.

5. Twistor lift forces M^4 to have generalize Kähler form and this in turn strongly suggests a generalization of the notion of calibration. At physics side the implication is the understanding of CP breaking and matter anti-matter asymmetry.

6. $M^8 - H$ duality requires that the dynamics of space-time surfaces in H is equivalent with the algebraic dynamics in M^8. The effective reduction to almost topological dynamics implied

by the minimal surface property implies this. String world sheets (partonic 2-surfaces) in H would be images of complex (co-complex sub-manifolds) of $X^4 \subset M^8$ in H. This should allows to understand why the partial derivatives of embedding space coordinates can be discontinuous at these edges/folds but there is no flow between interior and singular surface implying that string world sheets are minimal surfaces (so that one has conformal invariance).

The analogy with foams in 3-D space deserves to be noticed.

1. Foams can be modelled as 2-D minimal surfaces with edges meeting at vertices. TGD space-time could be seen as a dynamically generated foam in 4-D many-sheeted space-time consisting of 2-D minimal surfaces such that also the 4-D complement is a minimal surface. The counterparts for vertices would be light-like curves at light like orbits of partonic 2-surfaces from which several string world sheets can emanate.

2. Can one imagine something more analogous to the usual 3-D foam? Could the light-like orbits of partonic 2-surfaces define an analog of ordinary foam? Could also partonic 2-surfaces have edges consisting of 2-D minimal surfaces joined along edges representing strings connecting fermions inside partonic 2-surface?

   For years ago I proposed what I called as symplectic QFT (SQFT) as an analog of conformal QFT and as part of quantum TGD [K16]. SQFT would have symplectic transformations as symmetries, and provide a description for the symplectic dynamics of partonic 2-surfaces. SQFT involves an analog of triangulation at partonic 2-surfaces and Kähler magnetic fluxes associated with them serve as observables. The problem was how to fix this kind of network. Partonic foam could serve as a concrete physical realization for the symplectic network and have fundamental fermions at vertices. The edges at partonic 2-surfaces would be space-like geodesics. The outcome would be a calibration involving objects of all dimensions $0 \leq D \leq 4$ - a physical analog of homology theory.

6.10 Could metaplectic group have some role in TGD framework?

Metaplectic group appears as a covering group of linear symplectic group $Sp(2n, F)$ for any number field and its representations cam be regarded as analog of spinor representations of the rotation group. Since infinite-D symplectic group of $\delta M^4_+ \times CP_2$, where δM^4_+ is light-cone boundary, appears as an excellent candidate for the isometries of "world of classical worlds" in zero energy ontology (ZEO), one can ask whether and how the notion of metaplectic group generalizes to TGD framework [K39, **?**, K73, K53, K20, K19].

The condition for the existence of metaplectic structure is same as those for the spinor structure and not met in the case of CP_2. One however expects that also in the case of metaplectic structure the modified metaplectic structure exists is one couples spinors to an odd integer multiple of Kähler gauge potential. For triality 1 representation assignable to quarks one has $n = 1$. The fact that the center of $SU(3)$ is Z_3 suggests that metaplectic group for CP_2 is 3- or 6-fold covering of symplectic group instead of 2-fold covering.

Besides the ordinary representations of $SL(2, C)$ also the possibly existing analogs of metaplectic representations of $SL(2, C) = Sp(2, C)$ acting on wave functions at hyperbolic space H_3 at $a^2 = t^2 - r^2$ hyperbololoid of M^4_+ are cosmologically interesting since the many-sheeted space-time in number theoretic vision allows quantum coherence in even cosmological scales and there are indications for periodic redshift suggests tessellations of H_3 analogous to lattices in E^3 and defined by discrete subgroup of $Sl(2, C)$.

6.10.1 Heisenberg group, symplectic group, and metaplectic group

The following gives a brief summary of basics related to Heisenberg group, symplectic group, and metaplectic group.

Heisenberg group

1. The matrix representation of the simplest Heisenberg group http://tinyurl.com/y2fomegs is given by matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \tag{6.10.1}$$

   A 3-D Lie group is in question. The multiplication for group elements (a_1, b_1, c_1) and (a_2, b_2, c_2) is given by $(a_1, b_1, c_1) \circ (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 - a_1 b_2)$. The coefficients (a,b,c) can be belong to any ring sin the inverse can be expressed using only product and sum as $(-a, -b, ab - c)$. In particular, discrete variants of Heisenberg group such as those associated with extensions of rationals, exist. For odd primes one can define Heisenberg group modulo p as group of order p^3 in finite field F_p.

   $2n + 1$-D Heisenberg group consists of upper triangular with unit matrix at diagonal.

2. Continuous Heisenberg group is a nilpotent Lie group of dimension $d = 3$. Nilpotency means that it is Lie algebra elements are nilpotent. The Lie algebra is generated by upper-diagonal matrices and the commuation relations for the Lie algebra basis are $[X, Y] = Z$, $[X, Z] = 0$, $[Y, Z] = 0$. The coordinate $X = q$ and differential operator $Y = p = \hbar\partial_q$, $Z = i\hbar 1$satisfying $[p, q] = i\hbar Id$, define a concrete representation of the Lie algebra of the simplest 3-D Heisenberg group in the space of functions $f(q)$. By introducing n pairs of coordinates commuting to unit matrix one obtain $2n + 1$-D Heisenberg group.

Symplectic group

Symplectic group acts as automorphisms of Heisenberg group. Symplectic group leaves acts in function algebra of function $H(p, q)$ leaving invariant Poisson bracket $\{H_1, H_2\} = \partial_q H_1 \partial_p H_2 - \partial_q H_2 \partial_p H_1$. The Poisson bracket $\{p, q\} = 1$ giving the element of $J_{p,q} = 1$ symplectic form remaining invariant under symplectic transformations. Exponentiation of any Hamiltonian $H(p, q)$ acting as Hamiltonian generates symplectic flows. Symplectic group is infinite-D.

3-D linear symplectic group $Sp(2, F)$ is obtained as a special case. In continuous case Hamiltoniansare linear functions of p and q so that the action by Poisson bracket is linear. General linear symplectic group$Sp(2n, F)$ acts in $2n$-D space spanned spanned by the analogs of (q_p, p_i). When symplectic form is accompanied by complex structure and Kähler form symplectic isometries define a finite-D subgroup of symplectic group. For instance, in case of CP_2 symplectic isometries define group $SU(3)$.

Metaplectic group

Metaplectic group $Mp_m(2n, F)$ (see http://tinyurl.com/y5mpswy8 and http://tinyurl.com/y4kjys3e) is an m-fold covering of the linear symplectic group $Sp(2n, F)$. Metaplectic group like also linear symplectic group metaplectic grop is defined for all number fields, in particular p-adic number fields and even adeles. All representations of the metapelectic group are infinite-D (non-compactness is not the only reason: even finite-D non-unitary matrix representations fail to exist).

$Sp(2, R)$ co-incides with a covering group the special linear group $Sl(2, R)$ acting as real Möbius transformations in upper half-plane. Metaplectic group does not allow finite-D matrix representations and all representations are infinite-dimensional. Metaplectic group can be regarded as m-fold cover of symplectic group and in Weil representation the cover can be chosen to be 2-fold cover.

The elements for the metaplectic group $M_2(2,R)$ as 2-fold covering of $Sp(2,R)$ have representation as pairs (g,ϵ) with g a Möbius transformation represented by matrix $(a,b;c,d)$ with unit determinant acting as $z \to (az+b)/(cz+d)$ and with $\epsilon(z)^2 = cz+d$. The product of group elements is given by $(g_1, epsilon_1)(g_2,\epsilon_2) = (g_1g_2,\epsilon)$, $\epsilon(z) = \epsilon_1(g_2(z))\epsilon_2(z)$. The entities transforming in this manner are not functions but analogous to spinors and one can speak of symplectic spinors.

3. One can generalize the notion of symplectic structure to that of metaplectic structure. The topological conditions (the second Stiefel-Withney class vanishes) for the existence of metaplectic structure for given symplectic manifold are same as for the spinor structure.

   Interestingly, in the case of CP_2 this condition is not satisfied and the problem is circumvented by coupling CP_2 spinors to an odd multiple of Kähler gauge potential giving rise to Kähler form: this is essential for obtain electroweak couplings correctly for the induced spinor structure at space-time surface. Since Kähler form relates so closely to symplectic structure, it is reasonable to expect that also in case of CP_2(CP_{2n}) symplectic spinors exist.

   The center of isometry group $SU(3)$ of CP_2 is Z_3 acting trivial on CP_2 coordinates. The action is analogous to that of Möbius transformations being induced by linear action oof $SU(3)$ on projective coordinates (z_1,z_2,z_3) and by the projective map such as $(z_1,z_2,z_3) \to (z_1/z_3, z_2/z_3, 1)$ in given coordinate patch defined by a choice of two complex coordinates (z_i,z_j) now $(z_1/z_3, z_2/z_3)$. Do symplectic spinors spinors transform like CP_2 spinors under metaplectic action of $SU(3)$?

   CP_2 spinors with unit coupling to Kähler gauge potential allow triality $t = \pm 1$ partial impossible without the coupling making possible spinor structure and presumabley also metaplectic struture. Does this mean that in the case of CP_2 the metaplectic group must be identified as 3-fold or possibly 6-fold covering of symplectic group. The holomy group is electroweak $U(2)$ and acts like $SU(2) \times U(1)$. Does holonomy group acts as double covering of $SO(3)$ and as 3-fold covering of $U(1)$ giving 6-fold covering of tangent space group $SO(4)$?

6.10.2 Symplectic group in TGD

In TGD the symplectic transformations of $\delta M^4_+ \times CP_2$, where δM^4_+ is light-cone boundary, and generated by Hamiltonian algebra, are central and act in the "world of classical worlds" (WCW) [K39, **?**, K73, K53, K20, K19].

1. WCW is formed by pairs of 3-surfaces with members at opposite boundaries of causal diamond $CD = cd \times CP_2$ of embedding space $H = M^4 \times CP_2$. cd is causal diamond of M^4 defined as intersection of future and past directed light-cones. The members of the pair are connected by preferred extremal of action defined by twistor lift of TGD: it is sum of Kähler action and volume term. Preferred extremal is analogous to Bohr orbit.

2. The obvious question is whether also infinite-D symplectic group of $\delta M^4_+ \times CP_2$ allows metaplectic variant. Second question is how symplectic spinors relate to ordinary spinors. Are ordinary spinors of H symplectic spinors as one might expect?

3. In TGD the spinors of "world of classical worlds" (WCW) [K39, **?**, K73] should have interpretation as symplectic spinors. Spinors of WCW are fermionic Fock states created by quark oscillator operators replacing theta parameters in super-coordinates and in super-spinors of super variant of embedding space H. Their local composites appear as monomials with vanishing quark number in hermitian super-coordinates of super-variant of H and in super-quark-spinors of super-H containing only monomials with odd quark number. These super-fields differ from those of standard SUSY since monomials of theta parameters are replaced with monomials of quark oscillator operators and Majorana spinors are not in question.

   Infinite-D metaplectic group $\delta M^4_+ \times CP_2$ should act on WCW spinor fields and the action should be induced from action in H.

6.10.3 Kac-Moody type approach to representations of symplectic/metaplectic group

Representations of the symplectic/metaplectic group. Kac-Moody type approach is strongly suggested physically. Kac-Moody group has Lie-algebra which is central extension of the Lie-algebra of local gauge transformation. Kac-Moody algebra elements are labelled by elements with conformal weight $n \in Z$ but also the variant $n \geq 0$ ("half-algebra" exists as sub-algebra is clear from the commutation relations.

1. Let r denote the radial light-like coordinate of light-cone boundary $\delta M^4_+ \times CP_2$. $\delta M^4_+ = S^2 \times R^+$ is metrically 2-sphere S^2 and this implies extension of usual conformal invariance for S^2 to conformal invariance localized with respect to r and explains why 4-D Minkowski space is physically unique.

   Radially local conformal transformations $z \rightarrow f(r,z)$of light-cone boundary with scaling $r \rightarrow |df(r,z,zbar)/dz|^{-1} \times r$ in light-cone radial coordinate r compensating for the conformal scaling factor $|df(r,z,zbar)/dz|^2$ as isometries of light-cone boundary as also color rotation local with respect to r. One has radially local $S = SO(3) \times SU(3)$ as isometries of light-cone boundary. This would serve as the TGD variant of color gauge symmetry.

2. Effective localization of the symplectic algebra of $S^2 \times CP_2$ with respect to the radial light-like coordinate r. Denote the radial conformal weight h.

   Option 1: Radial waves of form r^h, $h = -1/2 + iy$ (something to do with zeros of zeta) behave like plane waves with wave vector y for in inner product defined by integration measure dr. Orthogonal plane-wave basis effectively.

   Restriction to causal diamond CD defined as intersection of future and past directed light-cones implies $r \leq r_{max}$ defining the size of CD and periodic boundary conditions for a discrete basis r^h. If $h = -1/2 + iy$ corresponds to a zero of zeta, the size of CD determined by r_{max} is quantized. For instance, $sin(yln(r_{max})) = 0$ would imply $ln(r_{max}) = n \times \pi/y$. Also $cos(yln(r_{max})) = 0$ can be considered.

   Option 2: One can include the real part of h to the integration measure of inner product defined as $d\mu = dr/r$. This is dimensionless and very natural by scaling invariance. For this choice one has $h = iy$ and the connection with Riemann zeta is not anymore natural. $r_{max} = exp(n \times \pi/y)$ would give periodic boundary conditions.

   For $y = k\pi$ one would have $r_{max} = exp(1/k)$, k integer. This conforms with the adelic picture since the infinite-D extension of rationals generated by $e^{1/k}$ induces finite-D extension of p-adic numbers since e^p is ordinary p-adic number.

   $y = k\pi/log(p)$ gives $r_{max} = p^{n/k}$ and one can construct finite-D extensions of rationals allowing roots of p.

3. Super-symplectic algebra is assumed to have fractal structure . There is a hierarchy of isomorphic super-sympletcic sub-algebras SSA_n, $n = 1, 2...$, for which conformal weights n-multiples of the weights for the entire algebra.

   Option 1: One would have also conformal weights $n(-1/2 + iy)$ for these radial waves however inner product using dmu= dr as integration measures does reduce to inner product for plane waves but to $\int r^{-n+1} exp(in(y_1 - y_2))du$, $u = log(r/r_0)$. This leads out from the original state space. The modification of the integration measure to $d\mu = r^(n-1)dr$ does not seem plausible.

   Option 2: Identify the conformal weight as $h = iy$ and include the real part -1/2 to the dimensionless integration measure $d\mu = dr/r$. This allows fractal hierarchy $h = niy$. This seems to be the only elegant option so that the connection with Riemann zeta seems artificial

This picture leads to some conjectures and questions.

1. Sub-algebra SSA_n and its commutator with entire algebra SSA represented trivially for physical states. Also classical Noether charges vanish: this gives strong conditions on preferred

extremals and makes them analogs of Bohr orbits: only preferred pairs of 3-surfaces at opposite boundaries of CD are connected by preferred extremal. Hierarchy of state spaces is the outcome.

This would be generalization of Super Virasoro conditions for which only the entire algebra would act trivially apart from the scaling generator L_0.

2. Could the hierarchies of extensions of rationals with dimensions $n_1|n_2|...$ (| is for "divides") correspond to hierarchies of inclusions of hyper-finite factors.

3. Could the hierarchies of SSA_n with $n_1|n_2|...$ correspond to hierarchies of extensions of extensions of.... of rationals with dimensions $n_1|n_2|....$

$\delta M^4_+ \times CP_2$ is metrically $S^2 \times CP_2$ and this leads to some questions.

1. Could one have Kac-Moody type representation of the symplectic algebra of $S^2 \times CP_2$, which is radially local and involves central extension? This is physically suggestive.

2. Symplectic isometries of $S^2 \times CP_2$ local with respect to r would define a sub-representation.

   Hamiltonians products of $\delta M^4_+ \times CP_2$ Hamiltonians for δM^4_+ and CP_2 labelled by angular momentum j and by the 2 Casimirs of triality $t = 0$ color representations.

   Isometry algebras $SO(3)$ and $SU(3)$ are sub-algebras of symplectic algebra determined by Hamiltonians at light-cone boundary in given representation to themselves. There are no higher-D sub-algebras so that one cannot consider hierarchy analogous to the hierarchy of sub-algebras labelled by radial conformal weights as n-multiples of weights of the entire algebra.

This in turn leads to a series of questions concerning what happens if one takes gauge symmetry and Kac-Moody symmetry as its analog as a physical guideline.

1. The metaplectic group of $SL(2, R)$ has only infinite-D representations but no matrix representations. Can this be true also for the metaplectic representation of infinite-D for $SO(3) \times SU(3)$ which is compact and allow finite-D unitary ordinary representations. $SO(3)$ must be lifted to $SU(2)$ and this is natural for quark spinors. SU(3) allows only triality $t = 0$ partial waves.

   Since $SU(3)$ has Z_3 as center one expects that the notion of metaplectic representation in this case generalizes so that one has 3-fold covering of function space instead of 2-fold one. Quark spinors indeed allow CP_2 partial waves which are in $t = 1$ representations. As already noticed CP_2 allows does not allow metaplectic structure in standard sense but the coupling to the Kähler gauge potential probably makes this possible since the condition for the existence of generalized metaplectic structure is same as for the existence of modified spinor structure.

2. Should one treat all S^2 Hamiltonians with $l > 1$ as gauge degrees of freedom? A possible interpretation would be in terms of finite measurement resolution and analog of Kac-Moody symmetry acting very much like gauge symmetry representing the finite measurement resolution. Symplectic group would effectively reduce to $SO(3) \times SU(3)$. If so, one would have $SO(3) \times SU(3)$ gauge theory with $l = 1$ states and spin 1/2 states with color as particles.

3. Only quark triplets and singlets of fermions and color octets of gluons are observed. Without any additional conditions TGD predicts infinite number of spinor harmonics. For CP_2 spinor harmonics there is a correlation for the color quantum numbers and electroweak quantum numbers of spinor harmonic. In QCD the color representation of quark does not however depend on electroweak quantum numbers. Also the masses of spinor harmonics depend on electroweak quantum numbers and are typically very large.

   Remark: One could of course ask whether quarks could move in different color partial waves but having $t = 1$. This however seems rather implausible.

   The proposal is that Kac-Moody type generators can be used to build massless states with have correct correlation between color represented as angular momentum like quantum number and electroweak quantum numbers. Could the experimental absence of higher color partial waves be due to the fact the gauge nature of higher excitations of symplectic algebra making higher color partial waves of quarks and leptons gauge degrees of freedom?

4. What about $l=1$ states assignable to $SO(3)$? Twistor lift of TGD predicts that also M^4 has analog of Kähler form and induced $U(1)$ gauge field analogous to induced Kähler form. The physical effects are weak and would be responsible for CP breaking and matter antimatter asymmetry. Could the $l=1$ triplet correspond to this $U(1)$ gauge boson somewhat like $SU(3)$ octet corresponds to gluon (gluon is identified as pair of quark and antiquark at different positions)?

5. How does this relate to the analog of metaplectic group for $SO(3)\times SU(3)$? What about the central extension of $SO(3)\times SU(3)$ assignable to spinor representations with weight $n=1/2$. If one adds to the Hamilton associated with rotation generator L_z around z-axis in $SO(3)$ and to hyper-charge generator Y of $SU(3)$ a constant, one obtains what looks like central extension at the level of Poisson brackets since right hand side of brackets receives an additive constant. In $SU(3)$ degrees of freedom one can have only $t=0$ color partial waves for scalars but for spinors one obtains the $t=1$ waves and can say that color partial waves possess and anomalous hyper-charge Y.

   The spectra of L_z and Y are shifted but Killing vector fields are not affected. The couplings of isometry generators are changed since there is coupling proportional to Hamiltonian. This does not seem to have have interpretation as a mere gauge transformation since it makes $t=1$ color partial waves possible for quarks.

6.10.4 Relationship to modular functions

The metaplectic representations involve in basic form $Sp(2n,F)$, F any number field.

1. $n=1$ is physically special: one has $Sp(2,C)=SL(2,C)$, which is double covering of Lorentz group. The so called modular representations giving rise to basic functions appearing in number theory are related to the representations of $SL(2,C)$ with the condition that SL(2,Z) or its discrete subgroup (there are infinite number of them) is represented either trivially or mere projective factor. In the representation realizing SL(2,C) as Möbius transformations $z\rightarrow (Az+B)/Cz+D)$ or upper half-plane one has $f(z)\rightarrow (Cz+D)^k f(z)$ when (A,B;C,D) represents element of SL(2,Z) or its subgroup G. k is integer or half integer. One has modular invariance apart from the projective factor.

   Although these nodularity conditions apply only to a discrete subgroup of $SL(2,R)$ they they imply projective invariance of the analytic functions involved so that projectively their support of the function reduces to $G\backslash H$, H upper complex plane analogous to unit cell. Could this kind of conditions correspond to the proposed analogs of Kac-Moody type gauge conditions proposed for symplectic symmetries of $\delta M^4_\pm \times CP_2$?

2. $SO(3,1)$ acts as isometries of the hyperbolic space H_3 identifiable as the hyperboloid H_3 as $a^2=t^2-r^2=constant$ surface of future light-cone M^4_+: a defines in TGD Lorentz invariant cosmic time and is natural embedding space coordinate in ZEO. Since $SL(2,Z)$ has infinite number if discrete subgroups, one has infinite number of tessellations of H_3 analogous to lattices in 3-D Euclidian space.

   In TGD quantum coherence is possible in even cosmological scales since TGD predicts hierarchy of effective values of Planck constants. Could one have quantum coherent structures represented as tessellations of the hyperboloid? The prediction would be quantization of redshift as reflection of quantization of distances from given point of tessellations to other points. Evidence for this kind of quantization has been observed.

3. Finite measurement resolution suggests consideration of tessellations as discretization of H_3 and assignable to extensions of rationals and also to subgroups of $SL(2,Z)$. This would mean discretized wave functions in the tessellation. This would be like wave function for particle in discrete lattice in E^3. On the other hand, modular functions with projective modular invariance would be analogs for wave functions of particles periodic symmetry implied by lattice but represented projectively.

Could one decompose the representation to products of modular forms as projective representations in coset space $SL(2,C)/\Gamma$, Γ a discrete subgroup of $SL(2,C)$ and of representations of discrete subgroup corresponding to finite measurement resolution. This would be like representation of wave functions as products of discrete lattice wave function and wave functions in the space of momenta modulo lattice momenta: Fermi sphere would be replaced by the coset space $SU(2)/G$.

4. The projective factor $\epsilon^2(Z) = (Cz + D)^k$ is essential for the projective representation of $Sp(2,C)$. Is it possible to generalize this factor acting on upper complex plane to the case of H_3? If subgroup $SO(3)$ is represented projectively, then one can use for H_3 coordinates (r,θ,ϕ), such that r as radius of sphere S^2 remains invariant under r and $SO(3)$ actsob the complex coordinate of S^2 transforming linearly under $SO(1)$ as $z \rightarrow (Az + B)/(Cz + D)$ so that the projective factor can be identified. These representations would be analogous to modular representations: the discrete subgroup of $SL(2,C)$ would be replaced with $SU(2)$.

   It would seem that it must be replaced with $SU(2)$ as subgroup. Could one generalize the notion of modular form invariant under discrete subgroup of $SL(2,C)$ so that the discrete subgroup would become discrete subgroup of $SO(3)$ ($SU(2)$).

   Platonic solids are lattices at S^2 and their isometries and finite subgroups $D(2n)$ appear in McKay correspondence relating discrete subgroups of $SU(2)$ and ADE Lie groups. Finite measurement resolution as dual interpretation. What about infinite discrete subgroups. Does invariance mean projective SU(2) invariance (the case when $n = 0$)

Chapter 7

Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory

7.1 Introduction

I have worked with the problem of understanding the construction of scattering amplitudes in the framework provided by Topological Geometrodynamics (TGD) for about four decades. It soon became clear that the naïve generalization of the path integral approach to a path integral over space-time surfaces did not work because of the horrible non-linearities involved. Around 1985 I started to work with the notion that I later called the "world of classical worlds" (WCW). Eventually I apprehended that the realization of general coordinate invariance (GCI) forces to assign to a 3-surface possibly unique space-time surface (X^4) at which the general coordinate transformations act [K39, K21]. Holography would reduce to GCI. The intuitive expectation is that either space-like 3-surfaces or light-like partonic orbits defining boundaries between Minkowskian and Euclidian space-time regions should be enough to determine X^4 as an analog of Bohr orbit. This leads to strong form of holography (SH) stating that data at partonic 2-surfaces and string world sheets code for X^4.

It should be possible to geometrize the entire quantum physics in terms of WCW geometry and associated spinor structure identifying WCW spinors as fermionic Fock states. A geometrization of the hermitian conjugation essential in quantum theory is needed. This fixed the WCW geometry to be Kähler geometry determined by Kähler function and defining Kähler form providing a realization of the imaginary unit as an antisymmetric tensor [K39]. The existence of Riemann connection fixes the Kähler geometry uniquely already in the case of loop spaces [A54]: maximal isometry group is required. In TGD framework it would correspond supersymplectic transformations of $\delta M^4_\pm \times CP_2$, where $\delta M^4_\pm$,denotes future or past light-cone [K21].

Classical physics becomes an exact part of quantum physics if the space-time surfaces are preferred extremals for some action and therefore analogous to Bohr orbits. Spinor fields should obey the modified Dirac equation (MDE). Modified Dirac action (MDA) is determined by the bosonic action via supersymmetry condition. Kähler function identified as the action for the preferred extremal associated with the 3-surface defines in complex coordinates the Kähler metric and Kähler form via its second derivatives of type $(1,1)$.

The natural looking identification of the action was as Kähler action - a non-linear generalization of Maxwell action replacing Maxwell field and metric with induced Kähler form and metric. It possessed a huge vacuum degeneracy interpreted as spin glass degeneracy and for a long time I looked this feature as something positive despite the fact that the WCW metric becomes degenerate at the vacuum extremals and classical determinism is lost. The addition of volume term having interpretation in terms of cosmological constant would have been a possible cure but would have broken conformal invariance bringing in an *ad hoc* dimensional coupling.

Decades later the proposal for a twistor lift of TGD led to the identification of fundamental action as an analog of Kähler action for 6-D twistor spaces having X^4 as base space and S^2 as fiber [L71]. The induction of the twistor structure from that for the 6+6-D product of twistor spaces of M^4 and CP_2 (these spaces are the only 4-spaces allowing twistor space with Kähler structure [A74] so that TGD is unique) to the 6-surface forces a dimensional reduction reducing 6-D Kähler action to a sum of 4-D Kähler action and volume them. The counterpart of the cosmological constant emerges dynamically. Λ depends on the p-adic length scale characterizing space-time surfaces and approaches to zero in long length scales [L71].

The ontology of standard quantum theory in which 3-D $t = constant$ slice of space-time contains the quantum states, does not fit nicely with TGD framework. Space-time surfaces in 1-1 correspondence with 3-surfaces are more natural objects to consider. This conforms also with the notion of holography implied by GCI: actually SH is highly suggestive and means that 2-D data at partonic 2-surfaces and string world sheets determined the X^4 as a preferred extremal. In particular, various anomalies suggest that the arrow of time need not be fixed.

Eventually this led to zero energy ontology (ZEO) [L68] in which quantum states are essentially superpositions of preferred extremals inside causal diamond (CD): space-time surfaces have ends at the boundaries of CD and these pairs of 3-surfaces or equivalently the 4-surfaces are the basic objects. CDs form a hierarchy: there are CDs with CDs and CDs can also intersect. They would form an analog of atlas of coordinate charts. Each CD would serve as a correlate for a conscious entity so that the charts can be said to be conscious.

ZEO leads to a quantum measurement theory and allows avoiding the basic problems of the standard quantum measurement theory. Zero energy states correspond to state pairs at opposite boundaries of CD or equivalently, superpositions of deterministic time evolutions. In state function reduction (SFR) as a superposition of classical deterministic time evolutions is replaced with a new one.

"Big" and "small" state function reduction - BSFR and SSFR - are the basic notions. In SSFRs as analogs of "weak" measurements following a unitary time evolution, the size of CD increases in statistical sense. The members of the state pairs associated with the passive boundary of CD do not change during SSFRs: this gives rise to the analog of Zeno effect. The active boundary and the states at it change. Active boundary also shifts farther from the passive one. BSFRs correspond to ordinary state function reductions and in BSFRs the arrow of time changes. One could speak of a death of a conscious entity in universal sense and reincarnation with an opposite arrow of time. For instance, the findings of Minev *et al* [L60] provide support for the time reversal [L60].

7.1.1 How to construct the TGD counterpart of unitary S-matrix?

The concrete construction of scattering amplitudes remained a challenge from very beginning. During years I have proposed several proposals and many important aspects of the problem are understood but simple rules are still lacking.

1. The time evolutions assignable to SSFRs should be describable by a unitary S-matrix or its analog.

2. The counterpart of S-matrix should have the huge super-symplectic algebra (SSA) and Kac-Moody algebras related to isometries of H as symmetries. These symmetries, extended further to Yangian symmetries and quantum groups with both algebra and co-algebra structure, are expected to be a key element in the construction of the counterpart of S-matrix. In particular, product and co-product in the super-symplectic algebra define excellent candidates for vertices. What has been missing was a concrete guiding principle.

3. Feynman (or twistor) diagrammatics should generalize. Point-like particles are replaced with 3-surfaces and topologically incoming and outgoing many-particle states correspond to disjoint unions of 3-surfaces at the boundaries of CD. The first guess is that the vertices

correspond to 3-surfaces at which 4-D lines of the analog of Feynman diagram meet. SH and M^8-H duality [L65] however suggest that the lines of the diagrams should correspond to 3-D light-like orbits of partonic 2-surfaces defining boundaries between space-time regions with Euclidian and Minkowskian signature of the induced metric. Also string world sheets connecting them and also serving as carriers of information in SH should be considered. The 1-D light-like intersections of strings world sheets with partonic orbits would define carriers of fermion number.

4. The identification of fermionic anti-commutation relations was a longstanding challenge. It turned out that the induction of second quantized free fermion fields from H to X^4 fixes the anti-commutations of the induced spinor fields and allows to calculate fermionic propagators. Therefore quantum algebra would give what is needed to calculate scattering amplitudes: the interaction vertices assignable to partonic 2-surfaces and fermionic propagators would result from the induction procedure. 8-D fermions have however 7-D delta functions as anti-commutators and normal ordering of fermions can produce divergences already at the level of the MDA.

   The problem disappears if the MDA is made bilocal [L83]: in this article a more detailed discussion is given and leads to a rather detailed picture about MDA.

5. M^8-H duality [L65, L43, L44, L45] allows to concretize this picture. One can regard X^4 either as a surface in the complexified M^8 or in H. M^8-H duality maps space-time surfaces from M^8 to H. Space-time surfaces in the complexified M^8 correspond to algebraic 4-surfaces determined by real polynomials with real (rational if one requires p-adicization) coefficients. Also rational and even analytic functions can be considered, in which case polynomials could be seen as approximations. The roots of the real polynomial dictate the space-time surfaces as quaternionic/associative 4-surfaces in complexified octonionic M^8. Holography becomes discrete.

   The algebraic equations defining space-time surfaces also have special solutions, in particular 6-spheres. These analogs of 6-branes have as M^4 projections in both M^8 and $H=M^4\times CP_2 t=r_n$ hyperplanes, where r_n corresponds to a root of a real polynomial defining X^4 in complexified M^8 The interpretation of these hyper-planes is in TGD inspired consciousness is as "very special moments in the life of self".

   The solutions of the analog of Dirac equation in M^8 as algebraic equation [L84] are localized to 3-D light-like surfaces and mapped to light-like 3-surfaces in H identifiable as orbits of partonic 2-surfaces. Partonic 2-surfaces serving as vertices of topological analogs of Feynman diagrams would reside at the above described $t=r_n$ hyperplanes of $H=M^4\times CP_2$. Scattering amplitudes would have partonic 2-surfaces as vertices and their 3-D light-like orbits as lines. The intersections of string world sheets with the partonic orbits would be 1-D lines and could be interpreted as fermion lines so that also the point particle description would be part of the picture.

   CDs inside CD would define the regions inside which particle reactions occur and this suggests a fractal hierarchy of CDs within CDs as a counterpart for the hierarchy of radiative corrections.

What is still missing is the general principle allowing a bird's eye of view about the counterpart of S-matrix. Wheeler was the first to introduce the notion of unitary S-matrix, which generalizes probability conservation to an infinite number of conditions. Could one challenge the unitary principle and consider something else instead of it?

1. Unitary time evolution is natural in non-relativistic quantum mechanics but is already problematic in quantum field theory (QFT), in particular in twistor Grassmannian approach [B32]. The idea about the reduction of physics to Kähler geometry inspires the question whether Kähler geometry of WCW could provide a general principle for the construction of the scattering amplitudes and perhaps even an explicit formulas for them.

   Kähler metric defines a complex inner product. Complex inner products also define scattering amplitudes. Usually metric is regarded as defining length and angle measurements. Could the Kähler metric of state space code the counterpart of S-matrix and even unitary S-matrix? Also the Kähler metric satisfies conditions analogous to unitarity conditions.

An amazingly simple argument demonstrates that one could construct scattering probabilities from the matrix elements of Kähler metric and assign to the Kähler metric the analog of a unitary S-matrix by assuming that some additional conditions guaranteeing that the probabilities are real and non-negative are satisfied.

(a) If the probabilities are identified as the real parts of complex analogs $p^c_{i,j} = g_{i,\bar{j}} g^{\bar{j},i}$ of probabilities, it is enough to require $Re(p^c_{i,j}) \geq 0$. The complex analogs of $ip^c_{i,j}$) would define the analog of Teichmueller matrix [A43, A64, A52] (`https://en.wikipedia.org/wiki/Teichm\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{u\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\relax\let\ignorespaces\relax\accent127u\egroup\spacefactor\accent@spacefactorller_space`) for which imaginary parts of matrix elements are non-negative. Teichmueller space parameterizes the complex structures of Riemann surface: could the allowed WCW Kähler metrics - or rather the associated complex probability matrices - correspond to complex structures for some space? By SH, the most natural candidate would be Cartesian product of Teichmueller spaces of partonic 2 surfaces with punctures and string world sheets.

(b) By positing the condition that $g_{i,\bar{j}}$ and $g^{\bar{j},i}$ have opposite phases, one can assign to Kähler metric a unitary S-matrix but this does not seem to be necessary. The experience with loop spaces suggests that for infinite-D Hilbert spaces the existence of non-flat Kähler metric requires a maximal group of isometries. Hence one expects that the counterpart of S-matrix is highly unique. These solutions would be special case of Teichmueller solutions: Teichmueller matrix would be purely imaginary. The condition looks too restrictive. For instance, for torus, this would correspond to a metric conformally equivalent with a flat metric.

2. This inspires the idea that quantum physics could be geometrized by the same way as Einstein geometrized gravitation. Take a flat Hilbert space bundle (in the case of TGD) and replace its flat Kähler metric both base space and fiber with a non-flat Kähler metric. The replacement of flat metric with a curved one would lead from a non-interacting quantum theory to an interacting one. Quantum theory would be gravitation at the level of this Hilbert bundle! This replacement is completely universal.

   In the TGD framework the world of classical worlds (WCW) has Kähler geometry allowing spinor structure. WCW spinors correspond to Fock states for second quantized spinors at X^4 and induced from second quantized spinors of the embedding space. Scattering amplitudes would be determined by the Kähler metric for the Hilbert space bundle of WCW spinor fields realized in ZEO and satisfying Teichmueller condition guaranteeing non-negative probabilities.

   WCW geometry is also characterized by zero modes corresponding to non-complex coordinates for WCW giving no contribution to WCW metric. This is self-evident from SH. The zero modes would be in 1-1 correspondence with Teichmueller parameters and WCW Kähler metrics.

   Equivalence Principle (EP) generalizes to level of WCW and its spinor bundle. In ZEO one can assign also to the Kähler space of zero energy states spinor structure and this suggests an infinite hierarchy of second quantizations starting from space-time level, continuing at the level of WCW, and continuing further at the level of the space of zero energy states. This would give a possible interpretation for an old idea about infinite primes as an infinite hierarchy of second quantizations of an arithmetic QFT [K80].

There is also challenge of constructing the Kähler metric and associated spinor structure for the spinor bundle of WCW. This would mean a specification of the analogs of Feynman rules so that instead of two problems one would have only one problem.

1. WCW gamma matrices can be identified as superpositions of fermionic oscillator operators associated with quark spinors [L69]. One can consider two approaches to the quantization

of these spinors: one studies induced spinor fields obeying MDE and quantizes this or one generalizes the induction of spinors from H to the induction of second quantized spinor fields in H: this would mean simply projecting the spinor fields to X^4. The latter option is extremely simple. It seems possible to avoid divergence problems if the anti-commutators are assigned to different 3-surfaces at different boundaries of CD. This would allow the identification of the Dirac propagator. As a matter of fact, the two approaches are equivalent.

2. WCW gamma matrices would allow the identification as super generators of SSA identified as contractions of gamma matrices SSA with Killing vectors. Quantum states would be created by bosonic and fermionic SSA generators.

3. I have proposed a further supersymmetrization of both H coordinates and spinors by replacing them with expansions in powers of local composites of oscillator operators for quarks and antiquarks [L69]. This however requires Kronecker delta type anti-commutators natural for cognitive representations defining unique discretization of X^4: this allows to avoid normal ordering divergences. Induction of the H spinor fields would lead to 8-D delta function type divergences. This suggests that local composites are not quite local but states consisting of quarks and antiquarks at opposite throats of wormhole contacts identifiable as partonic 2-surfaces. One would obtain leptons as 3-quark states with quarks at the same partonic 2-surface but not at the same point anymore as in the proposal of [L69].

4. The matrix elements of the Kähler metric of WCW Hilbert bundle correspond to scattering amplitudes analogous to Feynman diagrams. What are the Feynman rules? Partonic two surfaces and their orbits correspond to vertices and propagators topologically. The TGD counterpart for $F\overline{F}B$ vertex would correspond to a bosonic wormhole contact with a fermion and antifermion at opposite wormhole throats and representing SCA generator which decomposes to two partonic 2-surfaces carrying fermions at opposite throats representing fermionic SCA generators. This allows avoiding of normal ordering divergences.

   The vertex would correspond to a product or co-product, which can be said to be time reversals of each other. The structure constants of SCA extended to quantum algebra would fix the vertices and thus the analogs of Feynman diagrams completely. Their number is presumably finite for a X^4 with fixed 3-surfaces at its ends and summation over Feynman diagrams would correspond to integration in WCW.

Before discussing them current proposal in detail, the complementary way to overview TGD as either WCW geometry or as number theory are discussed below. Readers might skip these sections at their first reading and choose to read the section discussing the basic idea in more detail.

In the sequel the basic idea about representation of scattering amplitudes as elements of Kähler metric satisfying what I call "Teichmueller condition", is discussed in TGD framework.

The detailed formulation allows a formulation of conditions for the cancellation of normal ordering divergences and also other divergences. The induction of the second quantized free spinor field from H to space-time surface fixes the propagators at the space-time level. If the creation and annihilation operators are at different space-time sheets - say at throats of wormhole contacts, divergences are avoided. ZEO suggests an alternative but not exclusive option that the annihilation operators correspond to creation operators for conjugated Dirac vacuum associated with the opposite half-cone of CD or sub-CD.

The fact that the Dirac propagators for massive particles in the TGD sense reduce in a good approximation to massless propagators when the propagation takes place along light-like distances, allows to considerable insight to why physical particles are so light although the spinor harmonics for CP_2 correspond to CP_2 mass scale.

Of course, one must not forget that this proposal is only an interesting thought game. It is quite possible that zero energy ontology allows to define a natural way a unitary S-matrix or a more general isometric map between the states spaces associated with the extensions of rationals with different algebraic dimensions assignable naturally to to space-time regions inside causal diamonds. The huge symmetries of WCW generalized to Yangian symmetries could lead to a unique S-matrix and number theoretic conditions pose extremely powerful constraints. In [L113], a proposal along these lines was developed 3 years after writing this.

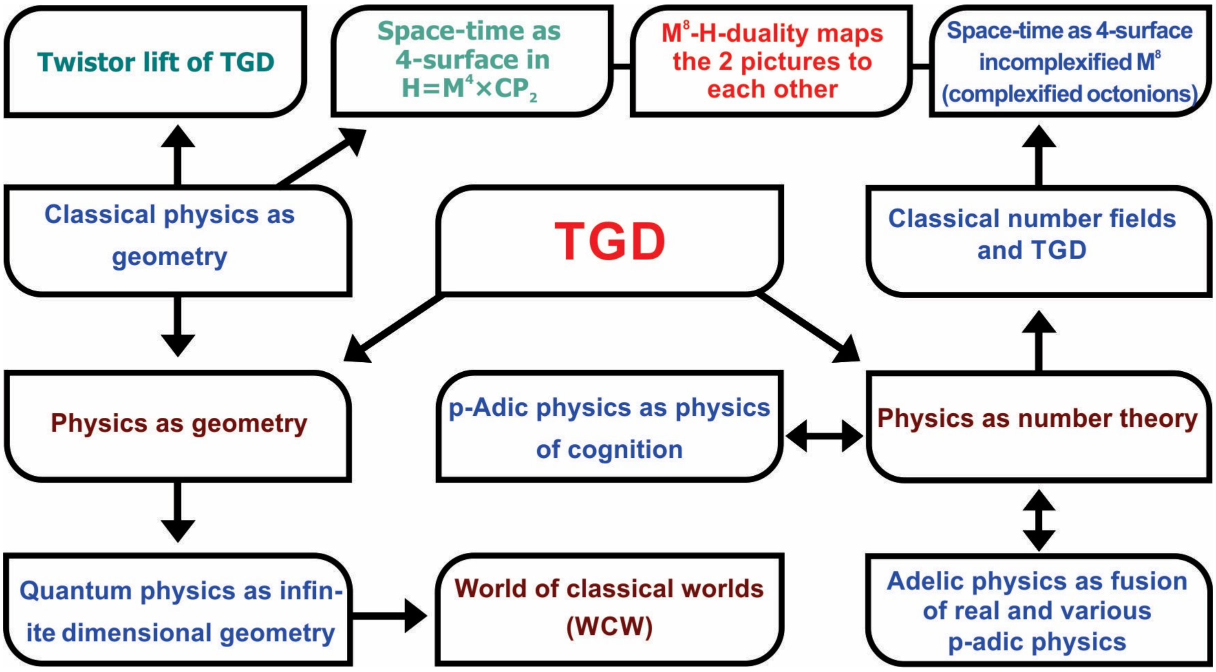

Figure 7.1: TGD is based on two complementary visions: physcs as geometry and physics as number theory.

7.2 Physics as geometry

One can end up with TGD in two ways (see **Fig. 7.2**). Either as a solution of energy problem of GRT realizing Einstein's dream about geometrization of classical physicsor as a generalization of hadronic string model or of superstring theory [B42]. In case of hadronic string model the generalization of string to 3-surface would allow to get rid of spontaneous compactification and the landscape catastrophe implied by it.

At fundamental level TGD could be seen as a hybrid of GRT and SRT: the notion of force does not disappear and can be defined as rate for an exchange of conserved quantity which can be Poincare or color charge. This connection with Newtonian limit is more clear than in GRT, where the conservation laws are lost.

7.2.1 Classical physics as sub-manifold geometry

The new elements are many-sheeted space-time topologically non-trivial in all scales, and topological field quantization implying that physical systems have field identity, field body, in particular magnetic body (MB) central in applications [L11, L10] (see **Fig. 7.3**).

Induction procedure

One ends up to a geometrization of gravitational field and gauge fields of the standard model as induced fields. Induction means induction of bundle structure is in question. Parallel translation at X^4 is carried out by using spinor connection of H and distances are measured using the metric

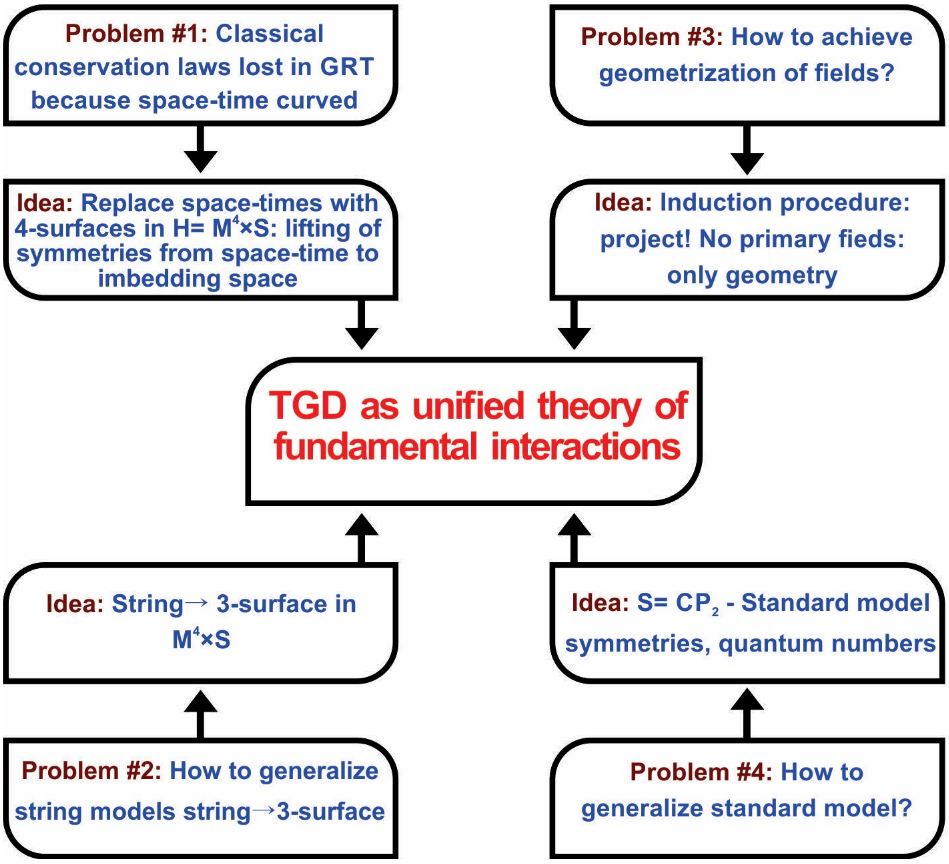

Figure 7.2: The problems leading to TGD as their solution.

of H. The components of induced gauge potentials and metric are projections to X^4. Color gauge potentials are identified as projections of Killing vector fields of CP_2 and one can define for them gauge algebra structure. The components of the induced color field are proportional to $H_A J$, where H_A is the Hamiltonian of color isometry and J induced Kähler form. For details see [L12] or the material at my homepage.

The induction of spinor structure allows to avoid the problems related to the definition of spinor structure for general 4-geometry encountered in GRT. For the induced spinor structure induction means projection of gamma matrices to X^4. The definition of gamma matrices is modified when classical action defining the space-time dynamics contains besides volume term also Kähler action with the projection of CP_2 Kähler form defining the analog of Maxwell field. Modified gamma matrices are contractions $T^{\alpha k}\gamma_k$ of the embedding space gamma matrices γ_k with canonical momentum currents $T^{\alpha k}$ associated with the action: this is required by the hermiticity of the modified Dirac action and means existence of infinite number of super currents labelled by the modes of the modified Dirac action.

Spacetime is topologically complex

Locally the theory is extremely simple: by GCI there are only 4 field-like variables corresponding to a suitable identification of embedding space coordinates as space-time coordinates. The possibility to choose the coordinates in this manner means enormous simplification since the problems caused by GCI in GRT disappear. It is however obvious that 4 field-like variables does not conform with standard model and GRT. This simplicity is compensated by topological complexity in all scales implied by the many-sheeted space-time. The QFT-GRT limit explained in introduction gives the space-time of gauge theories and GRT.

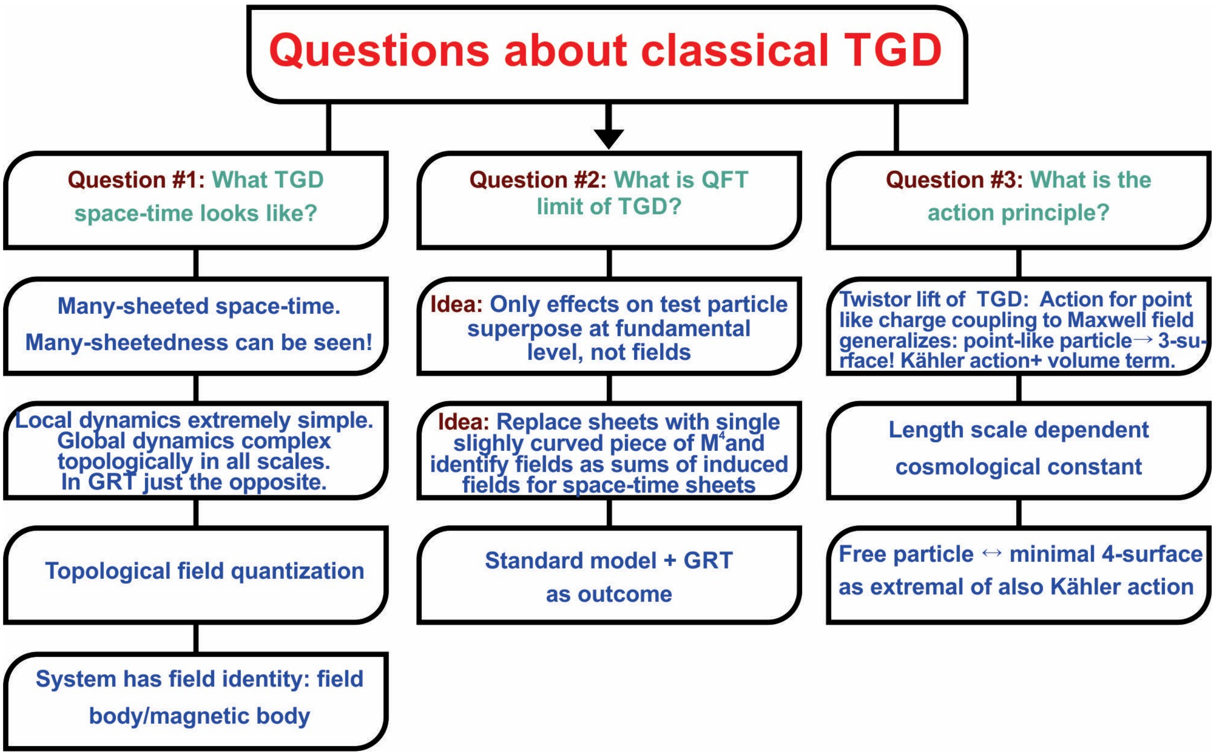

Figure 7.3: Questions about classical TGD.

Geometrically the QFT limit for space-time surfaces having 4-D M^4 projection is obtained by replacing the sheets of many-sheeted space-time with slightly curved region of M^4 and identifying gauge potentials and gravitational field (deviation of the metric from M^4 metric) as superpositions of induced fields at various space-time sheets. Einstein's equations hold true as a remnant of the Poincare invariance.

The presence of space-time regions with M^4 projection of dimension $D < 4$ must be described at QFT limit as particle- or string-like entities. Particle-like entities correspond to CP_2 type extremals having Euclidian signature of induced metric and light-like M^4 projection. 3-D light-like surfaces serve as boundaries between them and Minkowskian space-time regions: the identification is as partonic orbits carrying fermion number serving as building bricks of elementary particles [L52].

The topology of partonic 2-surface is characterized by its genus (number of handless attached to sphere) and is propose to explain family replication for fermions. Also for bosons 3 families are predicted. The existence of 3 light fermion families is understood in terms of the fact that only 3 lowest genera have global Z_2 as conformal symmetry making possible bound state of 2 handles. For the higher genera handles would behave like particles and mass spectrum would be continuum.

Cosmic strings are fundamental objects of this kind and appear as two different species. Those carrying monopole flux mean deviation from Maxwell's theory. They are unstable against perturbations making their M^4 projection 4-D and transforming them to magnetic flux tubes playing a key role in TGD inspired cosmology.

Twistor lift

One could end up with the twistor lift of TGD from problems of twistor Grassmannian approach originally due to Penrose [B56] and developed to a powerful computational tool in $\mathcal{N} = 4$ SYM [B27, B20, B35, B13, B29].

Twistor lift of TGD [L37, L77, L78] generalizes the ordinary twistor approach [L58, L59] (see **Fig. 7.4**). The 4-D masslessness implying problems in twistor approach is replaced with 8-D masslessness so that masses can be non-vanishing in 4-D sense.

The basic recipe is simple: replaced fields with surfaces. Twistors as field configurations are replaced with 6-D surfaces in the 12-D product $T(M^4) \times T(CP_2)$ of 6-D twistor spaces $T(M^4)$ and $T(CP_2)$ having the structure of S^2 bundle and analogous to twistor space $T(X^4)$. Bundle structure requires dimensional reduction. The induction of twistor structure allows to avoid the problems with the non-existence of twistor structure for arbitrary 4-geometry encountered in GRT.

The pleasant surprise is that twistor space has Kähler structure only for M^4 and CP_2 [A74]: this had been discovered already when started to develop TGD! Since Kähler structure is necessary for the twistor lift of TGD, TGD is unique. One outcome is length scale dependent cosmological constant Λ assignable to any system - even hadron - taking a central role in the theory. At long length scales Λ approaches zero and this solves the basic problem associated with it. At this limit action reduces to Kähler action, which for a long time was the proposal for the variational principle.

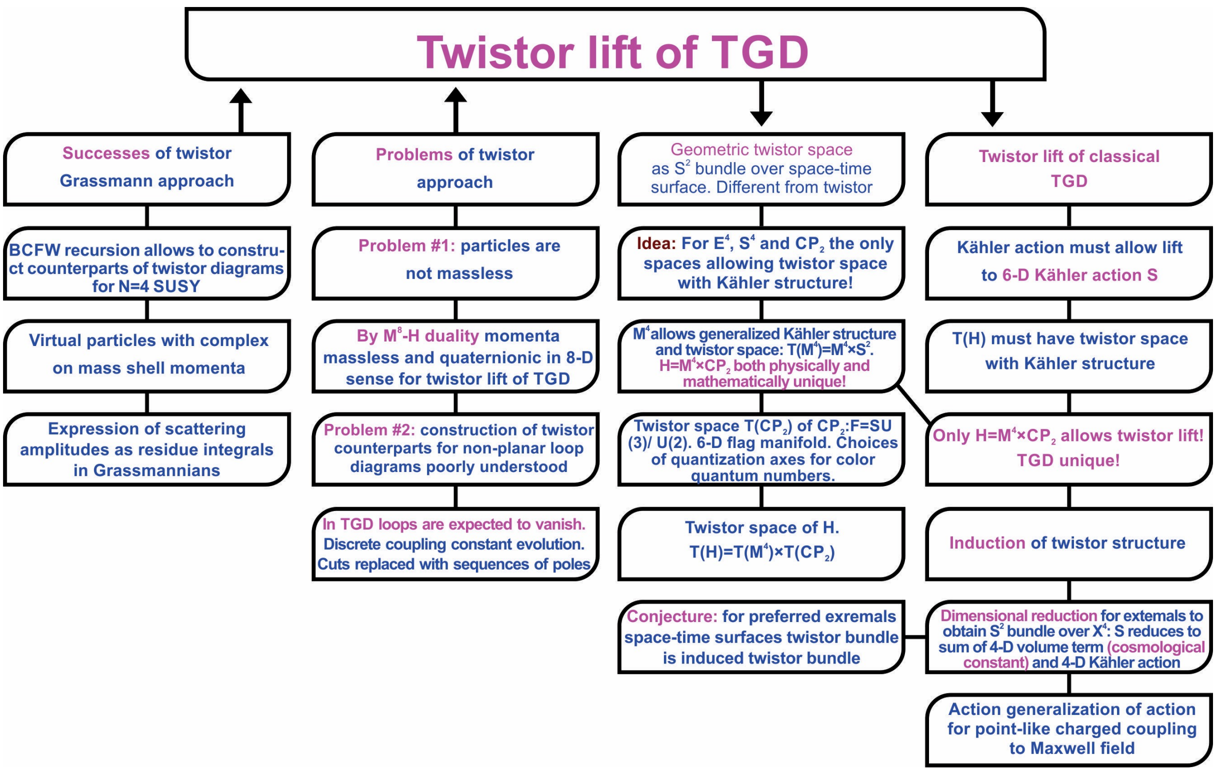

Figure 7.4: Twistor lift

7.2.2 Quantum physics as WCW geometry

WCW as an analog of Wheeler's superspace

Quantum TGD replaces Wheeler's superspace of 3-geometries with the "World of Classical Worlds" (WCW) as the space of 3-surfaces (see **Fig. 7.5**). The holography forced by general coordinate invariance (GCI) implies their 1-1 correspondence with space-time surfaces identified as preferred extremals (PEs) of the basic variational principle analogous to Bohr orbits. Classical physics becomes an exact part of quantum physics [L15, L14]. Einstein's geometrization of classical physics extends to that of quantum physics.

The geometry of infinite-D WCW (see **Fig. 7.5**) and physics is highly unique from its mere existence requiring maximal group of isometries: a result proved by Freed for loop spaces [A54]. The group of WCW isometries is identified as the group of symplectic (contact) transformations of $\delta M^3_+ \times CP_2$ having the light-like radial coordinate in the role of complex variable z in conformal field theories

Remark: The geometric properties of boundary of 4-D light-cone are unique by its metric 2-dimensionality. In particular, the ordinary 2-D conformal symmetries involving local scaling of the radial light-like coordinate give rise to isometries).

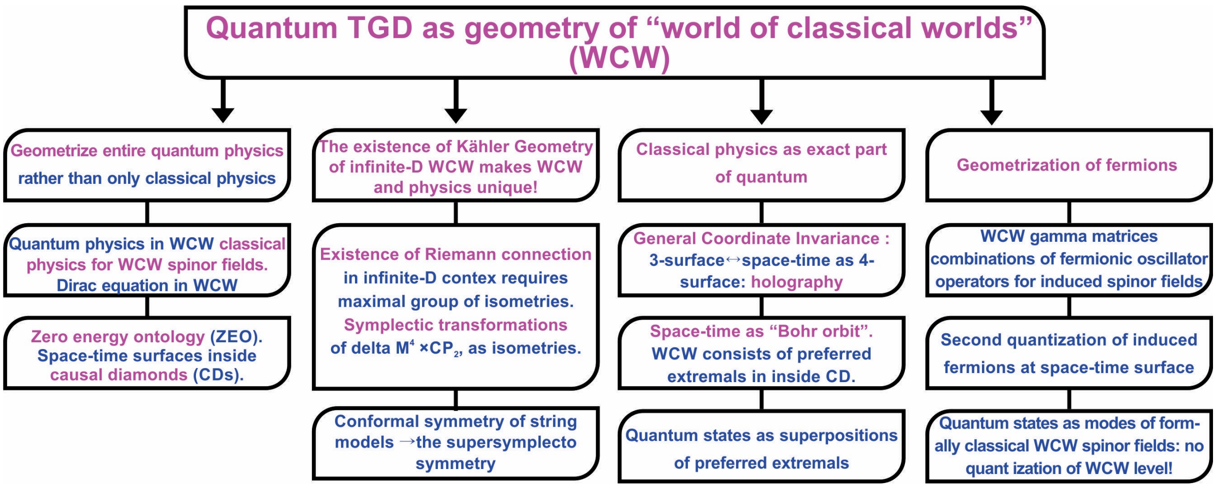

Figure 7.5: Geometrization of quantum physics in terms of WCW

The assumption that space-time surfaces as preferred extremals (PEs) are fundamental entities leads to zero energy ontology (ZEO) in which quantum superpositions of pairs (X^3_1, X^3_2) of 3-surfaces at opposite boundaries of causal diamond (CD) and connected by PE represent quantum states [L92]. This leads to a solution of the basic problem of quantum measurement theory due to the conflict between the determinism of field equations and non-determinism of state function reduction (SFR) and quantum measurement theory extends to a theory of consciousness bringing observer a part of the physisal system.

Quantum states are identified as modes of classical WCW spinor fields so that apart from quantum jump the theory is formally classical. WCW spinor structure involves complexified gamma matrices expressible as superpositions of second quantized oscillator operators of the induced spinor fields at space-time so that a geometrization of fermionic statistics is achieved [L29, L79, L83]. The simplest formulation assumes only quark spinors and would predict that lepton are local composites of 3 quarks.

7.2.3 Super-symplectic group as isometries of WCW

The work of Freed related to the geometrization of loop spaces [A54] demonstrated that the Kähler metric allows awell-defined Riemann connection onlyif it has a maximal group of isometries. This fixes the metric completely. The natural conjecture isthat this is true also in 3-D case and that the group consists of symplectic (contact) transformations at $\delta M^4_\pm \times CP_2$. Here $\delta M^4_\pm$ is future/past directed lightcone boundary containing the "upper"/"lower" boundary of a causal diamond of M^4.

WCW allows as infinitesimal isometries huge super-symplectic algebra (SSA) [K39, K21] acting on space-like 3-surfaces at the ends of space-time surfaces inside causal diamond (CD) and also generalization of Kac-Moody and conformal symmetries acting on the 3-D light-like orbits of partonic 2-surfaces (partonic super-conformal algebra (PSCA)). These symmetry algebras have a fractal structure containing a hierarchy of sub-algebras isomorphic to the full algebra. Even ordinary conformal algebras with non-negative conformal weights have similar fractal structure as also Yangian. In fact, quantum algebras are formulated in terms of these half algebras.

The proposal is that physical states are annihilated by a sub-algebra SSA_n of SSA (with non-negative conformal weights), $n = 1, 2, ...,$ with conformal weights coming as n-multiples of those for SSA and thus isomorphic to the entire SSA, and by the commutator $[SSA_n, SSA]$. What remains seems to be a finite-D Kac-Moody algebra as an effective "coset" algebra obtained. Note that the resulting analog of a normal sub-group could actually be a quantum group.s There is a direct analogy with the decomposition of the Galois group Gal to a product of sub-group and normal subgroup H. If the normal subgroup H acts trivially on the representation of Gal reduces to that of the group Gal/H. Now one works at Lie algebra level: Gal is replaced with SSA and H with its sub-algebra with conformal weights multiples of those for SSA. These two hierarchies of subgroups could correspond to each other and to the hierarchy of inclusions of hyperfinite factors of type II_1 (HFFs) [K95, K30]. These conditions would guarantee preferred extremal property of the space-time surface and holography or even its strong form.

Holography from GCI

Gravitational holography has been one of the dominating themes in recent day theoretical physics. It was originally proposed by Susskind [B49], and formulated by Maldacena as AdS/CFT correspondence [B46]. One application is by Preskill *et al* to quantum error correcting codes [B37].

By holography implied by GCI the basic variational problem can be seen either as boundary value problem with 3-surfaces at opposite boundaries of CD or as initial value problem caused by PE property. Ordinary 3-D holography is thus forced by general coordinate invariance (GCI) and loosely states that the data at 3-surface at either boundary of CD allows to determine $X^4 \subset H$. In ZEO 3-surfaces correspond to pairs of 3-surfaces with members at the opposite light-like boundaries of causal diamond (CD) and are analogous to initial and final states of deterministic time evolution as Bohr orbit.

Holography poses additional strong conditions on X^4.

1. The conjecture is that these conditions state the vanishing of super-symplectic Noether charges for a sub-algebra of super-symplectic algebra SSA_n with radial conformal weights coming as n-multiples of those for the entire algebra SSA and its commutator $[SSA_n, SSSA]$ with the entire algebra: these conditions generalize super conformal conditions and one obtains a hierarchy of realizations. An open question is whether this hierarchy corresponds to the hierarchy of EQs with n identifiable as dimension of the extension.

2. Second conjecture is that PEs are extremals of both the volume term and Kähler action term of the action resulting by dimensional reduction making possible the induction of twistor

structure from the product of twistor spaces of M^4 and CP_2 to 6-D S^2 bundle over X^4 defining the analog of twistor space. These twistor spaces must have Kähler structure since action for 6-D surfaces is Kähler action - it exists only in these two cases [A74] so that TGD is unique.

Strong form of holography

Strong form of holography (SH) is a strengthening of 3-D holography. Strong form of GCI requires that one can use either the data associated

1. either with light-like 3-surfaces defining partonic orbits as surfaces at which signature of the induced metric changes from Euclidian to Minkowskian,

2. or the space-like 3-surfaces at the ends of CD to determine X^4 as PE (in case that it exists),

This suggests that the data at the intersections of these 2-surfaces defined by partonic 2-surfaces might be enough for holography. A slightly weaker form of SH is that also string world sheets intersecting partonic orbits along their 1-D boundaries is needed and this form seems more realistic.

SH allows to weaken the strong form of M^8-H duality [L72] mapping $X^4 \subset M^8$ to $X4 \subset H = M^4 \times CP_2$ that it allows to map only certain 2-D sub-manifolds $X^2 \subset X^4 \subset M^8$: SH allows to determine $X^4 \subset H$ from this 2-D data.

Further generalizations

This picture about WCW is not general enough.

1. M^8-H duality [L72] suggests that the notion of WCW applies also M^8 picture. The parameters of polynomials defining $X^4 \subset M^8$ are assumed to be rational. The points of M^8 counterpart of WCW have the rational coefficients of these polynomials as coordinates so that WCW should be discrete in real topology. This should be the case also for H counterpart of WCW. Could one see real and p-adic variants of WCW as completions of this discrete WCW.

2. Adelic physics inspires the question whether p-adic and adelic variants of WCW make sense or is it enough to have number theoretically universal cognitive representations to define unique discretized variants of X^4 and correspondingly discretized WCW.

3. For TGD variant of SUSY [L76, L75] super coordinates for H correspond to hermitian local composites of quark oscillator operators. For super-quarks they correspond to local components with fixed quark number. Leptons can be understand as local composites of quarks - super field components [L83]. SUSY replaces modes of super-field with super-surfaces so that the components of super-field correspond to sets of disjoint 4-surfaces. This is true also for the points of super WCW.

7.3 Physics as number theory

Number theoretical vision is second thread of TGD. It decomposes to 3 threads corresponding to various p-adic physics [L16] fusing to adelic physics [L52], classical number fields [L17], and infinite primes [L18] (not discussed in the sequel).

7.3.1 p-Adic and adelic physics and extensions of rationals (EQs)

p-Adic number fields would serve as correlates of cognition and imagination (see **Fig. 7.6**) . Space-time is replaced with a book like structure having both real and various p-adic space-time sheets as pages. The outcome is adelic physics as fusion of various p-adic physics [L48, L52] (see `http://tinyurl.com/ycbhse5c`). The EQ induces extensions of p-adic numbers fields and of adele giving rise to a hierarchy of physics having interpretation in terms of evolution induced by the increase of the complexity of the EQ.

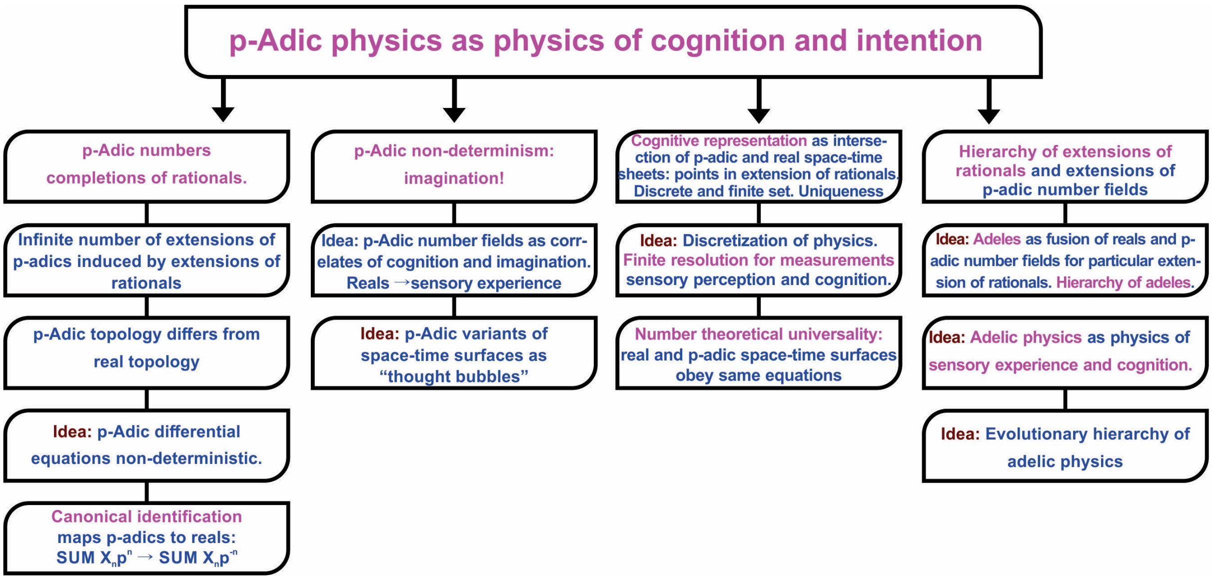

Figure 7.6: p-Adic physics as physics of cognition and imagination.

Adelic physics leads also the hierarchy of Planck constants $h_{eff}/h_0 = n$ with n identified as dimension of EQ labelling phases of ordinary matter behaving like dark mater, and making possible quantum coherence in arbitrarily long time scales essential for understanding living matter.

EQs are characterized by discriminant D assignable to a polynomial giving rise to the extension (for second order polynomials D has expression familiar from school days). Now polynomials with rationals (equivalently integer) valued coefficients are interesting. The primes dividing the discriminant are known as ramified primes and they have a property that for p-adic variant of polynomial degenerate roots appear in $O(p) = 0$ approximation [L74]. The interpretation could be in terms of quantum criticality and physically preferred p-adic primes are identified as ramified primes of extension [L82].

Remark: One can also consider polynomials with algebraic coefficients. The notion of Galois group make sense also for real coefficients.

The hierarchy of EQs labelling levels of dark matter hierarchy and of hierarchy of adelic physics follows from $M^8 - H$ duality allowing to identify $X^4 \subset M^8$ as a projection of $X^4_c \subset M^8_c$ - identified as complexified octonions O_c - and satisfying algebraic equations associated with a polynomial of degree n.

Real and p-adic physics are strongly correlated and mass calculations represent the most important application of p-adic physics [K46]. Elementary particles seem to correspond to p-adic primes near powers 2^k (there are also indicatons for powers of 3). Corresponding p-adic length - and time scales would come as half-octaves of basic scale if all integers k are allowed. For odd values of k one would have octaves as analog for period doubling. In chaotic systems also the generalization of period doubling in which prime $p = 2$ is replaced by some other small prime appear and there is indeed evidence for powers of $p = 3$ (period tripling as approach to chaos) [?, ?]. Many elementary particles and also hadron physics and electroweak physics seem to correspond to Mersenne primes and Gaussian Mersennes which are maximally near to powers of 2 and the challenge is to understand this [L38].

7.3.2 Classical number fields

Second aspect of number theoretical vision are classical number fields: reals, complex numbers, quaternions and octonions and their complexifications by a commuting imaginary unit i (see **Fig. 7.7**).

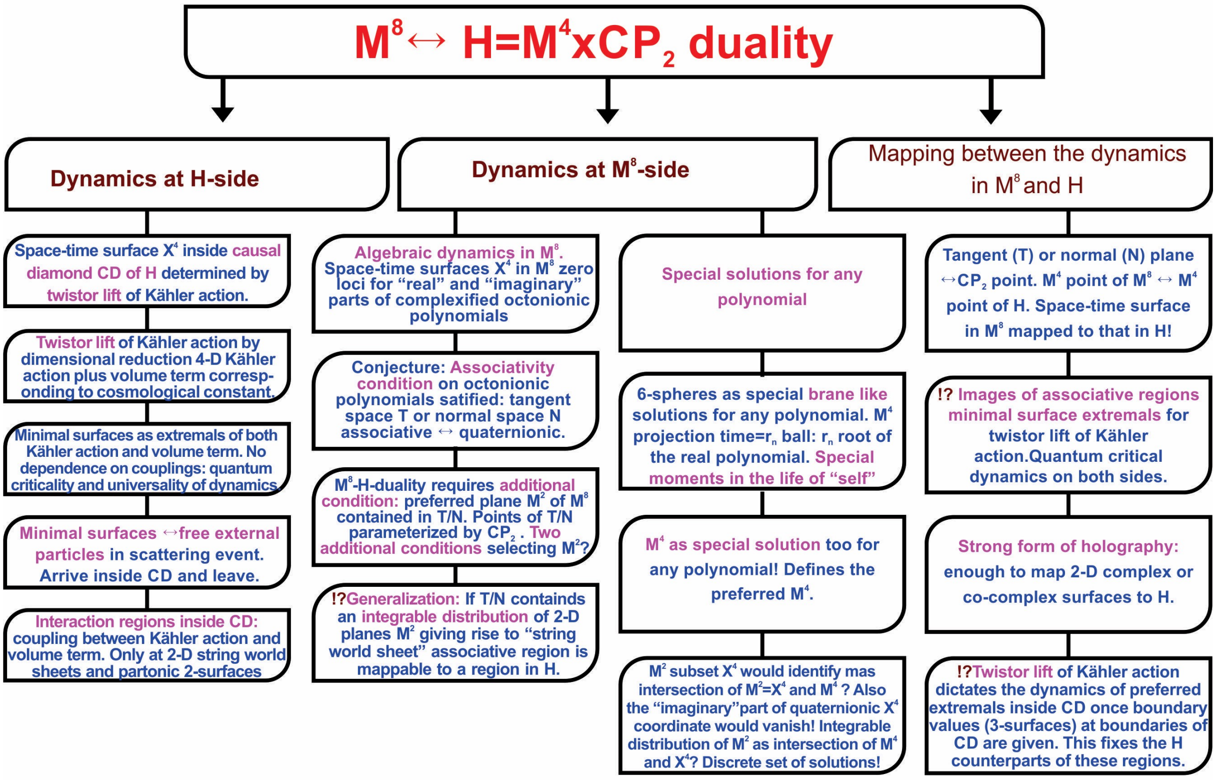

Figure 7.7: $M^8 - H$ duality

Space-time as 4-surface in $M^8_c = O_c$

One can regard real space-time surface $X^4 \subset M^8$ as a M^8-–projection of $X^4_c \subset M^8_c = O_c$. M^4_c is identified as complexified quaternions H_c [L72, L82]. The dynamics is purely algebraic and therefore local.

1. The basic condition is associativity of $X^4 \subset M^8$ in the sense that either the tangent space or normal space is associative - that is quaternionic. This would be realized if X^4_c as a root for the quaternion-valued "real" or "imaginary part" for the O_c algebraic continuation of real analytic function $P(x)$ in octonionic sense. Number theoretical universality requires that the Taylor coefficients are rational numbers and that only polynomials are considered.

   The 4-surfaces with associative normal space could correspond to elementary particle like entities with Euclidian signature (CP_2 type extremals) and those with associative tangent space to their interaction regions with Minkowskian signature. These two kinds space-time surfaces could meet along these 6-branes suggesting that interaction vertices are located at these branes.

2. The conditions allow also exceptional solutions for any polynomial for which both "real" and "imaginary" parts of the octonionic polynomial vanish. Brane-like solutions correspond to 6-spheres S^6 having $t = r_n$ 3-ball B^3 of light-cone as M^4 projection: here r_n is a root of the real polynomial with rational coefficients and can be also complex - one reason for complexification by commuting imaginary unit i. For scattering amplitudes the topological vertices as 2-surfaces would be located at the intersections of X^4_c with 6-brane. Also Minkowski space M^4 is a universal solution appearing for any polynomial and would provide a universal reference space-time surface.

3. Polynomials with rational coefficients define EQs and these extensions form a hierarchy realized at the level of physics as evolutionary hierarchy. Given extension induces extensions of p-adic number fields and adeles and one obtains a hierarchy of adelic physics. The dimension n of extension allows interpretation in terms of effective Planck constant $h_{eff} = n \times h_0$. The phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ behave like dark matter and galactic dark matter could correspond to classical energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes. It is not completely clear whether number galactic dark matter must have $h_{eff} > h$. Dark energy in would correspond to the volume part of the energy of the flux tubes.

   There are good arguments in favor of the identification $h = 6h_0$ [**?**] "Effective" means that the actual value of Planck constant is h_0 but in many-sheeted space-time n counts the number of symmetry related space-time sheets defining X^4 as a covering space locally. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is nh_0.

   The ramified primes of extension in turn are identified as preferrred p-adic primes. The moduli for the time differences $|t_r - t_s|$ have identification as p-adic time scales assignable to ramified primes [L82]. For ramified primes the p-adic variants of polynomials have degenerate zeros in $O(p) = 0$ approximation having interpretation in terms of quantum criticality central in TGD inspired biology.

4. During the preparation of this article I made a trivial but overall important observation. Standard Minkowski signature emerges as a prediction if conjugation in O_c corresponds to the conjugation with respect to commuting imaginary unit i rather than octonionic imaginary units as though earlier. If X^4 corresponds to the projection $O_c \rightarrow M^8 \rightarrow M^4$ with real time coordinate and imaginary spatial coordinates the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

How to realize $M^8 - H$ duality?

$M^8 - H$ duality (see **Fig. 7.7**) allows to $X^4 \subset M^8$ to $X^4 \subset H$ so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in M^8 and as minimal surfaces with 2-D preferred 2-surfaces defining holography making possible $M^8 - H$ duality and possibly appearing as singularities in H. The dynamics of minimal surfaces, which are also extremals of Kähler action, reduces for known extremals to purely algebraic conditions analogous to holomorphy conditions in string models and thus involving only gradients of coordinates. This condition should hold generally and should induce the required huge reduction of degrees of freedom proposed to be realized also in terms of the vanishing of super-symplectic Noether charges already mentioned [L79].

Twistor lift allows several variants of this basic duality [L77, L78]. M^8_H duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of EQs defining an evolutionary hierarchy. This forms the basics for the number theoretical vision about TGD.

As already noticed, $X^4 \subset M^8$ would satisfy an infinite number of additional conditions stating vanishing of Noether charges for a sub-algebra $SSA_n \subset SSA$ of super-symplectic algebra SSA actings as isometries of WCW.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions [L72].

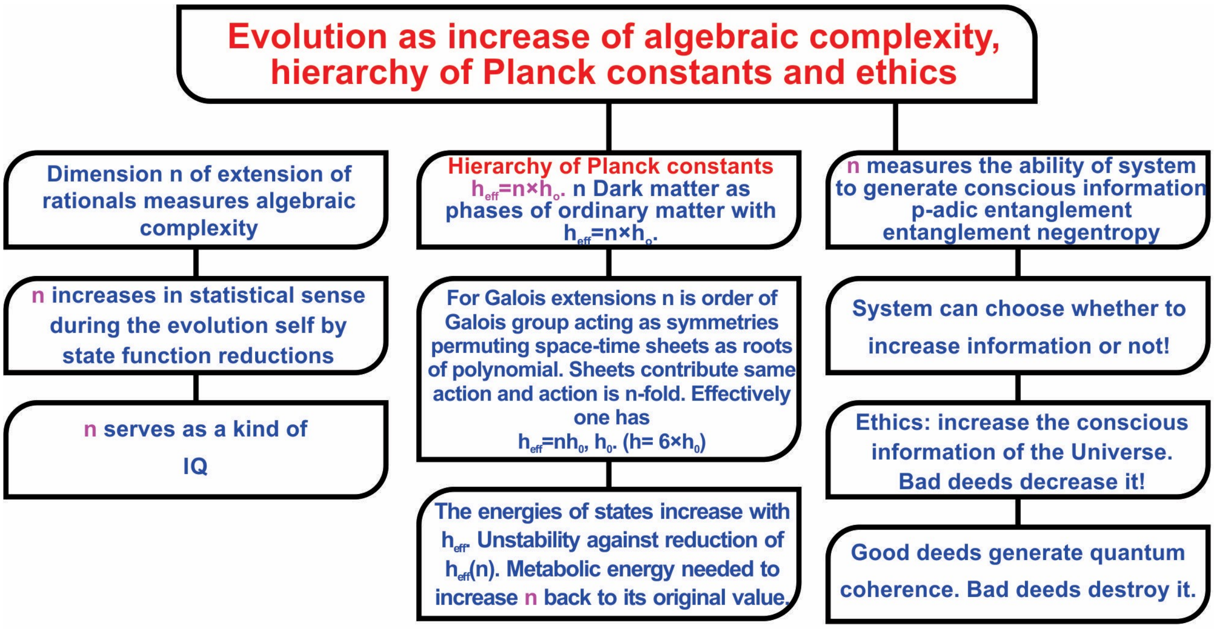

Figure 7.8: Number theoretic view about evolution

1. Associativity condition for tangent-/normal space is the first essential condition for the existence of M^8-H duality and means that tangent - or normal space is associative/quaternionic.

2. Each tangent space of X^4 at x must contain a preferred $M_c^2(x) \subset M_c^4$ such that $M_c^2(x)$ define an integrable distribution and therefore complexified string world sheet in M_c^4. This gives similar distribution for their orthogonal complements $E^2c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X_c^2 \subset X_c^4$ in R_c sense. $E_c^2(x)$ would correspond to partonic 2-surface. This condition generalizes for X^4 with quaternionic normal space.

   One can imagine two realizations for this condition.

Option I: Global option states that the distributions $M_c^2(x)$ and $E_c^2(x)$ define a slicing of X_c^4.

Option II: Only discrete set of 2-surfaces satisfying the conditions exist, they are mapped to H, and strong form of holography (SH) applied in H allows to deduce $X^4 \subset H$. This would be the minimal option.

It seems that only **Option II** can be realized.

1. The basic observation is that X_c^2 can be fixed by posing to the non-vanishing H_c-valued part of octonionic polynomial P condition that the C_c valued "real" or "imaginary" part in C_c sense for P vanishes. M_c^2 would be the simplest solution but also more general complex sub-manifolds $X_c^2 \subset M_c^4$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

These surfaces would be like the families of curves in complex plane defined by $u = 0$ an $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. SH makes possible $M^8 - H$ duality assuming that associativity conditions hold true only at 2-surfaces including partonic 2-surfaces or string world sheets or perhaps both. Thus one can give up the conjecture that the polynomial ansatz implies the additional condition globally.

   SH indeed states that PEs are determined by data at 2-D surfaces of X^4. Even if the conditions defining X^2_c have only a discrete set of solutions, SH at the level of H could allow to deduce the PEs from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. The existence of $M^2(x)$ would be required only at the 2-D surfaces.

3. There is however a delicacy involved: the X^2 might be only metrically 2-D but not topologically. The partonic orbits are 3-D light-like surfaces with metric dimension $D = 2$. The 4-metric degenerates to 2-D metric at them. Therefore their pre-images would be natural candidates for the singularities at which the dimension of the quaternionic tangent or normal space reduces to 2 [L73] . If this happens, SH would not be quite so strong as expected. The study of fermionic variant of $M^8 - H$ correspondence indeed leads to this conclusion.

One can generalize the condition selecting X^2_c so that it selects 1-D surface inside X^2_c. By assuming that R_c-valued "real" or "imaginary" part of complex part of P at this 2-surface vanishes. One obtains preferred M^1_c or E^1_c containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as a complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \to C_c \to H_c \to O_c$ realized as surfaces.

What about $M^8 - H$ duality in the fermionic sector?

During the preparation of this article I became aware of the fact that the realization $M^8 - H$ duality in the fermionic sector has remained poorly understood. This led to a considerable integration of the ideas about $M^8 - H$ duality also in the bosonic sector and the existing phenomenological picture follows now from $M^8 - H$ duality. There are powerful mathematical guidelines available.

1. Octonionic spinors

By supersymmetry, octonionicity should have also a fermionic counterpart.

1. The interpretation of M^8_c as complexified octonions suggests that one should use complexified octonionic spinors in M^8_c. This is also suggested by SO(1,7) triality unique for dimension $d = 8$ and stating that the dimensions of vector representation, spinor representation and its conjugate are same and equal to $D = 8$. I have already earlier considered the possibility to interpret M^8 spinors as octonionic [L30]. Both octonionic gamma matrices and spinors have interpretation as octonions and gamma matrices satisfy the usual anti-commutation rules. The product for gamma matrices and spinors is replaced with the non-associative octonionic product.

2. Octonionic spinors allow only one M^8-chirality, which conforms with the assumption of TGD inspired SUSY that only quarks are fundamental fermions and leptons are their local composites [L76, L75].

3. The decomposition of $X^2 \subset X^4 \subset M^8$ corresponding to $R \subset C \subset Q \subset O$ should have an analog for the O_c spinors as a tensor product decomposition. The special feature of dimension $D = 8$ is that the dimensions of spinor spaces associated with these factors are indeed 1, 2, 4, and 8 and correspond to dimensions for the surfaces!

   One can define for the octonionic spinors associative/co-associative sub-spaces as quaternionic/co-quaternionic spinors by posing chirality conditions. For $X^4 \subset M^8_c$ one could define the analogs of projection operators $P_\pm = (1 \pm \gamma_5)/2$ as projection operators to either factor of the spinor space as tensor product of spinor space associated with the tangent and normal spaces of X^4: the analog of γ_5 would correspond to tangent or normal space depending on whether tangent or normal space is associative. For the spinors with definite chirality there would be no entanglement between the tensor factors. The condition would generalize the chirality condition for massless M^4 spinors to a condition holding for the local M^4 appearing as tangent/normal space of X^4.

4. The chirality condition makes sense also for $X^2 \subset X^4$ identified as a complex/co-complex surface of X^4. Now γ_5 is replaced with γ_3 and states that the spinor has well-defined spin in the direction of axis defined by the decomposition of X^2 tangent space to $M^1 \times E^1$ with M^1 defining real octonion axis and selecting rest frame. Interpretation in terms of quantum measurement theory is suggestive.

What about the sigma matrices associated with the octonionic gamma matrices? The surprise is that the commutators of M^4 sigma matrices and those of E^4 sigma matrices close to the same $SO(3)$ algebra allowing interpretation as representation for quaternionic automorphisms. Lorentz boosts are represented trivially, which conforms with the fact that octonion structure fixes unique rest system. Analogous result holds in E^4 degrees of freedom. Besides this one has unit matrix assignable to the generalized spinor structure of CP_2 so that also electroweak $U(1)$ factor is obtained.

One can understand this result by noticing that octonionic spinors correspond to 2 copies of a tensor products of the spinor doublets associated with spin and weak isospin. One has $2 \otimes 2 = 3 \oplus 1$ so that one must have $1 \oplus 3 \oplus 1 \oplus 3$. The octonionic spinors indeed decompose like $1 + 1 + 3 + \overline{3}$ under $SU(3)$ representing automorphisms of the octonions. $SO(3)$ could be interpreted as $SO(3) \subset SU(3)$. $SU(3)$ would be represented as tangent space rotations.

2. Dirac equation as partial differential equation must be replaced by an algebraic equation

Algebraization of the dynamics should be supersymmetric. The modified Dirac equation in H is linear partial differential equation and should correspond to a linear algebraic equation in M^8.

1. The key observation is that for the ordinary Dirac equation the momentum space variant of Dirac equation for momentum eigenstates is algebraic! Could the interpretation for $M^8 - H$ duality as an analog of momentum-position duality of wave mechanics considered already earlier make sense! This could also have something to do with the dual descriptions of twistorial scattering amplitudes in terms of either twistor and momentum twistors. Already the earlier work excludes the interpretation of the octonionic coordinate o as 8-momentum. Rather, $P(o)$ has this interpretation and o corresponds to the embedding space coordinates.

2. The first guess for the counterpart of the modified Dirac equation at the level of $X^4 \subset M^8$ is $P\Psi = 0$, where Ψ is octonionic spinor and the octonionic polynomial P defining X^4 can be seen as a generalization of momentum space Dirac operator with octonion units representing gamma matrices. If associativity/co-associativity holds true, the equation becomes quaternionic/co-quaternionic and reduces to the 4-D analog of massless Dirac equation and of modified Dirac equation in H. Associativity holds true if also Ψ satisfies associativity/co-associativity condition as proposed above.

3. What about the square of the Dirac operator? There are 3 conjugations involved: quaternionic conjugation assumed in the earlier work, conjugation with respect to i, and their combination. The analog of octonionic norm squared defined as the product $o_c o_c^*$ with conjugation with respect to i only, gives Minkowskian metric $m_{kl} o^k \overline{o}^l$ as its real part. The imaginary part of the norm squared is vanishing for the projection $O_c \rightarrow M^8 \rightarrow M^4$ so that time coordinate is real and spatial coordinates imaginary. Therefore Dirac equation allows solutions only for the M^4 projection X^4 and M^4 ($M8$) signature of the metric can be said to be an outcome of quaternionicity (octonionicity) alone in accordance with the duality between metric and algebraic pictures.

   Both $P^\dagger P$ and PP should annihilate Ψ. $P^\dagger P\Psi = 0$ gives $m_{kl} P^k \overline{P}^l = 0$ as the analog of vanishing mass squared in M^4 signature in both associative and co-associative cases. $PP\Psi = 0$ reduces to $P\Psi = 0$ by masslessness condition. One could perhaps interpret the projection $X_c^4 \rightarrow M^8 \rightarrow M^4$ in terms of Uncertainty Principle.

There is a $U(1)$ symmetry involved: instead of the plane M^8 one can choose any plane obtained by a rotation $exp(i\phi)$ from it. Could it realize quark number conservation in the M^8 picture?

For $P = o$ having only $o = 0$ as root $Po = 0$ reduces to $o^\dagger o = 0$ and o takes the role of momentum, which is however vanishing. 6-D brane like solutions S^6 having $t = r_n$ balls $B^3 \subset CD_4$ as M^4 projections one has $P = 0$ so that the Dirac equation trivializes and does not pose conditions on Ψ. o would have interpretation as space-time coordinates and $P(o)$ as position dependent momentum components P^k.

The variation of P at the mass shell of M^8_c (to be precise) could be interpreted in terms of the width of the wave packet associated with a particle. Since the light-like curve at partonic 2-surface for fermion at X^3_L is not a geodesic, mass squared in M^4 sense is not vanishing. Could one understand mass squared and the decay width of the particle geometrically? Note that mass squared is predicted also by p-adic thermodynamics [K46].

4. The masslessness condition restricts the spinors at 3-D light-cone boundary in $P(M^8)$. M^8-H duality [L72] suggests that this boundary is mapped to $X^3_L \subset H$ defining the light-like orbit of the partonic 2-surface in H. The identification of the images of $P_kP^k = 0$ surfaces as X^3_L gives a very powerful constraint on SH and $M^8 - H$ duality.

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6. The variant Dirac equation would hold true also at 2-surfaces $X^2 \subset X^4$ and should commute with the corresponding chirality condition. Now $D^\dagger D\Psi = 0$ defines a 2-D variant of masslessness condition with 2-momentum components represented by those of P. 2-D masslessness locates the spinor to a 1-D curve X^1_L. Its H-image would naturally contain the boundary of the string word sheet at X^3_L assumed to carry fermion quantum numbers and also the boundary of string world sheet at the light-like boundary of CD_4. The interior of the string world sheet in H would not carry an induced spinor field.

7. The general solution for both 4-D and 2-D cases can be written as $\Psi = P\Psi_0$, Ψ_0 a constant spinor - this is in a complete analogy with the solution of modified Dirac equation in H. P depends on position: the WKB approximation using plane waves with position dependent momentum seems to be nearer to reality than one might expect.

3. The phenomenological picture at H-level follows from the M^8-picture

Remarkably, the partly phenomenological picture developed at the level of H is reproduced at the level of M^8. Whether the induced spinor fields in the interior of X^4 are present or not, has been a long standing question since they do not seem to have any role in the physical picture. The proposed picture answers this question.

Consider now the explicit realization of $M^8 - H$-duality for fermions.

1. SH and the expected analogy with the bosonic variant of $M^8 - H$ duality lead to the first guess. The spinor modes in $X^4 \subset M^8$ restricted to X^2 can be mapped by $M^8 - H$-duality to those at their images $X^2 \subset H$, and define boundary conditions allowing to deduce the solution of the modified Dirac equation at $X^4 \subset H$. X^2 would correspond to string world sheets having boundaries X^1_L at X^3_L.

The guess is not quite correct. Algebraic Dirac equation requires that the solutions are restricted to the 3-D and 1-D mass shells $P_kP^k = 0$ in M^8. This should remain true also in H and X^3_L and their 1-D intersections X^1_L with string world sheets remain. Fermions would live at boundaries. This is just the picture proposed for the TGD counterparts of the twistor amplitudes and corresponds to that used in the twistor Grassmann approach!

For 2-D case constant octonionic spinors Ψ_0 and gamma matrix algebra are equivalent with the ordinary Weyl spinors and gamma matrix algebra and can be mapped as such to H. This gives one additional reason for why SH must be involved.

2. At the level of H the first guess is that the modified Dirac equation $D\Psi = 0$ is true for D based on the modified gamma matrices associated with both volume action and Kähler action. This would select preferred solutions of modified Dirac equation and conform with the vanishing of super-symplectic Noether charges for SSA_n for the spinor modes. The guess is not quite correct. The restriction of the induced spinors to X^3_L requires that Chern-Simons action at X^3_L defines the modified Drac action.

3. The question has been whether the 2-D modified Dirac action emerges as a singular part of 4-D modified Dirac action assignable to singular 2-surface or can one assign an independent 2-D Dirac action assignable to 2-surfaces selected by some other criterion. For singular surfaces M^8-H duality fails since tangent space would reduce to 2-D space so that only their images can appear in SH at the level of H.

   This supports the view that singular surfaces are actually 3-D mass shells M^8 mapped to X^3_L for which 4-D tangent space is 2-D by the vanishing of $\sqrt{g_4}$ and light-likeness. String world sheets would correspond to non-singular $X^2 \subset M^8$ mapped to H and defining data for SH and their boundaries $X^1_L \subset X^3_L$ and $X^1_L \subset CD_4$ would define fermionic variant of SH.

What about the modified Dirac operator D in H?

1. For X^3_L modified Dirac equation $D\Psi = 0$ based on 4-D action S containing volume and Kähler term is problematic since the induced metric fails to have inverse at X^3_L. The only possible action is Chern-Simons action S_{CS} used in topological quantum field theories and now defined as sum of C-S terms for Kähler actions in M^4 and CP_2 degrees of freedom. The presence of M^4 part of Kähler form of M^8 is forced by the twistor lift, and would give rise to small CP breaking effects explaining matter antimatter asymmetry [L76, L75]. S_{C-S} could emerge as a limit of 4-D action.

   The modified Dirac operator D_{C-S} uses modified gamma matrices identified as contractions $\Gamma^\alpha_{CS} = T^{\alpha k}\gamma_k$, where $T^{\alpha k} = \partial L_{CS}/\partial(\partial_\alpha h^k)$ are canonical momentum currents for S_{C-S} defined by a standard formula.

2. CP_2 part would give conserved Noether currents for color in and M^4 part Poincare quantum numbers: the apparently small CP breaking term would give masses for quarks and leptons! The bosonic Noether current $J_{B,A}$ for Killing vector j^k_A would be proportional to $J^\alpha_{B,A} = T^\alpha_k j_A k$ and given by $J_{B,A} = \epsilon^{\alpha\beta\gamma}[J_{\beta\gamma}A_k + A_\beta J_{\gamma k}]j^k_A$.

   Fermionic Noether current would be $J_{F,A} = \overline{\Psi}J^\alpha\Psi$ 3-D Riemann spaces allow coordinates in which the metric tensor is a direct sum of 1-D and 2-D contributions and are analogous to expectation values of bosonic Noether currents. One can also identify also finite number of Noether super currents by replacing $\overline{\Psi}$ or Ψ by its modes.

3. In the case of X^3_L the 1-D part light-like part would vanish. If also induced Kähler form is non-vanishing only in 2-D degrees of freedom, the Noether charge densities J^t reduce to $J^t = JA_k j^k_A$, $J = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}$ defining magnetic flux. The modified Dirac operator would reduce to $D = JA_k\gamma^k D_t$ and 3-D solutions would be covariantly constant spinors along the light-like geodesics parameterized by the points 2-D cross section. One could say that the number of solutions is finite and corresponds to covariantly constant modes continued from X^1_L to X^3_L. This picture is just what the twistor Grassmannian approach led to [L58, L59].

7.4 Could Kähler metric of state space replace S-matrix?

In the sequel a more detailed view about the reduction of S-matrix to a non-flat Kähler geometry of Hilbert space consisting of WCW spinor fields is considered. The proposal is novel in the sense that the state space would codes interactions to its geometry just like space-time geometry codes gravitational interaction in general relativity.

7.4.1 About WCW spinor fields

Induction of second quantized spinor fields from H

There are two approaches to the quantization of induced spinors at space-time surfaces, and these approaches are equivalent.

1. Induction means that gamma matrices are determined by Kähler action as analogs for projections of embedding space gamma matrices and space-time spinor field χ is simply the restriction of H spinor field Ψ. For a given action determining X^4, supersymmetry allows the identification of the modified Dirac operator D and finding of the modes of the induced H spinor field as solutions of the modified Dirac equation (MDE) $D\chi = 0$. Second quantization would replace their coefficients with oscillator operators. However, it is not clear what the anti-commutation relations for the oscillator operators are.

2. One can generalize the classical induction of spinors Ψ to an induction of second quantized spinor fields in H as a restriction of the second quantized Ψ in H to the X^4. One must however get rid of normal ordering diverges due the fact that the anti-commutators for coinciding points give 7-D delta functions. One gets rid of them, if the Ψ and $\overline{\Psi}$ are assigned to disjoin space-time regions. This leads to bi-local modified Dirac action (MDA), implying automatically the classical field equations for the action determining D.

   What does $D\Psi = 0$ really mean when Ψ is quantum field? One can develop the restrictions of the c-valued modes of Ψ in terms of modes of χ satisfying $D\chi = 0$, and obtain an expression for Ψ at X^4 in terms of these modes each satisfying MDE. The operator valued coefficients of Ψ modes contributing to a given mode of χ would define the corresponding oscillator value fermionic oscillator operators at X^4.

   Also the generalizations of the variants of MDE restricted to sub-manifolds of X^4 make sense and are needed. The beauty is that there is no need to introduce spinor fields at lower-D surfaces as independent dynamical degrees of freedom. For instance, one only a variant of a modified Dirac action defined by Cherns-Simons analog of Kähler action makes sense at light-like partonic orbits so that one has an analog of a topological quantum field theory (TQFT).

How to avoid normal ordering divergences from fermionic oscillator operators?

The normal ordering divergences due to the anti-commutators of fermionic fields at the same point are really serious since induce spinor fields of 8-D $H = M^4 \times CP_2$ so that normal ordering singularities are proportional to 7-D delta function $\delta^7(0)$. They are encountered already for the ordinary MDA giving rise to bosonic SCA charges as Noether charges, which also are plagued by these divergences. Normal ordering for the oscillators in the Noether charges associated with MDA would allow to get rid of the divergences but is a mere trick. The proposal considered in [L83] is to make MDA bi-local at the space-time level.

Consider the general constraints on bi-locality coming from the cancellation of the normal ordering divergences.

1. Consider first 4-D variant of MDA. The most general option for MDA is that there is an integral over the entire X^4 for both Ψ and $\overline{\Psi}$ separately sothat one has 2 4-D integrations. One obtains potential normal ordering divergences proportional to $\delta^7(0)d^8x$. If one has two space-time sheets which in the generic case intersect transversally at discrete set of points, one obtains a vanishing result. However, the self-pairing of a given space-time sheet gives a divergence as a 4-D volume integral of $\delta^3(0)$. The definition of the self-pairing as a limit of separate space-time sheets approaching each other to get rid of the divergences looks like a trick.

   This suggests that the pairing can occur only between disjoint space-time regions, most naturally space-time sheets. For instance, parallel space-time sheets with overlapping M^4 projections. Allowing pairing only between disjoint regions eliminates also the divergences associated with the bosonic Noether charges deduced from MDA and involving 3+4-D integral instead of 3-D integral.

What could be the precise definition for these disjoint regions? M^8-H duality suggests that they correspond to different roots of the octonionic polynomial defined by real polynomials. When 2 roots coincide, one obtains a term of type $\delta^7(0)d^7x$ giving a finite result. What if the number of coinciding roots is higher than 2? This case will be discussed later in number theoretic context.

What about space-like regions, in particular the wormhole contacts expected to be small deformations of a warped embedding of CP_2 having light-like M^4 projection but having same Kähler metric and Kähler form as CP_2 [K5]? There is no pairing with a parallel space-time sheet now. It seems that the pairing must be between different wormhole contacts. This pairing could be essential for the understanding of string like entities as paired wormhole contacts providing a model for elementary particles.

2. For the bilinear MDA, the variation of the 4+4-D modified Dirac action with respect to $\overline{\Psi}$ and Ψ yields both the modified Dirac equation $D\Psi = 0$ plus the field equations for the preferred extremal. This gives the modes of the induced spinor field. In the standard picture the hermiticity condition for the Dirac action yields the same outcome and has interpretation as a supersymmetry between classical and fermionic degrees of freedom.

3. Both the phenomenological picture developed during years and M^8-H duality strongly suggest that spinors can be restricted also to lower-D surfaces. For the lower-D variants of MDA the normal ordering divergences appear already for tranversal intersections. For instance, for 3-D variant of MDA one has $\delta^7(0)d^7x$ type divergences. The only possible manner to avoid them is to require that paired regions are disjoint. For the 3-D Chern-Simons-Kähler action associated with the light-like partonic orbits the paired space-time sheets are very naturally the opposite wormhole throats so that fermions and antifermions would reside at opposite wormhole throats.

   Physical picture also suggests the assignment of actions to 2-D string world sheets and 1-D light-boundaries defining their intersections with partonic orbits.

4. Also 6-D brane-like solutions having the topology of S^6 and $t = r_n$ hyper-plane as intersection with M^4 are of physical interest. Different 4-D space-time surfaces could be glued together along 3-surfaces or 2-D partonic 2-surfaces at S^6. Arguments similar to those already discussed exclude the pairing of various objects with these 6-branes as also their self-pairing.

   Also M^4 and CP_2 define special solutions to the algebraic equations in M^8. MDA reduces to ordinary massless Dirac equation in M^4. In the case of CP_2 one has a massless Dirac equation in CP_2 and only the right-handed neutrino ν_R is possible as a solution. If only quarks are allowed, this solution is excluded. What happens for the deformations of CP_2? Could it be that quarks cannot reside inside wormhole contacts as 4-D entities? Or could one allow solutions of $D\Psi = 0$ as analytic functions of CP_2 coordinates finite in the region in which they are defined - wormhole contact does not span the entire CP_2?

Cognitive representations provide additional insights to the problem of normal ordering divergences, and it could be even argued that they are the only possible manner to define scattering amplitudes as a sequences of improving approximation natural in the approach based on hyper-finite factors of type II_1 (HFFs).

1. For a given extension of rationals determined by the polynomial defining the space-time region in M^8, the space-time surfaces inside CD are replaced with their discretizations consisting of points of M^8 in the extension considered. This surface and cognitive representation are mapped to H by M^8-H correspondence [L65]. For cognitive representations one can perform discretization by replacing the integrals defining SCA generators with discrete sums over points of the cognitive representation. This replacement is very natural since in the p-adic context the counterpart of the Riemann integral does not exist.

2. The Galois group of extension serves as a symmetry group and one can form analogs of group algebra elements - wave functions in discrete Galois group - acting on the cognitive representation and giving rise to discrete representation of quantum states. This state space has as its dimension the dimension n of the Galois group which for Galois extensions coincides

with the dimension of extension [L39, L87]. This group algebra-like structure can be given Kähler metric and also spinor structure and this spinor structure could discretize the spinor structure of WCW if gamma matrices are identified as fermionic oscillator operators.

3. Also now one can avoid divergences if the paired space-time regions, say space-time sheets, in MDA are disjoint. It can however happen that n separate points at the orbit of the Galois group approach each other and coincide: this would correspond to the touching of space-time sheets meaning coinciding roots of the octonionic polynomial. In this situation a subgroup of the Galois group would leave the intersection point invariant.

   The possible normal ordering divergence comes from different pairs of the m points, which coincide. In 4-D case, the situation corresponds to transversal space-time sheets so that the divergence vanishes. For lower-dimensional surfaces, say partonic orbits, the intersections do not occur in the generic situation but if they occur, the divergence is multiplied by a sum over the values of wave function at coinciding branches and vanishes if the representation is *non-singlet*. It would thus seem that the non-singlet character of Galois representations must be posed as an additional condition.

4. This cancellation mechanism works even without discretization since the notions of Galois group and its representations make sense for arbitrary polynomial surfaces without a restriction to rational or algebraic polynomial coefficients so that the cancellation occurs for non-singlet representations when the space-time sheets intersect.

Are fermions 4-D in H but 3-D in M^8?

M^8-H duality suggests the restriction of the induced spinor fields to light-like 3-surfaces having 2-D partonic surfaces as ends. M^8-H duality reduces space-time surfaces in M^8 to algebraic surfaces defined by polynomials of real variable. The coefficients can be complex. Concerning p-adicization real rationals defines the most attractive option. This leads to a picture in which a hierarchy of extensions of rationals defines evolutionary and cognitive hierarchies. The extensions provide cognitive representations as unique discretizations of the X^4 with embedding space coordinates in extension of rationals and the one can formulate quantum TGD in finite measurement resolution at least using these representations.

The fermionic variant of M^8-H duality [L84] leads to the conclusion that spinor modes in M^8 are restricted at 3-D light-like surfaces obeying an algebraic equations analogs to the momentum space variant of massless Dirac equation. Are H fermions also always restricted to the 3-D light-like orbits of the partonic 2-surfaces at which the signature of the induced metric changes?

On the other hand, the picture deduced at the level of H from the cancellation of the normal ordering divergences allows 4-D fermions, and also implies field equations for X^4 itself. Can one say that free fermions can reside in 4-D space-time but reside only at the 3-D mass shell in momentum space. M^8-H duality would be analogous to the duality between space-time and momentum space descriptions of particles.

Even more, string world sheets have light-like boundaries at the parton orbits. Also fermions in H would be naturally located at string boundaries and behave like point-like particles. One would obtain a picture resembling that provided by twistor Grassmannian approach. Also the cancellation of normal ordering divergences supports this picture and leads to a detailed form of bi-linear modified Dirac action. Also strong form of holography (SH) stating that 2-surfaces carry all information needed to construct the X^4 supports this view. This is actually the same as the phenomenological picture that has been applied.

M^8-H duality predicts also "very special moments in the life self" to have as correlates 6-branes with M^4 time defining in M^8 octonionic real axis (unique rest system) having as values roots of the polynomial defining the space-time surfaces. These surfaces should contain the partonic 2-surfaces defining the reaction vertices. If there is a non-determinism associated with these surfaces it should preserve classical charges and also SSA charge.

Is the proposed counterpart of QFT supersymmetry only an approximate symmetry?

The proposal for the cancellation of the normal ordering divergences allows overviewing leptons as three quark composites with 3 quarks at the same wormhole throat. This option is strongly suggested by the conceptual economy since quarks are enough for WCW spinor structure.

An interesting question is whether TGD allows a counterpart of QFT supersymmetry (SUSY). This was proposed in [L69]. The idea was that both embedding space coordinates and spinors can be expanded as polynomials in the local compositesof quark and antiquark oscillator operators - rather than anticommuting hermitian theta parameters leading to problems with fermion number conservation - with a well-defined quark number.

The proposal was that leptons are purely local 3-quark-composite analogous to a superpartner of quark: note however that quark superspinor would have quark number one so that precise spartner interpretation fails. This option and only its slightly local variant is possible only for the TGD view about color as angular momentum rather than spin-like quantum number.

This proposal was based on discrete cognitive representations as unique discretizations of the X^4 and on the crucial assumption that fermionic oscillator operators obey Kronecker delta type anticommutations rather than the 8-D anticommutations giving $\delta^7(0)$ anti-commutator singularities for the induced second quantized quark field in H. Can the notion of super-field based on local composites of quarks and antiquarks with a definite fermion number avoid normal ordering divergences for the induced anticommutation relations? One can of course think of a normal ordering of monomials but one expects problems with vertices.

This suggests that the super coordinates of H and superspinors can be only approximate notions. Superfield components would correspond to states with a fixed quark number but quarks and antiquarks would reside at opposite wormhole throats rather than forming exactly local composites. Since the throat is expected to have CP_2 size, these states would be for all practical purposes strictly local composites.

7.4.2 Kähler metric as the analog of S-matrix

Kähler metric defines a complex inner product. Complex inner products also define scattering amplitudes. Usually metric is regarded as defining length and angle measurement. Could the Kähler metric define unitary S-matrix? Under simple additional conditions this is true!

The analogs of unitarity conditions

The following little arguments show that given Kähler metric defines an analog of unitary S-matrix giving rise positive transition probabilities, and under additional conditions also a unitary S-matrix between states with quantum numbers labeling basis of complex vectors or of complexified gamma matrices. This defines an S-matrix like entity and under some additional conditions even an unitary S-matrix.

1. The defining conditions for unitary S-matrix and Kähler metric are very similar. S and $S^\dagger$ would correspond to the covariant metric $g_{m\overline{n}}$ and contravariant metric $g_{\overline{m}n}$. Unitary for S-matrix corresponds to the conditions

$$S_{mr}S^\dagger_{rn} = S_{mr}S_{nr} = \delta_{m,n} \quad .$$

(there is summation over repeated indices). The rows of S-matrix are orthonormalized. The definition of the contravariant metriccorresponds the conditions

$$g_{m\overline{r}}g^{\overline{r}n} = \delta_{m,n} \quad .$$

The complex rows of metric tensor and contravariant metric are orthonormalized also now and rows are orthonormal

2. For S-matrix the probabilities are given by $p_{mn} = S_{mn}S^\dagger_{nm} = S_{mn}S_{mn}*$ and are real and non-negative and their sum is equal to one. Also for the Kähler metric the complex analogs of probabilities defined by

$$p^c_{mn} = g_{m\overline{r}}g^{\overline{r}n}$$

sum up to unity. Hence the real parts $Re(p^c_{mn})$ of p^c_{mn} sum up to unity whereas the imaginary parts sum up to zero.

3. p^c_{mn} are not however automatically real and non-negative and it is not clear how to interpret complex or even real but negative probabilities physically. One can however pose the positivity of the real parts of p^c_{mn} as an additional condition on the phase factors $U_{m\overline{n}} = exp(\Phi_{m\overline{n}})$ and $V_{m\overline{n}} = exp(\Psi_{m\overline{n}})$ associated with $g_{m\overline{n}} = R_{mn}U_{m\overline{n}}$ and $g^{\overline{n}m} = S_{nm}V_{\overline{n}m}$. The condition for positivity is

$$U_{m\overline{n}}V_{\overline{n}m} = cos(\Phi_{\overline{n}m} - \Psi_{\overline{n}m} \geq 0$$

and is rather mild requiring the angle difference to be in the range $(-\pi/2, \pi/2)$. This is true of the angles are in the range $(\pi/4, \pi/4)$. The condition $Re(p^c_{mn}) \geq 0$ is equivalent with the condition $Im(ip^c_{mn}) \geq 0$, and characterizes the coefficients of Teichmueller matrices [A43, A64, A52] [K18]: the meaning of this condition will be discussed below.

4. Under what conditions p^c_{mn} reduce to non-negative real numbers? One can express the probabilities as $p_{mn} = g_{m\overline{n}} \times cof(g_{m\overline{n}})/det(g)$. Note that $Z = det(g)$ is constant depending only on the point of the Kähler manifold. One can express g_{mn} as $g_{m\overline{n}} = A_{mn}U_{m\overline{n}}$ and $cof(g_{m\overline{n}})$ as $cof(g_{m\overline{n}}) = B_{mn}V_{m\overline{n}}$. The reality condition implies

$$U_{m\overline{n}} = \overline{V_{m\overline{n}}} \quad .$$

The phases of $g_{m\overline{n}}$ and $cof(g_{m\overline{n}})$ are opposite.

This gives additional conditions. Kähler metric involves $N_{tot} = 2N^2$ real parameters There are $(N^2-N)/2$ elements in say upper diagonal and by hermiticity they are complex conjugates of the lower diagonal. This is the number N_{cond} of conditions coming from the reality. There is also one additional condition due to the fact that the probabilities do not depend on the normalization of g. The total number of real parameters is

$$N_{param} = N_{tot} - N_{cond} - 1 = N(N-1) - 1 \quad .$$

For instance, for $N \in \{2,3,4\}$ one has $N_{param} =\in \{1,5,11\}$. Unitary matrix allows $N_{unit} = N^2$ real parameters and the ratio $N_{param}/N_{unit} = (N(N-1)-1)/N^2$ approaches unity for large values of N. Note that a unitary matrix with real diagonals has $N^2 - N$ parameters so that the number of parameters is the same as for a hermitian metric with unit determinant.

5. Could one transform the metric defining non-negative probabilities to a unitary matrix by a suitable scaling? One can indeed define a matrix S as a matrix $S_{mn} = \sqrt{A_{mn}B_{mn}/Z}U_{mn}$. One has $S_{mn}S_{mn}* = A_{mn}B_{mn}/Z$ given also by the product of $g_{m\overline{n}}g^{\overline{n}m}$ so that the probabilities are the same. The unitarity conditions reduce to $g_{m\overline{r}}g^{\overline{r}n} = \delta^n_m$.

   In infinite dimensional case problems might be produced by the appearance of the square root of determinant expected to be infinite. However, also the cofactors are expected to diverge, and one can express them as partial derivatives of the metric determinant with respect to the corresponding element of the metric. This is expected to give a finite value for the elements of the contravariant metric. Note that the ratios of the probabilities do not depend on the metric determinant.

Can one distinguish between the descriptions based on Kähler metric and S-matrix?

For the Teichmueller option the proposed analog for S-matrix involves imaginary part. Does it have some physically observable consequences?

Could one imagine a physical situation allowing to test whether the S-matrix description or its TGD variant is nearer to truth? One can indeed imagine an analog of a Markov process characterized by a matrix p of transition probabilities p_{mn} at a given step. For a two-step process the transition matrix would be p^2_{mn}.

In the TGD context one would have $p_{mn} = Re(p^c_{mn})$. What happens in a two-step process? Should one use use p^2_{mn} or $Re((p^c)^2)_{mn} = Re((p^c)2)_{mn} - Im(p^c)^2_{mn}$? If both options are possible, what could distinguish physically between them?

Could the correct interpretation be that p^2_{mn} describes the process when the outcome is measured in both steps, and $Re((p^c)^2)_{mn}$ the process in which only initial and final states are measured? This picture would generalize to n-step processes and predict a deviations from the ordinary Markov process and perhaps allow to compare the predictions of the TGD view and standard view and deduce $Im(p^c)$.

S-matrix and its Hermitian conjugate correspond in standard physics to situations related by CPT symmetry defined as the product of charge conjugation C, spatial reflection P and time reversal T. The transition probabilities would remain invariant in this transformation although transition amplitudes are replaced with their complex conjugates.

What happens to CPT in TGD framework? In TGD framework CPT induces a hermitian conjugation $g_{m\overline{n}} \rightarrow g_{\overline{n}m} =$

7.5 The role of fermions

In this section the role of fermions (quarks as it seems) is discussed in more detail. In particular, the conditions on the scattering amplitudes from the cancellation of normal ordering divergences and co-associative octonionic spinors at the level of M^8 are discussed. Also the formulation of scattering amplitudes the level of M^8 is briefly considered.

7.5.1 Some observations about Feynman propagator for fundamental quark field

In the sequel the divergence cancellation mechanism and the properties of Dirac propagator are discussed in detail. The surprise is that the massive propagators with CP_2 mass scale reduce essentially to massless propagators for light-like separations. This allows understanding of why quarks can give rise to light elementary particles.

The second quantized free quark field Ψ in H defines fundamental fermions appearing as a building brick of elementary particles. The Feynman propagator for Ψ appears in the analogs of Feynman diagrams. Apart from the right handed neutrino (present only as a 3 quark composite at partonic 2-surface if only quarks are involved) the modes of Ψ are extremely massive. Elementary particles are light. How can one understand this?

In p-adic thermodynamics the generation of small mass was assumed to involve a generation of a negative, "tachyonic", ground state conformal weight encountered also in string models. $M^8 - H$ correspondence allows a more sophisticated description based on the choice of $M^4 \subset M^8$ mapped to $M^4 \subset H$. By 8-D Lorentz invariance the 4-D mass squared of ground state massless in 8-D sense, depends on the choice of $M^4 \subset H$, and with a proper re-choice of M^4 the particle having large M^4 mass becomes massless.

The action of the generators of super-conformal algebra creates states with a well-defined conformal weight, which are massless for a proper choice of $M^4 \subset M^8$. In p-adic thermodynamics the choice of $M^4 \subset M^8$ would correspond to a generation of negative ground state conformal weight.

The states can however mix slightly with states having higher value of conformal weight, and since one cannot choose M^4 separately for these states, a small mass is generated and described by p-adic thermodynamics. The classical space-time correlate for the almost masslessness is minimal surface property, which provides a non-linear geometrization for massless fields as surfaces. The non-linearity at the classical level leads to a generation of small mass in 4-D sense for which p-adic thermodynamics provides a model.

The propagators for the fundamental quarks in H correspond to CP_2 mass scale. Can this be consistent with the proposed picture? The following simple observations about the properties of predicted fermion propagator and anticommutator for the induced spinor fields lead to a result, which was a surprise to me. The propagators and anti-commutators of massive quarks at light cone boundary are in excellent approximation massless for light-like distances. This makes it possible to understand why elementary fermions are light.

This mechanism does not work in QFT defined in M^4 since inverse propagator is $\gamma^k p_k + m$ so that M^4 chiralities mix for massive states. In TGD picture H-chirality is fixed by 8-D masslessnes and the product of M^4 and CP_2 chiralities for spinors equals to the H chirality. The inverse progator is proportional to the operator $p^k\gamma + D_{CP_2}$, where D_{CP2} is CP_2 part of Dirac operator.

General form of the Dirac propagator in H

Second quantized quark field Ψ restricted to the space-time surface determines the Feynman propagator fundamental quark. The propagator can be expressed as a sum of left- and right-handed propagators as

$$S_F = S_{F,L} + S_{F,R} = D_L G_{F,L} + D_R G_{F,R} \ .$$

Here D_L and D_R are the left- and right-handed parts of a massless (in 8-D sense) Dirac operator D in H involving couplings to CP_2 spinor connection depending on CP_2 chirality in accordance with electroweak parity breaking. $G_{F,L}$ *resp.* $G_{F,R}$ is the propagator for a massless (in 8-D sense) scalar Laplacian in H coupling to the spinor connection assignable to left *resp.* right handed modes. G_F can be expressed by generalizing the formula from 4-D case

$$G_{F,I} \sum_n \int d^4p \frac{1}{p^2 - M_{n,I}^2} exp(ip\cdot(m_1 - m_2))\Phi^*_{n,I}(s_1)\Phi_{n,I}(s_2) \ .$$

Here one has $I \in \{L, R\}$ and the mass spectra are different for these modes. Here m_i denote points of M^4 and s_i points of CP_2. n, I, $I \in \{L, R\}$, labels the modes $\Phi_{n,I}$ of a scalar field in CP_2 associated with right and left handed modes having mass squared $M_{n,R}$. Since H-chirality is fixed to be quark chirality, there is a correlation between M^4 - and CP_2 chiralities. Apart from ν_R all modes are massive (ν_R is need not be present as a fundamental fermion) and the mass M_n, which is of order CP_2 mass about 10^{-4} Planck masses, is determined by the CP_2 length scale and depends on CP_2 chirality.

$G_{F,I}$ reduces to a superposition over massive propagators with mass $M_{n,I}$:

$$G_{F,I} = \sum_n G_F(m_1 - m_2|M_n)\Phi^*_{n,I}(s_1)\Phi_{n,I}(s_2)P_I \ .$$

Here P_I, $I \in \{L, R\}$ is a projector to the left/right handed spinors. One can express $S_{F,I}$ as a sum of the free M^4 part and interaction term proportional to the left - or right-handed part of CP_2 spinor connection:

$$S_{F,I} = D(M^4)G_{F,I} + A_I G_{F,I} \ .$$

A_I, $I \inf\{L, R\}$ acts either on s_1 or s_2 but the outcome should be the same. The first term gives sum over terms proportional to massive free Dirac propagator in M^4 allowing to get a good idea about the behavior of the propagator.

About the behavior of the quark propagator

The quark propagator reduces to left- and right-handed contributions corresponding to various mass values $M_{n,I}$. To get view about the behaviour of the quark propagator it is useful to study the behavior of $G_F(x, y|M)$ for a given mass as well as the behaviors of free and interacting parts of S_F its free part

From the explicit expression of $G_F(m_1 - m_2|M_n)$ one can deduce the behavior of the corresponding contribution to the Feynman propagator. Only ν_R could give a massless contribution to the progator. Explicit formula for G_F can be found from Wikipedia [A9] (`https://en.wikipedia.org/wiki/Propagator#Feynman_propagator`):

$$G_F(x, y|m) = \begin{cases} -\frac{1}{4\pi}\delta(s) + \frac{m}{8\pi\sqrt{s}} H_1^{(1)}(m\sqrt{s}), & s \geq 0 \\ -\frac{im}{4\pi^2\sqrt{-s}} K_1^{(1)}(m\sqrt{-s}), & s \leq 0 \ . \end{cases}$$

Here $H_1^{(1)}(x)$ is Hankel function of first kind and $K_1^{(1)}$ is modified Bessel function [A1](`https://en.wikipedia.org/wiki/Bessel_function`). Note that for massless case the Hankel term vanishes.

Consider first Hankel function.

1. Hankel function $H^{(1)}_{\alpha}(x)$ [A1, A9] obeys the defining formula

$$H^{(1)}_{\alpha}(x) = \frac{J_{-\alpha}(x) - exp(i\alpha\pi i)J_{\alpha}(x)}{isin(\alpha\pi)} \quad .$$

For integer values of α one has $J_{-n}(x) = (-1)^n J_n(x)$ so that $\alpha = n$ case gives formally 0/0 and the limit must be obtained using Hospital's rule.

2. Hankel function $H^{(1)}_1(x)$ can be expressed as sum of Bessel functions of first and second kind

$$H^{(1)}_1(x) = J_1(x) + iY_1(x) \quad .$$

J_1 vanishes at origin whereas Y_1 diverges like $1/x$ at origin.

3. The behaviors of Bessel functions and their variants near origin and asymptotically are easy to understand by utilizing Schrödinger equation inside a cylinder as a physical analogy. The asymptotic behaviour of Hankel function for large values of x is

$$H^{(1)}_{\alpha}(x) = \frac{2}{\pi x} exp(i(x - 3\pi/4)) \ ,$$

4. The asymptotic behavior of Hankel function implies that the massive Feynman propagator an oscillatory behavior as a function of $m\sqrt{s}$. Modulus decreases like $1/\sqrt{m\sqrt{s}}$. The asymptotic behavior for the real and imaginary parts corresponds to that for Bessel functions of first kind (J_1) and second kind (Y_1). At origin $H^{(1)}_{\alpha}(x)$ diverges like $Y_1(x) \sim \frac{(x/2)^{-n}}{\pi}$ near origin. For large values of x $K_1(x)$ decreases exponentially like $exp(-x)\frac{\sqrt{\pi}}{2x}$. At origin $K_1(x)$ diverges.

5. In the recent case the quark propagator would oscillate extremely rapidly leaving only the $\delta(s)$ part so that the propagator behaves like massless propagator!

   The localization of quarks to the partonic surfaces with a size scale of CP_2 radius implies that that the oscillation does not lead to a vanishing of the Hankel contribution to the scattering amplitudes. For induced spinor fields in the interior of space-time surfaces destructive interference is however expected to occur so that behavior is like that for a massless particle. This should explain why the observed particles are light although the fundamental fermions are extremely massive. The classical propagation would be essentially along light-like rays.

   The long range correlations between quarks would come from the $\delta(s)$ part of the propagator, and would not depend on quark mass so that it would effectively behave like a massless particle. Also the action of Dirac operator on $G_F(x, y)$ in M^4 degrees of freedom is that of a massless Dirac propagator coupling to induced gauge potentials. The quarks inside hadrons and also elementary particles associated with the wormhole throats of flux tubes could be understood as quarks at different partonic 2-surfaces at the boundary of CD having light-like distance in an excellent approximation.

6. The above argument is for the Feynman propagator but should generalize also for anti-commutator. The anticommutator for Dirac operator D in M^4 can be expressed as $D\Delta(x, y)$, where D is a scalar field propagator.

$$\Delta(x, y|m) \propto \left\{ \begin{array}{ll} \frac{m}{8\pi\sqrt{s}} H^{(1)}_1(m\sqrt{s}), & s \geq 0 \\ -\frac{m}{\sqrt{-s}} K^{(1)}_1(m\sqrt{-s}), & s \leq 0 \end{array} \right. .$$

   Apart from possible proportionality constants the behavior is very similar to that for Feynman propagator except that the crucial $\delta(s)$ term making possible effectively massless propagation is absent. At light-cone boundary however $\sqrt{s}$ is zero along light rays, and this gives long range correlations between fermions at different partonic 2-surfaces intersected by light rays from the origin. Hence one could have a non-vanishing Hermitian inner product for 3-D states at boundaries of CD.

   Rather remarkably, these results provide a justification for twistor-diagrams identified as polygons consisting of light-like segments.

Possible normal ordering divergences

Concerning the cancellation of normal ordering divergences the singularities of the propagators G_F are crucial. The bi-linearity of the modified Dirac action forcing anticommuting quark and antiquark oscillator operators at different throats of wormhole contacts but this need not guarantee the absence of the divergence since the free quark propagator in M^4 contains mass independent $\delta(s)$ part plus the divergent part from Hankel function behaving like $1/\sqrt{sm}$. For the massless propagator assignable to ν_R the propagator would reduce to M^4 propagator and only the $\delta(s)$ would contribute.

$s = 0$ condition tells that the distance between fermion and anti-fermion is light-like and is possible to satisfy at the light-like boundary of CD. Paired quark and antiquark at the wormhole throats must reside at the same light-like radial ray from the tip of cd (cd corresponds to causal diamond in M^4). Since partonic surfaces are 2-D this condition selects discrete pairs of points at the pair of the partonic surfaces. The integration over the position of the end of the propagator line over paired partonic 2-surfaces should smooth out the divergences and yield a finite result. This would be crucial for having an inner product for states at the boundary of the light-cone.

This applies also to the point pairs at opposite throats of wormhole contact. Time-ordered product vanishing for $t_1 = t_2$ so that the points must have different values of t and this is possible. The two 2-D integrations are expected to smooth out the singularities and eliminate divergences also now.

7.6 Conclusions

TGD predicts revolution in quantum theory based on three new principles.

1. ZEO solving the basic paradox of quantum measurement theory. Ordinary ("big") state function reduction involves time reversal forcing a generalization of thermodynamics and leading to a theory of quantum self-organization and self-organized quantum criticality (homeostasis in living matter).

2. Phases of ordinary matter labelled by effective Planck constant $h_{eff} = nh_0$ identified as dark matter and explaining the coherence of living matter in terms of dark matter at magnetic body serving as a master, and predicting quantum coherence in all scales at the level of magnetic bodies. $h_{eff}/h_0 = n$ has interpretation as the dimension for an extension of rationals and is a measure of algebraic complexity. Evolution corresponds to the increase of n.

   Extensions of rationals are associated with adelic physics providing description of sensory experience in terms of real physics and of cognition in terms of p-adic physics. Central notion is cognition representation providing unique discretization of X^4 in terms of points with embedding space coordinates in the extension of rationals considered $M^8 - H$ duality realizes the hierarchy of rational extensions and assigns them to polynomials defining space-time regions at the level of M^8 and mapped to minimal surfaces in H by $M8 - H$ duality.

3. The replacement of the unitary S-matrix with the Kähler metric of the Kähler space defined by WCW spinor fields satisfying the analog of unitarity and predicting positive definite transition probabilities defining matrix in Teichmueller space. Einstein's geometrization of classical physics extends to the level of state space, Equivalence Principle generalizes, and interactions are coded by the geometry of the state space rather than by an *ad hoc* unitary matrix. Kähler geometry for the spinor bundle of WCW has Riemann connection only for a maximal group of isometries identified as super-symplectic transformations (SS). This makes the theory unique and leads to explicit analogs of Feynman rules and to a proof that theory is free of divergences.

In this work the third principle, which is new, is formulated and some of its consequences are discussed. The detailed formulation allows understanding of how normal ordering divergences and other divergences cancel. The key idea is to induce the second quantized free spinor field from H to space-time surface. This determines the propagators at the space-time level. The condition that creation and annihilation operators are at different space-time sheets - say at throats of wormhole contacts is enough. An alternative but not exclusive option suggested by ZEO is that the annihilation operators correspond to creation operators for conjugated Dirac vacuum associated with the opposite half-cone of CD or sub-CD.

A further observation is that the Dirac propagators for particles reduce in a good approximation to massless propagators when the propagation takes place along light-like distances: this provides a considerable insight to why physical particles are so light although the spinor harmonics for CP_2 correspond to CP_2 mass scale.

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Chapter 8

Breakthrough in understanding of $M^8 - H$ duality

8.1 Introduction

$M^8 - H$ duality [L65, L63, L64, L84] has become a cornerstone of quantum TGD but several aspects of this duality are still poorly understood.

8.1.1 Development of the idea about $M^8 - H$ duality

A brief summary about the development of the idea is in order.

1. The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent/normal space and contains a complex co-complex surface, it can be mapped to a 4-surface in $M^4 \times CP_2$.

2. Later emerged the idea that octonionic analyticity realized in terms of a real polynomials P algebraically continued to polynomials of complexified octonion might realize the dream [L43, L44, L45]. The original idea was that the vanishing condition for the real/imaginary part of P in quaternion sense could give rise to co-assocative/associative sense.

   $M^8 - H$ duality concretizes number theoretic vision [L48, L47] summarized as adelic physics fusing ordinary real number based physics for the correlates of sensory experience and various p-adic physics ($p = 2, 3, ...$) as physics for the correlates of cognition. The polynomials of real variable restricted to be rational valued defines an extension or rationals via the roots of the polynomials and one obtains an evolutionary hierachy associated with these extensions increasing in algebraic complexity. These extensions induce extensions of p-adic numbers and the points of space-time surface in M^8 with coordinates in the extension of rationals define cognitive representations as unique discretizations of the space-time surface.

3. The realization of the general coordinate invariance in TGD framework [K39, K21, K73, L89] [L86] motivated the idea that strong form of holography (SH) in H could allow realizing $M^8 - H$ duality by assuming associativity/co-associativity conditions only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits at which the signature of the induced metric changes from Minkowskian to Euclidian.

8.1.2 Critical re-examination of the notion

In this article $M^8 - H$ duality is reconsidered critically.

1. The healthy cold shower was the learning that quaternion (associative) sub-spaces of quaternionic spaces are geodesic manifolds [A46]. The distributions of quaternionic normal spaces are however always integrable. Hence, co-associativity remains the only interesting option.

Also the existence of co-commutative sub-manifolds of space-time surface demanding the existence of a 2-D integrable sub-distribution of subspaces is possible. This learning experience motivated a critical examination of the $M^8 - H$ duality hypothesis.

2. The basic objection is that for the conjectured associative option, one must assign to each state of motion of a particle its own octonionic structure since the time axis would correspond to the octonionic real axis. It was however clear from the beginning that there is an infinite number of different 4-D planes O_c in which the number theoretical complex valued octonion inner product reduces to real - the number theoretic counterpart for Riemann metric. In the co-associative case this is the only option. Also the Minkowski signature for the real projection turns out to be the only physically acceptable option. The mistake was to assume that Euclidian regions are co-associative and Minkowskian regions associative: both must be co-associative.

3. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [L43, L44, L45] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [L63, L68].

4. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X = 0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.

   The expectation was wrong! The equations $X = 0$ and $Y = 0$ involve the same(!) complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X = 0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.

   There are solutions $X = 0$ and $Y = 0$ only if the two polynomials considered have a common a_c^2 as a root! Also now the solutions are complex mass shells H^7_c.

5. How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .

   A co-associative X^4_c is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines $M^8 - H$ duality even if the co-associativity in geometric sense were not satisfied.

   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

It this, still speculative, picture is correct, it would fulfil the original dream about solving classical TGD exactly in terms of roots for real/imaginary parts of octonionic polynomials in M^8 and by mapping the resulting space-time surfaces to H by $M^8 - H$ duality. In particular, strong form of holography (SH) would not be needed at the level of H, and would be replaced with a dramatically stronger number theoretic holography.

Octonionic Dirac equation, which is purely algebraic equation and the counterpart for ordinary Dirac equation in momentum space, serves as a second source of information.

1. The first implication is that O_c has interpretation as an analog of momentum space for quarks: this has profound implications concerning the interpretation. The space-time surface in M^8 would be analog of Fermi ball. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.

2. Cognitive representations are defined by points of M^8 with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta of quarks. In the generic case the cognitive representations are finite. If the points of M^8 correspond to quark momenta, momentum conservation is therefore expected to make the scattering trivial.

   However, a dramatic implication of the reduction of the co-associativity conditions to the vanishing of ordinary polynomials Y is that by the Lorentz invariance of roots of P, the 3-D mass shells ofd M^4 have an infinite number of points in a cognitive representation defined by points with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta. This is what makes interesting scattering amplitudes for massive quarks possible.

3. What is the situation for the images of M^4 points under the effective local CP_2 element defined by local $SU(3)$ element g preserving the mass squared and mapping H^3 to $g(H^3)$? If g is expressible in terms of rational functions with rational coefficients, algebraic points are mapped to algebraic points. This is true also in the interior of M^4.

   This would mean a kind of cognitive explosion for massive quark momenta. Without the symmetry one might have only forward scattering in the interior of X^4_r. Note that massless quarks can however arrive at the boundary of CD which also allows cognitive representation with an infinite number of points.

4. In the number theoretic approach, kinematics becomes a highly non-trivial part of the scattering. The physically allowed momenta would naturally correspond to algebraic integers in the extension E of rationals defined by P. Momentum conservation and on-mass-shell conditions together with the condition that momenta are algebraic integers in E are rather strong. The construction of Pythagorean squared generalize to the case of quaternions provides a general solutions to the conditions: the solutions to the conditions are combinations of momenta which correspond to squares of quaternions having algebraic integers as components.

5. The original proposal was that local $G_{2,c}$ element $g(x)$ defines a vanishing holomorphic gauge field and its restriction to string world sheet or partonic 2-surface defines conserved current. $M^8 - H$ duality however requires that local $SU(3)$ element with the property that image point is invariant under $U(2)$ is required by $M^8 - H$ duality defines $X^4 \subset M^8$.

   In any case, these properties suggest a Yangian symmetry assignable to string world sheets and partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The generators of the Yangian algebra have a representation as Hamiltonians which are in involution. They define conserved charges at the orbits for a Hamiltonian evolution defined by any combination of these the Hamiltonians. ZEO suggests a concrete representation of this algebra in terms of quark and antiquark oscillator operators. This algebra extends also to super-algebra. The co-product of the associated Yangian would give rise to zero energy states defining as such the scattering amplitudes.

8.1.3 Octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of the ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = qq = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of $p\,p$ are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

Before continuing, I must apologize for the still fuzzy organization of the material related to $M^8 - H$ duality. The understanding of its details has been a long and tedious process, which still continues, and there are unavoidably inaccuracies and even logical inconsistencies caused by the presence of archeological layers present.

8.2 The situation before the cold shower

The view about $M^8 - H$ duality before the cold shower - leading to what I dare to call a breakthrough - helps to gain idea about the phenomenological side of $M^8 - H$ duality. Most of the phenomenology survives the transition to a more precise picture. This section is however not absolutely necessary for what follows it.

8.2.1 Can one deduce the partonic picture from $M^8 - H$ duality?

The M^8 counterparts for partons and their light like orbits in H can be understood in terms of octonionic Dirac equation in M^8 as an analog for the algebraic variant of ordinary Dirac equation at the level of momentum space [L84, L83] but what about the identification of partonic 2-surfaces as interaction vertices at which several partonic orbits meet? Can one deduce the phenomenological view about elementary particles as pairs of wormhole contacts connected by magnetic flux tubes from $M^8 - H$ duality? There is also the question whether partonic orbits correspond to their own sub-CDs as the fact that their rest systems correspond to different octonionic real axes suggests.

There are also some questions which have become obsolote. For instance: qhy should the partonic vertices reside at $t = r_n$ branes? This became obsolste with the realization that M^8 is analogous to momentum space so that the identification as real octonionic coordinate corresponds now to a component of 8-momentum identifiable as energy. Furthermore, the assumption the associativity of the 4-surface in M^8 had to be replaced with-co-associtivity and octonionic real coordinate does not have interpretation as time coordinate is associative surface

$M^8 - H$ duality indeed conforms with the phenomenological picture about scattering diagrams in terms of partonic orbits [L89, L88] [L88, L89] [L89], and leads to the view about elementary particles as pairs of Euclidian wormhole contacts associated with flux tubes carrying monopole flux.

8.2.2 What happens to the "very special moments in the life of self"?

The original title was "What happens at the "very special moments in the life of self?" but it turned out that "at" must be replaced with "to". The answer to the new question would be "They disappear from the glossary".

The notion of "very special moments in the life of self" (VPM) [L63, L68] makes sense if M^8 has interpretation as an 8-D space-time. M^4 projections of VPMs were originally identified as hyperplanes $t = r_n$, where t is time coordinate and r_n is a root of the real polynomial defining octonionic polynomial as its algebraic continuation.

The interpretation of M^8 as cotangent space of H was considered from the beginning but would suggest the interpretation of M^8 as the analog of momentum space. It is now clear that this interpretation is probably correct and that $M^8 - H$ duality generalizes the momentum-position duality of wave mechanics. Therefore one should speak of $E = r_n$ plane and simply forget the misleading term VMP. VPMs would correspond to constant values of the M^8 energy assignable to M^4 time coordinate.

The identification of space-time surface as co-associative surface with quaternionic normal space containing integrable distribution of 2-D commutative planes essential for $M^8 - H$ duality is also in conflict with the original interpretation. Also the modification of $M^8 - H$ duality in M^4

degrees of freedom forced by Uncertainty Principle [L99] has led to the conclusion that VMPs need not have a well-defined images in H.

8.2.3 What does SH mean and its it really needed?

SH has been assumed hitherto but what is its precise meaning?

1. Hitherto, SH at the level of H is believed to be needed: it assumes that partonic 2-surfaces and/or string world sheets serve as causal determinants determining X^4 via boundary conditions.

   (a) The normal or tangent space of X^4 at partonic 2-surfaces and possibly also at string world sheets has been assumed to be associative that is quaternionic. This condition should be true at the entire X^4.

   (b) Tangent or normal space has been assumed to contain preferred M^2 which could be replaced by an integrable distribution of $M^2(x) \subset M^4$. At string world sheets only the tangent space can be associative. At partonic 2-surfaces also normal space could be associative. This condition would be true only at string world sheets and partonic 2-surfaces so that only these can be mapped to H by $M^8 - H$ duality and continued to space-time surfaces as preferred extremals satisfying SH.

   The current work demonstrates that although SH could be used at the level of SH, this is not necessary. Co-associativity together with co-commutativity for string world sheets allows the mapping of the real space-time surfaces in M^8 to H implying exact solvability of the classical TGD.

8.2.4 Questions related to partonic 2-surfaces

There are several questions related to partonic 2-surfaces.

Q1: What are the M^8 pre-images of partons and their light-like partonic orbits in H?

It will be found that the octonionic Dirac equation in M^8 implies that octo-spinors are located to 3-D light-like surfaces Y^3_r - actually light-cone boundary and its 3-D analogs at which number theoretic norm squared is real and vanishes - or to the intersections of X^3_r with the 6-D roots of P in which case Dirac equation trivializes and massive states are allowed. They are mapped to H by $M^8 - H$ duality.

Remark: One can ask whether the same is true in H in the sense that modified Dirac action would be localized to 3-D light-like orbits and 3-D ends of the space-time surfaces at the light-like boundaries of CD having space-like induced metric. Modified Dirac action would be defined by Cherm-Simons term and would force the classical field equations for the bosonic Chern-Simons term. If the interior part of the modified Dirac action is absent, the bosonic action is needed to define the space-time surfaces as extremals. They would be minimal surfaces and universal by their holomorphy and would not depend on coupling parameters so that very general actions can have them as preferred extremals. This issue remains still open.

The naïve - and as it turned out, wrong - guess was that the images of the light-like surfaces should be light-like surfaces in H at the boundaries of Minkowskian and Euclidian regions (wormhole contacts). In the light-like case Y^3_r corresponds to the light-cone boundary so that this would be the case. X^3_r however turns out to correspond to a hyperboloid in M^4 as an analog of a mass shell and is not identifiable as a partonic orbit.

It turned out that the complex surface X^4_c allows real sections in the sense that the number theoretic complex valued metric defined as a complex continuation of Minkowski norm is real at 4-D surfaces: call them Z^4_r. They are bounded by a 3-D region at Z^3_r at which the value of norm squared vanishes. This surface is an excellent candidate for the pre-image of the light-like orbit of partonic 2-surface serving as a topological vertex. One has therefore strings worlds sheets, partonic 2-surfaces and their light-like orbits and they would connect the "mass shells" at X^4_r. All ingredients for SH would be present.

The intersections of Z^3_r with X^3_r identifiable as the section of X^4_r $a = constant$ hyperboloid would give rise to partonic 2-surfaces appearing as topological reaction vertices.

The assumption that the 4-D tangent space at these light-like 3-surfaces is co-associative, would give an additional condition determining the image of this surface in H, so that the boundary conditions for SH would become stronger. One would have boundary conditions at light-like partonic orbits. Note that string world sheets are assumed to have light-like boundaries at partonic orbits.

Q2: Why should partonic 2-surfaces appear as throats of wormhole contact in H? Wormhole contacts do not appear in M^8.

1. In M^8 light-like orbits are places where the Minkowskian signature changes to Euclidian. Does $M^8 - H$ duality map the images of these coinciding roots for Euclidian and Minkowskian branches to different throats of the wormhole contact in H so that the intersection would disappear?

2. This is indeed the case. The intersection of Euclidian and Minkowskian branches defines a single 3-surface but the tangent and normal spaces of branches are different. Therefore their H images under $M^8 - H$ duality for the partonic 2-surface are different since normal spaces correspond to different CP_2 coordinates. These images would correspond to the two throats of wormhole contact so that the H-image by SH is 2-sheeted. One would have wormhole contacts in H whereas in M^8 the wormhole contact would reduce to a single partonic 2-surface.

3. The wormhole contact in H can have only Euclidian signature of the induced metric. The reason is that the M^4 projections of the partonic surfaces in H are identical so that the points with same M^4 coordinates have different CP_2 coordinates and their distance is space-like.

Q3: In H picture the interpretation of space-time surfaces as analogs of Feynman graphs assumes that several partonic orbits intersect at partonic 2-surfaces. This assumption could be of course wrong.This raises questions.

What the pre-images of partonic 2-surfaces are in M^8? Why should several partonic orbits meet at a given partonic 2-surface? Is this needed at all?

The space-time surface X^4_r associated intersects the surface X^6_r associated with different particle - say with different value of mass along 2-D surface. Could this surface be identified as partonic 2-surface X^2_r? This occurs symmetrically so that one has a pair of 2-surfaces X^2_r. What does this mean? Could these surface map to the throats of wormhole contact in H?

Why several partonic surfaces would co-incide in topological reaction vertex at the level of H? At this moment is is not clear whether this is forced by M^8 picture.

Octonionic Dirac equation implies that M^8 has interpretation as analog of momentum space so that interaction vertices are replaced by multilocal vertices representing momenta and propagators become local being in this sense analogous to vertices of QFT. One could of course argue that without the gluing along ends there would be no interactions since the interactions in X^6_r for two 3-surfaces consist in the generic case of a discrete set of points. One could also ask whether the surfaces Y^3_r associated with the space-time surfaces X^4_r associated with incoming particles must intersect along partonic 2-surface rather than at discrete set of points.

The meeting along ends need not be true at the level of M^8 since the momentum space interpretation would imply that momenta do not differ much so that particles should have identical masses: for this to make sense one should assume that the exchanged virtual particles are massless. One other hand, if momenta are light-like for Y^3_r, this might be the case.

Q4: Why two wormhole contacts and monopole flux tubes connecting them at the level of H? Why monopole flux?

1. The tangent spaces of the light-like orbits have different light-like direction. Intuitively, this corresponds to different directions of light-like momenta. Momentum conservation requires

more than one partonic orbit changing its direction meeting at partonic 2-surface. By light-likeness, the minimum is 2 incoming and two outgoing lines giving a 4-vertex. This allows the basic vertices involving Ψ and $\overline{Psi}$ at opposite throats of wormhole contacts. Also a higher number of partonic orbits is possible.

2. A two-sheeted closed monopole flux tube having wormhole contacts as its "ends" is suggested by elementary particle phenomenology. Since M^8 homology is trivial, there is no monopole field in M^8. If $M^8 - H$ duality is continuous it maps homologically trivial partonic 2-surfaces to homologically trivial 2-surfaces in H. This allows the wormhole throats in H to have opposite homology charges. Since the throats cannot correspond to boundaries there must be second wormhole contact and closed flux tube.

3. What does the monopole flux for a partonic 2-surface mean at the level of M^8? The distribution of quaternionic 4-D tangent/normal planes containing preferred M^2 and associated with partonic 2-surface in M^8 would define a homologically on-trivial 2-surface in CP_2. The situation is analogous to a distribution of tangent planes or equivalently normal vectors in S^2.

Q4: What is the precise form of $M^8 - H$ duality: does it apply only to partonic 2-surfaces and string world sheets or to the entire space-time surfaces?

$M^8 - H$ duality is possible if the X^4 in M^8 contains also integrable distribution of complex tangent or normal 2-planes at which 4-D tangent space is quaternionic/associative. String world sheets and partonic 2-surfaces define these distributions.

The minimum condition allowed by SH in H is that string world sheets and there is a finite number of partonic 2-surfaces and string world sheets. In this case only these 2-surfaces can be mapped to H and SH assigns to them a 4-D space-time surface. The original hypothesis was that these surfaces define global orthogonal slicings of the X^4 so that $M^8 - H$ duality could be applied to the entire X^4. This condition is probably too strong.

8.3 Challenging $M^8 - H$ duality

$M^8 - H$ duality involves several alternative options and in the following arguments possibly leading to a unique choice are discuses.

1. Are both associativity and co-associativity possible or is only either of these options allowed? Is it also possible to pose the condition guaranteeing the existence of 2-D complex sub-manifolds identifiable as string world sheets necessary to map the entire space-time surface from M^8 to H? In other words, is the strong form of holography (SH) needed in M^8 and/or H or is it needed at all?

2. The assignment of the space-time surface at the level of M^8 to the roots of real or imaginary part (in quaternionic sense) of octonionic polynomial P defined as an algebraic continuation of real polynomial is an extremely powerful hypothesis in adelic physics [L47, L48] and would mean a revolution in biology and consciousness theory.

   Does P fix the space-time surface with the properties needed to realize $M^8 - H$ duality or is something more needed? Does the polynomial fix the space-time surface uniquely - one would have extremely strong number theoretic holography - so that one would have number theoretic holography with coefficients of a real polynomial determining the space-time surface?

3. $M^8 - H$ duality involves mapping of $M^4 \subset M^8$ to $M^4 \subset H$. Hitherto it has been assumed that this map is direct identification. The form of map should however depend on the interpretation of M^8. In octonionic Dirac equation M^8 coordinates are in the role of momenta [L84]. This suggests the interpretation of M^8 as an analog of 8-D momentum space. If this interpretation is correct, Uncertainty Principles demands that the map $M^4 \subset M^8 \to M^4 \subset H$ is analogous to inversion mapping large momenta to small distances.

4. Twistor lift of TGD [K86] is an essential part of the TGD picture. Ttwistors and momentum twistors provide dual approaches to twistor Grassmann amplitudes. Octonionic Dirac

equation suggests that M^8 and H are in a similar dual relation. Could $M^8 - H$ duality allow a generalization of twistorial duality to TGD framework?

8.3.1 Explicit form of the octonionic polynomial

What does the identification of the octonionic polynomial P as an octonionic continuation of a polynomial with real or complexified coefficients imply? In the following I regard M^8_c as O^8_c and consider products for complexified octonions.

Remark: In adelic vision the coefficients of P must be rationals (or at most algebraic numbers in some extension of rationals).

One interesting situation corresponds to the real subspace of O_c spanned by $\{I_0, iI_k\}, = 1, ..7$, with a number theoretic metric signature $(1, -1, -1..., -1)$ of M^8 which is complex valued except at in various reals subspaces.This subspace is associative. The original proposal was that Minkowskian space-time regions as projections to this signature are associative whereas Euclidian regions are co-associative. It however turned out that associative space-time surfaces are physically uninteresting.

The canonical choice $(iI_0, I_1, I_2, iI_3, I_4, iI_5, I_6, iI_7)$ defining the complexification of the tangent space represents a co-associative sub-space realizing Minkowski signature. It turns out that both Minkowskian and Euclidian space-time regions must be co-associative .

Surprises

The explicit calculation of the octonionic polynomial yielded a chilling result. If one poses (co-)associativity conditions as vanishing of the imaginary or real part in quaterionic sense: $Im_Q(P) = 0$ or $Re_Q(P) = 0$, the outcome is that the space-time surface is just M^4 or E^4. Second chilling result is that quaternionic sub-manifolds are geodesic sub-manifolds. This led to the question how to modify the (co-)associativity hypothesis.

The vision has been that space-time surfaces can be identified as roots for the imaginary (co-associative) part $Im_Q(O)$ or real part $Im_Q(O)$ of octonionic polynomial using the standard decomposition $(1, e_1, e_2, e_3)$.

1. The naïve counting of dimensions suggests that one obtains 4-D surfaces. The surprise was that also 6-D brane like entities located at the boundary of M^8 light-cone and with topology of 6-sphere S^6 are possible. They correspond to the roots of a real polynomial $P(o)$ for the choice $(1, iI_1, ..., iI_7)$. The roots correspond to the values of the real octonion coordinate interpreted as values of linear M^4 time in the proposal considered. Also for the canonical proposal one obtains a similar result. In O_c they correspond to 12-D complex surfaces X^6_c satisfying the same condition conditions $x_0^2 + r^2 = 0$ and $P(x_0) = 0$.

2. There was also another surprise. As already described, the general form for the octonionic polynomial $P(o)$ induced from a real polynomial is extremely simple and reduces to $X(t^2, r^2)I_0 + iY(t^2, r^2)Im(o)$. There are only two complex variables t and r^2 involved and the solutions of $P = 0$ are 12-D complex surfaces X^6_c in O_c. Also the special solutions have the same dimension.

3. In the case of co-associativity 8 conditions are needed for $Re_Q(P) = 0$: note that $X = 0$ is required. The naive expectation is that this gives a complex manifold X^4_c with 4-D real projection X^4_r as an excellent candidate for a co-associative surface.

   The expectation turned out to be wrong: in absence of any additional conditions the solutions are complex 7-dimensional mass shells! This is due to the symmetries of the octonionic polynomials as algebraic continuation of a real polynomial.

4. The solution of the problem is to change the interpretation completely. One must assign to the 7-D complex mass shell H^7_c a 3-D complex mass shell H^3_c.

   One can do this by assuming space-time surface is surface intersecting the7-D mass shell obtained as a deformation of $M^4_c \subset M^8_c$ by acting with local $SU(3)$ gauge transformation and requiring that the image point is invariant under $U(2)$. If the 4-D complex mass squared remains invariant in this transformation, X^4_c intersects H^7_c.

With these assumptions, a local CP_2 element defines X^4_c and X^4_r is obtained as its real projection in M^4. This definition assigns to each point of M^4 a point of CP_2 so that M^8-H duality is well-defined.

One obtains holography in which the fixing of 3-D mass shells fixes the 4-surface and also assigns causal diamond with the pair of mass shells with opposite energies. If the space-time surface is analog of Bohr orbit, also its preimage under M^8-H duality should be such and P would determine 4-surface highly uniquely [L101] and one would have number theoretic holography.

General form of P and of the solutions to $P=0$, $Re_Q(P)=0$, and $Im_Q(P)=0$

It is convenient to introduce complex coordinates for O_c since the formulas obtained allow projections to various real sections of O_c.

1. To see what happens, one can calculate o_c^2. Denote o_c by $o_c = tI_0 + \overline{o_c}$ and the norm squared of $\overline{o}$ by r^2, where $r^2 = \sum o_k^2$ where o_k are the complex coordinates of octonion. Number theoretic norm squared for o_c is t^2+r^2 and reduces to a real number in the real sections of O_c. For instance, in the section $(I_1, iI3, iI_5, iI_7)$ the norm squared is $-x_1^2+x_3^2+x_5^2+x_7^2$ and defines Minkowskian norm squared.

   For o^2 one has:

   $$o^2 = t^2 - r^2 + 2t\overline{o} \equiv X_2 + \overline{Y}_2 \ .$$

   For o^3 one obtains

   $$o^3 = tX_2 - \overline{o}\cdot\overline{Y}_2 + t\overline{Y}_2 + X_2\overline{o} \ .$$

   Clearly, $Im_Q(o^n)$ has always the same direction as $Im_Q(o)$. Hence one can write in the general case

   $$o^n = X + Y\overline{o} \ . \tag{8.3.1}$$

   This trivial result was obtained years ago but its full implications became evident only while preparing the current article. The point is that the solutions to associativity/co-associativity conditions by putting $Re(Q(P)=0$ or $Im_Q(P)=0$ are trivial: just M^4 or E^4. What goes wrong with basic assumptions, will be discussed later.

   Remark: In M^8 sub-space one has imaginary $\overline{o}$ is proportional to the commuting imaginary unit.

2. It is easy to deduce a recursion formula for the coefficients for X and Y for n:th power of o_c. Denote by t the coordinate associated with the real octonion unit (not time coordinate). One obtains

   $$\begin{aligned} o_c^n &= X_nI_0 + Y_n\overline{o} \ , \\ X_n &= tX_{n-1} - rY_{n-1} \ , \\ Y_n &= tY_{n-1} + rX_{n-1} \ . \end{aligned} \tag{8.3.2}$$

   In the co-associative case one has $t=0$ or possibly constant $t=T$ (note that in the recent interpretation t does not have interpretation as time coordinate). The reason is that the choice of octonionic coordinates is unique apart from translation along the real axis from the condition that the coefficients of P remain complex numbers in powers of the new variable.

3. The simplest option correspond to $t = 0$. One can criticize this option since the quaternionicity of normal space should not be affected if t is constant different from zero. In any case, for $t = 0$ the recursion formula gives for the polynomial $P(o_c)$ the expression

$$P(o_c) = \sum (-1)^n r^{2n}(p_{2n-1}I_0 + p_{2n}\overline{o}) \ . \tag{8.3.3}$$

Denoting the even and of odd parts of P by P_{even} and P_{odd}, the roots $r_{k,odd}$ of $X = Re(P(o_c))$ are roots P_{odd} and roots $r_{k,even}$ of $Y = Im(P(o_c))$ are roots of P_{even}. Co-associativity gives roots of X and the roots of P as simultaneous roots of P_{odd} and P_{even}. The interpretation of roots is as in general complex mass squared values.

In the general case, the recursion relation would give the solution

$$\begin{gathered} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = A^n \begin{pmatrix} t \\ r \end{pmatrix} \\ A = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \end{gathered} \tag{8.3.4}$$

One can diagonalize the matrix appearing in the iteration by solving the eigenvalues $\lambda_\pm = t \pm ir$ and eigenvectors $X_\pm = (\pm i, 1)$ and by expressing $(X_1, Y_1) = (t, r)$ in terms of the eigenvectors as $(t, r) = ((it + r)X_+ + (r - it)X_-)/2$. This gives

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (t+ir)^n i - (t-ir)^n i \\ (t+ir)^n + (t-ir)^n \end{pmatrix} \tag{8.3.5}$$

This gives

$$\begin{aligned} P(o_c) &= P_1 I_0 + P_2 \overline{o} \ , \\ P_1(r) &= \sum X_n p_n r^{2n} \ , \\ P_2(r) &= \sum Y_n p_n r^{2n} \ . \end{aligned} \tag{8.3.6}$$

For the restriction to M_c^4, r^2 reduces to complex 4-D mass squared given by the root r_n. In general case r^2 corresponds to complex 8-D mass squared. All possible signatures are obtained by assuming M_c^8 coordinates to be either real or imaginary (the number theoretical norm squared is real with this restriction).

How does one obtain 4-D space-time surfaces?

Contrary to the naive expections, the solutions of the vanishing conditions for the $Re_Q(P)$ ($Im_Q(P)$) (real (imaginary) part in quaternionic sense) are 7-D complex mass shells $r^2 = r_{n,1}$ as roots of $P_1(r) = 0$ or $r^2 = r_{n,2}$ of $P_2(r) = 0$ rather than 4-D complex surfaces (for a detailed discussion see [K15]) A solution of both conditions requires that P_1 and P_2 have a common root but the solution remains a 7-D complex mass shell! This was one of the many cold showers during the development of the ideas about $M^8 - H$ duality! It seems that the adopted interpretation is somehow badly wrong. Here zero energy ontology (ZEO) and holography come to the rescue.

1. Could the roots of P_1 or P_2 define only complex mass shells of the 4-D complex momentum space identifiable as M_c^4? ZEO inspires the question whether a proper interpretation of mass shells could be as pre-images of boundaries of cd:s (intersections of future and past directed light-cones) as pairs of mass shells with opposite energies. If this is the case, the challenge would be to understand how X_c^4 is determined if P does not determine it.

   Here holography, considered already earlier, suggests itself: the complex 3-D mass shells belonging to X_c^4 would only define the 3-D boundary conditions for holography and the real mass shells would be mapped to the boundaries of cds. This holography can be restricted to X_R^4. Bohr orbit property at the level of H suggests that the polynomial P defines the 4-surface more or less uniquely.

2. Let us take the holographic interpretation as a starting point. In order to obtain an X_c^4 mass shell from a complex 7-D light-cone, 4 complex degrees of freedom must be eliminated. $M^8 - H$ duality requires that X_c^4 allows M_c^4 coordinates.

   Note that if one has $X_c^4 = M_c^4$, the solution is trivial since the normal space is the same for all points and the H image under $M^8 - H$ duality has constant $CP_2 = SU(3)/U(2)$ coordinates. X_c^4 should have interpretation as a non-trivial deformation of M_c^4 in M^8.

3. By $M^8 - H$ duality, the normal spaces should be labelled by $CP_2 = SU(3)/U(2)$ coordinates. $M^8 - H$ duality suggests that the image $g(p)$ of a momentum $p \in M_c^4$ is determined essentially by a point $s(p)$ of the coset space $SU(3)/U(2)$. This is achieved if M_c^4 is deformed by a local $SU(3)$ transformation $p \rightarrow g(p)$ in such a way that each image point is invariant under $U(2)$ and the mass value remains the same: $g(p)^2 = p^2$ so that the point represents a root of P_1 or P_2.

   Remark: I have earlier considered the possibility of G_2 and even $G_{2,c}$ local gauge transformation. It however seems that that local $SU(3)$ transformation is the only possibility since G_2 and $G_{2,c}$ would not respect $M^8 - H$ duality. One can also argue that only real $SU(3)$ maps the real and imaginary parts of the normal space in the same manner: this is indeed an essential element of $M^8 - H$ duality.

4. This option defines automatically $M^8 - H$ duality and also defines causal diamonds as images of mass shells $m^2 = r_n$. The real mass shells in H correspond to the real parts of r_n. The local $SU(3)$ transformation g would have interpretation as an analog of a color gauge field. Since the H image depends on g, it does not correspond physically to a local gauge transformation but is more akin to an element of Kac-Moody algebra or Yangian algebra which is in well-defined half-algebra of Kac-Moody with non-negative conformal weights.

The following summarizes the still somewhat puzzling situation as it is now.

1. The most elegant interpretation achieved hitherto is that the polynomial P defines only the mass shells so that mass quantization would reduce to number theory. Amusingly, I started to think about particle physics with a short lived idea that the d'Alembert equation for a scalar field could somehow give the mass spectrum of elementary particles so that the issue comes full circle!

2. Holography assigns to the complex mass shells complex 4-surfaces for which $M^8 - H$ duality is well-defined even if these surfaces would fail to be 4-D co-associative. These surfaces are expected to be highly non-unique unless holography makes them unique. The Bohr orbit property of their images in H indeed suggests this apart from a finite non-determinism [L101]. Bohr orbit property could therefore mean extremely powerful number theoretical duality for which the roots of the polynomial determine the space-time surface almost uniquely. $SU(3)$ as color symmetry emerges at the level of M^8. By $M^8 - H$ duality, the mass shells are mapped to the boundaries of CDs in H.

3. Do we really know that X_r^4 co-associative and has distribution of 2-D commuting subspaces of normal space making possible $M^8 - H$ duality? The intuitive expectation is that the

answer is affirmative [A46]. In any case, $M^8 - H$ duality is well-defined even without this condition.

4. The special solutions to $P = 0$, discovered already earlier, are restricted to the boundary of CD_8 and correspond to the values of energy (rather than mass or mass squared) coming as roots of the real polynomial P. These mass values are mapped by inversion to "very special moments in the life of self" (a misleading term) at the level of H as special values of light-cone proper time rather than linear Minkowski time as in the earlier interpretation [L63]. The new picture is Lorenz invariant.

Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \to M^4 \subset H$ from identification to inversion. The equations for $Re_Q(P) = 0$ reduce to simultaneous roots of the real polynomials defined by the odd and even parts of P having interpretation as complex values of mass squared mapped to light-cone proper time constant surfaces in H. This leads to the idea that the formulation of scattering amplitudes at M^8 levels provides the counterpart of momentum space description of scattering whereas the formulation at the level of H provides the counterpart of space-time description.

This picture combined with zero energy ontology (ZEO) leads also to a view about quantum TGD at the level of M^8. Local $SU(3)$ element has properties suggesting a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

8.3.2 The input from octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = q\overline{q} = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of $p\,p$ are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

One of the big surprises was that the cognitive representations for both light-like boundary and X_r^4 are not generic meaning that they would consist of a finite set of points but infinite due to the Lorentz symmetry: a kind of cognitive explosion would happen by the Lorentz symmetry. The natural assumption is that for a suitable momentum unit, physical momenta satisfying mass shell conditions are algebraic integers in the extension of rationals defined by P. Periodic boundary conditions in turn suggest that for the physical states the total momenta are ordinary integers and this leads to Galois confinement as a universal mechanism for the formation of bound states.

Hamilton-Jacobi structure and Kähler structure of $M^4 \subset H$ and their counterparts in $M^4 \subset M^8$

The Kähler structure of $M^4 \subset H$, forced by the twistor lift of TGD, has deep physical implications and seems to be necessary. It implies that for Dirac equation in H, modes are eigenstates of only the longitudinal momentum and in the 2 transversal degrees of freedom one has essentially harmonic oscillator states [L99, L97], that is Gaussians determined by the 2 longitudinal momentum components. For real longitudinal momentum the exponents of Gaussians are purely imaginary or purely real.

The longitudinal momentum space $M^2 \subset M^4$ and its orthogonal complement E^2 is in a preferred role in gauge theories, string models, and TGD. The localization of this decomposition leads to the notion of Hamilton-Jacobi (HJ) structure of M^4 and the natural question is how this relates to Kähler structures of M^4. At the level of H spinors fields only the Kähler structure corresponding to constant decomposition $M^2 \oplus E^2$ seems to make sense and this raises the question how the H-J structure and Kähler structure relate. TGD suggests the existence of two geometric structure in M^4: HJ structure and Kähler structure. It has remained unclear whether HJ structure and Kähler structure with covariantly constant self-dual Kähler form are equivalent notions or whether there several H-J structures accompaning the Kähler structure.

In the following I argue that H-J structures correspond to different choices of symplectic coordinates for M^4 and that the properties of $X^4 \subset H$ determined bt $M^- H$ duality make it natural to to choose particular symplectic coordinates for M^4.

Consider first what H-J structure and Kähler structure could mean in H.

1. The H-J structure of $M^4 \subset H$ would correspond to an integrable distribution of 2-D Minkowskian sub-spaces of M^4 defining a distribution of string world sheets $X^2(x)$ and orthogonal distribution of partonic 2-surfaces $Y^2(x)$. Could this decomposition correspond to self-dual covariantly Kähler form in M^4?

   What do we mean with covariant constancy now? Does it mean a separate covariant constancy for the choices of $M^2(x)$ and $Y^2(x)$ or only of their sum, which in Minkowski coordinates could correspond to a constant electric and magnetic fields orthogonal to each other?

2. The non-constant choice of $M^2(x)$ ($E^2(x)$) cannot be covariantly constant. One can write $J(M^4) = J(M^2(x)) \oplus J(E^2(x)$ corresponding to decomposition to electric and magnetic parts. Constancy of $J(M^2(x)$ would require that the gradient of $J(M^2(x)$ is compensated by the gradient of an antisymmetric tensor with square equal to the projector to $M^2(x)$. Same condition holds true for $J(E^2(x))$. The gradient of the antisymmetric tensor would be parallel to itself implying that the tensor is constant.

3. H-J structure can only correspond to a transformation acting on J but leaving $J_{kl}dm^kdm^l$ invariant. One should find analogs of local gauge transformations leaving J invariant. In the case of CP_2, these correspond to symplectic transformations and now one has a generalization of the notion. The M^4 analog of the symplectic group would parameterize various decompositions of $J(M^4)$.

   Physically the symplectic transformations define local choices of 2-D space $E^2(x)$ of transversal polarization directions and longitudinal momentum space M^2 emerging in the construction of extremals of Kähler action.

4. For the simplest Kähler form for $M^4 \subset H$, this decomposition in Minkowski coordinates would be constant: orthogonal constant electric and magnetic fields. This Kähler form extends to its number theoretical analog in M^8. The local $SU(3)$ element g would deform M^4 to $g(M^4)$ and define an element of local CP_2 defining $M^8 - H$ duality. g should correspond to a symplectic transformation of M^4.

Consider next the number theoretic counterparts of H-J- and Kähler structures of $M^4 \subset H$ in $M^4 \subset M^8$.

1. In M^4 coordinates H-J structure would correspond to a constant $M^2 \times E^2$ decomposition. In M^4 coordinates Kähler structure would correspond to constant E and B orthogonal to each other. Symplectic transformations give various representations of this structure as H-J structures.

2. The number theoretic analog of H-J structure makes sense also for $X^4 \subset M^8$ as obtained from the distribution of quaternionic normal spaces containing 2-D commutative sub-space

at each point by multiplying then by local unit $I_4(x)$ orthogonal to the quaternionic units $\{1, I_1 = I_2 = I_3\}$ with respect to octonionic inner product. There is a hierarchy of CDs and the choices of these structures would be naturally parameterized by G_2.

This would give rise to a number theoretically defined slicing of $X_c^4 \subset M_c^8$ by complexified string world sheets X_c^2 and partonic 2-surfaces Y_c^2 orthogonal with respect to the octonionic inner product for complexified octonions.

3. In $M^8 - H$ duality defined by $g(p) \subset SU(3)$ assigns a point of CP_2 to a given point of M^4. $g(p)$ maps the number theoretic H-J to H-J in $M^4 \subset M^8$. The space-time surface itself - that is $g(p)$ - defines these symplectic coordinates and the local $SU(3)$ element g would naturally define this symplectic transformation.

4. For $X^4 \subset M^8$ g reduces to a constant color rotation satisfying the condition that the image point is $U(2)$ invariant. Unit element is the most natural option. This would mean that g is constant at the mass and energy shells corresponding to the roots of P and the mass shell is a mass shell of M^4 rather than some deformed mass shell associated with images under $g(p)$.

   This alone does not yet guarantee that the 4-D tangent space corresponds to M^4. The additional physically very natural condition on g is that the 4-D momentum space at these mass shells is the same. $M^8 - H$ duality maps these mass shells to the boundaries of these cd:s in M^4 (CD= $cd \times CP_2$). This conforms with the identification of zero energy states as pairs of 3-D states at the boundaries of CD.

This generalizes the original intuitive but wrong interpretation of the roots r_n of P as "very special moments in the life of self" [L63].

1. Since the roots correspond to mass squared values, they are mapped to the boundaries of cd with size $L = \hbar_{eff}/m$ by $M^8 - H$ duality in M^4 degrees of freedom. During the sequence of SSFRs the passive boundary of CD remains does not shift only changes in size, and states at it remain unaffected. Active boundary is shifted due to scaling of cd.

   The hyperplane at which upper and lower half-cones of CD meet, is shifted to the direction of geometric future. This defines a geometric correlate for the flow of experienced time.

2. A natural proposal is that the moments for SSFRs have as geometric correlates the roots of P defined as intersections of geodesic lines with the direction of 4-momentum p from the tip of CD to its opposite boundary (here one can also consider the possibility that the geodesic lines start from the center of cd). Also energy shells as roots $E = r_n$ of P are predicted. They decompose to a set of mass shells $m_{n.,k}$ with the same $E = r_n$: similar interpretation applies to them.

3. What makes these moments very special is that the mass and energy shells correspond to surfaces in M^4 defining the Lorentz quantum numbers. SSFRs correspond to quantum measurements in this basis and are not possible without this condition. At $X^4 \subset M^8$ the mass squared would remain constant but the local momentum frame would vary. This is analogous to the conservation of momentum squared in general relativistic kinematics of point particle involving however the loss of momentum conservation.

4. These conditions, together with the assumption that g is a rational function with real coefficients, strongly suggest what I have referred to as preferred extremal property, Bohr orbitology, strong form of holography, and number theoretical holography.

In principle, by a suitable choice of M^4 one can make the momentum of the system light-like: the light-like 8-momentum would be parallel to M^4. I have asked whether this could be behind the fact that elementary particles are in a good approximation massless and whether the small mass of elementary particles is due to the presence of states with different mass squares in the zero state allowed by Lorentz invariance.

The recent understanding of the nature of right-handed neutrinos based on M^4 Kähler structure [L97] makes this mechanism un-necessary but poses the question about the mechanism choosing some particular M^4. The conditions that $g(p)$ leaves mass shells and their 4-D tangent spaces invariant provides this kind of mechanism. Holography would be forced by the condition that the 4-D tangent space is same for all mass shels representing inverse images for very special moments of time.

What about string world sheets and partonic 2-surfaces?

One can apply the above arguments also to the identification of 2-D string world sheets and partonic 2-surfaces.

1. One has two kinds of solutions: M^2 and 3-D surfaces of X^4 as analogs of 6-brane. The interpretation for 3-D *resp.* 2-D branes as a light-like 3-surface associated with the octonionic Dirac equation representing mass shell condition *resp.* string world sheet is attractive.

2. M^2 would be replaced with an integrable distribution of $M^2(x)$ in local tangent space $M^4(x)$. The space for the choices of $M^2(x)$ would be S^3 corresponding to the selection of a preferred quaternion imaginary unit equal to the choices of preferred octonion imaginary unit.

   The choices of the preferred complex subspace $M^2(x)$ at a given point would be characterized by its normal vector and parameterized by sphere S^2: the interpretation as a quantization axis of angular momentum is suggestive. One would have space $S^3 \times S^2$. Also now the integrability conditions $de_A = 0$ would hold true.

3. String world sheets could be regarded as analogs of superstrings connecting 3-D brane like entities defined by the light-like partonic orbits. The partonic 2-surfaces at the ends of light-like orbits defining also vertices could correspond to the 3-surfaces at which quaternionic 4-surfaces intersect 6-branes.

8.3.3 Is (co-)associativity possible?

The number theoretic vision relying on the assumption that space-time surfaces are 8-D complex 4-surfaces in o_c^8 determined as algebraic surfaces for octonionic continuations of real polynomials, which for adelic physics would have coefficients which are rational or belong to an extension of rationals. The projections to subspaces Re^8 of o_c^8 defined as space for which given coordinate is purely real or imaginary so that complexified octonionic norm is real would give rise to real 4-D space-time surfaces. $M^8 - H$ duality would map these surfaces to geometric objects in $M^4 \times CP_2$. This vision involves several poorly understood aspects and it is good to start by analyzing them.

Challenging the notions of associativity and co-associativity

Consider first the notions of associativity *resp.* co-associativity equivalent with quaternionicity *resp.* co-quaternionicity. The original hope was that both options are possible for surfaces of real sub-spaces of O_c ("real" means here that complexified octonionic metric is real).

1. The original idea was that the associativity of the tangent space or normal space of a real space-time surface X^4 reduces the classical physics at the level of M^8 to associativity. Associativity/co-associativity of the space-time surface states that at each point of the tangent-/normal space of the real space-time surface in O is quaternionic. The notion generalizes also to $X_c^4 \subset O_c^8$. (Co-)associativity makes sense also for the real subspaces space of O with Minkowskian signature.

2. It has been however unclear whether (co-)associativity is possible. The cold shower came as I learned that associativity allows only for geodesic sub-manifolds of quaternionic spaces about which octonions provide an example [A46]. The good news was that the distribution of co-associative tangent spaces always defines an integrable distribution in the sense that one can find sub-manifold for which the associative normal space at a given point has tangent space as an orthogonal complement. Should the number theoretic dynamics rely on co-associativity rather than associativity?

3. Minkowskian space-time regions have been assumed to be associative and to correspond to the projection to the standard choice for basis as $\{1, iI_1, iI_2, iI_3\}$. The octonionic units $\{1, I_1, I_2, I_3\}$ define quaternionic units and associative subspace and their products with unit I_4 define the orthogonal co-associative subspace as $\{I_4, I_5 = I4I_1, I_6 = I_4I_2, I_7 = I_4I_3\}$. This result forces either to weaken the notion of associativity or to consider alternative identifications of Minkowskian regions, which can be co-associative: fortunately, there exists a large number of candidates.

The article [A46] indeed kills the idea about the associativity of the space-time surface. The article starts from a rather disappointing observation that associative sub-manifolds are geodesic sub-manifolds and therefore trivial. Co-associative quaternion sub-manifolds are however possible. With a motivation coming from this observation, the article discusses what the author calls RC quaternionic sub-manifolds of quaternion manifolds. For a quaternion manifold the tangent space allows a realization of quaternionic units as antisymmetric tensors. These manifolds are constant curvature spaces and typically homogeneous spaces.

1. Quaternion sub-manifold allows a 4-D integrable distribution of quaternion units. The normal complement of this distribution is expressible in terms of the second fundamental form and the condition that it is trivial implies that the second fundamental form is vanishing so that one has a geodesic submanifold. Quaternionic sub-manifolds are thus too trivial to be interesting. As a diametric opposite, one can also define totally real submanifolds for which the normal space contains a distribution of quaternion units. In this case the distribution is always integrable. This case is much more interesting from the TGD point of view.

2. Author introduces the notion of CR quaternion sub-manifold $N \subset M$, where M is quaternion manifold with constant sectional curvatures. N has quaternion distribution D in its tangent spaces if the action of quaternion units takes D to itself. $D^\perp$ is the co-quaternionic orthogonal complement D in the normal space N. D would take also $D^\perp$ to itself. $D^\perp$ can be expressed in terms of the components of the second fundamental form and vanishes for quaternion sub-manifolds.

3. Author deduces results about CR quaternion sub-manifolds, which are very interesting from the TGD point of view.

   (a) Sub-manifold is CR quaternion sub-manifold only if the curvature tensor of R_M of the embedding space satisfies $R_M(D, D, D^\perp, D\ \ ) = 0$. The condition is trivial if the quaternion manifold is flat. In the case of octonions this would be the case.

   (b) D is integrable only if the second fundamental form restricted to it vanishes meaning that one has a geodesic manifold. Totally real distribution $D^\perp$ is always integrable to a co-associative surface.

   (c) If $D^\perp$ integrates to a minimal surface then N itself is a minimal surface.

Could one consider RC quaternion sub-manifolds in TGD framework? Both octonions and their complexifixation can be regarded as quaternionic spaces. Consider the real case.

1. If the entire D is quaternionic then N is a geodesic sub-manifold. This would leave only E^4 and its Minkowskian variants with various signatures. One could have however 4-D totally real (co-associative) space-time surfaces. Simple arguments will show that the intersections of the conjectured quaternionic and co-quaternionic 4-surfaces have 2- and 3-D intersections with 6-branes.

   Should one replace associative space-time surfaces with CR sub-manifolds with $d \leq 3$ integrable distribution D whereas the co-quaternionic surfaces would be completely real having 4-D integrable $D^\perp$? Could one have 4-D co-associative surfaces for which $D^\perp$ integrates to $n \geq 1$-dimensional minimal surface (geodesic line) and the X^4 itself is a minimal surface?

   Partially associative CR manifold do not allow M^8H duality. Only co-associative surfaces allow it and also their signature must be Minkowskian: the original idea [L65, L43, L44, L45] about Euclidian (Minkowskian) signature for co-associative (associative) surfaces was wrong.

2. The integrable 2-D sub-distributions D defining a distribution of normal planes could define foliations of the X^4 by 2-D surfaces. What springs to mind is foliations by string world sheets and partonic 2 surfaces orthogonal to them and light-like 3-surfaces and strings transversal to them. This expectation is realized.

How to identify the Minkowskian sub-space of O_c?

There are several identifications of subspaces of O_c with Minkowskian signature. What is the correct choice has been far from obvious. Here symmetries come in rescue.

1. Any subspace of O^c with 3 (1) imaginary coordinates and 1 (3) real coordinates has Minkowskian signature in octonionic norm algebraically continued to O_c (complex valued continuation of real octonion norm instead of real valued Hilbert space norm for O_c). Minkowskian regions should have local tangent space basis consisting of octonion units which in the canonical case would be $\{I_1, iI_3, iI_5, iI_7\}$, where i is commutative imaginary unit. This particular basis is co-associative having whereas its complement $\{iI_0, I_2, I_4, I_6\}$ is associative and has also Minkowskian signature.

2. The size of the isometry group of the subspace of M^8_c depends on whether the tangent basis contains real octonion unit 1 or not. The isometry group for the basis containing I_0 is $SO(3)$ acting as automorphisms of quaternions and $SO(k, 3-k)$ when $3-k$ units are proportional to i. The reason is that G_2 (and its complexification $G_{2,c}$) and its subgroups do not affect I_0. For the tangent spaces built from 4 imaginary units I_k and iI_l the isometry group is $SO(k, 4-k) \subset G_{2,c}$.

   The choice therefore allows larger isometry groups and also co-associativity is possible for a suitable choice of the basis. The choice $\{I_1, iI_3, iI_5, iI_7\}$ is a representative example, which will be called canonical basis. For these options the isometry group is the desired $SO(1,3)$ as an algebraic continuation of $SO(4) \subset G_2$ acting in $\{I_1, I_3, I_5, I_7\}$, to $SO(1,3) \subset G_{2,c}$.

   Also Minkowskian signature - for instance for the original canonical choice $\{I_0, iI_1, iI_2, iI_3\}$ - can have only $SO(k, 3-k)$ as isometries. This is the basic objection against the original choice $\{I_0, iI_1, iI_2, iI_3\}$. This identification would force the realization of $SO(1,3)$ as a subgroup of $SO(1,7)$. Different states of motion for a particle require different octonion structure with different direction of the octonion real axis in M^8. The introduction of the notion of moduli space for octonion structures does not look elegant. For the option $\{I_1, iI_3, iI_5, iI_7\}$ only a single octonion structure is needed and $G_{2,c}$ contains $SO(1,3)$.

   Note that also the signatures $(4,0)$, $(0,4)$ and $(2,2)$ are possible and the challenge is to understand why only the signature (1,3) is realized physically.

Co-associative option is definitely the only physical alternative. The original proposal for the interpretation of the Minkowski space in terms of an associative real sub-space of M^4 had a serious problem. Since time axis was identified as octonionic real axis, one had to assign different octonion structure to particles with non-parallel moment: $SO(1,7)$ would relate these structures: how to glue the space-time surfaces with different octonion structures together was the problem. This problem disappears now. One can simply assign to particles with different state of motion real space-time surface defined related to each other by a transformation in $SO(1,3) \subset G_{2,c}$.

The condition that $M^8 - H$ duality makes sense

The condition that $M^8 - H$ duality makes sense poses strong conditions on the choice of the real sub-space of M^8_c.

1. The condition that tangent space of O_c has a complexified basis allowing a decomposition to representations of $SU(3) \subset G_2$ is essential for the map to $M^8 \to H$ although it is not enough. The standard representation of this kind has basis $\{\pm iI_0 + I_1\}$ behaving like $SU(3)$ singlets $\{I_2 + \epsilon iI_3, I_4 + \epsilon iI_5, \epsilon I_6 \pm iI_7\}$ behaves like SU(3) triplet 3 for $\epsilon = 1$ and its conjugate $\overline{3}$ for $\epsilon = -1$. $G_{2,c}$ provides new choices of the tangent space basis consistent with this choice. $SU(3)$ leaves the direction I_1 unaffected but more general transformations act as Lorentz

transformation changing its direction but not leaving the M^4 plane. Even more general $G_{2,c}$ transformations changing M^4 itself are in principle possible.

Interestingly, for the canonical choice the co-associative choice has $SO(1,3)$ as isometry group whereas the complementary choice failing to be associative correspond to a smaller isometry group $SO(3)$. The choice with M^4 signature and co-associativity would provide the highest symmetries. For the real projections with signature $(2,2)$ neither consistent with color structure, neither full associativity nor co-associativity is possible.

2. The second essential prerequisite of $M^8 - H$ duality is that the tangent space is not only (co-)associative but contains also (co-)complex - and thus (co-)commutative - plane. A more general assumption would be that a co-associative space-time surface contains an integrable distribution of planes $M^2(x)$, which could as a special case reduce to M^2.

   The proposal has been that this integrable distribution of $M^2(x)$ could correspond to string sheets and possibly also integrable orthogonal distribution of their co-complex orthogonal complements as tangent spaces of partonic 2-surfaces defining a slicings of the space-time surface. It is now clear that this dream cannot be realized since the space-time surface cannot be even associative unless it is just E^4 or its Minkowskian variants.

3. As already noticed, any distribution of the associative normal spaces integrates to a co-associative space-time surface. Could the normal spaces also contain an integrable distribution of co-complex planes defined by octonionic real unit 1 and real unit $I_k(x)$, most naturally I_1 in the canonical example? This would give co-commutative string world sheet. Commutativity would be realized at the 2-D level and associativity at space-time level. The signature of this plane could be Minkowskian or Euclidian. For the canonical example $\{I_1, iI_3, iI_5, iI_7\}$ the 2-D complex plane in quaternionic sense would correspond to $(a \times 1, +n_2I_2 + n_4I_6 + n_6I_6$, where the unit vector n_i has real components and one has $a = 1$ or $a = i$ is forced by the complexification as in the canonical example.

   Since the distribution of normal planes integrates to a 4-surface, one expects that its sub-distribution consting of commutative planes integrates to 2-D surface inside space-time surface and defines the counterpart of string worlds sheet. Also its normal complement could integrate to a counterpart of partonic 2-surface and a slicing of space-time surface by these surfaces would be obtained.

4. The simplest option is that the commutative space does not depend on position at X^4. This means a choice of a fixed octonionic imaginary unit, most naturally I_1 for the canonical option. This would make $SU(3)$ and its sub-group $U(2)$ independent of position. In this case the identification of the point of $CP_2 = SU(3)/U(2)$ labelling the normal space at a given point is unique.

   For a position dependent choice $SU(3)(x)$ it is not clear how to make the specification of $U(2)(x)$ unique: it would seem that one must specify a unique element of $G_2(x)$ relating $SU(3)(x)$ to a choice at special point x_0 and defining the conjugation of both $SU(3)(x)$ and $U(2)(x)$. Otherwise one can have problems. This would also mean a unique choice for the direction of time axis in O and fixing of $SO(1,3)$ as a subgroup of $G_{2,c}$. Also this distribution of associative normal spaces is integrable. Physically this option is attractive but an open question is whether it is consistent with the identification of space-time surfaces as roots $Re_Q(P) = 0$ of P.

Co-associativity from octonion analyticity or/and from G_2 holography?

Candidates for co-associative space-time surfaces X^4_r are defined as restrictions X^4_r for the roots X^4_c of the octonionic polynomials such that the O_c coordinates in the complement of a real co-associative sub-space of O_c vanish or are constant. Could the surfaces X^4_r or even X^4_c be co-associative?

1. X^4_r is analogous to the image of real or imaginary axis under a holomorphic map and defines a curve in complex plane preserving angles. The tangent vectors of X^4_r and X^4_c involve gradients of all coordinates of O_c and are expressible in terms of all octonionic unit vectors. It is not

obvious that their products would belong to the normal space of X_r^4 a strong condition would be that this is the case for X_c^4.

2. Could octonion analyticity in the proposed sense guarantee this? The products of octonion units also in the tangent space of the image would be orthogonal to the tangent space. Ordinary complex functions preserve angles, in particular, the angle between x- and y-axis is preserved since the images of coordinate curves are orthogonal. Octonion analyticity would preserve the orthogonality between tangent space vectors and their products.

3. This idea could be killed if one could apply the same approach to associative case but this is not possible! The point is that when the real tangent space of O_c contains the real octonion unit, the candidate for the 4-D space-time surface is a complex surface X_c^2. The number theoretic metric is real only for 2-D X_r^2 so that one obtains string theory with co-associativity replaced with co-commutativity and $M^4 \times CP_2$ with $M^2 \times S^2$. One could of course ask whether this option could be regarded as a "sub-theory" of the full theory.

My luck was that I did not realize the meaning of the difference between the two cases first and realized that one can imagine an alternative approach.

1. G_2 as an automorphism group of octonions preserves co-associativity. Could the image of a co-associative sub-space of O_c defined by an octonion analytic map be regarded as an image under a local G_2 gauge transformation. $SU(3) \subset G_2$ is an especially interesting subgroup since it could have a physical interpretation as a color gauge group. This would also give a direct connection with $M^8 - H$ duality since $SU(3)$ corresponds to the gauge group of the color gauge field in H.

2. One can counter-argue that an analog of pure gauge field configuration is in question at the level of M^8. But is a pure gauge configuration for $G_{2,c}$ a pure gauge configuration for G_2? The point is that the $G_{2,c}$ connection $g^{-1}\partial_\mu g$ trivial for $G_{2,c}$ contains by non-linearity cross terms from $g_2 g, c = g_{2,1} + ig_{2,2}$, which are of type $Re = X[g_{2,1}, g_{2,1}] - X[g_{2,2}, g_{2,2}] = 0$ and $Im = iZ[g_{2,1}, g_{2,2}] = 0$. If one puts $g_{2,2}$ contributions to zero, one obtains $Re = X[g_{2,1}, g_{2,1}]$, which does not vanish so that $SU(3)$ gauge field is non-trivial.

3. X_r^4 could be also obtained as a map of the co-associative M^4 plane by a local $G_{2,c}$ element. It will turn out that $G_{2,c}$ could give rise to the speculated Yangian symmetry [L41] at string world sheets analogous to Kac-Moody symmetry and gauge symmetry and crucial for the construction of scattering amplitudes in $M8$.

4. The decomposition of the co-associative real plane of O_c should contain a preferred complex plane for $M^8 - H$ duality to make sense. $G_{2,c}$ transformation should trivially preserve this property so that SH would not be necessary at H side anymore.

There is a strong motivation to guess that the two options are equivalent so that $G_{2,c}$ holography would be equivalent with octonion analyticity. The original dream was that octonion analyticity would realize both associative and co-associative dynamics but was exaggeration!

Does one obtain partonic 2-surfaces and strings at boundaries of ΔCD_8?

It is interesting to look for the dimensions of the intersections of the light-like branes at the boundary of CD_8 giving rise to the boundary of CD_4 in M^4 to see whether it gives justification for the existing phenomenological picture involving light-like orbits of partonic 2-surfaces connected by string world sheets.

1. Complex light-cone boundary has dimension $D = 14$. $P = 0$ as an additional condition at δCD_8 gives 2 complex conditions and defines a 10-D surface having 5-D real projections.

2. The condition $Im_Q(P) = 0$ gives 8 conditions and gives a 2-D complex surface with 1-D real projection. The condition $Re_Q(P) = 0$ gives 3 complex conditions since $X = 0$ is already satisfied and the solution is a 4-D surface having 2-D real projection. Could the interpretation be in terms of the intersection of the orbit of a light-like partonic surface with the boundary of CD_8?

3. Associativity is however not a working option. If only co-associative Minkowskian surfaces allowing mapping to H without SH are present then only 4-D space-time surfaces with Minkowskian signature, only partonic 2-surfaces and their light-like orbits would emerge from co-associativity.

   This option would not allow string world sheets for which there is a strong intuitive support. What could a co-complex 2-surface of a co-associative manifold mean? In the co-associative case the products of octonion imaginary units are in the normal space of space-time surface. Could co-complex surface $X_c^2 \subset X_c^4$ be defined by an integrable co-complex sub-distribution of co-associative distribution. The 4-D distribution of normal planes is always integrable.

   Could the 2-D sub-distributions of co-associative distribution integrate trivially and define slicings by string world sheets or partonic 2-surfaces. Could the distribution of string distributions and its orthogonal complement be both integrable and provide orthogonal slicings by string world sheets and partonic 2-surfaces? String world sheets with Minkowskian signature should intersect the partonic orbits with Euclidian signature along light-like lines. This brings in mind the orthogonal grid of flow lines defined by the $Re(f) = 0$ and $Im(f) = 0$ lines of an analytic function in plane.

4. In this picture the partonic 2-surfaces associated with light-like 3-surface would be physically unique and could serve as boundary values for the distributions of partonic 2-surfaces. But what about string world sheets connecting them? Why would some string world sheets be exceptional? String world sheets would have a light-like curve as an intersection with the partonic orbit but this is not enough.

   Could the physically special string world sheets connect two partonic surfaces? Could the string associated with a generic string world sheet be like a flow line in a hydrodynamic flow past an obstacle - the partonic 2-surface? The string as a flowline would go around the obstacle along either side but there would be one line which ends up to the object.

Interactions would correspond geometrically to the intersections of co-associative space-time surfaces X_r^4 associated with particles and corresponding to different real sub-spaces of O_c related by Lorentz boost in $SO(1,3) \subset G_{2,c}$. In the generic case the intersection would be discrete. In the case that X and Y have a common root the real surfaces $X_r^4 \subset X_r^6$ associated with quarks and depending on their state of motion would reside inside the same 6-D surface X_r^6 and have a 2-D surface X_r^2 as intersection. Could this surface be interpreted as a partonic 2-surface? One must however bear in mind that partonic 2-surfaces as topological vertices are assumed to be non-generic in the sense that the light-like partonic orbits meet at them. At the level of H, the intersections would be partonic 2-surfaces X^2 at which the four 3-D partonic orbits would meet along their ends. Does this hold true at the level of M^8? Or can it hold true even at the level H?

The simplest situation corresponds to 4 external quarks. There are 6 different intersections. Not all of them are realized since a given quark can belong only to a single intersection. One must have two disjoint pairs -say 12 and 34. Most naturally positive *resp.* negative energy quarks form a pair. These pairs are located in different half-cones. The intersections would give two partonic 2-surfaces and this situation would be generic. This suggests a modification of the description of particle reaction in M^8 . $M^8 - H$ duality suggests a similar description in H.

What could be the counterparts of wormhole contacts at the level of M^8?

The experience with H, in particular the presence of extremals with Euclidian signature of the induced metric and identified as building bricks of elementary particles, suggest that also the light-like 3-surfaces in M_c^8 could have a continuation with an Euclidian signature of the number theoretic metric with norm having real values only for the projections to planes allowing real coordinates.

The earlier picture has been that the wormhole contacts as CP_2 type extremals correspond to co-associative regions and their exteriors to associative regions. If one wants $M^8 - H$ duality in strong form and thus without need for SH, one should assume that both these regions are co-associative.

1. The simplest option is that the real Minkowskian time coordinate becomes imaginary. Instead of the canonical (I_1, iI_3, iI_5, iI_7) the basis would be (iI_1, iI_3, iI_5, iI_7) having Euclidian

signature and SO(4) as isometry group. The signature would naturally change at light-like 3-surface the time coordinate along light-like curves becomes zero - proper time for photon vanishes - and can ransforms continuously from real to imaginary.

2. Wormhole contacts in H behave like pairs of magnetic monopoles with monopole charges at throats. If one does not allow point-like singularity, the monopole flux must go to a parallel Minkowskian space-time sheet through the opposite wormhole throat. Wormhole contact with effective magnetic charge would correspond in M^8_c to a distribution of normal 4-planes at the partonic 2-surfaces analogous to the radial magnetic field of monopole at a sphere surrounding it. To avoid singularity of the distribution, there must be another light-like 3-surface M^8 such that its partonic throat has a topologically similar distribution of normal planes.

In the case of X^3_c dimension does not allow co-quaternion structure: could they allow 4-D co-associative sub-manifolds? It will be found that this option is not included since co-associative tangent space distributions in a quaternion manifold (now O) are always integrable.

8.3.4 Octonionic Dirac equation and co-associativity

Also the role of associativity concerning octonionic Dirac equation in M^8 must be understood. It is found that co-associativity allows very elegant formulation and suggests the identification of the points appearing as the ends of quark propagator lines in H as points of boundary of CD representing light-like momenta of quarks. Partonic vertices would involve sub-CDs and momentum conservation would have purely geometric meaning bringing strongly in mind twistor Grassmannian approach [B31, B29, ?]. I have discussed the twistor lift of TGD replacing twistors as fields with surfaces in twistor space having induced twistor structure in [K86, L50, L57] [L70, L71].

Octonionic Dirac equation

The following arguments lead to the understanding of co-associativity in the case of octonion spinors. The constant spinor basis includes all spinors but the gamma matrices appearing in the octonionic Dirac equation correspond to co-associative octonion units.

1. At the level of O_c the idea about massless Dirac equation as partial differential equation does not make sense. Dirac equation must be algebraic and the obvious idea is that it corresponds to the on mass shell condition for a mode of ordinary Dirac equation with well-define momentum: $p^k\gamma_k\Psi = 0$ satisfying $p^kp_k = 0$. This suggests that octonionic polynomial P defines the counterpart of $p^k\gamma_k$ so that gamma matrices γ_k would be represented as octonion components. Does this make sense?

2. Can one construct octonionic counterparts of gamma matrices? The imaginary octonion units I_k indeed define the analogs of gamma matrices as $\gamma_k \equiv iI_k$ satisfying the conditions $\{\gamma_k, \gamma_l\} = 2\delta_{kl}$ defining Euclidian gamma matrices. The problem is that one has $I_0I_lk + I_kI_0 = 2I_k$. One manner to solve the problem would be to consider tensor products $I_0\sigma_3$ and $I_k\sigma_2$ where σ_3 and $sigma_2$ are Pauli's sigma matrices with anti-commutation relations $\{\sigma_i, \sigma_j\} = \delta_{i,j}$. Note that I_k do not allow a matrix representation.

   Co-associativity condition suggests an alternative solution. The restriction of momenta to be co-associative and therefore vanishing component p^0 as octonion, would selects a sub-space spanned by say the canonical choice $\{I_2, iI3, iI_5, iI_7\}$ satisfying the anticommutation relations of Minkowskian gamma matrices. Octonion units do not allow a matrix representation because they are not associative. The products for a co-associative subset of octonion units are however associative $(a(bc) = (ab)c$ so that they can be mapped to standard gamma matrices in Minkowski space. Co-associativity would allow the representation of 4-D gamma matrices as a maximal associative subset of octonion units.

3. What about octonionic spinors. The modes of the ordinary Dirac equation with a well-defined momentum are obtained by applying the Dirac operator to an orthogonal basis of constant spinors u_i to give $\Psi = p^k\gamma_ku_i$. Now the counterparts of constant spinors u_i would naturally

be octonion units $\{I_0, I_k\}$: this would give the needed number 8 of real spinor components as one has for quark spinors.

Dirac equation reduces to light-likeness conditions $p^k p_k = 0$ and p_k must be chosen to be real - if pk are complex, the real and imaginary parts of momentum are parallel. One would obtain an entire 3-D mass shell of solution and a single mode of Dirac equation would correspond to a point of this mass shell.

Remark: Octonionic Dirac equation is associative since one has a product of form $(p_k\gamma_k)^2 u_i$ and octonion products of type $x^2 y$ are associative.

4. p^k would correspond to the restriction of $P(o_c)$ to M^4 as sub-space of octonions. Since co-associativity implies $P(o_c) = Y(o_c)o_c$ restricted to counterpart of M^4 (say subspace spanned by $\{I_2, iI3, iI_5, iI_7\}$), Dirac equation reduces to the condition $o^k o_k = 0$ in M^4 defining a light-cone of M^4. This light-cone is mapped to a curved light-like 3-surface X^3 in o_c as $o_c \rightarrow P(o_c) = Y o_c$. $M^8 - H$ duality maps points of space-time surface on M^8 H and therefore the light-cone of M^4 corresponds to either light-like boundary of CD. It seems that the image of X^3 in H has M^4 projection to the light-like boundary of CD.

   Co-associative space-time surfaces have 3-D intersections X^3 with the surface $P = 0$: the conjecture is that X^3 corresponds to a light-like orbit of partonic 2-surfaces in H at which the induced metric signature changes. At X^3 one has besides $X = 0$ also $Y = 0$ so that octonionic Dirac equation $P(o_c)\Psi = P^k I_k \Psi = Y p^k I_k \Psi = 0$ is trivially satisfied for all momenta $p^k = o^k$ defined by the M^4 projections of points of X^3 and one would have $P^k = Y p^k = 0$ so that the identification of P^k as 4-momentum would not allow to assign non-vanishing momenta to X^3. The direction of p^k is constrained only by the condition of belonging to X^3 and the momentum would be in general time-like since X^3 is inside future light-cone.

   $Y = 0$ condition conforms with the proposal that X^3 defines a boundary of Minkowskian and Euclidian region: Euclidian mass shell condition for real P^k requires $P^k = 0$. The general complex solution to $P^2 = 0$ condition is $P = P_1 + iP_2$ with $P_1^2 = P_2^2$.

A single mode of Dirac equation with a well-defined value of p^k as the analog of 4-momentum would correspond to a selection of single time-like point at X^3 or light-like point at the light-like boundary of CD. X^3 intersects light-cone boundary as part of boundary of 7-D light-cone. The picture about scattering amplitudes - consistent with the view about cognitive representations as a unique discretization of space-time surface - is that quarks are located at discrete points of partonic 2-surfaces representing the ends of fermionic propagator lines in H and that one can assign to them light-like momenta.

Challenging the form of $M^8 - H$ duality for the map $M^4 \subset M^8$ to $M^4 \subset H$

The assumption that the map $M^4 \subset M^8$ to $M^4 \subset H$ in $M^8 - H$ duality is a simple identification map has not been challenged hitherto.

1. Octonionic Dirac equation forces the identification of M^8 as analog of 8-D momentum space and the earlier simple identification is in conflict with Uncertainty Principle. Inversion allowed by conformal invariance is highly suggestive: what comes first in mind is a map $m^k \rightarrow \hbar_{eff} m^k / m^k m_k$.

   At the light-cone boundary the map is ill-defined. Here on must take as coordinate the linear time coordinate m^0 or equivalently radial coordinate $r_M = m^0$. In this case the map would be of fporm $t \rightarrow \hbar_{eff}/m^0$: m^0 has interpretation as energy of massless particle.

   The map would give a surprisingly precise mathematical realization for the intuitive arguments assigning to mass a length scale by Uncertainty Principle.

2. Additional constraints on $M^8 - H$ duality in M^4 degrees of freedom comes from the following argument. The two half-cones of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T - o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T - o)$: the space-time surfaces at half-cones would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - \overline{o}^2$ only, the condition is identically satisfied for both options.

There are two options for the identification of the coordinate t.

Option a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time.

At the level of M^8, The M^4 projection of CD_8 is a union of future and past directed light-cones with a common tip rather than CD_4. Both incoming and outgoing momenta have the same origin automatically. This identification is the natural one at the level of M^8.

Option b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The half-cone at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.

If **Option a)** is realized at the level of M^8 and **Option b)** at the level of H, as seems natural, a time translation $m^0 \rightarrow m^0 + 2T$ of the past directed light-cone in $M^4 \subset H$ is required in order to to give upper half-cone of CD_4.

3. The map of the momenta to embedding space points does not prevent the interpretation of the points of M^8 as momenta also at the level of H since this information is not lost. One cannot identify p^k as such as four-momentum neither at the level of M^8 nor H as suggested by the naïve identification of the Cartesian factors M^4 for M^8 and H. This problem is circumvented by a conjugation in M^8_c changing the sign of 3-momentum. The light-like momenta along the light-cone boundary are non-physical but transform to light-like momenta arriving into light-cone as the physical intuition requires.

   Therefore the map would have in the interior of light-cone roughly the above form but there is still a question about the precise form of the map. Does one perform inversion for the M^4 projection or does one take M^4 projection for the inversion of complex octonion. The inversion of M^4 projection seems to be the more plausible option. Denoting by $P(o_c)$ the real M^4 projection of X^4 point one therefore has:

$$P(o_c) \rightarrow \hbar_{eff} \frac{\overline{P(o_c)}}{P(o_c) \cdot P(o_c)} \quad . \tag{8.3.7}$$

   Note that the conjugation changes the direction of 3-momentum.

   At the light-cone boundary the inversion is ill-defined but Uncertainty Principle comes in rescue, and one can invert the M^4 time coordinate:

$$Re(m^0) = t \rightarrow \hbar_{eff} \frac{1}{t} \quad . \tag{8.3.8}$$

A couple of remarks are in order.

1. The presence of $\hbar_{eff}$ instead of $\hbar$ is required by the vision about dark matter. The value of $\hbar_{eff}/h_0$ is given by the dimension of extension of rationals identifiable as the degree of P.

2. The image points $\overline{p}^k$ in H would naturally correspond to the ends of the propagator lines in the space-time representation of scattering amplitudes.

The information about momenta is not lost in the map. What could be the interpretation of the momenta $\overline{p}^k$ at the level of H?

1. Super-symplectic generators at the partonic vertices in H do not involve momenta as labels. The modes of the embedding space spinor field assignable to the ground states of super-symplectic representations at the boundaries of CD have 4-momentum and color as labels. The identification of $\overline{p}^k$ as this momentum label would provide a connection with the classical picture about scattering events.

   At the partonic 2-surfaces appearing as vertices, one would have a sum over the ground states (spinor harmonics). This would give integral over momenta but $M^8 - H$ duality and number theoretic discretization would select a finite subset and the momentum integral would reduce to a discrete sum. The number of M^8 points with coordinates in a given extension of rationals is indeed finite.

2. $M^4 \subset M^8$ could be interpreted as the space of 4-momenta labeling the spinor harmonics of M^8. Same would apply at the level of H: spinor harmonics would correspond to the ground states of super-symplectic representations.

3. The interpretation of the points of M^4_c as complex 4-momenta inspires the question whether the interpretation of the imaginary part of the momentum squared in terms of decay decay width so that M^8 picture would code even information about the dynamics of the particles.

8.4 How to achieve periodic dynamics at the level of $M^4 \times CP_2$?

Assuming $M^8 - H$ duality, how could one achieve typical periodic dynamics at the level of H - at least effectively?

It seems that one cannot have an "easy" solution to the problem?

1. Irreducible polynomials which are products of monomials corresponding to roots r_n which are in good approximation evenly spaced $r_n = r_0 + nr_1\Delta r_n$ would give "very special moments in the life of self" as values of M^4 time which are evenly spaced [L65, L63]. This could give rise to an effective periodicity but it would be at the level of M^8, not H, where it is required.

2. Is it enough that the periodic functions are *only* associated with the spinor harmonics of H involved with the construction of scattering amplitudes in H [L88]? For the modified Dirac equation [K96] the periodic behavior is possible. Note also that the induced spinors defining ground states of super-symplectic representations are restrictions of second quantized spinors of H proportional to plane waves in M^4. These solutions do not guarantee quantum classical correspondence.

8.4.1 The unique aspects of Neper number and number theoretical universality of Fourier analysis

Could one assume more general functions than polynomials at the level of H? Discrete Fourier basis is certainly an excellent candidate in this respect but does it allow number theoretical universality?

1. Discrete Fourier analysis involves in the Euclidian geometry periodic functions $exp(2\pi x)$, n integer and in hyperbolic geometry exponential functions $exp(kx)$.

   Roots of unity $exp(i2\pi/n)$ allow to generalize Fourier analysis. The p-adic variants of $exp(ix)$ exist for rational values of $x = k2\pi/n$ for $n = K$ if $exp(i2\pi/K)$ belongs to the extension of rationals. $x = k = 2pi/n$ does not exist as a p-adic number but $exp(x) = exp(i2\pi/n)$ can exist as phase replacing x as coordinate in extension of p-adics. One can therefore define Fourier basis $\{exp(inx)|n \in Z\}$ which exist at discrete set of rational points $x = k/n$

   Neper number e is also p-adically exceptional in that e^p exists as a p-adic number for all primes p. One has a hierarchy of finite-D extensions of p-adic numbers spanned by the roots $e^{1/n}$. Finiteness of cognition might allow them. Hyperbolic functions $exp(nx)$, $n = 1, 2...$ would have values in extension of p-adic number field containing $exp(1/N)$ in a discrete set of points $\{x = k/N|k \in Z\}$.

2. (Complex) rationality guarantees number theoretical universality and is natural since CP_2 geometry is complex. This would correspond to the replacement $x \rightarrow exp(ix)$ or $x \rightarrow exp(x)$ for powers x^n. The change of the signature by replacing real coordinate x with ix would automatically induce this change.

3. Exponential functions are in a preferred position also group theoretically. Exponential map maps $g \rightarrow exp(itg)$ the points of Lie algebra to the points of the Lie group so that the tangent space of the Lie algebra defines local coordinates for the Lie group. One can say that tangent space is mapped to space itself. M^4 defines an Abelian group and the exponential map would mean replacing of the M^4 coordinates with their exponential, which are p-adically more natural. Ordinary Minkowski coordinates have both signs so that they would correspond to the Lie algebra level.

4. CP_2 is a coset space and its points are obtained as selected points of $SU(3)$ using exponentiation of a commutative subalgebra t in the decomposition $g = h + t + \bar{t}$ in the Lie-algebra of $SU(3)$. One could interpret the CP_2 points as exponentials and the emergence of exponential basis as a basis satisfying number theoretical universality.

8.4.2 Are CP_2 coordinates as functions of M^4 coordinates expressible as Fourier expansion

Exponential basis is not natural at the level of M^8. Exponential functions belong to dynamics, not algebraic geometry, and the level H represents dynamics.

It is the dependence of CP_2 coordinates on M^4 coordinates, where the periodicity is needed. The map of the tangent spaces of $X^4 \subset M^8$ to points of CP_2 is slightly local since it depends on the first derivatives crucial for dynamics. Could this bring in dynamics and exponential functions at the level of H?

These observations inspire the working hypothesis that CP_2 points as functions of M^4 coordinates are expressible as polynomials of hyperbolic and trigonometric exponentials of M^4 coordinates.

Consider now the situation in more detail.

1. The basis for roots of e would be characterized by integer K in $e^{1/K}$. This brings in a new parameter characterizing the extension of rationals inducing finite extensions of p-adic numbers. K is analogous to the dimension of extension of rationals: the p-adic extension has dimension $d = Kp$ depending on the p-adic prime explicitly.

2. If CD size T is given, $e^{-T/K}$ defines temporal and spatial resolution in H. K or possibly Kp could naturally correspond to the gravitational Planck constant [L53] [K6] [E1] $K = n_{gr} = \hbar_{gr}/h_0$.

3. In [L90] many-sheetedness with respect to CP_2 was proposed to correspond to flux tubebundles in M^4 forming quantum coherent structures. A given CP_2 point corresponds to several M^4 points with the same tangent space and their number would correspond to the number of the flux tubes in the bundle.

   Does the number of these points relate to K or Kp? p-Adic extension would have finite dimension $d = Kp$. Could $d = Kp$ be analogous to a degree of polynomial defining the dimension of extension of rationals? Could this be true in p-adic length scale resolution $O(p^2) = 0$ The number of points would be Kp and very large. For electron one has $p = M_{127} = 2^{127} - 1$.

4. The dimension n_A Abelian extension associated with EQ would naturally satisfy $n_A = K$ since the trigonometric and hyperbolic exponentials are obtained from each other by replacing a real coordinate with an imaginary one.

5. There would be two effective Planck constants. $h_{eff} = nh_0$ would be defined by the degree n of the polynomial P defining $X^4 \subset M^8$. $\hbar_{gr} = n_{gr}h_0$ would define infra-red cutoff in M^4 as the size scale of CD in $H = M^4 \times CP_2$. n *resp.* $n_{gr} = Kp$ would characterize many-sheetedness in M^4 *resp.* CP_2 degrees of freedom.

8.4.3 Connection with cognitive measurements as analogs of particle reactions

There is an interesting connection to the notion of cognitive measurement [L90, L91, L96].

1. The dimension n of the extension of rationals as the degree of the polynomial $P = P_{n_1} \circ P_{n_2} \circ ...$ is the product of degrees of degrees n_i: $n = \prod_i n_i$ and one has a hierarchy of Galois groups G_i associated with $P_{n_i} \circ ....$ G_{i+1} is a normal subgroup of G_i so that the coset space $H_i = G_i/G_{i+1}$ is a group of order n_i. The groups H_i are simple and do not have this kind of decomposition: simple finite groups appearing as building bricks of finite groups are classified. Simple groups are primes for finite groups.

2. The wave function in group algebra $L(G)$ of Galois group G of P has a representation as an entangled state in the product of simple group algebras $L(H_i)$. Since the Galois groups act on the space-time surfaces in M^8 they do so also in H. One obtains wave functions in the space of space-time surfaces. G has decomposition to a product (not Cartesian in general) of simple groups. In the same manner, $L(G)$ has a representation of entangled states assignable to $L(H_i)$ [L90, L96].

This picture leads to a model of analysis as a cognitive process identified as a cascade of "small state function reductions" (SSFRs) analogous to "weak" measurements.

1. Cognitive measurement would reduce the entanglement between $L(H_1)$ and $L(H_2)$, the between $L(H_2)$ and $L(H_3)$ and so on. The outcome would be an unentangled product of wave functions in $L(H_i)$ in the product $L(H_1) \times L(H_2) \times ....$ This cascade of cognitive measurements has an interpretation as a quantum correlate for analysis as factorization of a Galois group to its prime factors. Similar interpretation applies in M^4 degrees of freedom.

2. This decomposition could correspond to a replacement of P with a product $\prod_i P_i$ of polynomials with degrees $n = n_1 n_2 ...$, which is irreducible and defines a union of separate surfaces without any correlations. This process is indeed analogous to analysis.

3. The analysis cannot occur for simple Galois groups associated with extensions having no decomposition to simpler extensions. They could be regarded as correlates for irreducible primal ideas. In Eastern philosophies the notion of state empty of thoughts could correspondto these cognitive states in which SSFRs cannot occur.

4. An analogous process should make sense also in the gravitational sector and would mean the splitting of $K = n_A$ appearing as a factor $n_{gr} = Kp$ to prime factors so that the sizes of CDs involved with the resulting structure would be reduced. This process would reduce to a simultaneous measurement cascade in hyperbolic and trigonometric Abelian extensions. The IR cutoffs having interpretation as coherence lengths would decrease in the process as expected. Nature would be performing ordinary prime factorization in the gravitational degrees of freedom.

Cognitive process would also have a geometric description.

1. For the algebraic EQs, the geometric description would be as a decay of n-sheeted 4-surface with respect to M^4 to a union of n_i-sheeted 4-surfaces by SSFRs. This would take place for flux tubes mediating all kinds of interactions.

   In gravitational degrees of freedom, that is for trascendental EQs, the states with $n_{gr} = Kp$ having bundles of Kp flux tubes would deca to flux tubes bundles of $n_{gr,i} = K_i p$, where K_i is a prime dividing K. The quantity $log(K)$ would be conserved in the process and is analogous to the corresponding conserved quantity in arithmetic quantum field theories (QFTs) and relates to the notion of infinite prime inspired by TGD [K80].

2. This picture leads to ask whether one could speak of cognitive analogs of particle reactions representing interactions of "thought bubbles" i.e. space-time surfaces as correlates of cognition. The incoming and outgoing states would correspond to a Cartesian product of simple

subgroups: $G = \prod_i^{\times} H_i$. In this composition the order of factors does not matter and the situation is analogous to a many particle system without interactions. The non-commutativity in general case leads to ask whether quantum groups might provide a natural description of the situation.

3. Interestingly, Equivalence Principle is consistent with the splitting of gravitational flux tube structures to smaller ones since gravitational binding energies given by Bohr model in $1/r$ gravitational potential do not depend on the value of $\hbar_{gr}$ if given by Nottale formula $\hbar_{gr} = GMm/v_0$ [L100]. The interpretation would be in terms of spontaneous quantum decoherence taking place as a decay of gravitational flux tube bundles as the distance from the source increases.

8.4.4 Still some questions about $M^8 - H$ duality

There are still on questions to be answered.

1. The map $p^k \rightarrow m^k = \hbar_{eff} p^k / p \cdot p$ defining $M^8 - H$ duality is consistent with Uncertainty Principle but this is not quite enough. Momenta in M^8 should correspond to plane waves in H.

   Should one demand that the momentum eigenstate as a point of cognitive representation associated with $X^4 \subset M^8$ carrying quark number should correspond to a plane wave with momentum at the level of $H = M^4 \times CP_2$? This does not make sense since $X^4 \subset CD$ contains a large number of momenta assignable to fundamental fermions and one does not know which of them to select.

2. One can however weaken the condition by assigning to CD a 4-momentum, call it P. Could one identify P as

   (a) the total momentum assignable to either half-cone of CD

   (b) or the sum of the total momenta assignable to the half-cones?

The first option does not seem to be realistic. The problem with the latter option is that the sum of total momenta is assumed to vanish in ZEO. One would have automatically zero momentum planewave. What goes wrong?

1. Momentum conservation for a single CD is an ad hoc assumption in conflict with Uncertainty Principle, and does not follow from Poincare invariance. However, the sum of momenta vanishes for non-vanishing planewave when defined in the entire M^4 as in QFT, not for planewaves inside finite CDs. Number theoretic discretization allows vanishing in finite volumes but this involves finite measurement resolution.

2. Zero energy states represent scattering amplitudes and at the limit of infinite size for the large CD zero energy state is proportional to momentum conserving delta function just as S-matrix elements are in QFT. If the planewave is restricted within a large CD defining the measurement volume of observer, four-momentum is conserved in resolution defined by the large CD in accordance with Uncertainty Principle.

3. Note that the momenta of fundamental fermions inside half-cones of CD in H should be determined at the level of H by the state of a super-symplectic representation as a sum of the momenta of fundamental fermions assignable to discrete images of momenta in $X^4 \subset H$.

$M^8 - H$-duality as a generalized Fourier transform

This picture provides an interpretation for $M^8 - H$ duality as a generalization of Fourier transform.

1. The map would be essentially Fourier transform mapping momenta of zero energy as points of $X^4 \subset CD \subset M^8$ to plane waves in H with position interpreted as position of CD in H. CD and the superposition of space-time surfaces inside it would generalize the ordinary Fourier

transform . A wave function localized to a point would be replaced with a superposition of space-time surfaces inside the CD having interpretation as a perceptive field of a conscious entity.

2. $M^8 - H$ duality would realize momentum-position duality of wave mechanics. In QFT this duality is lost since space-time coordinates become parameters and quantum fields replace position and momentum as fundamental observables. Momentum-position duality would have much deeper content than believed since its realization in TGD would bring number theory to physics.

How to describe interactions of CDs?

Any quantum coherent system corresponds to a CD. How can one describe the interactions of CDs? The overlap of CDs is a natural candidate for the interaction region.

1. CD represents the perceptive field of a conscious entity and CDs form a kind of conscious atlas for M^8 and H. CDs can have CDs within CDs and CDs can also intersect. CDs can have shared sub-CDs identifiable as shared mental images.

2. The intuitive guess is that the interactions occur only when the CDs intersect. A milder assumption is that interactions are observed only when CDs intersect.

3. How to describe the interactions between overlapping CDs? The fact the quark fields are induced from second quantized spinor fields in in H *resp.* M^8 solves this problem. At the level of H, the propagators between the points of space-time surfaces belonging to different CDs are well defined and the systems associated with overlapping CDs have well-defined quark interactions in the intersection region. At the level of M^8 the momenta as discrete quark carrying points in the intersection of CDs can interact.

Zero energy states as scattering amplitudes and subjective time evolution as sequence of SSFRs

This is not yet the whole story. Zero energy states code for the ordinary time evolution in the QFT sense described by the S-matrix. What about subjective time evolution defined by a sequence of "small" state function reductions (SSFRs) as analogs of "weak" measurements followed now and then by BSFRs? How does the subjective time evolution fit with the QFT picture in which single particle zero energy states are planewaves associated with a fixed CD.

1. The size of CD increases at least in statistical sense during the sequence of SSFRs. This increase cannot correspond to M^4 time translation in the sense of QFTs. Single unitary step followed by SSFR can be identified as a scaling of CD leaving the passive boundary of the CD invariant. One can assume a formation of an intermediate state which is quantum superposition over different size scales of CD: SSFR means localization selecting single size for CD. The subjective time evolution would correspond to a sequence of scalings of CD.

2. The view about subjective time evolution conforms with the picture of string models in which the Lorentz invariant scaling generator L_0 takes the role of Hamiltonian identifiable in terms of mass squared operator allowing to overcome the problems with Poincare invariance. This view about subjective time evolution also conforms with super-symplectic and Kac-Moody symmetries of TGD.

   One could perhaps say that the Minkowski time T as distance between the tips of CDs corresponds to exponentiated scaling: $T = exp(L_0 t)$. If t has constant ticks, the ticks of T increase exponentially.

The precise dynamics of the unitary time evolutions preceding SSFRs has remained open.

1. The intuitive picture that the scalings of CDs gradually reveal the entire 4-surface determined by polynomial P in M^8: the roots of P as "very special moments in the life of self" would correspond to the values of time coordinate for which SSFRs occur as one new root emerges.

These moments as roots of the polynomial defining the space-time surface would correspond to scalings of the size of both half-cones for which the space-time surfaces are mirror images. Only the upper half-cone would be dynamical in the sense that mental images as sub-CDs appear at "geometric now" and drift to the geometric future.

2. The scaling for the size of CD does *not* affect the momenta associated with fermions at the points of cognitive representation in $X^4 \subset M^8$ so that the scaling is not a genuine scaling of M^4 coordinates which does not commute with momenta. Also the fact that L_0 for super symplectic representations corresponds to mass squared operator means that it commutes with Poincare algebra so that M^4 scaling cannot be in question.

3. The Hamiltonian defining the time evolution preceding SSFR could correspond to an exponentiation of the sum of the generators L_0 for super-symplectic and super-Kac Moody representations and the parameter t in exponential corresponds to the scaling of CD assignable to the replaced of root r_n with root r_{n+1} as value of M^4 linear time (or energy in M^8). L_0 has a natural representation at light cone boundaries of CD as scalings of light-like radial coordinate.

4. Does the unitary evolution create a superposition over all over all scalings of CD and does SSFR measure the scale parameter and select just a single CD?

   Or does the time evolution correspond to scaling? Is it perhaps determined by the increase of CD from the size determined by the root r_n as "geometric now" to the root r_{n+1} so that one would have a complete analogy with Hamiltonian evolution? The scaling would be the ratio r_{n+1}/r_n which is an algebraic number.

   Hamiltonian time evolution is certainly the simplest option and predicts a fixed arrow of time during SSFR sequence. L_0 identifiable essentially as a mass squared operator acts like conjugate for the logarithm of the logarithm of light-cone proper time for a given half-cone.

   One can assume that L_0 as the sum of generators associated with upper and lower half-cones if the fixed state at the lower half-cone is eigenstate of L_0.

How does this picture relate to p-adic thermodynamics in which thermodynamics isdetermined by partition function which would in real sector be regarded as a vacuum expectation value of an exponential $exp(iL_0t)$ of a Hamiltonian for imaginary time $t = i\beta$ $\beta = 1/T$ defined by temperature. L_0 is proportional to mass squared operator.

1. In p-adic thermodynamics temperature T is dimensionless parameter and $\beta = 1/T$ is integer valued. The partition function as exponential $exp(-H/T)$ is replaced with $p^{\beta L_0)}$, $\beta = n$, which has the desired behavior if L_0 has integer spectrum. The exponential form $e^{L_0/T_R)}$, $\beta_R = nlog(p)$ equivalent in the real sector does not make sense p-adically since the p-adic exponential function has p-adic norm 1 if it exists p-adically.

2. The time evolution operator $exp(-iL_0t)$ for SSFRs (t would be the scaling parameter) makes sense for the extensions of p-adic numbers if the phase factors for eigenstates are roots of unity belonging to the extension. $t = 2\pi k/n$ since L_0 has integer spectrum. SSFRs would define a clock. The scaling $exp(t) = exp(2\pi k/n)$ is however not consistent with the scaling by r_{n-1}/r_n.

   Both the temperature and scaling parameter for time evolution by SSFRs would be quantized by number theoretical universality. p-Adic thermodynamics could have its origins in the subjective time evolution by SSFRs.

3. In the standard thermodynamics it is possible to unify temperature and time by introducing a complex time variable $\tau = t + i\beta$, where $\beta = 1/T$ is inverse temperature. For the space-time surface in complexified M^8, M^4 time is complex and the real projection defines the 4-surface mapped to H. Could thermodynamics correspond to the imaginary part of the time coordinate?

   Could one unify thermodynamics and quantum theory as I have indeed proposed: this proposal states that quantum TGD can be seen as a "complex square root" of thermodynamics. The exponentials $U = exp(\tau L_0/2)$ would define this complex square root and thermo-dynamical partition function would be given by $UU^\dagger = exp(-\beta L_0)$.

8.5 Can one construct scattering amplitudes also at the level of M^8?

$M^8 - H$ duality suggests that the construction is possible both at the level of H and M^8. These pictures would be based on differential geometry on one hand and algebraic geometry and number theory on the other hand. The challenge is to understand their relationship.

8.5.1 Intuitive picture

H picture is phenomenological but rather detailed and M^8 picture should be its pre-image under $M^8 - H$ duality. The following general questions can be raised.

1. Can one construct the counterparts of the scattering amplitudes also at the level of M^8?

2. Can one use $M^8 - H$ duality to map scattering diagrams in M^8 to the level of H?

Consider first the notions of CD and sub-CD.

1. The intuitive picture is that at the level of H that one must surround partonic vertices with sub-CDs, and assign the external light-like momenta with the ends of propagator lines from the boundaries of CD and other sub-CDs. The incoming momenta $\overline{p}^k$ would be assigned to the boundary of sub-CD.

2. What about the situation in M^8? Sub-CDs must have different origin in the general case since the momentum spectrum would be shifted. Therefore the sub-CDs have the same tip - either upper or lower tip, and have as their boundary part of either boundary of CD. A hierarchy of CDs associated with the same upper or lower tip is suggestive and the finite maximal size of CD in H gives IR cutoff and the finite maximal size of CD in M^8 gives UV cutoff.

3. Momentum conservation at the vertices in M^8 could decompose the diagram to sub-diagrams for which the momentum conservation is satisfied. On the basis of QFT experience, one expects that there are some minimal diagrams from which one can construct the diagram: in the TGD framework this diagram would describe 4-quark scattering. The condition that the momenta belong to the extension of rationals gives extremely strong constraints and it is not clear that one obtains any solutions to the conditions unless one poses some conditions on the polynomials assigned with the two boundaries of CD.

   The two half-cones (HCs) of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T - o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T - o)$: the space-time surfaces at HCs would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - r^2$ only, the condition is identically satisfied for both options.

   There are two options for the identification of the coordinate t.

   Option (a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time. The M^4 projection of CD_8 gives a union of future and past directed light-cones with a common tip rather than CD_4 in M^4 at the level of M^8 . Both incoming and outgoing momenta have the same origin automatically. This identification seems to be the natural one at the level of M^8.

   Option (b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The HC at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.

   If Option (a) is realized at the level of M^8 and Option b) at the level of H, as seems natural, a time translation of the past directed light-cone by T in $M^4 \subset H$ is required to give CD_4. The momentum spectra of the two HCs differ only by sign and at least a scattering diagram in which all points are involved is possible. In fact all the pairs of subsets with opposite momenta are allowed. These however correspond to a trivial scattering. The decomposition to say 4-vertices with common points involving momentum space propagator suggests a decomposition into sub-CDs. The smaller the sub-CDs at the tips of the CD, the smaller the momenta are and the better is the IR resolution.

4. The proposal has been that one has a hierarchy of discrete size scales for the CDs. Momentum conservation gives a constraint on the positions of quarks at the ends of propagator lines in M^8 mapped to a constraint for their images in H: the sum of image points in H is however not vanishing since inversion is not a linear map.

5. QFT intuition would suggest that at the level of M^8 the scattering diagrams decompose to sub-diagrams for which momentum conservation is separately satisfied. If two such sub-diagrams A and B have common momenta, they correspond to internal lines of the diagram involving local propagator D_p, whose non-local counterpart at the level of H connects the image point to corresponding point of all copies of B.

   The usual integral over the endpoint of the propagator line $D(x,y)$ at space-time level should correspond to a sum in which the H image of B is shifted in M^4. Introduction of a large number of copies of H image of the sub-diagram looks however extremely ugly and challenges the idea of starting from the QFT picture.

   What comes in mind is that all momenta allowed by cognitive representation and summing up to zero define the scattering amplitude as a kind of super-vertex and that Yanigian approach allows this construction.

8.5.2 How do the algebraic geometry in M^8 and the sub-manifold geometry in H relate?

Space-time surfaces in H have also Euclidian regions - in particular wormhole contacts - with induced metric having Euclidian signature due to the large CP_2 contribution to the induced metric. They are separated from Minkowskian regions by a light-like 3-surfaces identifiable as partonic orbits at which the induced metric becomes degenerate.

1. The possible M^8 counterparts of these regions are expected to have Euclidian signature of the number theoretic metric defined by complexified octonion inner product, which must be real in these regions so that the coordinates for the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ are either imaginary or real. This allows several signatures.

2. The first guess is that the energy p^0 assignable to I_1 becomes imaginary. This gives tachyonic p^2. The second guess is that all components of 3-momentum $\{iI_3, iI_5, iI_7\}$ become imaginary meaning that the length of 3-momentum becomes imaginary.

3. One cannot exclude the other signatures, for instance the situation in which 1 or 2 components of the 3-momentum become imaginary. Hence the transition could occur in 3 steps as $(1,-1,-1-,1) \to (1,1,-1,-1) \to (1,1,1,-1) \to (1,1,1,1)$. The values of $p^2 \equiv Re(p_c^2)$ would be non-negative and also their images in $M^4 \subset H$ would be inside future light-cone. This could relate to the fact that all these signatures are possible inthe twistor Grassmannian approach.

4. These regions belong to the complex mass shell $p_c^2 = r_n = m_0^2 = r_n$ appearing as a root to the co-associativity condition $X = 0$. This gives the conditions

$$
\begin{aligned}
Re(p_c) \cdot Im(p_c^2) &= Im(r_n) \ , \\
Re(p_c^2) \equiv p^2 &= Im(p_c^2) + m_n^2 \ , \\
m_n^2 &\equiv Re(r_n) \geq 0 \ .
\end{aligned}
\tag{8.5.1}
$$

Consider first the case $(1, 1, 1, 1)$.

1. The components of p_c are either real or imaginary. Using the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ the components of p_c are real in the Minkowskian region and imaginary in the totally time-like Euclidian region. One has for the totally time-like momentum $p = (p_0, iIm(p_3))$ in the canonical basis.

   This would give

$$Re(p_c^2) \equiv p^2 = p_0^2 = -Im(p_3)^2 + m_n^2 \quad . \tag{8.5.2}$$

   The number theoretic metric is Euclidian and totally time-like but one has $p^2 \geq 0$ in the range $[m_0^2, 0]$. This region is a natural counterpart for an Euclidian space-time region in H. The region $p^2 \geq m_0^2$ has Minkowskian signature and counterpart for Minkowskian regions in H. The region $0 \leq p^2 < m_0^2$ is a natural candidate for an Euclidian region in M^4.

   Remark: A possible objection is that Euclidian regions in O_c are totally time-like and totally space-like in H.

2. The image of these regions under the map $Re(p^k) \to M^k$ under inversion plus octonionic conjugation defined as $p^k \to \hbar_{eff}\overline{p^k}/p^2$ (to be discussed in more detail in the sequel) consists of points M^k in the future light-cone of $M^4 \subset H$. The image of the real Euclidian region of O_c with $p^2 \in [0, m_0^2)$ is mapped to the region $M^k M_k < \hbar_{eff}^2/m_0^2$ of $M^4 \subset H$.

3. The contribution of CP_2 metric to the induced metric is space-like so that it can become Euclidian. This would naturally occur in the image of a totally time-like Euclidian region and this region would correspond to small scales $M^k M_k < \hbar_{eff}^2/m_0^2$. The change of the signature should take place at the orbits of partonic 2-surfaces and the argument does not say anything about this. The boundary of between the two regions corresponds to momenta $p = (p_0, 0)$ which is is a time-like line perhaps identifiable as the analog of the light-like geodesic defining the M^4 projection of CP_2 type extremal, which is an idealized solution to actual field equations.

This transition does not explain the M^8 counterpart of the 3-D light-like partonic orbit to which the light-light geodesic thickens in the real situation?

The above argument works also for the other signatures of co-associative real sub-spaces and provides additional insights. Besides the Minkowskian signature, 3 different situations with signatures $(1, 1, 1, 1)$, $(1, -1, 1, 1)$, and $(1, -1 - 1, 1)$ with non-space-like momentum squared are possible.

The following formulas list the signatures, the expressions of real momentum squared, and dimension D of the transition transition $Im(p_c^2) = 0$ as generalization of partonic orbit and the possible identification of the transition region.

$$\begin{array}{lll}
\textbf{Signature} & p^2, & D \\
(+,-,-,+): & (p^0)^2 - (p^1)^2 - (p^2)^2 = -Im(p^3)^2 + m_n^2 & 3 \ , \\
\textbf{Identification} & \text{partonic orbit} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,-,+,+): & (p^0)^2 - (p^1)^2 = -Im(p^2)^2 - Im(p^3)^2 + m_n^2 \ , & 2 \ , \\
\textbf{Identification} & \text{string world sheet} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,+,+,+): & (p^0)^2 = -Im(p^1)^2 - Im(p^2)^2 - Im(p^3)^2 + m_n^2 \ , & 1 \ . \\
\textbf{Identification} & \text{string boundary} & .
\end{array} \tag{8.5.3}$$

Since the map of the co-associative normal space to CP_2 does not depend on the signature, $M^8 - H$ duality is well defined for all these signatures. One can ask whether a single transition creates partonic orbit, two transitions a string world sheet and 3 transitions ends of string world sheet inside partonic orbit or even outside it.

8.5.3 Quantization of octonionic spinors

There are questions related to the quantization of octonionic spinors.

1. Co-associative gamma matrices identified as octonion units are associative with respect to their octonionic product so that matrix representation is possible. Do second quantized octonionic spinors in M^8 make sense? Is it enough to second quantize them in M^4 as induced octonionic spinors? Are the anti-commutators of oscillator operators Kronecker deltas or delta functions in which case divergence difficulties might be encountered? This is not needed since the momentum space propagators can be identified as those for E_c^8 restricted to X_r^4 as a subspace with real octonion norm.

   The propagators are just massless Dirac propagators for the choice of M^4 for which light-like M^8 momentum reduces to M^4 momentum. Could one formulate the scattering amplitudes using only massless inverse propagators as in the twistor Grassmannian approach?This does not seem to be the case.

2. Could the counterpart of quark propagator as inverse propagator in M^8 as the idea about defining momentum space integrals as residue integrals would suggest? This would allow on-mass-shell propagation like in twistor diagrams and would conform with the idea that inversion relates M^8 and H descriptions. This is suggested by the fact that no integration over intermediate virtual momenta appears in the graphs defined by the algebraic points of the pre-images of the partonic 2-surfaces X_r^2 .

How to identify external quarks? Note that bosons would consist of correlated quark-antiquark pairs with the propagator obtained as a convolution of quark propagators. The correlation would be present for the external states and possibly also for the states in the diagram and produced by topologically.

1. The polynomial P and the $P = 0$ surface with 6-D real projection X_r^6 is not affected by octonion automorphisms. Quarks with different states of motion would correspond to the same P but to different choices of M^4 as co-associative subspace for M_c^8. P could be seen as defining a class of scattering diagrams. P determines the vertices.

2. The space-time surface associated with a quark carrying given 4-momentum should be obtainable by a Lorenz transformation in $SO(3,1) \subset G_{2,c}$ to give it light-like M^4 so that complexified octonionic automorphisms would generate 3-surfaces representing particles. If $M^4 \subset M^8$ and thus the CD associated with the quark is chosen suitably, the quark is massless. Any incoming particle would be massless in this frame.

   Lorentz invariance however requires a common Lorentz frame provided by the CD. The momentum of a quark in CD would be obtained by $G_{2,c}$ transformation. In the frame of CD the external quark momenta arriving to the interior of CD at vertices associated with $X_r^3 \cap Y_r^3$ are time-like. Momentum conservation would hold in this frame. The difference between massive constituent quarks and massless current quarks could be understood as reflecting M^8 picture.

To sum up, the resulting picture is similar to that at the level of H these diagrammatic structures would be mapped to H by momentum inversion. Quantum classical correspondence would be very detailed providing both configuration space and momentum space pictures.

8.5.4 Does $M^8 - H$ duality relate momentum space and space-time representations of scattering amplitudes?

It would seem that the construction of the scattering amplitudes is possible also at the level of M^8 [L88]. M^8 picture would provide momentum representation of scattering diagrams whereas H picture would provide the space-time representation.

Consider first a possible generalization of QFT picture involving propagators and vertices.

1. At first it seems that it is not possible to talk about propagation at the level of momentum space: in positive energy ontology nothing propagates in momentum space if the propagator

is a free propagator D_p! In ZEO this is not quite so. One can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD): one has momentum space propagation from p to $-p$! The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. The momentum space propagator D_p would actually result from the pairing of creation creation operators with the opposite values of p and the notation $D(p,-p)$ would be more appropriate.

2. In QFT interaction vertices are local in space-time but non-local in momentum space. The n-vertex conserves the total momentum. Therefore one should just select points of M^8 and they are indeed selected by cognitive representation and assign scattering amplitude to this set of points. To each point one could assign momentum space propagator of quark in M^8_c but it would not represent propagation! The vertex would be a multilocal entity defined by the vertices defining the masses involved at light cone boundary and mass shells.

   The challenge would be to identify these vertices as poly-local entities. In the QFT picture there would be a set of n-vertices with some momenta common. What could this mean now? One would have subset sets of momenta summing up to zero as vertices. If two subsets have a common momentum this would correspond to a propagator line connecting them. Should one decompose the points of cognitive representation so that it represents momentum space variant of Feynman graph? How unique this decomposition is and do this kind of decompositions exist unless one poses the condition that the total momenta associated with opposite boundaries sum up to zero as done in ZEO. A given n-vertex in the decomposition means the presence of sub-CDs for which the external momenta sum up to zero. This poses very tight constraints on the cognitive representation, and one can wonder they can be satisfied if the cognitive representation is finite as it is in the generic case.

3. Note that for given a polynomial P allowing only points in cognitive representation, one would *not* have momentum space integrations as in QFT: they could however come from integrations over the polynomial coefficients and would correspond to integration of WCW. In adelic picture one allows only rational coefficients for the polynomials. This strongly suggests that the twistor Grasmmannian picture [B29, **?**, B60, B15] in which residue integral in the momentum space gives as residues inverse quark propagators at the poles. M^8 picture would represent the end result of this integration and only on mass shell quarks would be involved. One could even challenge the picture based on propagators and vertices and start from Yangian algebra based on the generalization of local symmetries to multilocal symmetries [A36, A90] [B24] [L41].

4. In the case of H restriction of the second quantized free quark field of H to space-time surface defines the propagators. In the recent case one would have a second quantized octonionic spinor field in M^8. The allowed modes of H spinor field are just the co-associative modes for fixed selection of M^4 analogous to momentum space spinors and restricted to Y^3_r. One could speak of wave functions at Y^3_r, which is very natural since they correspond to mass shells.

   The induced spinor field would have massless part corresponding to wave functions at the M^4 light-cone boundary and part corresponding to X^3 at which the modes would have definite mass. $P = 0$ would select a discrete set of masses. Could second quantization have the standard meaning in terms of anti-commutation relations posed on a free M^8 spinor field. In the case of M^8_c one avoids normal ordering problems since there is no Dirac action. The anti-commutators however have singularities of type 7-D delta function. The anti-commutators of oscillator operators at the same point are the problem. If only a single quark oscillator operator at a given point of M^8 is allowed since there is no local action in coordinate space with the interaction part producing the usual troubles.

5. Could one perform a second quantization for E^8 spinor field using free Dirac action? Could one restrict the expansion of the spinor field to co-associative space-time surfaces giving oscillator operators at the points of cognitive representation with the additional restriction to the pre-image of given partonic 2-surface, whose identification was already considered. Scattering amplitudes would involve n-vertices consisting of momenta summing up to zero and connected to opposite incoming momenta at the opposite sides of the HCs with the

same tip in M^8. Scattering amplitude would decompose to sub-diagrams defining a cluster decomposition, and would correspond to sub-CDs. The simplest option is that there are no internal propagator lines. The vanishing of the total momenta poses stringent conditions on the points of cognitive representation.

Normal ordering divergences can however produce problems for this option in the case of bosonic charges bilear in oscillator operators. At the level of H the solution came from a bilocal modified Dirac action leading to bilocal expressions for conserved charges. Now Yangian symmetry suggests a different approach: local vertices in momentum space can involve only commuting oscillator operators.

Indeed, in ZEO one can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD). The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. As already noticed, also the momentum space propagator $D_p = D(p, -p)$ would be also a bi-local object.

6. This is not enough yet. If there is only a single quark at given momentum, genuine particle creation is not possible and the particle reactions are only re-arrangements of quarks but already allowing formation of bosons as bound states of quarks and antiquarks. Genuine particle creation demands local composites of several quarks at the same point p having interpretation as a state with collinear momenta summing up to p and able to decay to states with the total momentum p. This suggests the analog of SUSY proposed in [L69]. Also Yangian approach is highly suggestive.

   To sum up, momentum conservation together with the assumption of finite cognitive representations is the basic obstacle requiring new thinking.

8.5.5 Is the decomposition to propagators and vertices needed?

One can challenge the QFT inspired picture.

1. As already noticed, the relationship $P_1(t) = P(2T - t)$ makes it possible to satisfy this condition at least for the entire set of momenta. This does not yet allow non-trivial interactions without posing additional conditions on the momentum spectrum. This does not look nice. One can ask whether there is a kind of natural selection leading to polynomials defining space-time surfaces allowing cognitive representations with vertex decompositions and polynomials $P(t)$ and $P_r(t)$ without this symmetry? This idea looks ugly. Or could evolution start from simplest surfaces allowing 4 vertices and lead to an engineering of more complex scattering diagrams from these?

2. The map of momentum space propagators regarded as completely local objects in M^8 to H propagators is second ugly feature. The beauty and simplicity of the original picture would be lost by introducing copies of sub-diagrams mapped to the various translations in H.

3. The Noether charges of the Dirac action in H fail to give rise to 4-fermion vertex operator. The theory would be naturally just free field theory if one assumes cognitive representations.

The first heretic question is whether the propagators are really needed at the level of momentum space. This seems to be the case.

1. In ZEO the propagators pair creation and operators with opposite 4-momenta assignable to the opposite HCs of CD having conjugate fermionic vacua (Dirac sea of negative energy fermions and Dirac sea of positive energy fermions) so that momentum space propagators $D(p, -p)$ are non-local objects. The propagators would connect positive and negative energy fermions at the opposite HCs and this should be essential in the formulation of scattering amplitudes. They cannot be avoided.

2. The propagators would result from the contractions of fermion oscillator operators giving a 7-D delta function at origin in continuum theory. This catastrophe is avoided in the

number theoretic picture. Since one allows only points with M^8 coordinates in an extension of rationals, one can assume Kronecker delta type anti-commutators. Besides cognitive representations, this would reflect the profound difference between momentum space and space-time.

This would also mean that the earlier picture about the TGD analog of SUSY based on local composites of oscillator operators [L69] makes sense at the level of M^8. The composites could be however local only for oscillator operators associated with the HC of CD. With the same restriction they could be local also in the$H$ picture.

What about vertices? Could Yangian algebra give directly the scattering amplitudes? This would simplify dramatically the $M^8 - H$ duality for transition amplitudes. For this option the $P_1(t) = P(2T - t)$ option required by continuity would be ideal.

1. Without vertices the theory would be a free field theory. The propagators would connect opposite momenta in opposite HCs of CD. Vertices are necessary and they should be associated with sub-CDs. Unless sub-CDs can have different numbers of positive and negative energy quarks at the opposite HCs, the total quark number is the same in the initial and final states if quarks and antiquarks associated with bosons as bound states of fermion and antiquark are counted. This option would require minimally 4-quark vertex having 2 fermions of opposite energies at the two hemi-spheres of the CD. A more general option looks more plausible. One obtains non-trivial scattering amplitudes by contracting fermions assigned to the boundary P (F) past (future) HC of CD to the past (future) boundary P_{sub} (F_{sub}) of a sub-CD. Sub-CD and CD must have an opposite arrow of time to get the signs of energies correctly.

   Sub-CDs would thus make particle creation and non-trivial scattering possible. There could be an arbitrary number of sub-CDs and they should be assignable to the pre-images of the partonic 2-surfaces X^2_r if the earlier picture is correct. The precise identification of the partonic 2-surfaces is still unclear as also the question whether light-like orbits of partonic 2-surfaces meet along their ends in the vertices.

2. As in the case of H, one could assign the analogs of n-vertices at pre-images of partonic 2-surfaces at X^2_r representing the momenta of massive modes of the octonionic Dirac equation and belonging to the cognitive representations. The idea is to use generators of super-Yangian algebra to be discussed later which are both bosonic and fermionic. The simplest construction would assign these generators to the vertices as points in cognitive representation.

   An important point is that Yangian symmetry would be a local symmetry at the level of momentum space and correspond to non-local symmetry at the level of space-time rather than vice versa as usually. The conserved currents would be local composites of quark oscillator operators with same momentum just as they are in QFTs at space-time level representing parallelly propagating quarks and antiquarks.

   The simplest but not necessary assumption is that they are linear and bilinear in oscillator operators associated with the same point of M^8 and thus carrying 8-momenta assignable to the modes of E^8 spinor field and restricted to the co-associative 4-surface. Their number of local composites is finite and corresponds to the number 8 of different states of 8-spinors of given chirality.

   Also a higher number of quarks is possible, and this was indeed suggested in [L69]. The proposal was that instance leptons would correspond to local composites of 3 quarks. The TGD based view about color allows this. These states would be analogous to the monomials of theta parameters in the expansion of super-field. The H picture allows milder assumptions: leptonic quarks reside at partonic 2-surface at different points but this is not necessary.

3. Instead of super-symplectic generators one has $G_{2,c}$ as the complexified automorphism group. Also the Galois group of the extension acts as an automorphism group and is proposed to have a central role in quantum TGD with applications to quantum biology [L39, L87]. As found, $G_{2,c}$ acts as an analog of gauge or Kac-Moody group. Yangian has analogous structure but the analogs of conformal weights are non-negative.

4. The identification of the analogs of the poly-local vertex operators as produces of charges generators associated with FHC anbd PHC is the basic challenge. They should consist of quark creation operators (annihilation operators being associated as creation operators at the opposite HC) and be generators of infinitesimal symmetries which in number theoretic physics would correspond instead of isometries of WCW to the octonionic automorphism group G_2 complexified to $G_{2,c}$ containing also the generators of $SO(4) \subset G_2$ and thus also those of Lorentz group $SO(1,3) \subset G_{2,c}$.

   The construction Noether charges of E^8 second quantized spinor field at momentum space representation gives bilinear expressions in creation and annihilation operators associated with opposite 3-momenta and would have a single fermion in a given HC. This is not enough: there should be at least 4 fermions.

What strongly suggests itself are Yangian algebras [A36] [L41] having poly-local generators and considered already earlier and appearing in the twistor Grassmannian approach [B29, ?]. The sums of various quantum numbers would vanish for the vertex operators. These algebras are quantum algebras and the construction of n-vertices could involve co-algebra operation. What is new as compared to Lie algebras is that Yangian algebras are quantum algebras having co-algebra structure allowing to construct n-local generators representing scattering amplitudes. It might be possible replace oscillator operators with the quantum group counterparts.

8.5.6 Does the condition that momenta belong to cognitive representations make scattering amplitudes trivial?

Yangian symmetry is associated with 2-D integrable QFTs which tend to be physically rather uninteresting. The scattering is in the forward direction and only phase shifts are induced. There is no particle creation. If the relationship $P_1(t) = P(2T - t)$ is applied the momentum spectra for FHC and PHC differ only by the sign. If all momenta are involved and the cognitive representations are finite, the situation would be the same! Also the existence of cluster compositions involving summations of subsets of momenta to zero is implausible. Something seems to go wrong!

The basic reason for the problem is the assumption that the momenta belong to cognitive representations assumed to be finite as they indeed are in the generic case. But are they finite in the recent situation involving symmetries?

1. The assumption that all possible momenta allowed by cognitive representation are involved, allows only forward scattering unless there are several subsets of momenta associated with either HC such that the momenta sum-up to the same total momentum. This would allow the change of the particle number. The subsets S_i with same total momentum p_{tot} in the final state could save as final states of subsets S_j with the same total momentum p in the initial state. What could be the number theoretical origin of this degeneracy?

2. In the generic case the cognitive representation contains only a finite set of points (Fermat theorem, in which one considers rational roots of $x^n + y^n = z^n$, $n > 2$ is a basic example of this) . There are however special cases in which this is not true. In particular, M^4 and its geodesic sub-manifolds provide a good example: all points in the extension of rationals are allowed in M^4 coordinates (note that there are preferred coordinates in the number theoretic context).

   The recent situation is indeed highly symmetric due to the Lorentz invariance of space-time surfaces as roots reducing the equations to ordinary algebraic equations for a single complex variable. $X = 0$ condition gives as a result $a_c^2 = constant$ complex hyperboloid with a real mass hyperboloid as a real projection. $a_c^2 = r_n$ is in the extension of rationals as a root of n:th order polynomial. One has the condition $Re(m^2)^2 - Im(m^2) = Re(r_n)$ giving X_r^4 a slicing by real mass hyperboloids. If $Im(m)$ and the spatial part of $Re(m)$ belongs to the extension, one has for real time coordinate $t = \sqrt{r_M^2 + Im(m^2) + r_n}$. If $r_M^2 + Im(m)^2 + r_n$ is a square in the extension also t belongs to the extension. Cognitive representation would contain an infinite number of points and the it would be possible to have non-trivial cluster decompositions. Scattering amplitude would be a sum over different choices of the momenta of the external particles satisfying momentum conservation condition.

As found, the intersection of X_r^4 and X_r^6 is either empty or X_r^4 belongs to X_r^6, Cognitive representations would have an infinite number of points also now by the previous argument. Partonic 2-surfaces at X_r^3 would be replaced with 3-D surfaces in X_r^4 in this situation and would contain a large number of roots. The partonic 2-surfaces would be still present and correspond to the intersections of incoming space-time surfaces of quarks inside X_r^6. These surfaces would also contain the vertices.

3. Could number theoretic evolution gradually select space-time surfaces for which the number theoretic dynamics involving massive quarks is possible? First would be generic polynomials for which X_r^3 would be empty and only massless quarks arriving at the light-cone boundary would be possible. After that surfaces allowing non-empty X_r^3 and massive quarks would appear. There is a strong resemblance with the view about cosmological evolution starting from massless phases and proceeding as a sequence of symmetry breakings causing particle massivation. Now the massivation would not be caused by Higgs like fields but have purely number theoretic interpretation and conform with the p-adic mass calculations [K46].

   Also a cognitive explosion would occur since these space-time surfaces would be cognitively superior after the emergence of massive quarks. If this picture has something to do with reality, the space-time surfaces contributing to the scattering amplitudes would be very special and interactions could be seen as a kind of number theoretical resonance phenomenon.

4. Even is not enough to obtain genuine particle reaction instead of re-arrangements: one must have also local composites of collinear quarks at the same momentum p identifiable as the sum of parallel momenta discussed in [L69]. This kind of situation is also encountered for on-mass-shell vertices in twistor Grassmannian approach. The local composites could decay to local composites with a smaller number of quarks but respecting momentum conservation. Here the representations of Yangian algebra would come in rescue.

8.5.7 Momentum conservation and on-mass-shell conditions for cognitive representations

Momentum conservation and on-mass shell-conditions are very powerful for cognitive representations, which in the generic case are finite. At mass shells the cognitive representations consist of momenta in the extension of rationals satisfying the condition $p^2 = Re(r_n)$, r_n a complex root of X, which is polynomial of degree n in p^2 defined by the odd part of P. If $\sqrt{Re(r_n)}$ does not belong to the extension defined by P, it can be extended to contain also $\sqrt{Re(r_n)}$.

For Pythagorean triangles in the field of rationals, mass shell condition gives for the momentum components in extension an equation analogous to the equation $k^2 + l^2 = m^2$, which can be most easily solved by noticing that the equation has rotation group $SO(2)$ consisting of rational rotation matrices as symmetries. The solutions are of form $(k = r^2 - s^2, l = 2rs, m = r^2 + s^2)$. By $SO(2)$ invariance, one can choose the coordinate frame so that one has $(k, l) = (r^2 + s^2, 0)$. By applying to this root a rational rotation with $cos(\phi) = (r^2 - s^2)/(r^2 + s^2), sin(\phi) = 2rs/(r^2 + s^2)$ to obtain the general solution $(k = r^2 - s^2, l = 2rs, n = r^2 + s^2)$. The expressions for k and l can be permuted, which means replacing ϕ with $\phi - pi/2$. For a more general case $k^2 + l^2 = n$ one can replace n with $\sqrt{n}$ so that one has an extension of rationals.

For the hyperbolic variants of Pythagorean triangles, one has $k^2 - l^2 = m^2$ or equivalently $l^2 + m^2 = k^2$ giving a Pythagorean triangle. The solution is $k = r^2 + s^2, l = r^2 - s^2, m^2 = 2rs$. The expressions for l and ma can be permuted. Rotation is replaced with 2-D Lorentz boost $cosh(\eta) = (r^2 + s^2)/(r^2 - s^2)$ and $sinh(\eta) = 2rs/(r^2 - s^2)$ with rational matrix elements.

Consider now the 4-D case.

1. The algebra behind the solution depends in no manner on the number field considered and makes sense even for the non-commutative case if m and n commute. Hence one can apply the Pythagorean recipe also in 4-D case to the extension of rationals defined by P by adding to it $\sqrt{r_n}$.

2. Assume that a Lorentz frame can be chosen to be the rest frame in which one has $p = (E = \sqrt{Rer_n}, 0)$ (this might not be possible always). As in the Pythagorean case, there must be a consistency condition. Now it would be of form $E = \sqrt{r_n} = p_0^2 - p_1^2 - p_2^2 - p_3^2$ in the

extension defined by $\sqrt{r_n}$. It is not clear whether this condition can be solved for all choices of momentum components in the extension or assuming that algebraic integers of extension are in question. One can also consider an option in which one has algebraic integer divided by some integer N. p-Adic considerations would suggest that prime powers $N = p^k$ might be interesting.

The solutions $\sqrt{r_n} = p_1^2 - p_2^2$ represent a special case. The general solution is obtained by making Lorenz transformation with a matrix with elements in the discrete subgroup of Lorentz group with matrix elements in the extension of rationals.

3. The solutions would define a discretization of the mass shell (3-D hyperbolic space) defined as the orbit of the infinite discrete subgroup of $SO(1,3)$ considered - perhaps the subgroup of $SL(2,C)$ with matrix elements identified as algebraic integers.

If the entire subgroup of SL(2,C) with matrix elements in the extension of rationals is realized, the situation would correspond effectively to a continuous momentum spectrum for infinite cognitive representations. The quantization of momenta is however physically a more realistic option.

1. An interesting situation corresponds to momenta with the same time component, in which case the group would be a discrete subgroup of $SO(3)$. The finite discrete symmetry subgroups act as symmetries of Platonic solids and polygons forming the ADE hierarchy associated to the inclusions of hyperfinite factors of type II_1 and proposed to provide description of finite measurement resolution in TGD framework.

2. The scattering would be analogous to diffraction and only to the directions specified by the vertices of the Platonic solid. Platonic solids, in particular, icosahedron appear also in TGD inspired quantum biology [L23, L85], and also in Nature. Could their origin be traced to $M^8 - H$ duality mapping the Platonic momentum solids to H by inversion?

A more general situation would correspond to the restriction to a discrete non-compact sub-group $\Gamma \subset SL(2,C)$ with matrix elements in the extension of rationals. $Sl(2,C)$ has a representation as Möbius transformations of upper half-plane H^2 of complex plane acting as conformal transformations whereas the action in H^3 is as isometries. The Möbius transformation acting as isometries of H^2 corresponds to $SL(2,Z)$ having also various interesting subgroups, in particular congruence subgroups.

1. Subgroups Γ of the modular group $SL(2,Z)$ define tessellations (analogs of orindary lattices in a curved space) of both H^2 and H^4. The fundamental domain [A11] (`https://cutt.ly/ahBrtT5`) of the tessellation defined by $\Gamma \subset SL(2,C)$ contains exactly one point at from each orbit of Γ. The fundamental domain is analogous to lattice cell for an Euclidian 3-D lattice.

   Γ must be small enough since the orbits would be otherwise dense just like rationals are a dense sub-set of reals. In the case of rationals this leaves into consideration tje modular subgroup $SL(2,Z)$ or its subgroups. In the recent situation an extension of the modular group allowing matrix elements to be algebraic integers of the extension is natural. Physically this would correspond to the quantization of momentum components as algebraic integers. The tessellation in M^8 and its image in H would correspond to reciprocal lattice and lattice in condensed matter physics.

2. So called uniform honeycombs [A23, A15, A33] (see `https://cutt.ly/xhBwTph`, `https://cutt.ly/lhBwPRc`, and `https://cutt.ly/OhBwUOO`) in H^3 assignable to $SL(2,Z)$ can be regarded as polygons in 4-D space and H^3 takes the roles of sphere S^2 for platonic solids for which the tessellation defined by faces is finite.

   The four regular compact honeycombs in H^3 for which the faces and vertex figures (the faces meeting the vertex) are finite are of special interest physically. In the Schönflies notation characterizing polytopes (tessellations are infinite variants of them) they are labelled by (p,q,r), where p is the number of vertices of face, q is the number of faces meeting at vertex, and s is the number of cells meeting at edge.

The regular compact honeycombs are listed by (5,3,4), (4,3,5), (3,5,3), (5,3,5). For Platonic solids (5,3) characterizes dodecahedron, (4,3) cube, and (3,5) for icosahedron so that these Platonic solids serve as basic building bricks of these tessellations. Rather remarkably, icosahedral symmetries central in the TGD based model of genetic code [L23, L85], characterize cells for 3 uniform honeycombs.

Consider now the momentum conservation conditions explicitly assuming momenta to be algebraic integers. It is natural to restrict the momenta to algebraic integers in the extension of rationals defined by the polynomial P. This allows linearization of the constraints from momentum conservation quite generally.

Pythagorean case allows to guess what happens in 4-D case.

1. One can start from momentum conservation in the Pythagorean case having interpretation in terms of complex integers $p = (r+is)^2 = r^2 - s^2 + 2irs$. The momenta in the complex plane are squares of complex integers $z = r+is$ obtained by map $z \to w = z^2$ and complex integers. One picks up in the w-plane integer momenta for the incoming and outgoing states satisfying the conservation conditions $\sum_i P_{out,i} = \sum_k P_{in,k}$: what is nice is that the conditions are linear in w-plane. After this one checks whether the inverse images $\sqrt{P_{out,i}}$ and $\sqrt{P_{in,i}}$ are also complex integers.

2. To get some idea about constraints, one can check what CM system for a 2-particle system means (it is not obvious whether it is always possible to find a CM system: one could have massive particles which cannot form a rest system). One must have opposite spatial momenta for $P_1 = (r_1+is_1)^2$ and $P_2 = (r_2+is_2)^2$. This gives $r_{s1} = r_2s_2$. The products r_is_i correspond to different compositions of the same integer N to factors. The values of $r_i^2+si^2$ are different.

3. In hyperbolic case one obtains the same conditions since the roles of $r^2 - s^2$ and $r^2 + s^2$ in the conditions are changed so that $r^2 - s^2$ corresponds now to mass mass mass and differs for different decomposition of N to factors. The linearization of the conservation conditions generalizes also to the algebraic extensions of rationals with integers replaced by algebraic integers.

The generalization to the 4-D case is possible in terms of octonions.

1. Replace complex numbers by quaternions $q = q_0+\overline{q}$. The square of quaternion is $q^2 = q_0^2 - \overline{q}\cdot\overline{q} + 2iq_0\overline{q}$. Allowed momenta for given mass correspond to points in q^2-plane. Conservation conditions in the q^2 plane are linear and satisfied by quaternionic integers, which are squares. So that in the q^2 plane the allowed momenta form an integer lattice and the identification as a square selects a subset of this lattice. This generalizes also to the algebraic integers in the extension of rationals.

2. What about the co-associative case corresponding to the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$? Momenta would be as co-associative octonion o but o^2 is a quaternion in the plane defined by $\{I_0, iI_2, iI_4, iI_6\}$. o representable in terms of a complexified quaternion $q = q_0 + i\overline{q}$ as $o = I_4q$ and the in general complex values norm squared is give by $o\overline{o}$ with conjugation of octonionic imaginary units but not i: this gives Minkowskian norm squared. This reduces the situation to the quaternionic case.

3. In this case the CM system for two-particle case corresponds to the conditions $q_{1,0}\overline{q}_1 = q_{2,0}\overline{q}_2$ implying that q_1 and q_2 have opposite directions and $q_{1,0}|\overline{q}_1| = q_{2,0}|\overline{q}_2|$. The ratio of the lengths of the momenta is integer. Now the squares $q_{i,0}|\overline{q}_i|^2$, $i = 1,2$ are factorizations of the same integer N. Masses are in general different.

4. The situation generalizes also to complexified quaternions - the interpretation of the imaginary part of momentum might be in terms of a decay width - and even to general octonions since associativity is not involved with the conditions.

8.5.8 Further objections

The view about scattering amplitudes has developed rather painfully by objections creating little shocks. The representation of scattering amplitudes is based on quark oscillator operator algebra. This raises two further objections.

The non-vanishing contractions of the oscillator operators are necessary for obtaining non-trivial scattering amplitudes but is this condition possible to satisfy.

1. One of the basic deviations of TGD from quantum field theories (QFTs) is the hypothesis that all elementary particles, in particular bosons, can be described as bound states of fermions, perhaps only quarks. In TGD framework the exchange of boson in QFT would mean an emission of a virtual quark pair and its subsequent absorption. In ZEO in its basic form this seems to be impossible.

2. If scattering corresponds to algebra morphism mapping products to products of co-products - the number of quarks in say future HC is higher than in the past HC as required. But how to obtain non-vanishing scattering amplitudes? There should be non-vanishing counterparts of propagators between points of FHC but this is not possible if only creation operators are present in a given HC as ZEO requires. All particle reactions would be re-arrangements of quarks and antiquarks to elementary fermions and bosons (OZI rule of the hadronic string model: `https://en.wikipedia.org/wiki/OZI_rule`). The emission of virtual or real bosons requires the creation of quark antiquark pairs and seems to be in conflict with the OZI rule.

3. It would be natural to assign to quarks and bosons constructed as their bound states non-trivial inner product in a given HC of CD. Is this possible if the counterparts of annihilation operators act as creation operators in the opposite HC? Can one assign inner product to a given boundary of CD by assuming that hermitian conjugates of quark oscillator operators act in the dual Hilbert space of the quark Fock space? Could this dual Hilbert space relate to the Drinfeld's double?

How could one avoid the OZI rule?

1. Is it enough to also allow annihilation operators in given HC? Bosonic $G_{2,c}$ generators could involve them. The decay of boson to quark pair would still correspond to re-arrangement but one would have inner product for states at given HC. The creation of bosons would still be a problem. Needless to say, this option is not attractive.

2. A more plausible solution for this problem is suggested by the phenomenological picture in which quarks at the level of H are assigned with partonic 2-surfaces and their orbits, string world sheets, and their boundaries at the orbits of partonic 2-surfaces. By the discussion in the beginning of this section, these surfaces could correspond at the level of M^8 to space-time regions of complexified space-time surface with real number theoretic metric having signature (+,+,-,-), (+,+,+,-), (+,+,+,+) having 2,3, or 4 time-like dimensions. They would allow non-negative values of mass squared and would be separated from the region of Minkowskian signature by a transition region space-time region with dimension $D \in \{3,2,1\}$ mapped to CP$_2$.

   In these regions one would have 1, 2, or 3 additional energy like momentum components $p_i = E_i$. E_i. Could the change of sign for E_i transform creation operator to annihilation operator as would look natural. This would give bosonic states with a non-vanishing norm and also genuine boson creation. What forces to take this rather radical proposal seriously that it conforms with the phenomenological picture.

   In this region one could have a non-trivial causal diamond CD with signature (+,+,-,-), (+,+,+,-). For the signature (+,+,+,+) CD reduces to a point with a vanishing four-momentum and would correspond to CP_2 type extremals (wormhole contacts). Elementary fermions and bosons would consist of quarks in regions with signature (+,+,-,-) and (+,+,+,-). It would seem that the freedom to select signature in twistorial amplitude is not mere luxury but has very deep physical content.

One can invent a further objection. Suppose that the above proposal makes sense and allows to assign propagators to a given HC. Does Yangian co-product allow a construction of zero energy states giving rise to scattering amplitudes, which typically have a larger number of particles in the future HC (FHC) than in past HC (PHC) and represent a genuine creation of quark pairs?

1. One can add to the PHC quarks and bosons one-by-one by forming the product super $G(2,c)$ generators assignable to the added particles. To the FHC one would add the product of co-products of these super $G(2,c)$ generators (co-product of product is product of co-products as an algebra morphism).

2. By the basic formula of co-product each addition would correspond to a superposition of two states in FHC. The first state would be the particle itself having suffered a forward scattering. Second state would involve 2 generators of super $G_{2,c}$ at different momenta summing up to that for the initial state, and represent a scattering $q \rightarrow q+b$ for a quark in the initial state and scattering $b \rightarrow 2b$, $b \rightarrow 2b$, or $b \rightarrow 2q$ for a boson in the initial state.

   Number theoretic momentum conservation assuming momenta to be algebraic integers should allow processes in which quark oscillator operators are contracted between the states in FHC and PHC or between quarks in the FHC.

3. Now comes the objection. Suppose that the state in PC consists of fundamental quarks. Also the FC containing the product of co-products of quarks must contain these quarks with the same momenta. But momentum conservation does not allow anything else in FC! The stability of quarks is a desirable property in QFTs but something goes wrong! How to solve the problem?

   Also now phenomenological picture comes to the rescue and tells that elementary particles - as opposed to fundamental fermions - are composites of fundamental fermions assignable to flux tubes like structures involving 2 wormhole contacts. In particular, quarks as elementary particles would involve quark at either throat of the first wormhole contact and quark-antiquark pair associated with the second wormhole contact. The state would correspond to a quantum superposition of different multilocal momentum configurations defining multi-local states at M^8 level. The momentum conservation constraint could be satisfied without trivializing the scattering amplitudes since the contractions could occur between different components of the superposition - this would be essential.

   Note also that at H level there can be several quarks at a given wormhole throat defining a multilocal state in M^8: one could have a superposition of these states with different momenta and again different components of the wave function could contract. By Uncertainty Principle the almost locality in H would correspond to strong non-locality in M^8. This could be seen as an approximate variant of the TGD variant of H variant of SUSY considered in [L69].

   Could the TGD variant of SUSY proposed in [L69] but realized at the level of momentum space help to circumvent the objection? Suppose that the SUSY multiplet in M^8 can be created by a local algebraic product possessing a co-product delocalizing the local product of oscillator operators at point p in PC and therefore represents the decay of the local composite to factors with momenta at p_1 and $p - p_1$ in FC. This would not help to circumvent the objection. Non-locality and wave functions in momentum space is needed.

8.6 Symmetries in M^8 picture

8.6.1 Standard model symmetries

Can one understand standard model symmetries in M^8 picture?

1. $SU(3) \subset G_2$ would respect a given choice of time axis as preferred co-associative set of imaginary units ($I_2 \subset \{I_2, iI_3, iIb, iI_7\}$ for the canonical choice). The labels would therefore correspond to the group $SU(3)$. $SU(3)_c$ would be analogous to the local color gauge group in

the sense that the element of local $SU(3)_c$ would generate a complecofied space-time surface from the flat and real M^4. The real part of pure $SU(3)_c$ gauge potential would not however reduce to pure $SU(3)$ gauge potential. Could the vertex factors be simply generators of $SU(3)$ or $SU(3)_c$?

2. What about electroweak quantum numbers in M^8 picture? Octonionic spinors have spin and isospin as quantum numbers and can be mapped to H spinors. Bosons would be bound states of quarks and antiquarks at both sides.

   How could electroweak interactions emerge at the level of M^8? At the level of H an analogous problem is met: spinor connection gives only electroweak spinor connection but color symmetries are isometries and become manifest via color partial waves. Classical color gauge potentials can be identified as projections of color isometry generators to the space-time surface.

   Could electroweak gauge symmetries at the level of M^8 be assigned with the subgroup $U(2) \subset SU(3)$ of $CP_2 = SU(3)/U(2)$ indeed playing the role of gauge group? There is a large number of space-time surfaces mapped to the same surface in H and related by a local $U(2)$ transformation. If this transformation acted on the octonionic spinor basis, it would be a gauge transformation but this is not the case: constant octonion basis serves as a gauge fixing. Also the space-time surface in M^8 changes but preserves its "algebraic shape".

8.6.2 How the Yangian symmetry could emerge in TGD?

Yangian symmetry [A36, A90] appears in completely 2-D systems. The article [B38] (`https://arxiv.org/pdf/1606.02947.pdf`) gives a representation which is easy to understand by a physicist like me whereas the Wikipedia article remains completely incomprehensible to me.

Yangian symmetry is associated with 2-D QFTs which tend to be physically rather uninteresting. The scattering is in forward direction and only phase shifts are induced. There is no particle creation. Yangian symmetry appears in 4-D super gauge theories [B24] and in the twistor approach to scattering amplitudes [B25, B33, B29, ?]. I have tried to understand the role of Yangian symmetry in TGD [L41].

Yangian symmetry from octonionic automorphisms

An attractive idea is that the Yangian algebra having co-algebra structure could allow to construct poly-local conserved charges and that these could define vertex operators in M^8.

1. Yangian symmetry appears in 2-D systems only. In TGD framework strings world sheets could be these systems as co-commutative 2-surfaces of co-associative space-time surface.

2. What is required is that there exists a conserved current which can be also regarded as a flat connection. In TGD the flat connection would a connection for $G_{2,c}$ or its subgroup associated with the map taking standard co-associative sub-space of O_c for which the number theoretic norm squared is real and has Minkowski signature (M^4 defined by the canonical choice $\{I_2, iI_3, iI_5, iI_7\}$.

   The recent picture about the solution of co-associativity conditions fixes the subgroup of G_2 to $SU(3)$. X^4 corresponds to element g of the local $SU(3)$ acting on preferred $M^4 \subset M^8_c$ with the additional condition that the 4-surface $X^4 \subset M^8$ is invariant under $U(2) \subset SU(3)$ so that each point of X^4 corresponds to a CP_2 point. At the mas shells as roots of a polynomial P, g reduces to unity and the 4-D tangent space is parallel to the preferred M^4 on which g acts.

   One can induce this flat connection to string world sheet and holomorphy of g at this surface would guarantee the conservation of the current given by $j_{0)} = g^{-1}dg$.

3. Under these conditions the integral of the time component of current along a space-like curve at string world sheets with varying end point is well-defined and the current

$$j_{1)}(x) = \epsilon \mu\nu j_{0),\nu}(x) - \frac{1}{2}[j^{\mu}_{0)}(x,t), \int^x j^0_{0)}(t,y)dy]$$

is conserved. This is called the current at first level. Note that the currents have values in the Lie algebra considered. It is essential that the integration volume is 1-D and its boundary is characterized by a value of single coordinate x.

4. One can continue the construction by replacing j_0 with j_1 in the above formula and one obtains an infinite hierarchy of conserved currents $j_{n)}$ defined by the formula

$$j_{n+1)}(x) \quad = \quad \epsilon\mu\nu j_{n),\nu}(x) - \frac{1}{2}[j^{\mu}_{n)}(x,t), \int^{x} j^{0}_{n)}(t,y)dy] \tag{8.6.1}$$

The corresponding conserved charges Q_n define the generators of Yangian algebra.

5. 2-D metric appears in the formulas. In the TGD framework one does not have Riemann metric - only the number theoretic metric which is real only at real space-time surfaces already discussed. Is the (effective) 2-dimensionality and holomorphy enough to avoid the possible problems? Holomorphy makes sense also number theoretically and implies that the metric disappears from the formulas for currents. Also current conservation reduces to the statement of that current is equivalent to complex differential form.

6. Conserved charges would however require a 1-D integral and number theory does not favor this. The solution of the problem comes from the observation that one can construct a slicing of string world sheet to time-like curves as Hamiltonian orbits with Hamiltonian belonging to the Yangian algebra and defined by the conserved current by standard formula $j^{\alpha} = J^{\alpha\beta}\partial_{\beta}H$ in terms of Kähler form defined by the 2-D Kähler metric of string world sheet. This generalizes to Minkowskian signature and also makes sense for partonic 2-surfaces. Hamiltonians become the classical conserved charges constant along the Hamiltonian orbit. This gives an infinite hierarchy of conserved Hamiltonian charges in involution. Hamiltonian can be any combination of the Hamiltonians in the hierarchy and labelled by a non-negative integer and the label of $G_{2,c}$ generator. This is just what integrability implied by Yangian algebra means. Co-associativity and co-commutativity would be the deeper number theoretic principles implying the Yangian symmetry.

7. Could one formulate this argument in dimension $D = 4$? Could one consider instead of local current the integral of conserved currents over 2-D surfaces labelled by single coordinate x for a given value of t? If the space-time surface in M^8 (analog of Fermi sphere) allows a slicing by orthogonal strings sheets and partonic 2-surfaces, one might consider the fluxes of the currents $g^{-1}dg$ over the 2-D partonic 2-surfaces labelled by string coordinates (t,x) as effectively 2-D currents, whose integrals over x would give the conserved charge. Induced metric should disappear from the expressions so that fluxes of holomorphic differential forms over partonic 2-surface at (t,x) should be in question. Whether this works is not clear.

One should interpret the above picture at the level of momentum space instead of ordinary space-time. The roles of momentum space and space-time are changed. At this point, one can proceed by making questions.

1. One should find a representation for the algebra of the Hamiltonians associated with $g(x)$ defining the space-time surface. The charges are associated with the slicings of string world sheets or partonic 2-surfaces by the orbits of Hamiltonian dynamics defined by a combination of conserved currents so that current conservation becomes charge conservation. These charges are labelled by the coordinate x characterizing the slices defined by the Hamiltonian orbits and from these one can construct a non-local basis discrete basis using Fourier transform.

2. What the quantization of these classical charges - perhaps using fermionic oscillator operators in ZEO picture for which the local commutators vanish - could mean (only the anti-commutators of creation operators associated with the opposite half-cones of CD with opposite momenta are non-vanishing)? Do the Yangian charges involve only creation operators of either type with the same 8-momentum as locality at M^8 level suggests? Locality is

natural l since these Yangian charges are analogous to charges constructed from local currents at space-time level.

3. Could the Yangian currents give rise to poly-local charges assignable to the set of vertices in a cognitive representation and labelled by momenta? Could the level n somehow correspond to the number n of the vertices and could the co-product Δ generate the charges? What does the tensor product appearing in the co-product really mean: do the sector correspond to different total quark numbers for the generators? Is it a purely local operation in M^8 producing higher monomials of creation operators with the same momentum label or is superposition over Hamiltonian slices by Fourier transform possibly involved ?

How to construct quantum charges

One should construct quantum charges. In the TGD framework the quantization of $g(x)$ is not an attractive idea. Could one represent the charges associated with g it in terms of quark oscillator operators induced from the second quantized E^8 spinors so that propagators would emerge in the second quantization? Analogs of Kac Moody representations but with a non-negative spectrum of conformal weights would be in question. Also super-symplectic algebra would have this property making the formulation of the analogs of gauge conditions possible, and realizing finite measurement resolution in terms of hierarchy of inclusions of hyper-finite factors of type II_1 [K95, K30]. The Yangian algebra for $G_{2,c}$ or its subgroup could be the counterpart for these symmetries at the level of H.

The following proposal for the construction for the charges and super-charges of Yangian algebra in terms of quark oscillator operators is the first attempt.

1. One knows the Lie-algebra part of Yangian from the Poisson brackets of Hamiltonians associated with string world sheet slicing and possibly also for a similar slicing for partonic 2-surfaces. One should construct a representation in terms of quark ocillator operators in ZEO framework for both Lie-algebra generators and their super-counterparts. Also co-product should be needed.

2. The oscillator operators of E^8 spinor field located at the points of X^4 are available. The charges must be local and describe states with non-linear quarks and antiquarks.

   One must construct conserved charges as currents associated with the Hamiltonian orbits. Bosonic currents are bilinear in quark and antiquark oscillator operators and their super counterparts linear in quark or antiquark oscillator operators.

3. Since the system is 2-D one can formally assume in Euclidian signature (partonic 2-surface) Kähler metric $g^{z\overline{z}}$ and Kähler form $J^{z\overline{z}} = igz\overline{z}$, which is antisymmetric and real in real coordinates ($J^{kl} = -J^{lk}$) knowing that they actually disappear from the formulas. One can also define gamma matrices $\Gamma_\alpha = \gamma_k \partial_\alpha p^k$ as projections of embedding space gamma matrices to the string world sheet. In the case of string world sheet one can introduce light-like coordinates (u, v) as analogous of complex coordinates and the only non-vanishing component of the metric is g^{uv}.

4. The claim is that the time components J_n^u the bosonic currents

$$J_n^\alpha = b_p^\dagger \overline{v}(p) \Gamma^\alpha H_n u(p) a^\dagger \tag{8.6.2}$$

   at the Hamiltonian curves with time coordinate t define conserved charges ($\alpha \in \{u, v\}$ at the string world sheet).

   Remark: v_p corresponds to momentum $-p$ for the corresponding plane wave in the Fourier expansion of quark field but the physical momentum is p and the point of M^8 that this state corresponds.

   Therefore one should have

$$\frac{J^u_n}{du} = 0 \quad (8.6.3)$$

One can check by a direct calculation what additional conditions are possibly required by this condition.

5. The first point is that H_n is constant if $v = constant$ coordinate line is a Hamiltonian orbit. Also oscillator operators creating fermions and antifermions are constant. The derivative of $u(p)$ is

$$\frac{du(p)}{du} = \frac{\partial u(p)}{\partial_{p^k}}\frac{dp^k}{du} \quad .$$

. u_p is expressible as $u_p = Du_a$, where D is a massless Dirac operator in M^8 and u_a is a constant 8-D quark spinor with fixed chirality. D is sum of M^4- and E^4 parts and M^4 part is given by $D(M^4) = \gamma^k p_k$ so that one has $dp^k/dt = \gamma_r dp^r/dt$.

This gives

$$\frac{d(\Gamma^u H_n u(p))}{du} = g^{uv}\gamma_k \partial_v p^k \frac{du(p)}{du} = g^{uv}\partial_u p \cdot \partial_v p \quad .$$

If the tangent curves of u and v are orthogonal in the induced metric and $v = 0$ constant lines are Hamiltonian orbits the bosonic charges are conserved.

One can perform a similar calculation for $d\frac{d\overline{v}(p)}{du}$ and the result is vanishing.

One must also have $dg^{uv}/du = 0$. This should reduce to the covariant constancy of g^{uv}. If the square root of the metric determinant for string world sheet is included it cancels g^{uv}.

6. From the bosonic charges one construct corresponding fermionic super charges by replacing the fermionic or anti-quark oscillator operator part with a constant spinor.

The simplest option is that partonic 2-surfaces contain these operators at points of cognitive representation. One can ask whether co-product could forces local operators having a higher quark number. What is clear that this number is limited to the number $n = 0$ of spin degrees of $n = 8$.

1. The commutators of bosonic and fermionic charges are fermionic charges and co-product would in this case be a superposition of tensor products of bosonic and fermionic charges, whose commutator gives bosonic charge. Now however the bosonic and fermionic charges commute in the same half-cone of CD. Does this mean that the tensor product in question must be tensor product for the upper and lower half-cones of CD?

   For instance, in the fermionic case one would obtain superposition over pairs of fermions at say lower half-cone and bosons at the upper half-cone. The momenta would be opposite meaning that a local bosonic generator would have total momentum $2p$ at point p and fermionic generator at opposite cone would have momentum $-p$. The commutator would have momentum p as required. In this manner one could create bosons in either half-cone.

2. One can also assign to the bosonic generators a co-product as a pair of bosonic generators in opposite half-cones commuting to the bosonic generator. Assume that bosonic generator is at lower half-cone. Co-product must have a local composite of 4 oscillator operators in the lower half-cone and composite of 2 oscillator operators in the upper half-cone. Their anti-commutator contracts two pairs and leaves an operator of desired form. It therefore seems.

   Statistics allows only generators with a finite number of oscillator operators corresponding to 8 spin indices, which suggests an interpretation in terms of the proposed SUSY [L69]. The roots of P are many-sheeted coverings of M^4 and this means that there are several 8-momenta with the same M^4 projection. This degree of freedom corresponds to Galois degrees of freedom.

3. Only momenta in cognitive representation are allowed and momentum is conserved. The products of generators appearing in the sum defining the co-product of a given generator T, which is a local composite of quarks, would commute or anti-commute to T, and their momenta would sum-up to the momentum associated with T. The co-product would be poly-local and receive contributions from the points of the cognitive representation. Also other quantum numbers are conserved.

About the physical picture behind Yangian and definition of co-product

The physical picture behind the definition of Yangian in the TGD framework differs from that adopted by Drinfeld, who has proposed - besides a general definition of the notion of quantum algebra - also a definition of Yangian. In the Appendix Drinfeld's definition is discussed in detail: this discussion appears almost as such in [L41].

1. Drinfeld proposes a definition in terms of a representation in terms of generators of a free algebra to which one poses relations [B51]. Yangian can be seen as an analog of Kac-Moody algebra but with generators labelled by integer $n \geq 0$ as an analog of non-negative conformal weight. Also super-symplectic algebra has this property and its Yangianization is highly suggestive. The generators of Yangian as algebra are elements J_n^A, $n \geq 0$, with $n = 0$ and $n = 1$. Elements J_0^A define the Lie algebra and elements J_1^A transform like Lie-algebra elements so that commutators at this level are fixed.

   Remark: I have normally used generator as synonym for the element of Lie algebra: I hope that this does not cause confusion

   The challenge is to construct higher level generators J_n^A. Their commutators with JA^0 with $J^A n$ are fixed and also the higher level commutators can be guessed from the additivity of n and the transformation properties of generators J_n^A. The commutators are very similar to those for Kac-Moody algebra. In the TGD picture the representation as Hamiltonians fixes these commutation relations as being induced by a Poisson bracket. The Lie-algebra part of Yangian can be therefore expressed explicitly.

2. The challenge is to understand the co-product Δ. The first thing to notice is that Δ is a Lie algebra homomorphism so that one has $\Delta(XY) = \Delta(X)\Delta(Y)$ plus formulas expressing linearity. The intuitive picture is that Δ adds a tensor factor and is a kind of time reversal of the product conserving total charges and the total value of the weight n. Already this gives a good overall view about the general structure of the co-commutation relations.

   The multiplication of generators by the unit element Id of algebra gives the generator itself so that $\Delta(J_A)$ should involve part $Id \otimes J^A \oplus J^A \otimes Id$. Generators are indeed additive in the ordinary tensor product for Lie-algebra generators - for instance, rotation generators are sums of those for the two systems. However, one speaks of interaction energy: could the notion of "interaction quantum numbers" make sense quite generally. Could this notion provide some insights to proton spin puzzle [C1] meaning that quark spins do not seem to contribute considerably to proton spin? A possible TGD based explanation is in terms of angular momentum associated with the color magnetic flux tubes [K50], and the formulation of this notion at M^8 level could rely on the notion of "interaction angular momentum".

   The time reversal rule applied to $[J_A^m, J_B^n] \propto f_{ABC} J_C^{m+n}$ suggests that $\Delta(T_A^n)$ contains a term proportional to $f_{CBA} J_C^m \otimes J_B^{n-m}$. This would suggest that co-product as a time reversal involves also in the case of J_A^0 the term $k_1 f_{CBA} J_C^0 \otimes J_B^0$, where k_1 as an analog of interaction energy.

   Drinfeld's proposal does not involve this term in accordance with Drinfeld's intuition that co-product represents a deformation of Lie-algebra proportional to a parameter denoted by $\hbar$, which need not (and cannot!) actually correspond to $\hbar$. This view could be also defended by the fact that J_0^A do not create physical states but only measures the quantum numbers generated by J_A^n, $n > 0$. TGD suggests interpretation as the analog of the interaction energy.

3. In Drinfeld's proposal, the Lie-algebra commutator is taken to be $[J_A^0, J_B^0] = k f_{ABC} J_C^0$, $k = 1$. Usually one thinks that generators have the dimension of $\hbar$ so that dimensional

consistency requires $k = \hbar$. It seems that Drinfeld puts $\hbar = 1$ and the $\hbar$ appearing in the co-product has nothing to do with the actual $\hbar$.

The conservation of dimension applied to the co-product would give $k_1 = 1/\hbar$! What could be the interpretation? The scattering amplitudes in QFTs are expanded in powers of gauge coupling strengths $\alpha = g^2/4\pi\hbar$. In ZEO co-product would be essential for obtaining non-trivial scattering amplitudes and the expansion in terms of $1/\hbar$ would emerge automatically from the corrections involving co-products - in path integral formalism this expansion emerges from propagors

This view would also conform with the vision that Mother Nature loves her theoreticians. The increase of $h_{eff}/h_0 = n$ as dimension of extension of rationals would be Mother Nature's way to make perturbation theory convergent [K29]. The increase of the degree of P defining the space-time surface increases the algebraic complexity of the space-time surface but reduces the value of α as a compensation.

4. Drinfeld gives the definition of Yangian in terms of relations for the generating elements with weight $n = 0$ and $n = 1$. From these one can construct the generators by applying Δ repeatedly. Explicit commutation relations are easier to understand by a physicist like me, and I do not know whether the really nasty looking representation relations - Drinfeld himself calls "horrible" [B38] - are the only manner to define the algebra. In the TGD framework the definition based on the idea about co-product as a strict time reversal of product would mean deviation in the $n = 0$ sector giving rise to an interaction term having natural interpretation as analog of interaction energy.

5. Drinfeld proposes also what is known as Drinfeld's double [A91] (see http://tinyurl.com/y7tpshkp) as a fusion of two Hopf algebras and allowing to see product and co-product as duals of each other. The algebra involves slight breaking of associativity characterized by Drinfeld's associator. ZEO suggests [L26] that the members of Drinfeld's double correspond to algebra and co-algebra located at the opposite half-cones and there are two different options. Time reversal occurring in "big" state functions reductions (BSFRs) would transform the members to each other and change the roles of algebra and co-algebra (fusion would become decay).

In the TGD framework there is also an additional degree of freedom related to the momenta in cognitive representation, which could be regarded also as a label of generators. The idea that commutators and co-commutators respect conservation of momentum allows the fixing of the general form of Δ. Co-product of a generator at momentum p ina given half-cone would be in the opposite half-cone and involve sum over all momentum pairs of generators at p_1 and p_2 with the constraint $p_1 + p_2 + p = 0$.

Summation does not make sense for momenta in the entire extension of rationals. The situation changes if the momenta are algebraic integers for the extension of rationals considered: quarks would be particles in a number theoretic box. In the generic case, very few terms - if any - would appear in the sum but for space-time surfaces as roots of octonionic polynomials this is not the case. The co-products would as such define the basic building bricks of the scattering amplitudes obtained as vacuum expectation reducing the pairs of fermions in opposite half-cones to propagators.

8.7 Appendix: Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

8.7.1 Yang-Baxter equation (YBE)

YBE has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A90]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [B24] (see http://tinyurl.com/mcvvcqp) and in conformal field theories and models for anyons. Yangian symmetry appears also in the twistor Grassmann approach to scattering amplitudes [B25, B33] and thus involves YBE. At the same time new invariants for links were discovered and a new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [B51] (see http://tinyurl.com/l4z6zyr, where one can also find a list of references). YBE was first discovered by McGuire (1964) and 3 years later by Yang in a quantum mechanical many-body problem involving a delta function potential $\sum_{i<j} \delta(x_i - x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as a building brick 2-particle scattering matrix - R-matrix. YBE emerged for the R-matrix as a consistency condition for factorization. Baxter discovered in 1972 a solution of the eight vertex model in terms of YBE. Zamolodchikov pointed out that the algebraic mechanism behind factorization of 2-D QFTs is the same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed a quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. The interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of http://tinyurl.com/kyw6xu6). In 2-D quantum field theories and in models for basic gate in topological quantum computation the R-matrix is unitary. R-matrix can be regarded as an endomorphism mapping $V_1 \otimes V_2$ to $V_2 \otimes V_1$ representing permutation of the particles.

YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v) = R_{12}(v)R_{13}(u+v)R_{23}(u) \tag{8.7.1}$$

having interpretation as associativity condition for quantum algebras.

At the limit $u, v \to \infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braiding operation replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on the basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V \otimes V$ the condition states that $R(0)$ is proportional to the permutation matrix P for the factors.

General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued to meromorphic functions in the complex plane with poles forming an Abelian group. R-matrices can be classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have a pole at origin and elliptic solutions have a lattice of poles. In [B51] (see http://tinyurl.com/l4z6zyr) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [B51] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.

3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anythingto do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of a Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be as interesting as Yangian: in this case co-product Δ does not seem to have a natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$. For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.

5. The article of Jimbo discusses also a fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs the R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

8.7.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \rightarrow A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \rightarrow C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Also ϵ satisfies associativity condition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [B51] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n+1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n + 1 \geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretations improve the understanding of classical correspondence (QCC).

Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian for $\mathcal{N}=4$ SUSY [B24], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

However, it must be emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC}J_0^B \times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \times J_1^A + f^A_{BC}J_0^B \times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n=0$ and $n=1$. The first half of these relations discussed in very clear manner in [B24] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB}J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB}J^{(1)C} \quad . \tag{8.7.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right] + \left[J^{(1)B},\left[J^{(1)C},\ J^A\right]\right] + \left[J^{(1)C},\left[J^{(1)A},\ J^B\right]\right] \\
&\qquad = \frac{1}{24}\ f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J_D,J_E,J_F\} \quad , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\ \left[J^C,J^{(1)D}\right]\right] + \left[\left[J^{(1)C},J^{(1)D}\right],\ \left[J^A,J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24}(f^{AGL}f^{BEM}f_K^{CD} \\
&\qquad + f^{CGL}f^{DEM}f_K^{AB})f^{KFN}f_{LMN}\{J_G,J_E,J_F\} \quad .
\end{aligned} \tag{8.7.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

The right hand side often has coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant and as noticed in the main text has dimension of $1/\hbar$. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A} = J^A$ the first Serre relation reduces to Jacobi identity and second to the antisymmetry of the Lie bracket. The right hand side involved completely symmetrized trilinears $\{J_D,J_E,J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra, whose elements are labeled by a non-negative integer n. The generators obtained in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purpose of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exist also for continuum one-dimensional index).

Under certain consistency conditions, a discrete one-dimensional lattice provides a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} = f^A_{BC} \sum_{i<j} J^B_i J^C_j \quad . \tag{8.7.4}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C \end{aligned} \tag{8.7.5}$$

Δ allows to imbed Lie algebra into the tensor product in a non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N} = 4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B24].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can involve besides the unit operator also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrized tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the following form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes the identity matrix and $PU(n|m)$ and $SU(n|m)$ are the same. This does not happen for $n = m$: this is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as a new physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'}J^{(1)C'} = g_{CC'}f^{C'}_{AB}\sum_{i<j}J_i^A J_j^B \\ &= g_{CC'}f^{C'}_{AB}g^{AA'}g^{BB'}\sum_{i<j}J^i_{A'}J^j_{B'} \quad . \end{aligned} \tag{8.7.6}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

8.8 Conclusions

$M^8 - H$ duality plays a crucial role in quantum TGD and this motivated a critical study of the basic assumptions involved.

8.8.1 Co-associativity is the only viable option

The notion of associativity of the tangent or normal space as a number theoretical counterpart of a variational principle. This is not enough in order to have $M^8 - H$ duality. The first guess was that the tangent space is associative and contains a commutative 2-D sub-manifold to guarantee $M^8 - H$ duality.

1. The cold shower came as I learned that 4-D associative sub-manifolds of quaternion spaces are geodesic manifolds and thus trivial. Co-associativity is however possible since any distribution of associative normal spaces integrates to a sub-manifold. Typically these sub-manifolds are minimal surfaces, which conforms with the physical intuitions. Therefore the surface X^4_r given by holography should be co-associative. By the same argument space-time surface contains string world sheets and partonic 2-surfaces as co-complex surfaces.

2. $X = Re_Q(o) = 0$ and $Y = Im_Q(P) = 0$ allow M^4 and its complement as associative/co-associative subspaces of O_c. The roots $P = 0$ for the complexified octonionic polynomials satisfy two conditions $X = 0$ and $Y = 0$.

   Surprisingly, universal solutions are obtained as brane-like entities X^6_c with real dimension 12, having real projection X^6_r ("real" means that the number theoretic complex valued octonion norm squared is real valued).

   Equally surprisingly, the non-universal solutions to the conditions to $X = 0$ correspond complex mass shells with real dimension 6 rather than 8. The solutions to $X = Y = 0$ correspond to common roots of the two polynomials involved and are also 6-D complex mass shells.

   The reason for the completely unexpected behavior is that the equations $X = 0$ and $Y = 0$ are reduced by Lorentz invariance to equations for the ordinary roots of polynomials for the complexified mass squared type variable. The intersection is empty unless X and Y have a common root and X^4_r belongs to X^6_r for a common root.

How to associate to the polynomial P a real 4-surface satisfying the conditions making $M8 - H$-duality?

1. P would fix complex mass shells in terms of its roots but not the 4-surfaces, contrary to the original expectations. The fact that the 3-D mass shells belong to the same M^4 and also their tangent spaces are parallel to M^4 together with rationality conditions for local $SU(3)$ element suggests number theoretical holography.

2. The key observation is that G_2 as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space $M^c \subset O_c$ gives a co-associative four-surface as a real projection. Also octonion analyticity allows G_2 gauge transformation. If X^4 is the image M^4 by a local $SU(3)$ element such that it also remains invariant under $SU(2)$ at each point, one obtains automatically $M^8 - H$ duality.

   The image of X^4 under $M^8 - H$ duality depends on g so that gauge invariance is not in question. The plausible interpretation in case of $SU(3)$ is in terms of Kac-Moody - or even Yangian symmetry. Note that at QFT limit the gauge potentials defined at H level as projections of Killing vector fields of $SU(3)$ are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

The study of octonionic Dirac equation shows that the solutions correspond to momenta at mass shells $m^2 = r_n$ obtained as roots of the polynomial P and that co-associativity is an essential for the octonionic Dirac equation. This conforms with the reduction of everything to algebraic conditions at the level of M^8.

8.8.2 Construction of the momentum space counter parts of scattering amplitudes in M^8

The construction of scattering amplitudes in M^8 was the main topic of this article. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. 0

1. The fact that $SU(3)$ gauge transformation with boundary conditions defined by the mass shells as roots of polynomial P defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differing from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anti-commutators since the cognitive representation is discrete.

2. The main objection is that the scattering amplitudes are trivial if quark momenta belong to cognitive representations, which are finite in the generic case. This would be the case also in 2-D integrable theories. The objection can be circumvented. First, the huge symmetries imply that cognitive representations can contain a very large - even an infinite - number of points. At partonic 2-surface this number could reduce to finite. Equally importantly, local composites of quark oscillation operators with collinear quark momenta are possible and would be realized in terms of representations of Yangian algebra for $G_{2,c}$ serving as the counterpart for super-symplectic and Kac-Moody algebras at the level of H.

3. ZEO leads to a concrete proposal for the construction of zero energy states - equivalently scattering amplitudes - by using a representation of Yangian algebra realized in terms of positive and negative energy quarks in opposite half-cones. Co-product plays a key role in the construction. Also the proposed local composites of quarks proposed in [L69] make sense.

4. Momentum conservation conditions and mass shell conditions combined with the requirement that the momenta are algebraic integers in the extension of rationals determined by the polynomial P look rather difficult to solve. These conditions however linearize in the sense that one can express the allowed momenta as squares of integer quaternions.

Also the construction of scattering amplitudes in M^8 is considered. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. The fact that $G_{2,c}$ gauge transformation defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differering from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anticommutators since the cognitive representation is discrete.

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Chapter 9

New findings related to the number theoretical view of TGD

9.1 Introduction

TGD could be seen as a holy trinity of three visions about quantum physics based on physics as geometry, physics as number theory, and physics as topology.

Quite recently I gave a talk on TGD and TGD inspired theory of consciousness and was asked about the motivations for the number theoretic vision. My response went roughly as follows.

1. The attempt to find a mathematical description for the physical correlates of cognition could have led to the vision of quantum TGD as a number theory. What are the possibly geometric/number theoretic/topological correlates of thought bubbles?

   A bold guess could have been p-adic numbers, $p = 2, 3, 5, 7, ...$ provide natural mathematical correlates for cognition. Rationals, algebraic extensions of rationals, and the extensions of p-adic number fields induced by them are natural candidates as also complex numbers, quaternions, and octonions. Also finite number fields emerged quite recently as natural ingredients of the number theoretic vision [K81, K82, K80] [L109].

   As a matter of fact, I ended up to a proposal that p-adic physics provides the correlates of cognition via a different route, by p-adic mass calculations based on p-adic thermodynamics, which turned out to have surprisingly high predictive power due to the number theoretic existence conditions [K46].

2. Sensory experience corresponds to real number based physics. There is a strong correlation between cognition and sensory experience, but it is not perfect. Sensations arouse thoughts, but cognition is also able to dream and imagine.

3. Cognition includes mathematical thought. The concretization of mathematical thinking as computation requires discretization. This suggests that discretization should correspond to what one might call a cognitive representation transforming thoughts to sensory percepts and it should have a number theoretic representation.

4. Mathematical thinking is able to imagine spaces with an arbitrary dimension, while the dimension of the perceptual world is fixed and is the dimension of three-space. How does cognition achieve this?

5. Cognition has evolved. Why and how can this be the case?

6. If the correlates of cognition are part of reality, then cognition must be optimally efficient. How?

This leads to the following questions and answers.

1. Could p-adic spacetime surfaces represent thought bubbles, equivalent to real 4-surfaces? They are a number-theoretic concept, they also involve a different topology than the sense-world, and p-adic space-time surfaces would be examples of algebraic geometry.

2. How is cognition able to imagine? p-Adic differential equations are non-deterministic: integration constants, which by definition have vanishing derivatives are only piecewise constants. Could this make imagination possible [K56]??

3. How the strong correlation between cognition and sensory experience could be realized? All p-adic number fields and their extensions must be allowed. Consider first the simplet book involving only reals and p-adic number fields. p-Adic number fields Q_p, $p = 2, 3, 5, ...$ can be combined into a book, an adele [L48, L47]. Different number fields as extensions of rationals represent the pages of this book. Real numbers correspond to sensory experience and various p-adic number fields to cognition. The back of the book corresponds to rational numbers that are common to all chapters.

   Every algebraic extension of rationals defines extensions of p-adic number fields. The p-adic pages of the algebraically extended book are algebraic extensions of various p-adic number fields. One obtains an infinite library with books labeled by algebraic extensions of rationals.

   Now the back of the book consists of algebraic numbers for the extension generated by the roots of a polynomial with integer coefficients. The back of the book gives a cognitive representation, a number theoretic discretization of the 4-surface that is unique for a given extension. The bigger the extension, the more accurate the discretization. Cognitive evolution would correspond to a refinement of cognitive representation induced by the increase of the order of the polynomial defining the extension.

4. How is cognition able to imagine higher-dimensional mathematical objects that do not exist at the level of sensory experience? algebraic extensions for p-adic numbers can have an arbitrarily high dimension if the corresponding polynomial has high enough degree. One can have p-adic 4-surfaces for which the associated algebraic dimension is arbitrarily high! p-Adic cognition is liberated from the chains of matter!

5. Why is evolution related to cognition? One gets an infinite number of books labeled algebraic extensions, a whole library. Does the evolution of cognition present a hierarchy? The bigger the algebraic extension, the better the approximation to real numbers and thus to sensory experience.

6. Can p-adic cognition be maximally effective? Here p-adic thermodynamics suggests the answer. p-Adic mass calculations assign to each elementary particle a p-adic prime. For instance, for electrons it is Mersenne prime $p = M_{127} = 2^{127} - 1 \simeq 10^{38}$. p-Adic mass squared value is expansion powers of p and its real counterpart is power series in negative powers of p. This series converges extremely rapidly for large primes such as $p \simeq 10^{38}$ and two lowest orders give a practically exact answer so all errors would be due to the assumptions of the model rather than due to computations.

How to realize number theoretic physics?

1. Number theoretic discretization does not resonate with the idea of general coordinate invariance. For $H = M^4 \times CP_2$ allows linear Minkowski coordinates but CP_2 coordinates are not linear although also now complex coordinates consistent with the isometries of $SU(3)$ are natural.

   What about M^8 or possibly its complexification suggested by twistorial considerations and also by the fact that classical TGD predicts that Euclidian space-time regions give an imaginary contribution to the conserved four momenta. M^8 allows highly unique linear Minkowski coordinates and the idea that M^8_c corresponds to complexified octonions is very natural. The automorphism group G_2 of octonions poses additional conditions.

2. This leads to the idea that number theoretic physics is realized at the level of M^8_c and that it is dual to the geometric physics realized at the level of H and that these physics are related

by $M^8 - H$ duality mapping 4-D surfaces in M^8 to H. TGD can be regarded as a wave mechanics for point-like particles replaced with 3-D surfaces in H, which, by the failure of complete determinism for holography, must be replaced by analogs of Bohr orbits. Wave mechanics is characterized by momentum-position duality, which naturally generalizes to $M^8 - H$ duality [L80, L81, L108, L109].

3. The physics in M^8_c should be purely algebraic as is also the ordinary physics at the level of momentum space for free fields. This physics should make sense also in all p-adic number fields. This suggests that polynomials with integer coefficients, in particular their roots, together with number theoretic holography based on associativity, partially characterize the 4-surfaces in M^8, which would make sense also as their p-adic variants.

   It is not clear whether the p-adicization is needed at the level of H: it might be enough to have it only at the level of M^8 so that only the p-adic variants of M^8 would be needed.

The geometric vision of TGD is rather well-understood (see for instance [L99]), but one need not think long to realize that there is still a lot of fog in the number theoretic vision (see for instance [K81, K82, K80] and [L80, L81, L93, L108, L109]).

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?

2. The detailed realization of $M^8 - H$ duality [L80, L81] involves uncertainties: in particular, how the complexification of M^8 to M^8_c can be consisted with the reality of $M^4 \subset H$.

3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?

4. How unique is the choice of 3-D surfaces at the mass shells $H^3_m \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

5. The understanding of 3-geometries is essential for the understanding of the holography in both M^8 and H. The mathematical understanding of 3-geometries is at a surprisingly high level: the prime 3-manifolds can be constructed using 8 building bricks. Do these building bricks, model geometries, have counterparts as prefered extremals of action in the TGD framework.

   The known extremals $X^4 \subset H$ satisfying holography should be analogues of Bohr orbits [L101]. They are proposed to satisfy a 4-D generalization of 2-D holomorphy and apart from lower-D singularities would be the same for any general coordinate invariant action based on induced geometry and spinor structure. They would be minimal surfaces both in H and M^8 except at singularities at which the details of the action principle would matter [L119]. This suggests that the preferred extremals could have maximal isometries and provide topological invariants as also do the model geometries in the classification of 3-geometries.

9.2 What does one mean with M^8 physics?

In TGD, the point-like particle is replaced by a 3-surface $X^3 \subset H = M^4 \times CP_2$, and the holography required by the general coordinate invariance requires the replacment of the 3-surfaces with the analogues of Bohr trajectories passing through them. The Bohr trajectories are not completely deterministic as already the case of hydrogen atoms demonstrates. The "World of Classical Worlds" (WCW) is thus the space of generalized Bohr orbits as the counterpart of the superspace of Wheeler which originally inspired the notion of WCW [L118, L119, L115, L116]).

In wave mechanics, the duality between the descriptions using momentum and position space applies in wave mechanics but does not generalize to field theory. The $M^8 - H$ duality [L80, L81] can be seen as a generalization of this duality. M^8 is the momentum space counterpart and $H = M^4 \times CP_2$ is the position space counterpart in this duality.

9.2.1 Physical interpretation of the 4-surfaces of the space M^8 and their singularities

The physical interpretation of 4-surfaces in the complexifixation of the momentum space M^8 is far from straightforward. There are many reasons for the complexification.

1. Complexified octonionicity requires that M^8, or equivalently E^8, is complexified: one has $M^8_c = E^8_c$ giving as its subspaces various real subspaces with various signatures of the number theoretical. The metric obtained from the Minkowski norm $\delta_{kl} z^k z^l$, where z^k are 8 complex coordinates. M^4 with signature $(1, -1, -1, -1)$ is in a special physical role and one can of course ask, whether also other signatures might be important.

2. If complex roots are allowed for polynomials P determining together with associativity the holography, complexification must be allowed. Virtual momenta could therefore be complex, but Galois confinement says that the total momenta of physical states are real and have integer components in the momentum scale determined by the size of the causal diamond (CD). Physical intuition suggests that the imaginary parts of the momenta code for the decay width of the particle. This is natural if the imaginary part is associated with the energy in the rest system.

3. The conserved momenta given by Noether's theorem at the level of H have real parts assignable to Minkowskian space-time regions. The fact that $\sqrt{g_4}$ appears in the integral defining a conserved quantity differs in Minkowski and Euclidean regions by an imaginary unit suggests that the contributions to momenta from the Euclidian regions are imaginary. The momenta from the Minkowskian space-time regions can be transferred to the light-like boundaries between Minkowskian and Euclidian regions identified as light-like partonic orbits. Quantum-classical correspondence requires that the classical total momenta, like all conserved quantities, correspond to the total momenta of the fermion state.

   Euclidian regions most naturally correspond to CP_2 type extremals as preferred extremals. They can be regarded as singularities resulting in the blow up of tip-like cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in M^8 . This would suggest that the real parts of momenta are associated with the Minkowskian regions of space-time surfaces and imaginary parts to the Euclidian regions. This applies also to other conserved quantities.

9.2.2 Number theoretic holography

Number theoretic holography has two forms.

1. The weak $3 \to 4$ form corresponds to the ordinary holography $Y^3 \subset M^8 \to Y^4 \subset M^8$, which is by $M^8 - H$ duality equivalent of the holography for $X^3 \subset H \to X^4 \subset H$ for space-time surfaces. The proposed interpretation of Y^3 is as a fundamental region of H^3/Γ.

2. For the strong $2 \to 4$ form of the holography Y^3 is determined by a 2-D data defined by the boundary of the fundamental region of H^3/Γ. The proposal to be considered is that the boundary of the fundamental region of H^3/Γ can be identified as 2-D hyperbolic space H^2/Γ.

Consider next the weak form of the holography.

1. The 4-surface $Y^4 \subset M^8_c$ is determined from number-theoretic dynamics and is an associative surface, i.e. its normal space is associative and therefore quaternionic.

2. There are also commutative 2-D surfaces that most naturally correspond to string world sheets, and for them commutativity of tangent space (as analog of associativity) as subspace of normal space of Y^2 defines holography. Holographic data corresponds now to strings connecting wormhole contacts assignable to Euclidian singularities inside $Y^3 \subset H^3_m$. One

can also consider the possibility that partonic 2-surfaces correspond to co-commutative 2-surfaces. The situation is not completely clear here.

3. One must also identify the 3-surfaces $Y^3 \subset H^3_m$ defining the holography. Holography is subject to very strong conditions and I have proposed that these surfaces are hyperbolic 3-manifolds X^3 obtained as coset spaces H^3/Γ, where G is suitably chosen discrete but infinite subgroup of $SL(2,C)$ acting as Lorentz transformations in H^3. The spaces H^3/Γ are fundamental domains of H^3 tessellations.

   $Y^3 = H^3/\Gamma$ is counterpart for the unit cells of a lattice in E^3, which effectively has this topology and geometry due to boundary conditions stating that G leaves various "field configurations" invariant. The situation is the same as in the case of ordinary condensed matter, where periodic boundary conditions for a cube as a unit cell make it effectively a 3-torus.

   Also the crystal-like structures consisting of a finite number of copies of the fundamental domain of H^3/Γ glued together are possible choices for Y^3. They would be analogous to the unit cells of the lattices of Euclidian space E^3 or finite crystals formed from them. Therefore the analog of solid state physics would be realized at the fundamental level.

   One can also consider closed 3-manifolds $Y^3 = H^3/\Gamma$ obtained by gluing two copies of the fundamental region with different S^3 coordinates connected together along their 2-D boundaries. The gluing could be performed by a cylinder of $S^3 \subset E^4 \subset M^4 \times E^4$ connecting the boundaries.

4. The quantum state at H^3_m consists of several Galois singlets assignable to 3-surfaces Y^3_i. The total momenta for X^3_i would be real and have integer valued components for the momentum unit defined by the size scale of CD involved. This condition is analogous to the periodic boundary conditions.

5. Quantum classical correspondence requires that the many-fermion state on H^3_m, characterized partially by momenta, which are in the algebraic extension of rationals associated with the polynomial P, determines $Y^3_i \subset H^3_m$. For a given Y^3_i, the accompanying fermions correspond to the points of H^3_m. The classical momentum of the state given by Noether theorem in H would the sum of the fermionic momenta.

   An attractive idea is that at least a subset of the fermionic momenta corresponds to cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`), which can be visualized geometrically as vertices of an algebraic surface at which the direction of normal space is ill-defined.

   The cusps correspond to parabolic subgroups $P \subset G \subset SL(2,C)$ (`https://rb.gy/b5t55`), where the $\Gamma \subset SL(2,C)$ defines the fundamental domain H^3/Γ of the tessellation. Parabolic subgroups P are automorphic to the subgroup of translations of upper half-plane H generated by $SL(2,C)$ matrix $(1a;0,1)$, a a real algebraic number. This particular P acts as Möbius transformations in H representing hyperbolic space H^2. The cusp singularities would encode at least a subset of fermionic momenta of the state into the hyperbolic geometry of Y^3_i. Each fermion would correspond to its own parabolic generator in the subgroup G.

   In the TGD view of hadron physics [L112], the fermions associated with the cusps could be identified as analogs of valence quarks. They would be associated with singularities identifiable as light-like 3-D partonic orbits serving as boundaries of 4-D CP_2 type extremals with Euclidian signature of the induced metric.

   Also fermionic momenta, which have algebraic integers as components but do not correspond to cusps, can be considered. These would be naturally associated with strings predicted to connect cusps at the throats of different wormhole contacts. The blow-up would be now 2-sphere relating to cusp singularity like line charge to point charge. It is not clear whether the sea partons could correspond to these string-like singularities. In any case, hyperbolic 3-manifolds have string-like singularities connecting cusps.

6. If Y^3_i corresponds to a Galois singlet, then its total 4-momentum is real and integer-valued and should be mapped to a discrete plane wave in the finite lattice defined by the crystal like structure formed by the copies $X^3(Y^3)$ in H^3_a given by inversion. Each Galois singlet Y^3_i

would define such a plane wave and one can imagine a hierarchy of such structures just as in the case of condensed matter with crystals of different sizes. The analogy with condensed matter physics suggests that Γ is a lattice. This follows also from the condition that H^3/Γ has a finite volume.

7. This picture would suggest that also $X^3(Y_i^3)$ is hyperbolic manifold of its fundamental region and perhaps isometric with Y^3. This would mean a geometric realization of Lorentz invariance analogous to the dual of conformal invariance encountered in the twistorialization.

9.2.3 Quantum classical correspondence for momenta

Quantum classical correspondence for momenta and also other conserved charges poses very strong conditions.

1. Noether charges for the classical action define momenta and other conserved charges. The classical contribution is a c-number. In addition, quantum contributions from fermions are included. They correspond to the momenta related to the second quantized spinor modes of H and from the orbital degrees of freedom associated with the "world of classical worlds" (WCW). The fermionic contributions are related to the ground states of the super symplectic representation characterized in terms of spinor modes for H spinor fields [K21, K73] [L109].

2. Are the classical contributions separate and do they add up to the total momentum? The fact that classical contributions are separately conserved, does not support this view.

3. Quantum classical correspondence would mean that the classical total momentum is the sum of the fermionic momenta determined by the multi-fermion state. This would hold quite generally for Cartan algebra of observables. For example, in the case of hadrons, the dominant classical contribution could correspond to the gluon sea, that is to multi-gluon state with gluons expressible in terms of quark-antiquark pairs. This picture is consistent with QCD and is therefore perhaps a more realistic guess.

4. Wormhole contact has Euclidean induce metric and the related classical conserved momentum is naturally imaginary. Could the sum of the imaginary parts of complex fermionic momenta of fermions for a wormhole throat correspond to the classical imaginary momentum assignable with the wormhole contact? Could the imaginary part of the fermionic momentum be assigned with the end of the euclidean string inside CP_2 type extremals, while the real momentum would be assigned with an end of a Minkowskian string?

5. Quantum-classical correspondence would be realized if the fermionic conserved four-momenta on the H side corresponded to M^8 points at hyperbolic 3-surfaces H^3_m. Their inversions in the M^8-H duality would be points of M^4_c of the spacetime surface $H^3_m \subset M^4_c \times CP_2$. It would seem that one must map only the real parts of the momenta at H^3_m to H^3_a.

   It would also seem that H^3_m must be associated with the M^4 projection of M^4_c. Whether the variant of H^3_m for complex valued m^2 makes even sense is not obvious.

9.2.4 The analog of time evolution in M^8 as a coupling constant evolution conserving dual quantum numbers

The proposal that 4-D surfaces appear at the level of M^8 suggests that it makes sense to talk about dynamics also in M^8 and the 4-D analogies of space-time surfaces make sense. This does not fit the usual classical intuition.

The twistor picture for conformal invariant field theories predicts that conformal invariance has a dual counterpart. This would mean that 4-momenta and other Poincare charges in H have dual counterparts in M^8. In TGD, the dual counterparts would be obtained by inversion from the defining the M^8-H duality and mapped to points of the space-time surface at the mass shells H^3_a in H. They would be analogs of the representation of lattice momenta as points of the heavenly sphere in crystal diffraction.

1. In zero energy ontology (ZEO), the time evolution at the level of H by "small" state function reductions (SSFRs), which are analogous to the so called weak measurements introduced by quantum opticians, would correspond to time evolution in terms of scalings rather than time translations. They would scale the size of the causal diamond (CD) and leave the passive boundary of CD invariant. These analogs of time evolutions would be generated by the scaling generator L_0. This would naturally also apply to M^8. This time evolution would be induced by scalings of the mass-scale, which need not be identical.

2. Could the "energy evolution", by identifying the square of the mass as the counterpart of time, correspond to the development related to the renormalization group? M^8-H duality would map the renormalization group evolution from M^8 to time evolution in H. This is quite a strong prediction.

3. Mass squared values for the fundamental fermions would not define particle masses but mass scales. 4-momenta for physical particles would correspond to total momenta for many fermion states, which obey Galois confinement, which requires that the momentum components are integers, when the mass unit is defined by the size scale of CD.

4. What would be the interpretation of the mass shells M^4_c determined by the roots of the polynomial P in the coupling constant evolution? Could the related hyperbolic 3-manifolds correspond to fixed points for the coupling constant evolution? With these mass values, something special would happen. Could H^3_a correspond to critical moments of light-cone proper time a when the SSFRs occur and a new unitary time evolution begins and ends with the next SSFR, as I have suggested?

5. What about the M^8 side? Could one talk about conserved quantities with respect to the evolution determined by scalings? Could these dual charges, dual momenta, and. also the charges related to E^4 isometries, be invariants of the renormalization group evolution?

   I have proposed that the $SO(4)$ symmetry of of hadrons in old-fashioned hadron physics involving notions like conserved vector current (CVC) and partially conserved axial current (PCAC) could correspond to the color symmetry of higher energy hadron physics by M^8-H duality in which the natural conserved charges on M^8 side are associated with the product of isometry groups of M^4 and E^4 and perhaps even with $SO(1,7)\times T^8$ or G_2 as automorphism group of octonions. At H side one would have a product of Poincare group and color group. Also holonomy groups are involved. At least $SO(4)$ symmetry could define invariants of the coupling constant evolution in M^8.

Consider now a more detailed interpretation of 4-surfaces $Y^4 \subset M^8$ in terms of a generalized coupling constant evolution.

1. The changes $m_i^2 \rightarrow m_{i+1}^2$ for the roots of P would define a discrete evolution in both M^8 and H. Discrete coupling constant evolution affects the mass resolution and brings in or deletes details and therefore would induce changes for the representations of the states. The 4-surfaces in M^8 would represent renormalization group flows. The failure of a complete determinism is expected and could be interpreted in terms of phase transitions occurring at critical masses.

2. A given mass shell m_i^2 determined by a root of P would define a discrete mass scale for the evolution having perhaps an interpretation as a fixed point or a critical point of the coupling constant evolution. It would be natural to assume that the evolution induced by the change of resolution conserves other total quantum numbers than 4-momentum.

3. What about the conservation of 4-momentum? At $m^2=m_{i+1}^2$ the value of mass squared for fundamental fermions defining the mass scale changes. The structure of the state must change in $m_i^2 \rightarrow m_{i+1}^2$ if 4-momentum conservation is assumed.

   The normalization group evolution for the mass m^2 of the physical state, is typically logarithmic in QFTs, and must be distinguished from the discrete evolution for the mass scale m_i^2. Hence the change of m^2 in $m_i^2 \rightarrow m_{i+1}^2$ is expected to be small. This could be realized if n corresponds to a (possibly normal) subgroup of the Galois group of P perhaps spanned by the roots $m_k^2 \leq m_i^2$.

Could the phase transition $m_i^2 \to m_{i+1}^2$ change the rest energy of the state? Does the change require an energy feed between the CD and its environment as ordinary phase transitions require? This is not the case if CD is interpreted as a perceptive field rather than a physical system.

4. Does it make sense to talk about the conservation of dual momentum $X^k = \sum_i X_i^k$, $X_i^k = Re(\hbar_{eff} p^k/p_{k,i} p_i^k) = Re(\hbar_{eff} p^k/m_i^2)$? The conservation of momentum p^k does not imply the conservation of dual momentum since it is proportional to $1/m_i^2$: X^k would scale as $1/m_i^2$. The size of the CD is assumed to increase in statistical sense during the sequence of small state function reductions (SSFRs). The increase of the size of the CD would gradually make the mass shells inside it visible.

   $M^4 \subset H$ center of mass position X^k therefore changes $m_i^2 \to mi + 1^2$ unless h_{eff} is not scaled to compensate the change $m_i^2 \to mi + 1^2$ in the formula for X_i^k. The integer n in $h_{eff} = nh_0$ is assumed to correspond to the order of the Galois group of P. It could also correspond to the order of a subgroup of the Galois group of P defined by the roots m_k^2, $k \leq i$. If so, h_{eff} would increase in evolution and one can even imagine a situation in which $Re(h_{eff}/m_i^2)$ remains constant.

9.3 $M^8 - H$ duality

The proposed $M^8 - H$ duality maps 4-surfaces $Y^4 \subset M_+^4 \subset M^8 = M^4 \times E^4$ to space-time surfaces $X^4 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$.

9.3.1 $M^8 - H$ duality as inversion

$M^8 - H$ duality relates also the hyperbolic spaces $H_m^3 \subset M_+^4 \subset M^8 = M^4 \times E^4$ and $H_a^3 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$. The hyperbolic space $H_m^3 \subset M_+^4 \subset M^8$ corresponds to the mass shell for which mass squared value m^2 is a root of a polynomial P. The hyperbolic space $H_a^3 \subset M^4 \subset= M^4 \subset M^4 \times CP_2$ corresponds to light-cone proper time constant surface $t^2 - r^2 = a^2$.

1. The root of P is in general a complex algebraic number. The first guess is that $M^8 - H$ duality is defined by inversion $p^k \to m^k = \hbar_{eff} p^k/p^l p_l$. Or briefly, $p \to \hbar_{eff} p/m^2$. The light-cone proper time $a = h_{eff}/m$ characterized the hyperboloid $H_a^3 \subset M^4$. $H_m^3 \to H_a^3$ is consistent with the Uncertainty Principle. In this case the image would be complex. This creates interpretational problems. There is no need for the complexification of CP_2, which also suggests that the image un H_a^3 must be real.

2. One can consider the possibility that only the real projection of the complex variant of H_m^3 to $H_{Re(m)}^3$ is involved. The image of the real part $Re(p^k)$ in H_a^3 obtained by inversion would be real but would not code information about the imaginary part $Im(p^k)$.

3. One could however take the real part of a complex inversion to get a point in H_a^3. $Re(\hbar_{eff} p^k/p^2$ would code information about the imaginary value of m^2.

Inversion fails at the light-cone boundary with $m = 0$. In this case, the inversion must be defined as the inversion of the energy of the massless state: $p^k \to \hbar_{eff} Re(p^k/E^2)$.

9.3.2 The technical problems posed $M^8 - H$ duality the complexification of M^8

Complexification of M^8 is highly desirable in the number-theoretic vision. But how to deal with the fact that fermion momenta, for which with components are algebraic integers in the algebraic extension determined by a polynomial P, are in general complex?

1. Without additional conditions, the mass shells in $M^4_c \subset M^8_c$ for complex m^2 as a root of P are 6-D. There are 2 conditions coming from the conditions fixing the value of $Re(m^2) = Re(p^2) - Im(p^2)$ and $Im(m^2) = 2Re(p) \cdot Im(p)$. If one only energy is complex, the dimension of the mass shell is 3. This looks natural. The preferred time axis would be determined by the rest system for massive states. A possible interpretation for the imaginary part is as decay width in the rest system.

2. The complexified mass shells of complexified $M^4 \subset H$ must be considered. Does this make sense? Since the CP_2 point labelling tangent space of Y^4 does not depend on complexification there is no need to consider complexification of CP_2. Therefore the natural conclusion is that also the $M^4 \subset H$ images should be real.

The inversion $p^k \to p^k/m_r p_r p_s$ is the simplest realization for $M^8 - H$ duality and would naturally fit into a generalization of 2-D conformal invariance to 4-D context. $h_{eff} = nh_0$ hierarchy comes along in a natural way. The polynomial P determines the algebraic extension and the value of h_{eff}. The size of the CD would scale like h_{eff} on the H side. There would be no scaling on the M^8 side.

1. The first thing to notice is that one could understand classically complex momenta. On the H side, Euclidean regions could give an imaginary contribution to the classical momentum.

2. The complex inversion $p^k \to p^k/m_r p_r p_s$ maps complex H^3_m to complex H^3_a. What would be the interpretation of the complex M^4_c coordinates? The same problem is also encountered in twistorization. One can ask whether a complex time coordinate corresponds to, for example, the inverse temperature?

   However, since no complexification is needed for CP_2, it seems that the only natural option is that the $M^4 \subset H$ image is real.

3. One can consider 3 options guaranteeing the reality.

   (a) Only the real parts of the complex $M^4 \subset M^8$ momenta are mapped to H. The information about the imaginary parts would be lost.

   (b) The complex algebraic integer valued momenta are allowed and the real part $Re(p^k \to p^k/m_r p_r p_s)$ of the complex inversion defines the image points in H. The $M^4 \subset H$ complexification would not be needed for this option but the information about the imaginary part of the momenta would not be lost.

   (c) By Galois confinement, the physical multiparticle states consist of momenta with integer value components using the momentum unit assignable to CD at the M^8 level of space with mass shells. These would define the 3-D data for the holography, which determines the 4-surface $Y^4 \subset M^8$ through the associativity of the normals space of Y^4. Only the real, integer-valued momenta of Galois confined states would be mapped from the mass shells of $M8$ to their images in H.

   The information about the fermion composition of the many-particle states would be lost completely. Therefore this option does not look realistic.

   The realization of this view might be possible however. 4-momenta determine the 3-surfaces Y^3 with real mass shells H^3 as data for associative holography. Momenta could correspond to point-like singularities on $Y^3 \subset H^3$ and these should be assigned as CP_2 type extremals at H side as blow-ups of the singularities.

To conclude, the option $p^k \to Re(p^k \to p^k/m_r p_r p_s)$ seems to be the physically realistic option.

9.3.3 Singularities and $M^8 - H$ duality

Consider next the description of the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in $M^8 - H$ duality. The condition that information is not lost, requires that the map is given by $p^k \to Re(p^k/m_{rs} p^r p^s)$.

1. The cusps in M^8-H duality would be mapped to a 3-D surface of CP_2. It would correspond to the 3-D section CP_2 of the extremum, which corresponds to a wormhole contact associated with fermions at its throats.

   At H^3_m there is a temptation to assign to the cusp singularity, identified as a blow-up, the 3-D sphere $S^3 \subset E^4$ of the normal space E^4 defined by the mass shell condition. The simplest option is that this sphere is mapped to $U(2)$ invariant sphere $S^3 \subset CP_2$ for which the radius would be fixed by the mass squared value.

   The metric of H^3/Γ is singular at the cusp. The elimination of the singularity requires that one must allow a hole Z^3 around the cusp. The boundary of X^3 can have any genus. The size scale of the hole should be determined by the mass squared value.

   This view conforms with the physical picture of the CP_2 type extremal as an orbit of an Euclidian wormhole contact connecting two Minkowskian space-time sheets. S^3 would be replaced with $S^3 \setminus Z^3$ mapped to CP_2, where it corresponds to a wormhole throat having arbitrary genus.

   This picture would suggest that a given cusp singularity can correspond only to a single wormhole throat. This is not in conflict with the recent view of what elementary particles having wormhole contacts as composites should be. Composite, involving 2 wormhole contacts (required by the conservation of the monopole flux forming a loop involving two space-time sheets) and therefore 2 wormhole throats, can have spin varying from 0 to 2 which conforms with the popular wisdom that elementary particle spins vary in this range.

2. In the case of string-like objects $Y^2 \times R \subset H^3_m$, that is $S^2 \times R$ and their higher genus counterparts $H^2/\Gamma \times R$, the counterpart of the blow-up would be $Y^2 \subset S^3 \subset E^4$. Y^2 would be mapped to $X^2 \subset CP_2$ such that the radius assignable to S^2 or the size scale assigned to H^2/Γ would correspond to the mass squared.

3. Fermion trajectories at the partonic orbits would be light-like curves defining boundaries of string world sheets. CP_2 extremal would be associated with a fermionic cusp by holography and M^8-H duality.

   Fundamental fermion as an analog of valence quark [L112] could correspond to a cusp at the boundary of the string world sheet. Cusps would be related to the boundary of X^3_a composed of partonic 2-surfaces.

4. In principle, fermion momenta in the interior of $Y^3 \subset H^3_m$ are also possible. The picture given by hadron physics would indicate that the interior contribution corresponds to the sea partons. They can also be associated with string world sheets and correspond to virtual bosons appearing as fermion-antifermion pairs. These singularities would be string-like objects of the form $X^2 \times R$ and $X^2 \subset CP_2$ would replace the sphere of CP_2. One could say that fermions are delocalized at string.

9.3.4 Realization of the Uncertainty Principle

Inversion alone is not enough to realize Uncertainty Principle (UP), which requires that M^8-H duality is analogous to the Fourier transform. However, with the help of H^3 tessellations, it is possible to understand how the UP is realized in a finite measurement resolution.

The invariance of points of H^3/Γ under the subgroup $G \subset SL(2,C)$ is analogous to the periodic boundary conditions that replace the cubic unit cell of a crystal lattice with a torus. Now the tessellation of H^3, which quantizes the momenta, would be replaced by H^3/Γ. The momentum lattice having the fundamental region for H^3_m/G as unit cell would be mapped by inversion to a position lattice having the fundamental region H^3_a/G as a unit cell. A point in H^3_m would correspond to an analog of plane wave as a superposition of all positions of $X^3(Y^3)$ in a part of the tessellation in H^3_a: a wave function in finite crystal. One would have a superposition of 3 surfaces X^3_a corresponding to different lattice points multiplied by the phase factor. For a multi-fermion state the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) assigned to the momenta of fermions would characterize H^3/G so that the information about ("valence") fermion state would be code geometrically. Similar coding would be realized also for the string-like entities $H^2/\Gamma \times R$ at H^3/Γ. What is new and surprising, and also challenges the interpretation, is that the genus of H^2/Γ would code for the momenta of many-fermion states. Does the number of fermion-antifermion pairs correlate with the genus which in turn is proposed to label fermion families? There would be one fermion-antifermion pair per single handle. This would conform with the quantum classical correspondence. The proposed explanation [K18] for the number of observed fermion families would be in terms of hyper-ellipticity meaning that Z_2 acts as a conformal symmetry for all genera smaller than 3. Genus two would correspond to a formation of a bound state of two handles. Could this mean a formation of a graviton-like bound state of 2 fermion pairs and that higher spin states are not possible as bound states of handle. If fermions correspond to cusp singularities surrounded by holes, this picture might make sense: fermion antifermion pair would correspond to two holes connected by a handle.

The $M^8 - H$ duality maps the surface $Y^4 \subset M^8$ to the space-time surface $X^4 \subset H$. The point of $M^4 \subset H$ is obtained as the real part of the inversion of the point of the M^4 projection of the surface Y^4.

There would be a direct analogy to the physics of condensed matter.

A hyperbolic 3-manifold would correspond to a fundamental domain of a tessellation. It would be the equivalent of a unit cell both in position space and momentum space. These unit cells would correspond to each other at the H^3 level by $M^8 - H$ duality. Both would involve discretization. By finite momentum and position resolution UP would be reduced to the interior of the finite tessellation analogous to finite crystal. Quantum-classical correspondence and inversion are consistent with the realization of the UP related to Bohr's orbitology. Momenta in H^3_m would be mapped to equivalents of plane waves, i.e. superpositions of positions of the fundamental region in the tessellation. This picture generalizes to the multi-fermion states. Each fermion momentum defines a cusp and fermionic statistics makes it possible to avoid several cusps at the same points. Fermions for which other quantum numbers, such as spin differ, can however have the same momentum. They should correspond to the same cusp. How can this make sense? Could S^3 be involved somehow. Could they correspond to different holes in S^3 whose sizes and locations correlate with the other quantum numbers somehow? I have considered this problem earlier in the twistor picture where spin corresponds to a geometric degree of freedom in twistor space, which has identification at the level of M^8. The space of causal diamonds (CDs) as a kind of spine of WCW is discussed in [L118]. Lorentz transformations also occur at the level of CDs. The moduli of CD correspond to cm degrees of freedom in H. The finite volume of CD allows states for which Poincare quantum numbers are not exactly opposite for the boundaries of CD. Therefore the values of the total Poincare quantum numbers can be assigned to the CD. Only at the limit of infinitely large CDs does the zero energy property become exact. Therefore the CD wave function carries genuine information. At the p-adic level, translations and Lorentz transformations have the same effect as transformations of a compact group. Translations or Lorentz transformations of order $O(p^n)$ do not increase the p-adic norm of a point.

9.4 Holography

4-D general coordinate invariance forces holography at the level of $H = M^4 \times CP_2$ and one can regard space-time surfaces as analogues of Bohr orbits determined almost uniquely by 3-D surfaces. Quantum TGD is therefore very much like wave mechanics with point-like particles replaced with 3-surfaces in turn replaced with 4-D Bohr orbits. In fact, a wave-mechanical toy model for TGD would replace electron wave functions in atoms with wave functions in the space of its Bohr orbits.

M^8 is analogous to the momentum space in wave mechanics and the 4-surfaces in M^8 obey number theoretical holography based on associativity.

9.4.1 What does one mean with holography?

Consider now a more precise definition of holography.

1. The standard form of holography as $3 \rightarrow 4$ assigning to a 3-surface at the boundary of causal diamond (CD) an almost unique 4-D surface is the weakest form of holography. The non-uniqueness of the holography forces zero energy ontology (ZEO) in which analogues of Bohr orbits are basic geometrical objects.

2. $2 \rightarrow 4$ holography is the strongest form of holography. I have called it strong holography (SH). The 2-D partonic surfaces and possibly also the string world sheets would encode the data about the 4-surface and also the data about quantum holography. The strong form of holography could be realized as super symplectic and super-Kac Moody invariance and super-conformal invariance being minimally broken. Only the scaling generator L_0 would not annihilate the states. This condition is however too strong.

3. For the weaker form of SH super-symplectic and conformal symmetries are broken such that the algebras A_n (there are several of them), whose conformal weights are n-multiples of the conformal weights of the entire algebra A, and $[A_n, A]$ annihilate the physical states [L109, L67]. This requires half-algebra with non-negative conformal weights.

   The breaking hierarchy labelled by the values of n makes sense also for the ordinary conformal invariance but to my best albeit non-professional knowledge is not considered as a physical option. Hierarchies corresponding to the inclusion hierarchies of rational extensions and HFFs are obtained.

Both holographies set very strong conditions for the 3-surfaces appearing as holographic data.

Role of polynomials

At the level of M^8 physics is algebraic as it is also for the momentum space in the case of free field theory and reduces to algebraic conditions like mass shell condition and orthogonality of polarization vector and momentum. Polynomials P having integer coefficients determined the physics.

1. P as such does not fix the 4-surface nor even the 3-surface defining the data for number theoretic holography.

2. The polynomial P must have integer coefficients to guarantee number theoretical universality in the sense that they make sense also in p-adic number fields. If the coefficients are smaller than the degree of P, also finite fields become natural mathematical structures in TGD so that all number fields are involved. The roots of P give rise to the mass shells in M^8_c with mass squared values defined by the roots of P. The roots define an extension of rationals.

3. Polynomials are also characterized by ramified primes as the divisors of the discriminant of the polynomial determined by the product of the differences of its roots [L119]. They are not a property of the algebraic extensions. They depend on P and the exponent of the classical action is proposed to correspond to the discriminant D. Ramified primes are identified as p-adic primes playing a central role in p-adic mass calculations [K46].

The role of fermions

Quantum classical correspondence requires that the 3-surfaces Y^3 at the mass shells are determined by the quantum numbers of fermions associated with the quantum states. What assumptions could provide this additional data and how could this data be coded to the geometry of Y^3?

The data in question correspond to fermion momenta, spins and electroweak quantum numbers. Color does not appear as spin-like quantum numbers but corresponds to color partial waves in CP_2. Consider next how momenta are coded to the properties of 3-surfaces.

1. I have proposed that the 3-surfaces Y^3 in H^3_m could correspond to the fundamental domains of tessellations of H^3. The unit cell of ordinary crystal in E^3 serves as an analog for the fundamental domain of a tessellation in H^3. The disjoint components of Y^3 would naturally

correspond to surfaces Y_i^3 at H_m^3 and would correspond to fundamental domains of analogs of finite crystals formed by gluing them together.

The points of $E^4 \subset M^4 \times E^4$ correspond to 3-sphere with radius determined by mass m and for given Y_i^3 the values of E^4 coordinates would be constant. A stronger condition would be that the values are the same for all Y_i^3. At the cusp points the point would be replaced by S^3, which could touch two disjoint sheets with different values of S^3 coordinates. Since the metric becomes singular at cusp, a natural proposal is that a small hole is drilled around the point and to S^3 and they are glued along their boundaries. The scale of the hole would be determined by the mass.

2. TGD predicts as basic objects also string-like objects $X^2 \times R \subset H$ and their deformations to magnetic flux tubes. By $M^8 - H$ duality they are expected to be present also in M^8 in particular at the hyperboloids H_m^3.

   There are two kinds of string-like objects depending on whether their CP_2 projection is homologically trivial or not. In the latter case the string carries monopole flux.

Quantum classical correspondence suggests that the momenta of fermions as points of $H_m^3 \subset M^8$ are coded into the geometry of Y^3 as singularities. $M^8 - H$ duality based on inversion in turn maps the momenta to singular points of H_a^3.

1. Singularities would be naturally cusps as analogs of tips of algebraic surfaces allowing all normal spaces of Y^4 at the singularity: M^8 duality would assign a 3-D subset of CP_2 to the tip.

2. Is it possible to have singularities, where the throats of the opposite wormhole throats touch? Or could the wormhole throats of the incoming partons fuse to single throat? This could occur in the topological counterpart of 3-vertex describing pair annihilation to a single particle. The singularities emerging in this way could relate to the description of the creation of fermion-antifermion pairs and would also define defects essential for exotic differential structures occurring only in dimension $D = 4$ [L107].

Can one code also spin to geometry or should it be regarded as a fermionic quantum number?

1. At the level of H one would have a product of twistor spaces $T(M^4)$ and $T(CP_2)$: these twistor spaces are unique in the sense that they have Kähler structure. This makes H a unique choice for the embedding space.

   Twistorialization replaces space-time surface with 6-D surface $X^6 \subset T(M^4) \times T(CP_2)$ having S^2-bundle structure as possessed also by $T(M^4)$ and $T(CP_2)$. Spinor description of spin and electroweak isospin is replaced by a wave function in twistor spheres S^2.

   The embedding corresponds to dimensional reduction producing X^6 as S^2 bundle. The twistor spheres associated with M^4 and CP_2 must be identified by the embedding of $X^6 \subset T(M^4) \times T(CP_2)$.

   The identification of the twistor spheres forces spin doublets to correlate almost completely with electroweak spin doublets apart from the directions of the two spins. This picture allows only spin- and electroweak spin doublest. Does this force a complete correlation between the values of spin and electroweak to be identical or do the details of the identification for the the embedding of $X^6 \subset T(M^4) \times T(CP_2)$ allow to regard spin and electroweak spin as independent?

   The identification of the two twistor spheres is not unique. The spin rotations and possibly also electroweak spin rotations (, which are not isometries) changes the identification of the two twistor spheres. This would make spin and electroweak spin independent quantum numbers.

   One can argue that only the relative rotation of the two spheres matters. Could this mean that electroweak spin axes can be thought of being completely fixed. Electroweak quantization axes are indeed completely fixed physically.

2. Something similar could happen at the level of M^8. Now one must consider twistor spaces of M^4 and E^4 and similar embedding of a 6-D surface $X^6 \subset M^8$ as twistor space with S^2 bundle structure.

   In M^8 one would have an algebraic description of spin and electroweak spin instead of a wave function at S^2. A direction of S^2 would define a quantization axis and the diametrically opposite points of S^2 associated with it would provide a geometric correlate for the spin and electroweak spin values of fermion. The relative rotations of the twistor spheres of M^4 and E^4 associated with their identification are also now possible so that the two quantum numbers can be regarded as independent but with the electroweak quantization axes fixed.

In the twistorial picture one would have $5 \rightarrow 6$ weak holography or even $4 \rightarrow 6$ strong almost unique holography.

9.4.2 What kind of 3-geometries are expected in the TGD framework?

To get a wider perspective, it is good to have an overall view of the Geometrization conjecture of Thurston `https://rb.gy/9x3pm` proven by Perelman by studying Ricci flows. Geometrization theorem implies Poincare conjecture and so called spherical space conjecture.

The inspiration comes from the classification of 2-D manifolds expressed by uniformization theorem (`https://rb.gy/ts8va`). There are only 3 closed simply connected Riemann manifolds: sphere, disk, and hyperbolic plane. These are constant curvature spaces with corresponding Lie groups of isometries. One can obtain connected closed 2-manifolds with a nontrivial fundamental group by identifying the points related by a discrete subgroup of isometries.

In the case of the hyperbolic plane the isometry group is infinite and gives rise to a non-trivial fundamental group. For the hyperbolic plane, one obtains 2-manifolds with nonvanishing genus allowing a negative constant curvature. Constant curvature can be normalized to be -1, 1 or 0 in various cases. For non-vanishing curvature, the area serves as a topological invariant. For torus this is not the case.

The following provides the summary of my non-professional understanding of the 3-D case. The TGD inspired comments rely on what I know from the universal preferred extremals of practically any variational principle which is general coordinate invariant and can be constructed from the induced geometric quantities. They are always minimal surfaces outside 3- or lower-dimensional singularities at which the field equations depend on the action. The known extremals are discussed in [K5, K10, L101].

1. Thurston's conjecture `https://rb.gy/9x3pm` states that every oriented and closed irreducible (prime) 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with a finite volume which becomes a topological invariant in geometric topology. For instance, knots give rise to a 3-manifold in this way.

   An important difference is that the closed 3-manifold decomposes to a union of different types of 3-manifolds rather than only of single type as in the 2-D case.

2. The notion of model geometry is essential. There exists a diffeomorphism to X/Γ for some model geometry such that Γ is a discrete subgroup of a Lie group of isometries acting in G. There are 8 types of model geometries.

3. Irreducible 3-manifolds appear as building bricks of 3-manifolds using connected sum. There are 8 types of model geometries for closed prime 3-manifolds, which by definition do not allow a connected sum decomposition. These geometries are E^3, S^3, H^3, $S^2 \times R$, $H^2 \times R$, $SL(2,"R")$, Nil, and $Solve$.

4. The model geometries allow a constant curvature metric. The finite volume of the manifold becomes a topological characteristic if one has constant curvature equal to ± 1.

5. All types except one, $S^2 \times R$, which corresponds to a string-like objects in TGD, allow a 3-D Lie group as subgroup of isometries (Bianchi group).

6. All model geometries except hyperbolic manifold (`https://rb.gy/snpft`) and Solv manifold are Seifert fiber spaces (`https://rb.gy/uxszk`), which are fibered by S^1 fiber. Hyperbolic manifolds are atoroidal but have an infinite fundamental group since Γ must be infinite from the finite volume property. Atoroidality means that there is no embedding of torus which would not be parallel to the boundary of the hyperbolic manifold.

   The finite volume property of H^3/Γ also requires that Γ is a lattice: this implies a deep analogy with condensed matter physics. The group elements in the TGD frame-work be $SL(2,C)$ matrices with elements which are algebraic integers in an extension of rationals defined by the polynomial P defining 4-surface in M^8. Note that also momentum components are predicted to be algebraic integers using a unit defined by the scale of the causal diamond (CD).

   TGD leads to the proposal [K77] that the H^3 lattices could appear in cosmological scales and explain "God's fingers" [E2] discovered by Halton Arp. They are astrophysical objects appearong along a line and having quantized redshifts.

7. One can form the spaces of the orbits for a discrete subgroup $\Gamma \subset G$ to obtain 3-manifolds with non-trivial fundamental group or orbifolds as in the case of S^3 and $S^2 \times R$. For hyperbolic 3-manifolds, the fundamental group is infinite and generated by elements of parabolic subgroups of G (`https://rb.gy/b5t55`). Cusp point and cusp neighborhood (`https://rb.gy/fd4dz`) are related to the infinite part of the fundamental group. Since parabolic subgroups $P \subset \Gamma$ are infinite groups, the fundamental group of the hyperbolic manifold is infinite.

8. One can decompose an irreducible 3-manifold to pieces, which are either Seifert manifolds or atoroidal. All 8 model geometries except hyperbolic geometries and so called Solv manifolds are Seifert manifolds.

   Solv manifolds are fiber spaces over a circle with 2-D plane with Minkowski signature as a fiber. In TGD solv manifolds could correspond to the so-called massless extremals [K5, K10, K60] serving representing classical radiation fields having only Fourier components with wave vectors in a single direction: laser beam is a good analog for them. They are not embeddable to H^3.

   In Ricci flows the hyperbolic pieces expand whereas other pieces contract so that asymptotically the manifold becomes hyperbolic. In fact, the collapse occurs in some caes in a finite time as found already by Richard S. Hamilton. The flow "kills" the positive curvature geometries S^3 and $S^2 \times R$ in the connected sum. What is left at large times is "thick-thin" decomposition. The "thick" piece is a hyperbolic geometry whereas the "thin" piece is a so-called graph manifold.

Hyperbolic manifolds and Seifert fiber spaces

Hyperbolic space and Seifert fiber space (`https://rb.gy/uxszk`) are in a central role in the TGD framework and therefore deserve short discussion. The following just gives the basic definitions and brief TGD inspired comments.

1. Hyperbolic manifolds

A hyperbolic n-manifold (`https://rb.gy/2esup`) is a complete Riemannian n-manifold of constant sectional curvature. Every complete, connected, simply-connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space H^n. As a result, the universal cover of any closed manifold M of constant negative curvature -1.

Every hyperbolic manifold (`https://rb.gy/snpft`) can be written as H^n/Γ, where Γ is a torsion-free discrete group of isometries on H^n. That is, Γ is a discrete subgroup of $SO^+_{1,n}$. The manifold has a finite volume if and only if Γ is a lattice.

Its "thick–thin" decomposition has a "thin" part consisting of tubular neighborhoods of closed geodesics and ends which are the product of a Euclidean $(n-1)$-manifold and the closed half-ray. The manifold is of finite volume if and only if its "thick" part is compact.

In the TGD framework, the lattice structure is natural and would mean that the elements of the matrices of Γ *are algebraic extensions in the extension of rational defined by the polynomial* P *determining* Y^4*. The tubular neighborhoods of the "thin" part would correspond to string-like objects (tubular neighborhoods) as geodesics whereas the ends would correspond to cusp singularities inducing blow-up as 3-surface* $S^3 \subset E^3$*.*
At the level of H *the tubular neighborhoods correspond to a string-like object and their ends to* CP_2 *type extremals serving as building bricks of elementary particles. Hadronic strings would represent examples of string-like objects and all elementary particles would involve them as monopole flux tubes connecting wormhole contacts.*

For $n > 2$ the hyperbolic structure on a finite volume hyperbolic n-manifold is unique by Mostow rigidity theorem and so geometric invariants are in fact topological invariants. One of these geometric invariants used as a topological invariant is the hyperbolic volume of a knot or link complement, which can allow us to distinguish two knots from each other by studying the geometry of their respective manifolds.

The identification of geometric invariants as topological invariants conforms with the TGD vision about "holy trinity" geometry-number theory-topology. Number theory would leak in through the identification of Γ *as a lattice determined by the polynomial* P*.*

2. Seifert fiber spaces

A Seifert manifold `https://rb.gy/uxszk` is a closed 3-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

A standard fibered torus corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$ (with the natural fibering by circles). If $a = 1$ the middle fiber is called ordinary, while if $a > 1$ the middle fiber is called exceptional. A compact Seifert fiber space has only a finite number of exceptional fibers.

The set of fibers forms a 2-dimensional orbifold, denoted by B and called the base — also called the orbit surface — of the fibration. It has an underlying 2-dimensional surface B_0, but may have some special orbifold points corresponding to the exceptional fibers.

The definition of Seifert fibration can be generalized in several ways. The Seifert manifold is often allowed to have a boundary (also fibered by circles, so it is a union of tori). When studying non-orientable manifolds, it is sometimes useful to allow fibers to have neighborhoods that look like the surface bundle of a reflection (rather than a rotation) of a disk, so that some fibers have neighborhoods looking like fibered Klein bottles, in which case there may be one-parameter families of exceptional curves. In both of these cases, the base B of the fibration usually has a non-empty boundary. 6 of the 8 basic geometries of Thurston are Seifert fiber spaces.

In the TGD framework, the Seifert fiber spaces would correspond to string-like objects, which appear as several variants.

The eight simply connected 3-geometries appearing in the Thurston's conjecture from the TGD point of view

This section contains as almost verbatim the description of the 8 Thurston geometries provided by the Wikipedia article `https://rb.gy/9x3pm`. There is a good reason for this: I am not a professional and do not understand the technical details. There is a good reason for not giving a mere Wikipedia link: I have added comments relating to the TGD based identification of these model geometries as 3-surfaces.

It turns out that the geometries could correspond to fundamental regions of H^3, to energy $E = constant$ (M^4 time $t = constant$ in H) surfaces $D^3 \subset M^4_+ \subset M^8$, to string-like objects $X^2 \times R$ allowing Seifert fiber space structure, or to masless extremals with structure $M^2 \times E^2$ with M^2 and E^2 corresponding to the orthogonal planes defined by light-like momentum and polarization vector.

First some definitions:

1. A model geometry is a simply connected smooth manifold X together with a transitive action of a Lie group G on X having compact stabilizers (the isotropy group of a point is compact).

2. A model geometry is called maximal if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.This condition can be aso included in the definition of a model geometry.

3. A geometric structure on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X; this is a special case of a complete (G, X)-structure. If a given manifold admits a geometric structure, then it admits a structure, whose model is maximal.

   One can say that the spaces X provide the raw material from which one obtains various 3-geometries by identifications using a discrete subgroup of G.

A 3-dimensional model geometry X is relevant for the geometrization conjecture if it is maximal and if there is at least one compact manifold with a geometric structure modelled on X. Thurston classified the 8 model geometries satisfying these conditions; they are listed below and are sometimes called Thurston geometries. (There are also uncountably many model geometries without compact quotients.)

There is a connection with the Bianchi groups, which are the 3-dimensional Lie groups. Most Thurston geometries can be realized as a left invariant metric on a Bianchi group. However, $S^2 \times R$ does not allow Bianchi geometry; Euclidean space corresponds to two different Bianchi groups; and there are an uncountable number of solvable non-unimodular Bianchi groups, most of which give model geometries having no compact representatives.

1. Spherical geometry S^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $O(4, R)$, with 2 components. The corresponding manifolds are exactly the closed 3-manifolds with a finite fundamental group. Examples include the 3-sphere, the Poincaré homology sphere, and lens spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type IX. Manifolds with this geometry are all compact, orientable, and have the structure of a Seifert fiber space (often in several ways). The complete list of such manifolds is given in the article on spherical 3-manifolds. Under Ricci flow, manifolds with this geometry collapse to a point in finite time.

> *In the TGD framework S^3 geometry could be associated with cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) of hyperbolic 3-manifold and represent the blow-up of a the cusp to S^3 which can be regarded as sphere in $E^4 \subset M^8 = M^4 \times E^4$. This is mapped to a 3-sphere of CP_2 in $M^8 - H$-duality.*

2. Euclidean geometry E^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $R^3 \times O(3, R)$, with 2 components. Examples are the 3-torus, and more generally the mapping torus of a finite-order automorphism of the 2-torus; see torus bundle. There are exactly 10 finite closed 3-manifolds with this geometry, 6 orientable and 4 non-orientable. This geometry can be modeled as a left invariant metric on the Bianchi groups of type I or VII_0.

Finite volume manifolds with this geometry are all compact, and have the structure of a Seifert fiber space (sometimes in two ways). The complete list of such manifolds is given in the article on Seifert fiber spaces. Under Ricci flow, manifolds with Euclidean geometry remain invariant.

> *In M^8 one has two kinds of roots of polynomials. For the first option they correspond mass square values defining mass shells H^3 . For the second option applying to the light-cone boundary as mass shell, energy E replaces mass and roots correspond to discrete energies. E = constant surface corresponds to E^3 as 3-balls inside the light-cone.*

3. Hyperbolic geometry H^3

The point stabilizer is $O(3,R)$, and the group G is the 6-dimensional Lie group $O^+(1,3,R)$, with 2 components. There are enormous numbers of examples of these, and their classification is not completely understood. The example with the smallest volume is the Weeks manifold. Other examples are given by the Seifert–Weber space, or "sufficiently complicated" Dehn surgeries on links, or most Haken manifolds.

The geometrization conjecture implies that a closed 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, and has an infinite fundamental group. This geometry can be modeled as a left invariant metric on the Bianchi group of type V or $\mathrm{VII}_{h\neq 0}$. Under Ricci flow, manifolds with hyperbolic geometry expand.

In TGD H^3 has an interpretation as a mass shell in $M^4 \subset M^8$ determined by the roots of the polynomial P or as a light-cone proper time constant hyperboloid in M^4.

This geometry does not allow Seifert fiber space structure unlike most other geometries.

4. The geometry of $S^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O(3,R) \times R \times Z/2Z$, with 4 components. The four finite volume manifolds with this geometry are: $S^2 \times S^1$, the mapping torus of the antipode map of S^2, the connected sum of two copies of 3-dimensional projective space, and the product of S^1 with two-dimensional projective space.

The first two are mapping tori of the identity map and antipode map of the 2-sphere, and are the only examples of 3-manifolds that are prime but not irreducible. The third is the only example of a non-trivial connected sum with a geometric structure. This is the only model geometry that cannot be realized as a left invariant metric on a 3-dimensional Lie group.

Finite volume manifolds with this geometry are all compact and have the structure of a Seifert fiber space (often in several ways). Under normalized Ricci flow manifolds with this geometry converge to a 1-dimensional manifold.

In the TGD framework, these surfaces could correspond to the simplest string-like objects for which S^2 corresponds to a geodesic sphere (homologically trivial or non-trivial) with a finite length connecting fundamental regions of H^3 or finite tessellations formed by them. S^2, which would correspond to a 2-D surface in CP_2 would be the base and string the fiber. One might argue that S^2 is more natural as fiber.

5. The geometry of $H^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O^+(1,2,R) \times R \times Z/2Z$, with 4 components. Examples include the product of a hyperbolic surface with a circle, or more generally the mapping torus of an isometry of a hyperbolic surface.

Finite volume manifolds with this geometry have the structure of a Seifert fiber space if they are orientable. (If they are not orientable the natural fibration by circles is not necessarily a Seifert fibration: the problem is that some fibers may "reverse orientation"; in other words their neighborhoods look like fibered solid Klein bottles rather than solid tori.) The classification of such (oriented) manifolds is given in the article on Seifert fiber spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type III. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

In the TGD context, these geometries would correspond to closed string-like objects for which the CP_2 projection is a 2-surface with genus $g > 0$. Seifert fiber space property corresponds to closed strings.

6. The geometry of the universal cover of $SL(2,"R")$

The universal cover of $SL(2,R)$ is denoted $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. It fibers over H^2, and the space is sometimes called "Twisted $H^2 \times R$". The group G has 2 components. Its identity component has the structure $(\mathbf{R} \times \widetilde{\mathrm{SL}}_2(\mathbf{R}))/\mathbf{Z}$. The point stabilizer is $O(2,R)$.

Examples of these manifolds include: the manifold of unit vectors of the tangent bundle of a hyperbolic surface, and more generally the Brieskorn homology spheres (excepting the 3-sphere and the Poincare dodecahedral space). This geometry can be modeled as a left invariant metric on the Bianchi group of type VIII or III. Finite volume manifolds with this geometry are orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

Also now the interpretation as a closed string-like entity is possible in TGD.

7. Nil geometry

This fibers over E^2, and so is sometimes known as "Twisted $E2\times\ R$". It is the geometry of the Heisenberg group. The point stabilizer is $O(2,R)$. The group G has 2 components, and is a semidirect product of the 3-dimensional Heisenberg group by the group $O(2,R)$ of isometries of a circle. Compact manifolds with this geometry include the mapping torus of a Dehn twist of a 2-torus, or the quotient of the Heisenberg group by the "integral Heisenberg group". This geometry can be modeled as a left invariant metric on the Bianchi group of type II.

Finite volume manifolds with this geometry are compact and orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow, compact manifolds with this geometry converge to R^2 with the flat metric. *In TGD also this geometry might be assigned with a closed string-like object as all Seifert fiber spaces.*

8. Sol geometry

This geometry (also called Solv geometry) fibers over the line with fiber the plane, and is the geometry of the identity component of the group G. The point stabilizer is the dihedral group of order 8. The group G has 8 components, and is the group of maps from 2-dimensional Minkowski space to itself that are either isometries or multiply the metric by - 1. The identity component has a normal subgroup R^2 with quotient R, where R acts on R^2 with 2 (real) eigenspaces, with distinct real eigenvalues of product 1.

This is the Bianchi group of type VI_0 and the geometry can be modeled as a left invariant metric on this group. All finite volume manifolds with solv geometry are compact. The compact manifolds with solv geometry are either the mapping torus of an Anosov map of the 2-torus (such a map is an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct, such as

$$\left(\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}\right)) \ ,$$

or quotients of these by groups of order at most 8. The eigenvalues of the automorphism of the torus generate an order of a real quadratic field, and the solv manifolds can be classified in terms of the units and ideal classes of this order. Under normalized Ricci flow compact manifolds with this geometry converge (rather slowly) to R^1.

Unlike in the case of Seifert fiber spaces, a plane or disk appears as a fiber. Could one consider the possibility whether boundary conditions guaranteeing conservation laws could allow string-like objects for which the cross section is disk rather than a closed 2-surface? The appearance of isometries of 2-D Minkowski space suggests that the disk X^2 must have Minkowski signature so that the embedding to H^3 would not be possible.
Could one assign this structure to massless extremals [K5, K60], which in the TGD framework define the representations for classical radiation fields, which involve the decomposition $M^4 = M^2\times D^2$. The circle $S^1\subset D^2$ defining its boundary would define the base space. Boundaries would be light-like and might allow to solve the boundary conditions. It is not clear how to obtain the counterparts of massless extremals at the M^8 level.

9.4.3 $3\rightarrow 4$ form of holography

One can consider two forms of holography. The first, weak, form corresponds to the ordinary $3\rightarrow$ holography in which 3-D boundaries provide the data defining the 4-surface. The second, strong form, corresponds to $2\rightarrow 4$ holography in which conformal boundaries provide the data defining the 4-surface. In this section the $3\rightarrow 4$ form of the holography is considered.

Fundamental domains of hyperbolic tessellations as data for $3 \to 4$ holography

Good candidates for the surfaces Y_i^3 are fundamental domains assignable to hyperbolic 3-manifolds H^3/Γ_i represented as surfaces in $H^3 \subset M^4 \subset M^8$ (or its complexification). In the case of string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. The treatment of this case is a rather straightforward modification of the first case so that the discussion is restricted to H^3/Γ_i.

The surfaces Y_i^3 would correspond topologically to many-particle states of free particles. Holography would induce topological interactions in the interior of Y^4 and $X^4(Y^4)$. The momenta (positions) of the fermions analogous to valence quarks correspond to the cusp singularities.

For fundamental fermions momenta would have components, which are algebraic integers. Galois confinement states that the momenta for many-fermion states are ordinary integers. This poses a condition for H^3/Γ and it would be interesting to understand what the condition means.

The degrees of freedom orthogonal to H^3 correspond to a complexified sphere S^3 of E^4, whose radius squared corresponds to the square of the complex mass squared.

1. Hyperbolicity is a generic property of 3-manifolds and probably preserved in small enough deformations. In other words, deformations of hyperbolic 3-manifold X_i^3 probably allow a hyperbolic metric although the induced metric for the deformation is not in general hyperbolic.

   Deformation of the hyperbolic manifold (`https://rb.gy/snpft`) could take place in its evolution defining Y_i^4 and $X^4(Y_i^4)$ and could lead to, for example, to singularities such as the touching of different surfaces and interaction vertices at which partonic 2-surfaces meet.

2. There is an interesting connection to the geometrization conjecture of Thurston (`https://rb.gy/9x3pm`), especially with the work of the Russian mathematician Grigori Perelman, who studied 3-D Ricci flows (`https://rb.gy/n6qlv`) for metrics and proved that, apart from scaling, they lead to hyperbolic geometries.

   Interestingly, hyperbolic manifolds decompose into "thin" and "thick" pieces and the "thin" piece corresponds to cusp neighborhoods (`https://rb.gy/fd4dz`). This decomposition brings in mind the notions of valence partons and sea partons with sea partons, in particular gluons assignable to the interior of Y_i^3 and giving the dominant contribution to the hadron mass.

Consider now what one can assume about Y_i^3.

1. The simplest assumption is that the S^3 coordinates are constant for Y_i^3 identified as the fundamental domain of a tessellation defined by H^3/Γ. It would represent a piece of H^3.

   Could one consider the allowance of S^3 deformations H_d^3 of H^3 in the direction of S^3, which are invariant under G so that the space H_d^3/G would exist. They would define what mathematicians would call a model of hyperbolic geometry.

2. Can one allow for a given Y_i^3 a multiple covering of H^3 by copies of Y_i^3 with different constant values of S^3 coordinates? Could this state correspond topologically to a many-sheeted covering naturally associated with the polynomial P?

   An interesting possibility is that Galois symmetry implies the existence of several copies of Y_i^3 with different S^3 coordinates as the orbit of the Galois group or its sub-group. Z_2 would be the simplest Galois group and give two sheets.

3-D data for $3 \to 4$ holography with 3-surfaces as hyperbolic 3-manifolds

It is good to start with questions.

1. Could the 3-surfaces X^3 associated with the mass shells $H_m^3 \subset M^8$ appearing as holographic data be fundamental domains (analogs of unit cell for crystals) of the tessellation H^3/Γ? Could a fermionic many-particle state for an algebraic extension determined by a given polynomial P assign a singularity to the fundamental domain and fix it?

   The TGD view of hadron physics provides some clues. Gluon sea consists of gluons identifiable as fermion-antifermion pairs and fermions and antifermions. Here is the data for the given hyperbolic 3-manifold of singularities. The valence fermions could reside at throats and virtual sea gluons could be associated with strings $Y^2 \times R$ inside flux tubes and would give to the classical string tension?

Hyperbolic 3-manifolds also have string-like singularities connecting the cusp singularities. In the physical picture of TGD, these would correspond to strings connecting wormhole throats of different wormhole contacts which in turn would correspond to blow-ups of cusps.

2. Is the situation the same in M^8 and H? Could 4-surfaces assignable to the X_i^3 be minimal surfaces in both H and M^8 having a generalization of holomorphic structure to dimension 4? It would be possible to map X_i^3 to each other by inversion. Note that M^8-H correspondence would map the $M^4 \subset M^8$ projections of the points of Y^4 by inversion to H also in the interior of 4-surface.

   Could this realize the dual conformal invariance proposed by the twistors, which would therefore be behind the analogy of Langlands duality and $M^8 - H$ duality?

9.4.4 Strong form of the hyperbolic holography

Holography roughly means an assignment of, not necessarily a unique 4-surface, to a set of 3-surfaces at the mass shells defined by roots of the polynomial P. The 4-surface is analogous to Bohr orbit.

A stronger form of the holography would be approximate $2 \rightarrow 4$ holography suggesting that the 3-surfaces allow $2 \rightarrow 3$ holography, which need not be completely deterministic. To understand what is involved one must have an idea about what kind of 3-surfaces could be involved.

1. Irreducible closed 3-surfaces Y_i^3 at H_m^3 consist of regions of 8 different types. Could these regions correspond to model geometries or at least have the symmetries of model geometries?

   This conjecture is natural if the 3-surfaces $Y_i^3 \subset H_m^3 \subset M^4 \subset M^8$ belong to (possibly complex) mass shells of M^4. In this case, the composites of fundamental regions of hyperbolic manifolds (`https://rb.gy/snpft`) as analogs of finite crystals would be natural.

   The interiors of these regions would correspond to the "thick" part of the 3-manifold whereas the cusp singularities and string singularities as boundaries of string world sheets would correspond to the "thin" part. The blow-ups of cusp singularities would give rise to 3-D regions of CP_2.

2. Also monopole flux tubes connecting hyperbolic regions to form a network should be involved. Here the natural model geometries would be of type $S^2 \times R$ or $H^2 \times R$ with the ends of R at the two hyperbolic regions. By replacing H^2 with H^2/Γ, one would obtain higher flux tubes with a cross section having a higher genus.

The natural idea is that hyperbolic holography gives rise to $2 \rightarrow 3$ holography. In the case of H^3/Γ, the holography would assign H^3/Γ its fundamental region Y^3 to H^2/Γ.

In the case of H^2/Γ, applying for string-like objects, holography would assign $Y^2 \times R$ to a union of circles H^1/Γ defining its boundary. The rule would be simple: Y^n/Γ is a union of fundamental regions Y_i^n having H^{n-1}/Γ as boundary.

Hyperbolic holography from H^2/G to the fundamental domain of H^3/Γ

The representation of M^4 momenta in terms of bispinors is possible only for massless particles. This raises the question whether one must assume a strong form of holography in which 2-D surfaces at the boundaries of H_m^3 dictate the 4-D surface almost completely. The hyperbolic 2-manifold H^2/G should define the boundary for Y_i^3 identifiable as a fundamental domain Y^3 of a hyperbolic 3-manifold H^3/Γ.

1. This would conform with the proposed realization of super-symplectic invariance and Kac-Moody type symmetries for light-like partonic orbits meaning that the interior degrees of freedom associated with the 3-surfaces X_i^3 and light-like orbits of partonic 2-surfaces are eliminated with a suitable gauge choice formulated in terms of a generalization the Virasoro and Kac-Moody conditions [L108, L109].

2. Physically this would mean that the fermion momenta at cusp point are light-like. This would conform with the view that fermions move along light-like curves inside the light-like partonic orbit.

3. If hyperbolic holography makes sense, the above formulation for H^2 would generalize to the case of H^3. Cusp neighborhood U/P as a projection $U \to H^2/G$ has a counterpart for H^3/Γ and the fundamental domain for H^2/G extends to a fundamental domain for H^3/Γ. For H^2/G as boundary it would correspond to the condition $p_3 > 0$ for the momentum component in the chosen direction.

4. The cusp singularity is analogous to a cusp of an algebraic surface. This suggests that near the cusp point of H^3/Γ the metric behaves like the induced metric of 3-D cusp in 4-D space. Near the cusp one has $t = k\sqrt{\rho}$ where t and ρ are vertical coordinate and transversal coordinates of the cusp in 4-D space. The radial component of the induced metric orthogonal to tip direction should behave like $g_{\rho\rho} = 1 + k^2/4\rho$ and the radial distance from the tip would diverge logarithmically. One cold say that this point is missing so that the hyperbolic manifold is compact but not closed since it has boundaries. The singularity of the metric is a good motivation for cutting off a small ball around the singularity in M^4 and a small ball from S^3 and for gluing the two together along boundaries. At the level of H this would correspond to wormhole throat.

9.4.5 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex

Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. M^8-H duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. Is this deformation of H^3 in imaginary time direction equivalent with a region of H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \ . \quad (9.4.1)$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

1. This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

2. A physically motivated proposal is that the real parts of $SU(3)_c$ gauge potential and color gauge field can be lifted to H and the lifts are equal to the classical gauge potentials and color gauge field proposed in H. Color gauge potentials in H are proportional to the isometry generators of the color gauge field and the components of the color gauge field are proportional to the products of color Hamiltonians with the induced Kähler form.

3. The color gauge field $Re(G)$ obeys the formula $Re(G) = dRe(A)+[Re(A), Re(A)] = [Re(A), Re(A)]$ and does not vanish since the contribution of $[Im(A), Im(A)]$ cancelling the real part is absent. The lift of $A_R = g^{-1}dg$ to H is determined by g using M^4 coordinates for Y^4 . The

M^4 coordinates $p^k(M^8)$ having interpretation as momenta are mapped to the coordinates m^k of H by the inversion

$$I : m^k = \hbar_{eff} Re(\frac{p^k}{p^2}) \ , \ \ p^2 \equiv p^k p_k \ ,$$

where p^k is complex momentum. $Re(A)_H$ is obtained by the action of the Jacobian

$$dI^k_l = \frac{\partial p^k}{\partial m^l} \ ,$$

as

$$A_H = dI \cdot Re(A_{M^8}) \ .$$

dI^k_l can be calculated as the inverse of the Jacobian $\partial m^k/\partial Re(p)^l$. Note that $Im(p^k)$ is expressible in terms of $Re(p^k)$.

For $Im(p^k) = 0$ the Jacobian for I reduces to that for $m^k = \hbar_{eff}\frac{p^k}{p^2}$ and one has

$$\frac{\partial m^k}{\partial p^l} = \frac{\hbar_{eff}}{p^2}(\delta^k_l - \frac{p^k p_l}{p^2}) \ .$$

This becomes singular for $m^2 = 0$. The nonvanishing of $Im(p^k)$ however saves from the singularity.

4. The $M^8 - H$ duality obeys a different formula at the light-cone boundaries associated with the causal diamond: now one has $p^0 = \hbar_{eff}/m^0$. This formula should be applied for $m^2 = 0$ if this case is encountered. Note that number theoretic evolution for masses and classical color gauge fields is directly coded by the mass squared values and holography.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M^8_c$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically. In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be

non-vanishing at the mass shells but outside the mass shells one would have nonvanishing G_2 gauge fields.

There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X_i^3.

9.5 Singularities, quantum classical correspondence, and hyperbolic holography

The point-like fermions and their 1-D trajectories appear as singularities of the minimal surfaces [L101]. Strings that connect fermions located at their ends, and string world sheets in the interior of X^4 appear also as singularities. Also partonic 2-surfaces separating Minkowskian and Euclidian regions should correspond to singularities of X_i^3 and their light-like radii.

There would therefore be singularities in dimensions $D = 0, 1, 2, 3$. These singularities should relate to the fundamental domains $Y_i^3 \subset H_m^3 \subset M^8$ and holography would suggest that they correspond to the singularities of 3-D hyperbolic manifolds (`https://rb.gy/snpft`).

9.5.1 Cusp singularities and fermionic point singularities

The singularities should be associated with hyperbolic manifolds Y_i^3 identified as fundamental domains of coset spaces H^3/Γ , that is, as effective geometries H^3/Γ defined by the boundary conditions for various "fields". In the same way as, for example, a torus geometry appears in condensed matter physics for a unit cell of lattice.

Cusp singularity (`https://rb.gy/fd4dz`) is a natural candidate for a point-like singularity and geometrically corresponds to a cusp. For abstract Riemann geometry, the cusp property would correspond to a singularity of the metric for a cusp (tip) and mean that the radial component of the metric diverges at the tip.

Consider first the basic concepts and ideas in the case of 2-D hyperbolic space H^2 and corresponding hyperbolic manifold H^2/G case.

1. Riemann surface can be regarded as a coset space H^2/G, which is represented as a fundamental region for a tessellation of H^2.

2. Cusp singularities of H^2/G correspond to parabolic subgroups P (`https://rb.gy/b5t55`) generated by a parabolic element for $G \subset SL(2, C)$. Parabolic subgroup P is isomorphic to a discrete group of translations along, say, the real axis as a boundary of the upper half-plane and is noncompact. It is represented as Möbius transformations induced by the matrices $(1, n : 0, n)$. P can be regarded as a subgroup generated by a Lorentz boost in a fixed direction.

The cusp singularity results from the identification of points related by the elements of P, which form a non-compact group. Let U denote the set with $Im(z) > 1$ which corresponds to the set $p^3 > 0$ in momentum space. U and $P(U)$ are disjoint. The cusp neighborhood (`https://rb.gy/fd4dz`) can be identified as the set U/P which is the projection of U to H/G.

3. In the simplest situation, one has $G \subset SL(2,Z) \subset SL(2,R) \subset SL(2,C))$, where $S(2,R)$ leaves the real axis invariant. Z could be replaced by an algebraic extension for rationals of algebraic integers in this extension.

$SL(2,C)$ and therefore also $SL(2,R)$ acts in M^4 as Lorentz transformations.

1. A given M^4 momentum has the representation $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2)$. The representation is unique apart from a complex scaling of z_i so that $z = z_1/z_2$ can be taken as a complex coordinate for the plane and $SL(2,C)$ acts as Möbius transformation. $SL(2,R)$ leaves the real axis invariant.

   The automorphism of sigma matrices induced by $SL(2,C)$ transformation in turn induces Lorentz transformation in momentum space.

2. Under what conditions can bi-spinors correspond to M^4 coordinates? Bi-spinor can be assumed to be of the form $(z_1, z_2) = (z, 1)$. From the formula $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2) = (z, 1)$ one can deduce an expression of the condition $Im(z_1/z_2) = Im(z_1) > 1$ in terms of p^k. The condition implies that the z-component of momentum satisfies $p^z = z\overline{z} - 1 > 0$.

   The description of M^4 momenta in terms of bi-spinors and H^2 identified as upper half-plane, denoted by H, is possible only for massless particles.

What happens at cusp singularity

What happens at the cusp singularity?

1. The normal space of the singularity is completely ill-defined as the direction of the electric field of a point-like charge. If so, CP_2 would always be a companion to the cusp. CP_2 would be a blow-up of the cusp points of X_i^3 as a hyperbolic manifold (`https://rb.gy/snpft`). One would have $X_i^3 \subset H^3$ and the cusp points would correspond to a 3-D sub-manifold of CP_2 defined by the normal spaces at the cusp singularity.

2. In the interior of the space-time surface the 3-D submanifold of CP_2 would extend to CP_2 type extremal with a light-like M^4 projection or its deformation. Several cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) could be associated with a single CP_2 type extremal representing wormhole contact. This corresponds to the view that wormhole throats can carry more than one fermion although the recent model assumes only a single fermion.

3. CP_2 type extremal defines a wormhole contact connecting two Minkowskian space-time sheets in $H = M^4 \times CP_2$. This would mean that the 3-D submanifold of CP_2 as a blow up is deformed to CP_2 type extremal with 2 throats at opposite sheets: at them the Euclidian induced metric transforms to Minkowskian signature.

   The conservation of monopole flux indeed forces the presence of two Minkowskian space-time sheets in the picture based on H. If the throat as a boundary of the 3-D region of $X^4 \subset H$ involves an incoming radial monopole flux, there must be another throat CP_2, where this flux runs to the other space-time sheet.

4. How could the throats connecting the two Minkowskian space-time sheets emerge in the M^8 picture? Should one allow several copies of Y_i^3 with the same H^3 projection but different constant S^3 coordinates and with a common cusp point. The blow-up would give several copies of 3-D regions of CP_2, and in holography they would define wormhole contact with 2 or even more throats.

   The simplest view is that quarks are the only fundamental fermions and leptons correspond to wormhole contacts carrying three antiquarks. They could have three throats associated with the same CP_2 type extremal but this is not the only possibility.

The singularities associated with string-like objects

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ, Γ a finite non-trivial subgroup of $SO(3)$ are orbifolds: the faces of Platonic solids are basic examples. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ are orbifolds if Γ is finite non-trivial subgroup of $SO(3)$: the triangular, quadrilateral, and pentagonal faces of Platonic solids are key examples. From these one can build finite lattices at S^2. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with a negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

What could be the quantal counterpart for the geometric holography? This has been a long standing open question. Suppose that the strong form of holography is realized.

1. In [L112], I considered quantal holography as a counterpart of geometric holography discussed in this article. This led to a suggestion that valence quarks at the wormhole throats could pair with pairs of dark quark and antiquark at strings associated with magnetic flux tubes in the interior of the hadronic 3-surfaces. Could these strings correspond to string-like singularities assignable to geodesic lines inside fundamental regions of $H^3/Gamma$?

2. The flux tubes were proposed to have an effective Planck constant $h_{eff} > h$. The correspondence between valence quarks and dark quarks was proposed to be holographic. The spin and electroweak quantum numbers of dark antiquark would be opposite to those of valence quark and dark quark would have quantum numbers valence quark. There would be entanglement in color degrees of freedom for valence quark and dark antiquark to form color single: this would screen the color of valence quark and transfer it to the magnetic body. The holography in this way would allow a convergent perturbation theory. Nature would be theoretician friendly: a phase transition increasing h_{eff}, transferring color to dark quarks, and reducing color coupling strength to $\alpha_s = 2^2/4\pi\hbar_{eff}$ would occur.

   Whether the dark quark-antiquark pairs as analog for gluon pairs as explanation for hadron mass could explain most of hadron mass remained open: if the classical conserved quantities are identical with the quantum contribution from fermions for Cartan algebra, this could be the case. Whether they could correspond to sea gluons remains also an open question.

3. Quantal holography allowing to obtain a convergent perturbation theory might be realized quite generally, also for leptons which correspond to color partial waves in CP_2 neutralized by super symplectic generator [K46, L57] [L67].

It should be noticed that leptonic dark holography would be very natural if leptons consist of 3 antiquarks [L94]. This option would explain matter-antimatter asymmetry in a new way. Antimatter would be identifiable leptons. For the simplest option, the 3 antiquarks would be associated with a single single wormhole throat. The generalized Kähler structure assignable to M^4 in twistor lift [L102, L103] allows a CP violation, which could favor the condensation of quarks to baryons and antiquarks to leptons.

There are however objections against this idea. The considerations of this article inspire the question whether a single wormhole throat can carry only a single quark assignable to the cusp singularity, as suggested already earlier. Two wormhole contacts would be required. This is required also by the fact that stable wormhole contact must carry a monopole flux and monopole flux flux loops must be closed. Uncertainty Principle would suggest that the flux tube must have length of order lepton Compton length. Can this be consistent with the point-like nature of leptons? These arguments favor the option in which leptons and quarks as opposite H chiralities of H-spinors are the fundamental fermions.

Other kinds of point-like singularities and analogy with Fermi surface

Point-like singularities as cusps would naturally correspond to fundamental fermions at the light-like orbits of partonic 2-surfaces.

1. The 2-D boundaries of the fundamental region Y_i^3 associated with H^3/Γ would be analogues for 2-D pieces of the Fermi surface corresponding to atomic energy levels as energy bands.

   In condensed matter physics, the energy shells can deform and the components of the Fermi surface can touch. These singularities are central to topological physics.

2. At M^8 level the 2-D boundaries of Y_i^3 are analogues of energy bands. The evolution defined by the number theoretic holography, identifiable as a coupling constant evolution at the level of M^8, induces deformations of Y_i^3. One expects that this kind of touching singularities take place.

   At the level of H this would correspond to simple touching of the outer boundaries of the physical objects. In particular, these touchings could take place at the partonic 2-surfaces identified as vertices at which severa partonic orbits meet as the partonic surfaces as their ends are glued to single surface just like the ends of lines of a vertex of Feynman diagram are glued together along their ends.

   Could the meeting of fermion and antifermion cusp singularity in this way relate to an annihilation to a boson regarded as a fermion antifermion pair?

3. One can of course challenge the assumption that all fermions correspond to cusps, which correspond to parabolic subgroups of $G \subset SL(2,C)$ (`https://rb.gy/b5t55`). The proposal that all momenta, whose components are algebraic integers for the extension defined by P, are possible. What could be the interpretation of fermions which do not correspond to cusp.

   What the addition of a fermion to a particular allowed momentum could mean? Could it mean that its momentum defines a parabolic subgroup of G? Or is it true only for the "thin" part of Y_i^3 perhaps representing analogs of valence quarks.

   Or could the non-singular momenta correspond to the momenta for the analogues of sea partons, in particular analogs of sea gluons as fermion-antifermion pairs so that their total momentum would dominate in the total momentum of hadron. These would correspond to the "thick" part of Y_i^3. Could these interior momenta correspond to states delocalized at the string world sheets in the interior of monopole flux tubes and also states delocalized in the interiors of the flux tubes. Are these fermions present too?

   The presence of these states should be coded by the geometry of the hyperbolic manifold H^3/Γ (`https://rb.gy/snpft`) and Y_i^3 as its fundamental domain. Somehow the group $G \subset SL(2,C)$ should be responsible for this coding.

9.5.2 About the superconformal symmetries for the light-like orbits of partonic 2-surfaces

Are the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) giving rise to CP_2 type extremals and the fermion momenta inside string world sheets and flux tubes associated with Y_i^3 sufficient to fix the 3-surfaces Y_i^3 in turn fixing number-theoretic holography?

1. The total energy for the classical action associated with these two kinds of fermions should correspond to the "sea" (thick part) and "valence fermions" assigned to the cusps (thin part).

2. Supersymplectic invariance and generalized conformal and Kac-Moody invariance assignable to light-like partonic orbits allows a large number of alternatives for the light-like surfaces [L99, L109, L108]. If supersymplectic and Kac-Moody symmetries act as gauge symmetries, the surfaces related by these symmetries are physically equivalent.

   The proposal is that these symmetries are partially broken and there is a hierarchy of breakings labelled by subalgebras $A_n \subset A$ of these algebras. The vanishing conditions for classical and quantal charges for A_n and $[A_n, A]$ serve as gauge conditions and also select the partonic 3-surfaces.

Interpretation of the partially broken gauge symmetries giving rise to dynamical symmetries is in terms of number theoretical measurement resolution and inclusion of hyperfinite factors of type II_1. These hierarchies relate to the hierarchies of extensions of rationals defined by the polynomials P defining the space-time surfaces apart from the effect of fermions.

If the preferred extremal property means generalization of holomorphy from 2-D case to 4-D case, one can conclude that the preferred extremals differ only at the singularities of space-time surfaces such as partonic orbits where the entire action comes into play. The regions outside the singularities would be universal: the minimal surface property would realize the 4-D generalization of the holomorphy.

3. Could different choices of the classical action, which determine the expressions of the classical and fermionic (quantal) Noether charges in terms of the modified Dirac action, correspond to different gauge choices selecting singular surfaces, in particular the CP_2 type extremals differently?

   The standard view would suggest that the change of the parameters of the action at the level of H corresponds to coupling constant evolution, which in the TGD framework is discrete and in terms of p-adic length scales. On the other hand, the existence of dual M^4 conformal invariance suggests that the coupling constant evolution at the level of M^8 is realized as "energy" evolution by using associativity as a dynamical principle. Can these two views be consistent?

   Note that the discriminant of the polynomial P is proposed to correspond to the exponent of action [L102, L104, L109, L108, L106]. The discriminant should change if the action changes. Does this mean that the change of the (effective) action in the discrete coupling constant evolution changes the polynomial?

9.6 Birational maps as morphisms of cognitive structures

https://en.wikipedia.org/wiki/Birational$_{g}eometry$ and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself.

1. In the TGD framework, the algebraic extensions E are defined by rational polynomials P at the level of M^8_c identifiable as complexified octonions. E defines a unique discretization for the number theoretically preferred coordinates of M^8_c by the condition that the M^8 coordinates have values in E: I call these discretizations cognitive representations. They make sense also in the extensions of p-adic number fields induced by E serving as correlates of cognition in TGD inspired theory of conscious experience. Birational maps respect the extension E associated with the cognitive representations and map cognitive representations to cognitive representation of same kind. They are clearly analogous to morphisms in category theory.

2. M^8-H duality [L80, L81, L117, L121] is a number theoretic analogue of momentum-position duality. M^8_c serves as the analog of momentum space and $H = M^4 \times CP_2$ as the analog of position space. $M^8 - H$ duality maps the 4-surface defined in M^8_c by number theoretic holography based on 3-D data to a 4-D space-time surface in H.

3. Should $M^8 - H$ duality respect the algebraic extension? If so, it would map the cognitive representation defined by points belonging to 4-D surface $Y^4 \subset M^8$ with the values of preferred coordinates in E to points of $M^4 \subset H$ with coordinate values in E. One could say that $M^8 - H$ duality respects the number theoretical character of cognitive representations. The precise meaning of this intuition is however far from clear.

There are also questions related to the choice of preferred coordinates in which the cognitive representation is defined.

1. Number theoretic constraints fix the preferred coordinates at M^8 side rather uniquely and this induces a preferred choice also on $M^4 \subset H$. For hyperbolic spaces (mass shells) a cognitive explosion happens and a natural question whether cognitive explosion happens also

for the light-like curves assignable to the partonic orbits. If the light-like curve is geodesic, the explosion indeed occurs. For more general light-like curves this is not the case always: could these more general light-like curves be related by a birational map to light-like geodesics?

2. At the H side one can also imagine besides standard Minkowski coordinates also other physically preferred choices of coordinates: are they also theoretically preferred? The notion of Hamilton-Jacobi structure [L114] suggests that in the case of M^4 Hamilton-Jacobi coordinates are very natural for the holomorphic realization of holography. If these are allowed, a natural condition would be that the Hamilton-Jacobi coordinates are related to each other by birational maps mapping the point of E to points of E so that cognitive representations are mapped to cognitive representations.

9.6.1 M^8-H duality, holography as holomorphy, Hamilton-Jacobi structures, and birational maps as cognitive morphisms

In the sequel the questions raised in the introduction are considered. The basic notions are M^8-H duality [L80, L81, L117, L121], holography as a generalized holomorphy [L111, L120], Hamilton-Jacobi structures [L114], and birational maps as cognitive morphisms.

About more precise definitions of the basic concepts

Consider first more precise definitions of various notions involved.

1. What are the preferred coordinates of M^8_c in which the cognitive representation is constructed? M^8_c has a number theoretic interpretation in terms of complexified octonions and physical interpretation as 8-D momentum space. Linear Minkowski coordinates are number-theoretically preferred since octonionic multiplication and other arithmetic operations have a very simple form in these coordinates. Also the number theoretic automorphisms respect the arithmetic operations. The allowed automorphisms correspond to the group G_2 which is a subgroup of $SO(1,7)$. Physically Minkowski space coordinates are preferred coordinates in the momentum space and also in $M^4 \subset H$.

2. How the algebraic extension of rationals, call it E, is determined? The proposal is that rational polynomials characterize partially the 3-D data for number theoretic holography [L117]. The roots of a rational polynomial P define an algebraic extension of rationals, call it E. A stronger, physically motivated, condition on P is that its coefficients are integers smaller than the degree of P.

   The roots of P define mass shells $H^3_c \subset M^4_c \subset M^8_c$, which in turn assign to the mass shells a 4-D surface Y^4 of M^8_c going through the mass shells by associative holography requiring that the normal space of Y^4 is associative, that is quaternionic. It has been be assumed that the roots are complex although also the condition that the roots are real can be considered. The imaginary unit i associated with the roots commutes with the octonionic imaginary units.

3. How the cognitive representation is defined? The points of $Y^4 \subset M^8_c$ with M^4 coordinates in E define a unique discretization of Y^4, called a cognitive representation, making sense also in the extensions of p-adic number fields induced by E. In general, the number of algebraic points in the interior of Y^4 is discrete and even finite but at the mass shells H^3 a cognitive explosion takes place. All points of H^3 with coordinates in E are algebraic.

   The algebraic points with coordinates, which are algebraic *integers* are physically and cognitively in very special role in number theoretic physics and make sense also as points of various p-adic number fields making possible number theoretical universality. The points of H^3 have interpretation as momenta and for physical states the total momentum as sum of momenta at mass shells defined by the roots of P has components which are integers, called Galois confinement [L102, L103], would define fundamental mechanism for the formation of bound states.

4. M^8-H duality maps the points of $H^3_c \subset M^4_c \subset M^8_c$ to points of $H^3 \subset M^4 \subset M^4 \times CP_2 = H$ by a map, which is essentially an inversion: this form is motivated by Uncertainty Principle:

for the most recent formulation of the duality see [L121]. This map is a birational map and takes points of E points of E. Also the points of cognitive representation belonging to the interior of $Y^4 \subset M_c^8$ are mapped to the interior of $X^4 \subset M_c^8$. One can ask whether the discrete set of points of cognitive representations in the interiors are of special physical interest, say having interpretation as interaction vertices.

Questions to be pondered

There are many questions to be considered.

1. Also partonic orbits in $X^4 \subset H$ define 3-D holographic data in H. What are these partonic orbits? The simplest partonic orbits have light-like M^4 projection but also more general light-like H projection can be considered (note the analogy with a 2-D rigid body rotating along a light-like geodesic of H). A general light-like geodesic of H is a combination of time-like geodesic of M^4 and space-like geodesic of CP_2.

   A point of the light-like partonic orbit correspond at the level of M^8 to the 3-D blowup of a point of M^8 at which the quaternionic normal space parametrized by CP_2 point is not unique so that the normal spaces for a 3-D section of CP_2, whose union along (probably light-like) geodesic is CP_2 with two holes corresponding to the ends of the partonic orbit. This singularity is highly analogous to the singularity of the electric field of a point charge. Partonic orbits define part of the 3-D holographic data.

2. Could one associate cognitive representations also to the partonic orbits? Could also partonic orbits allow a cognitive explosion? The simplest way to guarantee light-likeness for the H projection is as a light-like geodesic and this indeed allows an infinite number of algebraic points in Minkowski coordinates. Same applies to more general light-like orbits. One would have at least 1-D explosion of the cognitive representation.

3. What can one say about the CP_2 and M^4 projections of the partonic 2-surface? Could also these projections to X^2 and Y^2 allow an infinite number of points with coordinates in E or do these kinds of points have some special physical meaning, say as vertices for particle reactions? Concerning cognitive representation, the blow-up would mean that the point has an infinite but discrete set of quaternionic normal spaces at the level of M^8. Since the partonic surface can have an arbitrary complex sub-manifold as CP_2, there is indeed information to be cognitively represented.

The most general cognitively preferred coordinate choices for space-time surfaces and H?

In the case of M_c^8, number theoretical considerations fix the preferred coordinates highly uniquely. In the case of H the situation is not so obvious and one cannot exclude alternative coordinate choices related by a birational map.

A possible motivation comes from the following argument.

1. String world sheets are candidates for singularities analogous to partonic orbits. At a given point of the string world sheet a blow up to a 2-D complex sub-manifold of CP_2 would occur. This would mean that the normal spaces of the point in M_c^8 form this sub-manifold. Cosmic strings are the simplest objcts of this kind. Monopole flux tubes are deformations of the cosmic strings and allow also an interpretation in terms of maps from M^4 to CP_2.

   If string world sheets define part of the data needed to define holography, one could argue that it makes sense to assign cognitive explosion to the string world sheet.

2. Cognitive explosion takes place if the string world sheets are 2-D geodesic submanifolds of H. Planes $M^2 \subset M^4$ represent the simplest example. A more complex example is obtained by taking a space-like geodesic in H and rotating it along a time-like geodesic of H. One can also take a light-like geodesic in H and rotate it along a light-like geodesic in dual light-like

direction (ruler surface would be in question). In which case the gluing of the string world sheet along the boundary to the partonic orbit could be possible.

One might perhaps think of building string world sheets by gluing these kinds of ultrasimple regions along their boundaries so that one would have edges. An interpretation as a discretization would be appropriate. One might even go further and argue that the cognitive explosion explains why we are able to think of these kinds of regions in terms of simple formulas. One might argue that number theoretic physics realizes exactly what is usually regarded as approximation. One can however wonder whether life is so simple.

This argument encourages to consider a more complex option allowing more general string world sheets.

1. In the case of M^4 projection, the notion of the Hamilton-Jacobi structure [L114], generalizing the notion of ordinary complex structure, is highly interesting in this respect. It involves a generalization of complex coordinates involving local decompositions $M^4(x) = M^2(x) \times E^2(x)$ of the 4-D tangent space of M^4. The integrable distribution of $E^2(x)$ corresponds to complex coordinates $(w, \overline{w}$ integrating to a partonic 2-surface whereas the integrable distribution of $M^2(x)$ to light-like coordinate pairs (u, v) integrating to a string world sheet in M^4.

   Cognitive representation mean that the discretized values of the Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ are in E. Hamilton-Jacobi structure generalizes also to the level of $X^4 \subset H$ and now Y^2 can also correspond to CP_2 projection as in the case of cosmic strings and magnetic flux tubes. Note that in TGD one can use a subset of H coordinates as coordinates of X^4.

2. The simplest assumption is that the 1-D parton orbit corresponds to a light-like geodesic but could one map light-like geodesics to more general light-like curves by a birational map? Hamilton-Jacobi structure gives rise to a pair of curved (u, v) of light-like coordinates: could it relate to the standard flat light-like coordinates of M^2 by a birational map? Could a birational map relate standard complex coordinates of E^2 to the pair $(w, \overline{w})$? Could one also consider more general birational maps of $M^4 \to M^4$? If so, the Hamilton-Jacobi structures would be related by maps respecting algebraic extensions and cognitive representations. This would give a very powerful constraint on the Hamilton-Jacobi structures.

In the case of CP_2, projective coordinates are group-theoretically highly unique and determined apart from color rotations. Could one require that the CP_2 projection Y^2 associated with the partonic 2-surface and cosmic string or magnetic flux tube involves cognitive explosion. Are the allowed M^4 and CP_2 projections related by birational maps? Note that color rotations are birational maps.

These considerations suggest the following speculative view.

1. $M^8 - H$ duality, when restricted to 3-D holographic data at both sides, is analogous to a birational map expressible in terms of rational functions and respects the number theoretical character of cognitive representations.

2. Cognitive explosion occurs for the holographic data (this is very natural from the information theoretic perspective): this includes also string world sheets. Hamilton-Jacobi structures in the same cognitive class, partially characterized by the extension E of rationals, are related by a birational map.

3. $M^8 - H$ duality maps the quaternionic normal spaces to points of CP_2 and is an example of a birational map in M^4 degrees of freedom. It is not however easy to guess how the number theoretic holography is realized explicitly and how the 4-surfaces in M^8 are mapped to holomorphic 4-surfaces in H.

4. An interesting additional aspect relates to the non-determinism of partonic orbits due to the non-determinism of the light-likeness condition deriving from the fact that the action is Chern-Simons-Kähler action. The deformation of the partonic orbit induces the deformation of time derivatives of H coordinates at the boundary of $\delta M^4_+ \times CP_2$ to guarantee that boundary conditions at the orbit are realized. This suggests a strong form of holography [L120].

Already the 3-surfaces at $\delta M^4_+ \times CP_2$ or partonic orbits would be enough as holographic data. This in turn suggests that the analog of birational cognitive correspondence between the holographic data at $\delta M^4_+ \times CP_2$ and at partonic orbits.

9.6.2 Appendix: Some facts about birational geometry

Birational geometry has as its morphisms birational maps: both the map and its inverse are expressible in terms of rational functions. The coefficients of polynomials appearing in rational functions are in the TGD framework rational. They map rationals to rationals and also numbers of given extension E of rationals to themselves (one can assign to each space-time region an extension defined by a polynomial).

Therefore birational maps map cognitive representations, defined as discretizations of the space-time surface such that the points have physically/number theoretically preferred coordinates in E, to cognitive representations. They therefore respect cognitive representations and are morphisms of cognition. They are also number-theoretically universal, making sense for all p-adic number fields and their extensions induced by E. This makes birational maps extremely interesting from the TGD point of view.

The following lists basic facts about birational geometry as I have understood them on the basis of Wikipedia articles about birational geometry and Enriques-Kodaira classification. I have added physics inspired associations with TGD.

Birational geometries are one central approach to algebraic geometry.

1. They provide classification of complex varieties to equivalence classes related by birational maps. The classification complex curves (real dimension 2) reduces to the classification of projective curves of projective space CP_n determined as zeros of a homogeneous polynomial. Complex surfaces (real dimension 4) are of obvious interest in TGD: now however the notion of complex structure is generalized and one has Hamilton-Jacobi structure.

2. In TGD, a generalization of complex surfaces of complex dimension 2 in the embedding space $H = M^4 \times CP_2$ of complex dimension 4 is considered. What is new is the presence of the Minkowski signature requiring a combination of hypercomplex and complex structures to the Hamilton-Jacobi structure. Note however the space-time surfaces also have counterparts in the Euclidean signature $E^4 \times CP_2$: whether this has a physical interpretation, remains an open question. Second representation is provided as 4-surfaces in the space M^8_c of complexified octonions and an attractive idea is that $M^8 - H$ duality corresponds to a birational mapping of cognitive representations to cognitive representations.

3. Every algebraic variety is birationally equivalent with a sub-variety of CP_n so that their classification reduces to the classification of projective varieties of CP_n defined in terms of homogeneous polynomials. $n = 2$ (4 real dimensions) is of special relevance from the TGD point of view. A variety is said to be rational if it is birationally equivalent to some projective variety: for instance CP_2 is rational.

4. A concrete example of birational equivalence is provided by stereographic projections of quadric hypersurfaces in n+1-D linear space. Let p be a point of quadric. The stereographic projection sends a point q of the quadric to the line going through p and q, that is a point of CP_n in the complex case. One can select one point on the line as its representative. Another exammple is provided by Möbius transformations representing Lorentz group as transformations of complex plane.

The notion of a minimal model is important.

1. The basic observation is that it is possible to eliminate or add singularities by using birational maps of the space in which the surface is defined to some other spaces, which can have a higher dimension. The zeros of a birational map can be used to eliminate singularities of the algebraic surface of dimension n by blowups replacing the singularity with CP_n. Poles in turn create singularities. Peaks and self-intersections are examples of singularities.

   The idea is to apply birational maps to find a birationally equivalent surface representation, which has no singularities. There is a very counter-intuitive formal description for this. For instance, complex curves of CP_2 have intersections since their sum of their real dimensions is 4. The same applies to 4-surfaces in H. My understanding is as follows: the blowup for CP_2 makes it possible to get rid of an intersection with intersection number 1. One can formally say that the blow up by gluing a CP_1 defines a curve which has negative intersection number -1.

2. In the TGD framework, wormhole contacts have the same metric and Kähler structure as CP_2 and light-like M^4 projection (or even H projection). They appear as blowups of singularities of 4-surfaces along a light-like curve of M^8. The union of the quaternionic/associative normal spaces along the curve is not a line of CP_2 but CP_2 itself with two holes corresponding to the ends of the light-like curve. The 3-D normal spaces at the points of the light-like curve are not unique and form a local slicing of CP_2 by 3-D surfaces. This is a Minkowskian analog of a blow-up for a point and also an analog of cut of analytic function.

The Italian school of algebraic geometry has developed a rather detailed classification of these surfaces. The main result is that every surface X is birational either to a product $\mathbb{P}^1 \times C forsomecurve$ C$ortoaminimalsurface$Y$.Preferredextremalsareindeedminimalsurfacessothatspace- timesurfacesmightdefine minimalmodels.Theabsenceofsingularities(typicallypeaksorself-intersections)cha$

There are several birational invariants listed in the Wikipedia article. Many of them are rather technical in nature. The canonical bundle K_X for a variety of complex dimension n corresponds to n:th exterior power of complex cotangent bundle that is holomorphic n-forms. For space-time surfaces one would have $n = 2$ and holomorphic 2-forms.

1. Plurigenera corresponds to the dimensions for the vector space of global sections $H_0(X, K_X^d)$ for smooth projective varieties and are birational invariants. The global sections define global coordinates, which define birational maps to a projective space of this dimension.

2. Kodaira dimension measures the complexity of the variety and characterizes how fast the plurigenera increase. It has values $-\infty, 0, 1, ..n$ and has 4 values for space-time surfaces. The value $-\infty$ corresponds to the simplest situation and for $n = 2$ characterizes CP_2 which is rational and has vanishing plurigenera.

3. The dimensions for the spaces of global sections of the tensor powers of complex cotangent bundle (holomorphic 1-forms) define birational invariants. In particular, holomorphic forms of type $(p, 0)$ are birational invariants unlike the more general forms having type (p, q). Betti numbers are not in general birational invariants.

4. Fundamental group is birational invariant as is obvious from the blowup construction. Other homotopy groups are not birational invariants.

5. Gromow-Witten invariants are birational invariants. They are defined for pseudo-holomorphic curves (real dimension 2) in a symplectic manifold X. These invariants give the number of curves with a fixed genus and 2-homology class going through n marked points. Gromow-Witten invariants have also an interpretation as symplectic invariants characterizing the symplectic manifold X.

   In TGD, the application would be to partonic 2-surfaces of given genus g and homology charge (Kähler magnetic charge) representatable as holomorphic surfaces in $X = CP_2$ containing n marked points of CP_2 identifiable as the loci of fermions at the partonic 2-surface. This number would be of genuine interest in the calculation of scattering amplitudes.

What birational classification could mean in the TGD framework?

1. Holomorphic ansatz gives the space-time surfaces as Bohr orbits. Birational maps give new solutions from a given solution. It would be natural to organize the Bohr orbits to birational equivalence classes, which might be called cognitive equivalence classes. This should induce similar organization at the level of M^8_c.

2. An interesting possibility is that for certain space-time surfaces CP_2 coordinates can be expressed in terms of preferred M^4 coordinates using birational functions and vice versa. Cognitive representation in M^4 coordinates would be mapped to a cognitive representation in CP_2 coordinates.

3. The interpretation of M^8-H duality as a generalization of momentum position duality suggests information theoretic interpretation and the possibility that it could be seen as a cognitive/birational correspondence. This is indeed the case M^4 when one considers linear M^4 coordinates at both sides.

4. An intriguing question is whether the pair of hypercomplex and complex coordinates associated with the Hamilton-Jacobi structure could be regarded as cognitively acceptable coordinates. If Hamilton-Jacobi coordinates are cognitively acceptable, they should relate to linear M^4 coordinates by a birational correspondence so that M^8-H duality in its basic form could be replaced with its composition with a coordinate transformation from the linear M^4 coordinates to particular Hamilton-Jacobi coordinates. The color rotations in CP_2 in turn define birational correspondences between different choices of Eguchi-Hanson coordinates.

   If this picture makes sense, one could say that the entire holomorphic space-time surfaces, rather than only their intersections with mass shells H^3 and partonic orbits, correspond to cognitive explosions. This interpretation might make sense since holomorphy has a huge potential for generating information: it would make TGD exactly solvable.

Chapter 10

New findings related to the number theoretical view of TGD

10.1 Introduction

TGD could be seen as a holy trinity of three visions about quantum physics based on physics as geometry, physics as number theory, and physics as topology.

Quite recently I gave a talk on TGD and TGD inspired theory of consciousness and was asked about the motivations for the number theoretic vision. My response went roughly as follows.

1. The attempt to find a mathematical description for the physical correlates of cognition could have led to the vision of quantum TGD as a number theory. What are the possibly geometric/number theoretic/topological correlates of thought bubbles?

   A bold guess could have been p-adic numbers, $p = 2, 3, 5, 7, ...$ provide natural mathematical correlates for cognition. Rationals, algebraic extensions of rationals, and the extensions of p-adic number fields induced by them are natural candidates as also complex numbers, quaternions, and octonions. Also finite number fields emerged quite recently as natural ingredients of the number theoretic vision [K81, K82, K80] [L109].

   As a matter of fact, I ended up to a proposal that p-adic physics provides the correlates of cognition via a different route, by p-adic mass calculations based on p-adic thermodynamics, which turned out to have surprisingly high predictive power due to the number theoretic existence conditions [K46].

2. Sensory experience corresponds to real number based physics. There is a strong correlation between cognition and sensory experience, but it is not perfect. Sensations arouse thoughts, but cognition is also able to dream and imagine.

3. Cognition includes mathematical thought. The concretization of mathematical thinking as computation requires discretization. This suggests that discretization should correspond to what one might call a cognitive representation transforming thoughts to sensory percepts and it should have a number theoretic representation.

4. Mathematical thinking is able to imagine spaces with an arbitrary dimension, while the dimension of the perceptual world is fixed and is the dimension of three-space. How does cognition achieve this?

5. Cognition has evolved. Why and how can this be the case?

6. If the correlates of cognition are part of reality, then cognition must be optimally efficient. How?

This leads to the following questions and answers.

1. Could p-adic spacetime surfaces represent thought bubbles, equivalent to real 4-surfaces? They are a number-theoretic concept, they also involve a different topology than the sense-world, and p-adic space-time surfaces would be examples of algebraic geometry.

2. How is cognition able to imagine? p-Adic differential equations are non-deterministic: integration constants, which by definition have vanishing derivatives are only piecewise constants. Could this make imagination possible [K56]??

3. How the strong correlation between cognition and sensory experience could be realized? All p-adic number fields and their extensions must be allowed. Consider first the simplet book involving only reals and p-adic number fields. p-Adic number fields Q_p, $p = 2, 3, 5, ...$ can be combined into a book, an adele [L48, L47]. Different number fields as extensions of rationals represent the pages of this book. Real numbers correspond to sensory experience and various p-adic number fields to cognition. The back of the book corresponds to rational numbers that are common to all chapters.

   Every algebraic extension of rationals defines extensions of p-adic number fields. The p-adic pages of the algebraically extended book are algebraic extensions of various p-adic number fields. One obtains an infinite library with books labeled by algebraic extensions of rationals.

   Now the back of the book consists of algebraic numbers for the extension generated by the roots of a polynomial with integer coefficients. The back of the book gives a cognitive representation, a number theoretic discretization of the 4-surface that is unique for a given extension. The bigger the extension, the more accurate the discretization. Cognitive evolution would correspond to a refinement of cognitive representation induced by the increase of the order of the polynomial defining the extension.

4. How is cognition able to imagine higher-dimensional mathematical objects that do not exist at the level of sensory experience? algebraic extensions for p-adic numbers can have an arbitrarily high dimension if the corresponding polynomial has high enough degree. One can have p-adic 4-surfaces for which the associated algebraic dimension is arbitrarily high! p-Adic cognition is liberated from the chains of matter!

5. Why is evolution related to cognition? One gets an infinite number of books labeled algebraic extensions, a whole library. Does the evolution of cognition present a hierarchy? The bigger the algebraic extension, the better the approximation to real numbers and thus to sensory experience.

6. Can p-adic cognition be maximally effective? Here p-adic thermodynamics suggests the answer. p-Adic mass calculations assign to each elementary particle a p-adic prime. For instance, for electrons it is Mersenne prime $p = M_{127} = 2^{127} - 1 \simeq 10^{38}$. p-Adic mass squared value is expansion powers of p and its real counterpart is power series in negative powers of p. This series converges extremely rapidly for large primes such as $p \simeq 10^{38}$ and two lowest orders give a practically exact answer so all errors would be due to the assumptions of the model rather than due to computations.

How to realize number theoretic physics?

1. Number theoretic discretization does not resonate with the idea of general coordinate invariance. For $H = M^4 \times CP_2$ allows linear Minkowski coordinates but CP_2 coordinates are not linear although also now complex coordinates consistent with the isometries of $SU(3)$ are natural.

   What about M^8 or possibly its complexification suggested by twistorial considerations and also by the fact that classical TGD predicts that Euclidian space-time regions give an imaginary contribution to the conserved four momenta. M^8 allows highly unique linear Minkowski coordinates and the idea that M^8_c corresponds to complexified octonions is very natural. The automorphism group G_2 of octonions poses additional conditions.

2. This leads to the idea that number theoretic physics is realized at the level of M^8_c and that it is dual to the geometric physics realized at the level of H and that these physics are related

by $M^8 - H$ duality mapping 4-D surfaces in M^8 to H. TGD can be regarded as a wave mechanics for point-like particles replaced with 3-D surfaces in H, which, by the failure of complete determinism for holography, must be replaced by analogs of Bohr orbits. Wave mechanics is characterized by momentum-position duality, which naturally generalizes to $M^8 - H$ duality [L80, L81, L108, L109].

3. The physics in M_c^8 should be purely algebraic as is also the ordinary physics at the level of momentum space for free fields. This physics should make sense also in all p-adic number fields. This suggests that polynomials with integer coefficients, in particular their roots, together with number theoretic holography based on associativity, partially characterize the 4-surfaces in M^8, which would make sense also as their p-adic variants.

   It is not clear whether the p-adicization is needed at the level of H: it might be enough to have it only at the level of M^8 so that only the p-adic variants of M^8 would be needed.

The geometric vision of TGD is rather well-understood (see for instance [L99]), but one need not think long to realize that there is still a lot of fog in the number theoretic vision (see for instance [K81, K82, K80] and [L80, L81, L93, L108, L109]).

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?

2. The detailed realization of $M^8 - H$ duality [L80, L81] involves uncertainties: in particular, how the complexification of M^8 to M_c^8 can be consisted with the reality of $M^4 \subset H$.

3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?

4. How unique is the choice of 3-D surfaces at the mass shells $H_m^3 \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

5. The understanding of 3-geometries is essential for the understanding of the holography in both M^8 and H. The mathematical understanding of 3-geometries is at a surprisingly high level: the prime 3-manifolds can be constructed using 8 building bricks. Do these building bricks, model geometries, have counterparts as prefered extremals of action in the TGD framework.

   The known extremals $X^4 \subset H$ satisfying holography should be analogues of Bohr orbits [L101]. They are proposed to satisfy a 4-D generalization of 2-D holomorphy and apart from lower-D singularities would be the same for any general coordinate invariant action based on induced geometry and spinor structure. They would be minimal surfaces both in H and M^8 except at singularities at which the details of the action principle would matter [L119]. This suggests that the preferred extremals could have maximal isometries and provide topological invariants as also do the model geometries in the classification of 3-geometries.

10.2 What does one mean with M^8 physics?

In TGD, the point-like particle is replaced by a 3-surface $X^3 \subset H = M^4 \times CP_2$, and the holography required by the general coordinate invariance requires the replacment of the 3-surfaces with the analogues of Bohr trajectories passing through them. The Bohr trajectories are not completely deterministic as already the case of hydrogen atoms demonstrates. The "World of Classical Worlds" (WCW) is thus the space of generalized Bohr orbits as the counterpart of the superspace of Wheeler which originally inspired the notion of WCW [L118, L119, L115, L116]).

In wave mechanics, the duality between the descriptions using momentum and position space applies in wave mechanics but does not generalize to field theory. The $M^8 - H$ duality [L80, L81] can be seen as a generalization of this duality. M^8 is the momentum space counterpart and $H = M^4 \times CP_2$ is the position space counterpart in this duality.

10.2.1 Physical interpretation of the 4-surfaces of the space M^8 and their singularities

The physical interpretation of 4-surfaces in the complexifixation of the momentum space M^8 is far from straightforward. There are many reasons for the complexification.

1. Complexified octonionicity requires that M^8, or equivalently E^8, is complexified: one has $M_c^8 = E_c^8$ giving as its subspaces various real subspaces with various signatures of the number theoretical. The metric obtained from the Minkowski norm $\delta_{kl} z^k z^l$, where z^k are 8 complex coordinates. M^4 with signature $(1, -1, -1, -1)$ is in a special physical role and one can of course ask, whether also other signatures might be important.

2. If complex roots are allowed for polynomials P determining together with associativity the holography, complexification must be allowed. Virtual momenta could therefore be complex, but Galois confinement says that the total momenta of physical states are real and have integer components in the momentum scale determined by the size of the causal diamond (CD). Physical intuition suggests that the imaginary parts of the momenta code for the decay width of the particle. This is natural if the imaginary part is associated with the energy in the rest system.

3. The conserved momenta given by Noether's theorem at the level of H have real parts assignable to Minkowskian space-time regions. The fact that $\sqrt{g_4}$ appears in the integral defining a conserved quantity differs in Minkowski and Euclidean regions by an imaginary unit suggests that the contributions to momenta from the Euclidian regions are imaginary. The momenta from the Minkowskian space-time regions can be transferred to the light-like boundaries between Minkowskian and Euclidian regions identified as light-like partonic orbits. Quantum-classical correspondence requires that the classical total momenta, like all conserved quantities, correspond to the total momenta of the fermion state.

   Euclidian regions most naturally correspond to CP_2 type extremals as preferred extremals. They can be regarded as singularities resulting in the blow up of tip-like cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in M^8 . This would suggest that the real parts of momenta are associated with the Minkowskian regions of space-time surfaces and imaginary parts to the Euclidian regions. This applies also to other conserved quantities.

10.2.2 Number theoretic holography

Number theoretic holography has two forms.

1. The weak $3 \rightarrow 4$ form corresponds to the ordinary holography $Y^3 \subset M^8 \rightarrow Y^4 \subset M^8$, which is by $M^8 - H$ duality equivalent of the holography for $X^3 \subset H \rightarrow X^4 \subset H$ for space-time surfaces. The proposed interpretation of Y^3 is as a fundamental region of H^3/Γ.

2. For the strong $2 \rightarrow 4$ form of the holography Y^3 is determined by a 2-D data defined by the boundary of the fundamental region of H^3/Γ. The proposal to be considered is that the boundary of the fundamental region of H^3/Γ can be identified as 2-D hyperbolic space H^2/Γ.

Consider next the weak form of the holography.

1. The 4-surface $Y^4 \subset M_c^8$ is determined from number-theoretic dynamics and is an associative surface, i.e. its normal space is associative and therefore quaternionic.

2. There are also commutative 2-D surfaces that most naturally correspond to string world sheets, and for them commutativity of tangent space (as analog of associativity) as subspace of normal space of Y^2 defines holography. Holographic data corresponds now to strings connecting wormhole contacts assignable to Euclidian singularities inside $Y^3 \subset H_m^3$. One

can also consider the possibility that partonic 2-surfaces correspond to co-commutative 2-surfaces. The situation is not completely clear here.

3. One must also identify the 3-surfaces $Y^3 \subset H^3_m$ defining the holography. Holography is subject to very strong conditions and I have proposed that these surfaces are hyperbolic 3-manifolds X^3 obtained as coset spaces H^3/Γ, where G is suitably chosen discrete but infinite subgroup of $SL(2,C)$ acting as Lorentz transformations in H^3. The spaces H^3/Γ are fundamental domains of H^3 tessellations.

   $Y^3 = H^3/\Gamma$ is counterpart for the unit cells of a lattice in E^3, which effectively has this topology and geometry due to boundary conditions stating that G leaves various "field configurations" invariant. The situation is the same as in the case of ordinary condensed matter, where periodic boundary conditions for a cube as a unit cell make it effectively a 3-torus.

   Also the crystal-like structures consisting of a finite number of copies of the fundamental domain of H^3/Γ glued together are possible choices for Y^3. They would be analogous to the unit cells of the lattices of Euclidian space E^3 or finite crystals formed from them. Therefore the analog of solid state physics would be realized at the fundamental level.

   One can also consider closed 3-manifolds $Y^3 = H^3/\Gamma$ obtained by gluing two copies of the fundamental region with different S^3 coordinates connected together along their 2-D boundaries. The gluing could be performed by a cylinder of $S^3 \subset E^4 \subset M^4 \times E^4$ connecting the boundaries.

4. The quantum state at H^3_m consists of several Galois singlets assignable to 3-surfaces Y^3_i. The total momenta for X^3_i would be real and have integer valued components for the momentum unit defined by the size scale of CD involved. This condition is analogous to the periodic boundary conditions.

5. Quantum classical correspondence requires that the many-fermion state on H^3_m, characterized partially by momenta, which are in the algebraic extension of rationals associated with the polynomial P, determines $Y^3_i \subset H^3_m$. For a given Y^3_i, the accompanying fermions correspond to the points of H^3_m. The classical momentum of the state given by Noether theorem in H would the sum of the fermionic momenta.

   An attractive idea is that at least a subset of the fermionic momenta corresponds to cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`), which can be visualized geometrically as vertices of an algebraic surface at which the direction of normal space is ill-defined.

   The cusps correspond to parabolic subgroups $P \subset G \subset SL(2,C)$ (`https://rb.gy/b5t55`), where the $\Gamma \subset SL(2,C)$ defines the fundamental domain H^3/Γ of the tessellation. Parabolic subgroups P are automorphic to the subgroup of translations of upper half-plane H generated by $SL(2,C)$ matrix $(1a;0,1)$, a a real algebraic number. This particular P acts as Möbius transformations in H representing hyperbolic space H^2. The cusp singularities would encode at least a subset of fermionic momenta of the state into the hyperbolic geometry of Y^3_i. Each fermion would correspond to its own parabolic generator in the subgroup G.

   In the TGD view of hadron physics [L112], the fermions associated with the cusps could be identified as analogs of valence quarks. They would be associated with singularities identifiable as light-like 3-D partonic orbits serving as boundaries of 4-D CP_2 type extremals with Euclidian signature of the induced metric.

   Also fermionic momenta, which have algebraic integers as components but do not correspond to cusps, can be considered. These would be naturally associated with strings predicted to connect cusps at the throats of different wormhole contacts. The blow-up would be now 2-sphere relating to cusp singularity like line charge to point charge. It is not clear whether the sea partons could correspond to these string-like singularities. In any case, hyperbolic 3-manifolds have string-like singularities connecting cusps.

6. If Y^3_i corresponds to a Galois singlet, then its total 4-momentum is real and integer-valued and should be mapped to a discrete plane wave in the finite lattice defined by the crystal like structure formed by the copies $X^3(Y^3)$ in H^3_a given by inversion. Each Galois singlet Y^3_i

would define such a plane wave and one can imagine a hierarchy of such structures just as in the case of condensed matter with crystals of different sizes. The analogy with condensed matter physics suggests that Γ is a lattice. This follows also from the condition that H^3/Γ has a finite volume.

7. This picture would suggest that also $X^3(Y_i^3)$ is hyperbolic manifold of its fundamental region and perhaps isometric with Y^3. This would mean a geometric realization of Lorentz invariance analogous to the dual of conformal invariance encountered in the twistorialization.

10.2.3 Quantum classical correspondence for momenta

Quantum classical correspondence for momenta and also other conserved charges poses very strong conditions.

1. Noether charges for the classical action define momenta and other conserved charges. The classical contribution is a c-number. In addition, quantum contributions from fermions are included. They correspond to the momenta related to the second quantized spinor modes of H and from the orbital degrees of freedom associated with the "world of classical worlds" (WCW). The fermionic contributions are related to the ground states of the super symplectic representation characterized in terms of spinor modes for H spinor fields [K21, K73] [L109].

2. Are the classical contributions separate and do they add up to the total momentum? The fact that classical contributions are separately conserved, does not support this view.

3. Quantum classical correspondence would mean that the classical total momentum is the sum of the fermionic momenta determined by the multi-fermion state. This would hold quite generally for Cartan algebra of observables. For example, in the case of hadrons, the dominant classical contribution could correspond to the gluon sea, that is to multi-gluon state with gluons expressible in terms of quark-antiquark pairs. This picture is consistent with QCD and is therefore perhaps a more realistic guess.

4. Wormhole contact has Euclidean induce metric and the related classical conserved momentum is naturally imaginary. Could the sum of the imaginary parts of complex fermionic momenta of fermions for a wormhole throat correspond to the classical imaginary momentum assignable with the wormhole contact? Could the imaginary part of the fermionic momentum be assigned with the end of the euclidean string inside CP_2 type extremals, while the real momentum would be assigned with an end of a Minkowskian string?

5. Quantum-classical correspondence would be realized if the fermionic conserved four-momenta on the H side corresponded to M^8 points at hyperbolic 3-surfaces H^3_m. Their inversions in the $M^8 - H$ duality would be points of M^4_c of the spacetime surface $H^3_m \subset M^4_c \times CP_2$. It would seem that one must map only the real parts of the momenta at H^3_m to H^3_a.

   It would also seem that H^3_m must be associated with the M^4 projection of M^4_c. Whether the variant of H^3_m for complex valued m^2 makes even sense is not obvious.

10.2.4 The analog of time evolution in M^8 as a coupling constant evolution conserving dual quantum numbers

The proposal that 4-D surfaces appear at the level of M^8 suggests that it makes sense to talk about dynamics also in M^8 and the 4-D analogies of space-time surfaces make sense. This does not fit the usual classical intuition.

The twistor picture for conformal invariant field theories predicts that conformal invariance has a dual counterpart. This would mean that 4-momenta and other Poincare charges in H have dual counterparts in M^8. In TGD, the dual counterparts would be obtained by inversion from the defining the $M^8 - H$ duality and mapped to points of the space-time surface at the mass shells H^3_a in H. They would be analogs of the representation of lattice momenta as points of the heavenly sphere in crystal diffraction.

1. In zero energy ontology (ZEO), the time evolution at the level of H by "small" state function reductions (SSFRs), which are analogous to the so called weak measurements introduced by quantum opticians, would correspond to time evolution in terms of scalings rather than time translations. They would scale the size of the causal diamond (CD) and leave the passive boundary of CD invariant. These analogs of time evolutions would be generated by the scaling generator L_0. This would naturally also apply to M^8. This time evolution would be induced by scalings of the mass-scale, which need not be identical.

2. Could the "energy evolution", by identifying the square of the mass as the counterpart of time, correspond to the development related to the renormalization group? M^8-H duality would map the renormalization group evolution from M^8 to time evolution in H. This is quite a strong prediction.

3. Mass squared values for the fundamental fermions would not define particle masses but mass scales. 4-momenta for physical particles would correspond to total momenta for many fermion states, which obey Galois confinement, which requires that the momentum components are integers, when the mass unit is defined by the size scale of CD.

4. What would be the interpretation of the mass shells M^4_c determined by the roots of the polynomial P in the coupling constant evolution? Could the related hyperbolic 3-manifolds correspond to fixed points for the coupling constant evolution? With these mass values, something special would happen. Could H^3_a correspond to critical moments of light-cone proper time a when the SSFRs occur and a new unitary time evolution begins and ends with the next SSFR, as I have suggested?

5. What about the M^8 side? Could one talk about conserved quantities with respect to the evolution determined by scalings? Could these dual charges, dual momenta, and. also the charges related to E^4 isometries, be invariants of the renormalization group evolution?

   I have proposed that the $SO(4)$ symmetry of of hadrons in old-fashioned hadron physics involving notions like conserved vector current (CVC) and partially conserved axial current (PCAC) could correspond to the color symmetry of higher energy hadron physics by M^8-H duality in which the natural conserved charges on M^8 side are associated with the product of isometry groups of M^4 and E^4 and perhaps even with $SO(1,7)\times T^8$ or G_2 as automorphism group of octonions. At H side one would have a product of Poincare group and color group. Also holonomy groups are involved. At least $SO(4)$ symmetry could define invariants of the coupling constant evolution in M^8.

Consider now a more detailed interpretation of 4-surfaces $Y^4\subset M^8$ in terms of a generalized coupling constant evolution.

1. The changes $m_i^2\rightarrow m_{i+1}^2$ for the roots of P would define a discrete evolution in both M^8 and H. Discrete coupling constant evolution affects the mass resolution and brings in or deletes details and therefore would induce changes for the representations of the states. The 4-surfaces in M^8 would represent renormalization group flows. The failure of a complete determinism is expected and could be interpreted in terms of phase transitions occurring at critical masses.

2. A given mass shell m_i^2 determined by a root of P would define a discrete mass scale for the evolution having perhaps an interpretation as a fixed point or a critical point of the coupling constant evolution. It would be natural to assume that the evolution induced by the change of resolution conserves other total quantum numbers than 4-momentum.

3. What about the conservation of 4-momentum? At $m^2=m_{i+1}^2$ the value of mass squared for fundamental fermions defining the mass scale changes. The structure of the state must change in $m_i^2\rightarrow m_{i+1}^2$ if 4-momentum conservation is assumed.

   The normalization group evolution for the mass m^2 of the physical state, is typically logarithmic in QFTs, and must be distinguished from the discrete evolution for the mass scale m_i^2. Hence the change of m^2 in $m_i^2\rightarrow m_{i+1}^2$ is expected to be small. This could be realized if n corresponds to a (possibly normal) subgroup of the Galois group of P perhaps spanned by the roots $m_k^2 \leq m_i^2$.

Could the phase transition $m_i^2 \rightarrow m_{i+1}^2$ change the rest energy of the state? Does the change require an energy feed between the CD and its environment as ordinary phase transitions require? This is not the case if CD is interpreted as a perceptive field rather than a physical system.

4. Does it make sense to talk about the conservation of dual momentum $X^k = \sum_i X_i^k$, $X_i^k = Re(\hbar_{eff} p^k/p_{k,i}p_i^k) = Re(\hbar_{eff} p^k/m_i^2)$? The conservation of momentum p^k does not imply the conservation of dual momentum since it is proportional to $1/m_i^2$: X^k would scale as $1/m_i^2$. The size of the CD is assumed to increase in statistical sense during the sequence of small state function reductions (SSFRs). The increase of the size of the CD would gradually make the mass shells inside it visible.

   $M^4 \subset H$ center of mass position X^k therefore changes $m_i^2 \rightarrow mi + 1^2$ unless h_{eff} is not scaled to compensate the change $m_i^2 \rightarrow mi + 1^2$ in the formula for X_i^k. The integer n in $h_{eff} = nh_0$ is assumed to correspond to the order of the Galois group of P. It could also correspond to the order of a subgroup of the Galois group of P defined by the roots m_k^2, $k \leq i$. If so, h_{eff} would increase in evolution and one can even imagine a situation in which $Re(h_{eff}/m_i^2)$ remains constant.

10.3 $M^8 - H$ duality

The proposed $M^8 - H$ duality maps 4-surfaces $Y^4 \subset M_+^4 \subset M^8 = M^4 \times E^4$ to space-time surfaces $X^4 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$.

10.3.1 $M^8 - H$ duality as inversion

$M^8 - H$ duality relates also the hyperbolic spaces $H_m^3 \subset M_+^4 \subset M^8 = M^4 \times E^4$ and $H_a^3 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$. The hyperbolic space $H_m^3 \subset M_+^4 \subset M^8$ corresponds to the mass shell for which mass squared value m^2 is a root of a polynomial P. The hyperbolic space $H_a^3 \subset M^4 \subset= M^4 \subset M^4 \times CP_2$ corresponds to light-cone proper time constant surface $t^2 - r^2 = a^2$.

1. The root of P is in general a complex algebraic number. The first guess is that $M^8 - H$ duality is defined by inversion $p^k \rightarrow m^k = \hbar_{eff} p^k/p^l p_l$. Or briefly, $p \rightarrow \hbar_{eff} p/m^2$. The light-cone proper time $a = h_{eff}/m$ characterized the hyperboloid $H_a^3 \subset M^4$. $H_m^3 \rightarrow H_a^3$ is consistent with the Uncertainty Principle. In this case the image would be complex. This creates interpretational problems. There is no need for the complexification of CP_2, which also suggests that the image un H_a^3 must be real.
2. One can consider the possibility that only the real projection of the complex variant of H_m^3 to $H_{Re(m)}^3$ is involved. The image of the real part $Re(p^k)$ in H_a^3 obtained by inversion would be real but would not code information about the imaginary part $Im(p^k)$.
3. One could however take the real part of a complex inversion to get a point in H_a^3. $Re(\hbar_{eff} p^k/p^2$ would code information about the imaginary value of m^2.

Inversion fails at the light-cone boundary with $m = 0$. In this case, the inversion must be defined as the inversion of the energy of the massless state: $p^k \rightarrow \hbar_{eff} Re(p^k/E^2)$.

10.3.2 The technical problems posed $M^8 - H$ duality the complexification of M^8

Complexification of M^8 is highly desirable in the number-theoretic vision. But how to deal with the fact that fermion momenta, for which with components are algebraic integers in the algebraic extension determined by a polynomial P, are in general complex?

1. Without additional conditions, the mass shells in $M^4_c \subset M^8_c$ for complex m^2 as a root of P are 6-D. There are 2 conditions coming from the conditions fixing the value of $Re(m^2) = Re(p^2) - Im(p^2)$ and $Im(m^2) = 2Re(p) \cdot Im(p)$. If one only energy is complex, the dimension of the mass shell is 3. This looks natural. The preferred time axis would be determined by the rest system for massive states. A possible interpretation for the imaginary part is as decay width in the rest system.

2. The complexified mass shells of complexified $M^4 \subset H$ must be considered. Does this make sense? Since the CP_2 point labelling tangent space of Y^4 does not depend on complexification there is no need to consider complexification of CP_2. Therefore the natural conclusion is that also the $M^4 \subset H$ images should be real.

The inversion $p^k \rightarrow p^k/m_r p_r p_s$ is the simplest realization for $M^8 - H$ duality and would naturally fit into a generalization of 2-D conformal invariance to 4-D context. $h_{eff} = nh_0$ hierarchy comes along in a natural way. The polynomial P determines the algebraic extension and the value of h_{eff}. The size of the CD would scale like h_{eff} on the H side. There would be no scaling on the M^8 side.

1. The first thing to notice is that one could understand classically complex momenta. On the H side, Euclidean regions could give an imaginary contribution to the classical momentum.

2. The complex inversion $p^k \rightarrow p^k/m_r p_r p_s$ maps complex H^3_m to complex H^3_a. What would be the interpretation of the complex M^4_c coordinates? The same problem is also encountered in twistorization. One can ask whether a complex time coordinate corresponds to, for example, the inverse temperature?

   However, since no complexification is needed for CP_2, it seems that the only natural option is that the $M^4 \subset H$ image is real.

3. One can consider 3 options guaranteeing the reality.

   (a) Only the real parts of the complex $M^4 \subset M^8$ momenta are mapped to H. The information about the imaginary parts would be lost.

   (b) The complex algebraic integer valued momenta are allowed and the real part $Re(p^k \rightarrow p^k/m_r p_r p_s)$ of the complex inversion defines the image points in H. The $M^4 \subset H$ complexification would not be needed for this option but the information about the imaginary part of the momenta would not be lost.

   (c) By Galois confinement, the physical multiparticle states consist of momenta with integer value components using the momentum unit assignable to CD at the M^8 level of space with mass shells. These would define the 3-D data for the holography, which determines the 4-surface $Y^4 \subset M^8$ through the associativity of the normals space of Y^4. Only the real, integer-valued momenta of Galois confined states would be mapped from the mass shells of $M8$ to their images in H.

   The information about the fermion composition of the many-particle states would be lost completely. Therefore this option does not look realistic.

   The realization of this view might be possible however. 4-momenta determine the 3-surfaces Y^3 with real mass shells H^3 as data for associative holography. Momenta could correspond to point-like singularities on $Y^3 \subset H^3$ and these should be assigned as CP_2 type extremals at H side as blow-ups of the singularities.

To conclude, the option $p^k \rightarrow Re(p^k \rightarrow p^k/m_r p_r p_s)$ seems to be the physically realistic option.

10.3.3 Singularities and $M^8 - H$ duality

Consider next the description of the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in $M^8 - H$ duality. The condition that information is not lost, requires that the map is given by $p^k \rightarrow Re(p^k/m_{rs} p^r p^s)$.

1. The cusps in M^8-H duality would be mapped to a 3-D surface of CP_2. It would correspond to the 3-D section CP_2 of the extremum, which corresponds to a wormhole contact associated with fermions at its throats.

   At H^3_m there is a temptation to assign to the cusp singularity, identified as a blow-up, the 3-D sphere $S^3 \subset E^4$ of the normal space E^4 defined by the mass shell condition. The simplest option is that this sphere is mapped to $U(2)$ invariant sphere $S^3 \subset CP_2$ for which the radius would be fixed by the mass squared value.

   The metric of H^3/Γ is singular at the cusp. The elimination of the singularity requires that one must allow a hole Z^3 around the cusp. The boundary of X^3 can have any genus. The size scale of the hole should be determined by the mass squared value.

   This view conforms with the physical picture of the CP_2 type extremal as an orbit of an Euclidian wormhole contact connecting two Minkowskian space-time sheets. S^3 would be replaced with $S^3 \setminus Z^3$ mapped to CP_2, where it corresponds to a wormhole throat having arbitrary genus.

   This picture would suggest that a given cusp singularity can correspond only to a single wormhole throat. This is not in conflict with the recent view of what elementary particles having wormhole contacts as composites should be. Composite, involving 2 wormhole contacts (required by the conservation of the monopole flux forming a loop involving two space-time sheets) and therefore 2 wormhole throats, can have spin varying from 0 to 2 which conforms with the popular wisdom that elementary particle spins vary in this range.

2. In the case of string-like objects $Y^2 \times R \subset H^3_m$, that is $S^2 \times R$ and their higher genus counterparts $H^2/\Gamma \times R$, the counterpart of the blow-up would be $Y^2 \subset S^3 \subset E^4$. Y^2 would be mapped to $X^2 \subset CP_2$ such that the radius assignable to S^2 or the size scale assigned to H^2/Γ would correspond to the mass squared.

3. Fermion trajectories at the partonic orbits would be light-like curves defining boundaries of string world sheets. CP_2 extremal would be associated with a fermionic cusp by holography and M^8-H duality.

   Fundamental fermion as an analog of valence quark [L112] could correspond to a cusp at the boundary of the string world sheet. Cusps would be related to the boundary of X^3_a composed of partonic 2-surfaces.

4. In principle, fermion momenta in the interior of $Y^3 \subset H^3_m$ are also possible. The picture given by hadron physics would indicate that the interior contribution corresponds to the sea partons. They can also be associated with string world sheets and correspond to virtual bosons appearing as fermion-antifermion pairs. These singularities would be string-like objects of the form $X^2 \times R$ and $X^2 \subset CP_2$ would replace the sphere of CP_2. One could say that fermions are delocalized at string.

10.3.4 Realization of the Uncertainty Principle

Inversion alone is not enough to realize Uncertainty Principle (UP), which requires that M^8-H duality is analogous to the Fourier transform. However, with the help of H^3 tessellations, it is possible to understand how the UP is realized in a finite measurement resolution.

The invariance of points of H^3/Γ under the subgroup $G \subset SL(2,C)$ is analogous to the periodic boundary conditions that replace the cubic unit cell of a crystal lattice with a torus. Now the tessellation of H^3, which quantizes the momenta, would be replaced by H^3/Γ. The momentum lattice having the fundamental region for H^3_m/G as unit cell would be mapped by inversion to a position lattice having the fundamental region H^3_a/G as a unit cell. A point in H^3_m would correspond to an analog of plane wave as a superposition of all positions of $X^3(Y^3)$ in a part of the tessellation in H^3_a: a wave function in finite crystal. One would have a superposition of 3 surfaces X^3_a corresponding to different lattice points multiplied by the phase factor. For a multi-fermion state the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) assigned to the momenta of fermions would characterize H^3/G so that the information about ("valence") fermion state would be code geometrically. Similar coding would be realized also for the string-like entities $H^2/\Gamma \times R$ at H^3/Γ. What is new and surprising, and also challenges the interpretation, is that the genus of H^2/Γ would code for the momenta of many-fermion states. Does the number of fermion-antifermion pairs correlate with the genus which in turn is proposed to label fermion families? There would be one fermion-antifermion pair per single handle. This would conform with the quantum classical correspondence. The proposed explanation [K18] for the number of observed fermion families would be in terms of hyper-ellipticity meaning that Z_2 acts as a conformal symmetry for all genera smaller than 3. Genus two would correspond to a formation of a bound state of two handles. Could this mean a formation of a graviton-like bound state of 2 fermion pairs and that higher spin states are not possible as bound states of handle. If fermions correspond to cusp singularities surrounded by holes, this picture might make sense: fermion antifermion pair would correspond to two holes connected by a handle.

The $M^8 - H$ duality maps the surface $Y^4 \subset M^8$ to the space-time surface $X^4 \subset H$. The point of $M^4 \subset H$ is obtained as the real part of the inversion of the point of the M^4 projection of the surface Y^4.

There would be a direct analogy to the physics of condensed matter.

A hyperbolic 3-manifold would correspond to a fundamental domain of a tessellation. It would be the equivalent of a unit cell both in position space and momentum space. These unit cells would correspond to each other at the H^3 level by $M^8 - H$ duality. Both would involve discretization. By finite momentum and position resolution UP would be reduced to the interior of the finite tessellation analogous to finite crystal. Quantum-classical correspondence and inversion are consistent with the realization of the UP related to Bohr's orbitology. Momenta in H^3_m would be mapped to equivalents of plane waves, i.e. superpositions of positions of the fundamental region in the tessellation. This picture generalizes to the multi-fermion states. Each fermion momentum defines a cusp and fermionic statistics makes it possible to avoid several cusps at the same points. Fermions for which other quantum numbers, such as spin differ, can however have the same momentum. They should correspond to the same cusp. How can this make sense? Could S^3 be involved somehow. Could they correspond to different holes in S^3 whose sizes and locations correlate with the other quantum numbers somehow? I have considered this problem earlier in the twistor picture where spin corresponds to a geometric degree of freedom in twistor space, which has identification at the level of M^8. The space of causal diamonds (CDs) as a kind of spine of WCW is discussed in [L118]. Lorentz transformations also occur at the level of CDs. The moduli of CD correspond to cm degrees of freedom in H. The finite volume of CD allows states for which Poincare quantum numbers are not exactly opposite for the boundaries of CD. Therefore the values of the total Poincare quantum numbers can be assigned to the CD. Only at the limit of infinitely large CDs does the zero energy property become exact. Therefore the CD wave function carries genuine information. At the p-adic level, translations and Lorentz transformations have the same effect as transformations of a compact group. Translations or Lorentz transformations of order $O(p^n)$ do not increase the p-adic norm of a point.

10.4 Holography

4-D general coordinate invariance forces holography at the level of $H = M^4 \times CP_2$ and one can regard space-time surfaces as analogues of Bohr orbits determined almost uniquely by 3-D surfaces. Quantum TGD is therefore very much like wave mechanics with point-like particles replaced with 3-surfaces in turn replaced with 4-D Bohr orbits. In fact, a wave-mechanical toy model for TGD would replace electron wave functions in atoms with wave functions in the space of its Bohr orbits.

M^8 is analogous to the momentum space in wave mechanics and the 4-surfaces in M^8 obey number theoretical holography based on associativity.

10.4.1 What does one mean with holography?

Consider now a more precise definition of holography.

1. The standard form of holography as $3 \rightarrow 4$ assigning to a 3-surface at the boundary of causal diamond (CD) an almost unique 4-D surface is the weakest form of holography. The non-uniqueness of the holography forces zero energy ontology (ZEO) in which analogues of Bohr orbits are basic geometrical objects.

2. $2 \rightarrow 4$ holography is the strongest form of holography. I have called it strong holography (SH). The 2-D partonic surfaces and possibly also the string world sheets would encode the data about the 4-surface and also the data about quantum holography. The strong form of holography could be realized as super symplectic and super-Kac Moody invariance and super-conformal invariance being minimally broken. Only the scaling generator L_0 would not annihilate the states. This condition is however too strong.

3. For the weaker form of SH super-symplectic and conformal symmetries are broken such that the algebras A_n (there are several of them), whose conformal weights are n-multiples of the conformal weights of the entire algebra A, and $[A_n, A]$ annihilate the physical states [L109, L67]. This requires half-algebra with non-negative conformal weights.

   The breaking hierarchy labelled by the values of n makes sense also for the ordinary conformal invariance but to my best albeit non-professional knowledge is not considered as a physical option. Hierarchies corresponding to the inclusion hierarchies of rational extensions and HFFs are obtained.

Both holographies set very strong conditions for the 3-surfaces appearing as holographic data.

Role of polynomials

At the level of M^8 physics is algebraic as it is also for the momentum space in the case of free field theory and reduces to algebraic conditions like mass shell condition and orthogonality of polarization vector and momentum. Polynomials P having integer coefficients determined the physics.

1. P as such does not fix the 4-surface nor even the 3-surface defining the data for number theoretic holography.

2. The polynomial P must have integer coefficients to guarantee number theoretical universality in the sense that they make sense also in p-adic number fields. If the coefficients are smaller than the degree of P, also finite fields become natural mathematical structures in TGD so that all number fields are involved. The roots of P give rise to the mass shells in M_c^8 with mass squared values defined by the roots of P. The roots define an extension of rationals.

3. Polynomials are also characterized by ramified primes as the divisors of the discriminant of the polynomial determined by the product of the differences of its roots [L119]. They are not a property of the algebraic extensions. They depend on P and the exponent of the classical action is proposed to correspond to the discriminant D. Ramified primes are identified as p-adic primes playing a central role in p-adic mass calculations [K46].

The role of fermions

Quantum classical correspondence requires that the 3-surfaces Y^3 at the mass shells are determined by the quantum numbers of fermions associated with the quantum states. What assumptions could provide this additional data and how could this data be coded to the geometry of Y^3?

The data in question correspond to fermion momenta, spins and electroweak quantum numbers. Color does not appear as spin-like quantum numbers but corresponds to color partial waves in CP_2. Consider next how momenta are coded to the properties of 3-surfaces.

1. I have proposed that the 3-surfaces Y^3 in H^3_m could correspond to the fundamental domains of tessellations of H^3. The unit cell of ordinary crystal in E^3 serves as an analog for the fundamental domain of a tessellation in H^3. The disjoint components of Y^3 would naturally

correspond to surfaces Y_i^3 at H_m^3 and would correspond to fundamental domains of analogs of finite crystals formed by gluing them together.

The points of $E^4 \subset M^4 \times E^4$ correspond to 3-sphere with radius determined by mass m and for given Y_i^3 the values of E^4 coordinates would be constant. A stronger condition would be that the values are the same for all Y_i^3. At the cusp points the point would be replaced by S^3, which could touch two disjoint sheets with different values of S^3 coordinates. Since the metric becomes singular at cusp, a natural proposal is that a small hole is drilled around the point and to S^3 and they are glued along their boundaries. The scale of the hole would be determined by the mass.

2. TGD predicts as basic objects also string-like objects $X^2 \times R \subset H$ and their deformations to magnetic flux tubes. By $M^8 - H$ duality they are expected to be present also in M^8 in particular at the hyperboloids H_m^3.

   There are two kinds of string-like objects depending on whether their CP_2 projection is homologically trivial or not. In the latter case the string carries monopole flux.

Quantum classical correspondence suggests that the momenta of fermions as points of $H_m^3 \subset M^8$ are coded into the geometry of Y^3 as singularities. $M^8 - H$ duality based on inversion in turn maps the momenta to singular points of H_a^3.

1. Singularities would be naturally cusps as analogs of tips of algebraic surfaces allowing all normal spaces of Y^4 at the singularity: M^8 duality would assign a 3-D subset of CP_2 to the tip.

2. Is it possible to have singularities, where the throats of the opposite wormhole throats touch? Or could the wormhole throats of the incoming partons fuse to single throat? This could occur in the topological counterpart of 3-vertex describing pair annihilation to a single particle. The singularities emerging in this way could relate to the description of the creation of fermion-antifermion pairs and would also define defects essential for exotic differential structures occurring only in dimension $D = 4$ [L107].

Can one code also spin to geometry or should it be regarded as a fermionic quantum number?

1. At the level of H one would have a product of twistor spaces $T(M^4)$ and $T(CP_2)$: these twistor spaces are unique in the sense that they have Kähler structure. This makes H a unique choice for the embedding space.

   Twistorialization replaces space-time surface with 6-D surface $X^6 \subset T(M^4) \times T(CP_2)$ having S^2-bundle structure as possessed also by $T(M^4)$ and $T(CP_2)$. Spinor description of spin and electroweak isospin is replaced by a wave function in twistor spheres S^2.

   The embedding corresponds to dimensional reduction producing X^6 as S^2 bundle. The twistor spheres associated with M^4 and CP_2 must be identified by the embedding of $X^6 \subset T(M^4) \times T(CP_2)$.

   The identification of the twistor spheres forces spin doublets to correlate almost completely with electroweak spin doublets apart from the directions of the two spins. This picture allows only spin- and electroweak spin doublest. Does this force a complete correlation between the values of spin and electroweak to be identical or do the details of the identification for the the embedding of $X^6 \subset T(M^4) \times T(CP_2)$ allow to regard spin and electroweak spin as independent?

   The identification of the two twistor spheres is not unique. The spin rotations and possibly also electroweak spin rotations (, which are not isometries) changes the identification of the two twistor spheres. This would make spin and electroweak spin independent quantum numbers.

   One can argue that only the relative rotation of the two spheres matters. Could this mean that electroweak spin axes can be thought of being completely fixed. Electroweak quantization axes are indeed completely fixed physically.

2. Something similar could happen at the level of M^8. Now one must consider twistor spaces of M^4 and E^4 and similar embedding of a 6-D surface $X^6 \subset M^8$ as twistor space with S^2 bundle structure.

   In M^8 one would have an algebraic description of spin and electroweak spin instead of a wave function at S^2. A direction of S^2 would define a quantization axis and the diametrically opposite points of S^2 associated with it would provide a geometric correlate for the spin and electroweak spin values of fermion. The relative rotations of the twistor spheres of M^4 and E^4 associated with their identification are also now possible so that the two quantum numbers can be regarded as independent but with the electroweak quantization axes fixed.

In the twistorial picture one would have $5 \rightarrow 6$ weak holography or even $4 \rightarrow 6$ strong almost unique holography.

10.4.2 What kind of 3-geometries are expected in the TGD framework?

To get a wider perspective, it is good to have an overall view of the Geometrization conjecture of Thurston `https://rb.gy/9x3pm` proven by Perelman by studying Ricci flows. Geometrization theorem implies Poincare conjecture and so called spherical space conjecture.

The inspiration comes from the classification of 2-D manifolds expressed by uniformization theorem (`https://rb.gy/ts8va`). There are only 3 closed simply connected Riemann manifolds: sphere, disk, and hyperbolic plane. These are constant curvature spaces with corresponding Lie groups of isometries. One can obtain connected closed 2-manifolds with a nontrivial fundamental group by identifying the points related by a discrete subgroup of isometries.

In the case of the hyperbolic plane the isometry group is infinite and gives rise to a non-trivial fundamental group. For the hyperbolic plane, one obtains 2-manifolds with nonvanishing genus allowing a negative constant curvature. Constant curvature can be normalized to be -1, 1 or 0 in various cases. For non-vanishing curvature, the area serves as a topological invariant. For torus this is not the case.

The following provides the summary of my non-professional understanding of the 3-D case. The TGD inspired comments rely on what I know from the universal preferred extremals of practically any variational principle which is general coordinate invariant and can be constructed from the induced geometric quantities. They are always minimal surfaces outside 3- or lower-dimensional singularities at which the field equations depend on the action. The known extremals are discussed in [K5, K10, L101].

1. Thurston's conjecture `https://rb.gy/9x3pm` states that every oriented and closed irreducible (prime) 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with a finite volume which becomes a topological invariant in geometric topology. For instance, knots give rise to a 3-manifold in this way.

   An important difference is that the closed 3-manifold decomposes to a union of different types of 3-manifolds rather than only of single type as in the 2-D case.

2. The notion of model geometry is essential. There exists a diffeomorphism to X/Γ for some model geometry such that Γ is a discrete subgroup of a Lie group of isometries acting in G. There are 8 types of model geometries.

3. Irreducible 3-manifolds appear as building bricks of 3-manifolds using connected sum. There are 8 types of model geometries for closed prime 3-manifolds, which by definition do not allow a connected sum decomposition. These geometries are E^3, S^3, H^3, $S^2 \times R$, $H^2 \times R$, $SL(2,"R")$, Nil, and $Solve$.

4. The model geometries allow a constant curvature metric. The finite volume of the manifold becomes a topological characteristic if one has constant curvature equal to ± 1.

5. All types except one, $S^2 \times R$, which corresponds to a string-like objects in TGD, allow a 3-D Lie group as subgroup of isometries (Bianchi group).

6. All model geometries except hyperbolic manifold (`https://rb.gy/snpft`) and Solv manifold are Seifert fiber spaces (`https://rb.gy/uxszk`), which are fibered by S^1 fiber. Hyperbolic manifolds are atoroidal but have an infinite fundamental group since Γ must be infinite from the finite volume property. Atoroidality means that there is no embedding of torus which would not be parallel to the boundary of the hyperbolic manifold.

   The finite volume property of H^3/Γ also requires that Γ is a lattice: this implies a deep analogy with condensed matter physics. The group elements in the TGD frame-work be $SL(2,C)$ matrices with elements which are algebraic integers in an extension of rationals defined by the polynomial P defining 4-surface in M^8. Note that also momentum components are predicted to be algebraic integers using a unit defined by the scale of the causal diamond (CD).

   TGD leads to the proposal [K77] that the H^3 lattices could appear in cosmological scales and explain "God's fingers" [E2] discovered by Halton Arp. They are astrophysical objects appearong along a line and having quantized redshifts.

7. One can form the spaces of the orbits for a discrete subgroup $\Gamma \subset G$ to obtain 3-manifolds with non-trivial fundamental group or orbifolds as in the case of S^3 and $S^2 \times R$. For hyperbolic 3-manifolds, the fundamental group is infinite and generated by elements of parabolic subgroups of G (`https://rb.gy/b5t55`). Cusp point and cusp neighborhood (`https://rb.gy/fd4dz`) are related to the infinite part of the fundamental group. Since parabolic subgroups $P \subset \Gamma$ are infinite groups, the fundamental group of the hyperbolic manifold is infinite.

8. One can decompose an irreducible 3-manifold to pieces, which are either Seifert manifolds or atoroidal. All 8 model geometries except hyperbolic geometries and so called Solv manifolds are Seifert manifolds.

   Solv manifolds are fiber spaces over a circle with 2-D plane with Minkowski signature as a fiber. In TGD solv manifolds could correspond to the so-called massless extremals [K5, K10, K60] serving representing classical radiation fields having only Fourier components with wave vectors in a single direction: laser beam is a good analog for them. They are not embeddable to H^3.

   In Ricci flows the hyperbolic pieces expand whereas other pieces contract so that asymptotically the manifold becomes hyperbolic. In fact, the collapse occurs in some caes in a finite time as found already by Richard S. Hamilton. The flow "kills" the positive curvature geometries S^3 and $S^2 \times R$ in the connected sum. What is left at large times is "thick-thin" decomposition. The "thick" piece is a hyperbolic geometry whereas the "thin" piece is a so-called graph manifold.

Hyperbolic manifolds and Seifert fiber spaces

Hyperbolic space and Seifert fiber space (`https://rb.gy/uxszk`) are in a central role in the TGD framework and therefore deserve short discussion. The following just gives the basic definitions and brief TGD inspired comments.

1. Hyperbolic manifolds

A hyperbolic n-manifold (`https://rb.gy/2esup`) is a complete Riemannian n-manifold of constant sectional curvature. Every complete, connected, simply-connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space H^n. As a result, the universal cover of any closed manifold M of constant negative curvature -1.

Every hyperbolic manifold (`https://rb.gy/snpft`) can be written as H^n/Γ, where Γ is a torsion-free discrete group of isometries on H^n. That is, Γ is a discrete subgroup of $SO^+_{1,n}$. The manifold has a finite volume if and only if Γ is a lattice.

Its "thick–thin" decomposition has a "thin" part consisting of tubular neighborhoods of closed geodesics and ends which are the product of a Euclidean $(n-1)$-manifold and the closed half-ray. The manifold is of finite volume if and only if its "thick" part is compact.

In the TGD framework, the lattice structure is natural and would mean that the elements of the matrices of Γ *are algebraic extensions in the extension of rational defined by the polynomial* P *determining* Y^4. *The tubular neighborhoods of the "thin" part would correspond to string-like objects (tubular neighborhoods) as geodesics whereas the ends would correspond to cusp singularities inducing blow-up as 3-surface* $S^3 \subset E^3$.
At the level of H *the tubular neighborhoods correspond to a string-like object and their ends to* CP_2 *type extremals serving as building bricks of elementary particles. Hadronic strings would represent examples of string-like objects and all elementary particles would involve them as monopole flux tubes connecting wormhole contacts.*

For $n > 2$ the hyperbolic structure on a finite volume hyperbolic n-manifold is unique by Mostow rigidity theorem and so geometric invariants are in fact topological invariants. One of these geometric invariants used as a topological invariant is the hyperbolic volume of a knot or link complement, which can allow us to distinguish two knots from each other by studying the geometry of their respective manifolds.

The identification of geometric invariants as topological invariants conforms with the TGD vision about "holy trinity" geometry-number theory-topology. Number theory would leak in through the identification of Γ *as a lattice determined by the polynomial* P.

2. Seifert fiber spaces

A Seifert manifold `https://rb.gy/uxszk` is a closed 3-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

A standard fibered torus corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$ (with the natural fibering by circles). If $a = 1$ the middle fiber is called ordinary, while if $a > 1$ the middle fiber is called exceptional. A compact Seifert fiber space has only a finite number of exceptional fibers.

The set of fibers forms a 2-dimensional orbifold, denoted by B and called the base — also called the orbit surface — of the fibration. It has an underlying 2-dimensional surface B_0, but may have some special orbifold points corresponding to the exceptional fibers.

The definition of Seifert fibration can be generalized in several ways. The Seifert manifold is often allowed to have a boundary (also fibered by circles, so it is a union of tori). When studying non-orientable manifolds, it is sometimes useful to allow fibers to have neighborhoods that look like the surface bundle of a reflection (rather than a rotation) of a disk, so that some fibers have neighborhoods looking like fibered Klein bottles, in which case there may be one-parameter families of exceptional curves. In both of these cases, the base B of the fibration usually has a non-empty boundary. 6 of the 8 basic geometries of Thurston are Seifert fiber spaces.

In the TGD framework, the Seifert fiber spaces would correspond to string-like objects, which appear as several variants.

The eight simply connected 3-geometries appearing in the Thurston's conjecture from the TGD point of view

This section contains as almost verbatim the description of the 8 Thurston geometries provided by the Wikipedia article `https://rb.gy/9x3pm`. There is a good reason for this: I am not a professional and do not understand the technical details. There is a good reason for not giving a mere Wikipedia link: I have added comments relating to the TGD based identification of these model geometries as 3-surfaces.

It turns out that the geometries could correspond to fundamental regions of H^3, to energy $E = constant$ (M^4 time $t = constant$ in H) surfaces $D^3 \subset M^4_+ \subset M^8$, to string-like objects $X^2 \times R$ allowing Seifert fiber space structure, or to masless extremals with structure $M^2 \times E^2$ with M^2 and E^2 corresponding to the orthogonal planes defined by light-like momentum and polarization vector.

First some definitions:

1. A model geometry is a simply connected smooth manifold X together with a transitive action of a Lie group G on X having compact stabilizers (the isotropy group of a point is compact).

2. A model geometry is called maximal if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.This condition can be aso included in the definition of a model geometry.

3. A geometric structure on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X; this is a special case of a complete (G, X)-structure. If a given manifold admits a geometric structure, then it admits a structure, whose model is maximal.

   One can say that the spaces X provide the raw material from which one obtains various 3-geometries by identifications using a discrete subgroup of G.

A 3-dimensional model geometry X is relevant for the geometrization conjecture if it is maximal and if there is at least one compact manifold with a geometric structure modelled on X. Thurston classified the 8 model geometries satisfying these conditions; they are listed below and are sometimes called Thurston geometries. (There are also uncountably many model geometries without compact quotients.)

There is a connection with the Bianchi groups, which are the 3-dimensional Lie groups. Most Thurston geometries can be realized as a left invariant metric on a Bianchi group. However, $S^2 \times R$ does not allow Bianchi geometry; Euclidean space corresponds to two different Bianchi groups; and there are an uncountable number of solvable non-unimodular Bianchi groups, most of which give model geometries having no compact representatives.

1. Spherical geometry S^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $O(4, R)$, with 2 components. The corresponding manifolds are exactly the closed 3-manifolds with a finite fundamental group. Examples include the 3-sphere, the Poincaré homology sphere, and lens spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type IX. Manifolds with this geometry are all compact, orientable, and have the structure of a Seifert fiber space (often in several ways). The complete list of such manifolds is given in the article on spherical 3-manifolds. Under Ricci flow, manifolds with this geometry collapse to a point in finite time.

> *In the TGD framework S^3 geometry could be associated with cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) of hyperbolic 3-manifold and represent the blow-up of a the cusp to S^3 which can be regarded as sphere in $E^4 \subset M^8 = M^4 \times E^4$. This is mapped to a 3-sphere of CP_2 in $M^8 - H$-duality.*

2. Euclidean geometry E^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $R^3 \times O(3, R)$, with 2 components. Examples are the 3-torus, and more generally the mapping torus of a finite-order automorphism of the 2-torus; see torus bundle. There are exactly 10 finite closed 3-manifolds with this geometry, 6 orientable and 4 non-orientable. This geometry can be modeled as a left invariant metric on the Bianchi groups of type I or VII_0.

Finite volume manifolds with this geometry are all compact, and have the structure of a Seifert fiber space (sometimes in two ways). The complete list of such manifolds is given in the article on Seifert fiber spaces. Under Ricci flow, manifolds with Euclidean geometry remain invariant.

> *In M^8 one has two kinds of roots of polynomials. For the first option they correspond mass square values defining mass shells H^3 . For the second option applying to the light-cone boundary as mass shell, energy E replaces mass and roots correspond to discrete energies. E = constant surface corresponds to E^3 as 3-balls inside the light-cone.*

3. Hyperbolic geometry H^3

The point stabilizer is $O(3,R)$, and the group G is the 6-dimensional Lie group $O^+(1,3,R)$, with 2 components. There are enormous numbers of examples of these, and their classification is not completely understood. The example with the smallest volume is the Weeks manifold. Other examples are given by the Seifert–Weber space, or "sufficiently complicated" Dehn surgeries on links, or most Haken manifolds.

The geometrization conjecture implies that a closed 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, and has an infinite fundamental group. This geometry can be modeled as a left invariant metric on the Bianchi group of type V or $\mathrm{VII}_{h\neq 0}$. Under Ricci flow, manifolds with hyperbolic geometry expand.

In TGD H^3 has an interpretation as a mass shell in $M^4 \subset M^8$ determined by the roots of the polynomial P or as a light-cone proper time constant hyperboloid in M^4.

This geometry does not allow Seifert fiber space structure unlike most other geometries.

4. The geometry of $S^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O(3,R) \times R \times Z/2Z$, with 4 components. The four finite volume manifolds with this geometry are: $S^2 \times S^1$, the mapping torus of the antipode map of S^2, the connected sum of two copies of 3-dimensional projective space, and the product of S^1 with two-dimensional projective space.

The first two are mapping tori of the identity map and antipode map of the 2-sphere, and are the only examples of 3-manifolds that are prime but not irreducible. The third is the only example of a non-trivial connected sum with a geometric structure. This is the only model geometry that cannot be realized as a left invariant metric on a 3-dimensional Lie group.

Finite volume manifolds with this geometry are all compact and have the structure of a Seifert fiber space (often in several ways). Under normalized Ricci flow manifolds with this geometry converge to a 1-dimensional manifold.

In the TGD framework, these surfaces could correspond to the simplest string-like objects for which S^2 corresponds to a geodesic sphere (homologically trivial or non-trivial) with a finite length connecting fundamental regions of H^3 or finite tessellations formed by them. S^2, which would correspond to a 2-D surface in CP_2 would be the base and string the fiber. One might argue that S^2 is more natural as fiber.

5. The geometry of $H^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O^+(1,2,R) \times R \times Z/2Z$, with 4 components. Examples include the product of a hyperbolic surface with a circle, or more generally the mapping torus of an isometry of a hyperbolic surface.

Finite volume manifolds with this geometry have the structure of a Seifert fiber space if they are orientable. (If they are not orientable the natural fibration by circles is not necessarily a Seifert fibration: the problem is that some fibers may "reverse orientation"; in other words their neighborhoods look like fibered solid Klein bottles rather than solid tori.) The classification of such (oriented) manifolds is given in the article on Seifert fiber spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type III. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

In the TGD context, these geometries would correspond to closed string-like objects for which the CP_2 projection is a 2-surface with genus $g > 0$. Seifert fiber space property corresponds to closed strings.

6. The geometry of the universal cover of $SL(2,"R")$

The universal cover of $SL(2,R)$ is denoted $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. It fibers over H^2, and the space is sometimes called "Twisted $H^2 \times R$". The group G has 2 components. Its identity component has the structure $(\mathbf{R} \times \widetilde{\mathrm{SL}}_2(\mathbf{R}))/\mathbf{Z}$. The point stabilizer is $O(2,R)$.

Examples of these manifolds include: the manifold of unit vectors of the tangent bundle of a hyperbolic surface, and more generally the Brieskorn homology spheres (excepting the 3-sphere and the Poincare dodecahedral space). This geometry can be modeled as a left invariant metric on the Bianchi group of type VIII or III. Finite volume manifolds with this geometry are orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

Also now the interpretation as a closed string-like entity is possible in TGD.

7. Nil geometry

This fibers over E^2, and so is sometimes known as "Twisted $E2\times\ R$". It is the geometry of the Heisenberg group. The point stabilizer is $O(2,R)$. The group G has 2 components, and is a semidirect product of the 3-dimensional Heisenberg group by the group $O(2,R)$ of isometries of a circle. Compact manifolds with this geometry include the mapping torus of a Dehn twist of a 2-torus, or the quotient of the Heisenberg group by the "integral Heisenberg group". This geometry can be modeled as a left invariant metric on the Bianchi group of type II.

Finite volume manifolds with this geometry are compact and orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow, compact manifolds with this geometry converge to R^2 with the flat metric. *In TGD also this geometry might be assigned with a closed string-like object as all Seifert fiber spaces.*

8. Sol geometry

This geometry (also called Solv geometry) fibers over the line with fiber the plane, and is the geometry of the identity component of the group G. The point stabilizer is the dihedral group of order 8. The group G has 8 components, and is the group of maps from 2-dimensional Minkowski space to itself that are either isometries or multiply the metric by - 1. The identity component has a normal subgroup R^2 with quotient R, where R acts on R^2 with 2 (real) eigenspaces, with distinct real eigenvalues of product 1.

This is the Bianchi group of type VI_0 and the geometry can be modeled as a left invariant metric on this group. All finite volume manifolds with solv geometry are compact. The compact manifolds with solv geometry are either the mapping torus of an Anosov map of the 2-torus (such a map is an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct, such as

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}) \ ,$$

or quotients of these by groups of order at most 8. The eigenvalues of the automorphism of the torus generate an order of a real quadratic field, and the solv manifolds can be classified in terms of the units and ideal classes of this order. Under normalized Ricci flow compact manifolds with this geometry converge (rather slowly) to R^1.

Unlike in the case of Seifert fiber spaces, a plane or disk appears as a fiber. Could one consider the possibility whether boundary conditions guaranteeing conservation laws could allow string-like objects for which the cross section is disk rather than a closed 2-surface? The appearance of isometries of 2-D Minkowski space suggests that the disk X^2 must have Minkowski signature so that the embedding to H^3 would not be possible.
Could one assign this structure to massless extremals [K5, K60], which in the TGD framework define the representations for classical radiation fields, which involve the decomposition $M^4=M^2\times D^2$. The circle $S^1\subset D^2$ defining its boundary would define the base space. Boundaries would be light-like and might allow to solve the boundary conditions. It is not clear how to obtain the counterparts of massless extremals at the M^8 level.

10.4.3 $3\rightarrow 4$ form of holography

One can consider two forms of holography. The first, weak, form corresponds to the ordinary $3\rightarrow$ holography in which 3-D boundaries provide the data defining the 4-surface. The second, strong form, corresponds to $2\rightarrow 4$ holography in which conformal boundaries provide the data defining the 4-surface. In this section the $3\rightarrow 4$ form of the holography is considered.

Fundamental domains of hyperbolic tessellations as data for $3 \rightarrow 4$ holography

Good candidates for the surfaces Y_i^3 are fundamental domains assignable to hyperbolic 3-manifolds H^3/Γ_i represented as surfaces in $H^3 \subset M^4 \subset M^8$ (or its complexification). In the case of string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. The treatment of this case is a rather straightforward modification of the first case so that the discussion is restricted to H^3/Γ_i.

The surfaces Y_i^3 would correspond topologically to many-particle states of free particles. Holography would induce topological interactions in the interior of Y^4 and $X^4(Y^4)$. The momenta (positions) of the fermions analogous to valence quarks correspond to the cusp singularities.

For fundamental fermions momenta would have components, which are algebraic integers. Galois confinement states that the momenta for many-fermion states are ordinary integers. This poses a condition for H^3/Γ and it would be interesting to understand what the condition means.

The degrees of freedom orthogonal to H^3 correspond to a complexified sphere S^3 of E^4, whose radius squared corresponds to the square of the complex mass squared.

1. Hyperbolicity is a generic property of 3-manifolds and probably preserved in small enough deformations. In other words, deformations of hyperbolic 3-manifold X_i^3 probably allow a hyperbolic metric although the induced metric for the deformation is not in general hyperbolic.

   Deformation of the hyperbolic manifold (`https://rb.gy/snpft`) could take place in its evolution defining Y_i^4 and $X^4(Y_i^4)$ and could lead to, for example, to singularities such as the touching of different surfaces and interaction vertices at which partonic 2-surfaces meet.

2. There is an interesting connection to the geometrization conjecture of Thurston (`https://rb.gy/9x3pm`), especially with the work of the Russian mathematician Grigori Perelman, who studied 3-D Ricci flows (`https://rb.gy/n6qlv`) for metrics and proved that, apart from scaling, they lead to hyperbolic geometries.

   Interestingly, hyperbolic manifolds decompose into "thin" and "thick" pieces and the "thin" piece corresponds to cusp neighborhoods (`https://rb.gy/fd4dz`). This decomposition brings in mind the notions of valence partons and sea partons with sea partons, in particular gluons assignable to the interior of Y_i^3 and giving the dominant contribution to the hadron mass.

Consider now what one can assume about Y_i^3.

1. The simplest assumption is that the S^3 coordinates are constant for Y_i^3 identified as the fundamental domain of a tessellation defined by H^3/Γ. It would represent a piece of H^3.

   Could one consider the allowance of S^3 deformations H_d^3 of H^3 in the direction of S^3, which are invariant under G so that the space H_d^3/G would exist. They would define what mathematicians would call a model of hyperbolic geometry.

2. Can one allow for a given Y_i^3 a multiple covering of H^3 by copies of Y_i^3 with different constant values of S^3 coordinates? Could this state correspond topologically to a many-sheeted covering naturally associated with the polynomial P?

   An interesting possibility is that Galois symmetry implies the existence of several copies of Y_i^3 with different S^3 coordinates as the orbit of the Galois group or its sub-group. Z_2 would be the simplest Galois group and give two sheets.

3-D data for $3 \rightarrow 4$ holography with 3-surfaces as hyperbolic 3-manifolds

It is good to start with questions.

1. Could the 3-surfaces X^3 associated with the mass shells $H_m^3 \subset M^8$ appearing as holographic data be fundamental domains (analogs of unit cell for crystals) of the tessellation H^3/Γ? Could a fermionic many-particle state for an algebraic extension determined by a given polynomial P assign a singularity to the fundamental domain and fix it?

   The TGD view of hadron physics provides some clues. Gluon sea consists of gluons identifiable as fermion-antifermion pairs and fermions and antifermions. Here is the data for the given hyperbolic 3-manifold of singularities. The valence fermions could reside at throats and virtual sea gluons could be associated with strings $Y^2 \times R$ inside flux tubes and would give to the classical string tension?

Hyperbolic 3-manifolds also have string-like singularities connecting the cusp singularities. In the physical picture of TGD, these would correspond to strings connecting wormhole throats of different wormhole contacts which in turn would correspond to blow-ups of cusps.

2. Is the situation the same in M^8 and H? Could 4-surfaces assignable to the X_i^3 be minimal surfaces in both H and M^8 having a generalization of holomorphic structure to dimension 4? It would be possible to map X_i^3 to each other by inversion. Note that M^8-H correspondence would map the $M^4 \subset M^8$ projections of the points of Y^4 by inversion to H also in the interior of 4-surface.

   Could this realize the dual conformal invariance proposed by the twistors, which would therefore be behind the analogy of Langlands duality and $M^8 - H$ duality?

10.4.4 Strong form of the hyperbolic holography

Holography roughly means an assignment of, not necessarily a unique 4-surface, to a set of 3-surfaces at the mass shells defined by roots of the polynomial P. The 4-surface is analogous to Bohr orbit.

A stronger form of the holography would be approximate $2 \rightarrow 4$ holography suggesting that the 3-surfaces allow $2 \rightarrow 3$ holography, which need not be completely deterministic. To understand what is involved one must have an idea about what kind of 3-surfaces could be involved.

1. Irreducible closed 3-surfaces Y_i^3 at H_m^3 consist of regions of 8 different types. Could these regions correspond to model geometries or at least have the symmetries of model geometries?

   This conjecture is natural if the 3-surfaces $Y_i^3 \subset H_m^3 \subset M^4 \subset M^8$ belong to (possibly complex) mass shells of M^4. In this case, the composites of fundamental regions of hyperbolic manifolds (`https://rb.gy/snpft`) as analogs of finite crystals would be natural.

   The interiors of these regions would correspond to the "thick" part of the 3-manifold whereas the cusp singularities and string singularities as boundaries of string world sheets would correspond to the "thin" part. The blow-ups of cusp singularities would give rise to 3-D regions of CP_2.

2. Also monopole flux tubes connecting hyperbolic regions to form a network should be involved. Here the natural model geometries would be of type $S^2 \times R$ or $H^2 \times R$ with the ends of R at the two hyperbolic regions. By replacing H^2 with H^2/Γ, one would obtain higher flux tubes with a cross section having a higher genus.

The natural idea is that hyperbolic holography gives rise to $2 \rightarrow 3$ holography. In the case of H^3/Γ, the holography would assign H^3/Γ its fundamental region Y^3 to H^2/Γ.

In the case of H^2/Γ, applying for string-like objects, holography would assign $Y^2 \times R$ to a union of circles H^1/Γ defining its boundary. The rule would be simple: Y^n/Γ is a union of fundamental regions Y_i^n having H^{n-1}/Γ as boundary.

Hyperbolic holography from H^2/G to the fundamental domain of H^3/Γ

The representation of M^4 momenta in terms of bispinors is possible only for massless particles. This raises the question whether one must assume a strong form of holography in which 2-D surfaces at the boundaries of H_m^3 dictate the 4-D surface almost completely. The hyperbolic 2-manifold H^2/G should define the boundary for Y_i^3 identifiable as a fundamental domain Y^3 of a hyperbolic 3-manifold H^3/Γ.

1. This would conform with the proposed realization of super-symplectic invariance and Kac-Moody type symmetries for light-like partonic orbits meaning that the interior degrees of freedom associated with the 3-surfaces X_i^3 and light-like orbits of partonic 2-surfaces are eliminated with a suitable gauge choice formulated in terms of a generalization the Virasoro and Kac-Moody conditions [L108, L109].

2. Physically this would mean that the fermion momenta at cusp point are light-like. This would conform with the view that fermions move along light-like curves inside the light-like partonic orbit.

3. If hyperbolic holography makes sense, the above formulation for H^2 would generalize to the case of H^3. Cusp neighborhood U/P as a projection $U \rightarrow H^2/G$ has a counterpart for H^3/Γ and the fundamental domain for H^2/G extends to a fundamental domain for H^3/Γ. For H^2/G as boundary it would correspond to the condition $p_3 > 0$ for the momentum component in the chosen direction.

4. The cusp singularity is analogous to a cusp of an algebraic surface. This suggests that near the cusp point of H^3/Γ the metric behaves like the induced metric of 3-D cusp in 4-D space. Near the cusp one has $t = k\sqrt{\rho}$ where t and ρ are vertical coordinate and transversal coordinates of the cusp in 4-D space. The radial component of the induced metric orthogonal to tip direction should behave like $g_{\rho\rho} = 1 + k^2/4\rho$ and the radial distance from the tip would diverge logarithmically. One cold say that this point is missing so that the hyperbolic manifold is compact but not closed since it has boundaries. The singularity of the metric is a good motivation for cutting off a small ball around the singularity in M^4 and a small ball from S^3 and for gluing the two together along boundaries. At the level of H this would correspond to wormhole throat.

10.4.5 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \rightarrow f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex

Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. M^8-H duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. Is this deformation of H^3 in imaginary time direction equivalent with a region of H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \ . \qquad (10.4.1)$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \rightarrow Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \rightarrow 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for M^8-H duality?

The dream is an explicit formula for the M^8-H duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \rightarrow SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

1. This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

2. A physically motivated proposal is that the real parts of $SU(3)_c$ gauge potential and color gauge field can be lifted to H and the lifts are equal to the classical gauge potentials and color gauge field proposed in H. Color gauge potentials in H are proportional to the isometry generators of the color gauge field and the components of the color gauge field are proportional to the products of color Hamiltonians with the induced Kähler form.

3. The color gauge field $Re(G)$ obeys the formula $Re(G) = dRe(A) + [Re(A), Re(A)] = [Re(A), Re(A)]$ and does not vanish since the contribution of $[Im(A), Im(A)]$ cancelling the real part is absent. The lift of $A_R = g^{-1}dg$ to H is determined by g using M^4 coordinates for Y^4 . The

M^4 coordinates $p^k(M^8)$ having interpretation as momenta are mapped to the coordinates m^k of H by the inversion

$$I : m^k = \hbar_{eff} Re(\frac{p^k}{p^2}) \ , \ p^2 \equiv p^k p_k \ ,$$

where p^k is complex momentum. $Re(A)_H$ is obtained by the action of the Jacobian

$$dI^k_l = \frac{\partial p^k}{\partial m^l} \ ,$$

as

$$A_H = dI \cdot Re(A_{M^8}) \ .$$

dI^k_l can be calculated as the inverse of the Jacobian $\partial m^k/\partial Re(p)^l$. Note that $Im(p^k)$ is expressible in terms of $Re(p^k)$.

For $Im(p^k) = 0$ the Jacobian for I reduces to that for $m^k = \hbar_{eff}\frac{p^k}{p^2}$ and one has

$$\frac{\partial m^k}{\partial p^l} = \frac{\hbar_{eff}}{p^2}(\delta^k_l - \frac{p^k p_l}{p^2}) \ .$$

This becomes singular for $m^2 = 0$. The nonvanishing of $Im(p^k)$ however saves from the singularity.

4. The $M^8 - H$ duality obeys a different formula at the light-cone boundaries associated with the causal diamond: now one has $p^0 = \hbar_{eff}/m^0$. This formula should be applied for $m^2 = 0$ if this case is encountered. Note that number theoretic evolution for masses and classical color gauge fields is directly coded by the mass squared values and holography.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M^8_c$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically. In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be

non-vanishing at the mass shells but outside the mass shells one would have nonvanishing G_2 gauge fields.

There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X_i^3.

10.5 Singularities, quantum classical correspondence, and hyperbolic holography

The point-like fermions and their 1-D trajectories appear as singularities of the minimal surfaces [L101]. Strings that connect fermions located at their ends, and string world sheets in the interior of X^4 appear also as singularities. Also partonic 2-surfaces separating Minkowskian and Euclidian regions should correspond to singularities of X_i^3 and their light-like radii.

There would therefore be singularities in dimensions $D = 0, 1, 2, 3$. These singularities should relate to the fundamental domains $Y_i^3 \subset H_m^3 \subset M^8$ and holography would suggest that they correspond to the singularities of 3-D hyperbolic manifolds (`https://rb.gy/snpft`).

10.5.1 Cusp singularities and fermionic point singularities

The singularities should be associated with hyperbolic manifolds Y_i^3 identified as fundamental domains of coset spaces H^3/Γ , that is, as effective geometries H^3/Γ defined by the boundary conditions for various "fields". In the same way as, for example, a torus geometry appears in condensed matter physics for a unit cell of lattice.

Cusp singularity (`https://rb.gy/fd4dz`) is a natural candidate for a point-like singularity and geometrically corresponds to a cusp. For abstract Riemann geometry, the cusp property would correspond to a singularity of the metric for a cusp (tip) and mean that the radial component of the metric diverges at the tip.

Consider first the basic concepts and ideas in the case of 2-D hyperbolic space H^2 and corresponding hyperbolic manifold H^2/G case.

1. Riemann surface can be regarded as a coset space H^2/G, which is represented as a fundamental region for a tessellation of H^2.

2. Cusp singularities of H^2/G correspond to parabolic subgroups P (`https://rb.gy/b5t55`) generated by a parabolic element for $G \subset SL(2, C)$. Parabolic subgroup P is isomorphic to a discrete group of translations along, say, the real axis as a boundary of the upper half-plane and is noncompact. It is represented as Möbius transformations induced by the matrices $(1, n : 0, n)$. P can be regarded as a subgroup generated by a Lorentz boost in a fixed direction.

The cusp singularity results from the identification of points related by the elements of P, which form a non-compact group. Let U denote the set with $Im(z) > 1$ which corresponds to the set $p^3 > 0$ in momentum space. U and $P(U)$ are disjoint. The cusp neighborhood (`https://rb.gy/fd4dz`) can be identified as the set U/P which is the projection of U to H/G.

3. In the simplest situation, one has $G \subset SL(2,Z) \subset SL(2,R) \subset SL(2,C))$, where $S(2,R)$ leaves the real axis invariant. Z could be replaced by an algebraic extension for rationals of algebraic integers in this extension.

$SL(2,C)$ and therefore also $SL(2,R)$ acts in M^4 as Lorentz transformations.

1. A given M^4 momentum has the representation $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2)$. The representation is unique apart from a complex scaling of z_i so that $z = z_1/z_2$ can be taken as a complex coordinate for the plane and $SL(2,C)$ acts as Möbius transformation. $SL(2,R)$ leaves the real axis invariant.

   The automorphism of sigma matrices induced by $SL(2,C)$ transformation in turn induces Lorentz transformation in momentum space.

2. Under what conditions can bi-spinors correspond to M^4 coordinates? Bi-spinor can be assumed to be of the form $(z_1, z_2) = (z, 1)$. From the formula $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2) = (z, 1)$ one can deduce an expression of the condition $Im(z_1/z_2) = Im(z_1) > 1$ in terms of p^k. The condition implies that the z-component of momentum satisfies $p^z = z\overline{z} - 1 > 0$.

   The description of M^4 momenta in terms of bi-spinors and H^2 identified as upper half-plane, denoted by H, is possible only for massless particles.

What happens at cusp singularity

What happens at the cusp singularity?

1. The normal space of the singularity is completely ill-defined as the direction of the electric field of a point-like charge. If so, CP_2 would always be a companion to the cusp. CP_2 would be a blow-up of the cusp points of X^3_i as a hyperbolic manifold (`https://rb.gy/snpft`). One would have $X^3_i \subset H^3$ and the cusp points would correspond to a 3-D sub-manifold of CP_2 defined by the normal spaces at the cusp singularity.

2. In the interior of the space-time surface the 3-D submanifold of CP_2 would extend to CP_2 type extremal with a light-like M^4 projection or its deformation. Several cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) could be associated with a single CP_2 type extremal representing wormhole contact. This corresponds to the view that wormhole throats can carry more than one fermion although the recent model assumes only a single fermion.

3. CP_2 type extremal defines a wormhole contact connecting two Minkowskian space-time sheets in $H = M^4 \times CP_2$. This would mean that the 3-D submanifold of CP_2 as a blow up is deformed to CP_2 type extremal with 2 throats at opposite sheets: at them the Euclidian induced metric transforms to Minkowskian signature.

   The conservation of monopole flux indeed forces the presence of two Minkowskian space-time sheets in the picture based on H. If the throat as a boundary of the 3-D region of $X^4 \subset H$ involves an incoming radial monopole flux, there must be another throat CP_2, where this flux runs to the other space-time sheet.

4. How could the throats connecting the two Minkowskian space-time sheets emerge in the M^8 picture? Should one allow several copies of Y^3_i with the same H^3 projection but different constant S^3 coordinates and with a common cusp point. The blow-up would give several copies of 3-D regions of CP_2, and in holography they would define wormhole contact with 2 or even more throats.

   The simplest view is that quarks are the only fundamental fermions and leptons correspond to wormhole contacts carrying three antiquarks. They could have three throats associated with the same CP_2 type extremal but this is not the only possibility.

The singularities associated with string-like objects

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ, Γ a finite non-trivial subgroup of $SO(3)$ are orbifolds: the faces of Platonic solids are basic examples. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ are orbifolds if Γ is finite non-trivial subgroup of $SO(3)$: the triangular, quadrilateral, and pentagonal faces of Platonic solids are key examples. From these one can build finite lattices at S^2. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with a negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

What could be the quantal counterpart for the geometric holography? This has been a long standing open question. Suppose that the strong form of holography is realized.

1. In [L112], I considered quantal holography as a counterpart of geometric holography discussed in this article. This led to a suggestion that valence quarks at the wormhole throats could pair with pairs of dark quark and antiquark at strings associated with magnetic flux tubes in the interior of the hadronic 3-surfaces. Could these strings correspond to string-like singularities assignable to geodesic lines inside fundamental regions of $H^3/Gamma$?

2. The flux tubes were proposed to have an effective Planck constant $h_{eff} > h$. The correspondence between valence quarks and dark quarks was proposed to be holographic. The spin and electroweak quantum numbers of dark antiquark would be opposite to those of valence quark and dark quark would have quantum numbers valence quark. There would be entanglement in color degrees of freedom for valence quark and dark antiquark to form color single: this would screen the color of valence quark and transfer it to the magnetic body. The holography in this way would allow a convergent perturbation theory. Nature would be theoretician friendly: a phase transition increasing h_{eff}, transferring color to dark quarks, and reducing color coupling strength to $\alpha_s = 2^2/4\pi\hbar_{eff}$ would occur.

   Whether the dark quark-antiquark pairs as analog for gluon pairs as explanation for hadron mass could explain most of hadron mass remained open: if the classical conserved quantities are identical with the quantum contribution from fermions for Cartan algebra, this could be the case. Whether they could correspond to sea gluons remains also an open question.

3. Quantal holography allowing to obtain a convergent perturbation theory might be realized quite generally, also for leptons which correspond to color partial waves in CP_2 neutralized by super symplectic generator [K46, L57] [L67].

It should be noticed that leptonic dark holography would be very natural if leptons consist of 3 antiquarks [L94]. This option would explain matter-antimatter asymmetry in a new way. Antimatter would be identifiable leptons. For the simplest option, the 3 antiquarks would be associated with a single single wormhole throat. The generalized Kähler structure assignable to M^4 in twistor lift [L102, L103] allows a CP violation, which could favor the condensation of quarks to baryons and antiquarks to leptons.

There are however objections against this idea. The considerations of this article inspire the question whether a single wormhole throat can carry only a single quark assignable to the cusp singularity, as suggested already earlier. Two wormhole contacts would be required. This is required also by the fact that stable wormhole contact must carry a monopole flux and monopole flux flux loops must be closed. Uncertainty Principle would suggest that the flux tube must have length of order lepton Compton length. Can this be consistent with the point-like nature of leptons? These arguments favor the option in which leptons and quarks as opposite H chiralities of H-spinors are the fundamental fermions.

Other kinds of point-like singularities and analogy with Fermi surface

Point-like singularities as cusps would naturally correspond to fundamental fermions at the light-like orbits of partonic 2-surfaces.

1. The 2-D boundaries of the fundamental region Y_i^3 associated with H^3/Γ would be analogues for 2-D pieces of the Fermi surface corresponding to atomic energy levels as energy bands.

   In condensed matter physics, the energy shells can deform and the components of the Fermi surface can touch. These singularities are central to topological physics.

2. At M^8 level the 2-D boundaries of Y_i^3 are analogues of energy bands. The evolution defined by the number theoretic holography, identifiable as a coupling constant evolution at the level of M^8, induces deformations of Y_i^3. One expects that this kind of touching singularities take place.

   At the level of H this would correspond to simple touching of the outer boundaries of the physical objects. In particular, these touchings could take place at the partonic 2-surfaces identified as vertices at which severa partonic orbits meet as the partonic surfaces as their ends are glued to single surface just like the ends of lines of a vertex of Feynman diagram are glued together along their ends.

   Could the meeting of fermion and antifermion cusp singularity in this way relate to an annihilation to a boson regarded as a fermion antifermion pair?

3. One can of course challenge the assumption that all fermions correspond to cusps, which correspond to parabolic subgroups of $G \subset SL(2,C)$ (`https://rb.gy/b5t55`). The proposal that all momenta, whose components are algebraic integers for the extension defined by P, are possible. What could be the interpretation of fermions which do not correspond to cusp.

   What the addition of a fermion to a particular allowed momentum could mean? Could it mean that its momentum defines a parabolic subgroup of G? Or is it true only for the "thin" part of Y_i^3 perhaps representing analogs of valence quarks.

   Or could the non-singular momenta correspond to the momenta for the analogues of sea partons, in particular analogs of sea gluons as fermion-antifermion pairs so that their total momentum would dominate in the total momentum of hadron. These would correspond to the "thick" part of Y_i^3. Could these interior momenta correspond to states delocalized at the string world sheets in the interior of monopole flux tubes and also states delocalized in the interiors of the flux tubes. Are these fermions present too?

   The presence of these states should be coded by the geometry of the hyperbolic manifold H^3/Γ (`https://rb.gy/snpft`) and Y_i^3 as its fundamental domain. Somehow the group $G \subset SL(2,C)$ should be responsible for this coding.

10.5.2 About the superconformal symmetries for the light-like orbits of partonic 2-surfaces

Are the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) giving rise to CP_2 type extremals and the fermion momenta inside string world sheets and flux tubes associated with Y_i^3 sufficient to fix the 3-surfaces Y_i^3 in turn fixing number-theoretic holography?

1. The total energy for the classical action associated with these two kinds of fermions should correspond to the "sea" (thick part) and "valence fermions" assigned to the cusps (thin part).

2. Supersymplectic invariance and generalized conformal and Kac-Moody invariance assignable to light-like partonic orbits allows a large number of alternatives for the light-like surfaces [L99, L109, L108]. If supersymplectic and Kac-Moody symmetries act as gauge symmetries, the surfaces related by these symmetries are physically equivalent.

   The proposal is that these symmetries are partially broken and there is a hierarchy of breakings labelled by subalgebras $A_n \subset A$ of these algebras. The vanishing conditions for classical and quantal charges for A_n and $[A_n, A]$ serve as gauge conditions and also select the partonic 3-surfaces.

Interpretation of the partially broken gauge symmetries giving rise to dynamical symmetries is in terms of number theoretical measurement resolution and inclusion of hyperfinite factors of type II_1. These hierarchies relate to the hierarchies of extensions of rationals defined by the polynomials P defining the space-time surfaces apart from the effect of fermions.

If the preferred extremal property means generalization of holomorphy from 2-D case to 4-D case, one can conclude that the preferred extremals differ only at the singularities of space-time surfaces such as partonic orbits where the entire action comes into play. The regions outside the singularities would be universal: the minimal surface property would realize the 4-D generalization of the holomorphy.

3. Could different choices of the classical action, which determine the expressions of the classical and fermionic (quantal) Noether charges in terms of the modified Dirac action, correspond to different gauge choices selecting singular surfaces, in particular the CP_2 type extremals differently?

   The standard view would suggest that the change of the parameters of the action at the level of H corresponds to coupling constant evolution, which in the TGD framework is discrete and in terms of p-adic length scales. On the other hand, the existence of dual M^4 conformal invariance suggests that the coupling constant evolution at the level of M^8 is realized as "energy" evolution by using associativity as a dynamical principle. Can these two views be consistent?

   Note that the discriminant of the polynomial P is proposed to correspond to the exponent of action [L102, L104, L109, L108, L106]. The discriminant should change if the action changes. Does this mean that the change of the (effective) action in the discrete coupling constant evolution changes the polynomial?

10.6 Birational maps as morphisms of cognitive structures

https://en.wikipedia.org/wiki/Birational$_geometry$ and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself.

1. In the TGD framework, the algebraic extensions E are defined by rational polynomials P at the level of M^8_c identifiable as complexified octonions. E defines a unique discretization for the number theoretically preferred coordinates of M^8_c by the condition that the M^8 coordinates have values in E: I call these discretizations cognitive representations. They make sense also in the extensions of p-adic number fields induced by E serving as correlates of cognition in TGD inspired theory of conscious experience. Birational maps respect the extension E associated with the cognitive representations and map cognitive representations to cognitive representation of same kind. They are clearly analogous to morphisms in category theory.

2. M^8-H duality [L80, L81, L117, L121] is a number theoretic analogue of momentum-position duality. M^8_c serves as the analog of momentum space and $H = M^4 \times CP_2$ as the analog of position space. $M^8 - H$ duality maps the 4-surface defined in M^8_c by number theoretic holography based on 3-D data to a 4-D space-time surface in H.

3. Should $M^8 - H$ duality respect the algebraic extension? If so, it would map the cognitive representation defined by points belonging to 4-D surface $Y^4 \subset M^8$ with the values of preferred coordinates in E to points of $M^4 \subset H$ with coordinate values in E. One could say that $M^8 - H$ duality respects the number theoretical character of cognitive representations. The precise meaning of this intuition is however far from clear.

There are also questions related to the choice of preferred coordinates in which the cognitive representation is defined.

1. Number theoretic constraints fix the preferred coordinates at M^8 side rather uniquely and this induces a preferred choice also on $M^4 \subset H$. For hyperbolic spaces (mass shells) a cognitive explosion happens and a natural question whether cognitive explosion happens also

for the light-like curves assignable to the partonic orbits. If the light-like curve is geodesic, the explosion indeed occurs. For more general light-like curves this is not the case always: could these more general light-like curves be related by a birational map to light-like geodesics?

2. At the H side one can also imagine besides standard Minkowski coordinates also other physically preferred choices of coordinates: are they also theoretically preferred? The notion of Hamilton-Jacobi structure [L114] suggests that in the case of M^4 Hamilton-Jacobi coordinates are very natural for the holomorphic realization of holography. If these are allowed, a natural condition would be that the Hamilton-Jacobi coordinates are related to each other by birational maps mapping the point of E to points of E so that cognitive representations are mapped to cognitive representations.

10.6.1 M^8-H duality, holography as holomorphy, Hamilton-Jacobi structures, and birational maps as cognitive morphisms

In the sequel the questions raised in the introduction are considered. The basic notions are M^8-H duality [L80, L81, L117, L121], holography as a generalized holomorphy [L111, L120], Hamilton-Jacobi structures [L114], and birational maps as cognitive morphisms.

About more precise definitions of the basic concepts

Consider first more precise definitions of various notions involved.

1. What are the preferred coordinates of M^8_c in which the cognitive representation is constructed? M^8_c has a number theoretic interpretation in terms of complexified octonions and physical interpretation as 8-D momentum space. Linear Minkowski coordinates are number-theoretically preferred since octonionic multiplication and other arithmetic operations have a very simple form in these coordinates. Also the number theoretic automorphisms respect the arithmetic operations. The allowed automorphisms correspond to the group G_2 which is a subgroup of $SO(1,7)$. Physically Minkowski space coordinates are preferred coordinates in the momentum space and also in $M^4 \subset H$.

2. How the algebraic extension of rationals, call it E, is determined? The proposal is that rational polynomials characterize partially the 3-D data for number theoretic holography [L117]. The roots of a rational polynomial P define an algebraic extension of rationals, call it E. A stronger, physically motivated, condition on P is that its coefficients are integers smaller than the degree of P.

   The roots of P define mass shells $H^3_c \subset M^4_c \subset M^8_c$, which in turn assign to the mass shells a 4-D surface Y^4 of M^8_c going through the mass shells by associative holography requiring that the normal space of Y^4 is associative, that is quaternionic. It has been be assumed that the roots are complex although also the condition that the roots are real can be considered. The imaginary unit i associated with the roots commutes with the octonionic imaginary units.

3. How the cognitive representation is defined? The points of $Y^4 \subset M^8_c$ with M^4 coordinates in E define a unique discretization of Y^4, called a cognitive representation, making sense also in the extensions of p-adic number fields induced by E. In general, the number of algebraic points in the interior of Y^4 is discrete and even finite but at the mass shells H^3 a cognitive explosion takes place. All points of H^3 with coordinates in E are algebraic.

   The algebraic points with coordinates, which are algebraic *integers* are physically and cognitively in very special role in number theoretic physics and make sense also as points of various p-adic number fields making possible number theoretical universality. The points of H^3 have interpretation as momenta and for physical states the total momentum as sum of momenta at mass shells defined by the roots of P has components which are integers, called Galois confinement [L102, L103], would define fundamental mechanism for the formation of bound states.

4. M^8-H duality maps the points of $H^3_c \subset M^4_c \subset M^8_c$ to points of $H^3 \subset M^4 \subset M^4 \times CP_2 = H$ by a map, which is essentially an inversion: this form is motivated by Uncertainty Principle:

for the most recent formulation of the duality see [L121]. This map is a birational map and takes points of E points of E. Also the points of cognitive representation belonging to the interior of $Y^4 \subset M^8_c$ are mapped to the interior of $X^4 \subset M^8_c$. One can ask whether the discrete set of points of cognitive representations in the interiors are of special physical interest, say having interpretation as interaction vertices.

Questions to be pondered

There are many questions to be considered.

1. Also partonic orbits in $X^4 \subset H$ define 3-D holographic data in H. What are these partonic orbits? The simplest partonic orbits have light-like M^4 projection but also more general light-like H projection can be considered (note the analogy with a 2-D rigid body rotating along a light-like geodesic of H). A general light-like geodesic of H is a combination of time-like geodesic of M^4 and space-like geodesic of CP_2.

   A point of the light-like partonic orbit correspond at the level of M^8 to the 3-D blowup of a point of M^8 at which the quaternionic normal space parametrized by CP_2 point is not unique so that the normal spaces for a 3-D section of CP_2, whose union along (probably light-like) geodesic is CP_2 with two holes corresponding to the ends of the partonic orbit. This singularity is highly analogous to the singularity of the electric field of a point charge. Partonic orbits define part of the 3-D holographic data.

2. Could one associate cognitive representations also to the partonic orbits? Could also partonic orbits allow a cognitive explosion? The simplest way to guarantee light-likeness for the H projection is as a light-like geodesic and this indeed allows an infinite number of algebraic points in Minkowski coordinates. Same applies to more general light-like orbits. One would have at least 1-D explosion of the cognitive representation.

3. What can one say about the CP_2 and M^4 projections of the partonic 2-surface? Could also these projections to X^2 and Y^2 allow an infinite number of points with coordinates in E or do these kinds of points have some special physical meaning, say as vertices for particle reactions? Concerning cognitive representation, the blow-up would mean that the point has an infinite but discrete set of quaternionic normal spaces at the level of M^8. Since the partonic surface can have an arbitrary complex sub-manifold as CP_2, there is indeed information to be cognitively represented.

The most general cognitively preferred coordinate choices for space-time surfaces and H?

In the case of M^8_c, number theoretical considerations fix the preferred coordinates highly uniquely. In the case of H the situation is not so obvious and one cannot exclude alternative coordinate choices related by a birational map.

A possible motivation comes from the following argument.

1. String world sheets are candidates for singularities analogous to partonic orbits. At a given point of the string world sheet a blow up to a 2-D complex sub-manifold of CP_2 would occur. This would mean that the normal spaces of the point in M^8_c form this sub-manifold. Cosmic strings are the simplest objcts of this kind. Monopole flux tubes are deformations of the cosmic strings and allow also an interpretation in terms of maps from M^4 to CP_2.

   If string world sheets define part of the data needed to define holography, one could argue that it makes sense to assign cognitive explosion to the string world sheet.

2. Cognitive explosion takes place if the string world sheets are 2-D geodesic submanifolds of H. Planes $M^2 \subset M^4$ represent the simplest example. A more complex example is obtained by taking a space-like geodesic in H and rotating it along a time-like geodesic of H. One can also take a light-like geodesic in H and rotate it along a light-like geodesic in dual light-like

direction (ruler surface would be in question). In which case the gluing of the string world sheet along the boundary to the partonic orbit could be possible.

One might perhaps think of building string world sheets by gluing these kinds of ultrasimple regions along their boundaries so that one would have edges. An interpretation as a discretization would be appropriate. One might even go further and argue that the cognitive explosion explains why we are able to think of these kinds of regions in terms of simple formulas. One might argue that number theoretic physics realizes exactly what is usually regarded as approximation. One can however wonder whether life is so simple.

This argument encourages to consider a more complex option allowing more general string world sheets.

1. In the case of M^4 projection, the notion of the Hamilton-Jacobi structure [L114], generalizing the notion of ordinary complex structure, is highly interesting in this respect. It involves a generalization of complex coordinates involving local decompositions $M^4(x) = M^2(x) \times E^2(x)$ of the 4-D tangent space of M^4. The integrable distribution of $E^2(x)$ corresponds to complex coordinates $(w, \overline{w}$ integrating to a partonic 2-surface whereas the integrable distribution of $M^2(x)$ to light-like coordinate pairs (u, v) integrating to a string world sheet in M^4.

   Cognitive representation mean that the discretized values of the Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ are in E. Hamilton-Jacobi structure generalizes also to the level of $X^4 \subset H$ and now Y^2 can also correspond to CP_2 projection as in the case of cosmic strings and magnetic flux tubes. Note that in TGD one can use a subset of H coordinates as coordinates of X^4.

2. The simplest assumption is that the 1-D parton orbit corresponds to a light-like geodesic but could one map light-like geodesics to more general light-like curves by a birational map? Hamilton-Jacobi structure gives rise to a pair of curved (u, v) of light-like coordinates: could it relate to the standard flat light-like coordinates of M^2 by a birational map? Could a birational map relate standard complex coordinates of E^2 to the pair $(w, \overline{w})$? Could one also consider more general birational maps of $M^4 \to M^4$? If so, the Hamilton-Jacobi structures would be related by maps respecting algebraic extensions and cognitive representations. This would give a very powerful constraint on the Hamilton-Jacobi structures.

In the case of CP_2, projective coordinates are group-theoretically highly unique and determined apart from color rotations. Could one require that the CP_2 projection Y^2 associated with the partonic 2-surface and cosmic string or magnetic flux tube involves cognitive explosion. Are the allowed M^4 and CP_2 projections related by birational maps? Note that color rotations are birational maps.

These considerations suggest the following speculative view.

1. $M^8 - H$ duality, when restricted to 3-D holographic data at both sides, is analogous to a birational map expressible in terms of rational functions and respects the number theoretical character of cognitive representations.

2. Cognitive explosion occurs for the holographic data (this is very natural from the information theoretic perspective): this includes also string world sheets. Hamilton-Jacobi structures in the same cognitive class, partially characterized by the extension E of rationals, are related by a birational map.

3. $M^8 - H$ duality maps the quaternionic normal spaces to points of CP_2 and is an example of a birational map in M^4 degrees of freedom. It is not however easy to guess how the number theoretic holography is realized explicitly and how the 4-surfaces in M^8 are mapped to holomorphic 4-surfaces in H.

4. An interesting additional aspect relates to the non-determinism of partonic orbits due to the non-determinism of the light-likeness condition deriving from the fact that the action is Chern-Simons-Kähler action. The deformation of the partonic orbit induces the deformation of time derivatives of H coordinates at the boundary of $\delta M^4_+ \times CP_2$ to guarantee that boundary conditions at the orbit are realized. This suggests a strong form of holography [L120].

Already the 3-surfaces at $\delta M^4_+ \times CP_2$ or partonic orbits would be enough as holographic data. This in turn suggests that the analog of birational cognitive correspondence between the holographic data at $\delta M^4_+ \times CP_2$ and at partonic orbits.

10.6.2 Appendix: Some facts about birational geometry

Birational geometry has as its morphisms birational maps: both the map and its inverse are expressible in terms of rational functions. The coefficients of polynomials appearing in rational functions are in the TGD framework rational. They map rationals to rationals and also numbers of given extension E of rationals to themselves (one can assign to each space-time region an extension defined by a polynomial).

Therefore birational maps map cognitive representations, defined as discretizations of the space-time surface such that the points have physically/number theoretically preferred coordinates in E, to cognitive representations. They therefore respect cognitive representations and are morphisms of cognition. They are also number-theoretically universal, making sense for all p-adic number fields and their extensions induced by E. This makes birational maps extremely interesting from the TGD point of view.

The following lists basic facts about birational geometry as I have understood them on the basis of Wikipedia articles about birational geometry and Enriques-Kodaira classification. I have added physics inspired associations with TGD.

Birational geometries are one central approach to algebraic geometry.

1. They provide classification of complex varieties to equivalence classes related by birational maps. The classification complex curves (real dimension 2) reduces to the classification of projective curves of projective space CP_n determined as zeros of a homogeneous polynomial. Complex surfaces (real dimension 4) are of obvious interest in TGD: now however the notion of complex structure is generalized and one has Hamilton-Jacobi structure.

2. In TGD, a generalization of complex surfaces of complex dimension 2 in the embedding space $H = M^4 \times CP_2$ of complex dimension 4 is considered. What is new is the presence of the Minkowski signature requiring a combination of hypercomplex and complex structures to the Hamilton-Jacobi structure. Note however the space-time surfaces also have counterparts in the Euclidean signature $E^4 \times CP_2$: whether this has a physical interpretation, remains an open question. Second representation is provided as 4-surfaces in the space M^8_c of complexified octonions and an attractive idea is that $M^8 - H$ duality corresponds to a birational mapping of cognitive representations to cognitive representations.

3. Every algebraic variety is birationally equivalent with a sub-variety of CP_n so that their classification reduces to the classification of projective varieties of CP_n defined in terms of homogeneous polynomials. $n = 2$ (4 real dimensions) is of special relevance from the TGD point of view. A variety is said to be rational if it is birationally equivalent to some projective variety: for instance CP_2 is rational.

4. A concrete example of birational equivalence is provided by stereographic projections of quadric hypersurfaces in n+1-D linear space. Let p be a point of quadric. The stereographic projection sends a point q of the quadric to the line going through p and q, that is a point of CP_n in the complex case. One can select one point on the line as its representative. Another exammple is provided by Möbius transformations representing Lorentz group as transformations of complex plane.

The notion of a minimal model is important.

1. The basic observation is that it is possible to eliminate or add singularities by using birational maps of the space in which the surface is defined to some other spaces, which can have a higher dimension. The zeros of a birational map can be used to eliminate singularities of the algebraic surface of dimension n by blowups replacing the singularity with CP_n. Poles in turn create singularities. Peaks and self-intersections are examples of singularities.

   The idea is to apply birational maps to find a birationally equivalent surface representation, which has no singularities. There is a very counter-intuitive formal description for this. For instance, complex curves of CP_2 have intersections since their sum of their real dimensions is 4. The same applies to 4-surfaces in H. My understanding is as follows: the blowup for CP_2 makes it possible to get rid of an intersection with intersection number 1. One can formally say that the blow up by gluing a CP_1 defines a curve which has negative intersection number -1.

2. In the TGD framework, wormhole contacts have the same metric and Kähler structure as CP_2 and light-like M^4 projection (or even H projection). They appear as blowups of singularities of 4-surfaces along a light-like curve of M^8. The union of the quaternionic/associative normal spaces along the curve is not a line of CP_2 but CP_2 itself with two holes corresponding to the ends of the light-like curve. The 3-D normal spaces at the points of the light-like curve are not unique and form a local slicing of CP_2 by 3-D surfaces. This is a Minkowskian analog of a blow-up for a point and also an analog of cut of analytic function.

The Italian school of algebraic geometry has developed a rather detailed classification of these surfaces. The main result is that every surface X is birational either to a product $\mathbb{P}^1 \times C for some curve \mathrm{C} or to a minimal surface \mathrm{Y}. Preferred extremals are indeed minimal surfaces so that space-time surfaces might define\ minimal models. The absence of singularities (typically peaks or self-intersections) cha$

There are several birational invariants listed in the Wikipedia article. Many of them are rather technical in nature. The canonical bundle K_X for a variety of complex dimension n corresponds to n:th exterior power of complex cotangent bundle that is holomorphic n-forms. For space-time surfaces one would have $n = 2$ and holomorphic 2-forms.

1. Plurigenera corresponds to the dimensions for the vector space of global sections $H_0(X, K_X^d)$ for smooth projective varieties and are birational invariants. The global sections define global coordinates, which define birational maps to a projective space of this dimension.

2. Kodaira dimension measures the complexity of the variety and characterizes how fast the plurigenera increase. It has values $-\infty, 0, 1, ..n$ and has 4 values for space-time surfaces. The value $-\infty$ corresponds to the simplest situation and for $n = 2$ characterizes CP_2 which is rational and has vanishing plurigenera.

3. The dimensions for the spaces of global sections of the tensor powers of complex cotangent bundle (holomorphic 1-forms) define birational invariants. In particular, holomorphic forms of type $(p, 0)$ are birational invariants unlike the more general forms having type (p, q). Betti numbers are not in general birational invariants.

4. Fundamental group is birational invariant as is obvious from the blowup construction. Other homotopy groups are not birational invariants.

5. Gromow-Witten invariants are birational invariants. They are defined for pseudo-holomorphic curves (real dimension 2) in a symplectic manifold X. These invariants give the number of curves with a fixed genus and 2-homology class going through n marked points. Gromow-Witten invariants have also an interpretation as symplectic invariants characterizing the symplectic manifold X.

   In TGD, the application would be to partonic 2-surfaces of given genus g and homology charge (Kähler magnetic charge) representatable as holomorphic surfaces in $X = CP_2$ containing n marked points of CP_2 identifiable as the loci of fermions at the partonic 2-surface. This number would be of genuine interest in the calculation of scattering amplitudes.

What birational classification could mean in the TGD framework?

1. Holomorphic ansatz gives the space-time surfaces as Bohr orbits. Birational maps give new solutions from a given solution. It would be natural to organize the Bohr orbits to birational equivalence classes, which might be called cognitive equivalence classes. This should induce similar organization at the level of M_c^8.

2. An interesting possibility is that for certain space-time surfaces CP_2 coordinates can be expressed in terms of preferred M^4 coordinates using birational functions and vice versa. Cognitive representation in M^4 coordinates would be mapped to a cognitive representation in CP_2 coordinates.

3. The interpretation of $M^8 - H$ duality as a generalization of momentum position duality suggests information theoretic interpretation and the possibility that it could be seen as a cognitive/birational correspondence. This is indeed the case M^4 when one considers linear M^4 coordinates at both sides.

4. An intriguing question is whether the pair of hypercomplex and complex coordinates associated with the Hamilton-Jacobi structure could be regarded as cognitively acceptable coordinates. If Hamilton-Jacobi coordinates are cognitively acceptable, they should relate to linear M^4 coordinates by a birational correspondence so that M^8-H duality in its basic form could be replaced with its composition with a coordinate transformation from the linear M^4 coordinates to particular Hamilton-Jacobi coordinates. The color rotations in CP_2 in turn define birational correspondences between different choices of Eguchi-Hanson coordinates.

   If this picture makes sense, one could say that the entire holomorphic space-time surfaces, rather than only their intersections with mass shells H^3 and partonic orbits, correspond to cognitive explosions. This interpretation might make sense since holomorphy has a huge potential for generating information: it would make TGD exactly solvable.

Chapter 11

Homology of "world of classical worlds" in relation to Floer homology and quantum homology

11.1 Introduction

One of the mathematical challenges of TGD is the construction of the homology of "world of classical worlds" (WCW). With my rather limited mathematical skills, I had regarded this challenge as a mission impossible. The popular article in Quanta Magazine with title "Mathematicians transcend the geometric theory of motion" (see `https://cutt.ly/vO4eb5V` however stimulated the attempts to think whether it might be possible to say something interesting about WWC homology.

The article told about a generalization of Floer homology by Abouzaid and Blumberg [A37] (`https://cutt.ly/ZPeOTSc`) published as 400 page article with the title "Arnold Conjecture and Morava K-theory". This theory transcends my mathematical skills but the article stimulated the idea WCW homology might be obtained by an appropriate generalization of the basic ideas of Floer homology (`https://cutt.ly/VO4dSPD`).

The construction of WCW homology as a generalization of Floer homology looks rather straightforward in the zero ontology (ZEO) based view about quantum TGD. The notions of ZEO and causal diamond (CD) [L68] [K99], the notion of preferred extremal (PE) [L99] [K5], and the intuitive connection between the failure of strict non-determinism and criticality pose strong conditions on the possible generalization of Floer homology.

WCW homology group could be defined in terms of the free group formed by preferred extremals $PE(X^3, Y^3)$ for which X^3 is a *stable* maximum of Kähler function K associated with the *passive* boundary of CD and Y^3 associated with the *active* boundary of CD is a more general critical point.

The stability of X^3 conforms with the TGD view about state function reductions (SFRs) [L68]. The sequence of "small" SFRs (SSFRs) at the active boundary of CD as a locus of Y^3 increases the size of CD and gradually leads to a PE connecting X^3 with stable 3-surface Y^3. Eventually "big" SFR (BSFR) occurs and changes the arrow of time and the roles of the boundaries of the CD changes. The sequence of SSFRs is analogous to a decay of unstable state to a stable final state.

The identification of PEs as minimal surfaces with lower-dimensional singularities as loci of instabilities implying non-determinism allows to assign to the set $PE(X^3, Y_i^3)$ numbers $n(X^3, Y_i^3 \rightarrow Y_j^3)$ as the number of instabilities of singularities leading from Y_i^3 to Y_j^3 and define the analog of criticality index (number of negative eigenvalues of Hessian of function at critical point) as number $n(X^3, Y_i^3) = \sum_j n(X^3, Y_i^3 \rightarrow Y_j^3)$. The differential d defining WCW homology is defined in terms of $n(X^3, Y_i^3 \rightarrow Y_j^3)$ for pairs Y_i^3, Y_j^3 such that $n(X^3, Y_j^3) - n(X^3, Y_i^3) = 1$ is satisfied. What is nice is that WCW homology would have direct relevance for the understanding of quantum criticality.

The proposal for the WCW homology also involves a generalization of the notion of quantum connectivity crucial for the definition of Gromow-Witten invariants. Two surfaces (say branes) can be said to intersect if there is a string world sheet connecting them generalizes. In ZEO quantum connectivity translates to the existence of a preferred extremal (PE), which by the weak form of holography is almost unique, such that it connects the 3-surfaces at the opposite boundaries of causal diamond (CD).

11.2 Some background

In this section some background, including Morse theory, Floer homology, its generalization by Abouzaid and Blumberg, and the basic ideas of TGD proposal, is discussed.

11.2.1 The basic ideas of Morse theory

Torus as a 2-D example helps to understand the idea of homology and Morse theory. Homologically non-trivial surfaces are surfaces without boundary but are not boundaries themselves. Entire torus represents the element of H^2, the 2 homologically non-trivial circles, and points indeed have vanishing boundaries without being boundaries. The basic homological operation d represents the operation of forming a boundary: the boundary of a boundary is empty and this corresponds to $d^2 = 0$. d reduces degree of homology by one unit: $H_n \rightarrow H_{n-1}$.

How to understand the homology of torus? Morse theory based on the notion of Morse function provides the tool.

1. Consider the embedding of torus to 3-space. The height-coordinate h defines a Morse function at torus and one can assign to it $h = constant$ level surfaces. It has 4 critical points: h_0, h_1, h_2, h_3 at which the topology of level suraace changes.

   h has maximum h_3 at the top of torus and minimum h_0 at the bottom of the torus. h_3 corresponds to the entire torus, element of homology group H_2 and h_0 to a point as element of H_0.

   h has saddle points h_1, h_2 at the top and bottom of the "hole" of the torus. The level surfaces $h = h_1$ and $h = h_2$ correspond to two touching circles: the topology of the intersection changes from a contractible circle to a union of oppositely oriented incontractible small circles representing elements of the homology group H_1. That they have opposite orientations states conservation of homology charge in the topological reaction in which the level circle splits to two: $0 = 1 - 1$.

   Outside the critical points the topology of the $h = constant$ level surface is a circle or two disjoint circles. The critical points of h clearly code part of the homology of torus. What however remains missing is the homology group element, which corresponds to the large circle around the torus. This element of H_1 would be obtained if the height function h were a horizontal coordinate.

2. One can deform the torus and also add handles to it to get 2-D topologies with a higher genus. Morse function also helps to understand the homology of higher-dimensional spaces for which visual intuition fails.

   This situation is finite-D and too simple to apply in the case of the space of orbits of a Hamiltonian system. Now the point of torus is replaced with a loop as a single orbit in phase space. The loop space is infinite-dimensional and the Morse theory does not generalize as such. In Floer homology one studies even the homology of infinite-dimensional spaces.

Homology involves also the d operation. d can be indeed visualized in terms of dynamics of a gradient flow. Assume that torus is in the gravitational potential of Earth proportional to h. Gravitation defines a downwards directed gradient force. One can speak of critical directions as directions in which the particle forced to stay at the torus can fall downwards when subjected to an infinitesimal push.

1. At the top $h = h_3$ of the torus there are 2 critical directions: either along a small or large incontractible circle of torus. This number corresponds to the dimension $d = 2$ of torus as

the element of the homology group H_2. At the bottom $h = h_0$ there are 0 critical directions and one has a point as an element of H_0. At the saddle points h_1, h_2 there is 1 critical direction and it corresponds to a nontrivial circle as an element of H_1. The number n of critical directions corresponds to the dimension for elements of the homology group H_n.

2. The particle at the top h_3 has 2 critical directions (criticality 2), and can fall to the saddle point h_2, having criticality 1, by moving along the small homologically non-trivial circle. Criticality decreases by 1 unit so that one has a map $H_2 \rightarrow H_1$. The particle can also move along the large circle to the bottom, in which case criticality decreases by 2 units.

   The particle at critical point h_2 moves to h_1 along a circle homologous to the large circle without a change in criticality and the particle at h_1 moves to h_0 also the small circle: the criticality changes by 1 unit so that one has a map $H_1 \rightarrow H_0$.

   Therefore the elements of the homology group correspond to critical points for the gradient flow defined by the gravitational field and the effect of the map d can be represented dynamically as a motion in the gravitational field reducing the criticality by one unit.

   The representability of homology elements as critical points of Morse function and the representation of d-operation in terms of gradient dynamics is extremely useful in higher dimensional spaces, where geometric intuition does not help much. In Floer homology this dynamics is applied as a tool.

11.2.2 The basic ideas of Floer homology

Consider first the motivations and ideas of Floer homology (`https://cutt.ly/lO6EMp6`). The original goal was to prove Arnold's conjecture. One considers a symplectic manifold with symplectic form ω. Arnold conjectured that the number of fixed points of a Hamiltonian symplectomorphism generated by an exponentiation of a Hamiltonian H, is bounded below by the number of critical points of a smooth function on M.

The goal is to generalize Morse theory.

1. Morse theory involves the height function h in a finite-D manifold M and the critical points of h correspond to elements of homology groups H_n. The number n of negative eigenvalues of Hessian of f at critical points defines the index of criticality f and one can associate with the critical point an element of the homology group H_n. $n = 0$ corresponds to maximum of f. Note that in infinite-D case, Morse theory need not work since n can be arbitrarily large and if the convention for criticality is changed so that $n = 0$ corresponds to minimum, a different theory is obtained.

2. In Morse homology, the n-simplices of the simplicial homology are replaced by critical points with criticality index n and the homology groups are replaced with the Abelian group defined by the critical points and graded by the criticality index n. The gradient flow lines connecting critical points with $\Delta n = 1$ allow to define an analog of the exterior derivative d: it is defined by the number of flow lines connecting critical points with $\Delta n = 1$.

11.2.3 Floer homology

The motivation for the symplectic Floer homology is the conjecture by Arnold related to the Hamiltonian systems. These systems are defined in phase space, whose points are pairs of position and momentum of the particle. This notion is extremely general in classical physics.

1. One considers compact symplectic manifolds M and symplectic action $S = \oint p_i dq_i$ and its critical points, which are loops. Note that symplectic action has interpretation as an area. The general case $S = \oint (p_i dq_i/dt - H)dt$ is not considered in the Floer homology.

   Remark: A more general question is whether there exist closed orbits, kind of islands of order, in the middle of oceans of chaos consisting of non-closed chaotic orbits. This is indeed the case: there is a fractal structure formed by islands of order in oceans of disorder. Hamiltonian chaos differs from dissipative chaos in that the fractal has the same dimension as the symplectic manifold since symplectic transformations preserve area and high 2n-dimensional volumes.

2. Arnold's conjecture was that the number of critical points of a given criticality index of a symplectomorphism has as an upper bound the number of critical points for a generic function. The inspiration behind the Floer homology is the intuition that a generalization of Morse theory to the loop space $L(M)$ allows us to understand the homology. The conjecture is that the closed orbits serve as minimal area representatives for the homology of $L(M)$. These closed orbits would be critical points of S defining the area closed by the curve.

The goal is to understand the homology of a finite-dimensional compact symplectic manifold M and Floer homology provides the needed tool. Floer homology for the infinite-D loop space $L(M)$ serves as a tool to achieve this goal and the proof of Arnold's conjecture follows as an outcome.

In symplectic Floer homology, one is interested in closed loops as orbits of a symplectic flow in a compact symplectic space M. One wants to identify them as critical points of an analog of Morse function in the loop space $L(M)$.

1. In the symplectic Floer homology, M is a finite-D symplectic manifold and one deduces information about it from the homology of loop space $L(M)$ by generalizing Morse homology to the homology of $L(M)$.

2. The counterpart of the Morse function is unique and defined by the symplectic action functional $S = \oint p_i dq_i$ in $L(M)$. Note that S depends only on M. S defines the counterpart of free action with a vanishing Hamiltonian H. For a general Hamiltonian one would have $S = \oint (p_i dq_i/dt - H)dt$. Note that closed orbits are possible if M is compact. For a generic H the dynamic becomes chaotic.

   Closed loops for free flows define the analogs of critical points of Morse function. For instance, for 2-torus the closed orbits correspond to loops with winding numbers n_1, n_2.

3. One must identify the counterpart for the gradient flow lines connecting the critical points with $\Delta n = 1$ in order to define d. Here one considers a deformation of the system by a time dependent Hamiltonian H and hopes that the predictions do not depend on the choices of H. This gives to orbits of the closed loops in the loop space giving rise to cylinders in M.

   These cylinders define pseudoholomorphic curves and define the counterparts of the gradient flows connecting critical points as closed loops in X. The differential d for the Floer homology is defined in terms of the numbers of these curves between critical points with the property that the criticality index increases by one unit.

4. The basic result is a proof for the Arnold conjecture and roughly states that for the ranks of homology groups of M are smaller than the Floer homology groups defined by arbitrary Hamilton.

Floer homology has a rich variety of applications discussed in the Wikipedia article (`https://cutt.ly/lO6EMp6`). One application relates to the Lagrangian manifolds of a symplectic manifold. Now the chain complex is generated by the intersection points of Lagrangian manifolds intersecting transversely.

A further application is associated with Yang- Mills theory. The action is the Chern-Simons action defining a topological quantum field theory. Its critical points are topologically non-trivial gauge connections with a trivial curvature form. Topological non-triviality means that the group defined by the parallel translations along closed curves is non-trivial. The counterpart of the gradient flow is defined by Yang-Mills action and the flow lines correspond to instantons approach at the ends of the counterpart of mapping cylinder trivial connections.

11.2.4 The generalization of Floer homology by Abouzaid and Blumberg

The work of mathematicians Abouzaid and Blumberg [A37] (`https://cutt.ly/ZPeOTSc`), which represents the generalization of Floer homology which, using popular terms, allows to "count holes" in the infinite-D space of loops.

The abstract of the article of Abouzaid and Blumberg is following.

> We prove that the rank of the cohomology of a closed symplectic manifold with coefficients in a field of characteristic p is smaller than the number of periodic orbits of any non-degenerate Hamiltonian flow.
> Following Floer, the proof relies on constructing a homology group associated to each such flow, and comparing it with the homology of the ambient symplectic manifold. The proof does not proceed by constructing a version of Floer's complex with characteristic p coefficients, but uses instead the canonical (stable) complex orientations of moduli spaces of Floer trajectories to construct a version of Floer homology with coefficients in Morava's K-theories, and can thus be seen as an implementation of Cohen, Jones, and Segal's vision for a Floer homotopy theory. The key feature of Morava K-theory that allows the construction to be carried out is the fact that the corresponding homology and cohomology groups of classifying spaces of finite groups satisfy Poincare duality.

I try to express what I understand as a physicist about this highly technical summary.

1. The main emphasis is in the homology of finite-D symplectic manifolds and the homology of the infinite-D loop space is only a tool to obtain this information.

2. The generalization of Arnold's conjecture is expressed in the first paragraph. For closed symplectic manifolds the cohomology groups of a closed symplectic manifold have rank smaller than the number of periodic orbits of *any* non-degenerate Hamiltonian flow.

   Therefore Hamiltonian flows give information about the cohomology and by Poincare duality also about homology of the symplectic manifold.

3. The coefficients of homology can be chosen in very many ways: rationals, integers, finite fields, p-adic number fields. Integers are however the natural ones in the situation in which one counts concrete objects. The homology has coefficients in finite field F_p, integers modulo prime p: for instance, the numbers of flow lines of gradient flow connecting the critical points of symplectic action are counted modulo p.

4. Time dependent Hamiltonians enter into the picture as perturbations of the symplectic action. One replaces the free symplectic action $S = \oint p_i dq_i/dt$ in loop space with $S = \oint (p_i dq_i/dt - H)dt$ playing a role analogous to that of Morse function. This is like adding an interaction term to free action. It is essential that the symplectic space is compact so that closed orbits as critical points of S are possible.

11.2.5 Gromow-Witten invariants

The proposed TGD based generalization of the notion of "being connected" by a flow line of gradient flow resonates with the definition of Gromow-Witten (G-W) invariant. G-W invariant emerges in enumerative geometry, which is essentially counting of particular kinds of points of enumerative geometry which is a branch of algebraic geometry.

G-W invariants (`http://tinyurl.com/y9b5vbcw`) are rational number valued topological invariants useful in algebraic and symplectic geometry. These quantum invariants give information about these geometries not provided by classical invariants. Despite being rational numbers in the general case G-W invariants in some sense give the number of string world sheets connecting given branes.

The definition of G-W invariant involves a non-locality, which is completely analogous to the non-locality in the proposed definition of WCW homology. In TGD, the string world sheet as connector of branes is replaced with PE as a connector of the boundaries of opposite boundaries of CD taking the role of brane.

Here is the definition of G-W invariants with some TGD induced coloring taken from [L43, K48].

1. One considers a collection of n surfaces ("branes") with even dimensions in some symplectic manifold X of dimension $D = 2k$ (say Kähler manifold) and pseudo-holomorphic curves ("string world sheets") X^2, which have the property that they connect these n surfaces in the sense that they intersect the "branes" in the marked points x_i, $i = 1, .., n$.

"Connect" does not reduce to intersection in a topologically stable sense since connecting is possible also for branes with dimension smaller than $D-2$. One allows all surfaces X^2 that intersect the n surfaces at marked points if they are pseudo-holomorphic even if the basic dimension rule is not satisfied. In the 4-dimensional case this does not seem to have implications since the partonic 2-surfaces automatically satisfy the dimension rule. The n branes intersect or touch in a quantum sense: there is no concrete intersection but intersection with the mediation of "string world sheet".

2. Pseudo-holomorphy means that the Jacobian df of the embedding map $f: X^2 \to X$ commutes with the symplectic structures j *resp.* J of X^2 *resp.* X: i.e. one has $df(jT) = Jdf(T)$ for any tangent vector T at given point of X^2. For $X^2 = X = C$ this gives Cauchy-Riemann conditions.

   In the symplectic case X^2 is characterized topologically by its genus g and homology class A as the surface of X. In algebraic geometry context the degree d of the polynomial defining X^2 replaces A. In TGD X^2 corresponds to a string world sheet having also a boundary. X^2 has also n marked points $x_1, ..., x_n$ corresponding to intersections with the n surfaces.

3. G-W invariant $GW_{g,n}^{X,A}$ gives the number of pseudo-holomorphic 2-surfaces X^2 connecting n given surfaces in X - each at single marked point. In TGD these surfaces would be partonic 2-surfaces and marked points would be carriers of sparticles.

11.3 About the generalization of Floer homology in the TGD framework

A generalization of homotopy and homology groups could help to understand WCW topology. One of the intuitive visions behind TGD has indeed been that, despite the explicit appearance of metric, TGD in a certain sense is a topological quantum theory. A mathematical motivation for this intuition comes from the fact that minimal surfaces provide representations for homological equivalence classes. Floer homology suggests concrete ideas, which might help to understand the homology of WCW.

11.3.1 Key ideas behind WCW homology

The encounter with Floer homology inspired the question whether one could say something interesting about WCW homology by an appropriate generalization of the concepts involved with it.

Preferred extremals (PEs) as counterparts of critical points

PEs are an obvious candidate for the counterparts of critical points. ZEO however implies some important delicacies crucial for WCW homology.

1. In the TGD Universe, space-time is a 4-surface in $H = M^4 \times CP_2$, in a loose sense an orbit of 3-surface. General Coordinate Invariance (GCI) requires that the dynamics associates to a given 3-surface a highly unique 4-surface at which the 4-D general coordinate transformations act. This 4-surface is a PE of the action principle determing space-time surfaces in H and analogous to Bohr orbit. GCI gives Bohr orbitology as an exact part of quantum theory and also holography.

   These PEs as 4-surfaces are analogous to the closed orbits in Hamiltonian systems about which Arnold speculated. In the TGD Universe, only these PEs would be realized and would make TGD an integrable theory. The theorem of Abouzaid and Blumberg allows to prove Arnold's conjecture in homologies based on cyclic groups Z_p. Maybe it could also have use also in the TGD framework.

2. WCW generalizes the loop space considered in Floer's approach. Very loosely, loop or string is replaced by a 3-D surface, which by holography induced is more or less equivalent with

4-surface. In TGD just these minimal representatives for homology as counterparts of closed orbits would matter.

3. Symplectic structure and Hamiltonian are central notions also in TGD. Symplectic (or rather, contact) transformations assignable to the product $\delta M^4_+ \times CP_2$ of the light-cone boundary and CP_2 act as the isometries of the infinite-D "world of classical worlds" (WCW) consisting of these PEs, or more or less equivalently, corresponding 3-surfaces. Hamiltonian flows as 1-parameter subgroups of isometries of WCW are symplectic flows in WCW with symplectic structure and also Kaehler structure.

4. The space-time surfaces are 4-D minimal surfaces in H with singularities analogous to frames of soap films. Minimal surfaces are known to define representatives for homological equivalence classes of surfaces. This has inspired the conjecture that TGD could be seen as a topological/homological quantum theory in the sense that space-time surfaces served as unique representatives or their homological classes.

5. There is also a completely new element involved. TGD can be seen also as number theoretic quantum theory. M^8-H duality can be seen as a duality of a geometric vision in which space-times are 4-surfaces in H an of a number theoretic vision in which one consideres 4-surfaces in octonionic complexified M^8 determined by polynomials with dynamics reducing to the condition that the normal space of 4-surface is associative (quaternionic). M^8 is analogous to momentum space so that a generalization of momentum-position duality of wave mechanics is in question.

The first sketch for WCW homology

A suitable generalization of Floer's theory might allow us to define WCW homology.

1. The PEs would correspond to the critical points of an analog of Morse function in the infinite-D context. In TGD the Kähler function K defining the Kahler geometry of WCW is the unique candidate for the analog of Morse function.

   The space-time surfaces for which the exponent $exp(-K)$ of the Kähler function is stationary (so that the vacuum functional is maximum) would define PEs. Also other space-time surfaces could be allowed and it seems that the continuity of WCW requires this. However the maxima or perhaps extrema would provide an excellent approximation and number theoretic vision would give an explicit realization for this approximation.

   It is however important to notice that the K for, in general non-unique, preferred external $PE(X^3, Y^3)$ can be maximum for X^3 and a more general critical point for Y^3. This option conforms with the ZEO view about SFRs in which the passive boundary of CD is stable and a sequence of SSFRs takes place at the active boundary and increases its size. The homology would be assigned to the criticality of the active boundary of CD.

   This would require a varying CD size, which should therefore be determined by PE and appear as a parameter in PE. By M^8-H duality the boundary of CD corresponds to the image of a mass shell H^3 in M^3. Perhaps this property at the active end of PE codes for the size scale of the CD. The size scale of CD, not necessarily the size, should correspond to the p-adic length scale L_p determined by the largest ramified prime of the polynomial coding for PE. Does this mean that L_p remains the same during the entire sequence of SSFRs or can it increase? The size could increase by factor $\sqrt{p}$ with change ibn L_p and for large p-adic primes such as $M_{127} = 2^{127}-1$ this would mean very large scaling.

   Remark: Since WCW Kähler geometry has an infinite number of zero modes, which do not appear in the line element as coordinate differentials but only as parameters of the metric tensor, one expects an infinite number of maxima.

2. The PEs would correspond by M^8-H duality to roots of polynomials P in the complexified octonionic M^8 so that a connection with number theory emerges. M^8-H duality strongly strongly suggests that $exp(-K)$ is equal to the image of the discriminant D of P under canonical identification $I : \sum x_n p^n \rightarrow \sum x_n p^{-n}$ mapping p-adic numbers to reals. The prime p would correspond to the largest ramified prime dividing D [L102, L103].

3. The number theoretic vision could apply only to the critical points of $exp(-K)$ with respect to both ends of PE and give rise to what I call a hierarchy of p-adic physics as correlates of cognition. Everything would be discrete and one could speak of a generalization of computationalism allowing also the hierarchy of extensions of rationals instead of only rationals as in Turing's approach. The real-number based physics would also include the non-maxima via a perturbation theory involving a functional integral around the maxima. Here Kähler geometry allows to get rid of ill-defined metric and Gaussian determinants.

G-W invariants and ZEO

Enumerative geometry is also a central element of adelic physics.

1. M^8-H duality involves the notion of cognitive representations consisting of special points of 4-surface, in particular, points of 3-D mass shell $H^3 \subset M^4_c \subset M^8_c$. The "active" points containing quark are identified as quark momenta. A generalization of momentum-position duality is in question.

2. The points of the cognitive representation, having interpretation as four-momenta [L80, L81, L102, L103], are identified are algebraic integers in the extensions defined by the real polynomial P with rational coefficients continued to a polynomial of a complexified octonion. P defines mass shells as its roots with $m^2 = r_n$ defining the spectrum of virtual mass squared values for quarks. The finite number of mass shells guarantees the absence of divergences due to momentum space integrations.

3. By the symmetries of H^3, the number of points in cognitive representations is especially high at the mass shells. Physical states correspond to Galois singlets (Galois confinement implying conformal confinement) for which the sum of quark momenta is an ordinary integer as one uses as unit the p-adic mass scale defined by the largest ramified prime associated with P.

4. The mass shells associated with a given polynomial P are connected by a 4-surface X^4 as a deformation of M^4_c, which defines M^8-H duality by assigning to $X^4 \subset M^8$ space-time surface in $H = M^4 \times CP_2$. This surface is a minimal surface with singularities analogous to frames of a soap film. M^8-H duality maps the points of cognitive representation to $X^4 \subset H$ [L101].

The TGD view about WCW homology could perhaps be regarded as a generalization of the quantum connectedness behind G-W invariants. The role of the string world-sheet as a quantum connector is taken by PE so that there is no need to introduce gradient dynamics separately. The quantum connection between X^3_1 and X^3_2 at the boundary A of CD exists if $X^3_1 = CPT(Y^3_1)$ is true for a PE having X^3_1 and Y^3_1 as ends. $\Delta n = \pm 1$ translates to an appearance or disappearance of minimal number of critical directions. The attribute "quantum" is well-deserved since the classical non-determinism serves as a space-time correlate for quantum jumps at WCW level [L101, L87, L102, L103].

11.3.2 A more concrete proposal for WCW homology as a generalization of the Floer homology

Consider first the notion of "world of classical worlds" (WCW).

1. In TGD, point-like particles are replaced by 3-surfaces. Zero energy ontology (ZEO) is assumed, which means that space-time surfaces X^4 as "orbits" of 3-surfaces are inside causal diamonds. These 4-surfaces are PEs of the action principle. For the exact holography, 3-surface at either boundary of CD would determine X^4 uniquely but determinism is expected to be slightly violated so that there are several PEs associated with a given X^3 at either boundary of CD. The failure of strict determinism is analogous to the failure of determinism for soap films with frames.

   Let PE have X^3 *resp.* Y^3 as its ends at the opposite boundaries A *resp.* B of CD.

2. WCW is identified as the space of PEs. One could regard WCW also as covering a space such that for a given X^3 at (say) A, the fiber contains the PEs having X^3 as the first end. WCW has symplectic and even a Kähler structure and symplectic transformations at the light-like boundaries of CD are conjectured to define isometries of WCW but not symmetries of S_K.

3. Kähler function K, serving as the analog of symplectic action, defines Kähler form and symplectic structure. K corresponds to 4-D Kähler action S_K plus volume term for a PE. This action is obtained as a dimensional reduction of 6-D Kähler action for the 6-D surface X^6 in the 6+6-D twistor space of $T(M^4) \times T(CP_2)$. X^6 carries induced twistor structure and has X^4 as base space and S^2 as fiber.

WCW homology based on minimal surfaces with singularities

The challenge is to identify the counterpart of gradient flow as a counterpart of quantum connectivity. This should not bring anything new to the existing picture. The following proposal is perhaps the simplest one and conforms with the physical intuition.

1. Morse theory and Floer homology would suggest that one should consider the Hessian of Kähler function $K(PE(X^3)$ of WCW as functional of preferred extremal $PE(X^3, Y^3)$. One could calculate the numbers n_+ resp. n_- of positive and negative eigenvalues of Hessian and identify n_- as the criticality index and number of unstable directions.

2. There are several problems. The identification of the analog of gradient flow seems very difficult. However, by the weak holography due the failure of strict determinism, for a given X^3, there are several 3-surfaces Y_i^3 at the opposite boundary of CD defining PEs. The meaning of criticality is far from obvious since instability for a given time direction looks like stability in the opposite time direction. This is a potential problem since in ZEO [L68] [K99] both arrows of time are possible. There should be a clear distinction between the ends of a CD.

3. By the failure of the strict determinism, the basic objects in ZEO are pairs (X_i^3, Y_j^3) connected by $PE(X_i^3, Y_j^3)$ identifiable as critical points of K with respect to variations of at least one end. The physical picture suggests that criticality is possible for both ends and that a maximum for the passive boundary of CD and criticality for the opposite active boundary of CD (where quantum fluctuations due to "small" state function reductions (SSFRs) are located) is possible. The instabilities associated with criticality at active end would correspond to a definite time direction. It is however difficult to proceed without a more concrete picture.

4. WCW homology could also involve a generalization of the notion of quantum connectivity crucial for the definition of Gromow-Witten invariants. The idea is that two surfaces (say branes) can be said to intersect when there is a string world sheet connecting them, generalizes.

   In ZEO this translates to the existence of a preferred extremal (PE), which by the weak form of holography is almost unique, such that it connects the boundaries of causal diamond (CD), which plays the role of brane.

The identification of PEs as minimal surfaces [L101] allows us to make this picture more concrete and gives a direct connection to quantum criticality as it would be realized in terms of classical non-determinism. One would not count critical directions but critical transitions assignable to singularities of minimal surfaces.

1. PEs are identified as minimal surfaces with singularities analogous to the frames of soap film. At the singularities the minimal surface property fails and the Kähler action and volume term couple together in field equations so that conservation laws are satisfied.

2. The singular surfaces have dimension $d < 4$ and and can be regarded as loci of instability leading to non-determinism. By suitably perturbing the singularities, one can generate new

preferred extremals $PE(X^3, Y_j^3)$ from $PE(X^3, Y_i^3)$. The maximum property of K with respect to the variations of X^3 would suggest that one cannot replace X^3 with a new maximum in this way.

3. For each Y_i^3, one can count the number of deformations of the singularities leading to $PE(X^3, Y_j^3)$ and call this number $n(X^3, Y_i^3 \rightarrow Y_j^3)$ as an the analog for the number of gradient lines between given critical points in Floer homology.

   One can define the analog of criticality index $n(X^3, Y_i^3)$ as $n(X^3, Y_i^3) = \sum_j n(X^3, Y_i^3 \rightarrow Y_j^3)$ as the analog of n_- of the negative eigenvalues of Hessian. One defines an Abelian group as the complex formed by $PE(X_i^3, Y_j^3)$. $n(X^3, Y_i^3)$ defines the grading for $PE(X^3, Y_j^3)$ as an element of this complex.

   The differential d for WCW homology can be defined in the same way as in Floer homology. Assume $n(X^3, Y_j^3) - n(X^3, Y_i^3) = 1$. Define the action of d as $d(PE(X^3, Y^i)) = \sum_j n(X^3, Y_i^3 \rightarrow Y_j^3) PE(X^3, Y^j)$.

The non-determinism of 6-D Kähler and 4-D action would be essential as also the asymmetry between the active and passive boundaries of CD crucial for TGD based quantum measurement theory. Nondeterminism is also essential for the non-triviality of scattering amplitudes since quantum non-determinism in WCW degrees of freedom has classical non-determinism as a space-time correlate [L101]. If the determinism were exact the homology groups H_n would correspond directly to the groups C_n and one would have a Cartesian product of spaces with the homology group $H_n = C_n$. Interesting questions relate to the interpretation of PE pairs with $\Delta n \neq 1$.

Could CPT allow the concretization of quantum connectedness

The quantum connectedness in some sense identifies the 3-surfaces connected by $PE(X_1^3, Y_1^3)$ such that X_1^3 and Y_1^3 are at opposite boundaries of $CD = cd \times CP_2$. If one could assign to Y_1^3 at B a 3-surface X_2^3 at A, quantum connectedness would become more concrete. There is no compelling reason to effectively for this but can ask whether PE could allow to achieve this formally.

1. This formal connection is achieved if there is a discrete symmetry mapping the boundaries A and B of CD to each other. This symmetry must involve time reflection T with respect to the center point of cd. If one requires that the symmetry is an exact symmetry of quantum theory, CPT remains the only candidate. C would act as charge conjugation, realized as a complex conjugation in CP_2.

   CPT maps the boundaries of CDs to each other and therefore also the positive and negative energy parts of zero energy states. The 3-surfaces $(X_1^3\ X_2^3)$ at a given boundary of CD are quantum connected if one has $X_2^3 = CPT(Y_1^3)$ for a PE connecting X_1^3 and Y_1^3.

2. Critical points of K must be mapped to critical points so that CPT should act as a symmetry of the variational principle. If M^4 has Kähler structure the self-dual covariantly constant Kähler form of M^4, strongly suggested by the twistor lift of TGD, must be invariant under CPT and this is indeed the case. The Kähler gauge potential would be also fixed apart from the decomposition $M^4 = M^2 \times E^2$ defined by electric and magnetic parts of $J(M^4)$.

Part II

RELATED TOPICS

Chapter 12

The Classical Part of the Twistor Story

12.1 Introduction

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D embedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the embedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD [K87, K58, K72] and classical TGD [K17] defined by the extremals of Kähler action.

In the following I summarize first the basic results and problems of the twistor approach. After that I describe some of the mathematical background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams having as lines space-time surfaces with Euclidian signature of induced metric and having wormhole contacts as basic building bricks.

There is also a very close analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds [A3, A99] and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry [B16]emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. Twistor space has space-time as base-space rather than forming with it Cartesian factors of a 10-D space-time. The Calabi-Yau landscape is replaced with the space of twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings [B28]. The space of twistor spaces is the lift of the "world of classical worlds" (WCW) by adding the CP_1 fiber to the space-time surfaces so that the analog of landscape has beautiful geometrization.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach.

1. The notion of quaternion analyticity extending the notion of ordinary analyticity to 4-D context is highly attractive but has remained one of the long-standing ideas difficult to take quite seriously but equally difficult to throw to paper basked. Four-manifolds possess almost quaternion structure. In twistor space context the formulation of quaternion analyticity becomes possible and relies on an old notion of tri-holomorphy about which I had not been aware

earlier. The natural formulation for the preferred extremal property is as a condition stating that various charges associated with generalized conformal algebras vanish for preferred extremals. This leads to ask whether Euclidian space-time regions could be quaternion-Kähler manifolds for which twistor spaces are so called Fano spaces. In Minkowskian regions so called Hamilton-Jacobi property would apply.

2. The generalization of Witten's twistor theory to TGD framework is a natural challenge and the 2-surfaces studied defining scattering amplitudes in Witten's theory could correspond to partonic 2-surfaces identified as algebraic surfaces characterized by degree and genus. Besides this also string world sheets are needed. String worlds have 1-D lines at the light-like orbits of partonic 2-surfaces as their boundaries serving as carriers of fermions. This leads to a rather detailed generalization of Witten's approach using the generalization of twistors to 8-D context.

3. The generalization of the twistor Grassmannian approach to 8-D context is second fascinating challenge. If one requires that the basic formulas relating twistors and four-momentum generalize one must consider the situation in tangent space M^8 of embedding space ($M^8 - H$ duality) and replace the usual sigma matrices having interpretation in terms of complexified quaternions with octonionic sigma matrices.

   The condition that octonionic spinors are are equivalent with ordinary spinors has strong consequences. Induced spinors must be localized to 2-D string world sheets, which are (co-)commutative sub-manifolds of (co-)quaternionic space-time surface. Also the gauge fields should vanish since they induce a breaking of associativity even for quaternionic and complex surface so that CP_2 projection of string world sheet must be 1-D. If one requires also the vanishing of gauge potentials, the projection is geodesic circle of CP_2 so that string world sheets are restricted to Minkowskian space-time regions. Although the theory would be free in fermionic degrees of freedom, the scattering amplitudes are non-trivial since vertices correspond to partonic 2-surfaces at which partonic orbits are glued together along common ends. The classical light-like 8-momentum associated with the boundaries of string world sheets defines the gravitational dual for 4-D momentum and color quantum numbers associated with imbedding space spinor harmonics. This leads to a more detailed formulation of Equivalence Principle which would reduce to $M^8 - H$ duality basically.

   Number theoretic interpretation of the positivity of Grassmannians is highly suggestive since the canonical identification maps p-adic numbers to non-negative real numbers. A possible generalization is obtained by replacing positive real axis with upper half plane defining hyperbolic space having key role in the theory of Riemann surfaces. The interpretation of scattering amplitudes as representations of permutations generalizes to interpretation as braidings at surfaces formed by the generalized Feynman diagrams having as lines the light-like orbits of partonic surfaces. This because 2-fermion vertex is the only interaction vertex and induced by the non-continuity of the induced Dirac operator at partonic 2-surfaces. OZI rule generalizes and implies an interpretation in terms of braiding consistent with the TGD as almost topological QFT vision. This suggests that non-planar twistor amplitudes are constructible as analogs of knot and braid invariants by a recursive procedure giving as an outcome planar amplitudes.

4. Yangian symmetry is associated with twistor amplitudes and emerges in TGD from completely different idea interpreting scattering amplitudes as representations of algebraic manipulation sequences of minimal length (preferred extremal instead of path integral over space-time surfaces) connecting given initial and final states at boundaries of causal diamond. The algebraic manipulations are carried out in Yangian using product and co-product defining the basic 3-vertices analogous to gauge boson absorption and emission. 3-surface representing elementary particle splits into two or vice versa such that second copy carries quantum numbers of gauge boson or its super counterpart. This would fix the scattering amplitude for given 3-surface and leave only the functional integral over 3-surfaces.

12.2 Background And Motivations

In the following some background plus basic facts and definitions related to twistor spaces are summarized. Also reasons for why twistor are so relevant for TGD is considered at general level.

12.2.1 Basic Results And Problems Of Twistor Approach

The author describes both the basic ideas and results of twistor approach as well as the problems.

Basic results

There are three deep results of twistor approach besides the impressive results which have emerged after the twistor resolution.

1. Massless fields of arbitrary helicity in 4-D Minkowski space are in 1-1 correspondence with elements of Dolbeault cohomology in the twistor space CP_3. This was already the discovery of Penrose..The connection comes from Penrose transform. The light-like geodesics of M^4 correspond to points of 5-D sub-manifold of CP_3 analogous to light-cone boundary. The points of M^4 correspond to complex lines (Riemann spheres) of the twistor space CP_3: one can imagine that the point of M^4 corresponds to all light-like geodesics emanating from it and thus to a 2-D surface (sphere) of CP_3. Twistor transform represents the value of a massless field at point of M^4 as a weighted average of its values at sphere of CP_3. This correspondence is formulated between open sets of M^4 and of CP_3. This fits very nicely with the needs of TGD since causal diamonds which can be regarded as open sets of M^4 are the basic objects in zero energy ontology (ZEO).

2. Self-dual instantons of non-Abelian gauge theories for SU(n) gauge group are in one-one correspondence with holomorphic rank-N vector bundles in twistor space satisfying some additional conditions. This generalizes the correspondence of Penrose to the non-Abelian case. Instantons are also usually formulated using classical field theory at four-sphere S^4 having Euclidian signature.

3. Non-linear gravitons having self-dual geometry are in one-one correspondence with spaces obtained as complex deformations of twistor space satisfying certain additional conditions. This is a generalization of Penrose's discovery to the gravitational sector.

Complexification of M^4 emerges unavoidably in twistorial approach and Minkowski space identified as a particular real slice of complexified M^4 corresponds to the 5-D subspace of twistor space in which the quadratic form defined by the SU(2,2) invariant metric of the 8-dimensional space giving twistor space as its projectivization vanishes. The quadratic form has also positive and negative values with its sign defining a projective invariant, and this correspond to complex continuations of M^4 in which positive/negative energy parts of fields approach to zero for large values of imaginary part of M^4 time coordinate.

Interestgingly, this complexification of M^4 is also unavoidable in the number theoretic approach to TGD: what one must do is to replace 4-D Minkowski space with a 4-D slice of 8-D complexified quaternions. What is interesting is that real M^4 appears as a projective invariant consisting of light-like projective vectors of C^4 with metric signature (4,4). Equivalently, the points of M^4 represented as linear combinations of sigma matrices define hermitian matrices.

Basic problems of twistor approach

The best manner to learn something essential about a new idea is to learn about its problems. Difficulties are often put under the rug but the thesis is however an exception in this respect. It starts directly from the problems of twistor approach. There are two basic challenges.

1. Twistor approach works as such only in the case of Minkowski space. The basic condition for its applicability is that the Weyl tensor is self-dual. For Minkowskian signature this leaves only Minkowski space under consideration. For Euclidian signature the conditions are not

quite so restrictive. This looks a fatal restriction if one wants to generalize the result of Penrose to a general space-time geometry. This difficulty is known as "googly" problem.

According to the thesis MHV construction of tree amplitudes of $\mathcal{N}=4$ SYM based on topological twistor strings in CP_3 meant a breakthrough and one can indeed understand also have analogs of non-self-dual amplitudes. The problem is however that the gravitational theory assignable to topological twistor strings is conformal gravity, which is generally regarded as non-physical. There have been several attempts to construct the on-shell scattering amplitudes of Einstein's gravity theory as subset of amplitudes of conformal gravity and also thesis considers this problem.

2. The construction of quantum theory based on twistor approach represents second challenge. In this respect the development of twistor approach to $\mathcal{N}=4$ SYM meant a revolution and one can indeed construct twistorial scattering amplitudes in M^4.

12.2.2 Results About Twistors Relevant For TGD

First some background.

1. The twistors originally introduced by Penrose (1967) have made breakthrough during last decade. First came the twistor string theory of Edward Witten [B28] proposed twistor string theory and the work of Nima-Arkani Hamed and collaborators [B31] led to a revolution in the understanding of the scattering amplitudes of scattering amplitudes of gauge theories [B21, B20, B33]. Twistors do not only provide an extremely effective calculational method giving even hopes about explicit formulas for the scattering amplitudes of $\mathcal{N}=4$ supersymmetric gauge theories but also lead to an identification of a new symmetry: Yangian symmetry [A36], [B24, B25], which can be seen as multilocal generalization of local symmetries.

   This approach, if suitably generalized, is tailor-made also for the needs of TGD. This is why I got seriously interested on whether and how the twistor approach in empty Minkowski space M^4 could generalize to the case of $H = M^4 \times CP_2$. The twistor space associated with H should be just the cartesian product of those associated with its Cartesian factors. Can one assign a twistor space with CP_2?

2. First a general result [A74] deserves to be mentioned: any oriented manifold X with Riemann metric allows 6-dimensional twistor space Z as an almost complex space. If this structure is integrable, Z becomes a complex manifold, whose geometry describes the conformal geometry of X. In general relativity framework the problem is that field equations do not imply conformal geometry and twistor Grassmann approach certainly requires conformal structure.

3. One can consider also a stronger condition: what if the twistor space allows also Kähler structure? The twistor space of empty Minkowski space M^4 (and its Euclidian counterpart S^4 is the Minkowskian variant of $P_3 = SU(2,2)/SU(2,1)\times U(1)$ of 3-D complex projective space $CP_3 = SU(4)/SU(3)\times U(1)$ and indeed allows Kähler structure.

   The points of the Euclidian twistor space $CP_3 = SU(4)/SU(3)\times U(1)$ can be represented by any column of the 4×4 matrix representing element of $SU(4)$ with columns differing by phase multiplication being identified. One has four coordinate charts corresponding to four different choices of the column. The points of its Minkowskian variant $CP_{2,1} = SU(2,2)/SU(2,1)\times U(1)$ can be represented in similar manner as U(1) gauge equivalence classes for given column of SU(3,1) matrices, whose rows and columns satisfy orthonormality conditions with respect to the hermitian inner product defined by Minkowskian metric $\epsilon = (1,1,-1,-1)$.

   Rather remarkably, there are *no other space-times* with Minkowski signature allowing twistor space with Kähler structure [A74]. Does this mean that the empty Minkowski space of special relativity is much more than a limit at which space-time is empty?

   This also means a problem for GRT. Twistor space with Kähler structure seems to be needed but general relativity does not allow it. Besides twistor problem GRT also has energy problem: matter makes space-time curved and the conservation laws and even the definition of energy and momentum are lost since the underlying symmetries giving rise to the conservation laws through Noether's theorem are lost. GRT has therefore two bad mathematical problems which might explain why the quantization of GRT fails. This would not be surprising since quantum theory is to high extent representation theory for symmetries and symmetries are lost. Twistors would extend these symmetries to Yangian symmetry but GRT does not allow them.

4. What about twistor structure in CP_2? CP_2 allows complex structure (Weyl tensor is self-dual), Kähler structure plus accompanying symplectic structure, and also quaternion structure. One of the really big personal surprises of the last years has been that CP_2 twistor space indeed allows Kähler structure meaning the existence of antisymmetric tensor representing imaginary unit whose tensor square is the negative of metric in turn representing real unit.

   The article by Nigel Hitchin, a famous mathematical physicist, describes a detailed argument identifying S^4 and CP_2 as the only compact Riemann manifolds allowing Kählerian twistor space [A74]. Hitchin sent his discovery for publication 1979. An amusing co-incidence is that I discovered CP_2 just this year after having worked with S^2 and found that it does not really allow to understand standard model quantum numbers and gauge fields. It is difficult to avoid thinking that maybe synchrony indeed a real phenomenon as TGD inspired theory of consciousness predicts to be possible but its creator cannot quite believe. Brains at different side of globe discover simultaneously something closely related to what some conscious self at the higher level of hierarchy using us as instruments of thinking just as we use nerve cells is intensely pondering.

   Although 4-sphere S^4 allows twistor space with Kähler structure, it does not allow Kähler structure and cannot serve as candidate for S in $H = M^4 \times S$. As a matter of fact, S^4 can be seen as a Wick rotation of M^4 and indeed its twistor space is CP_3.

   In TGD framework a slightly different interpretation suggests itself. The Cartesian products of the intersections of future and past light-cones - causal diamonds (CDs) - with CP_2 - play a key role in ZEO (ZEO) [K4]. Sectors of "world of classical worlds" (WCW) [K39, K21] correspond to 4-surfaces inside $CD \times CP_2$ defining a the region about which conscious observer can gain conscious information: state function reductions - quantum measurements - take place at its light-like boundaries in accordance with holography. To be more precise, wave functions in the moduli space of CDs are involved and in state function reductions come as sequences taking place at a given fixed boundary. This kind of sequence is identifiable as self and give rise to the experience about flow of time. When one replaces Minkowski metric with Euclidian metric, the light-like boundaries of CD are contracted to a point and one obtains topology of 4-sphere S^4.

5. Another really big personal surprise was that there are *no other* compact 4-manifolds with Euclidian signature of metric allowing twistor space with Kähler structure! The embedding space $H = M^4 \times CP_2$ is not only physically unique since it predicts the quantum number spectrum and classical gauge potentials consistent with standard model but also mathematically unique!

   After this I dared to predict that TGD will be the theory next to GRT since TGD generalizes string model by bringing in 4-D space-time. The reasons are many-fold: TGD is the only known solution to the two big problems of GRT: energy problem and twistor problem. TGD is consistent with standard model physics and leads to a revolution concerning the identification of space-time at microscopic level: at macroscopic level it leads to GRT but explains some of its anomalies for which there is empirical evidence (for instance, the observation that neutrinos arrived from SN1987A at two different speeds different from light velocity [**?**] has natural explanation in terms of many-sheeted space-time). TGD avoids the landscape problem of M-theory and anthropic non-sense. I could continue the list but I think that this is enough.

6. The twistor space of CP_2 is 3-complex dimensional flag manifold $F_3 = SU(3)/U(1) \times U(1)$ having interpretation as the space for the choices of quantization axes for the color hypercharge and isospin. This choice is made in quantum measurement of these quantum numbers and a means localization to single point in F_3. The localization in F_3 could be higher level measurement leading to the choice of quantizations for the measurement of color quantum numbers.

F_3 is symmetric space meaning that besides being a coset space with $SU(3)$ invariant metric it also has involutions acting as a reflection at geodesics through a point remaining fixed under the involution. As a symmetric space with Fubini-Study metric F_3 is positive constant curvature space having thus positive constant sectional curvatures. This implies Einstein space property. This also conforms with the fact that F_3 is CP_1 bundle over CP_2 as base space (for more details see `http://tinyurl.com/ychdeqjz`).

The points of flag manifold $SU(3)/U(1)\times U(1)$ can be represented locally by identifying $SU(3)$ matrices for which columns differ by multiplication from left with exponential $e^{i(aY+bI_3)}$, a and b arbitrary real numbers. This transformation allows what might be called a "gauge choice". For instance, first two elements of the first row can be made real in this manner. These coordinates are not global.

7. Analogous interpretation could make sense for M^4 twistors represented as points of P_3. Twistor corresponds to a light-like line going through some point of M^4 being labelled by 4 position coordinates and 2 direction angles: what higher level quantum measurement could involve a choice of light-like line going through a point of M^4? Could the associated spatial direction specify spin quantization axes? Could the associated time direction specify preferred rest frame? Does the choice of position mean localization in the measurement of position? Do momentum twistors relate to the localization in momentum space? These questions remain fascinating open questions and I hope that they will lead to a considerable progress in the understanding of quantum TGD.

8. It must be added that the twistor space of CP_2 popped up much earlier in a rather unexpected context [K33]: I did not of course realize that it was twistor space. Topologist Barbara Shipman [A45] has proposed a model for the honeybee dance leading to the emergence of F_3. The model led her to propose that quarks and gluons might have something to do with biology. Because of her position and specialization the proposal was forgiven and forgotten by community. TGD however suggests both dark matter hierarchies and p-adic hierarchies of physics [K29, ?]. For dark hierarchies the masses of particles would be the standard ones but the Compton scales would be scaled up by $h_{eff}/h = n$ [?]. Below the Compton scale one would have effectively massless gauge boson: this could mean free quarks and massless gluons even in cell length scales. For p-adic hierarchy mass scales would be scaled up or down from their standard values depending on the value of the p-adic prime.

12.2.3 Basic Definitions Related To Twistor Spaces

One can find from web several articles explaining the basic notions related to twistor spaces and Calabi-Yau manifolds. At the first look the notions of twistor as it appears in the writings of physicists and mathematicians don't seem to have much common with each other and it requires effort to build the bridge between these views. The bridge comes from the association of points of Minkowski space with the spheres of twistor space: this clearly corresponds to a bundle projection from the fiber to the base space, now Minkowski space. The connection of the mathematician's formulation with spinors remains still somewhat unclear to me although one can understand CP_1 as projective space associated with spinors with 2 complex components. Minkowski signature poses additional challenges. In the following I try my best to summarize the mathematician's view, which is very natural in classical TGD.

There are many variants of the notion of twistor depending on whether how powerful assumptions one is willing to make. The weakest definition of twistor space is as CP_1 bundle of almost complex structures in the tangent spaces of an orientable 4-manifold. Complex structure at given point means selection of antisymmetric form J whose natural action on vector rotates a vector in the plane defined by it by $\pi/2$ and thus represents the action of imaginary unit. One must perform this kind of choice also in normal plane and the direct sum of the two choices defines the full J. If one chooses J to be self-dual or anti-self-dual (eigenstate of Hodge star operation), one can fix J uniquely. Orientability makes possible the Hodge star operation involving 4-dimensional permutation tensor.

The condition $i^1 = -1$ is translated to the condition that the tensor square of J equals to $J^2 = -g$. The possible choices of J span sphere S^2 defining the fiber of the twistor spaces. This is not quite the complex sphere CP_1, which can be thought of as a projective space of spinors with two complex components. Complexification must be performed in both the tangent space of X^4 and of S^2. Note that in the standard approach to twistors the entire 6-D space is projective space P_3 associated with the C^8 having interpretation in terms of spinors with 4 complex components.

One can introduce almost complex structure also to the twistor space itself by extending the almost complex structure in the 6-D tangent space obtained by a preferred choice of J by identifying it as a point of S^2 and acting in other points of S^2 identified as antisymmetric tensors. If these points are interpreted as imaginary quaternion units, the action is commutator action divided by 2. The existence of quaternion structure of space-time surfaces in the sense as I have proposed in TGD framework might be closely related to the twistor structure.

Twistor structure as bundle of almost complex structures having itself almost complex structure is characterized by a hermitian Kähler form ω defining the almost complex structure of the twistor space. Three basic objects are involved: the hermitian form h, metric g and Kähler form ω satisfying $h = g + i\omega$, $g(X,Y) = \omega(X,JY)$.

In the base space the metric of twistor space is the metric of the base space and in the tangent space of fibre the natural metric in the space of antisymmetric tensors induced by the metric of the base space. Hence the properties of the twistor structure depend on the metric of the base space.

The relationship to the spinors requires clarification. For 2-spinors one has natural Lorentz invariant antisymmetric bilinear form and this seems to be the counterpart for J?

One can consider various additional conditions on the definition of twistor space.

1. Kähler form ω is not closed in general. If it is, it defines symplectic structure and Kähler structure. S^4 and CP_2 are the only compact spaces allowing twistor space with Kähler structure [A74].

2. Almost complex structure is not integrable in general. In the general case integrability requires that each point of space belongs to an open set in which vector fields of type (1, 0) or (0, 1) having basis ∂/∂_{z^k} and $\partial/\partial_{\overline{z}^k}$ expressible as linear combinations of real vector fields with complex coefficients commute to vector fields of same type. This is non-trivial conditions since the leading names for the vector field for the partial derivatives does not yet guarantee these conditions.

   This necessary condition is also enough for integrability as Newlander and Nirenberg have demonstrated. An explicit formulation for the integrability is as the vanishing of Nijenhuis tensor associated with the antisymmetric form J (see (`http://tinyurl.com/ybp9vsa5` and `http://tinyurl.com/y8j36p4m`). Nijenhuis tensor characterizes Nijenhuis bracket generalizing ordinary Lie bracket of vector fields (for detailed formula see `http://tinyurl.com/y83mbnso`).

3. In the case of twistor spaces there is an alternative formulation for the integrability. Curvature tensor maps in a natural manner 2-forms to 2-forms and one can decompose the Weyl tensor W identified as the traceless part of the curvature tensor to self-dual and anti-self-dual parts W^+ and W^-, whose actions are restricted to self-dual resp. antiself-dual forms (self-dual and anti-self-dual parts correspond to eigenvalue + 1 and -1 under the action of Hodge * operation: for more details see `http://tinyurl.com/ybkhj4np`). If W^+ or W^- vanishes - in other worlds W is self-dual or anti-self-dual - the assumption that J is self-dual or anti-self-dual guarantees integrability. One says that the metric is anti-self-dual (ASD). Note that the vanishing of Weyl tensor implies local conformal flatness (M^4 and sphere are obviously conformally flat). One might think that ASD condition guarantees that the parallel translation leaves J invariant.

   ASD property has a nice implication: the metric is balanced. In other words one has $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$.

4. If the existence of complex structure is taken as a part of definition of twistor structure, one encounters difficulties in general relativity. The failure of spin structure to exist is similar difficulty: for CP_2 one must indeed generalize the spin structure by coupling Kähler gauge potential to the spinors suitably so that one obtains gauge group of electroweak interactions.

5. One could also give up the global existence of complex structure and require symplectic structure globally: this would give $d\omega = 0$. A general result is that hyperbolic 4-manifolds allow symplectic structure and ASD manifolds allow complex structure and hence balanced metric.

12.2.4 Why Twistor Spaces With Kähler Structure?

I have not yet even tried to answer an obvious question. Why the fact that M^4 and CP_2 have twistor spaces with Kähler structure could be so important that it could fix the entire physics? Let us consider a less general question. Why they would be so important for the classical TGD - exact part of quantum TGD - defined by the extremals of Kähler action [K10] ?

1. Properly generalized conformal symmetries are crucial for the mathematical structure of TGD [K21, K96, K20, K86]. Twistor spaces have almost complex structure and in these two special cases also complex, Kähler, and symplectic structures (note that the integrability of the almost complex structure to complex structure requires the self-duality of the Weyl tensor of the 4-D manifold).

   For years ago I considered the possibility that complex 3-manifolds of $CP_3 \times CP_3$ could have the structure of S^2 fiber space and have space-time surfaces as base space. I did not realize that these spaces could be twistor spaces nor did I realize that CP_2 allows twistor space with Kähler structure so that $CP_3 \times F_3$ looks a more plausible choice.

   The expectation was that the Cartesian product $CP_3 \times F_3$ of the two twistor spaces with Kähler structure is fundamental for TGD. The obvious wishful thought is that this space makes possible the construction of the extremals of Kähler action in terms of holomorphic surfaces defining 6-D twistor sub-spaces of $CP_3 \times F_3$ allowing to circumvent the technical problems due to the signature of M^4 encountered at the level of $M^4 \times CP_2$. It would also make the magnificent machinery of the algebraic geometry so powerful in string theories a tool of TGD. Here CP_3 could be replaced with its non-compact form and the problem is that one can have only compactification of M^4 for which metric is defined only modulo conformal scaling. There is however a problem: the compactified Minkowski space or its complexification has a metric defined only modulo conformal factor. This is not a problem in conformally invariant theories but becomes a problem if one wants to speak of induced metric.

   The next realization was that M^4 allows twistor bundle also in purely geometric sense and this bundle is just $T(M^4) = M^4 \times CP_2$. The two variants of twistor space would naturally apply at the level of momentum space and embedding space.

2. Every 4-D orientable Riemann manifold allows a twistor space as 6-D bundle with CP_1 as fiber and possessing almost complex structure. Metric and various gauge potentials are obtained by inducing the corresponding bundle structures. Hence the natural guess is that the twistor structure of space-time surface defined by the induced metric is obtained by induction from that for $T(M^4) \times F_3$ by restricting its twistor structure to a 6-D (in real sense) surface of $T(M^4) \times F_3$ with a structure of twistor space having at least almost complex structure with CP_1 as a fiber. For the embedding of the twistor space of space-time this requires the identification of S^2 fibers of $T(M^4)$ and F_3. If so then one can indeed identify the base space as 4-D space-time surface in $M^4 \times CP_2$ using bundle projections in the factors $T(M^4)$ and F_3.

3. There might be also a connection to the number theoretic vision about the extremals of Kähler action. At space-time level however complexified quaternions and octonions could allow alternative formulation. I have indeed proposed that space-time surfaces have associative of co-associative meaning that the tangent space or normal space at a given point belongs to quaternionic subspace of complexified octonions.

12.3 The Identification Of 6-D Twistor Spaces As Sub-Manifolds Of 12-D Twistor Space

How to identify the 6-D sub-manifolds with the structure of twistor space? Is this property all that is needed? Can one find a simple solution to this condition? What is the relationship of twistor spaces to the Calabi-Yau manifolds of super string models? In the following intuitive considerations of a simple minded physicist. Mathematician could probably make much more interesting comments.

12.3.1 Conditions For Twistor Spaces As Sub-Manifolds

Consider the conditions that must be satisfied using local trivializations of the twistor spaces. It will be assumed that the twistor space $T(M^4)$ is CP_3 or its Minkowskian variant. It has turned out that a more reasonable option $T(M^4) = M^4 \times CP_1$ is possible. The following consideration is however for CP_3 option. Before continuing let us introduce complex coordinates $z_i = x_i + iy_i$ resp. $w_i = u_i + iv_i$ for CP_3 resp. F_3.

1. 6 conditions are required and they must give rise by bundle projection to 4 conditions relating the coordinates in the Cartesian product of the base spaces of the two bundles involved and thus defining 4-D surface in the Cartesian product of compactified M^4 and CP_2.

2. One has Cartesian product of two fiber spaces with fiber CP_1 giving fiber space with fiber $CP_1^1 \times CP_1^2$. For the 6-D surface the fiber must be CP_1. It seems that one must identify the two spheres CP_1^i. Since holomorphy is essential, holomorphic identification $w_1 = f(z_1)$ or $z_1 = f(w_1)$ is the first guess. A stronger condition is that the function f is meromorphic having thus only finite numbers of poles and zeros of finite order so that a given point of CP_1^i is covered by CP_1^{i+1}. Even stronger and very natural condition is that the identification is bijection so that only Möbius transformations parametrized by SL(2, C) are possible.

3. Could the Möbius transformation $f : CP_1^1 \to CP_1^2$ depend parametrically on the coordinates z_2, z_3 so that one would have $w_1 = f_1(z_1, z_2, z_3)$, where the complex parameters a, b, c, d $(ad - bc = 1)$ of Möbius transformation depend on z_2 and z_3 holomorphically? Does this mean the analog of local SL(2, C) gauge invariance posing additional conditions? Does this mean that the twistor space as surface is determined up to SL(2, C) gauge transformation?

   What conditions can one pose on the dependence of the parameters a, b, c, d of the Möbius transformation on (z_2, z_3)? The spheres CP_1 defined by the conditions $w_1 = f(z_1, z_2, z_3)$ and $z_1 = g(w_1, w_2, w_3)$ must be identical. Inverting the first condition one obtains $z_1 = f^{-1}(w_1, z_2, z_3)$. If one requires that his allows an expression as $z_1 = g(w_1, w_2, w_3)$, one must assume that z_2 and z_3 can be expressed as holomorphic functions of (w_2, w_3): $z_i = f_i(w_k)$, $i = 2, 3$, $k = 2, 3$. Of course, non-holomorphic correspondence cannot be excluded.

4. Further conditions are obtained by demanding that the known extremals - at least non-vacuum extremals - are allowed. The known extremals [K10] can be classified into CP_2 type vacuum extremals with 1-D light-like curve as M^4 projection, to vacuum extremals with CP_2 projection, which is Lagrangian sub-manifold and thus at most 2-dimensional, to massless extremals with 2-D CP_2 projection such that CP_2 coordinates depend on arbitrary manner on light-like coordinate defining local propagation direction and space-like coordinate defining a local polarization direction, and to string like objects with string world sheet as M^4 projection (minimal surface) and 2-D complex sub-manifold of CP_2 as CP_2 projection, . There are certainly also other extremals such as magnetic flux tubes resulting as deformations of string like objects. Number theoretic vision relying on classical number fields suggest a very general construction based on the notion of associativity of tangent space or co-tangent space.

5. The conditions coming from these extremals reduce to 4 conditions expressible in the holomorphic case in terms of the base space coordinates (z_2, z_3) and (w_2, w_3) and in the more general case in terms of the corresponding real coordinates. It seems that holomorphic ansatz

is not consistent with the existence of vacuum extremals, which however give vanishing contribution to transition amplitudes since WCW ("world of classical worlds") metric is completely degenerate for them.

The mere condition that one has CP_1 fiber bundle structure does not force field equations since it leaves the dependence between real coordinates of the base spaces free. Of course, CP_1 bundle structure alone does not imply twistor space structure. One can ask whether non-vacuum extremals could correspond to holomorphic constraints between (z_2, z_3) and (w_2, w_3).

6. The metric of twistor space is not Kähler in the general case. However, if it allows complex structure there is a Hermitian form ω, which defines what is called balanced Kähler form [A98] satisfying $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$: ordinary Kähler form satisfying $d\omega = 0$ is special case about this. The natural metric of compact 6-dimensional twistor space is therefore balanced. Clearly, mere CP_1 bundle structure is not enough for the twistor structure. If the Kähler and symplectic forms are induced from those of $CP_3 \times Y_3$, highly non-trivial conditions are obtained for the embedding of the twistor space, and one might hope that they are equivalent with those implied by Kähler action at the level of base space.

7. Pessimist could argue that field equations are additional conditions completely independent of the conditions realizing the bundle structure! One cannot exclude this possibility. Mathematician could easily answer the question about whether the proposed CP_1 bundle structure with some added conditions is enough to produce twistor space or not and whether field equations could be the additional condition and realized using the holomorphic ansatz.

12.3.2 Twistor Spaces By Adding CP_1 Fiber To Space-Time Surfaces

The physical picture behind TGD is the safest starting point in an attempt to gain some idea about what the twistor spaces look like.

1. Canonical embeddings of M^4 and CP_2 and their disjoint unions are certainly the natural starting point and correspond to canonical embeddings of CP_3 and F_3 to $CP_3 \times F_3$.

2. Deformations of M^4 correspond to space-time sheets with Minkowskian signature of the induced metric and those of CP_2 to the lines of generalized Feynman diagrams. The simplest deformations of M^4 are vacuum extremals with CP_2 projection which is Lagrangian manifold.

   Massless extremals represent non-vacuum deformations with 2-D CP_2 projection. CP_2 coordinates depend on local light-like direction defining the analog of wave vector and local polarization direction orthogonal to it.

   The simplest deformations of CP_2 are CP_2 type extremals with light-like curve as M^4 projection and have same Kähler form and metric as CP_2. These space-time regions have Euclidian signature of metric and light-like 3-surfaces separating Euclidian and Minkowskian regions define parton orbits.

   String like objects are extremals of type $X^2 \times Y^2$, X^2 minimal surface in M^4 and Y^2 a complex sub-manifold of CP_2. Magnetic flux tubes carrying monopole flux are deformations of these.

   Elementary particles are important piece of picture. They have as building bricks wormhole contacts connecting space-time sheets and the contacts carry monopole flux. This requires at least two wormhole contacts connected by flux tubes with opposite flux at the parallel sheets.

3. Space-time surfaces are constructed using as building bricks space-time sheets, in particular massless exrremals, deformed pieces of CP_2 defining lines of generalized Feynman diagrams as orbits of wormhole contacts, and magnetic flux tubes connecting the lines. Space-time surfaces have in the generic case discrete set of self intersections and it is natural to remove them by connected sum operation. Same applies to twistor spaces as sub-manifolds of $CP_3 \times F_3$ and this leads to a construction analogous to that used to remove singularities of Calabi-Yau spaces [A98].

Physical intuition suggests that it is possible to find twistor spaces associated with the basic building bricks and to lift this engineering procedure to the level of twistor space in the sense that the twistor projections of twistor spaces would give these structure. Lifting would essentially mean assigning CP_1 fiber to the space-time surfaces.

1. Twistor spaces should decompose to regions for which the metric induced from the $CP_3 \times F_3$ metric has different signature. In particular, light-like 5-surfaces should replace the light-like 3-surfaces as causal horizons. The signature of the Hermitian metric of 4-D (in complex sense) twistor space is (1, 1, -1, -1). Minkowskian variant of CP_3 is defined as projective space $SU(2,2)/SU(2,1) \times U(1)$. The causal diamond (CD) (intersection of future and past directed light-cones) is the key geometric object in ZEO (ZEO) and the generalization to the intersection of twistorial light-cones is suggestive.

2. Projective twistor space has regions of positive and negative projective norm, which are 3-D complex manifolds. It has also a 5-dimensional sub-space consisting of null twistors analogous to light-cone and has one null direction in the induced metric. This light-cone has conic singularity analogous to the tip of the light-cone of M^4.

   These conic singularities are important in the mathematical theory of Calabi-You manifolds since topology change of Calabi-Yau manifolds via the elimination of the singularity can be associated with them. The S^2 bundle character implies the structure of S^2 bundle for the base of the singularity (analogous to the base of the ordinary cone).

3. Null twistor space corresponds at the level of M^4 to the light-cone boundary (causal diamond has two light-like boundaries). What about the light-like orbits of partonic 2-surfaces whose light-likeness is due to the presence of CP_2 contribution in the induced metric? For them the determinant of induced 4-metric vanishes so that they are genuine singularities in metric sense. The deformations for the canonical embeddings of this sub-space (F_3 coordinates constant) leaving its metric degenerate should define the lifts of the light-like orbits of partonic 2-surface. The singularity in this case separates regions of different signature of induced metric.

   It would seem that if partonic 2-surface begins at the boundary of CD, conical singularity is not necessary. On the other hand the vertices of generalized Feynman diagrams are 3-surfaces at which 3-lines of generalized Feynman digram are glued together. This singularity is completely analogous to that of ordinary vertex of Feynman diagram. These singularities should correspond to gluing together 3 deformed F_3 along their ends.

4. These considerations suggest that the construction of twistor spaces is a lift of construction space-time surfaces and generalized Feynman diagrammatics should generalize to the level of twistor spaces. What is added is CP_1 fiber so that the correspondence would rather concrete.

5. For instance, elementary particles consisting of pairs of monopole throats connected buy flux tubes at the two space-time sheets involved should allow lifting to the twistor level. This means double connected sum and this double connected sum should appear also for deformations of F_3 associated with the lines of generalized Feynman diagrams. Lifts for the deformations of magnetic flux tubes to which one can assign CP_3 in turn would connect the two F_3s.

6. A natural conjecture inspired by number theoretic vision is that Minkowskian and Euclidian space-time regions correspond to associative and co-associative space-time regions. At the level of twistor space these two kinds of regions would correspond to deformations of CP_3 and F_3. The signature of the twistor norm would be different in this regions just as the signature of induced metric is different in corresponding space-time regions.

   These two regions of space-time surface should correspond to deformations for disjoint unions of CP_3s and F_3s and multiple connected sum form them should project to multiple connected sum (wormhole contacts with Euclidian signature of induced metric) for deformed CP_3s. Wormhole contacts could have deformed pieces of F_3 as counterparts.

There are interesting questions related to the detailed realization of the twistor spaces of space-time surfaces.

1. In the case of CP_2 J would naturally correspond to the Kähler form of CP_2. Could one identify J for the twistor space associated with space-time surface as the projection of J? For deformations of CP_2 type vacuum extremals the normalization of J would allow to satisfy the condition $J^2 = -g$. For general extremals this is not possible. Should one be ready to modify the notion of twistor space by allowing this?

2. Or could the associativity/co-associativity condition realized in terms of quaternionicity of the tangent or normal space of the space-time surface guaranteeing the existence of quaternion units solve the problem and J could be identified as a representation of unit quaternion? In this case J would be replaced with vielbein vector and the decomposition 1+3 of the tangent space implied by the quaternion structure allows to use 3-dimensional permutation symbol to assign antisymmetric tensors to the vielbein vectors. Also the triviality of the tangent bundle of 3-D space allowing global choices of the 3 imaginary units could be essential.

3. Does associativity/co-associativity imply twistor space property or could it provide alternative manner to realize this notion? Or could one see quaternionic structure as an extension of almost complex structure. Instead of single J three orthogonal J: s (3 almost complex structures) are introduced and obey the multiplication table of quaternionic units? Instead of S^2 the fiber of the bundle would be $SO(3) = S^3$. This option is not attractive. A manifold with quaternionic tangent space with metric representing the real unit is known as quaternionic Riemann manifold and CP_2 with holonomy U(2) is example of it. A more restrictive condition is that all quaternion units define closed forms: one has quaternion Kähler manifold, which is Ricci flat and has in 4-D case Sp(1)=SU(2) holonomy. (see `http://tinyurl.com/y9qtoebe`).

4. Anti-self-dual property (ASD) of metric guaranteeing the integrability of almost complex structure of the twistor space implies the condition $\omega \wedge d\omega = 0$ for the twistor space. What does this condition mean physically for the twistor spaces associated with the extremals of Kähler action? For the 4-D base space this property is of course identically true. ASD property need of course not be realized.

12.3.3 Twistor Spaces As Analogs Of Calabi-Yau Spaces Of Super String Models

CP_3 is also a Calabi-Yau manifold in the strong sense that it allows Kähler structure and complex structure. Witten's twistor string theory considers 2-D (in real sense) complex surfaces in twistor space CP_3 or its Minkowskian variant. This choice does not however seem to be natural from the point of view of the induced geometry although it looks natural at the level of momentum space. It is less well-known that M^4 allows also second twistor space $T(M^4) = M^4 \times CP_1$, and this looks very natural concerning twistor lift of TGD replacing space-time surfaces in H with their twistor spaces in $T(H) = T(M^4) \times T(CP_2)$.

The original identification $T(M^4)$ with CP_3 or its Minkowskian variant has been given up bit it inspired some questions discussed in the sequel.

1. Could TGD in twistor space formulation be seen as a generalization of this theory?

2. General twistor space is not Calabi-Yau manifold because it does does not have Kähler structure. Do twistor spaces replace Calabi-Yaus in TGD framework?

3. Could twistor spaces be Calabi-Yau manifolds in some weaker sense so that one would have a closer connection with super string models.

Consider the last question.

1. One can indeed define non-Kähler Calabi-Yau manifolds by keeping the hermitian metric and giving up symplectic structure or by keeping the symplectic structure and giving up hermitian metric (almost complex structure is enough). Construction recipes for non-Kähler Calabi-Yau manifold are discussed in [A98]. It is shown that these two ways to give up Kähler structure correspond to duals under so called mirror symmetry [B16] which maps complex and symplectic structures to each other. This construction applies also to the twistor spaces.

2. For the modification giving up symplectic structure, one starts from a smooth Kähler Calabi-Yau 3-fold Y, such as CP_3. One assumes a discrete set of disjoint rational curves diffeomorphic to CP_1. In TGD framework work they would correspond to special fibers of twistor space.

   One has singularities in which some rational curves are contracted to point - in twistorial case the fiber of twistor space would contract to a point - this produces double point singularity which one can visualize as the vertex at which two cones meet (sundial should give an idea about what is involved). One deforms the singularity to a smooth complex manifold. One could interpret this as throwing away the common point and replacing it with connected sum contact: a tube connecting the holes drilled to the vertices of the two cones. In TGD one would talk about wormhole contact.

3. Suppose the topology looks locally like $S^3 \times S^2 \times R_\pm$ near the singularity, such that two copies analogous to the two halves of a cone (sundial) meet at single point defining double point singularity. In the recent case S^2 would correspond to the fiber of the twistor space. S^3 would correspond to 3-surface and $R_\pm$ would correspond to time coordinate in past/future direction. S^3 could be replaced with something else.

   The copies of $S^3 \times S^2$ contract to a point at the common end of R_+ and R_- so that both the based and fiber contracts to a point. Space-time surface would look like the pair of future and past directed light-cones meeting at their tips.

   For the first modification giving up symplectic structure only the fiber S^2 is contracted to a point and $S^2 \times D$ is therefore replaced with the smooth "bottom" of S^3. Instead of sundial one has two balls touching. Drill small holes two the two S^3s and connect them by connected sum contact (wormhole contact). Locally one obtains $S^3 \times S^3$ with k connected sum contacts.

   For the modification giving up Hermitian structure one contracts only S^3 to a point instead of S^2. In this case one has locally two CP_3: s touching (one can think that CP_n is obtained by replacing the points of C^n at infinity with the sphere CP_1). Again one drills holes and connects them by a connected sum contact to get k-connected sum of CP_3.

   For k CP_1s the outcome looks locally like to a k-connected sum of $S^3 \times S^3$ or CP_3 with $k \geq 2$. In the first case one loses symplectic structure and in the second case hermitian structure. The conjecture is that the two manifolds form a mirror pair.

   The general conjecture is that all Calabi-Yau manifolds are obtained using these two modifications. One can ask whether this conjecture could apply also the construction of twistor spaces representable as surfaces in $CP_3 \times F_3$ so that it would give mirror pairs of twistor spaces.

4. This smoothing out procedures isa actually unavoidable in TGD because twistor space is sub-manifold. The 6-D twistor spaces in 12-D $T(M^4) \times F_3$ have in the generic case self intersections consisting of discrete points. Since the fibers CP_1 cannot intersect and since the intersection is point, it seems that the fibers must contract to a point. In the similar manner the 4-D base spaces should have local foliation by spheres or some other 3-D objects with contract to a point. One has just the situation described above.

   One can remove these singularities by drilling small holes around the shared point at the two sheets of the twistor space and connected the resulting boundaries by connected sum contact. The preservation of fiber structure might force to perform the process in such a way that local modification of the topology contracts either the 3-D base (S^3 in previous example or fiber CP_1 to a point.

The interpretation of twistor spaces is of course totally different from the interpretation of Calabi-Yaus in superstring models. The landscape problem of superstring models is avoided and the multiverse of string models is replaced with generalized Feynman diagrams! Different twistor spaces correspond to different space-time surfaces and one can interpret them in terms of generalized Feynman diagrams since bundle projection gives the space-time picture. Mirror symmetry means that there are two different Calabi-Yaus giving the same physics. Also now twistor space for a given space-time surface can have several embeddings - perhaps mirror pairs define this kind of embeddings.

To sum up, the construction of space-times as surfaces of H lifted to those of (almost) complex sub-manifolds in $T(M^4) times F_3$ with induced twistor structure shares the spirit of the vision that induction procedure is the key element of classical and quantum TGD. It also gives deep connection with the mathematical methods applied in super string models and these methods should be of direct use in TGD.

12.4 Witten's Twistor String Approach And TGD

The twistor Grassmann approach has led to a phenomenal progress in the understanding of the scattering amplitudes of gauge theories, in particular the $\mathcal{N}=4$ SUSY.

As a non-specialist I have been frustrated about the lack of concrete picture, which would help to see how twistorial amplitudes might generalize to TGD framework. A pleasant surprise in this respect was the proposal of a particle interpretation for the twistor amplitudes by Nima Arkani Hamed *et al* in the article "Unification of Residues and Grassmannian Dualities" [B34] (see `http://tinyurl.com/y86mad5n`)

In this interpretation incoming particles correspond to spheres CP_1 so that n-particle state corresponds to $(CP_1)^n/Gl(2)$ (the modding by $Gl(2)$ might be seen as a kind of formal generalization of particle identity by replacing permutation group S_2 with $Gl(2)$ of 2×2 matrices). If the number of "wrong" helicities in twistor diagram is k, this space is imbedded to $CP_{k-1}^n/Gl(k)$ as a surface having degree $k-1$ using Veronese map to achieve the embedding. The embedding space can be identified as Grassmannian $G(k,n)$. This surface defines the locus of the multiple residue integral defining the twistorial amplitude.

The particle interpretation brings in mind the extension of single particle configuration space E^3 to its Cartesian power E^{3n}/S_n for n-particle system in wave mechanics. This description could make sense when point-like particle is replaced with 3-surface or partonic 2-surface: one would have Cartesian product of WCWs divided my S_n. The generalization might be an excellent idea as far calculations are considered but is not in spirit with the very idea of string models and TGD that many-particle states correspond to unions of 3-surfaces in H (or light-like boundaries of causal diamond (CD) in Zero Energy Ontology (ZEO).

Witten's twistor string theory [B28] is more in spirit with TGD at fundamental level since it is based on the identification of generalization of vertices as 2-surfaces in twistor space.

1. There are several kinds of twistors involved. For massless external particles in eigenstates of momentum and helicity null twistors code the momentum and helicity and are pairs of 2-spinor and its conjugate. More general momenta correspond to two independent 2-spinors.

   One can perform twistor Fourier transform for the conjugate 2-spinor to obtain twistors coding for the points of compactified Minkowski space. Wave functions in this twistor space characterized by massless momentum and helicity appear in the construction of twistor amplitudes. BCFW recursion relation [B20] allows to construct more complex amplitudes assuming that intermediate states are on mass shells massless states with complex momenta.

   One can perform twistor Fourier transformation (there are some technical problems in Minkowski signature) also for the second 2-spinor to get what are called momentum twistors providing in some aspects simpler description of twistor amplitudes. These code for the four-momenta propagating between vertices at which the incoming particles arrive and the differences if two subsequent momenta are equal to massless external momenta.

2. In Witten's theory the interactions of incoming particles correspond to amplitudes in which the twistors appearing as arguments of the twistor space wave functions characterized by momentum and helicity are localized to complex curves X^2 of twistor space CP_3 or its Minkowskian counterpart. This can be seen as a non-local twistor space variant of local interactions in Minkowski space.

   The surfaces X^2 are characterized by their degree d (of the polynomial of complex coordinates defining the algebraic 2-surface) the genus g of the algebraic surface, by the number k of "wrong" (helicity violating) helicities, and by the number of loops of corresponding diagram of SUSY amplitude: one has $d=k-1+l$, $g\leq l$. The interaction vertex in twistor space is not anymore completely local but the n particles are at points of the twistorial surface X^2.

In the following a proposal generalizing Witten's approach to TGD is discussed.

1. The fundamental challenge is the generalization of the notion of twistor associated with massless particle to 8-D context, first for $M^4 = M^4 \times E^4$ and then for $H = M^4 \times CP_2$. The notion of twistor space solves this question at geometric level. As far as construction of the TGD variant of Witten's twistor string is considered, this might be quite enough.

2. $M^8 - H$ duality and quantum-classical correspondence however suggest that M^8 twistors might allow tangent space description of four-momentum, spin, color quantum numbers and electroweak numbers and that this is needed. What comes in mind is the identification of fermion lines as light-like geodesics possessing M^8 valued 8-momentum, which would define the long sought gravitational counterparts of four-momentum and color quantum numbers at classical point-particle level. The M^8 part of this four-momentum would be equal to that associated with embedding space spinor mode characterizing the ground state of super-conformal representation for fundamental fermion.

   Hence one might also think of starting from the 4-D condition relating Minkowski coordinates to twistors and looking what it could mean in the case of M^8. The generalization is indeed possible in $M^8 = M^4 \times E^4$ by its flatness if one replaces gamma matrices with octonionic gamma matrices.

   In the case of $M^4 \times CP_2$ situation is different since for octonionic gamma matrices $SO(1,7)$ is replaced with G_2, and the induced gauge fields have different holonomy structure than for ordinary gamma matrices and octonionic sigma matrides appearing as charge matrices bring in also an additional source of non-associativity. Certainly the notion of the twistor Fourier transform fails since CP_2 Dirac operator cannot be algebraized.

   Algebraic twistorialization however works for the light-like fermion lines at which the ordinary and octonionic representations for the induced Dirac operator are equivalent. One can indeed assign 8-D counterpart of twistor to the particle classically as a representation of light-like hyper-octonionic four-momentum having massive M^4 and CP_2 projections and CP_2 part perhaps having interpretation in terms of classical tangent space representation for color and electroweak quantum numbers at fermionic lines.

   If all induced electroweak gauge fields - rather than only charged ones as assumed hitherto - vanish at string world sheets, the octonionic representation is equivalent with the ordinary one. The CP_2 projection of string world sheet should be 1-dimensional: inside CP_2 type vacuum extremals this is impossible, and one could even consider the possibility that the projection corresponds to CP_2 geodesic circle. This would be enormous technical simplification. What is important that this would not prevent obtaining non-trivial scattering amplitudes at elementary particle level since vertices would correspond to re-arrangement of fermion lines between the generalized lines of Feynman diagram meeting at the vertices (partonic 2-surfaces).

3. In the fermionic sector one is forced to reconsider the notion of the induced spinor field. The modes of the embedding space spinor field should co-incide in some region of the space-time surface with those of the induced spinor fields. The light-like fermionic lines defined by the boundaries of string world sheets or their ends are the obvious candidates in this respect. String world sheets is perhaps too much to require.

   The only reasonable identification of string world sheet gamma matrices is as induced gamma matrices and super-conformal symmetry requires that the action contains string world sheet area as an additional term just as in string models. String tension would correspond to gravitational constant and its value - that is ratio to the CP_2 radius squared, would be fixed by quantum criticality.

4. The generalization of the Witten's geometric construction of scattering amplitudes relying on the induction of the twistor structure of the embedding space to that associated with space-time surface looks surprisingly straight-forward and would provide more precise formulation of the notion of generalized Feynman diagrams forcing to correct some wrong details. One of the nice outcomes is that the genus appearing in Witten's formulation naturally corresponds to family replication in TGD framework.

12.4.1 Basic Ideas About Twistorialization Of TGD

The recent advances in understanding of TGD motive the attempt to look again for how twistor amplitudes could be realized in TGD framework. There have been several highly non-trivial steps of progress leading to a new more profound understanding of basic TGD.

1. $M^4 \times CP_2$ is twistorially unique [K86] in the sense that its factors are the only 4-D geometries allowing twistor space with Kähler structure (M^4 corresponds to S^4 in Euclidian signature) [A74]. The twistor spaces in question are CP_3 for S^4 and its Minkowskian variant for M^4 (I will use P^3 as short hand for both twistor spaces) and the flag manifold $F = SU(3)/U(1) \times U(1)$ parametrizing the choices of quantization axes for color group $SU(3)$ in the case of CP_2. Recall that twistor spaces are S^2 bundles over the base space and that all orientable four-manifolds have twistor space in this sense. Note that space-time surfaces allow always almost quaternionic structure and that preferred extremals are suggested to be quaternionic [K86].

2. The light-likeness condition for twistors in M^4 is fundamental in the ordinary twistor approach. In 8-D context light-likeness holds in generalized sense for the spinor harmonics of H: the square of the Dirac operator annihilates spinor modes. In the case M^8 one can indeed define twistors by generalizing the standard approach by replacing ordinary gamma matrices with octonionic ones [**?**] For H octonionic and ordinary gamma matrices are equivalent at the fermionic lines defined by the light-like boundaries of string world sheets and at string world sheets if they carry vanishing induced electro-weak gauge fields that is have 1-D CP_2 projection.

3. Twistor spaces emerge in TGD framework as lifts of space-time surfaces to corresponding twistor spaces realized as 6-surfaces in the lift of $M^4 \times CP_2$ to $T(H) = P^3 \times F$ having as base spaces space-time surfaces. This implies that that generalized Feynman diagrams and also generalized twistor diagrams can be lifted to diagrams in T and that the construction of twistor spaces as surfaces of T has very concrete particle interpretation.

   The modes of the embedding space spinor field defining ground states of the extended conformal algebras for which classical conformal charges vanish at the ends of the space-time surface (this defines gauge conditions realizing strong form of holography [K96]) are lifted to the products of modes of spinor fields in $T(H)$ characterized by four-momentum and helicity in M^4 degrees of freedom and by color quantum numbers and electroweak quantum numbers in F degrees of freedom. Thus twistorialization provides a purely geometric representation of spin and electro-weak spin and it seems that twistorialization allows to a formulation without H-spinors.

   What is especially nice, that twistorialization extends to from spin to include also electroweak spin. These two spins correspond correspond to M^4 and CP_2 helicities for the twistor space amplitude, and are non-local properties of these amplitudes. In TGD framework only twistor amplitudes for which helicities correspond to that for massless fermion and antifermion are possible and by fermion number conservation the numbers of positive and negative helicities are identical and equal to the fermion number (or antifermion number). Separate lepton and baryon number conservation realizing 8-D chiral symmetry implies that M^4 and CP_2 helicities are completely correlated.

   For massless fermions in M^4 sense helicity is opposite for fermion and antifermion and conserved. The contributions of initial and final states to k are same and equal to $k_i = k_f = 2(n(F) - n(\overline{F})$. This means restriction to amplitudes with $k = 2(n(F) - n(\overline{F})$. If fermions are massless only in M^8 sense, chirality mixing occurs and this rule does not hold anymore. This holds true in quark and lepton sector separately.

4. All generalized Feynman graphs defined in terms of Euclidian regions of space-time surface are lifted to twistor spaces [K20]. Incoming particles correspond quantum mechanically to twistor space amplitudes defined by their momenta and helicities and and classically to the entire twistor space of space-time surface as a surface in the twistor space of H. Of course, also the Minkowskian regions have this lift. The vertices of Feynman diagrams correspond to regions of twistor space in which the incoming twistor spaces meet along their 5-D ends having also S^2 bundle structure over space-like 3-surfaces. These space-like 3-surfaces correspond to

ends of Euclidian and Minkowskian space-time regions separated from each other by light-like 3-surfaces at which the signature of the metric changes from Minkowskian to Euclidian. These "partonic orbits" have as their ends 2-D partonic surfaces. By strong form of General Coordinate Invariance implying strong of holography, these 2-D partonic surfaces and their 4-D tangent space data should code for quantum physics. Their lifts to twistor space are 4-D S^2 bundles having partonic 2-surface X^2 as base.

5. The well-definedness of em charge for the spinor modes demands that they are localized at 2-D string world sheets [K96] and also these world sheets are lifted to sub-spaces of twistor space of space-time surface. If one demands that octonionic Dirac operator makes sense at string world sheets, they must carry vanishing induced electro-weak gauge fields and string world sheets could be minimal surfaces in $M^4 \times S^1$, $S^1 \subset CP_2$ a geodesic circle.

   The boundaries of string world sheets at partonic orbits define light-like curves identifiable as carriers of fermion number and they define the analogs of lines of Feynman diagrams in ordinary sense. The only purely fermionic vertices are 2-fermion vertices at the partonic 2-surfaces at which the end of space-time surface meet. As already explained, the string world sheets can be seen as correlates for the correlations between fermion vertices at different wormhole throats giving rise to the counterpart of bosonic propagator in quantum field theories.

   The localization of spinor fields to 2-D string world sheets corresponds to the localization of twistor amplitudes to their 4-D lifts, which are S^2 bundles and the boundaries of string world sheets are lifted to 3-D twistor lifts of fermion lines. Clearly, the localization of spinors to string world sheets would be absolutely essential for the emergence of twistor description.

6. All elementary particles are many particle bound states of massless fundamental fermions: the non-collinearity (and possible complex character) of massless momenta explains massivation. The fundamental fermions are localized at wormhole throats defining the light-like orbits of partonic 2-surfaces. Throats are associated with wormhole contacts connecting two space-time sheets. Stability of the contact is guaranteed by non-vanishing monopole magnetic flux through it and this requires the presence of second wormhole contact so that a closed magnetic flux tube carrying monopole flux and involving the two space-time sheets is formed. The net fermionic quantum numbers of the second throat correspond to particle's quantum numbers and above weak scale the weak isospins of the throats sum up to zero.

7. Fermionic 2-vertex is the only *local* many-fermion vertex [K20] being analogous to a mass insertion. The non-triviality of fundamental 4-fermion vertex is due to classical interactions between fermions at opposite throats of worm-hole. The non-triviality of the theory involves also the analog of OZI mechanism: the fermionic lines inside partonic orbits are redistributed in vertices. Lines can also turn around in time direction which corresponds to creation or annihilation of a pair. 3-particle vertices are obtained only in topological sense as 3 space-time surfaces are glued together at their ends. The interaction between fermions at different wormhole throats is described in terms of string world sheets.

8. The earlier proposal was that the fermions in the internal fermion lines are massless in M^4 sense but have non-physical helicity so that the algebraic M^4 Dirac operator emerging from the residue integration over internal four-momentum does not annihilate the state at the end of the propagator line. Now the algebraic induced Dirac operator defines the propagator at fermion lines. Should one assume generalization of non-physical helicity also now?

9. All this stuff must be lifted to twistorial level and one expects that the lift to S^2 bundle allows an alternative description of fermions and spinor structure so that one can speak of induced twistor structure instead of induced spinor structure. This approach allows also a realization of M^4 conformal symmetries in terms of globally well-defined linear transformations so that it might be that twistorialization is not a mere reformulation but provides a profound unification of bosonic and fermionic degrees of freedom.

12.4.2 The Emergence Of The Fundamental 4-Fermion Vertex And Of Boson Exchanges

The emergence of the fundamental 4-fermion vertex and of boson exchanges deserves a more detailed discussion.

1. I have proposed that the discontinuity of the Dirac operator at partonic two-surface (corner of fermion line) defines both the fermion boson vertex and TGD analog of mass insertion (not scalar but embedding space vector) giving rise to mass parameter having interpretation as Higgs vacuum expectation and various fermionic mixing parameters at QFT limit of TGD obtained by approximating many-sheeted space-time of TGD with the single sheeted region of M^4 such that gravitational field and gauge potentials are obtained as sums of those associated with the sheets.

2. Non-trivial scattering requires also correlations between fermions at different partonic 2-surfaces. Both partonic 2-surfaces and string world sheets are needed to describe these correlations. Therefore the string world sheets and partonic 2-surfaces cannot be dual: both are needed and this means deviation from Witten's theory. Fermion vertex corresponds to a "corner" of a fermion line at partonic 2-surface at which generalized 4-D lines of Feynman diagram meet and light-like fermion line changes to space-like one. String world sheet with its corners at partonic 2-surfaces (wormhole throats) describes the momentum exchange between fermions. The space-like string curve connecting two wormhole throats serves as the analog of the exchanged gauge boson.

3. Two kinds of 4-fermion amplitudes can be considered depending on whether the string connects throats of single wormhole contact (CP_2 scale) or of two wormhole contacts (p-adic length scale - typically of order elementary particle Compton length). If string worlds sheets have 1-D CP_2 projection, only Minkowskian string world sheets are possible. The exchange in Compton scale should be assignable to the TGD counterpart of gauge boson exchange and the fundamental 4-fermion amplitude should correspond to single wormhole contact: string need not to be involved now. Interaction is basically classical interaction assignable to single wormhole contact generalizing the point like vertex.

4. The possible TGD counterparts of BCFW recursion relations [B20] should use the twistorial representations of fundamental 4-fermion scattering amplitude as seeds. Yangian invariance poses very strong conditions on the form of these amplitudes and the exchange of massless bosons is suggestive for the general form of amplitude.

   The 4-fermion amplitude assignable to two wormhole throats defines the analog of gauge boson exchange and is expressible as fusion of two fundamental 4-fermion amplitudes such that the 8-momenta assignable to the fermion and anti-fermion at the opposite throats of exchanged wormhole contact are complex by BCFW shift acting on them to make the exchanged momenta massless but complex. This entity could be called fundamental boson (not elementary particle).

5. Can one assume that the fundamental 4-fermion amplitude allows a purely formal composition to a product of $F\overline{F}B_v$ amplitudes, B_v a purely fictive boson? Two 8-momenta at both $F\overline{F}B_v$ vertices must be complex so that at least two external fermionic momenta must be complex. These external momenta are naturally associated with the throats of the a wormhole contact defining virtual fundamental boson. Rather remarkably, without the assumption about product representation one would have general four-fermion vertex rather than boson exchange. Hence gauge theory structure is not put in by hand but emerges.

12.4.3 What About SUSY In TGD?

Extended super-conformal symmetry is crucial for TGD and extends to quaternion-super-conformal symmetry giving excellent hopes about calculability of the theory. $\mathcal{N} = 4$ space-time supersymmetry is in the key role in the approach of Witten and others.

In TGD framework space-time SUSY could be present as an approximate symmetry.

1. The many fermion states at partonic surfaces are created by oscillator operators of fermionic Clifford algebra having also interpretation as a supersymmetric algebra but in principle having $\mathcal{N} = \infty$. This SUSY is broken since the generators of SUSY carry four-momentum.

2. More concrete picture would be that various SUSY multiplets correspond to collinear many-fermion states at the same wormhole throat. By fermionic statistics only the collinear states for which internal quantum numbers are different are realized: other states should have antisymmetric wave function in spatial degrees of freedom implying wiggling in CP_2 scale so that the mass of the state would be very high. In both quark and lepton sectors one would have $\mathcal{N} = 4$ SUSY so that one would have the analog $\mathcal{N} = \forall$ SUSY (color is not spin-like quantum number in TGD).

   At the level of diagrammatics single line would be replaced with "line bundle" representing the fermions making the many-fermion state at the light-like orbit of the partonic 2-surface. The fusion of neighboring collinear multifermion stats in the twistor diagrams could correspond to the process in which partonic 2-surfaces and single and many-fermion states fuse.

3. Right handed neutrino modes, which are not covariantly constant, are also localized at the fermionic lines and defines the least broken $\mathcal{N} = 2$ SUSY. The covariantly constant mode seems to be a pure gauge degree of freedom since it carriers no quantum numbers and the SUSY norm associated with it vanishes. The breaking would be smallest for $\mathcal{N} = 2$ variant assignable to right-handed neutrino having no weak and color interactions with other particles but whose mixing with left-handed neutrino already induces SUSY breaking.

Why this SUSY has not been observed? One can consider two scenarios [K75].

1. The first scenario relies on the absence of weak and color interactions: one can argue that the bound states of fermions with right-handed neutrino are highly unstable since only gravitational interaction so that sparticle decays very rapidly to particle and right-handed or left-handed neutrino. By Uncertainty Principle this makes sparticle very massive, maybe having mass of order CP_2 mass. Neutrino mixing caused by the mixing of M^4 and CP_2 gamma matrices in induced gamma matrices is the weak point of this argument.

2. The mixing of left and right-handed neutrinos could be characterized by the p-adic mass scale of neutrinos and be long. Sparticles would have same p-adic mass scale as particles and would be dark having non-standard value of Planck constant $h_{eff} = n \times h$: this would scale up the lifetime by factor n and correlate with breaking of conformal symmetry assignable to the mixing [K75].

What implications the approximate SUSY would have for scattering amplitudes?

1. $k = 2(n(F) - n(\overline{F})$ condition reduces the number of amplitudes dramatically if the fermions are massless in M^4 sense but the presence of weak iso-spin implies that the number of amplitudes is 2^n as in non-supersymmetric gauge theories. One however expects broken SUSY with generators consisting of fermionic oscillator operators at partonic 2-surfaces with symmetry breaking taking place only at the level of physical particles identifiable as many particle bound states of massless (in 8-D sense) particles. This motivates the guess that the formal $F\overline{F}B_v$ amplitudes defining fundamental 4-fermion vertex are expressible as those associated with $\mathcal{N} = 4$ SUSY in quark and lepton sectors respectively. This would reduce the number of independent amplitudes to just one.

2. Since SUSY and its breaking emerge automatically in TGD framework, super-space can provide a useful technical tool but is not fundamental.

   Side note: The number of external fermions is always even suggesting that the super-conformal anomalies plaguing the amplitudes with odd n (`http://tinyurl.com/yb85tnvc`) [B52] are absent.

12.4.4 What Does One Really Mean With The Induction Of Embedding Space Spinors?

The induction of spinor structure is a central notion of TGD but its detailed definition has remained somewhat obscure. The attempt to generalize Witten's approach has made it clear that the mere restriction of spinor fields to space-time surfaces is not enough and that one must understand in detail the correspondence between the modes of embedding space spinor fields and those of induced spinor fields.

Even the identification of space-time gamma matrices is far from obvious at string world sheets.

1. The simplest notion of the space-time gamma matrices is as projections of embedding space gamma matrices to the space-time surface - induced gamma matrices. If one assumes that induced spinor fields are defined at the entire space-time surfaces, this notion fails to be consistent with fermionic super-conformal symmetry unless one replaces Kähler action by space-time volume. This option is certainly unphysical.

2. The notion of Kähler-Dirac matrices in the interior of space as gamma matrices defined as contractions of canonical momentum densities of Kähler with embedding space gamma matrices allows to have conformal super-symmetry with fermionic super charges assignable to the modes of the induced spinor field. Also Chern-Simons action could define gamma matrices in the same manner at the light-like 3-surfaces between Minkowskian and Euclidian space-time regions and at space-like 3-surfaces at the ends of space-time surface. Chern-Simons-Dirac matrices would involve only CP_2 gamma matrices.

It is however not quite clear whether the spinor fields in the interior of space-time surface are needed at all in the twistorial approach and they seem to be only an un-necessary complication. For instance, their modes would have well-defined electromagnetic charge only when induced W gauge fields vanish, which implies that CP_2 projection is 2-dimensional. This forces to consider very seriously the possibility that induced spinor fields reside at string world sheets only (their ends are at partonic 2-surfaces). This option supported also by strong form of holography and number theoretic universality.

What about the space-time gamma matrices at string world sheets and their boundaries?

1. The first option would be reduction of Kähler-Dirac gamma matrices by requiring that they are parallel to the string world sheets. This however poses additional conditions besides the vanishing of W fields already restricting the dimension to two in the generic case. The conditions state that the embedding space 1-forms defined by the canonical momentum densities of Kähler action involve only 2 linearly independent ones and that they are proportional to embedding space coordinate gradients: this gives Frobenius conditions. These conditions look first too strong but one can also think that one fixes first string world sheets, partonic 2-surfaces, and perhaps also their light-like orbits, requires that the normal components of canonical momentum currents at string world sheets vanish, and deduces space-time surface from this data. This would be nothing but strong form of holography!

   For this option the string world sheets could emerge in the sense that it would be possible to express Kähler action as an area of string world sheet in the effective metric defined by the anticommutator of K-D gamma matrices appearing also in the expressions for the matrix elements of WCW metric. Gravitational constant would be a prediction of the theory.

2. Second possibility is to use induced gamma matrices automatically parallel to the string world sheet so that no additional conditions would result. This would also conform with the ordinary view about string world sheets and spinors.

   Supersymmetry would require the addition of the area of string world sheet to the action defining Kähler function in Euclidian regions and its counterpart in Minkowskian regions. This would bring in gravitational constant, which otherwise remains a prediction. Quantum criticality could fix the ratio of $\hbar G/R^2$ (R is CP_2 radius). The vanishing of induced weak gauge fields requires that string world sheets have 1-D CP_2 projection and are thus restricted to Minkowskian regions with at most 3-D CP_2 projection. Even stronger condition would be that string world sheets are minimal surfaces in $M^4 \times S^1$, S^1 a geodesic sphere of CP_2.

There are however grave objections. The presence of a dimensional parameter G as fundamental coupling parameter does not encourage hopes about the renomalizibility of the theory. The idea that strings connecting partonic 2-surfaces gives rise to the formation of gravitationally bound states is suggested by AdS/CFT correspondence. The problem is that the string tension defined by gravitational constant is so large that only Planck length sized bound states are feasible. Even the replacement $\hbar \rightarrow \hbar_{eff}$ fails to allow gravitationally bound states with length scale of order Schwartschild radius. For the K-D option the string tension behaves like $1/\hbar^2$ and there are no problems in this respect.

At this moment my feeling is that the first option - essentially the original view - is the correct one. The short belief that the second option is the correct choice was a sidetrack, which however helped to become convinced that the original vision is indeed correct, and to understand the general mechanism for the formation of bound states in terms of strings connection partonic 2-surfaces (in the earlier picture I talked about magnetic flux tubes carrying monopole flux: the views are equivalent).

Both options have the following nice features.

1. Induced gammas at the light-like string boundaries would be light-like. Massless Dirac equation would assign to spinors at these lines a light-like space-time four-momentum and twistorialize it. This four-momentum would be essentially the tangent vector of the light-like curve and would not have a constant direction. Light-likeness for it means light-likeness in 8-D sense since light-like curves in H correspond to non-like momenta in M^4. Both M^4 mass squared and CP_2 mass would be conserved. Even four-momentum could be conserved if M^4 projection of stringy curve is geodesic line of M^4.

2. A new connection with Equivalence Principle (EP) would emerge. One could interpret the induced four-momentum as gravitational four-momentum which would be massless in 4-D sense and correspond to inertial 8-momentum. EP wold state in the weakest form that only the M^4 masses associated with the two momenta are identical. Stronger condition would be that the Minkowski parts of the two momenta co-incide at the ends of fermion lines at partonic 2-surfaces. Even stronger condition is that the 8-momentum is 8-momentum is conserved along fermion line. This is certainly consistent with the ordinary view about Feynman graphs. This is guaranteed if the light-like curve is light-like geodesic of embedding space.

The induction of spinor fields has also remained somewhat imprecise notion. It how seems that quantum-classical correspondence forces a unique picture.

1. Does the induced spinor field co-incide with embedding space spinor harmonic in some domain? This domain would certainly include the ends of fermionic lines at partonic 2-surfaces associated with the incoming particles and vertices. Could it include also the boundaries of string world sheets and perhaps also the string world sheets? The Kähler-Dirac equation certainly does not allow this for entire space-time surface.

2. Strong form of holography suggest that the light-like momenta for the Dirac equation at the ends of light-like string boundaries has interpretation as 8-D light-like momentum has M^4 projection equal to that of H spinor-harmonic. The mass squared of M^4 momentum would be same as the CP_2 momentum squared in both senses. Unless the gravitational four-momentum assignable to the induced Dirac operato r is conserved one cannot pose stronger condition.

3. If the induced spinor mode equals to embedding space-spinor mode also at fermion line, the light like momentum is conserved. The fermion line would be also light-like geodesic of the embedding space so that twistor polygons would have very concrete interpretation. This condition would be clearly analogous to the conditions in Witten's twistor string theory. A stronger condition would be that the mode of the embedding space spinor field co-incides with induced spinor field at the string world sheet.

4. p-Adic mass calculations require that the massive excitations of embedding space spinor fields with CP_2 mass scale are involved. The thermodynamics could be for fermion line, wormhole throat carrying possible several fermions, or wormhole contact carrying fermion at both throats. In the case of fermions physical intuition suggests that p-adic thermodynamics must be associated with single fermionic line. The massive excitations would correspond to light-like geodesics of the embedding space.

To minimize confusion I must confess that until recently I have considered a different options for the momenta associated with fermionic lines.

1. The action of the Kähler-Dirac operator on the induced spinor field at the fermionic line equals to that of 4-D Dirac operator $p^k\gamma_k$ for a massless momentum identified as M^4 momentum [K20].

   Now the action reduces to that of 8-D massless algebraic Dirac operator for light-like string boundaries in the case of induced gamma matrices. Explicit calculation shows that in case of K-D gamma matrices and for light-like string boundaries it can happen that the 8-momentum of the mode can be tachyonic. Intriguingly, p-adic mass calculations require a tachyonic ground state?

2. For this option the helicities for virtual fermions were assumed to be non-physical in order to get non-vanishing fermion lines by residue integration: momentum integration for Dirac operator would replace Dirac propagators with Dirac operators. This would be the counterpart for the disappearance of bosonic propagators in residue integration.

3. This option has problems: quantum classical correspondence is not realized satisfactorily and the interpretation of p-adic thermodynamics is problematic.

12.4.5 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K86] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

The case of $M^8 = M^4\times E^4$

M^8-H duality [K82] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?

2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0=\sigma_z\times 1$, $\gamma_k=\sigma_y\times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.

4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K20].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?

2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.

3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

   (a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

   (b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

(c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

12.4.6 How To Generalize Witten's Twistor String Theory To TGD Framework?

The challenge is to lift the geometric description of particle like aspects of twistorial amplitudes involving only algebraic curves (2-surfaces) in twistor space to TGD framework.

1. External particles correspond to the lifts of H-spinor harmonics to spinor harmonics in the twistor space and are labeled by four-momentum, helicity, color, and weak helicity (isospin) so that there should be no need to included fermions explicitly. The twistorial wave functions would be superpositions of the eigenstates of helicity operator which would become a non-local property in twistor space. Light-likeness would hold true in 8-D sense for spinor harmonics as well as for the corresponding twistorial harmonics.

2. The surfaces X^2 in Witten's theory would be replaced with the lifts of partonic 2-surfaces X^2 to 4-D surfaces with bundle structure with X^2 as base and S^2 as fiber. S^2 would be non-dynamical. Whether X^2 or its lift to 4-surface allows identification as algebraic surface is not quite clear but it seems that X^2 could be the relevant object identifiable as surface of the base space of $T(X^4)$. If X^2 is the basic object one would have the additional constraint (not present in Witten's theory) that it belongs to the base space X^4. The genus of the lift of X^2 would be that of its base space X^2. One obtains a union of partonic 2-surfaces rather than single surface and lines connecting them as boundaries of string world sheets.

   The n points of given X^2 would correspond to the ends of boundaries of string world sheets at the partonic 2-surface X^2 carrying fermion number so that the helicities of twistorial spinor modes would be highly fixed unless M^4 chiralities mix making fermions massive in M^4 sense. This picture is in accordance with the fact that the lines of fundamental fermions should correspond to QFT limit of TGD.

3. In TGD genus g of the partonic 2-surface labels various fermion families and $g < 3$ holds true for physical fermions. The explanation could be that Z^2 acts as global conformal symmetry (hyper-ellipticity) for $g < 3$ surfaces irrespective of their conformal moduli but for $g > 3$ only in for special moduli. Physically for $g > 2$ the additional handles would make the partonic 2-surface to behave like many-particle state of free particles defined by the handles.

   This assumption suggests that assigns to the partonic surface what I have called modular invariant elementary particle vacuum functional (EVPF) in modular degrees of freedom such that for a particle characterized by genus g one has $l \geq g$ and $l > g$ amplitudes are possible because the EPVF leaks partially to higher genera [K18]. This would also induce a mixing of boundary topologies explaining CKM mixing and its leptonic counterpart. In this framework it would be perhaps more appropriate to define the number of loops as $l_1 = l - g$.

   A more precise picture is as follows. Elementary particles have actually four wormhole throats corresponding to the two wormhole contacts. In the case of fermions the wormhole throat carrying the electroweak quantum numbers would have minimum value g of genus characterized by the fermion family. Furthermore, the universality of the standard model physics requires that the couplings of elementary fermions to gauge bosons do not depend on genus. This is the case if one has quantum superposition of the wormhole contacts carrying the quantum numbers of observed gauge bosons at their opposite throats over the three lowest genera $g = 0, 1, 2$ with identical coefficients. This meas SU(3) singlets for the dynamical SU(3) associated with genus degeneracy. Also their exotic variants - say octets - are in principle possible.

4. This description is not complete although already twistor string description involves integration over the conformal moduli of the partonic 2-surface. One must integrate over the "world of classical worlds" (WCW) -roughly over the generalized Feynman diagrams and their complements consisting of Minkowskian and Euclidian regions. TGD as almost topological QFT reduces this integration to that of the boundaries of space-time regions.

   By quaternion conformal invariance [K86] this functional integral could reduce to ordinary integration over the quaternionic-conformal moduli space of space-time surfaces for which the moduli space of partonic 2-surfaces should be contained (note that strong form of holography suggests that only the modular invariants associated with the tangent space data should enter the description). One might hope that twistor space approach allows an elegant description of the moduli assignable to the tangent space data.

12.4.7 Yangian Symmetry

One of the victories of the twistor Grassmannian approach is the discovery of Yangian symmetry [A36], [B25, B33], [K86], whose variant associated with extended super-conformal symmetries is expected to be in key role in TGD.

1. The very nature of the residue integral implies that the complex surface serving as the locus of the integrand of the twistor amplitude is highly non-unique. Indeed, the Yangian symmetry [K86] acting as multi-local symmetry and implying dual of ordinary conformal invariance acting on momentum twistors, has been found to reduce to diffeomorphisms of $G(k,n)$ respecting positivity property of the decomposition of $G(k,n)$ to polyhedrons. It is quite possible that this symmetry corresponds to quaternion conformal symmetries in TGD framework.

2. Positivity of a given regions means parameterization by non-negative coordinates in TGD framework a possible interpretation is based on the observation that canonical identification mapping reals to p-adic number and vice versa is well-defined only for non-negative real numbers. Number theoretical Universality of spinor amplitudes so that they make sense in all number fields, would therefore be implied.

3. Could the crucial Yangian invariance generalizing the extended conformal invariance of TGD could reduce to the diffeomorphisms of the extended twistor space $T(H)$ respecting positivity. In the case of CP_2 all coordinates could be regarded as angle coordinates and be replaced by phase factors coding for the angles which do not make sense p-adically. The number theoretical existence of phase factors in p-adic case is guaranteed if they belong to some algebraic extension of rationals and thus also p-adics containing these phases as roots of unity. This implies discretization of CP_2.

   ZEO allows to reduce the consideration to causal diamond CD defined as an intersection of future and past directed light-cones and having two light-like boundaries. CD is indeed a natural counterpart for S^4. One could use as coordinates light-cone proper time a invariant under Lorentz transformations of either boundary of CD, hyperbolic angle η and two spherical angles (θ,ϕ). The angle variables allow representation in terms of finite algebraic extension. The coordinate a is naturally non-negative and would correspond to positivity. The diffeomorphisms perhaps realizing Yangian symmetry would respect causality in the sense that they do not lead outside CD.

   Quaternionic conformal symmetries the boundaries of $CD\times CP_2$ continued to the interior by translation of the light-cones serve as a good candidates for the diffeomorphisms in question since they do not change the value of the Minkowski time coordinate associated with the line connecting the tips of CD.

12.4.8 Does BCFW Recursion Have Counterpart In TGD?

Could BCFW recursion for tree diagrams and its generalization to diagrams with loops have a generalization in TGD framework? Could the possible TGD counterpart of BCFW recursion have a representation at the level of the TGD twistor space allowing interpretation in terms of geometry of partonic 2-surfaces and associated string world sheets? Supersymmetry is essential ingredient in obtaining this formula and the proposed SUSY realized at the level of amplitudes and broken at the level of states gives hopes for it. One could however worry about the fact that spinors are Dirac spinors in TGD framework and that Majorana property might be essential element.

How to produce Yangian invariants

Nima Arkani-Hamed *et al* [B33] (`http://tinyurl.com/y97rlzqb`) describe in detail various ways to form Yangian invariants defining the singular parts of the integrands of the amplitudes allowing to construct the full amplitudes. The following is only a rough sketch about what is involved using particle picture and I cannot claim of having understood the details.

1. One can *add* particle $((k,n) \rightarrow (k+1,n+1))$ to the amplitude by deforming the momentum twistors of two neighboring particles in a way depending on the momentum twistor of the added particle. One inserts the new particle between n-1:th and 1st particle, modifies their momentum twistors without changing their four-momenta, and multiplying the resulting amplitude by a twistor invariant known as $[n-2,n-1,n,1,2]$ so that there is dependence on the added n:th momentum twistor.
2. One can *remove* particle $((k,n) \rightarrow (k-1,n-1))$ by contour integrating over the momentum twistor variable of one particle.
3. One can *fuse* invariants simply by multiplying them.
4. One can *merge* invariants by identifying momentum twistors appearing in the two invariants. The integration over the common twistor leads to an elimination of particle.
5. One can form a *BCFW bridge* between n_1+1-particle invariant and n_2+1-particle invariant to get $n = n_1 + n_2$-particle invariant using the operations listed. One starts with the *fusion* giving the product $I_1(1,...,n_1,I)I_2(n_1+1,..n,I)$ of Yangian invariants followed by *addition* of n_1+1 to I_1 between n_1 and I and 1 to I_2 between I and n_1+1 (see the first item for details). After that follows the *merging* of lines labelled by I next to n_1 in I_1 and the precedessor of n_1+1 in I_2 reducing particle number by one unit and followed by residue integration over Z_I reducing particle number further by one unit so that the resulting amplitude is n-particle amplitude.
6. One can perform *entangled removal* of two particles. One could remove them one-by-one by independent contour integrations but one can also perform the contour integrations in such a way that one first integrates over two twistors at the same complex line and then over the lines: this operation adds to n-particle amplitude loop.

BCFW recursion formula

BCFW recursion formula allows to express n-particle amplitudes with l loops in terms of amplitudes with amplitudes having at most $l-1$ loops. The basic philosophy is that singularities serve as data allowing to deduce the full integrands of the amplitudes by generalized unitarity and other kinds of arguments.

Consider first the arguments behind the BCFW formula.

1. BCFW formula is derived by performing the canonical momentum twistor deformation $Z_n \rightarrow z_n + zZ_{n-1}$, multiplying by $1/z$ and performing integration along small curve around origin so that one obtains original amplitude from the residue inside the curve. One obtains also and alternative of the residue integral expression as sum of residues from its complement. The singularities emerge by residue integral from poles of scattering amplitudes and eliminate two lines so that the recursion formula for n-particle amplitude can involve at most $n+2$-particle amplitudes.

   It seems that one must combine all n-particle amplitudes to form a single entity defining the full amplitude. I do not quite understand what how this is done. In ZEO zero energy state involving different particle numbers for the final state and expressible in terms of S-matrix (actually its generalization to what I call M-matrix) might allow to understand this.

2. In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L + n_R = n + 2$, $k_L + k_R = k - 1$, and $l_L + l_R = l$.

3. The singularities are easy to understand in the case of tree amplitudes: they emerge from the poles of the conformally invariant quantities in the denominators of amplitudes. Physically this means that the sum of the momenta for a subset of particles corresponds to a complex pole (BCFW deformation makes two neighboring momenta complex). Hence one obtains sum over products of $j+1$-particle amplitudes BCFW bridged with $n-j$-particle amplitude to give n-particle amplitude by the merging process.

4. This is not all that is needed since the diagrams could be reduced to products of 1 loop 3-particle amplitudes which vanish by the triviality of coupling constant evolution in $\mathcal{N} = 4$ SUSY. Loop amplitudes serving as a kind of source in the recursion relation save the situation. There is indeed also a second set of poles coming from loop amplitudes.

   One-loop case is the simplest one. One begins from $n+2$ particle amplitude with $l-1$ loops. At momentum space level the momenta the neighboring particles have opposite light-like momenta: one of the particles is not scattered at all. This is called forward limit. This limit suffers from collinear divergences in a generic gauge theory but in supersymmetric theories the limit is well-defined. This forward limit defines also a Yangian invariant at the level of twistor space. It can be regarded as being obtained by entangled removal of two particles combined with merge operation of two additional particles. This operation leads from $(n+2, l-1)$ amplitude to (n, l) amplitude.

Does BCFW formula make sense in TGD framework?

In TGD framework the four-fermion amplitude but restricted so that two outgoing particles have (in general) complex massless 8-momenta is the basic building brick. This changes the character of BCFW recursion relations although the four-fermion vertex effectively reduces to $F\overline{F}B$ vertex with boson identified as wormhole contact carrying fermion and antifermion at its throats.

The fundamental 4-fermion vertices assignable to wormhole contact could be formally expressed in terms of the product of two $F\overline{F}B_v$ vertices (MHV expression), where B_v is purely formal gauge boson, using the analog of MHV expression and taking into account that the second $F\overline{F}$ pair is associated with wormhole contact analogous to exchanged gauge boson.

If the fermions at fermion lines of the same partonic 2-surface can be assumed to be collinear and thus to form single coherent particle like unit, the description as superspace amplitude seems appropriate. Consequently, the effective $F\overline{F}B_v$ vertices could be assumed to have supersymmetry defined by the fermionic oscillator operator algebra at the partonic 2-surface (Clifford algebra). A good approximation is to restrict this algebra to that generating various spinor components of embedding space spinors so that $\mathcal{N} = 4$ SUSY is obtained in leptonic and quark sector. Together these give rise to $\mathcal{N} = 8$ SUSY at the level of vertices broken however at the level of states.

Side note: The number of external fermions is always even suggesting that the super-conformal anomalies plaguing the SUSY amplitudes with odd n (`http://tinyurl.com/yb85tnvc`) [B52] are absent in TGD: this would be basically due to the decomposition of gauge bosons to fermion pairs.

The leading singularities of scattering amplitudes would naturally correspond to the boundaries of the moduli space for the unions of partonic 2-surfaces and string world sheets.

1. The tree contribution to the gauge boson scattering amplitudes with $k = 0, 1$ vanish as found by Parke and Taylor who also found the simple twistorial form for the $k = 2$ case (`http://tinyurl.com/y7nas26b`). In TGD framework, where lowest amplitude is 4-fermion amplitude, this situation is not encountered. According to Wikipedia article the so called CSW rules inspired by Witten's twistor theory have a problem due to the vanishing of $++-$ vertex which is not MHV form unless one changes the definition of what it is to be "wrong helicity". $++-$ is needed to construct $++++$ amplitude at one loop which does not vanish in YM theory. In SUSY it however vanishes.

   In TGD framework one does not encounter these problems since 4-fermion amplitudes are the basic building bricks. Fermion number conservation and the assumption that helicities do not mix (light-likeness in M^4 rather than only M^8-sense) implies $k = 2(n(F) - n(\overline{F})$.

In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L + n_R = n + 2$, $k_L + k_R = k - 1$. If the TGD counterpart of the bridge eliminates two antifermions with the same "wrong" helicity -1/2, and one indeed has $k_L + k_R = k - 1$ if fermions have well-defined M^4 helicity rather than being in superposition in completely correlated M^4 and CP_2 helicities.

2. In string theory loops correspond to handles of a string world sheet. Now one has partonic 2-surfaces and string world sheets and both can in principle have handles. The condition $l \geq g$ of Witten's theory suggests that $l - g$ defines the handle number for string world sheet so that l is the total number of handles.

   The identification of loop number as the genus of partonic 2-surface is second alternative: one would have $l = g$ and string world sheets would not contain handles. This might be forced by the Minkowskian signature of the induced metric at string world sheet. The signature of the induced metric would be presumably Euclidian in some region of string world sheet since the M^4 projection of either homology generator assignable with the handle would presumably define time loop in M^4 since the derivative of M^4 time coordinate with respect to string world sheet time should vanish at the turning points for M^4 time. Minimal surface property might eliminate Euclidian regions of the string world sheet. In any case, the area of string world sheet would become complex.

3. In the moduli space of partonic 2-surfaces first kind of leading singularities could correspond to pinches formed as n partonic 2-surfaces decomposes to two 2-surfaces having at least single common point so that moduli space factors into a Cartesian product. This kind of singularities could serve as counterparts for the merge singularities appearing in the BCFW bridging of amplitudes. The numbers of loops must be additive and this is consistent with both interpretations for l.

4. What about forward limit? One particle should go through without scattering and is eliminated by entangled removal. In ZEO one can ask whether there is also quantum entanglement between the positive and negative energy parts of this single particle state and state function reduction does not occur. The addition of particle and merging it with another one could correspond to a situation in which two points of partonic 2-surface touch. This means addition of one handle so that loop number l increases.

   It seems that analytically the loop is added by the entangled removal but at the level of partonic surface it is added by the merging. Also now both $l > g$ and $l = g$ options make sense.

12.4.9 Possible Connections Of TGD Approach With The Twistor Grassmannian Approach

For a non-specialist lacking the technical skills, the work related to twistors is a garden of mysteries and there are a lot of questions to be answered: most of them of course trivial for the specialist. The basic questions are following.

How the twistor string approach of Witten and its possible TGD generalization relate to the approach involving residue integration over projective sub-manifolds of Grassmannians $G(k, n)$?

1. In [B34] Nima *et al* argue that one can transform Grassmannian representation to twistor string representation for tree amplitudes. The integration over $G(k, n)$ translates to integration over the moduli space of complex curves of degree $d = k - 1 + l$, $l \geq g$ is the number of loops. The moduli correspond to complex coefficients of the polynomial of degree d and they form naturally a projective space since an overall scaling of coefficients does not change the surfaces. One can expect also in the general case that moduli space of the partonic 2-surfaces can be represented as a projective sub-manifold of some projective space. Loop corrections would correspond to the inclusion of higher degree surfaces.

2. This connection gives hopes for understanding the integration contours in $G(k, n)$ at deeper level in terms of the moduli spaces of partonic 2-surfaces possibly restricted by conformal gauge conditions.

Below I try to understand and relate the work of Nima Arkani Hamed *et al* with twistor Grassmannian approach to TGD.

The notion of positive Grassmannian

The notion of positive Grassmannian is one of the central notions introduced by Nima et al.

1. The claim is that the sub-spaces of the real Grassmannian $G(k,n)$ contributing to the amplitudes for $++--$ signature are such that the determinants of the $k\times k$ minors associated with ordered columns of the $k\times n$ matrix C representing point of $G(k,n)$ are positive. To be precise, the signs of all minors are positive or negative simultaneously: only the ratios of the determinants defining projective invariants are positive.

2. At the boundaries of positive regions some of the determinants vanish. Some k-volumes degenerate to a lower-dimensional volume. Boundaries are responsible for the leading singularities of the scattering amplitudes and the integration measure associated with $G(k,n)$ has a logarithmic singularity at the boundaries. These boundaries would naturally correspond to the boundaries of the moduli space for the partonic 2-surfaces. Here also string world sheets could contribute to singularities.

3. This condition has a partial generalization to the complex case: the determinants whose ratios serve as projectively invariant coordinates are non-vanishing. A possible further manner to generalize this condition would be that the determinants have positive real part so that apart from rotation by $\pi/2$ they would reside in the upper half plane of complex plane. Upper half plane is the hyperbolic space playing key role in complex analysis and in the theory of hyperbolic 2-manifolds for which it serves as universal covering space by a finite discrete subgroup of Lorentz group $SL(2,C)$. The upper half-plane having a deep meaning in the theory of Riemann surfaces might play also a key role in the moduli spaces of partonic 2-surfaces. The projective space would be based - not on projectivization of C^n but that of H^n, H the upper half plane.

Could positivity have some even deeper meaning?

1. In TGD framework the number theoretical universality of amplitudes suggests this. Canonical identification maps $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ p-adic number to non-negative reals. p-Adicization is possible for angle variables by replacing them by discrete phases, which are roots of unity. For non-angle like variables, which are non-negative one uses some variant of canonical identification involving cutoffs [K97]. The positivity should hold true for all structures involved, the $G(2,n)$ points defined by the twistors characterizing momenta and helicities of particles (actually pairs of orthogonal planes defined by twistors and their conjugates), the moduli space of partonic 2-surfaces, etc...

2. p-Adicization requires discretization of phases replacing angles so that they come as roots of unity associated with the algebraic extension used. The p-adic valued counterpart of Riemann or Lebesque integral does not make sense p-adically. Residue integrals can however allow to define p-adic integrals by analytic continuation of the integral and discretization of the phase factor along the integration contour does not matter (not however the p-adically troublesome factor 2π!).

3. TGD suggests that the generalization of positive real projectively invariant coordinates to complex coordinates of the hyperbolic space representable as upper half plane, or equivalently as unit disk obtained from the upper half plane by exponential mapping $w = exp(iz)$: positive coordinate α would correspond to the radial coordinate for the unit disk (Poincare hyperbolic disk appearing in Escher's paintings). The real measure $d\alpha/\alpha$ would correspond to $dz = dw/w$ restricted to a radial line from origin to the boundary of the unit disk. This integral should correspond to integral over a closed contour in complex case. This is the case if the integrand is discontinuity over a radial cut and equivalent with an integral over curve including also the boundary of the unit disk. This integral would reduce to the sum of the residues of poles inside the unit disk.

The notion of amplituhedron

The notion of amplituhedron is the latest step of progress in the twistor Grassmann approach [B15, B14]. What is so remarkable, is the simplicity of the expressions for all-loop amplitudes and the fact that positivity implies locality and unitarity for $\mathcal{N}=4$ SUSY.

Consider first tree amplitudes with general value of k.

1. The notion of amplituhedron relies on the mapping of $G(k,n)$ to $G_+(k,k+m)$ $n\geq k+m$. $G_+(k,k+m)$ is positive Grassmannian characterized by the condition that all $k\times k$- minors $k\times(k+m)$ matrix representing the point of $G_+(k,k+m)$ are non-negative and vanish at the boundaries $G_+(k,k+m)$. The value of m is $m=4$ and follows from the conditions that amplitudes come out correctly. The constraint $Y=C\cdot Z$, where Y corresponds to point of $G_+(k,k+4)$ and Z to the point of $G(k,n)$ performs this mapping, which is clearly many-to one. One can decompose $G_+(k,k+4)$ to positive regions intersecting only along their common boundary portions. The decomposition of a convex polygon in plane represent the basic example of this kind of decomposition.

2. Each decomposition defines a sum of contributions to the scattering amplitudes involving integration of a projectively invariant volume form over the positive region in question. The form has a logarithmic singularity at the boundaries of the integration region but spurious singularities cancel so that only the contribution of the genuine boundary of $G_+(k,k+4)$ remains. There are additional delta function constraints fixing the integral completely in real case.

3. In complex case one has residue integral. The proposed generalization to the complex case is by analytic continuation. TGD inspired proposal is that the positivity condition in the real case is generalized to the condition that the positive coordinates are replaced by complex coordinates of hyperbolic space representable as upper half plane or equivalently as the unit disk obtained from upper half plane by exponential mapping $w=exp(iz)$. The measure $d\alpha/\alpha$ would correspond to $dz=dw/w$. If taken over boundary circle labelled by discrete phase factors $exp(i\phi)$ given by roots of unity the integral would be numerically a discrete Riemann sum making no sense p-adically but residue theorem could allow to avoid the discretizaton and to define the p-adic variant of the integral by analytic continuation. These conditions would be completely general conditions on various projectively invariant moduli involved.

4. One must extend the bosonic twistors Z_a of external particles by adding k coordinates. Somewhat surprisingly, these coordinates are anticommutative super-coordinates expressible as linear combinations of fermionic parts of super-twistor using coefficients, which are also Grassmann numbers. Integrating over these one ends up with the standard expression of the amplitude using canonical integration measure for the regions in the decomposition of amplituhedron.

   What looks to me intriguing is that there is only super-integration involved over the additional k degrees of freedom. In Witten's approach $k-1$ corresponds to the minimum degree of the polynomial defining the string world sheet representing tree diagram. In TGD framework $k+1$ (rather than $k-1$) could correspond to the minimum degree of partonic 2-surface. In TGD approximate SUSY would correspond to Grassmann algebra of fermionic oscillator operators defined by the spinor basis for embedding space spinors. The interpretation could be that each fermion whose helicity differs from that allowed by light-likeness in M^4 sense (this requires non-vanishing M^4 mass), contributes $\Delta k=1$ to the degree of corresponding partonic 2-surface. Since the partonic 2-surface is common for all particles, one must have $d=k+1$ at least. The k-fold super integration would be basically integral over the moduli characterizing the polynomials of degree k realizing quantum classical correspondence in fermionic degrees of freedom.

BFCW recursion formula involves also loop amplitudes for which amplituhedron provides also a very nice representation.

1. The basic operation is the addition of a loop to get (n,k,l) amplitude from $(n+2,k,l-1)$ amplitude. That 2 particles must be removed for each loop is not obvious in $\mathcal{N}=4$ SUSY

but follows from the condition that positivity of the integration domain is preserved. This procedure removes from $(n+2, k, l-1)$-amplitude 2 particles with opposite four-momenta so that (n, k, l) amplitude is obtained. In the case of L-loops one extends $G(k, n)$ by adding its "complement" as a Cartesian factor $G(n-k, n)$ and imbeds to $G(n-k, n)$ 2-plane for each loop. Positivity conditions can be generalized so that they apply to $(k+2l) \times (k+2l)$-minors associated with matrices having as rows $0 \leq l \leq L$ ordered D_{i_k}:s and of C. The general expressions of the loop contributions are of the same form as for tree contributions: only the number of integration variables is $4 \times (k+L)$.

2. As already explained, in TGD framework the addition of loop would correspond to the formation of a handle to the partonic surface by fusing two points of partonic 2-surface and thus creating a surface intermediate between topologies with g and g+1 handles. g would correspond to the genus characterizing fermion family and one would have $L \geq g$. In elementary particle wave functionals loop [K18] contributions would correspond to higher genus contributions $l_1 = l - g > 0$ with basic contribution coming from genus g. For scattering amplitudes loop contributions would involve the change of the genus of the incoming wormhole throat so that they correspond to singular surfaces at the boundaries of their moduli space identifiable as loop corrections. $l_1 = l - g > 0$ would represent the number of pinches of the partonic 2-surface.

What about non-planar amplitudes?

Non-planar Feynman diagrams have remained a challenge for the twistor approach. The problem is simple: there is no canonical ordering of the extrenal particles and the loop integrand involving tricky shifts in integrations to get finite outcome is not unique and well-defined so that twistor Grasmann approach encounters difficulties.

Recently Nima Arkani-Hamed *et al* have considered also non-planar MHV diagrams [B35] (having minimal number of "wrong" helicities) of N=4 SUSY, and shown that they can be reduced to non-planar diagrams for different permutations of vertices of planar diagrams ordered naturally. There are several integration regions identified as positive Grassmannians corresponding to different orderings of the external lines inducing non-planarity. This does not however hold true generally.

At the QFT limit the crossings of lines emerges purely combinatorially since Feynman diagrams are purely combinatorial objects with the ordering of vertices determining the topological properties of the diagram. Non-planar diagrams correspond to diagrams, which do not allow crossing-free embedding to plane but require higher genus surface to get rid of crossings.

1. The number of the vertices of the diagram and identification of lines connecting them determines the diagram as a graph. This defines also in TGD framework Feynman diagram like structure as a graph for the fermion lines and should be behind non-planarity in QFT sense.

2. Could 2-D Feynman graphs exists also at geometric rather than only combinatorial level? Octonionization at embedding space level requires identification of preferred $M^2 \subset M^4$ defining a preferred hyper-complex sub-space. Could the projection of the Fermion lines defined concrete geometric representation of Feynman diagrams?

3. Despite their purely combinatorial character Feynman diagrams are analogous to knots and braids. For years ago [K41] I proposed the generalization of the construction of knot invariants in which one gradually eliminates the crossings of the knot projection to end up with a trivial knot is highly suggestive as a procedure for constructing the amplitudes associated with the non-planar diagrams. The outcome should be a collection of planar diagrams calculable using twistor Grassmannian methods. Scattering amplitudes could be seen as analogs of knot invariants. The reduction of MHV diagrams to planar diagrams could be an example of this procedure.

One can imagine also analogs of non-planarity, which are geometric and topological rather than combinatorial and not visible at the QFT limit of TGD.

1. The fermion lines representing boundaries of string world sheets at the light-like orbits of partonic 2-surfaces can get braided. The same can happen also for the string boundaries at

space-like 3-surfaces at the ends of the space-time surface. The projections of these braids to partonic 2-surfaces are analogs of non-planar diagrams. If the fermion lines at single wormhole throat are regarded effectively as a line representing one member of super-multiplet, this kind of braiding remains below the resolution used and cannot correspond to the braiding at QFT limit.

2. 2-knotting and 2-braiding are possible for partonic 2-surfaces and string world sheets as 2-surfaces in 4-D space-time surfaces and have no counterpart at QFT limit.

12.4.10 Permutations, Braidings, And Amplitudes

In [B30] Nima Arkani-Hamed demonstrates that various twistorially represented on-mass-shell amplitudes (allowing light-like complex momenta) constructible by taking products of the 3-particle amplitude and its conjugate can be assigned with unique permutations of the incoming lines. The article describes the graphical representation of the amplitudes and its generalization. For 3-particle amplitudes, which correspond to $++-$ and $+--$ twistor amplitudes, the corresponding permutations are cyclic permutations, which are inverses of each other. One actually introduces double cover for the labels of the particles and speaks of decorated permutations meaning that permutation is always a right shift in the integer and in the range $[1, 2\times n]$.

Amplitudes as representation of permutations

It is shown that for on mass shell twistor amplitudes the definition using on-mass-shell 3-vertices as building bricks is highly reducible: there are two moves for squares defining 4-particle sub-amplitudes allowing to reduce the graph to a simpler one. The first ove is topologically like the s-t duality of the old-fashioned string models and second one corresponds to the transformation black $\leftrightarrow$ white for a square sub-diagram with lines of same color at the ends of the two diagonals and built from 3-vertices.

One can define the permutation characterizing the general on mass shell amplitude by a simple rule. Start from an external particle a and go through the graph turning in in white (black) vertex to left (right). Eventually this leads to a vertex containing an external particle and identified as the image $P(a)$ of the a in the permutation. If permutations are taken as right shifts, one ends up with double covering of permutation group with $2\times n!$ elements - decorated permutations. In this manner one can assign to any any line of the diagram two lines. This brings in mind 2-D integrable theories where scattering reduces to braiding and also topological QFTs where braiding defines the unitary S-matrix. In TGD parton lines involve braidings of the fermion lines so that an assignment of permutation also to vertex would be rather nice.

BCFW bridge has an interpretation as a transposition of two neighboring vertices affecting the lines of the permutation defining the diagram. One can construct all permutations as products of transpositions and therefore by building BCFW bridges. BCFW bridge can be constructed also between disjoint diagrams as done in the BCFW recursion formula.

Can one generalize this picture in TGD framework? There are several questions to be answered.

(a) What should one assume about the states at the light-like boundaries of string world sheets? What is the precise meaning of the supersymmetry: is it dynamical or gauge symmetry or both?

(b) What does one mean with particle: partonic 2-surface or boundary line of string world sheet? How the fundamental vertices are identified: 4 incoming boundaries of string world sheets or 3 incoming partonic orbits or are both aspects involved?

(c) How the 8-D generalization of twistors bringing in second helicity and doubling the M^4 helicity states assignable to fermions does affect the situation?

(d) Does the crucial right-left rule relying heavily on the possibility of only 2 3-particle vertices generalize? Does M^4 massivation imply more than 2 3-particle vertices implying many-to-one correspondence between on-mass-shell diagrams and permutations? Or should one generalize the right-left rule in TGD framework?

Fermion lines for fermions massless in 8-D sense

What does one mean with particle line at the level of fermions?

(a) How the addition of CP_2 helicity and complete correlation between M^4 and CP_2 chiralities does affect the rules of $\mathcal{N}=4$ SUSY? Chiral invariance in 8-D sense guarantees fermion number conservation for quarks and leptons separately and means conservation of the product of M^4 and CP_2 chiralities for 2-fermion vertices. Hence only M^4 chirality need to be considered. M^4 massivation allows more 4-fermion vertices than $\mathcal{N}=4$ SUSY.

(b) One can assign to a given partonic orbit several lines as boundaries of string world sheets connecting the orbit to other partonic orbits. Supersymmetry could be understoond in two ways.

   i. The fermions generating the state of super-multiplet correspond to boundaries of different string world sheets which need not connect the string world sheet to same partonic orbit. This SUSY is dynamical and broken. The breaking is mildest breaking for line groups connected by string world sheets to same partonic orbit. Right handed neutrinos generated the least broken $\mathcal{N}=2$ SUSY.

   ii. Also single line carrying several fermions would provide realization of generalized SUSY since the multi-fermion state would be characterized by single 8-momentum and helicity. One would have $\mathcal{N}=4$ SUSY for quarks and leptons separately and $\mathcal{N}=8$ if both quarks and leptons are allowed. Conserved total for quark and antiquarks and leptons and antileptons characterize the lines as well.
   What would be the propagator associated with many-fermion line? The first guess is that it is just a tensor power of single fermion propagator applied to the tensor power of single fermion states at the end of the line. This gives power of $1/p^{2n}$ to the denominator, which suggests that residue integral in momentum space gives zero unless one as just single fermion state unless the vertices give compensating powers of p. The reduction of fermion number to 0 or 1 would simplify the diagrammatics enormously and one would have only 0 or 1 fermions per given string boundary line. Multi-fermion lines would represent gauge degrees of freedom and SUSY would be realized as gauge invariance. This view about SUSY clearly gives the simplest picture, which is also consistent with the earlier one, and will be assumed in the sequel

(c) The multiline containing n fermion oscillator operators can transform by chirality mixing in 2^n ways at 4-fermion vertex so that there is quite a large number of options for incoming lines with n_i fermions.

(d) In 4-D Dirac equation light-likeness implies a complete correlation between fermion number and chirality. In 8-D case light-likeness should imply the same: now chirality correspond to fermion number. Does this mean that one must assume just superposition of different M^4 chiralities at the fermion lines as 8-D Dirac equation requires. Or should one assume that virtual fermions at the end of the line have wrong chirality so that massless Dirac operator does not annihilate them?

Fundamental vertices

One can consider two candidates for fundamental vertices depending on whether one identifies the lines of Feynman diagram as fermion lines or as light-like orbits of partonic 2-surfaces. The latter vertices reduces microscopically to the fermionic 4-vertices.

(a) If many-fermion lines are identified as fundamental lines, 4-fermion vertex is the fundamental vertex assignable to single wormhole contact in the topological vertex defined by common partonic 2-surface at the ends of incoming light-like 3-surfaces. The discontinuity is what makes the vertex non-trivial.

(b) In the vertices generalization of OZI rule applies for many-fermion lines since there are no higher vertices at this level and interactions are mediated by classical induced gauge fields and chirality mixing. Classical induced gauge fields vanish if CP_2 projection is 1-dimensional for string world sheets and even gauge potentials vanish if the projection is to geodesic circle. Hence only the chirality mixing due to the mixing of M^4 and CP_2 gamma matrices is possible and changes the fermionic M^4 chiralities. This would dictate what vertices are possible.

(c) The possibility of two helicity states for fermions suggests that the number of amplitudes is considerably larger than in $\mathcal{N} = 4$ SUSY. One would have 5 independent fermion amplitudes and at each 4-fermion vertex one should be able to choose between 3 options if the right-left rule generalizes. Hence the number of amplitudes is larger than the number of permutations possibly obtained using a generalization of right-left rule to right-middle-left rule.

(d) Note however that for massless particles in M^4 sense the reduction of helicity combinations for the fermion and antifermion making virtual gauge boson happens. The fermion and antifermion at the opposite wormhole throats have parallel four-momenta in good approximation. In M^4 they would have opposite chiralities and opposite helicities so that the boson would be M^4 scalar. No vector bosons would be obtained in this manner.

In 8-D context it is possible to have also vector bosons since the M^4 chiralities can be same for fermion and anti-fermion. The bosons are however massive, and even photon is predicted to have small mass given by p-adic thermodynamics [K46]. Massivation brings in also the M^4 helicity 0 state. Only if zero helicity state is absent, the fundamental four-fermion vertex vanishes for $++++$ and $----$ combinations and one extend the right-left rule to right-middle-left rule. There is however no good reason for he reduction in the number of 4-fermion amplitudes to take place.

Partonic surfaces as 3-vertices

At space-time level one could identify vertices as partonic 2-surfaces.

(a) At space-time level the fundamental vertices are 3-particle vertices with particle identified as wormhole contact carrying many-fermion states at both wormhole throats. Each line of BCFW diagram would be doubled. This brings in mind the representation of permutations and leads to ask whether this representation could be re-interpreted in TGD framework. For this option the generalization of the decomposition of diagram to 3-particle vertices is very natural. If the states at throats consist of bound states of fermions as SUSY suggests, one could characterize them by total 8-momentum and helicity in good approximation. Both helicities would be however possible also for fermions by chirality mixing.

(b) A genuine decomposition to 3-vertices and lines connecting them takes place if two of the fermions reside at opposite throats of wormhole contact identified as fundamental gauge boson (physical elementary particles involve two wormhole contacts).

The 3-vertex can be seen as fundamental and 4-fermion vertex becomes its microscopic representation. Since the 3-vertices are at fermion level 4-vertices their number is greater than two and there is no hope about the generalization of right-left rule.

OZI rule implies correspondence between permutations and amplitudes

The realization of the permutation in the same manner as for $\mathcal{N} = 4$ amplitudes does not work in TGD. OZI rule following from the absence of 4-fermion vertices however implies much simpler and physically quite a concrete manner to define the permutation for external fermion lines and also generalizes it to include braidings along partonic orbits.

(a) Already $\mathcal{N}=4$ approach assumes decorated permutations meaning that each external fermion has effectively two states corresponding to labels k and $k+n$ (permutations are shifts to the right). For decorated permutations the number of external states is effectively 2^n and the number of decorated permutations is $2\times n!$. The number of different helicity configurations in TGD framework is 2^n for incoming fermions at the vertex defined by the partonic 2-surface. By looking the values of these numbers for lowest integers one finds $2n \geq 2^n$: for $n=2$ the equation is saturated. The inequality $log(n!) > nlog(n)/e)+1$ (see `http://tinyurl.com/2bjk5h`). gives

$$\frac{log(2n!)}{log(2^n)} \geq \frac{log(2)+1+nlog(n/e)}{nlog(2)} = log(n/e)/log(2)+O(1/n)$$

so that the desired inequality holds for all interesting values of n.

(b) If OZI rule holds true, the permutation has very natural physical definition. One just follows the fermion line which must eventually end up to some external fermion since the only fermion vertex is 2-fermion vertex. The helicity flip would map $k\rightarrow k+n$ or vice versa.

(c) The labelling of diagrams by permutations generalizes to the case of diagrams involving partonic surfaces at the boundaries of causal diamond containing the external fermions and the partonic 2-surfaces in the interior of CD identified as vertices. Permutations generalize to braidings since also the braidings along the light-like partonic 2-surfaces are allowed. A quite concrete generalization of the analogs of braid diagrams in integrable 2-D theories emerges.

(d) BCFW bridge would be completely analogous to the fundamental braiding operation permuting two neighboring braid strands. The almost reduction to braid theory - apart from the presence of vertices conforms with the vision about reduction of TGD to almost topological QFT.

To sum up, the simplest option assumes SUSY as both gauge symmetry and broken dynamical symmetry. The gauge symmetry relates string boundaries with different fermion numbers and only fermion number 0 or 1 gives rise to a non-vanishing outcome in the residue integration and one obtains the picture used hitherto. If OZI rule applies, the decorated permutation symmetry generalizes to include braidings at the parton orbits and $k\rightarrow k\pm n$ corresponds to a helicity flip for a fermion going through the 4-vertex. OZI rules follows from the absence of non-linearities in Dirac action and means that 4-fermion vertices in the usual sense making theory non-renormalizable are absent. Theory is essentially free field theory in fermionic degrees of freedom and interactions in the sense of QFT are transformed to non-trivial topology of space-time surfaces.

3. If one can approximate space-time sheets by maps from M^4 to CP_2, one expects General Relativity and QFT description to be good approximations. GRT space-time is obtained by replacing space-time sheets with single sheet - a piece of slightly deformed Minkowski space but without assupmtion about embedding to H. Induced classical gravitational field and gauge fields are sums of those associated with the sheets. The generalized Feynman diagrams with lines at various sheets and going also between sheets are projected to single piece of M^4. Many-sheetedness makes 1-homology non-trivial and implies analog of braiding, which should be however invisible at QFT limit.

A concrete manner to eliminate line crossing in non-planar amplitude to get nearer to non-planar amplitude could proceed roughly as follows. This is of course a pure guess motivated only by topological considerations. Professional might kill it in few seconds.

1. If the lines carry no quantum numbers, reconnection allows to eliminate the crossings. Consider the crossing line pair connecting AB in the initial state to CD in final state. The crossing lines are AD and BC. Reconnection can take place in two ways: AD and BC transform either to AB and CD or to AC and BD: neither line pair has crossing. The final state of the braid would be quantum superposition of the resulting more planar braids.

2. The crossed lines however carry different quantum numbers in the generic situation: for instance, they can be fermionic and bosonic. In this particular case the reconnection does not make sense since a line carrying fermion number would transform to a line carrying boson.

   In TGD framework all lines are fermion lines at fundamental level but the constraint due to different quantum numbers still remains and it is easy to see that mere reconnection is not enough. Fermion number conservation allows only one of the two alternatives to be realized. Conservation of quantum numbers forces to restrict gives an additional constraint which for simplest non-planar diagram with two crossed fermion lines forces the quantum numbers of fermions to be identical.

   It seems also more natural to consider pairs of wormhole contacts defining elementary particles as "lines" in turn consisting of fermion lines. Yangian symmetry allows to develop a more detailed view about what this decomposition could mean.

   Quantum number conservation demands that reconnection is followed by a formation of an additional internal line connecting the non-crossing lines obtained by reconnection. The additional line representing a quantum number exchange between the resulting non-crossing lines would guarantee the conservation of quantum numbers. This would bring in two additional vertices and one additional internal line. This would be enough to reduce planarity. The repeated application of this transformation should produced a sum of non-planar diagrams.

3. What could go wrong with this proposal? In the case of gauge theory the order of diagram increases by g^2 since two new vertices are generated. Should a multiplication by $1/g^2$ accompany this process? Or is this observation enough to kill the hypothesis in gauge theory framework? In TGD framework the situation is not understood well enough to say anything. Certainly the critical value of α_K implies that one cannot regard it as a free parameter and cannot treat the contributions from various orders as independent ones.

12.5 Could The Universe Be Doing Yangian Arithmetics?

One of the old TGD inspired really crazy ideas about scattering amplitudes is that Universe is doing some sort of arithmetics so that scattering amplitude are representations for computational sequences of minimum length. The idea is so crazy that I have even given up its original form, which led to an attempt to assimilate the basic ideas about bi-algebras, quantum groups [K8], Yangians [K86], and related exotic things. The work with twistor Grassmannian approach inspired a reconsideration of the original idea seriously with the idea that super-symplectic Yangian could define the arithmetics. I try to describe the background, motivation, and the ensuing reckless speculations in the following.

12.5.1 Do Scattering Amplitudes Represent Quantal Algebraic Manipulations?

I seems that tensor product $\otimes$ and direct sum $\oplus$ - very much analogous to product and sum but defined between Hilbert spaces rather than numbers - are naturally associated with the basic vertices of TGD. I have written about this a highly speculative chapter - both mathematically and physically [K59]. The chapter [K8] is a remnant of earlier similar speculations.

1. In $\otimes$ vertex 3-surface splits to two 3-surfaces meaning that the 2 "incoming" 4-surfaces meet at single common 3-surface and become the outgoing 3-surface: 3 lines of Feynman diagram meeting at their ends. This has a lower-dimensional shadow realized for partonic 2-surfaces. This topological 3-particle vertex would be higher-D variant of 3-vertex for Feynman diagrams.

2. The second vertex is trouser vertex for strings generalized so that it applies to 3-surfaces. It does not represent particle decay as in string models but the branching of the particle wave function so that particle can be said to propagate along two different paths simultaneously. In double slit experiment this would occur for the photon space-time sheets.

3. The idea is that Universe is doing arithmetics of some kind in the sense that particle 3-vertex in the above topological sense represents either multiplication or its time-reversal co-multiplication.

The product, call it $\circ$, can be something very general, say algebraic operation assignable to some algebraic structure. The algebraic structure could be almost anything: a random list of structures popping into mind consists of group, Lie-algebra, super-conformal algebra quantum algebra, Yangian, etc.... The algebraic operation $\circ$ can be group multiplication, Lie-bracket, its generalization to super-algebra level, etc...). Tensor product and thus linear (Hilbert) spaces are involved always, and in product operation tensor product $\otimes$ is replaced with $\circ$.

1. The product $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is analogous to a particle reaction in which particles A_k and A_l fuse to particle $A_k \otimes A_l \rightarrow C = A_k \circ A_l$. One can say that $\otimes$ between reactants is transformed to $\circ$ in the particle reaction: kind of bound state is formed.

2. There are very many pairs A_k, A_l giving the same product C just as given integer can be divided in many ways to a product of two integers if it is not prime. This of course suggests that elementary particles are primes of the algebra if this notion is defined for it! One can use some basis for the algebra and in this basis one has $C = A_k \circ A_l = f_{klm} A_m$, f_{klm} are the structure constants of the algebra and satisfy constraints. For instance, associativity $A(BC) = (AB)C$ is a constraint making the life of algebraist more tolerable and is almost routinely assumed.

   For instance, in the number theoretic approach to TGD associativity is proposed to serve as fundamental law of physics and allows to identify space-time surfaces as 4-surfaces with associative (quaternionic) tangent space or normal space at each point of octonionic embedding space $M^4 \times CP_2$. Lie algebras are not associative but Jacobi-identities following from the associativity of Lie group product replace associativity.

3. Co-product can be said to be time reversal of the algebraic operation $\circ$. Co-product can be defined as $C = A_k \rightarrow \sum_{lm} f_k^{lm} A_l \otimes A_m$, where f_k^{lm} are the structure constants of the algebra. The outcome is quantum superposition of final states, which can fuse to C (the "reaction" $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is possible). One can say that $\circ$ is replaced with $\otimes$: bound state decays to a superposition of all pairs, which can form the bound states by product vertex.

There are motivations for representing scattering amplitudes as sequences of algebraic operations performed for the incoming set of particles leading to an outgoing set of particles with particles identified as algebraic objects acting on vacuum state. The outcome would be analogous to Feynman diagrams but only the diagram with minimal length to which a preferred extremal can be assigned is needed. Larger ones must be equivalent with it.

The question is whether it could be indeed possible to characterize particle reactions as computations involving transformation of tensor products to products in vertices and co-products to tensor products in co-vertices (time reversals of the vertices). A couple of examples gives some idea about what is involved.

1. The simplest operations would preserve particle number and to just permute the particles: the permutation generalizes to a braiding and the scattering matrix would be basically unitary braiding matrix utilized in topological quantum computation.

2. A more complex situation occurs, when the number of particles is preserved but quantum numbers for the final state are not same as for the initial state so that particles must interact. This requires both product and co-product vertices. For instance, $A_k \otimes A_l \rightarrow f_{kl}^m A_m$ followed by $A_m \rightarrow f_m^{rs} A_r \otimes A_s$ giving $A_k \rightarrow f_{kl}^m f_m^{rs} A_r \otimes A_s$ representing 2-particle scattering. State function reduction in the final state can select any pair $A_r \otimes A_s$ in the final state. This reaction is characterized by the ordinary tree diagram in which two lines fuse to single line and defuse back to two lines. Note also that there is a non-deterministic element involved. A given final state can be achieved from a given initial state after large enough number of trials. The analogy with problem solving and mathematical theorem proving is obvious. If the interpretation is correct, Universe would be problem solver and theorem prover!

3. More complex reactions affect also the particle number. 3-vertex and its co-vertex are the simplest examples and generate more complex particle number changing vertices. For instance, on twistor Grassmann approach on can construct all diagrams using two 3-vertices. This encourages the restriction to 3-vertice (recall that fermions have only 2-vertices)

4. Intuitively it is clear that the final collection of algebraic objects can be reached by a large - maybe infinite - number of ways. It seems also clear that there is the shortest manner to end up to the final state from a given initial state. Of course, it can happen that there is no way to achieve it! For instance, if $\circ$ corresponds to group multiplication the co-vertex can lead only to a pair of particles for which the product of final state group elements equals to the initial state group element.

5. Quantum theorists of course worry about unitarity. How can avoid the situation in which the product gives zero if the outcome is element of linear space. Somehow the product should be such that this can be avoided. For instance, if product is Lie-algebra commutator, Cartan algebra would give zero as outcome.

12.5.2 Generalized Feynman Diagram As Shortest Possible Algebraic Manipulation Connecting Initial And Final Algebraic Objects

There is a strong motivation for the interpretation of generalized Feynman diagrams as shortest possible algebraic operations connecting initial and final states. The reason is that in TGD one does not have path integral over all possible space-time surfaces connecting the 3-surfaces at the ends of CD. Rather, one has in the optimal situation a space-time surface unique apart from conformal gauge degeneracy connecting the 3-surfaces at the ends of CD (they can have disjoint components).

Path integral is replaced with integral over 3-surfaces. There is therefore only single minimal generalized Feynman diagram (or twistor diagram, or whatever is the appropriate term). It would be nice if this diagram had interpretation as the shortest possible computation leading from the initial state to the final state specified by 3-surfaces and basically fermionic states at them. This would of course simplify enormously the theory and the connection to the twistor Grassmann approach is very suggestive. A further motivation comes from the observation that the state basis created by the fermionic Clifford algebra has an interpretation in terms of Boolean quantum logic and that in ZEO the fermionic states would have interpretation as analogs of Boolean statements $A \rightarrow B$.

To see whether and how this idea could be realized in TGD framework, let us try to find counterparts for the basic operations $\otimes$ and $\circ$ and identify the algebra involved. Consider first the basic geometric objects.

1. Tensor product could correspond geometrically to two disjoint 3-surfaces representing 3-particles. Partonic 2-surfaces associated with a given 3-surface represent second possibility. The splitting of a partonic 2-surface to two could be the geometric counterpart for co-product.

2. Partonic 2-surfaces are however connected to each other and possibly even to themselves by strings. It seems that partonic 2-surface cannot be the basic unit. Indeed, elementary particles are identified as pairs of wormhole throats (partonic 2-surfaces) with magnetic monopole flux flowing from throat to another at first space-time sheet, then through throat to another sheet, then back along second sheet to the lower throat of the first contact and then back to the thirst throat. This unit seems to be the natural basic object to consider. The flux tubes at both sheets are accompanied by fermionic strings. Whether also wormhole throats contain strings so that one would have single closed string rather than two open ones, is an open question.

3. The connecting strings give rise to the formation of gravitationally bound states and the hierarchy of Planck constants is crucially involved. For elementary particle there are just two wormhole contacts each involving two wormhole throats connected by wormhole contact. Wormhole throats are connected by one or more strings, which define space-like boundaries of corresponding string world sheets at the boundaries of CD. These strings are responsible for the formation of bound states, even macroscopic gravitational bound states.

12.5.3 Does Super-Symplectic Yangian Define The Arithmetics?

Super-symplectic Yangian would be a reasonable guess for the algebra involved.

1. The 2-local generators of Yangian would be of form $T_1^A = f^A_{BC} T^B \otimes T^C$, where f^A_{BC} are the structure constants of the super-symplectic algebra. n-local generators would be obtained by iterating this rule. Note that the generator T_1^A creates an entangled state of T^B and T^C with f^A_{BC} the entanglement coefficients. T_n^A is entangled state of T^B and T^C_{n-1} with the same coefficients. A kind replication of T^A_{n-1} is clearly involved, and the fundamental replication is that of T^A. Note that one can start from any irreducible representation with well defined symplectic quantum numbers and form similar hierarchy by using T^A and the representation as a starting point.

   That the hierarchy T_n^A and hierarchies irreducible representations would define a hierarchy of states associated with the partonic 2-surface is a highly non-trivial and powerful hypothesis about the formation of many-fermion bound states inside partonic 2-surfaces.

2. The charges T^A correspond to fermionic and bosonic super-symplectic generators. The geometric counterpart for the replication at the lowest level could correspond to a fermionic/bosonic string carrying super-symplectic generator splitting to fermionic/bosonic string and a string carrying bosonic symplectic generator T^A. This splitting of string brings in mind the basic gauge boson-gauge boson or gauge boson-fermion vertex.

   The vision about emission of virtual particle suggests that the entire wormhole contact pair replicates. Second wormhole throat would carry the string corresponding to T^A assignable to gauge boson naturally. T^A should involve pairs of fermionic creation and annihilation operators as well as fermionic and anti-fermionic creation operator (and annihilation operators) as in quantum field theory.

3. Bosonic emergence suggests that bosonic generators are constructed from fermion pairs with fermion and anti-fermion at opposite wormhole throats: this would allow to avoid the problems with the singular character of purely local fermion current. Fermionic and anti-fermionic string would reside at opposite space-time sheets and the whole structure would correspond to a closed magnetic tube carrying monopole flux. Fermions would correspond to superpositions of states in which string is located at either half of the closed flux tube.

4. The basic arithmetic operation in co-vertex would be co-multiplication transforming T_n^A to $T^A_{n+1} = f^A_{BC} T^B_n \otimes T^C$. In vertex the transformation of T^A_{n+1} to T^A_n would take place. The interpretations would be as emission/absorption of gauge boson. One must include also emission of fermion and this means replacement of T^A with corresponding fermionic generators F^A, so that the fermion number of the second part of the state is reduced by one unit. Particle reactions would be more than mere braidings and re-grouping of fermions and anti-fermions inside partonic 2-surfaces, which can split.

5. Inside the light-like orbits of the partonic 2-surfaces there is also a braiding affecting the M-matrix. The arithmetics involved would be therefore essentially that of measuring and "co-measuring" symplectic charges.

   Generalized Feynman diagrams (preferred extremals) connecting given 3-surfaces and many-fermion states (bosons are counted as fermion-anti-fermion states) would have a minimum number of vertices and co-vertices. The splitting of string lines implies creation of pairs of fermion lines. Whether regroupings are part of the story is not quite clear. In any case, without the replication of 3-surfaces it would not be possible to understand processes like e-e scattering by photon exchange in the proposed picture.

It is easy to hear the comments of the skeptic listener in the back row.

1. The attribute "minimal" - , which could translate to minimal value of Kähler function - is dangerous. It might be very difficult to determine what the minimal diagram is - consider only travelling salesman problem or the task of finding the shortest proof of theorem. It would be much nicer to have simple calculational rules.

The original proposal might help here. The generalization of string model duality was in question. It stated that it is possible to move the positions of the vertices of the diagrams just as one does to transform s-channel resonances to t-channel exchange. All loops of generalized diagrams could be be eliminated by transforming the to tadpoles and snipped away so that only tree diagrams would be left. The variants of the diagram were identified as different continuation paths between different paths connecting sectors of WCW corresponding to different 3-topologies. Each step in the continuation procedure would involve product or co-product defining what continuation between two sectors means for WCW spinors. The continuations between two states require some minimal number of steps. If this is true, all computations connecting identical states are also physically equivalent. The value of the vacuum functional be same for all of them. This looks very natural.

That the Kähler action should be same for all computational sequences connecting the same initial and final states looks strange but might be understood in terms of the vacuum degeneracy of Kähler action.

2. QFT perturbation theory requires that should have superposition of computations/continuations. What could the superposition of QFT diagrams correspond to in TGD framework?

   Could it correspond to a superposition of generators of the Yangian creating the physical state? After all, already quantum computer perform superpositions of computations. The fermionic state would not be the simplest one that one can imagine. Could AdS/CFT analogy allow to identify the vacuum state as a superposition of multi-string states so that single super-symplectic generator would be replaced with a superposition of its Yangian counterparts with same total quantum numbers but with a varying number of strings? The weight of a given superposition would be given by the total effective string world sheet area. The sum of diagrams would emerge from this superposition and would basically correspond to functional integration in WCW using exponent of Kähler action as weight. The stringy functional integral ("functional" if also wormhole contacts contain string portion, otherwise path integral) would give the perturbation theory around given string world sheet. One would have effective reduction of string theory.

12.5.4 How Does This Relate To The Ordinary Perturbation Theory?

One can of course worry about how to understand the basic results of the usual perturbation theory in this picture. How does one obtain a perturbation theory in powers of coupling constant, what does running coupling constant mean, etc...? I have already discussed how the superposition of diagrams could be understood in the new picture.

1. The QFT picture with running coupling constant is expected at QFT limit, when many-sheeted space-time is replaced with a slightly curved region of M^4 and gravitational field and gauge potentials are identified as sums of the deviations of induced metric from M^4 metric and classical induced gauge potentials associated with the sheets of the many-sheeted space-time. The running coupling constant would be due to the dependence of the size scale of CD, and p-adic coupling constant evolution would be behind the continuous one.

2. The notion of running coupling constant is very physical concept and should have a description also at the fundamental level and be due to a finite computational resolution, which indeed has very concrete description in terms of Noether charges of super-symplectic Yangian creating the states at the ends of space-time surface at the boundaries of CD. The space-time surface and the diagram associated with a given pair of 3-surfaces and stringy Noether charges associated with them can be characterized by a complexity measured in terms of the number of vertices (3-surface at which three 3-surfaces meet).

   For instance, 3-particle scattering can be possible only by using the simplest 3-vertex defined by product or co-product for pairs of 3-surfaces. In the generic case one has more complex diagram and what looks first 3-particle vertex has complex substructure rather than being simple product or co-product.

3. Complexity seems to have two separate aspects: the complexities of the positive and negative parts of zero energy state as many-fermion states and the complexity of associated 3-surfaces.

The generalization of AdS/CFT however suggests that once the string world sheets and partonic 2-surfaces appearing in the diagram have been fixed, the space-time surface itself is fixed. The principle also suggests that the fixing partonic 2-surface and the strings connecting them at the boundaries of CD fixes the 3-surface apart from the action of sub-algebra of Yangian acting as gauge algebra (vanishing classical Noether charges). If one can determine the minimal sequence of allowed algebraic operation of Yangian connecting initial and final fermion states, one knows the minimum number of vertices and therefore the topological structure of the connecting minimal space-time surface.

4. In QFT spirit one could describe the finite measurement resolution by introducing effective 3-point vertex, which is need not be product/co-produce anymore. 3-point scattering amplitudes in general involve microscopic algebraic structure involving several vertices. One can however give up the nice algebraic interpretation and just talk about effective 3-vertex for practical purposes. Just as the QFT vertex described by running coupling constant decomposes to sum of diagrams, product/co-product in TGD could be replaced with effective product/co-product expressible as a longer computation. This would imply coupling constant evolution.

   Fermion lines could however remain as such since they are massless in 8-D sense and mass renormalization does not make sense.

   Similar practical simplification could be done the initial and final states to get rid of superposition of the Yangian generators with different numbers of strings ("cloud of virtual particles"). This would correspond to wave function renormalization.

5. The number of vertices and wormhole contact orbits serves as a measure for the complexity of the diagram. Since fermion lines are associated with wormhole throats assignable with wormhole contacts identifiable as deformations CP_2 type vacuum extremals, one expects that the exponent of the Kähler function defining vacuum functional is in the first approximation the total CP_2 volume of wormhole contacts giving a measure for the importance of the contribution in functional integral. If it converges very rapidly only Gaussian approximation around maximum is needed.

6. Convergence depends on how large the fraction of volume of CP_2 is associated with a given wormhole contact. The volume is proportional to the length of the wormhole contact orbit. One expects exponential convergence with the number of fermion lines and their lengths for long lines. For short distances the exponential damping is small so that diagrams with microscopic structure of diagrams are needed and are possible. This looks like adding small scale details to the algebraic manipulations.

7. One must be of course be very cautious in making conclusions. The presence of $1/\alpha_K \propto h_{eff}$ in the exponent of Kähler function would suggest that for large values of h_{eff} only the 3-surfaces with smallest possible number of wormhole contact orbits contribute. On the other hand, the generalization of AdS/CFT duality suggests that Kähler action reducible to area of string world sheet in the effective metric defined by canonical momentum currents of Kähler action behaves as $\alpha_K^2 \propto 1/h_{eff}^2$. What does this mean?

To sum up, the identification of vertex as a product or co-product in Yangian looks highly promising approach. The Nother charges of the super-symplectic Yangian are associated with strings and are either linear or bilinear in the fermion field. The fermion fields associated with the partonic 2-surface defining the vertex are contracted with fermion fields associated with other partonic 2-surfaces using the same rule as in Wick expansion in quantum field theories. The contraction gives fermion propagator for each leg pair associated with two vertices. Vertex factor is proportional to the contraction of spinor modes with the operators defining the Noether charge or super charge and essentially Kähler-Dirac gamma matrix and the representation of the action of the symplectic generator on fermion realizable in terms of sigma matrices. This is very much like the corresponding expression in gauge theories but with gauge algebra replaced with symplectic algebra. The possibility of contractions of creation and annihilation operator for fermion lines associated with opposite wormhole throats at the same partonic 2-surface (for Noether charge bilinear in fermion field) gives bosonic exchanges as lines in which the fermion lines turns in time direction: otherwise only regroupings of fermions would take place.

12.5.5 This Was Not The Whole Story Yet

The proposed amplitude represents only the value of WCW spinor field for single pair of 3-surfaces at the opposite boundaries of given CD. Hence Yangian construction does not tell the whole story.

1. Yangian algebra would give only the vertices of the scattering amplitudes. On basis of previous considerations, one expects that each fermion line carries propagator defined by 8-momentum. The structure would resemble that of super-symmetric YM theory. Fermionic propagators should emerge from summing over intermediate fermion states in various vertices and one would have integrations over virtual momenta which are carried as residue integrations in twistor Grassmann approach. 8-D counterpart of twistorialization would apply.

2. Super-symplectic Yangian would give the scattering amplitudes for single space-time surface and the purely group theoretical form of these amplitudes gives hopes about the independence of the scattering amplitude on the pair of 3-surfaces at the ends of CD near the maximum of Kähler function. This is perhaps too much to hope except approximately but if true, the integration over WCW would give only exponent of Kähler action since metric and poorly defined Gaussian and determinants would cancel by the basic properties of Kähler metric. Exponent would give a non-analytic dependence on α_K.

   The Yangian supercharges are proportional to $1/\alpha_K$ since covariant Kähler-Dirac gamma matrices are proportional to canonical momentum currents of Kähler action and thus to $1/\alpha_K$. Perturbation theory in powers of $\alpha_K = g_K^2/4\pi\hbar_{eff}$ is possible after factorizing out the exponent of vacuum functional at the maximum of Kähler function and the factors $1/\alpha_K$ multiplying super-symplectic charges.

   The additional complication is that the characteristics of preferred extremals contributing significantly to the scattering amplitudes are expected to depend on the value of α_K by quantum interference effects. Kähler action is proportional to $1/\alpha_K$. The analogy of AdS/CFT correspondence states the expressibility of Kähler function in terms of string area in the effective metric defined by the anti-commutators of K-D matrices. Interference effects eliminate string length for which the area action has a value considerably larger than one so that the string length and thus also the minimal size of CD containing it scales as h_{eff}. Quantum interference effects therefore give an additional dependence of Yangian super-charges on h_{eff} leading to a perturbative expansion in powers of α_K although the basic expression for scattering amplitude would not suggest this.

3. Non-planar diagrams of quantum field theories should have natural counterpart and linking and knotting for braids defines it naturally. This suggests that the amplitudes can be interpreted as generalizations of braid diagrams defining braid invariants: braid strands would appear as legs of 3-vertices representing product and co-product. Amplitudes could be constructed as generalized braid invariants transforming recursively braided tree diagram to an un-braided diagram using same operations as for braids. In [L26] I considered a possible breaking of associativity occurring in weak sense for conformal field theories and was led to the vision that there is a fractal hierarchy of braids such that braid strands themselves correspond to braids. This hierarchy would define an operad with subgroups of permutation group in key role. Hence it seems that various approaches to the construction of amplitudes converge.

12.6 Appendix: Some Mathematical Details About Grassmannian Formalism

In the following I try to summarize my amateurish understanding about the mathematical structure behind the Grassmann integral approach. The representation summarizes what I have gathered from the articles of Arkani-Hamed and collaborators [B31, B33]. These articles are rather sketchy and the article of Bullimore provides additional details [B50] related to soft factors. The article of Mason and Skinner provides excellent introduction to super-twistors [B25] and dual superconformal invariance. I apologize for unavoidable errors.

Before continuing a brief summary about the history leading to the articles of Arkani-Hamed and others is in order. This summary covers only those aspects which I am at least somewhat familiar with and leaves out many topics about existence which I am only half-conscious.

1. It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle \lambda, \mu \rangle &= \epsilon_{ab}\lambda^a \mu^b \ , \\ \left[\tilde{\lambda}, \tilde{\mu}\right] &= \epsilon_{a'b'}\lambda^{a'}\mu^{b'} \ , \\ p \cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu}\right] \ , \quad (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ . \end{aligned} \tag{12.6.1}$$

If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \quad \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda}\right]} \ , \quad \text{negative helicity} \ . \end{aligned} \tag{12.6.2}$$

In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

2. Tree amplitudes are considered and it is convenient to drop the group theory factor $Tr(T_1 T_2 \cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity-taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{12.6.3}$$

When the sign of the helicities is changed $\langle .. \rangle$ is replaced with $[..]$.

3. The article of Witten [B28] proposed that twistor approach could be formulated as a twistor string theory with string world sheets "living" in 6-dimensional CP_3 possessing Calabi-Yau structure and defining twistor space. In this article Witten introduced what is known as half Fourier transform allowing to transform momentum integrals over light-cone to twistor integrals. This operation makes sense only in space-time signature $(2, 2)$. Witten also demonstrated that maximal helicity violating (MHV) twistor amplitudes (two gluons with negative helicity) with n particles with $k + 2$ negative helicities and l loops correspond in this approach to holomorphic 2-surfaces defined by polynomials defined by polynomials of degree $D = k - 1 + l$, where the genus of the surface satisfies $g \leq l$. AdS/CFT duality provides a second stringy approach to $\mathcal{N} = 4$ theory allowing to understand the scattering amplitudes in terms of Wilson loops with light-like edges: about this I have nothing to say. In any case, the generalization of twistor string theory to TGD context is highly attractive idea and will be considered later.

4. In the article [B21] Cachazo, Svrcek, and Witten propose the analog of Feynman diagrammatics in which MHV amplitudes can be used as analogs of vertices and ordinary $1/P^2$ propagator as propagator to construct tree diagrams with arbitrary number of negative helicity gluons. This approach is not symmetric with respect to the change of the sign of helicities since the amplitudes with two positive helicities are constructed as tree diagrams. The construction is non-trivial because one must analytically continue the on mass shell tree amplitudes to off mass shell momenta. The problem is how to assign a twistor to these momenta. This is achieved by introducing an arbitrary twistor $\eta^{a'}$ and defining λ_a as $\lambda_a = p_{aa'}\eta^{a'}$. This works for both massless and massive case. It however leads to a loss of the manifest Lorentz invariance. The paper however argues and the later paper [B20, B20] shows rigorously that the loss is only apparent. In this paper also BCFW recursion formula is introduced allowing to construct tree amplitudes recursively by starting from vertices with 2 negative helicity gluons. Also the notion which has become known as BCFW bridge representing the massless exchange in these diagrams is introduced. The tree amplitudes are not tree amplitudes in gauge theory sense where correspond to leading singularities for which 4 or more lines of the loop are massless and therefore collinear. What is important that the very simple MHV amplitudes become the building blocks of more complex amplitudes.

5. The nex step in the progress was the attempt to understand how the loop corrections could be taken into account in the construction BCFW formula. The calculation of loop contributions to the tree amplitudes revealed the existence of dual super-conformal symmetry which was found to be possessed also by BCFW tree amplitudes besides conformal symmetry. Together these symmetries generate infinite-dimensional Yangian symmetry [B25].

6. The basic vision of Arkani-Hamed and collaborators is that the scattering amplitudes of $\mathcal{N} = 4$ SYM are constructible in terms of leading order singularities of loop diagrams. These singularities are obtained by putting maximum number of momenta propagating in the lines of the loop on mass shell. The non-leading singularities would be induced by the leading singularities by putting smaller number of momenta on mass shell are dictated by these terms. A related idea serving as a starting point in [B31] is that one can define loop integrals as residue integrals in momentum space. If I have understood correctly, this means that one an imagine the possibility that the loop integral reduces to a lower dimensional integral for on mass shell particles in the loops: this would resemble the approach to loop integrals based on unitarity and analyticity. In twistor approach these momentum integrals defined as residue integrals transform to residue integrals in twistor space with twistors representing massless particles. The basic discovery is that one can construct leading order singularities for n particle scattering amplitude with $k+2$ negative helicities as Yangian invariants $Y_{n,k}$ for momentum twistors and invariants constructed from them by canonical operations changing n and k. The correspondence $k = l$ does not hold true for the more general amplitudes anymore.

12.6.1 Yangian Algebra And Its Super Counterpart

The article of Witten [B24] gives a nice discussion of the Yangian algebra and its super counterpart. Here only basic formulas can be listed and the formulas relevant to the super-conformal case are given.

Yangian algebra

Yangian algebra $Y(G)$ is associative Hopf algebra. The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [B24] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{12.6.4}$$

Besides this Serre relations are satisfied. These have more complex and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^{C}\right]\right]+\left[J^{(1)B},\left[J^{(1)C},J^{A}\right]\right]+\left[J^{(1)C},\left[J^{(1)A},J^{B}\right]\right] \\
&\qquad = \frac{1}{24}f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J_D,J_E,J_F\} \ , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^{C},J^{(1)D}\right]\right]+\left[\left[J^{(1)C},J^{(1)D}\right],\left[J^{A},J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24}(f^{AGL}f^{BEM}f_K^{CD} \\
&\qquad +f^{CGL}f^{DEM}f_K^{AB})f^{KFN}f_{LMN}\{J_G,J_E,J_F\} \ .
\end{aligned} \tag{12.6.5}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A=\sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} are given as

$$J^{(1)A} = f^A_{BC}\sum_{i<j}J_i^BJ_j^C \ . \tag{12.6.6}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R\otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N}=4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned}
\Delta(J^A) &= J^A\otimes 1+1\otimes J^A \ , \\
\Delta(J^{(1)A}) &= J^{(1)A}\otimes 1+1\otimes J^{(1)A}+f^A_{BC}J^B\otimes J^C
\end{aligned} \tag{12.6.7}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B24].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n = m$ is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f_{AB}^{C'} \sum_{i<j} J_i^A J_j^B \\ &= g_{CC'} f_{AB}^{C'} g^{AA'} g^{BB'} \sum_{i<j} J_{A'}^i J_{B'}^j \ . \end{aligned} \tag{12.6.8}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

Generators of super-conformal Yangian symmetries

The explicit formula for the generators of super-conformal Yangian symmetries in terms of ordinary twistors is given by

$$\begin{aligned} j_B^A &= \sum_{i=1}^{n} Z_i^A \partial_{Z_i^B} \ , \\ j_B^{(1)A} &= \sum_{i<j} (-1)^C \left[Z_i^A \partial_{Z_j^C} Z_j^C \partial_{Z_j^B} \right] \ . \end{aligned} \tag{12.6.9}$$

This formula follows from completely general formulas for the Yangian algebra discussed above and allowing to express the dual generators $j_N^{(1)}$ as quadratic expression of j_N involving structures constants. In this rather sketchy formula twistors are ordinary twistors. Note however that in the recent case the lattice is replaced with its finite cutoff corresponding to the external particles of the scattering amplitude. This probably corresponds to the assumption that for the representations considered only finite number of lattice points correspond to non-trivial quantum numbers or to cyclic symmetry of the representations.

In the expression for the amplitudes the action of transformations is on the delta functions and by partial integration one finds that a total divergence results. This is easy to see for the linear generators but not so for the quadratic generators of the dual super-conformal symmetries. A similar formula but with j_B^A and $j_B^{(1)A}$ interchanged applies in the representation of the amplitudes as Grassmann integrals using ordinary twistors. The verification of the generalization of Serre formula is also straightforward.

12.6.2 Twistors And Momentum Twistors And Super-Symmetrization

In [B25] the basics of twistor geometry are summarized. Despite this it is perhaps good to collect the basic formulas here.

Conformally compactified Minkowski space

Conformally compactified Minkowski space can be described as $SO(2,4)$ invariant (Klein) quadric

$$T^2+V^2-W^2-X^2-Y^2-Z^2=0 \ . \tag{12.6.10}$$

The coordinates (T,V,W,X,Y,Z) define homogenous coordinates for the real projective space RP^5. One can introduce the projective coordinates $X_{\alpha\beta}=-X_{\beta\alpha}$ through the formulas

$$\begin{array}{lll} X_{01}=W-V \ , & X_{02}=Y+iX \ , & X_{03}=\frac{i}{\sqrt{2}}T-Z \ , \\ X_{12}=-\frac{i}{\sqrt{2}}(T+Z) \ , & X_{13}=Y-iX \ , & X_{23}=\frac{1}{2}(V+W) \ . \end{array} \tag{12.6.11}$$

The motivation is that the equations for the quadric defining the conformally compactified Minkowski space can be written in a form which is manifestly conformally invariant:

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} \ = \ 0 \ per. \tag{12.6.12}$$

The points of the conformally compactified Minkowski space are null separated if and only if the condition

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta}=0 \tag{12.6.13}$$

holds true.

Correspondence with twistors and infinity twistor

One ends up with the correspondence with twistors by noticing that the condition is equivalent with the possibility to expression $X_{\alpha\beta}$ as

$$X_{\alpha\beta}=A_{[\alpha}B_{\beta]} \ , \tag{12.6.14}$$

where brackets refer to antisymmetrization. The complex vectors A and B define a point in twistor space and are defined only modulo scaling and therefore define a point of twistor space CP_3 defining a covering of 6-D Minkowski space with metric signature $(2,4)$. This corresponds to the fact that the Lie algebras of $SO(2,4)$ and $SU(2,2)$ are identical. Therefore the points of conformally compactified Minkowski space correspond to lines of the twistor space defining spheres CP_1 in CP_3.

One can introduce a preferred scale for the projective coordinates by introducing what is called infinity twistor (actually a pair of twistors is in question) defined by

$$I_{\alpha\beta} = \begin{pmatrix} \epsilon^{A'B'} & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{12.6.15}$$

Infinity twistor represents the projective line for which only the coordinate X_{01} is non-vanishing and chosen to have value $X_{01} = 1$.

One can define the contravariant form of the infinite twistor as

$$I^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} \quad . \tag{12.6.16}$$

Infinity twistor defines a representative for the conformal equivalence class of metrics at the Klein quadric and one can express Minkowski distance as

$$(x-y)^2 = \frac{X^{\alpha\beta} Y_{\alpha\beta}}{I_{\alpha\beta} X^{\alpha\beta} I_{\mu\nu} Y^{\mu\nu}} \quad . \tag{12.6.17}$$

Note that the metric is necessary only in the denominator. In twistor notation the distance can be expressed as

$$(x-y)^2 = \frac{\epsilon(A,B,C,D)}{\langle AB\rangle\langle CD\rangle} \quad . \tag{12.6.18}$$

Infinite twistor $I_{\alpha\beta}$ and its contravariant counterpart project the twistor to its primed and unprimed parts usually denoted by $\mu^{A'}$ and λ^A and defined spinors with opposite chiralities.

Relationship between points of M^4 and twistors

In the coordinates obtained by putting $X_{01} = 1$ the relationship between space-time coordinates $x^{AA'}$ and $X^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{A'B'} x^2 & -ix^{A'}_{\ \ B} \\ ix_{\ \ A}^{B'} & \epsilon_{A,B} \end{pmatrix} \quad , \quad X^{\alpha\beta} = \begin{pmatrix} \epsilon_{A'B}{}' x^2 & -ix_{\ \ A'}^{B} \\ ix^A_{B'} & -\frac{1}{2}\epsilon^{AB} x^2 \end{pmatrix} \quad , \tag{12.6.19}$$

If the point of Minkowski space represents a line defined by twistors (μ_U, λ_U) and (μ_V, λ_V), one has

$$x^{AC'} = i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)^{AC'}}{\langle UV\rangle} \tag{12.6.20}$$

The twistor μ for a given point of Minkowski space in turn is obtained from λ by the twistor formula by

$$\mu^{A'} = -ix^{AA'}\lambda_A \quad . \tag{12.6.21}$$

Generalization to the super-symmetric case

This formalism has a straightforward generalization to the super-symmetric case. CP_3 is replaced with $CP_{3|4}$ so that Grassmann parameters have four components. At the level of coordinates this means the replacement $[W_I] = [W_\alpha, \chi_\alpha]$. Twistor formula generalizes to

$$\mu^{A'} = -ix^{AA'}\lambda_A \quad , \quad \chi_\alpha = \theta^A_\alpha \lambda_A \quad . \tag{12.6.22}$$

The relationship between the coordinates of chiral super-space and super-twistors generalizes to

$$(x,\theta) \quad = \quad \left(i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)}{\langle UV\rangle}, \frac{(\chi_V\lambda_U - \chi_U\lambda_V)}{\langle UV\rangle}\right) \tag{12.6.23}$$

The above formulas can be applied to super-symmetric variants of momentum twistors to deduce the relationship between region momenta x assigned with edges of polygons and twistors assigned with the ends of the light-like edges. The explicit formulas are represented in [B25]. The geometric picture is following. The twistors at the ends of the edge define the twistor pair representing the region momentum as a line in twistor space and the intersection of the twistor lines assigned with the region momenta define twistor representing the external momenta of the graph in the intersection of the edges.

Basic kinematics for momentum twistors

The super-symmetrization involves replacement of multiplets with super-multiplets

$$\Phi(\lambda,\tilde{\lambda},\eta) = G^+(\lambda,\tilde{\lambda}) + \eta_i\Gamma^a\lambda,\tilde{\lambda}) + \cdots + \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^dG^-(\lambda,\tilde{\lambda}) \ . \tag{12.6.24}$$

Momentum twistors are dual to ordinary twistors and were introduced by Hodges. The light-like momentum of external particle a is expressed in terms of the vertices of the closed polygon defining the twistor diagram as

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = \lambda_i\tilde{\lambda}_i \ , \ \ \theta_i - \theta_{i+1} = \lambda_i\eta_i \ . \tag{12.6.25}$$

One can say that massless momenta have a conserved super-part given by $\lambda_i\eta_i$. The dual of the super-conformal group acts on the region momenta exactly as the ordinary conformal group acts on space-time and one can construct twistor space for dua region momenta.

Super-momentum conservation gives the constraints

$$\sum p_i = 0 \ \ , \ \ \sum\lambda_i\eta_i = 0 \ . \tag{12.6.26}$$

The twistor diagrams correspond to polygons with edges with lines carrying region momenta and external massless momenta emitted at the vertices.

This formula is invariant under overall shift of the region momenta x_a^μ. A natural interpretation for x_a^μ is as the momentum entering to the vertex where p_a is emitted. Overall shift would have interpretation as a shift in the loop momentum. x_a^μ in the dual coordinate space is associated with the line $Z_{a-1}Z_a$ in the momentum twistor space. The lines $Z_{a-1}Z_a$ and Z_aZ_{a+1} intersect at Z_a representing a light-like momentum vector p_a^μ.

The brackets $\langle abcd\rangle \equiv \epsilon_{IJKL}Z_a^IZ_b^JZ_c^KZ_d^L$ define fundamental bosonic conformal invariants appearing in the tree amplitudes as basic building blocks. Note that Z_a define points of 4-D complex twistor space to be distinguished from the projective twistor space CP_3. Z_a define projective coordinates for CP_3 and one of the four complex components of Z_a is redundant and one can take $Z_a^0 = 1$ without a loss of generality.

12.6.3 Brief Summary Of The Work Of Arkani-Hamed And Collaborators

The following comments are an attempt to summarize my far from complete understanding about what is involved with the representation as contour integrals. After that I shall describe in more detail my impressions about what has been done.

Limitations of the approach

Consider first the limitations of the approach.

1. The basis idea is that the representation for tree amplitudes generalizes to loop amplitudes. On other words, the amplitude defined as a sum of Yangian invariants expressed in terms of Grassmann integrals represents the sum of loops up to some maximum loop number. The problem is here that shifts of the loop momenta are essential in the UV regularization procedure. Fixing the coordinates $x_1, \cdots, x_n$ having interpretation as momenta associated with lines in the dual coordinate space allows to eliminate the non-uniqueness due to the common shift of these coordinates.

2. It is not however not possible to identify loop momentum as a loop momentum common to different loop integrals unless one restricts to planar loops. Non-planar diagrams are obtained from a planar diagram by permuting the coordinates x_i but this means that the unique coordinate assignment is lost. Therefore the representation of loop integrands as Grassmann integrals makes sense only for planar diagrams. From TGD point of view one could argue that this is one good reason for restricting the loops so that they are for on mass shell particles with non-parallel on mass shell four-momenta and possibly different sign of energies for given wormhole contact representing virtual particle.

3. IR regularization is needed even in $\mathcal{N} = 4$ for SYM given by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory.

What has been done?

The article proposes a generalization of the BCFW recursion relation for tree diagrams of $\mathcal{N} = 4$ for SYM so that it applies to planar diagrams with a summation over an arbitrary number of loops.

1. The basic goal of the article is to generalize the recursion relations of tree amplitudes so that they would apply to loop amplitudes. The key idea is following. One can formally represent loop integrand as a contour integral in complex plane whose coordinate parameterizes the deformations $Z_n \rightarrow Z_n + \epsilon Z_{n-1}$ and re-interpret the integral as a contour integral with oppositely oriented contour surrounding the rest of the complex plane which can be imagined also as being mapped to Riemann sphere. What happens only the poles which correspond to lower number of loops contribute this integral. One obtains a recursion relation with respect to loop number. This recursion seems to be the counterpart for the recursive construction of the loops corrections in terms of absorptive parts of amplitudes with smaller number of loop using unitarity and analyticity.

2. The basic challenge is to deduce the Grassmann integrands as Yangian invariants. From these one can deduce loop integrals by integration over the four momenta associated with the lines of the polygonal graph identifiable as the dual coordinate variables x_a. The integration over loop momenta can induce infrared divergences breaking Yangian symmetry. The big idea here is that the operations described above allow to construct loop amplitudes from the Yangian invariants defining tree amplitudes for a larger number of particles by removing external particles by fusing them to form propagator lines and by using the BCFW bridge to fuse lower-dimensional invariants. Hence the usual iterative procedure (bottom-up) used to construct scattering amplitudes is replaced with a recursive procedure (top-down). Of course, once lower amplitudes has been constructed they can be used to construct amplitudes with higher particle number.

3. The first guess is that the recursion formula involves the same lower order contributions as in the case of tree amplitudes. These contributions have interpretation as factorization of channels involving single particle intermediate states. This would however allow to reduce loop amplitudes to 3-particle loop amplitudes which vanish in $\mathcal{N} = 4$ SYM by the vanishing of coupling constant renormalization. The additional contribution is necessary and corresponds to a source term identifiable as a "forward limit" of lower loop integrand. These terms are obtained by taking an amplitude with two additional particles with opposite four-momenta

and forming a state in which these particles are entangled with respect to momentum and other quantum numbers. Entanglement means integral over the massless momenta on one hand. The insertion brings in two momenta x_a and x_b and one can imagine that the loop is represented by a branching of propagator line. The line representing the entanglement of the massless states with massless momentum define the second branch of the loop. One can of course ask whether only massless momentum in the second branch. A possible interpretation is that this state is expressible by unitarity in terms of the integral over light-like momentum.

4. The recursion formula for the loop amplitude $M_{n,k,l}$ involves two terms when one neglects the possibility that particles can also suffer trivial scattering (cluster decomposition). This term basically corresponds to the Yangian invariance of n arguments identified as Yangian invariant of $n-1$ arguments with the same value of k.

    (a) The first term corresponds to single particle exchange between particle groups obtained by splitting the polygon at two vertices and corresponds to the so called BCFW bridge for tree diagrams. There is a summation over different splittings as well as a sum over loop numbers and dimensions k for the Grassmann planes. The helicities in the two groups are opposite.

    (b) Second term is obtained from an amplitude obtained by adding of two massless particles with opposite momenta and corresponds to $n+2, k+1, l-1$. The integration over the light-like momentum together with other operations implies the reduction $n+2 \rightarrow n$. Note that the recursion indeed converges. Certainly the allowance of added zero energy states with a finite number of particles is necessary for the convergence of the procedure.

12.6.4 The General Form Of Grassmannian Integrals

If the recursion formula proposed in [B33] is correct, the calculations reduce to the construction of N^kMHV (super) amplitudes. MHV refers to maximal helicity violating amplitudes with 2 negative helicity gluons. For N^kMHV amplitude the number of negative helicities is by definition $k+2$ [B31]. Note that the total right handed R-charge assignable to 4 super-coordinates η_i of negative helicity gluons can be identified as $R=4k$. BCFW recursion formula [B20, B20] allows to construct from MHV amplitudes with arbitrary number of negative helicities.

The basic object of study are the leading singularities of color-stripped n-particle N^kMHV amplitudes. The discovery is that these singularities are expressible in terms Yangian invariants $Y_{n,k}(Z_1,\cdots,Z_n)$, where Z_i are momentum super-twistors. These invariants are defined by residue integrals over the compact $nk-1$-dimensional complex space $G(n,k) = U(n)/U(k) \times U(n-k)$ of k-planes of complex n-dimensional space. n is the number of external massless particles, k is the number negative helicity gluons in the case of N^kMHV amplitudes, and Z_a, $i=1,\cdots,n$ denotes the projective 4-coordinate of the super-variant $CP^{3|4}$ of the momentum twistor space CP_3 assigned to the massless external particles is following. $Gl(n)$ acts as linear transformations in the n-fold Cartesian power of twistor space. Yangian invariant $Y_{n,k}$ is a function of twistor variables Z^a having values in super-variant $CP_{3|3}$ of momentum twistor space CP_3 assigned to the massless external particles being simple algebraic functions of the external momenta.

It is also possible to define N^kMHV amplitudes in terms of Yangian invariants $L_{n,k+2}(W_1,\cdots,W_n)$ by using ordinary twistors W_a and identical defining formula. The two invariants are related by the formula $L_{n,k+2}(W_1,\cdots,W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1,\cdots,Z_n)$. Here M^{tree}_{MHV} is the tree contribution to the maximally helicity violating amplitude for the scattering of n particles: recall that these amplitudes contain two negative helicity gluons whereas the amplitudes containing a smaller number of them vanish [B21]. One can speak of a factorization to a product of n-particle amplitudes with $k-2$ and 2 negative helicities as the origin of the duality. The equivalence between the descriptions based on ordinary and momentum twistors states the dual conformal invariance of the amplitudes implying Yangian symmetry. It has been conjectured that Grassmannian integrals generate all Yangian invariants.

The formulas for the Grassmann integrals for twistors and momentum twistors appearing in the expressions of N^kMHV amplitudes are given by following expressions.

1. The integrals $L_{n,k}(W_1,\cdots,W_n)$ associated with $N^{k-2}MHV$ amplitudes in the description based on ordinary twistors correspond to k negative helicities and are given by

$$\begin{aligned} L_{n,k}(W_1,\cdots,W_n) &= \frac{1}{Vol(GL(2))}\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)}\times \\ &\times \prod_{\alpha=1}^{k} d^{4|4}Y_\alpha \prod_{i=1}^{n}\delta^{4|4}(W_i - C_{\alpha i}Y_\alpha)\ . \end{aligned} \tag{12.6.27}$$

Here $C_{\alpha a}$ denote the $n\times k$ coordinates used to parametrize the points of $G_{k,n}$.

2. The integrals $Y_{n,k}(W_1,\cdots,W_n)$ associated with N^kMHV amplitudes in the description based on momentum twistors are defined as

$$Y_{n,k}(Z_1,\cdots,Z_n) = \frac{1}{Vol(GL(k))}\times\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)}\times\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}Z_a)\ . \tag{12.6.28}$$

The possibility to select $Z_a^0 = 1$ implies $\sum_k C_{\alpha k} = 0$ allowing to eliminate $C_{\alpha n}$ so that the actual number of coordinates Grassman coordinates is $nk-1$. As already noticed, $L_{n,k+2}(W_1,\cdots,W_n) = M^{tree}_{MHV}\times Y_{n,k}(Z_1,\cdots,Z_n)$. Momentum twistors are obviously calculationally easier since the value of k is smaller by two units.

The $4k$ delta functions reduce the number of integration variables of contour integrals from nk to $(n-4)k$ in the bosonic sector (the definition of delta functions involves some delicacies not discussed here). The n quantities $(m,\cdots m+k)$ are $k\times k$-determinants defined by subsequent columns from m to $m+k-1$ of the $k\times n$ matrix defined by the coordinates $C_{\alpha a}$ and correspond geometrically to the k-volumes of the k-dimensional parallel-pipeds defined by these column vectors. The fact that the scalings of twistor space coordinates Z_a can be compensated by scalings of $C_{\alpha a}$ deforming integration contour but leaving the residue integral invariant so that the integral depends on projective twistor coordinates only.

Since the integrand is a rational function, a multi-dimensional residue calculus allows to deduce the values of these integrals as residues associated with the poles of the integrand in a recursive manner. The poles correspond to the zeros of the $k\times k$ determinants appearing in the integrand or equivalently to singular lower-dimensional parallel-pipeds. It can be shown that local residues are determined by $(k-2)(n-k-2)$ conditions on the determinants in both cases. The value of the integral depends on the explicit choice of the integration contour for each variable $C_{\alpha a}$ left when delta functions are taken into account. The condition that a correct form of tree amplitudes is obtained fixes the choice of the integration contours.

For the ordinary twistors W the residues correspond to projective configurations in CP_{k-1}, or more precisely in the space $CP^n_{k-1}/Gl(k)$, which is $(k-1)n-k^2$-dimensional space defining the support for the residues integral. $Gl(k)$ relates to each other different complex coordinate frames for k-plane and since the choice of frame does not affect the plane itself, one has $Gl(k)$ gauge symmetry as well as the dual $Gl(n-k)$ gauge symmetry.

CP_{k-1} comes from the fact that C_{α_k} are projective coordinates: the amplitudes are indeed invariant under the scalings $W_i\rightarrow t_iW_i$, $C_{\alpha i}\rightarrow tC_{\alpha i}$. The coset space structure comes from the fact that $Gl(k)$ is a symmetry of the integrand acting as $C_{\alpha i}\rightarrow \Lambda_\alpha^{\ \beta}C_{\beta_i}$. This analog of gauge symmetry allows to fix k arbitarily chosen frame vectors $C_{\alpha i}$ to orthogonal unit vectors. For instance, one can have $C_{\alpha i} = \delta_{\alpha_i}$ for $\alpha = i\in 1,\cdots,k$. This choice is discussed in detail in [B31]. The reduction to CP_{k-1} implies the reduction of the support of the integral to line in the case of MHV amplitudes and to plane in the case of NMHV as one sees from the expression $d\mu = \prod_\alpha d^{4|4}Y_\alpha\prod_{i=1}^n\delta^{4|4}(W_i - C_{\alpha i}Y_\alpha)$. For $(i_1,\cdots,i_k) = 0$ the vectors $i_1,..i_k$ belong to $k-2$-dimensional plane of CP_{k-1}. In the case of $NMHV$ (N^2MHV) amplitudes this translates at the level of twistors to the condition that the corresponding twistors $\{i_1,i_2,i_3\}$ ($\{i_1,i_2,i_3,i_4\}$) are collinear (in the same plane) in twistor space. This can be understood from the fact that the delta functions in $d\mu$ allow to express W_i in terms of $k-1$ Y_α: s in this case.

The action of conformal transformations in twistor space reduces to the linear action of $SU(2,2)$ leaving invariant Hermitian sesquilinear form of signature $(2,2)$. Therefore the conformal invariance of the Grassmannian integral and its dual variant follows from the possibility to perform a compensating coordinate change for $C_{\alpha a}$ and from the fact that residue integral is invariant under small deformations of the integration contour. The above described relationship between representations based on twistors and momentum twistors implies the full Yangian invariance.

12.6.5 Canonical Operations For Yangian Invariants

General l-loop amplitudes can be constructed from the basic Yangian invariants defined by N^kMHV amplitudes by various operations respecting Yangian invariance apart from possible IR anomalies. There are several operations that one can perform for Yangian invariants $Y_{n,k}$ and all these operations appear in the recursion formula for planar all loop amplitudes. These operations are described in [B33] much better than I could do it so that I will not go to any details. It is possible to add and remove particles, to fuse two Yangian invariants, to merge particles, and to construct from two Yangian invariants a higher invariant containing so called BCFW bridge representing single particle exchange using only twistorial methods.

Inverse soft factors

Inverse soft factors add to the diagram a massless collinear particles between particles a and b and by definition one has

$$O_{n+1}(a,c,b,\cdots) = \frac{\langle ab\rangle}{\langle ac\rangle\langle cb\rangle}O_n(a'b') \ . \tag{12.6.29}$$

At the limit when the momentum of the added particle vanishes both sides approach the original amplitude. The right-handed spinors and Grassmann parameters are shifted

$$\begin{array}{ll} \tilde{\lambda}'_a = \tilde{\lambda}_a + \frac{\langle cb\rangle}{\langle ab\rangle}\tilde{\lambda}_c \ , & \tilde{\lambda}'_b = \tilde{\lambda}_b + \frac{\langle ca\rangle}{\langle ba\rangle}\tilde{\lambda}_c \ , \\ \eta'_a = \eta_a + \frac{\langle cb\rangle}{\langle ab\rangle}\eta_c \ , & \eta'_b = \eta_b + \frac{\langle ca\rangle}{\langle ba\rangle}\eta_c \ . \end{array} \tag{12.6.30}$$

There are two kinds of inverse soft factors.

1. The addition of particle leaving the value k of negative helicity gluons unchanged means just the re-interpretation

$$Y'_{n,k}(Z_1,\cdots,Z_{n-1},Z_n) = Y_{n-1,k}(Z_1,\cdots,Z_{n-1}) \tag{12.6.31}$$

without actual dependence on Z_n. There is however a dependence on the momentum of the added particle since the relationship between momenta and momentum twistors is modified by the addition obtained by applying the basic rules relating region super momenta and momentum twistors (light-like momentum determines λ_i and twistor equations for x_i and λ_i,η_i determine (μ_i,χ_i)) is expressible assigned to the external particles [B50]. Modifications are needed only for the new vertex and its neighbors.

2. The addition of a particle increasing k with single unit is a more complex operation which can be understood in terms of a residue of $Y_{n,k}$ proportional to $Y_{n-1,k-1}$ and Yangian invariant $[z_1\cdots z_5]$ with five arguments constructed from basic Yangian invariants with four arguments. The relationship between the amplitudes is now

$$Y'_{n,k}(..,Z_{n-1}Z_n,Z_1\cdots) = [n-2\ n-1\ n\ 1\ 2]\times Y_{n-1,k-1}(\cdots\hat{Z}_{n-1},\hat{Z}_1,\cdots) \tag{12.6.32}$$

Here

$$[abcde] = \frac{\delta^{0|4}(\eta_a\langle bcde\rangle + cyclic)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle} . \tag{12.6.33}$$

denoted also by $R(a,b,c,d,e)$ is the fundamental R-invariant appearing in one loop corrections of MHV amplitudes and will appears also in the recursion formulas. $\langle abcd\rangle$ is the fundamental super-conformal invariant associated with four super twistors defined in terms of the permutation symbol.

$\hat{Z}_{n-1}, \hat{Z}_1$ are deformed momentum twistor variables. The deformation is determined from the relationship between external momenta, region momenta and momentum twistor variables. $\hat{Z}^1$ is the intersection $\hat{Z}^1 = (n-2\ n-1\ 2)\cap(12)$ of the line (12) with the plane $(n-2\ n-1\ 2)$ and $\hat{Z}^{n-1}$ the intersection $\hat{Z}^1 = (12n)\cap(n-2\ n-1)$ of the line $(n-2\ n-1)$ with the plane $(12n)$. The interpretation for the intersections at the level of ordinary Feynman diagrams is in terms of the collinearity of the four-momenta involved with the underlying box diagram with parallel on mass shell particles. These result from unitarity conditions obtained by putting maximal number of loop momenta on mass shell to give the leading singularities.

The explicit expressions for the momenta are

$$\begin{aligned} \hat{Z}^1 &\equiv (n-2\ n-1\ 2)\cap(12)Z_1 = \langle 2\ n-2\ n-1\ n\rangle + Z_2\langle n-2\ n-1\ n\ 1\rangle , \\ \hat{Z}^{n-1} &\equiv (12n)\cap(n-2\ n-1) = Z_{n-2}\langle n-2\ n-1\ n\ 2\rangle + Z_{n-1}\langle n\ 1\ 2\ n-2\rangle . \end{aligned} \tag{12.6.34}$$

These intersections also appear in the expressions defining the recursion formula.

Removal of particles and merge operation

Particles can be also removed. The first manner to remove particle is by integrating over the twistor variable characterizing the particle. This reduces k by one unit. Merge operation preserves the number of loops but removes a particle particle by identifying the twistor variables of neighboring particles. This operation corresponds to an integral over on mass shell loop momentum at the level of tree diagrams and by Witten's half Fourier transform can be transformed to twistor integral.

The product

$$Y'(Z_1,\cdots Z_n) = Y_1(Z_1,\cdots Z_m)\times Y_2(Z_{m+1},\cdots Z_n) \tag{12.6.35}$$

of two Yangian invariants is again a Yangian invariant. This is not quite trivial since the dependence of region momenta and momentum twistors on the momenta of external particles makes the operation non-trivial.

Merge operation allows to construct more interesting invariants from the products of Yangian invariants. One begins from a product of Yangian invariants (Yangian invariant trivially) represented cyclically as points of circle and identifies the last twistor argument of given invariant with the first twistor argument of the next invariant and performs integrals over the momentum twistor variables appearing twice. The soft k-increasing and preserving operations can be described also in terms of this operation for Yangian invariants such that the second invariant corresponds to 3-vertex. The cyclic merge operation applied to four MHV amplitudes gives NMHV amplitudes associated with on mass shell momenta in box diagrams. By applying similar operation to NMHV amplitudes and MHV amplitudes one obtains 2-loop amplitudes. In [B33] examples about these operations are described.

BCFW bridge

BCFW bridge allows to build general tree diagrams from MHV tree diagrams [B20, B20] and recursion formula of [B33] generalizes this to arbitrary diagrams. At the level of Feynman diagrams it corresponds to a box diagram containing general diagrams labeled by L and R and MHV and $\overline{MHV}$ 3-vertices ($\overline{MHV}$ 3-vertex allows expression in terms of MHV diagrams) with the lines of the box on mass shell so that the three momenta emanating from the vertices are parallel and give rise to a one-loop leading singularity.

At the level of Feynman diagrams BCFW bridge corresponds to so called "two-mass hard" leading singularities associated with box diagrams with light-like momenta at the four lines of the diagram [B31]. The motivation for the study of these diagrams comes from the hypothesis the leading order singularities obtained by putting as many particles as possible on mass shell contain the data needed to construct scattering amplitudes of $\mathcal{N}=4$ SYM completely. This representation of the leading singularities generalizes to arbitrary loops. The recent article is a continuation of this program to planar amplitudes.

Also BCFW bridge allows an interpretation as a particular kind fusion for Yang invariants and involves all the basic operations. One starts from the amplitudes $Y^L_{n_L,k_L}$ and $Y^R_{n_R,k_R}$ and constructs an amplitude $Y'_{n_L+n_R,k_L+k_R+1}$ representing the amplitude which would correspond to a generalization of the MHV diagrams with the two tree diagrams connected by the MHV propagator (BCFW bridge) replaced with arbitrary loop diagrams. Particle "1" *resp.* "j+1" is added by the soft k-increasing factor to Y_{n_L+1,k_L+1} *resp.* Y_{n_R+1,k_R+1} giving amplitude with $n+2$ particles and with k-charge equal to k_L+k_R+2. The subsequent operations must reduce k-charge by one unit. First repeated "1" and "j+1" are identified with their copies by k conserving merge operation, and after that one performs an integral over the twistor variable Z^I associated with the internal line obtained and reducing k by one unit. The soft k-increasing factors bring in the invariants $[n-1\ n\ 1\ I\ j+2]$ associated with Y_L and $[1\ I\ j+1\ j\ j-1]$ associated with Y_R. The integration contour is chosen so that it selects the pole defined by $\angle n-1\ n\ 1\ I\rangle$ in the denominator of $[n-1\ n\ 1\ I\ j+2]$ and the pole defined by $\langle 1\ I\ j+1\ j\rangle$ in the denominator of $[1\ I\ j+1\ j\ j-1]$.

The explicit expression for the BCFW bridge is very simple:

$$\begin{aligned}(Y_L\otimes_{BCFW} Y_R)(1,\cdots,n) &= [n-1\ n\ 1\ j\ j+1]\times Y_R(1,\cdots,j,I)Y_L(I,j+1,\cdots,n-1,\hat{n})\ ,\\ \hat{n} &= (n-1\ n)\cap(j\ j+1\ 1)\ ,\ \ I=(j\ j+1)\cap(n-1\ n\ 1)\end{aligned} \quad (12.6.36)$$

Single cuts and forward limit

Forward limit operation is used to increase the number of loops by one unit. The physical picture is that one starts from say 1-loop amplitude and cuts one line by assigning to the pieces of the line opposite light-like momenta having interpretation as incoming and outgoing particles. The resulting amplitude is called forward limit. The only reasonable interpretation seems to be that the loop integration is expressed by unitarity as forward limit meaning cutting of the line carrying the loop momentum. This operation can be expressed in a manifestly Yangian invariant way as entangled removal of two particles with the merge operation meaning the replacement $Z_n\to Z_{n-1}$. Particle $n+1$ is added adjacent to A,B as a k-increasing inverse soft factor and then A and B are removed by entangled integration, and after this merge operation identifies $n+1$ and 1.

Forward limit is crucial for the existence of loops and for Yangian invariants it corresponds to the poles arising from $\langle (AB)_q Z_n(z)Z_1)\rangle$ the integration contour Z_n+zZ_{n-1} around Z_b in the basic formula $M=\oint(dz/z)M_n$ leading to the recursion formula. A and B denote the momentum twistors associated with opposite light-like momenta. In the generalized unitarity conditions the singularity corresponds to the cutting of line between particles n and 1 with momenta q and $-q$, summing over the multiplet of stats running around the loop. Between particles n_2 and 1 one has particles $n-1$, n with momenta $q,-q$. $q=x_1-x_n=-x_n+x_{n-1}$ giving $x_1=x_{n-1}$. Light-likeness of q means that the lines $(71)=(76)$ and (15) intersect. At the forward limit giving rise to the pole Z_6 and Z_7 approach to the intersection point $(76)\cap(15)$. In a generic gauge theories the forward limits are ill-defined but in super-symmetric gauge theories situation changes.

The corresponding Yangian operation removes two external particles with opposite four-momenta and involves integration over two twistor variables Z_a and Z_b and gives rise to the following expression

$$\int_{GL(2)} Y(\cdots, Z_n, Z_A, Z_B, Z_1, \cdots) \ . \tag{12.6.37}$$

The integration over $GL(2)$ corresponds to integration over twistor variables associated Z_A and Z_B. This operation allows addition of a loop to a given amplitude. The line Z_aZ_b represents loop momentum on one hand and the dual x-coordinate identified as momentum propagating along the line on the other hand.

The integration over these variables is equivalent to an integration over loop momentum as the explicit calculation of [B33] (see pages 12-13) demonstrates. If the integration contours are products in the product of twistor spaces associated with a and b the and gives lower order Yangian invariant as answer. It is however also possible to choose the integration contour to be entangled in the sense that it cannot be reduced to a product of integration contours in the Cartesian product of twistor spaces. In this case the integration gives a loop integral. In the removal operation Yangian invariance can be broken by IR singularities associated with the integration contour and the procedure does not produce genuine Yangian invariant always.

What is highly interesting from TGD point of view is that this integral can be expressed as a contour integral over $CP_1 \times CP_1$ combined with integral over loop momentum. If TGD vision about generalized Feynman graps in zero energy ontology is correct, the loop momentum integral is discretized to an an integral over discrete mass shells and perhaps also to a sum over discretized momenta and one can therefore avoid IR singularities.

12.6.6 Explicit Formula For The Recursion Relation

Recall that the recursion formula is obtained by considering super-symmetric momentum-twistor deformation $Z_n \rightarrow Z_n + zZ_{n-1}$ and by integrating over z to get the identity

$$M_{n,k,l} = \oint \frac{dz}{z} \hat{M}_{n,k,l}(z) \ . \tag{12.6.38}$$

This integral equals to integral with reversed integration contour enclosing the exterior of the contour. The challenge is to deduce the residues contributing to the residue integral and the claim of [B33] is that these residues reduce to simple basic types.

1. The first residue corresponds to a pole at infinity and reduces the particle number by one giving a contribution $M_{n-1,k,l}(1, \cdots, n-1)$ to $M_{n,k,l}(1, \cdots, n-1, n)$. This is not totally trivial since the twistor variables are related to momenta in different manner for the two amplitudes. This gives the first contribution to the right hand side of the formula below.

2. Second pole corresponds to the vanishing of $\langle Z_n(z)Z_1Z_jZ_{j+1}\rangle$ and corresponds to the factorization of channels. This gives the second BCFW contribution to the right hand side of the formula below. These terms are however not enough since the recursion formula would imply the reduction to expressions involving only loop corrections to 3-loop vertex which vanish in $\mathcal{N} = 4$ SYM.

3. The third kind of pole results when $\langle (AB)_q Z_n(z)Z_1\rangle$ vanishes in momentum twistor space. $(AB)_q$ denotes the line in momentum twistor space associated with q: th loop variable.

The explicit formula for the recursion relation yielding planar all loop amplitudes is obtained by putting all these pieces together and reads as

$$\begin{aligned} M_{n,k,l}(1, \cdots, n) &= M_{n-1,k,l}(1, \cdots, n-1) \\ &+ \sum_{n_L,k_L,l_L;j} [j\ j+1\ n-1\ n\ 1]\, M^R_{n_R,k_R,l_R}(1, \cdots, j, I_j) \times M^L_{n_L,k_L,l_L}(I_j, j+1, \cdots, \hat{n}_j) \\ &+ \int_{GL(2)} [AB\ n-1\ n\ 1]\, M_{n+2,k+1,n,k-1}(1, \cdots, \hat{n}_{AB}, \hat{A}, B) \ , \\ n_L &+ n_R = n+2 \ , \ k_L + k_R = k-1 \ , \ l_R + l_L = l \ . \end{aligned} \tag{12.6.39}$$

The momentum super-twistors are given by

$$\begin{array}{ll} \hat{n}_j = (n-1\ n)\cap(j\ j+1\ 1)\ , & I_j = (j\ j+1\ 1)\cap(n-1\ n\ 1)\ , \\ \hat{n}_{AB} = (n-1\ n)\cap(AB\ 1)\ , & \hat{A} = (AB)\cap(n-1\ n\ 1)\ . \end{array} \tag{12.6.40}$$

The index l labels loops in $n+2$-particle amplitude and the expression is fully symmetrized with equal weight for all loop integration variables $(AB)_l$. A and B are removed by entangled integration meaning that $GL(2)$ contour is chosen to encircle points where both points A, B on the line (AB) are located at the intersection of the line (AB) with the plane $(n-1\ n\ 1)$. GL(2) integral can be done purely algebraically in terms of residues.

In [B33] and [B50] explicit calculations for N^kMHV amplitudes are carried out to make the formulas more concrete. For N^1MHV amplitudes second line of the formula vanishes and the integrals are rather simple since the determinants are 1×1 determinants.

Chapter 13

Unified Number Theoretical Vision

13.1 Introduction

Octonions, quaternions, quaternionic space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized. This summary involves several corrections to the picture which has been developing for a decade or so.

A brief updated view about M^8-H duality and twistorialization is in order. There is a beautiful pattern present suggesting that M^8-H duality makes sense and that $H=M^4\times CP_2$ is completely unique on number theoretical grounds.

1. M^8-H duality allows to deduce $M^4\times CP_2$ via number theoretical compactification. For the option with minimal number of conjectures the associativity/co-associativity of the space-time surfaces in M^8 guarantees that the space-time surfaces in M^8 define space-time surfaces in H. The tangent/normal spaces of quaternionic/hyper-quaternionic surfaces in M^8 contain also an integrable distribution of hyper-complex tangent planes $M^2(x)$.

   An important correction is that associativity/co-associativity does not make sense at the level of H since the spinor structure of H is already complex quaternionic and reducible to the ordinary one by using matrix representations for quaternions. The associativity condition should however have some counterpart at level of H. One could require that the induced gamma matrices at each point could span a *real*-quaternionic sub-space of complexified quaternions for quaternionicity and a purely imaginary quaternionic sub-space for co-quaternionicity. One might hope that it is consistent with - or even better, implies - preferred extremal property. I have not however found a viable definition of quaternionic "reality". On the other hand, it is possible assigne the tangent space M^8 of H with octonion structure and define associativity as in case of M^8.

   M^8-H duality could generalize to $H-H$ duality in the sense that also the image of the space-time surface under duality map is not only preferred extremal but also associative (co-associative) surface. The duality map $H\rightarrow H$ could be iterated and would define the arrow for the category formed by preferred extremals.

2. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. M^8 allows twistor space for octonionic spinor structure obtained by direct generalization of the standard construction for M^4. $M^4\times CP_2$ spinors can be regarded as tensor products of quaternionic spinors associated with M^4 and CP_2: this trivial observation forces to challenge the earlier rough vision, which however seems to stand up the challenge.

3. Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing 12-D $T(M^8)=G_2/U(1)\times U(1)$ to the 12-D twistor space $T(H)=CP_3\times SU^3/U(1)\times U(1)$. The interpretation of the twistor space in the case of M^8 is as the space of choices of quantization axes for the

2-D Cartan algebra of G_2 acting as octonionic automorphisms. For CP_2 one has space for the chocies of quantization axes for the 2-D $SU(3)$ Cartan algebra.

4. It is also possible that the dualities extend to a sequence $M^8 \to H \to H...$ by mapping the associative/co-associative tangent space to CP_2 and M^4 point to M^4 point at each step. One has good reasons to expect that this iteration generates fractal as the limiting space-time surface.

5. A fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically 7-sphere. Second analogous structure is the 7-D Lie algebra like structure defined by octonionic analogs of sigma matrices.

The analogy of quaternionicity of M^8 pre-images of preferred extremals and quaternionicity of the tangent space of space-time surfaces in H with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity at the level of M^8 indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program. Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L22].

13.2 Number Theoretic Compactification And $M^8 - H$ Duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally $M^8 - H$ duality was introduced as a number theoretic explanation for $H = M^4 \times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4 \subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M^8_c using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of $M^8 - H$ duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of $M^8 - H$ duality.

The strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M^8_c composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M^8_c identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M^8_c must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere Kḧler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3)\times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4\times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering M^8-H duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M^8_c=O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under multiplication. One can however however define the notion of associativity for the subspace of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q=q_0+q^kI_k$. Hyper-quaternions are expressible as $q=q_0+jq^kI_k$ and form a subspace of complexified quaternions $Q_c=Q\oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O\oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H=q_0+jq^kI_k+jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4\subset M^8$ implies unique decomposition $M^8=M^4\times E^4$ needed to define M^8-H duality uniquely. This applies also to M^8_c. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCW s associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M^8_c) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

   The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *embedding space level* and *space-time level*. One must have embedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \rightarrow H \rightarrow H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of Kähler-Dirac gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the Kähler-Dirac gamma matrices need not span the entire tangent space. The space spanned by the Kähler-Dirac gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the Kähler-Dirac gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

   If the space spanned by Kähler-Dirac gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For Kähler-Dirac gamma matrices the notion of co-associativity can produce problems since Kähler-Dirac gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by Kähler-Dirac gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the Kähler-Dirac gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the Kähler-Dirac gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

13.2.1 Basic Idea Behind $M^8 - M^4 \times CP_2$ Duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different ways to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M^8_c. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M^8_c) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \to S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \to M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M_c^8: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M_c^8$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a way to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [K10]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [K96] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of Kähler-Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent

plane makes it possible to define M^8-H duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

13.2.2 Hyper-Octonionic Pauli "Matrices" And The Definition Of Associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [K86, L33]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

   Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

   There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \rightarrow H \rightarrow H$.... This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and Kähler-Dirac gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

13.2.3 Are Kähler And Spinor Structures Necessary In M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the M^8-H duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naïve replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1+1+3+\overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1+kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2 + y^2 + z^2 + t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1+ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of embedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

13.2.4 How Could One Solve Associativity/Co-Associativity Conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \to H \to H...$ iteration generating new solutions from existing ones.

Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O \oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1 + io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M^8_c but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M^8_c (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a, b, c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a way that one of the tangent vectors corresponds to real unit (in the embedding map embedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the

octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition looks very simple - perhaps too simple!- since it involves only first derivatives of the embedding space vectors.

One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of embedding space coordinates (rather than involving embedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$\begin{aligned} e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\ A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\ e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\ \Gamma_k &= e^A_k \gamma_A \ . \end{aligned} \tag{13.2.1}$$

The very naïve idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{13.2.2}$$

One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see `http://tinyurl.com/ya7h3n9z`). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD} x^B y^C z^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [A82] (see **Fig. 13.1**) expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product e_1e_2 (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x)e_1e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$e_0 = a \times 1 + b^i e_i$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

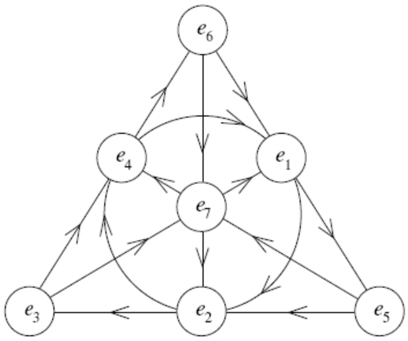

Figure 13.1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

13.2.5 Quaternionicity At The Level Of Embedding Space Quantum Numbers

From the multiplication table of octonions as illustrated by Fano triangle [A82] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2: s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

13.2.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of $M^8 - H$ duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the Kähler-Dirac gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k \gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the Kähler-Dirac gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection Kähler-Dirac gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2 \times S^2 \subset M^4 \times CP_2$. It seems that associativity is satisfied by all known extremals. Hence Kähler-Dirac gamma matrices are flexible enough to realize associativity in H.

3. Kähler-Dirac gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner $M^8 - H$ duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the Kähler-Dirac gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand $M^8 - H$ correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does $M^8 - H$ correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

$M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since WCW degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined

by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K55].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

13.2.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \rightarrow H \rightarrow H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

13.3 Quaternions and TGD

13.3.1 Are Euclidian Regions Of Preferred Extremals Quaternion- Kähler Manifolds?

In blog comments Anonymous gave a link to an article (see `http://tinyurl.com/y7j9hxr8`) about construction of 4-D quaternion-Kähler metrics with an isometry: they are determined by so called $SU(\infty)$ Toda equation. I tried to see whether quaternion-Kähler manifolds could be relevant for TGD.

From Wikipedia (see `http://tinyurl.com/yd8feoev`) one can learn that QK is characterized by its holonomy, which is a subgroup of $Sp(n) \times Sp(1)$: $Sp(n)$ acts as linear symplectic transformations of $2n$-dimensional space (now real). In 4-D case tangent space contains 3-D submanifold identifiable as imaginary quaternions. CP_2 is one example of QK manifold for which the subgroup in question is $SU(2) \times U(1)$ and which has non-vanishing constant curvature: the components of Weyl tensor represent the quaternionic imaginary units. QKs are Einstein manifolds: Einstein tensor is proportional to metric.

What is really interesting from TGD point of view is that twistorial considerations show that one can assign to QK a special kind of twistor space (twistor space in the mildest sense requires only orientability). Wiki tells that if Ricci curvature is positive, this (6-D) twistor space is what is known as projective Fano manifold with a holomorphic contact structure. Fano variety has the nice property that as (complex) line bundle (the twistor space property) it has enough sections to define the embedding of its base space to a projective variety. Fano variety is also complete: this is algebraic geometric analogy of topological property known as compactness.

QK manifolds and twistorial formulation of TGD

How the QKs could relate to the twistorial formulation of TGD?

1. In the twistor formulation of TGD [K86] the space-time surfaces are 4-D base spaces of 6-D twistor spaces in the Cartesian product of 6-D twistor spaces of M^4 and CP_2 - the only twistor spaces with Kähler structure. In TGD framework space-time regions can have either Euclidian or Minkowskian signature of induced metric. The lines of generalized Feynman diagrams are Euclidian.

2. Could the twistor spaces associated with the lines of generalized Feynman diagrams be projective Fano manifolds? Could QK structure characterize Euclidian regions of preferred extremals of Kähler action? Could a generalization to Minkowskian regions exist.

   I have proposed that so called Hamilton-Jacobi structure [K96] characterizes preferred extremals in Minkowskian regions. It could be the natural Minkowskian counterpart for the quaternion Kähler structure, which involves only imaginary quaternions and could make sense also in Minkowski signature. Note that unit sphere of imaginary quaternions defines the sphere serving as fiber of the twistor bundle.

Why it would be natural to have QK that is corresponding twistor space, which is projective contact Fano manifold?

1. QK property looks very strong condition but might be true for the preferred extremals satisfying very strong conditions stating that the classical conformal charges associated with various conformal algebras extending the conformal algebras of string models [K96], [L28]. These conditions would be essentially classical gauge conditions stating that strong form of holography implies by strong form of General Coordinate Invariance (GCI) is realized: that is partonic 2-surfaces and their 4-D tangent space data code for quantum physics.

2. Kähler property makes sense for space-time regions of Euclidian signature and would be natural is these regions can be regarded as small deformations of CP_2 type vacuum extremals with light-like M^4 projection and having the same metric and Kähler form as CP_2 itself.

3. Fano property implies that the 4-D Euclidian space-time region representing line of the Feynman diagram can be imbedded as a sub-manifold to complex projective space CP_n. This would allow to use the powerful machinery of projective geometry in TGD framework. This could also be a space-time correlate for the fact that CP_ns emerge in twistor Grassmann approach expected to generalize to TGD framework.

4. CP_2 allows both projective (trivially) and contact (even symplectic) structures. $\delta M^4_+ \times CP_2$ allows contact structure - I call it loosely symplectic structure. Also 3-D light-like orbits of partonic 2-surfaces allow contact structure. Therefore holomorphic contact structure for the twistor space is natural.

5. Both the holomorphic contact structure and projectivity of CP_2 would be inherited if QK property is true. Contact structures at orbits of partonic 2-surfaces would extend to holomorphic contact structures in the Euclidian regions of space-time surface representing lines of generalized Feynman diagrams. Projectivity of Fano space would be also inherited from CP_2 or its twistor space $SU(3)/U(1) \times U(1)$ (flag manifold identifiable as the space of choices for quantization axes of color isospin and hypercharge).

The article considers a situation in which the QK manifold allows an isometry. Could the isometry (or possibly isometries) for QK be seen as a remnant of color symmetry or rotational symmetries of M^4 factor of embedding space? The only remnant of color symmetry at the level of embedding space spinors is anomalous color hyper charge (color is like orbital angular momentum and associated with spinor harmonic in CP_2 center of mass degrees of freedom). Could the isometry correspond to anomalous hypercharge?

How to choose the quaternionic imaginary units for the space-time surface?

Parallellizability is a very special property of 3-manifolds allowing to choose quaternionic imaginary units: global choice of one of them gives rise to twistor structure.

1. The selection of time coordinate defines a slicing of space-time surface by 3-surfaces. GCI would suggest that a generic slicing gives rise to 3 quaternionic units at each point each 3-surface? The parallelizability of 3-manifolds - a unique property of 3-manifolds - means the possibility to select global coordinate frame as section of the frame bundle: one has 3 sections of tangent bundle whose inner products give rose to the components of the metric (now induced metric) guarantees this. The tri-bein or its dual defined by two-forms obtained by contracting tri-bein vectors with permutation tensor gives the quanternionic imaginary units. The construction depends on 3-metric only and could be carried out also in GRT context. Note however that topology change for 3-manifold might cause some non-trivialities. The metric 2-dimensionality at the light-like orbits of partonic 2-surfaces should not be a problem for a slicing by space-like 3-surfaces. The construction makes sense also for the regions of Minkowskian signature.

2. In fact, any 4-manifold (see `http://tinyurl.com/yb8l34b5`) [A101] allows almost quaternionic as the above slicing argument relying on parallelizibility of 3-manifolds strongly suggests.

3. In zero energy ntology (ZEO)- a purely TGD based feature - there are very natural special slicings. The first one is by linear time-like Minkowski coordinate defined by the direction of the line connecting the tips of the causal diamond (CD). Second one is defined by the light-cone proper time associated with either light-cone in the intersection of future and past directed light-cones defining CD. Neither slicing is global as it is easy to see.

The relationship to quaternionicity conjecture and $M^8 - H$ duality

One of the basic conjectures of TGD is that preferred extremals consist of quaternionic/ co-quaternionic (associative/co-associative) regions [K82]. Second closely related conjecture is $M^8 - H$ duality allowing to map quaternionic/co-quaternionic surfaces of M^8 to those of $M^4 \times CP_2$. Are these conjectures consistent with QK in Euclidian regions and Hamilton-Jacobi property in Minkowskian regions? Consider first the definition of quaternionic and co-quaternionic space-time regions.

1. Quaternionic/associative space-time region (with Minkowskian signature) is defined in terms of induced octonion structure obtained by projecting octonion units defined by vielbein of $H = M^4 \times CP_2$ to space-time surface and demanding that the 4 projections generate quaternionic sub-algebra at each point of space-time.

   If there is also unique complex sub-algebra associated with each point of space-time, one obtains one can assign to the tangent space-of space-time surface a point of CP_2. This allows to realize $M^8 - H$ duality [K82] as the number theoretic analog of spontaneous compactification (but involving no compactification) by assigning to a point of $M^4 = M^4 \times CP_2$ a point of $M^4 \times CP_2$. If the image surface is also quaternionic, this assignment makes sense also for space-time surfaces in H so that $M^8 - H$ duality generalizes to $H - H$ duality allowing to assign to given preferred extremal a hierarchy of extremals by iterating this assignment. One obtains a category with morphisms identifiable as these duality maps.

2. Co-quaternionic/co-associative structure is conjectured for space-time regions of Euclidian signature and 4-D CP_2 projection. In this case normal space of space-time surface is quaternionic/associative. A multiplication of the basis by preferred unit of basis gives rise to a quaternionic tangent space basis so that one can speak of quaternionic structure also in this case.

3. Quaternionicity in this sense requires unique identification of a preferred time coordinate as embedding space coordinate and corresponding slicing by 3-surfaces and is possible only in TGD context. The preferred time direction would correspond to real quaternionic unit.

Preferred time coordinate implies that quaternionic structure in TGD sense is more specific than the QK structure in Euclidian regions.

4. The basis of induced octonionic imaginary unit allows to identify quaternionic imaginary units linearly related to the corresponding units defined by tri-bein vectors. Note that the multiplication of octonionic units is replaced with multiplication of antisymmetric tensors representing them when one assigns to the quaternionic structure potential QK structure. Quaternionic structure does not require Kähler structure and makes sense for both signatures of the induced metric. Hence a consistency with QK and its possible analog in Minkowskian regions is possible.

5. The selection of the preferred imaginary quaternion unit is necessary for $M^8 - H$ correspondence. This selection would also define the twistor structure. For quaternion-Kähler manifold this unit would be covariantly constant and define Kähler form - maybe as the induced Kähler form.

6. Also in Minkowskian regions twistor structure requires a selection of a preferred imaginary quaternion unit. Could the induced Kähler form define the preferred imaginary unit also now? Is the Hamilton-Jacobi structure consistent with this?

   Hamilton-Jacobi structure involves a selection of 2-D complex plane at each point of space-time surface. Could induced Kähler magnetic form for each 3-slice define this plane? It is not necessary to require that 3-D Kähler form is covariantly constant for Minkowskian regions. Indeed, massless extremals representing analogs of photons are characterized by local polarization and momentum direction and carry time-dependent Kähler-electric and -magnetic fields. One can however ask whether monopole flux tubes carry covariantly constant Kähler magnetic field: they are indeed deformations of what I call cosmic strings [K10, K22] for which this condition holds true?

13.3.2 The Notion of Quaternion Analyticity

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals.

Basic idea

One can argue that quaternion analyticity is the more natural option in the sense that the local octonionic embedding space coordinate (or at least M^8 or E^8 coordinate, which is enough if M^8-H duality holds true) would for preferred extremals be expressible in the form

$$o(q) = u(q) + v(q) \times I \quad . \tag{13.3.1}$$

Here q is quaternion serving as a coordinate of a quaternionic sub-space of octonions, and I is octonion unit belonging to the complement of the quaternionic sub-space, and multiplies $v(q)$ from *right* so that quaternions and quaternionic differential operators acting from left do not notice these coefficients at all. A stronger condition would be that the coefficients are real. $u(q)$ and $v(q)$ would be quaternionic Taylor- of even Laurent series with coefficients multiplying powers of q from right for the same reason.

The signature of M^4 metric is a problem. I have proposed a complexification of M^8 and M^4 to get rid of the problem by assuming that the embedding space corresponds to surfaces in the space M^8 identified as octonions of form $o_8 = Re(o) + iIm(o)$, where o is imaginary part of ordinary octonion and i is commuting imaginary unit. M^4 would correspond to quaternions of form $q_4 = Re(q) + iIm(q)$. What is important is that powers of q_4 and o_8 belong to this sub-space (as follows from the vanishing of cross product term in the square of octonion/quaternion) so that powers of q_4 (o_8) has imaginary part proportional to $Im(q)$ ($Im(o)$)

I ended up to reconsider the idea of quaternion analyticity after having found two very interesting articles discussing the generalization of Cauchy-Riemann equations. The first article (see http://tinyurl.com/yb8l34b5) [A101] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article (see http://tinyurl.com/y7kww2o2) [A86] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

The first form of Cauchy-Riemann-Fueter conditions

Cauchy-Riemann-Fueter (CRF) conditions generalize Cauchy-Riemann conditions. These conditions are however not unique. Consider first the translationally invariant form of CRF conditions.

1. The translationally invariant form of CRF conditions is $\partial_{\overline{q}} f = 0$ or explicitly

$$\partial_{\overline{q}} f = d_1 f + d_2 f \equiv (\partial_t - \partial_x I) f - (\partial_y J + \partial_z K) f = 0 \quad . \tag{13.3.2}$$

   This form is not unique: one can perform SO(3) rotations of the quaternionic imaginary units acting as automorphisms of quaternions. This form does not allow quaternionic Taylor series as a solution. Note that the Taylor coefficients multiplying powers of the coordinate from right are arbitrary quaternions. What looks pathological is that even linear functions of q fail be solve this condition. What is however interesting that in flat space the equation is equivalent with Dirac equation for a pair of Majorana spinors [A101].

   Function $f = t + Ix - Jy - Kz$ is perhaps the simplest solution to the condition. One can define also other variants of $\overline{q}$, in particular the variant $\overline{q} = t + Ix - Jy - Kz$ giving $f = t + Ix + Jy + Kz$ as a solution.

2. The condition allows functions depending on complex coordinate z of some complex-plane only. It also allows functions satisfying two separate analyticity conditions, say $d_1 f = 0$ and $d_2 f = 0$, say

$$\begin{aligned} \partial_{\overline{u}} f &= (\partial_t - \partial_x I) f = 0 \quad , \\ \partial_{\overline{v}} f &= -(\partial_y J + \partial_z K) f = -J(\partial_y - \partial_z I) f = 0 \quad . \end{aligned} \tag{13.3.3}$$

   In the latter formula J multiplies from *left*! One has good hopes of obtaining holomorphic functions of two complex coordinates.

   The simplest solution to the conditions is complex value function $f(u = x + iy, v = y + iz)$ of two complex variables. The image of E^4 is 2-dimensional whereas for $f_0 = t + Ix - Jy - Kz$ it is 4-D.

   In Euclidian signature one obtains quaternion valued map if the Taylor coefficients a_{mn} in the series of $f(u, v)$ are quaternions and are taken to the right: $q = f(u, v) = \sum u^m v^n a_{mn}$ to avoid problems from non-commutativity. With this assumption the image would be 4-D in the generic case.

In TGD one must consider Minkowskian signature and it turns out that the situation changes dramatically, and the naïve view about quaternion analyticity must be given up. The experience about externals of Kähler action suggests a modification of the analyticity properties consistent with the signature but whether one should call this analyticity quaternion analyticity is a matter of taste.

Second form of CRF conditions

Second form of CRF conditions proposed in [A86] is tailored in order to realize the almost obvious manner to realize quaternion analyticity.

1. The ingenious idea is to replace preferred quaternionic imaginary unit by a imaginary unit which is in radial direction: $e_r = (xI + yJ + zK)/r$ and require analyticity with respect to the coordinate $t + e_r r$. The solution to the condition is power series in $t + e_r r = q$ so that one obtains quaternion analyticity.

2. The explicit form of the conditions is

$$(\partial_t - e_r\partial_r)f = (\partial_t - \frac{e_r}{r}r\partial_r)f = 0 \quad . \tag{13.3.4}$$

   This form allows both the desired quaternionic Taylor series and ordinary holomorphic functions of complex variable in one of the 3 complex coordinate planes as general solutions.

3. This form of CRF is neither Lorentz invariant nor translationally invariant but remains invariant under simultaneous scalings of t and r and under time translations. Under rotations of either coordinates or of imaginary units the spatial part transforms like vector so that quaternionic automorphism group $SO(3)$ serves as a moduli space for these operators.

4. The interpretation of the latter solutions inspired by ZEO would be that in Minkowskian regions r corresponds to the light-like radial coordinate of the either boundary of CD, which is part of $\delta M^4_\pm$. The radial scaling operator is that assigned with the light-like radial coordinate of the light-cone boundary. A slicing of CD by surfaces parallel to the $\delta M^4_\pm$ is assumed and implies that the line $r = 0$ connecting the tips of CD is in a special role. The line connecting the tips of CD defines coordinate line of time coordinate. The breaking of rotational invariance corresponds to the selection of a preferred quaternion unit defining the twistor structure and preferred complex sub-space.

   In regions of Euclidian signature r could correspond to the radial Eguchi-Hanson coordinate of CP_2 and $r = 0$ corresponds to a fixed point of $U(2)$ subgroup under which CP_2 complex coordinates transform linearly.

5. Also in this case one can ask whether solutions depending on two complex local coordinates analogous to those for translationally invariant CRF condition are possible. The remain imaginary units would be associated with the surface of sphere allowing complex structure.

Generalization of CRF conditions?

Could the proposed forms of CRF conditions be special cases of much more general CRF conditions as CR conditions are?

1. Ordinary complex analysis suggests that there is an infinite number of choices of the quaternionic coordinates related by the above described quaternion-analytic maps with 4-D images. The form of of the CRF conditions would be different in each of these coordinate systems and would be obtained in a straightforward manner by chain rule.

2. One expects the existence of large number of different quaternion-conformal structures not related by quaternion-analytic transformations analogous to those allowed by higher genus Riemann surfaces and that these conformal equivalence classes of four-manifolds are characterized by a moduli space and the analogs of Teichmüller parameters depending on 3-topology. In TGD framework strong form of holography suggests that these conformal equivalence classes for preferred extremals could reduce to ordinary conformal classes for the partonic 2-surfaces. An attractive possibility is that by conformal gauge symmetries the functional integral over WCW reduces to the integral over the conformal equivalence classes.

3. The quaternion-conformal structures could be characterized by a standard choice of quaternionic coordinates reducing to the choice of a pair of complex coordinates. In these coordinates the general solution to quaternion-analyticity conditions would be of form described for the linear ansatz. The moduli space corresponds to that for complex or hyper-complex structures defined in the space-time region.

Geometric formulation of the CRF conditions

The previous naïve generalization of CRF conditions treats imaginary units without trying to understand their geometric content. This leads to difficulties when when tries to formulate these conditions for maps between quaternionic and hyper-quaternionic spaces using purely algebraic representation of imaginary units since it is not clear how these units relate to each other.

In [A101] the CRF conditions are formulated in terms of the antisymmetric (1, 1) type tensors representing the imaginary units: they exist for almost quaternionic structure. One might hope that this so also for the almost hyper-quaternionic structure needed in Minkowskian signature.

The generalization of CRF conditions is proposed in terms of the Jacobian J of the map mapping tangent space TM to TN and antisymmetric tensors J_u and j_u representing the quaternionic imaginary units of N and M respectively. The generalization of CRF conditions reads as

$$J-\sum_u J_u\circ J\circ j_u=0 \ . \tag{13.3.5}$$

For $N = M$ it reduces to the translationally invariant algebraic form of the conditions discussed above. These conditions reduce to CR conditions in 2-D case when one has only single J_u. In quaternionic case this form is only replaced with sum over all 3 antisymmetric forms representing quaternionic units.

These conditions are not unique. One can perform an SO(3) rotation (quaternion automorphism) of the imaginary units mediated by matrix Λ^{uv} to obtain

$$J-\Lambda^{uv}J_u\circ J\circ j_v=0 \ . \tag{13.3.6}$$

The matrix Λ can depend on point so that one has a kind of gauge symmetry. The most general triholomorphic map allows the presence of Λ. Note that these conditions make sense on any coordinates and complex analytic maps generate new forms of these conditions.

In Minkowskian signature one would have 3 forms iJ_u serving as counterparts for iI, iJ, iK. The most natural possibility is that i is represented as algebraic unit and I, J, K as antisymmetry self-dual em fields with E and B constant and parallel to each other and normalize to have unit lengths. Their directions would correspond to 3 orthogonal coordinate axis. The twistor lift forces to introduce the generalization of Kähler form of M^4 and one can introduce all these 3 independent forms as counterpart of hyperquaternionic units: they are introduced also for ordinary twistor structure but one of them is selected as a preferred one. The only change in the conditions is change of sign of the sign of the sum coming from $i^2 = -1$ so that one has

$$J+\sum_u J_u\circ J\circ j_u=0 \ . \tag{13.3.7}$$

These conditions are therefore formally well-defined also when one maps quaternionic to hyper-quaternionic space or vice versa.

In 2-dimensional hypercomplex case the conditions allow to write hypercomplex map $X - Y = U = f(x-y)$ and $X + Y = V = f(x + y)$. In special case this solutions of massless d'Alembertian in M^2. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. It is however not clear whether one can write a Taylor expansion in hyper-quaternionic coordinate in the similar manner.

Covariant forms of structure constant tensors reduce to octonionic structure constants and this allows to write the conditions explicitly. The index raising of the second index of the structure constants is however needed using the metrics of M and N. This complicates the situation and spoils linearity: in particular, for surfaces induced metric is needed. Whether local SO(3) rotation can eliminate the dependence on induced metric is an interesting question. Since M^4 imaginary units differ only by multiplication by i, Minkowskian structure constants differ only by sign from the Euclidian ones.

In the octonionic case the geometric generalization of CRF conditions does not seem to make sense. By non-associativity of octonion product it is not possible to have a matrix representation for the matrices so that a faithful representation of octonionic imaginary units as antisymmetric 1-1 forms does not make sense. If this representation exists it it must map octonionic associators to zero. Note however that CRF conditions do not involve products of three octonion units so that they make sense as algebraic conditions at least.

Does residue calculus generalize?

CRF conditions allow to generalize Cauchy formula allowing to express value of analytic function in terms of its boundary values [A101]. This would give a concrete realization of the holography in the sense that the physical variables in the interior could be expressed in terms of the data at the light-like partonic orbits and the ends of the space-time surface. Triholomorphic function satisfies d'Alembert/Laplace equations - in induced metric in TGD framework- so that the maximum modulus principle holds true. The general ansatz for a preferred extremals involving Hamilton-Jacobi structure leads to d'Alembert type equations for preferred extremals [K96].

Could the analog of residue calculus exist? Line integral would become 3-D integral reducing to a sum over poles and possible cuts inside the 3-D contour. The space-like 3-surfaces at the ends of CDs could define natural integration contours, and the freedom to choose contour rather freely would reflect General Coordinate Invariance. A possible choice for the integration contour would be the closed 3-surface defined by the union of space-like surfaces at the ends of CD and by the light-like partonic orbits.

Poles and cuts would be in the interior of the space-time surface. Poles have co-dimension 2 and cuts co-dimension 1. Strong form of holography suggests that partonic 2-surfaces and perhaps also string world sheets serve as candidates for poles. Light-like 3-surfaces (partonic orbits) defining the boundaries between Euclidian and Minkowskian regions are singular objects and could serve as cuts. The discontinuity would be due to the change of the signature of the induced metric. There are CDs inside CDs and one can also consider the possibility that sub-CDs define cuts, which in turn reduce to cuts associated with sub-CDs.

13.3.3 Are Preferred Extremals Quaternion-Analytic in Some Sense?

At what level quaternion analyticity could appear in TGD framework? Does it appear only in the formulation of conformal algebras and replace loop algebra with double loop algebra (roughly $z^m \rightarrow u^m v^n$)? Or does it appear in some form also at the level of preferred extremals for which geometric form of quaternionicity is expected to appear - at least at the level of M^8?

Minimalistic view

Before continuing it is good to bring in mind the minimal assumptions and general vision.

1. If $M^8 - H$ duality [K82] holds true, the space-time surface $X^4 \subset M^8 = M^4 \times E^4$ is quaternionic surface in the sense that it have quaternionic tangent space and contains preferred $M^2 \subset M^4$ as part of their tangent space or more generally the 2-D hyper-complex subspaces $M^2(x)$ define and integrable distribution defining 2-D surface.

2. Quaternionicity in geometric sense in M^8 alone *implies* the interpretation as a 4-D surface in $H = M^4 \times CP_2$. There is *no need* to assume quaternionicity in geometric sense in H although it cannot be excluded and would have strong implications [K82]. This one should remember in order to avoid drowning to an inflation of speculations.

It is not at all clear what quaternion analyticity in Minkowskian signature really means or whether it is even possible. The skeptic inside me has a temptation to conclude that the direct extrapolation of quaternion analyticity from Euclidian to Minkowskian signature for space-time surfaces in H is not necessary and might be even impossible. On the other hand, the properties of the known extremals strongly suggest its analog. Quaternion analyticity could however appear at the tangent space level for various generalized conformal algebras transformed to double loop algebras for the proposed realization of the quaternion analyticity.

The naïve generalization of quaternion analyticity to Minkowski signature fails

Quaternion analyticity works nicely in Euclidian signature for maps $E^4 \rightarrow E^4$. One can also consider quaternion analytic maps $E^4 \rightarrow E^8$ with E^8 regarded as octonionic space of form $E^4 \oplus E^4 J$, where E^4 is quaternionic space and J is octonion unit in the complement of $E^4 \subset E^8$. The maps decompose to sums $f_1 \oplus f_2 J$ where f_i are quaternion analytic maps $E^4 \rightarrow E^4$. Consider maps $f : E^4 \rightarrow E^8$, whose graph should define Euclidian space-time surface.

1. One can construct octonion valued maps $f(u, v) = f_0 + \sum u^m v^n a_{mn} : E^4 \rightarrow E^8$ with E^4 identified as quaternionic sub-space of E^8. Recall that one has $u = t + Iz$, $v = (x + Iy)J$. a_{mn} can be octonions with the proposed definition of the Taylor series. Since each power $u^m v^n$ is analytic function, one still has quaternion analyticity in the proposed sense. The image would be 4-D in the general case.

2. By linearity the solutions obey linear superposition. They can be also multiplied if the product is defined as ordered product in such a way that only the powers $t+ix$ and $y+iz$ are multiplied together at left and coefficients a_{mn} are multiplied together at right. The analogy with quantum non-commutativity is obvious.

Can one generalize this ansatz to Minkowskian signature? One can try to look the ansatz for the embedding $X^4 \subset M^8 = M^4 \times E^4 J$ as sum $f = (f_1, f_2)$ of quaternion analytic maps $f_1 : X^4 \rightarrow M^4$ and $f_2 : X^4 \rightarrow E^4$. The general quaternion analytic ansatz for $X^4 \subset E^8$ fails due to the non-commutativity of quaternions.

The comparison of 2-dimensional hypercomplex case with 4-D hyperquaternionic case reveals the basic problem.

1. The analogs CR conditions allow to write hypercomplex map $X - Y = U = f(x - y)$ and $X + Y = V = f(x + y)$. In special case this gives the solutions of massless d'Alembertian in M^2 as sum of these solutions. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. The use of hypercomplex numbers and hypercomplex analyticity is equivalent with use of functions $f(x - y)$ or $f(x + y)$.

2. The essential point is that for M^2 regarded as a sub-space of "complexified" complex numbers $z_1 + iz_2$ consisting of points $x + iIy$, the multiplication of numbers of form $x + iIy$ does not lead out of M^2. For M^4 this is not anymore the case since $iI \times iJ = -K$ does not belong to the Minkowskian subspace of complexified quaternions. Hence there are no hopes about the existence of the analog of $f(z) = \sum a_n z^n$. For this reason also non-trivial powers $u^m v^n$ are excluded and one cannot build a Minkowskian generalization of quaternion analytic power series.

3. If one can allow the values of hyper-quaternion analytic functions to be in M_c^4 rather than M^4, there are no problems but if one wants to represents space-time surfaces as graphs of hyper-quaternion analytic maps $f : M^4 \rightarrow M^8$ one must pose strong restrictions on allowed functions.

The restrictions on the allowed hyper-quaternion analytic functions look rather obvious for what might be called hyper-quaternion analytic maps $M^4 \rightarrow M^4$.

1. Assume a decomposition $M^4 = M^2 \times E^2$ such one has $f = (f_1, f_2)$, where $f_1 : M^2 \rightarrow M^2$ is analytic in hyper-complex sense and $f_2 : E^2 \rightarrow E^2$ is analytic in complex sense. Both these options are possible. One can write the map as $f(u, v) = f_1(u = t + iIz) + f_2(v = x + Iy)iJ$

and it satisfies the usual conditions $\partial_{\overline{u}}f = 0$ and $\partial_{\overline{v}}f = 0$. Note that iJ is taken to the right so that the differential operators acting from left in the analyticity conditions does not "notice" it.

Linear superpositions of this kind of solutions with real coefficients are possible. One can multiply this kind of solutions if the multiplication is done separately in the Cartesian factors. Also functional composition is possible in the factors.

2. A generalization of the solution ansatz to integrable decompositions $M^4 = M^2(x) \oplus E^2(x)$ is rather plausible. This would mean a foliation of M^4 by pairs of 2-D surfaces. String world sheets and partonic 2-surfaces would be the physical counterpart for these foliations. I have called this kind of foliation Hamilton-Jacobi structure [K5] and it would serve as a generalization of the complex structure to 4-D Minkowskian case. In Euclidian signature it corresponds to ordinary complex structure in 4-D sense.

3. The analogy of double loop Lie algebra replacing powers z^m with $u^m v^n$ does does not however seem to be possible. Could this relate to SH forcing to code data using only functions of u or v and to select either string world sheet or partonic 2-surface (fixing the gauge)?

   On the other hand, the supersymplectic algebra (SSA) and extension of Kac-Moody algebras to light-like orbits of partonic 2-surfaces suggests strongly that functions of form $(t-z)^m v^n$ as basis associated with SSA and SKMAs must be allowed as basis at these 3-D light-like surfaces. These functions generate deformations of boundaries defining symmetries but the corresponding deformations in the interior of the preferred extremals are not expected to be of this form. Double loop algebra would not be lost but would have a nice separable form only at boundaries of CD and at light-like partonic orbits.

What can one conclude?

1. The general experience about the solutions of field equations conforms with this picture coded to the notion of Hamilton-Jacobi structure [K5]. Field equations and purely number theoretic conditions related to Minkowski signature force what might be called number theoretic spontaneous symmetry breaking. This symmetry breaking is analogous to a selection of single imaginary unit defining the analog of Kähler structure for M^4: this imaginary unit defines a new kind of $U(1)$ force in TGD explaining large scale breaking of CP,P, and T [L40]. This kind of selection occurs also for the quaternionic structure of CP_2 [L50].

2. The realistic form of analyticity condition abstractable from the properties known extremals seems to be following. For the Minkowskian space-time surfaces the complex coordinates of H are analytic functions of complex coordinates and of light-like coordinate assignable to space-time surface. These coordinates can be assigned to M^4 and define decomposition $M^4 = M^2 \times E^2$: this decomposition can be local but must be integrable (Hamilton-Jacobi structure). For Euclidian regions with 4-D CP_2 projection complex coordinate of E^2 is complex function of complex coordinates of CP_2 and M^2 light-like coordinate is function of real CP_2 coordinates and second light-like coordinate is constant.

3. The transition to Minkowskian signature by regarding M^4 as sub-space of complex-quaternionic M^4 does not respect the notion of quaternion analyticity in the naïvest sense. Both Euclidian and Minkowskian variants of quaternionic (associative) sub-manifold however makes sense as also co-quaternionic (co-associative) sub-manifold. An attractive hypothesis is that the geometric view about quaternionicity is consistent with the above view about analyticity. The known extremals are consistent with this form of analyticity. Analyticity in this sense should be consistent with the geometric quaternionicity of X^4 in Minkowskian signature and geometric co-quaternionicity in Euclidian signature.

4. The geometric form of quaternionicity (or associativity) requires that the associator $a(bc) - (ab)c$ for any 3 tangent space vectors vanishes. These conditions involve products of 3 partial derivates of embedding space coordinates. For co-associativity this holds true in the normal space. Again one must remember that these conditions might be needed only in M^8 but make sense also for H.

One must be however cautious: quaternionicity (associativity) in M^8 in the geometric sense *need not* imply even the above realistic form of quaternion analyticity condition in M^8 and even less so in H: this however seems to be the case.

Can the known extremals satisfy the realistic form of quaternion-analyticity?

To test the consistency the realistic form of quaternion analyticity, at the level of M^8 or even H, the best thing to do is to look whether quaternion analyticity is possible for the known extremals for the twistor lift of Kähler action.

Twistor lift drops away most vacuum extremals from consideration and leaves only minimal surfaces. The surviving vacuum extremals include CP_2 type extremals with light-like geodesic rather than arbitrary light-like curve as M^4 projection. Vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2 remain in the spectrum only if they are also minimal surfaces: this kind of minimal surfaces are known to exist.

Massless extremals (MEs) with 2-D CP_2 projection remain in the spectrum. Cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2 are extremals of both Kähler action and volume term. One can also check whether Hamilton-Jacobi structure of M^4 and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^4 and Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k\cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ - a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condensate of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure [K5]. The non-geodesic motion of photons with light-velocity and variation of the polarization direction would be due to the interactions with the space-time sheet to which it is topologically condensed.

   Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of the realistic form of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t-x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

   The dependence on the real coordinate $v = t - z$ does not conform with the proposed ansatz since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

   If this picture makes sense, one must modify the ansatz for the preferred extremals with Minkowskian signature. The E^4 and coordinates and perhaps even real CP_2 coordinates can depend on light-like coordinate u.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection survive if they are minimal surfaces. This property should guarantee the realistic form of quaternion analyticity. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals surviving as extremals the M^4 projection is light-like geodesic with $t + z = 0$ with suitable choice of light-like coordinates in M^2. $t - z$ can arbitrary function of CP_2 coordinates. Associativity of the normal space is the only possible option now.

   One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light- like coordinate would vary along random light-like geodesics providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined.

5. In the naïve generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [A86] both forms of the algebraic CRF conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

   Could octonion analytic maps of embedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

Quaternion analyticity and generalized conformal algebras

The realistic quaternionic analyticity should apply at the level of conformal algebras for conformal algebra is replaced with a direct sum of 2-D conformal and hyper-conformal algebra assignable to string world sheets and partonic 2-surfaces. This would conform with SH and the considerations above.

It is however too early to exclude the possibility that the powers z^n of conformal algebras are replaced by $u^m z^n$ ($u = t - z$ and $w = x + iy$) for symmetries restricted to the light-like boundaries of CD and to the light-like orbits of partonic 2-surfaces. This preferred form at boundaries would be essential for reducing degrees of freedom implied by SSA and SKMA gauge conditions. In the interior of space-time surfaces this simple form would be lost.

This would realize the Minkowskian analog of double loop algebras suggested by 4-dimensionality. This option is encouraged by the structure of super-symplectic algebra and generalization of Kac-Moody algebras for light-like orbits of partonic 2-surfaces. Again one must however remember that these algebras should have a realization at the level of M^8 but might not be necessary at the level of H.

1. The basic vision of quantum TGD is that string world sheets are replaced with 4-D surfaces and this forces a generalization of the notion of conformal invariance and one indeed obtains

generalized conformal invariances for both the light-like boundaries of CD and for the light-like 3-surfaces defining partonic orbits as boundaries between Minkowskian and Euclidian space-time regions. One can very roughly say that string string world sheets parameterized by complex coordinate are replaced by space-time surfaces parameterized by two complex coordinates. Quaternion analyticity in the sense discussed would roughly conform with this picture.

2. The recent work with the Yangians [L41] and so called n-structures related to the categorification of TGD [K9] suggest that double loop algebras for which string world sheets are replaced with 4-D complex spaces. Quantum groups and Yangians assignable to Kac-Moody algebras rather than Lie algebras should be also central. Also double quantum groups depending on 2 parameters with so called elliptic R-matrix seem to be important. This physical intuition agrees with the general vision of Russian mathematician Igor Frenkel, who is one of the pioneers of quantum groups. For the article summarizing the work of Frenkel see `http://tinyurl.com/y7eego8c`). This article tells also about the work of Frenkel related to quaternion analyticity, which he sees to be of physical relevance but as a mathematicians is well aware of the fact that the non-commutativity of quaternions poses strong interpretation problems and means the loss of many nice properties of the ordinary analyticity.

3. The twistor lift of TGD suggest similar picture [K31, L33, L50]. The 6-D twistor space of space-time surface would be 6-surface in the product $T(M^4) \times T(CP_2)$ of geometric twistor spaces of M^4 and CP_2 and have induced twistor structure. The detailed analysis of this statement strongly suggests that data given at surfaces with dimension not higher than $D = 2$ should fix the preferred extremals. For the twistor lift action contains besides Kähler action also volume term. Asymptotic solutions are extremals of both Kähler action and minimal surfaces and all non-vacuum extremals of Kähler action are minimal surfaces so that the only change is that vacuum extremals of Kähler action must be restricted to be minimal surfaces.

The article about the work of Igor Frenkel (see `http://tinyurl.com/y7eego8c`) explains the general mathematics inspired vision about 3-levelled hierarchy of symmetries.

1. At the lowest level are Lie algebras. Gauge theories are prime example about this level.

2. At the second level loop algebras and quantum groups (defined as deformations of enveloping algebra of Lie algebra) and also Yangians. Loop algebras correspond to string models and quantum groups to TQFTs formulated at 3-D spaces.

3. At the third level are double loop algebras, quantum variants of loop algebras (also Yangians), and double quantum quantum groups - deformations of Lie algebras for which the R-matrix is elliptic function and depends on 2 complex parameters.

The conjecture of Frenkel (see `http://tinyurl.com/y7eego8c`) based on mathematical intuition is that these levels are actually the only ones. The motivation for this claim is 2-dimensionality making possible braiding and various quantum algebras. The set of poles for the R-matrix forms Abelian group with respect to addition in complex plane and can have rank equal to 0, 1, or (single pole, poles along line, lattice of poles). Higher ranks are impossible in $D = 2$.

In TGD framework physical intuition leads to a similar vision.

1. The dimension $D = 4$ for space-time surface and the choice $H = M^4 \times CP_2$ have both number theoretical and twistorial motivations [L41]. The replacement of point like particle with partonic 2-surface implies that TGD corresponds to the third level since loop algebras are replaced with their double loop analogs. 4-dimensionality makes also possible 2-braids and reconnections giving rise to a new kind of topological physics.

   The double loop group would represent the most dynamical level and its singly and doubly quantized variants correspond to a reduction in degrees of freedom, which one cannot exclude in TGD.

   The interesting additional aspect relates to the adelic physics [L46] implying a hierarchy of physics labelled by extensions of rationals. For cognitive representations the dynamics is discretized [K9]. Light-like 3-surfaces as partonic orbits are part of the picture and Chern-Simons term is naturally associated with them. TGD as almost topological QFT has been one of the guiding ideas in the evolution of TGD.

2. Double loop algebras represent unknown territory of mathematical physics. Igor Frenkel has also considered a possible realization of double loop algebras (see `http://tinyurl.com/y7eego8c`). He starts from the work of Mickelson (by the way, my custos in my thesis defence in 1982!) giving a realization of loop algebras: the idea is clearly motivated by WZW model which is 2-D conformal field theory with action containing a term associated with a 3-ball having 2-sphere as boundary.

   Mickelson starts from a circle represented as a boundary of a disk at which the physical states of CFT are realized. CFT is obtained by gluing together two disks with the boundary circles identified. The sphere in turn can be regarded as a boundary of a ball. The proposal of Frenkel is to complexify all these structures: circle becomes a Riemann surface, disk becomes 4-D manifold possessing complex structure in some sense, and 3-ball becomes 6-D complex manifold in some sense conjectured to be Calabi-Yau manifold.

3. The twistor lift of TGD leads to an analogous proposal. Circle is replaced with partonic 2-surfaces and string world sheets. 2-D complex surface is replaced with space-time region with complex structure or Hamilton-Jacobi structure [K5] and possessing twistor structure. 6-D Calabi-Yau manifold is replaced with the 6-D twistor space of space-time surface (sphere bundle over space-time surface) represented as 6-surface in 12-D Cartesian product $T(H) = T(M^4) \times T(CP_2)$ of the geometric twistor spaces of M^4 and CP_2.

   Twistor structure is induced and this is conjectured to determine the dynamics to be that for the preferred extremals of Kähler action plus volume term. This vision would generalize Penrose's original vision by eliminating gauge fields as primary dynamical variables and replacing there dynamics with the geometrodynamics of space-time surface.

Do isometry currents of preferred extremals satisfy Frobenius integrability conditions?

During the preparation of the book I learned that Agostino Prastaro [A41, A42] has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action of its twistor lift in TGD. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis.

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could the natural topology in the parameter space of Noether charges zero modes of WCW metric) be p-adic and realize adelic physics at the level of WCW? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the 6 lowest dynamical homotopy/homology groups of WCW would be non-trivial. The Kähler structure of WCW suggests that only Π_0, Π_2, and Π_4 are non-trivial.

The interpretation of the analog of Π_1 as deformations of generalized Feynman diagrams with elementary cobordism snipping away a loop as a move leaving scattering amplitude invariant conforms with the number theoretic vision about scattering amplitude as a representation for a sequence of algebraic operation can be always reduced to a tree diagram. TGD would be indeed topological QFT: only the dynamical topology would matter.

A further outcome is an ansatz for generalizing the earlier proposal for preferred extremals and stating that non-vanishing conserved isometry currents satisfy Frobenius integrability conditions so that one could assign global coordinate with their flow lines. This ansatz looks very similar to the CRF conditions stating quaternion analyticity [L24].

Conclusions

To sum up, connections between different conjectures related to the preferred extremals - $M^8 - H$ duality, Hamilton-Jacobi structure, induced twistor space structure, quaternion-Kähler property and its Minkowskian counterpart, and perhaps even quaternion analyticity - albeit not in the naïve form , are clearly emerging. The underlying reason is strong form of GCI forced by the construction of WCW geometry and implying strong from of holography posing extremely powerful quantization conditions on the extremals of Kähler action in ZEO. Without the conformal gauge conditions the mutual inconsistency of these conjectures looks rather infeasible.

13.4 Octo-Twistors And Twistor Space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall $M^8 - H$ duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [K86]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic embedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

13.4.1 Two ways To Twistorialize Embedding Space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1)\times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably,

M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-definec operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4 \times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

   Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are "real" and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and embedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

13.4.2 Octotwistorialization Of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

13.4.3 Octonionicity, $SO(1,7)$, G_2, And Non-Associative Malcev Group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1, 7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \rightarrow G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (`http://tinyurl.com/ybd4lcpy`). The basic observation is that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14 + 14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a, b] = ab-ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}\left[\gamma_k, \gamma_l\right] \quad . \tag{13.4.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

This algebra has decomposition $g = h + t$, $[h, t] \subset t$, $[t, t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1, 7)$.

13.4.4 Octonionic Spinors In M^8 And Real Complexified-Quaternionic Spinors In H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.

2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

   The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

   One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc) - (ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

13.4.5 What The Replacement Of $SO(7, 1)$ Sigma Matrices With Octonionic Sigma Matrices Could Mean?

The basic implication of octonionization is the replacement of $SO(7, 1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4 \times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4 \times CP_2$ but interpret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ , \ \gamma^i = \gamma^i \otimes \sigma_2 \ , \ i = 1,..,7 \ . \tag{13.4.2}$$

   7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1,...,6 \ , \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ . \tag{13.4.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \otimes \sigma_1 \ , \ \gamma_i = e_i \otimes \sigma_2 \ . \tag{13.4.4}$$

   where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = je_i \times \sigma_3 \ , \ \Sigma_{ij} = jf_{ij}{}^k e_k \otimes 1 \ . \tag{13.4.5}$$

   Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A25] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I^3_L = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [L2]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the Kähler-Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8?

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://tinyurl.com/y93aerea`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of hyper-complex numbers) such that the second complexified Kähler-Dirac gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

Does this mean that the difference between quarks and leptons becomes visible only at the embedding space level where ground states of super-conformal representations correspond to embedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

Octo-spinors and their relation to ordinary embedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\Psi_{L,i} = e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ,$$
$$\Psi_{q,i} = e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{13.4.6}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{13.4.7}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic Kähler-Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

13.4.6 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K86] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

The case of $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K82] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?

2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.

4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K20].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?

2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.

3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions

should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

(a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

(b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

(c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

13.5 What Could Be The Origin Of Preferred P-Adic Primes And P-Adic Length Scale Hypothesis?

p-Adic mass calculations [K54] allow to conclude that elementary particles correspond to one or possible several preferred primes assigning p-adic effective topology to the real space-time sheets in discretization in some length scale range. TGD inspired theory of consciousness leads to the identification of p-adic physics as physics of cognition. The recent progress leads to the proposal that quantum TGD is adelic: all p-adic number fields are involved and each gives one particular view about physics. tgdquantum/tgdquantum Adelic approach [K42, K59] plus the view about evolution as emergence of increasingly complex extensions of rationals leads to a possible answer to th question of the title. The algebraic extensions of rationals are characterized by preferred rational primes, namely those which are ramified when expressed in terms of the primes of the extensions. These primes would be natural candidates for preferred p-adic primes. An argument relying on what I call weak form of NMP in turn allows to understand why primes near powers of 2 are preferred: as a matter of fact, also primes near powers of other primes are predicted to be favoured.

13.5.1 Earlier Attempts

How the preferred primes emerge in TGD framework? I have made several attempts to answer this question. As a matter fact, the question has been slightly different: what determines the p-adic prime assigned to elementary particle by p-adic mass calculations [K46]. The recent view assigns to particle entire adele but some p-adic number fields in it are different.

1. Classical non-determinism at space-time level for real space-time sheets could in some length scale range involving rational discretization for space-time surface itself or for parameters characterizing it as a preferred extremal correspond to the non-determinism of p-adic differential equations due to the presence of pseudo constants which have vanishing p-adic derivative. Pseudo- constants are functions depend on finite number of pinary digits of its arguments.

2. The quantum criticality of TGD [?] is suggested to be realized in in terms of infinite hierarchies of super-symplectic symmetry breakings in the sense that only a sub-algebra with

conformal weights which are n-ples of those for the entire algebra act as conformal gauge symmetries [K73]. This might be true for all conformal algebras involved. One has fractal hierarchy since the sub-algebras in question are isomorphic: only the scale of conformal gauge symmetry increases in the phase transition increasing n. The hierarchies correspond to sequences of integers $n(i)$ such tht $n(i)$ divides $n(i+1)$. These hierarchies would very naturally correspond to hierarchies of inclusions of hyper-finite factors and $m(i) = n(i+1)/n(i)$ could correspond to the integer n characterizing the index of inclusion, which has value $n \geq 3$. Possible problem is that $m(i) = 2$ would not correspond to Jones inclusion. Why the scaling by power of two would be different? The natural question is whether the primes dividing $n(i)$ or $m(i)$ could define the preferred primes.

3. Negentropic entanglement corresponds to entanglement for which density matrix is projector [K49]. For n-dimensional projector any prime p dividing n gives rise to negentropic entanglement in the sense that the number theoretic entanglement entropy defined by Shannon formula by replacing p_i in $log(p_i) = log(1/n)$ by its p-adic norm $N_p(1/n)$ is negative if p divides n and maximal for the prime for which the dividing power of prime is largest power-of-prime factor of n. The identification of p-adic primes as factors of n is highly attractive idea. The obvious question is whether n corresponds to the integer characterizing a level in the hierarchy of conformal symmetry breakings.

4. The adelic picture about TGD led to the question whether the notion of unitarity could be generalized. S-matrix would be unitary in adelic sense in the sense that $P_m = (SS^\dagger)_{mm} = 1$ would generalize to adelic context so that one would have product of real norm and p-adic norms of P_m. In the intersection of the realities and p-adicities P_m for reals would be rational and if real and p-adic P_m correspond to the same rational, the condition would be satisfied. The condition that $P_m \leq 1$ seems however natural and forces separate unitary in each sector so that this options seems too tricky.

These are the basic ideas that I have discussed hitherto.

13.5.2 Could Preferred Primes Characterize Algebraic Extensions Of Rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of ramification of primes (see `http://tinyurl.com/hddljlf`) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this language): As one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called integral closure (see `http://tinyurl.com/js6fpvr`) over integers of K decompose to products of prime ideals of L (prime is a more rigorous manner to express primeness).

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K and having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

2. There are two further basic notions related to ramification and characterizing it. Relative discriminant is the ideal divided by all ramified ideals in K and relative different is the ideal of L divided by all ramified P_i:s. Note that te general ideal is analog of integer and these ideas represent the analogous of product of preferred primes P of K and primes P_i of L dividing them.

3. A physical analogy is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, where e_i is the ramification index - the physical analog would be the number of elementary particles of type i in the state (see `http://tinyurl.com/h9528pl`). Could the ramified rational primes could define the physically preferred primes for a given elementary system?

In TGD framework the extensions of rationals (see `http://tinyurl.com/h9528pl`) and p-adic number fields (see`http://tinyurl.com/zq22tvb`) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would have gradually proceeded to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. IT would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (see `http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

4. Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n :th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified. The physical analog $P = P_0^n$ is Bose-Einstein condensate of n bosons. There is a strong temptation to identify the preferred primes of irreducible polynomials as preferred p-adic primes.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with $\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

5. What does this mean in p-adic context? The identity of the ideals can be stated by saying $P = P_0^n$ for the ideals defined by the primes satisfying the Eisenstein condition. Very loosely one can say that the algebraic extension defined by the root involves n:th root of p-adic prime p. This does not work! Extension would have a number whose n:th power is zero modulo p. On the other hand, the p-adic numbers of the extension modulo p should be finite field but this would not be field anymore since there would exist a number whose n:th power vanishes. The algebraic extension simply does not exist for preferred primes. The physical meaning of this will be considered later.

6. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \to x+1$ transforms $(x^n-1)/(x-1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

13.5.3 A Connection With Langlands Program?

In Langlands program (`http://tinyurl.com/ycej7s43`,`RecentAdvancesinLanglandsprogram`) [A57, A56] the great vision is that the n-dimensional representations of Galois groups G characterizing algebraic extensions of rationals or more general number fields define n-dimensional adelic representations of adelic Lie groups, in particular the adelic linear group $Gl(n,A)$. This would mean that it is possible to reduce these representations to a number theory for adeles. This would be highly relevant in the vision about TGD as a generalized number theory. I have speculated with this possibility earlier (`http://tinyurl.com/y9ee3lk6`) [K42] but the mathematics is so horribly abstract that it takes decade before one can have even hope of building a rough vision.

One can wonder whether the irreducible polynomials could define the preferred extensions K of rationals such that the maximal abelian extensions of the fields K would in turn define the adeles utilized in Langlands program. At least one might hope that everything reduces to the maximally ramified extensions.

At the level of TGD string world sheets with parameters in an extension defined by an irreducible polynomial would define an adele containing various p-adic number fields defined by the primes of the extension. This would define a hierarchy in which the prime ideals of previous level would decompose to those of the higher level. Each irreducible extension of rationals would correspond to some physically preferred p-adic primes.

It should be possible to tell what the preferred character means in terms of the adelic representations. What happens for these representations of Galois group in this case? This is known.

1. For Galois extensions ramification indices are constant: $e(i) = e$ and Galois group acts transitively on ideals P_i dividing P. One obtains an n-dimensional representation of Galois group. Same applies to the subgroup of Galois group G/I where I is subgroup of G leaving P_i invariant. This group is called inertia group. For the maximally ramified case G maps the ideal P_0 in $P = P_0^n$ to itself so that $G = I$ and the action of Galois group is trivial taking P_0 to itself, and one obtains singlet representations.

2. The trivial action of Galois group looks like a technical problem for Langlands program and also for TGD unless the singletness of P_i under G has some physical interpretation. One

possibility is that Galois group acts as like a gauge group and here the hierarchy of sub-algebras of super-symplectic algebra labelled by integers n is highly suggestive. This raises obvious questions. Could the integer n characterizing the sub-algebra of super-symplectic algebra acting as conformal gauge transformations, define the integer defined by the product of ramified primes? P_0^n brings in mind the n conformal equivalence classes which remain invariant under the conformal transformations acting as gauge transformations. . Recalling that relative discriminant is an of K ideal divisible by ramified prime ideals of K, this means that n would correspond to the relative discriminant for $K = Q$. Are the preferred primes those which are "physical" in the sense that one can assign to the states satisfying conformal gauge conditions?

If the Galois group corresponds to gauge symmetries for these primes, it is physically natural that the p-adic algebraic extension does not exists and that p-adic variant of the Galois group is absent. Nothing is lost from cognition since there is nothing to cognize!

13.5.4 What Could Be The Origin Of P-Adic Length Scale Hypothesis?

The argument would explain the existence of preferred p-adic primes. It does not yet explain p-adic length scale hypothesis [K57, K46] stating that p-adic primes near powers of 2 are favored. A possible generalization of this hypothesis is that primes near powers of prime are favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [**?**] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present, this is discussed in TGD inspired theory of music harmony and genetic code [K70].

The weak form of NMP might come in rescue here.

1. Entanglement negentropy for a negentropic entanglement [K49] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretic entanglement negentropy. If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real preferred extremal as p-adic preferred extermal (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. Weak form of NMP [K49, K91] suggests a more convincing explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. Weak form of NMP allows free will so that all dimensions $n - k$, $k = 0, 1, ...n - 1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n - k$ is power of prime. For $n = 2^k = M_k + 1$, where M_k is Mersenne prime $n - 1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n = 2^k - r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p = 2$ can be replaced by any prime.

This argument together with the hypothesis that preferred prime is ramified would correlate the character of the irreducible extension and character of super-conformal symmetry breaking. The integer n characterizing super-symplectic conformal sub-algebra acting as gauge algebra would depends on the irreducible algebraic extension of rational involved so that the hierarchy of quantum criticalities would have number theoretical characterization. Ramified primes could appear as divisors of n and n would be essentially a characteristic of ramification known as discriminant. An interesting question is whether only the ramified primes allow the continuation of string world sheet and partonic 2-surface to a 4-D space-time surface. If this is the case, the assumptions behind p-adic mass calculations would have full first principle justification.

13.5.5 A Connection With Infinite Primes?

Infinite primes are one of the mathematical outcomes of TGD [K80]. There are two kinds of infinite primes. There are the analogs of free many particle states consisting of fermions and bosons labelled by primes of the previous level in the hierarchy. They correspond to states of a supersymmetric arithmetic quantum field theory or actually a hierarchy of them obtained by a repeated second quantization of this theory. A connection between infinite primes representing bound statesandirreducible polynomials is highly suggestive.

1. The infinite prime representing free many-particle state decomposes to a sum of infinite part and finite part having no common finite prime divisors so that prime is obtained. The infinite part is obtained from "fermionic vacuum" $X = \prod_k p_k$ by dividing away some fermionic primes p_i and adding their product so that one has $X \rightarrow X/m + m$, where m is square free integer. Also $m = 1$ is allowed and is analogous to fermionic vacuum interpreted as Dirac sea without holes. X is infinite prime and pure many-fermion state physically. One can add bosons by multiplying X with any integers having no common denominators with m and its prime decomposition defines the bosonic contents of the state. One can also multiply m by any integers whose prime factors are prime factors of m.

2. There are also infinite primes, which are analogs of bound states and at the lowest level of the hierarchy they correspond to irreducible polynomials $P(x)$ with integer coefficients. At the second levels the bound states would naturally correspond to irreducible polynomials $P_n(x)$ with coefficients $Q_k(y)$, which are infinite integers at the previous level of the hierarchy.

3. What is remarkable that bound state infinite primes at given level of hierarchy would define maximally ramified algebraic extensions at previous level. One indeed has infinite hierarchy of infinite primes since the infinite primes at given level are infinite primes in the sense that they are not divisible by the primes of the previous level. The formal construction works as such. Infinite primes correspond to polynomials of single variable at the first level, polynomials of two variables at second level, and so on. Could the Langlands program could be generalized from the extensions of rationals to polynomials of complex argument and that one would obtain infinite hierarchy?

4. Infinite integers in turn could correspond to products of irreducible polynomials defining more general extensions. This raises the conjecture that infinite primes for an extension K of rationals could code for the algebraic extensions of K quite generally. If infinite primes correspond to real quantum states they would thus correspond the extensions of rationals to which the parameters appearing in the functions defining partonic 2-surfaces and string world sheets.

   This would support the view that partonic 2-surfaces associated with algebraic extensions defined by infinite integers and thus not irreducible are unstable against decay to partonic 2-surfaces which corresponds to extensions assignable to infinite primes. Infinite composite integer defining intermediate unstable state would decay to its composites. Basic particle physics phenomenology would have number theoretic analog and even more.

5. According to Wikipedia, Eisenstein's criterion (`http://tinyurl.com/47kxjz`) allows generalization and what comes in mind is that it applies in exactly the same form also at the higher levels of the hierarchy. Primes would be only replaced with prime polynomials and the there would be at least one prime polynomial $Q(y)$ dividing the coefficients of $P_n(x)$ except the highest one such that its square would not divide P_0. Infinite primes would give rise to an infinite hierarchy of functions of many complex variables. At first level zeros of function would give discrete points at partonic 2-surface. At second level one would obtain 2-D surface: partonic 2-surfaces or string world sheet. At the next level one would obtain 4-D surfaces. What about higher levels? Does one obtain higher dimensional objects or something else. The union of n 2-surfaces can be interpreted also as $2n$-dimensional surface and one could think that the hierarchy describes a hierarchy of unions of correlated partonic 2-surfaces. The correlation would be due to the preferred extremal property of Kähler action.

   One can ask whether this hierarchy could allow to generalize number theoretical Langlands to the case of function fields using the notion of prime function assignable to infinite prime. What this hierarchy of polynomials of arbitrary many complex arguments means physically is unclear. Do these polynomials describe many-particle states consisting of partonic 2-surface such that there is a correlation between them as sub-manifolds of the same space-time sheet representing a preferred extremals of Kähler action?

This would suggest strongly the generalization of the notion of p-adicity so that it applies to infinite primes.

1. This looks sensible and maybe even practical! Infinite primes can be mapped to prime polynomials so that the generalized p-adic numbers would be power series in prime polynomial - Taylor expansion in the coordinate variable defined by the infinite prime. Note that infinite primes (irreducible polynomials) would give rise to a hierarchy of preferred coordinate variables. In terms of infinite primes this expansion would require that coefficients are smaller than the infinite prime P used. Are the coefficients lower level primes? Or also infinite integers at the same level smaller than the infinite prime in question? This criterion makes sense since one can calculate the ratios of infinite primes as real numbers.

2. I would guess that the definition of infinite-P p-adicity is not a problem since mathematicians have generalized the number theoretical notions to such a level of abstraction much above of a layman like me. The basic question is how to define p-adic norm for the infinite primes (infinite only in real sense, p-adically they have unit norm for all lower level primes) so that it is finite.

3. There exists an extremely general definition of generalized p-adic number fields (see `http://tinyurl.com/y5zreeg`). One considers Dedekind domain D, which is a generalization of integers for ordinary number field having the property that ideals factorize uniquely to prime ideals. Now D would contain infinite integers. One introduces the field E of fractions consisting of infinite rationals.

   Consider element e of E and a general fractional ideal eD as counterpart of ordinary rational and decompose it to a ratio of products of powers of ideals defined by prime ideals, now those defined by infinite primes. The general expression for the p-adic norm of x is $x^{-ord(P)}$, where n defines the total number of ideals P appearing in the factorization of a fractional ideal in E: this number can be also negative for rationals. When the residue field is finite (finite field G(p,1) for p-adic numbers), one can take c to the number of its elements ($c = p$ for p-adic numbers.

   Now it seems that this number is not finite since the number of ordinary primes smaller than P is infinite! But this is not a problem since the topology for completion does not depend on the value of c. The simple infinite primes at the first level (free many-particle states) can be mapped to ordinary rationals and q-adic norm suggests itself: could it be that infinite-P p-adicity corresponds to q-adicity discussed by Khrennikov [A40]. Note however that q-adic numbers are not a field.

Finally a loosely related question. Could the transition from infinite primes of K to those of L takes place just by replacing the finite primes appearing in infinite prime with the decompositions?

The resulting entity is infinite prime if the finite and infinite part contain no common prime divisors in L. This is not the case generally if one can have primes P_1 and P_2 of K having common divisors as primes of L: in this case one can include P_1 to the infinite part of infinite prime and P_2 to finite part.

13.6 More About Physical Interpretation Of Algebraic Extensions Of Rationals

The number theoretic vision has begun to show its power. The basic hierarchies of quantum TGD would reduce to a hierarchy of algebraic extensions of rationals and the parameters - such as the degrees of the irreducible polynomials characterizing the extension and the set of ramified primes (see `http://tinyurl.com/hddljlf`) - would characterize quantum criticality and the physics of dark matter as large h_{eff} phases. The value of $h_{eff}/h = n$ would naturally correspond to the order of the Galois group of the extension.

The conjecture is that preferred p-adic primes correspond to ramified primes for extensions of rationals for which especially many number theoretic discretizations of the space-time surfaces allow strong form of holography as an algebraic continuation of string world sheets to space-time surfaces. The generalization of the p-adic length scale hypothesis as a prediction of NMP is another conjecture. What remains to be shown that the primes predicted by generalization p-adic length scale hypothesis indeed are preferred primes in the proposed sense.

By strong form of holography the parameters characterizing string world sheets and partonic 2-surfaces serve as WCW coordinates. By various conformal invariances, one expects that the parameters correspond to conformal moduli, which means a huge simplification of quantum TGD since the mathematical apparatus of superstring theories becomes available and number theoretical vision can be realized. Scattering amplitudes can be constructed for a given algebraic extension and continued to various number fields by continuing the parameters which are conformal moduli and group invariants characterizing incoming particles.

There are many un-answered and even un-asked questions.

1. How the new degrees of freedom assigned to the n-fold covering defined by the space-time surface pop up in the number theoretic picture? How the connection with preferred primes emerges?

2. What are the precise physical correlates of the parameters characterizing the algebraic extension of rationals? Note that the most important extension parameters are the degree of the defining polynomial and ramified primes.

13.6.1 Some Basic Notions

Some basic information about extensions are in order. I emphasize that I am not a specialist.

Basic facts

The algebraic extensions of rationals are determined by roots of polynomials. Polynomials be decomposed to products of irreducible polynomials, which by definition do not contain factors which are polynomials with rational coefficients. These polynomials are characterized by their degree n, which is the most important parameter characterizing the algebraic extension.

One can assign to the extension primes and integers - or more precisely, prime and integer ideals. Integer ideals correspond to roots of monic polynomials $P_n(x) = x^n + ..a_0$ in the extension with integer coefficients. Clearly, for $n = 0$ (trivial extension) one obtains ordinary integers. Primes as such are not a useful concept since roots of unity are possible and primes which differ by a multiplication by a root of unity are equivalent. It is better to speak about prime ideals rather than primes.

Rational prime p can be decomposed to product of powers of primes of extension and if some power is higher than one, the prime is said to be ramified and the exponent is called ramification index. Eisenstein's criterion (see `http://tinyurl.com/47kxjz` states that any polynomial $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_1x + a_0$ for which the coefficients a_i, $i < n$ are divisible by p and

a_0 is not divisible by p^2 allows p as a maximally ramified prime. The corresponding prime ideal is n:th power of the prime ideal of the extensions (roughly n:th root of p). This allows to construct endless variety of algebraic extensions having given primes as ramified primes.

Ramification is analogous to criticality. When the gradient potential function $V(x)$ depending on parameters has multiple roots, the potential function becomes proportional a higher power of $x - x_0$. The appearance of power is analogous to appearance of higher power of prime of extension in ramification. This gives rise to cusp catastrophe. In fact, ramification is expected to be number theoretical correlate for the quantum criticality in TGD framework. What this precisely means at the level of space-time surfaces, is the question.

Galois group as symmetry group of algebraic physics

I have proposed long time ago that Galois group (see `http://tinyurl.com/h9528pl`) acts as fundamental symmetry group of quantum TGD and even made clumsy attempt to make this idea more precise in terms of the notion of number theoretic braid. It seems that this notion is too primitive: the action of Galois group must be realized at more abstract level and WCW provides this level.

First some facts (I am not a number theory professional, as the professional reader might have already noticed!).

1. Galois group acting as automorphisms of the field extension (mapping products to products and sums to sums and preserves norm) characterizes the extension and its elements have maximal order equal to n by algebraic n-dimensionality. For instance, for complex numbers Galois group acs as complex conjugation. Galois group has natural action on prime ideals of extension mapping them to each other and preserving the norm determined by the determinant of the linear map defined by the multiplication with the prime of extension. For instance, for the quadratic extension $Q(\sqrt{5})$ the norm is $N(x + \sqrt{5}y) = x^2 - 5y^2$: not that number theory leads to Minkowkian metric signatures naturally. Prime ideals combine to form orbits of Galois group.

2. Since Galois group leaves the rational prime p invariant, the action must permute the primes of extension in the product representation of p. For ramified primes the points of the orbit of ideal degenerate to single ideal. This means that primes and quite generally, the numbers of extension, define orbits of the Galois group.

Galois group acts in the space of integers or prime ideals of the algebraic extension of rationals and it is also physically attractive to consider the orbits defined by ideals as preferred geometric structures. If the numbers of the extension serve as parameters characterizing string world sheets and partonic 2-surfaces, then the ideals would naturally define subsets of the parameter space in which Galois group would act.

The action of Galois group would leave the space-time surface invariant if the sheets co-incide at ends but permute the sheets. Of course, the space-time sheets permuted by Galois group need not co-incide at ends. In this case the action need not be gauge action and one could have non-trivial representations of the Galois group. In Langlands correspondence these representation relate to the representations of Lie group and something similar might take place in TGD as I have indeed proposed.

The value of effective Planck constant $h_{eff}/h = n$ corresponds to the number of sheets of some kind of covering space defined by the space-time surface. The discretization of the space-time surface identified as a monadic manifold [L34] with embedding space preferred coordinates in extension of rationals defining the adele has Galois group of extension as a group of symmetries permuting the sheets of the covering group. Therefore $n = h_{eff}/h$ would naturally correspond to the dimension of the extension dividing the order of its Galois group. Dark matter in TGD sense would correspond to number theoretic physics.

Remark: Strong form of holography supports also the vision about quaternionic generalization of conformal invariance implying that the adelic space-time surface can be constructed from the data associated with functions of two complex variables, which in turn reduce to functions of single variable.

If this picture is correct, it is possible to talk about quantum amplitudes in the space defined by the numbers of extension and restrict the consideration to prime ideals or more general integer ideals.

1. These number theoretical wave functions are physical if the parameters characterizing the 2-surface belong to this space. One could have purely number theoretical quantal degrees of freedom assignable to the hierarchy of algebraic extensions and these discrete degrees of freedom could be fundamental for living matter and understanding of consciousness.

2. The simplest assumption that Galois group acts as a gauge group when the ends of sheets co-incide at boundaries of CD seems however to destroy hopes about non-trivial number theoretical physics but this need not be the case. Physical intuition suggests that ramification somehow saves the situation and that the non-trivial number theoretic physics could be associated with ramified primes assumed to define preferred p-adic primes.

13.6.2 How New Degrees Of Freedom Emerge For Ramified Primes?

How the new discrete degrees of freedom appear for ramified primes?

1. The space-time surfaces defining singular coverings are n-sheeted in the interior. At the ends of the space-time surface at boundaries of CD however the ends co-incide. This looks very much like a critical phenomenon.

   Hence the idea would be that the end collapse can occur only for the ramified prime ideals of the parameter space - ramification is also a critical phenomenon - and means that some of the sheets or all of them co-incide. Thus the sheets would co-incide at ends only for the preferred p-adic primes and give rise to the singular covering and large h_{eff}. End-collapse would be the essence of criticality! This would occur, when the parameters defining the 2-surfaces are in a ramified prime ideal.

2. Even for the ramified primes there would be n distinct space-time sheets, which are regarded as physically distinct. This would support the view that besides the space-like 3-surfaces at the ends the full 3-surface must include also the light-like portions connecting them so that one obtains a closed 3-surface. The conformal gauge equivalence classes of the light-like portions would give rise to additional degrees of freedom. In space-time interior and for string world sheets they would become visible.

   For ramified primes n distint 3-surfaces would collapse to single one but the n discrete degrees of freedom would be present and particle would obtain them. I have indeed proposed number theoretical second quantization assigning fermionic Clifford algebra to the sheets with n oscillator operators. Note that this option does not require Galois group to act as gauge group in the general case. This number theoretical second quantization might relate to the realization of Boolean algebra suggested by weak form of NMP [K94].

13.6.3 About The Physical Interpretation Of The Parameters Characterizing Algebraic Extension Of Rationals In TGD Framework

It seems that Galois group is naturally associated with the hierarchy $h_{eff}/h = n$ of effective Planck constants defined by the hierarchy of quantum criticalities. n would naturally define the maximal order for the element of Galois group. The analog of singular covering with that of $z^{1/n}$ would suggest that Galois group is very closely related to the conformal symmetries and its action induces permutations of the sheets of the covering of space-time surface.

Without any additional assumptions the values of n and ramified primes are completely independent so that the conjecture that the magnetic flux tube connecting the wormhole contacts associated with elementary particles would not correspond to very large n having the p-adic prime p characterizing particle as factor ($p = M_{127} = 2^{127} - 1$ for electron). This would not induce any catastrophic changes.

TGD based physics could however change the situation and reduce number theoretical degrees of freedom: the intuitive hypothesis that p divides n might hold true after all.

1. The strong form of GCI implies strong form of holography. One implication is that the WCW Kähler metric can be expressed either in terms of Kähler function or as anti-commutators of super-symplectic Noether super-charges defining WCW gamma matrices. This realizes what can be seen as an analog of Ads/CFT correspondence. This duality is much more general. The following argument supports this view.

   (a) Since fermions are localized at string world sheets having ends at partonic 2-surfaces, one expects that also Kähler action can be expressed as an effective stringy action. It is natural to assume that string area action is replaced with the area defined by the effective metric of string world sheet expressible as anti-commutators of Kähler-Dirac gamma matrices defined by contractions of canonical momentum currents with embedding space gamma matrices. It string tension is proportional to h_{eff}^2, string length scales as h_{eff}.

   (b) AdS/CFT analogy inspires the view that strings connecting partonic 2-surfaces serve as correlates for the formation of - at least gravitational - bound states. The distances between string ends would be of the order of Planck length in string models and one can argue that gravitational bound states are not possible in string models and this is the basic reason why one has ended to landscape and multiverse non-sense.

2. In order to obtain reasonable sizes for astrophysical objects (that is sizes larger than Schwartschild radius $r_s = 2GM$) For $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ one obtains reasonable sizes for astrophysical objects. Gravitation would mean quantum coherence in astrophysical length scales.

3. In elementary particle length scales the value of h_{eff} must be such that the geometric size of elementary particle identified as the Minkowski distance between the wormhole contacts defining the length of the magnetic flux tube is of order Compton length - that is p-adic length scale proportional to $\sqrt{p}$. Note that dark physics would be an essential element already at elementary particle level if one accepts this picture also in elementary particle mass scales. This requires more precise specification of what darkness in TGD sense really means.

   One must however distinguish between two options.

   (a) If one assumes $n \simeq \sqrt{p}$, one obtains a large contribution to classical string energy as $\Delta \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/\sqrt{p}$, which is of order particle mass. Dark mass of this size looks un-feasible since p-adic mass calculations assign the mass with the ends wormhole contacts. One must be however very cautious since the interpretations can change.

   (b) Second option allows to understand why the minimal size scale associated with CD characterizing particle correspond to secondary p-adic length scale. The idea is that the string can be thought of as being obtained by a random walk so that the distance between its ends is proportional to the square root of the actual length of the string in the induced metric. This would give that the actual length of string is proportional to p and n is also proportional to p and defines minimal size scale of the CD associated with the particle. The dark contribution to the particle mass would be $\Delta m \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/p$, and completely negligible suggesting that it is not easy to make the dark side of elementary visible.

4. If the latter interpretation is correct, elementary particles would have huge number of hidden degrees of freedom assignable to their CDs. For instance, electron would have $n = 2^{127} - 1 \simeq 10^{38}$ hidden discrete degrees of freedom and would be rather intelligent system - 127 bits is the estimate- and thus far from a point-like idiot of standard physics. Is it a mere accident that the secondary p-adic time scale of electron is .1 seconds - the fundamental biorhythm - and the size scale of the minimal CD is slightly large than the circumference of Earth?

   Note however, that the conservation option assuming that the magnetic flux tubes connecting the wormhole contacts representing elementary particle are in $h_{eff}/h = 1$ phase can be considered as conservative option.

13.7 p-Adicization and adelic physics

This section is devoted to the challenges related to p-adicization and adelization of physics in which the correspondence between real and p-adic numbers via canonical identification serves as the basic building brick. Also the problems associated with p-adic variants of integral, Fourier analysis, Hilbert space, and Riemann geometry should be solved in a way respecting fundamental symmetries and their p-adic variants must be met. The notion of number theoretical universality (NTU) plays a key role here. One should also answer to questions about the origin of preferred primes and p-adic length scale hypothesis.

13.7.1 Challenges

The basic challenges encountered are construction of the p-adic variants of real number based physics, understanding their relationship to real physics, and the fusion of various physics to single coherent whole.

The p-adicization of real physics is not just a straightforward formal generalization of scattering amplitudes of existing theories but requires a deeper understanding of the physics involved. The interpretation of p-adic physics as correlate for cognition and imagination is an important guideline and will be discussed l in more detail in separate section.

Definite integral and Fourier analysis are basic elements of standard physics and their generalization to the p-adic context defines a highly non-trivial challenge. Also the p-adic variants of Riemann geometry and Hilbert space are suggestive. There are however problems.

1. There are problems associated with p-adic definite integral. Riemann sum does not make sense since it approaches zero if the p-adic norm of discretization unit approaches zero. The problems are basically due to the absence of well-orderedness essential for the definition of definite integral and differential forms and their integrals.

   Residue integration might make sense in finite angle resolution. For algebraic extension containing $e^{i\pi/n}$ the number theoretically universal approximation $i\pi = n(e^{i\pi/n} - 1)$ could be used. In twistor approach integrations reduce to multiple residue integrations and since twistor approach generalizes in TGD framework, this approach to integration is very attractive.

   Positivity is a central notion in twistor Grassmannian approach [B30]. Since canonical identification maps p-adic numbers to non-negative real numbers, there is a strong temptation to think that positivity relates to NTU [L27].

2. There are problems with Fourier analysis. The naïve generalization of trigonometric functions by replacing e^{ix} with its p-adic counterpart is not physical. Same applies to e^x. Algebraic extensions are needed to get roots of unity ad e as counterparts of the phases and discretization is necessary and has interpretation in terms of finite resolution for angle/phase and its hyperbolic counterpart.

3. The notion of Hilbert space is problematic. The naïve generalization of Hilbert space norm square $|x|^2 = \sum x_n \overline{x}_n$ for state $(x_1, x_2, ...)$ can vanish p-adically. Also here NTU could help. State would contain as coefficients only roots of e and unity and only the overall factor could be p-adic number. Coefficients could be restricted to the algebraic numbers generating the algebraic extension of rational numbers and would not contain powers of p or even ordinary p-adic numbers expect in the overall normalization factor.

Second challenge relates to the relationship between real and p-adic physics. Canonical identification (CI) $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ or some of its variants should play an important role. CI is expected to map the invariants appearing in scattering amplitudes to their real counterparts.

1. Real and p-adic variants of space-time surfaces should exist and relate to each other somehow. Is this relationship local and involve CI at space-time level or embedding space level? Or is it only a global and non-local assignment of preferred real extremals to their p-adic counterparts? Or is between these extreme options and involves algebraic discretization of the space-time surface weakening the strong form of SH as already proposed? How do real

and p-adic embedding spaces relate to each other and can this relationship induce local correspondence between preferred extremals (PEs) [K5, K10, L33]?

2. NTU in some sense is a highly suggestive approach to these questions and would suggest that canonical identification applies to isometry invariants whereas angles and hyperbolic angles, or rather the corresponding "phases" belonging to an extension of p-adics containing roots of e and roots of unity are mapped to themselves. Note that the roots of e define extensions of rationals, which induce finite dimensional algebraic extensions of p-adic numbers. This would make possible to define embedding space in accordance with NTU. Also the Hilbert space could be defined by requiring that its points correspond to number theoretically universal angles expressible in terms of roots of unity.

3. What about real and p-adic variants of WCW? Are they needed at all? Or could their existence be used as a powerful constraint on real physics? The representability of WCW as a union of infinite-dimensional symmetric spaces labelled by zero modes suggests that the same description applies at the level of WCW and embedding space.

   One cannot circumvent the question about how to generalize functional integral from real WCW to p-adic WCWs. In particular, what is the p-adic variant of the action defining the dynamics of space-time surfaces. In the case of exponent of action the p-adic variant could be defined by assuming algebraic universality: again the roots of e and of unity would be in central role. Also the Kähler structure of WCW implying that Gaussian and metric determinants cancel each other in functional integral, would be absolutely crucial.

   One must remember that the exponents of action for scattering amplitudes for the stationary phase extremal cancel from the path integral representation of scattering amplitudes. Also now this mechanism would allow to get rid of the poorly defined exponent for single minimum. If there is sum over scattering amplitudes assignable to different maxima, normalization sould give ratios of these exponents for different extrema/maxima and only these ratios should belong to the extension of rationals.

   The zero modes of WCW metric are invariants of supersymplectic group so that canonical identification could relate their real and p-adic variants. Zero modes could break NTU and would be behind p-adic thermodynamics and dependence of mass scale on p-adic prime.

The third challenge relates to the fusion of p-adic physics and real physics to a larger structure. Here a generalization of number concept obtained by glueing reals and various p-adics together along an extension of rational numbers inducing the extensions of p-adic numbers is highly suggestive. Adeles associated with the extension of rationals are highly attractive and closely related notion. Real and various p-adic physics would be correlates for sensory and cognitive aspects of the same universal physics rather than separate physics in this framework. One important implication of this view is that real entropy and p-adic negentropies characterize the same entanglement with coefficients in an extension of rationals.

NTU for hyperbolic and ordinary phases is definetely the central idea. How the invariance of angles under conformal transformations does relate to this? Could one perhaps define a discretized version of conformal symmetry preserving the phases defined by the angles between vectors assignable with the tangent spaces of discretized geometric structures and thus respecting NTU? Of should one apply conformal symmetry at Lie algebra level only?

13.7.2 NTU and the correspondence between real and p-adic physics

p-Adic real correspondence is certainly the basic problem of p-adicization and adelization. One can make several general questions about p-adic real correspondence and canonical identification inspired by p-adic mass calculations.

How generally p-adic real correspondence does apply? Could canonical identification for group invariants combined with direct identification of ordinary and hyperbolic phases identified as roots of unity and e apply at WCW and embedding space level having maximally symmetric geometries? Could this make sense even at space-time level as a correspondence induced from embedding space level [L34]? Does canonical identification apply locally for the discretizations of space-time surface or only globally for the parameters characterizing PEs (string world sheets and partonic 2-surfaces by SH), which are general coordinate invariant and Poincare invariant quantities?

The following vision seems to be the most feasible one found hitherto.

1. Preservation of symmetries and continuity compete. Lorenz transformations do not commute with canonical identification. This suggests that canonical identification applies only to Lorentz invariants formed from quantum numbers. This is enough in the case of scattering amplitudes. Canonical identification applies only to isometry invariants at the level of WCW and the phases/exponents of ordinary/hyperbolic angles correspond to numbers in the algebraic extension common to extensions of rationals and various p-adics.

2. Canonical identification applies at the level of momentum space and maps p-adic Lorentz invariants of scattering amplitudes to their real counterparts. Phases of angles and their hyperbolic counterparts should correspond to parameters defining extension and should be mapped as such to their p-adic counterparts.

3. The constraints coming from GCI and symmetries do not allow local correspondence but allow to consider its discretized version at space-time leve induced by the correspondence at the level of embedding space.

   This requires the restriction of isometries and other symmetries to algebraic subgroups defined by the extension of rationals. This would imply reduction of symmetry due to finite cognitive/measurement resolution and should be acceptable. If one wants to realize the ideas about imagination, discretization must be applied also for the space-time interior meaning partial breaking of SH and giving rise to dark matter degrees freedom in TGD sense. SH could apply in real sector for realizable imaginations only. Note that the number of algebraic points of space-time surface is expected to be relatively small.

The correspondence must be considered at the level of embedding space, space-time, and WCW.

1. At the level of embedding space p-adic–real correspondence is induced by points in extension of rationals and is totally discontinuous. This requires that space-time dimension is smaller than embedding space dimension.

2. At space-time level the correspondence involves field equations derivable from a local variational principle make sense also p-adically although the action itself is ill-defined as 4-D integral. The notion of p-adic PE makes sense by strong form of holography applied to 2-surfaces in the intersection. p-Adically however only the vanishing of Noether currents for a sub-algebra of the super-symplectic algebra might make sense. This condition is stronger than the vanishing of Noether charges defined by 3-D integrals.

3. Correspondence at the level of WCW can make sense and reduces to that for string world sheets and partonic 2-surfaces by SH. Real and p-adic 4-surfaces would be obtained by algebraic continuation as PEs from 2-surfaces by assuming that the space-time surface contains subset of points of embedding space belonging to the extension of rationals. p-Adic pseudo constants make p-adic continuation easy. Real continuation need not exist always. p-Adic WCW would be considerably larger than real WCW and make possible a predictive quantum theory of imagination and cognition.

   What I have called intersection of realities and p-adicities can be identified as the set of 2-surfaces plus algebraic discretization of space-time interior. Also the values of induced spinor fields at the points of discretization must be given. The parameters characterizing the extremals (say coefficients of polynomials) - WCW coordinates - would be in extension of rationals inducing a finite-D extension of p-adic number fields.

   The hierarchy of algebraic extensions induces an evolutionary hierarchy of adeles. The interpretation could be as a mathematical correlate for cosmic evolution realized at the level of the core of WCW defined by the intersection? 2-surfaces could be called space-time genes.

4. Also the p-adic variant Kähler action or at least the exponent of Kähler action defining vacuum functional should be obtainable by algebraic continuation. The weakest condition states that the ratios of action exponents for the maxima of Kähler function to the sum of action exponents for maxima belong to the extension. Without this condition the hopes of satisfying NTU seem rather meager.

13.7.3 NTU at space-time level

What about NTU at space-time level? NTU requires a correspondence between real and p-adic numbers and the details of this corresponds have been a long standing problem.

1. The recent view about the correspondence between real PEs to their p-adic counterparts does not demand discrete local correspondence assumed in the earlier proposal [K97]. The most abstract approach would give up the local correspondence at space-time level altogether, and restrict the preferred coordinates of WCW (having maximal group of isometries) to numbers in the extension of rationals considered. WCW would be discretized.

   Intuitively a more realistic view is a correspondence at space-time level in the sense that real and p-adic space-time sheets intersect at points belonging to the extension of rationals and defining "cognitive representations". Only some p-adic space-time surfaces would have real counterpart.

2. The strongest form of NTU would require that the allowed points of embedding space belonging an extension of rationals are mapped as such to corresponding extensions of p-adic number fields (no canonical identification). At embedding space level this correspondence would be extremely discontinuous. The "spines" of space-time surfaces would however contain only a subset of points of extension, and a natural resolution length scale could emerge and prevent the fluctuation. This could be also seen as a reason for why space-times surfaces must be 4-D. The fact that the curve $x^n + y^n = z^n$ has no rational points for $n > 2$, raises the hope that the resolution scale could emerge spontaneously.

3. The notion of monadic geometry discussed in detail in [L34] would realize this idea. Define first a number theoretic discretization of embedding space in terms of points, whose coordinates in group theoretically preferred coordinate system belong to the extension of rationals considered. One can say that these algebraic points are in the intersection of reality and various p-adicities. Overlapping open sets assigned with this discretization define in the real sector a covering by open sets. In p-adic sector compact-open-topology allows to assign with each point 8^{th} Cartesian power of algebraic extension of p-adic numbers. These compact open sets define analogs for the monads of Leibniz and p-adic variants of field equations make sense inside them.

   The monadic manifold structure of H is induced to space-time surfaces containing discrete subset of points in the algebraic discretization with field equations defining a continuation to space-time surface in given number field, and unique only in finite measurement resolution. This approach would resolve the tension between continuity and symmetries in p-adic–real correspondence: isometry groups would be replaced by their sub-groups with parameters in extension of rationals considered and acting in the intersection of reality and p-adicities.

   The Galois group of extension acts non-trivially on the "spines" of space-time surfaces. Hence the number theoretical symmetries act as physical symmetries and define the orbit of given space-time surface as a kind of covering space. The coverings assigned to the hierarchy of Planck constants would naturally correspond to Galois coverings and dark matter would represent number theoretical physics.

   This would give rise to a kind of algebraic hierarchy of adelic 4-surfaces identifiable as evolutionary hierarchy: the higher the dimension of the extension, the higher the evolutionary level.

13.7.4 NTU and WCW

p-Adic–real correspondence at the level of WCW

It has not been obvious whether one should perform p-adicization and adelization at the level of WCW. Minimalist could argue that scattering amplitudes are all we want and that their p-adicization and adelization by algebraic continuation can be tolerated only if it can give powerful enough constraints on the amplitudes.

1. The anti-commutations for fermionic oscillator operators are number theoretically universal. Supersymmetry suggests that also WCW bosonic degrees of freedom satisfy NTU. This could mean that the coordinates of p-adic WCW consist of super-symplectic invariants mappable by canonical identification to their real counterparts plus phases and their hyperbolic counterparts expressible as genuinely algebraic numbers common to all number fields. This kind of coordinates are naturally assignable to symmetric spaces [L34].

2. Kähler structure should be mapped from p-adic to real sector and vice versa. Vacuum functional identified as exponent of action should be NTU. Algebraic continuation defined by SH involves p-adic pseudo constants. All p-adic continuations by SH should correspond to the same value of exponent of action obtained by algebraic continuation from its real value. The degeneracy associated with p-adic pseudo-constants would be analogous to gauge invariance - imagination in TGD inspired theory of consciousness.

3. Ist it possible have NTU for WCW functional integration? Or is it enough to realize NTU for scattering amplitudes only. What seems clear that functional integral must reduce to a discrete sum. Physical intuition suggests a sum over maxima of Kähler function forming a subset of PEs representing stationary points. One cannot even exclude the possibility that the set of PEs is discrete and that one can sum over all of them.

   Restriction to maximum/stationary phase approximation gives rise to sum over exponents multiplied with Gaussian determinants. The determinant of Kähler metric however cancels the Gaussian determinants, and one obtains only a sum over the exponents of action.

   The breaking of strong NTU could happen: consider only p-adic mass calculations. This breaking is however associated with the parts of quantum states assignable to the boundaries of CD, not with the vacuum functional.

NTU for functional integral

Number theoretical vision relies on NTU. In fermionic sector NTU is necessary: one cannot speak about real and p-adic fermions as separate entities and fermionic anti-commutation relations are indeed number theoretically universal.

What about NTU in case of functional integral? There are two opposite views.

1. One can define p-adic variants of field equations without difficulties if preferred extremals are minimal surface extremals of Kähler action so that coupling constants do not appear in the solutions. If the extremal property is determined solely by the analyticity properties as it is for various conjectures, it makes sense independent of number field. Therefore there would be no need to continue the functional integral to p-adic sectors. This in accordance with the philosophy that thought cannot be put in scale. This would be also the option favored by pragmatist.

2. Consciousness theorist might argue that also cognition and imagination allow quantum description. The supersymmetry NTU should apply also to functional integral over WCW (more precisely, its sector defined by CD) involved with the definition of scattering amplitudes.

1. Key observations

The general vision involves some crucial observations.

1. Only the expressions for the scatterings amplitudes should should satisfy NTU. This does not require that the functional integral satisfies NTU.

2. Since the Gaussian and metric determinants cancel in WCW Kähler metric the contributions form maxima are proportional to action exponentials $exp(S_k)$ divided by the $\sum_k exp(S_k)$. Loops vanish by quantum criticality.

3. Scattering amplitudes can be defined as sums over the contributions from the maxima, which would have also stationary phase by the double extremal property made possible by the complex value of α_K. These contributions are normalized by the vacuum amplitude.

   It is enough to require NTU for $X_i = exp(S_i)/\sum_k exp(S_k)$. This requires that $S_k - S_l$ has form $q_1 + q_2 i\pi + q_3 log(n)$. The condition brings in mind homology theory without boundary operation defined by the difference $S_k - S_l$. NTU for both S_k and $exp(S_k)$ would only values of general form $S_k = q_1 + q_2 i\pi + q_3 log(n)$ for S_k and this looks quite too strong a condition.

4. If it is possible to express the 4-D exponentials as single 2-D exponential associated with union of string world sheets, vacuum functional disappears completely from consideration! There is only a sum over discretization with the same effective action and one obtains purely combinatorial expression.

2. What does one mean with functional integral?

The definition of functional integral in WCW is one of the key technical problems of quantum TGD [K94]. NTU states that the integral should be defined simultaneously in all number fields in the intersection of real and p-adic worlds defined by string world sheets and partonic 2-surfaces with WCW coordinates in algebraic extension of rationals and allowing by strong holography continuation to 4-D space-time surface. NTU is powerful constraint and could help in this respect.

1. Path integral is not in question. Rather, the functional integral is analogous to Wiener integral and perhaps allows identification as a genuine integral in the real sector. In p-adic sectors algebraic continuation should give the integral and here number theoretical universality gives excellent hopes. The integral would have exactly the same form in real and p-adic sector and expressible solely in terms of algebraic numbers characterizing algebraic extension and finite roots of e and roots of unity $U_n = exp(i2\pi/n)$ in algebraic extension of p-adic numbers.

   Since vacuum functional $exp(S)$ is exponential of complex action S, the natural idea is that only rational powers e^q and roots of unity and phases $exp(i2\pi q)$ are involved and there is no dependence on p-adic prime p! This is true in the integer part of q is smaller than p so that one does not obtain e^{kp}, which is ordinary p-adic number and would spoil the number theoretic universality. This condition is not possible to satisfy for all values of p unless the value of Kähler function is smaller than 2. One might consider the possibility that the allow primes are above some minimum value.

   The minimal solution to NTU conditions is that the ratios of action exponentials for maxima of Kähler function to the sum of these exponentials belong to the extension of rationals considered.

2. What does one mean with functional integral? TGD is expected to be an integrable in some sense. In integrable QFTs functional integral reduces to a sum over stationary points of the action: typically only single point contributes - at least in good approximation.

   For real α_K and Λ vacuum functional decomposes to a product of exponents of real contribution from Euclidian regions ($\sqrt{g_4}$ real) and imaginary contribution Minkowskian regions ($\sqrt{g_4}$ imaginary). There would be no exchange of momentum between Minkowskian and Euclidian regions. For complex values of α_K [K28] situation changes and Kähler function as real part of action receives contributions from both Euclidian and Minkowskian regions. The imaginary part of action has interpretation as analog of Morse function and action as it appears in QFTs. Now saddle points must be considered.

   PEs satisfy extremely strong conditions [K5, L33]. All classical Noether charges for a sub-algebra associated with super-symplectic algebra and isomorphic to the algebra itself vanish at both ends of CD. The conformal weights of this algebra are $n > 0$-ples of those for the entire algebra. What is fascinating that the condition that the preferred extremals are minimal surface extremals of Kähler action could solve these conditions and guarantee also NTU at the level of space-time surfaces. Supersymplectic boundary conditions at the ends of CD would however pose number theoretic conditions on the coupling parameters. In p-adic case the conditions should reduce to purely local conditions since p-adic charges are not well-defined as integrals.

3. In TGD framework one is constructing zero energy states rather calculating the matrix elements of S-matrix in terms of path integral. This gives certain liberties but a natural expectation is that functional integral as a formal tool at least is involved.

   Could one *define* the functional integral as a discrete sum of contributions of standard form for the preferred extremals, which correspond to maxima in Euclidian regions and associated stationary phase points in Minkowskian regions? Could one assume that WCW spinor field is concentrated along single maximum/stationary point.

   Quantum classical correspondence suggests that in Cartan algebra isometry charges are equal to the quantal charges for quantum states expressible in number theoretically universal manner in terms of fermionic oscillator operators or WCW gamma matrices? Even stronger condition would be that classical correlation functions are identical with quantal ones for allowed space-time surfaces in the quantum superposition. Could the reduction to a discrete sum be interpreted in terms of a finite measurement resolution?

4. In QFT Gaussian determinants produce problems because they are often poorly defined. In the recent case they could also spoil the NTU based on the exceptional properties of e. In the recent case however Gaussian determinant and metric determinant for Kähler metric cancel each other and this problem disappears. One could obtain just a sum over products of roots of e and roots of unity. Thus also Kähler structure seems to be crucial for the dream about NTU.

13.7.5 Breaking of NTU at the level of scattering amplitudes

NTU in strong sense could be broken at the level of scattering amplitudes. At space-time level the breaking does not look natural in the recent framework. Consider only p-adic mass calculations predicting that mass scale depends on p-adic prime. Also for the action strong form of NTU might fail for small p-adic primes since the value of the real part of action would be larger than than p. Should one allow this? What does one actually mean with NTU in the case of action?

Canonical identification is an important element of p-adic mass calculations and might also be needed to map p-adic variants of scattering amplitudes to their real counterparts. The breaking of NTU would take place, when the canonical real valued image of the p-adic scattering amplitude differs from the real scattering amplitude. The interpretation would be in terms of finite measurement resolution. By the finite measurement/cognitive resolution characterized by p one cannot detect the difference.

The simplest form of the canonical identification is $x = \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$. Product xy and sum $x + y$ do not in general map to product and sum in canonical identification. The interpretation would be in terms of a finite measurement resolution: $(xy)_R = x_R y_R$ and $(x+y)_R = x_R + y_R$ only modulo finite measurement resolution. p-Adic scattering amplitudes are obtained by algebraic continuation from the intersection by replacing algebraic number valued parameters (such as momenta) by general p-adic numbers. The real images of these amplitudes under canonical identification are in general not identical with real scattering amplitudes the interpretation being in terms of a finite measurement resolution.

In p-adic thermodynamics NTU in the strong sense fails since thermal masses depend on p-adic mass scale. NTU can be broken by the fermionic matrix elements in the functional integral so that the real scattering amplitudes differ from the canonical images of the p-adic scattering amplitudes. For instance, the elementary particle vacuum functionals in the space of Teichmueller parameters for the partonic 2-surfaces and string world sheets should break NTU [K18].

13.7.6 NTU and the spectrum of Kähler coupling strength

During years I have made several attempts to understand coupling evolution in TGD framework. The most convincing proposal has emerged rather recently and relates the spectrum of $1/\alpha_K$ to that for the zeros of Riemann zeta [K28] and to the evolution of the electroweak U(1) couplings strength.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K39]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.

   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K98] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking.

   The problem is that the dynamics in Minkowskian and Euclidian regions decouple completely and if Euclidian regions serve as space-time correlates for physical objects, there would be no exchanges of classical charges between physical objects. Should one conclude that α_K must be complex?

2. p-Adic mass calculations for 2 decades ago [K46] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity [L55]. Needless to say these attempts were premature and a hoc.

3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [**?**] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

   Quite recently the number theoretic interpretation of coupling constant evolution [K94] [L25] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K90]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets

add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.

   Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.

   The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s = 2$. The trivial poles for $s = 2n$, $n = 1, 2, ..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff} = n \times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s = 2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?

   It turns out that at p-adic length scale $k = 131$ ($p \simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K94]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k = 127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w = w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `http://tinyurl.com/gwjs85b`) with real coefficients (element of $GL(2, R)$) so that one as $\zeta_F((as + b)/(cs + d))$. Rather general arguments force it to be and element of $GL(2, Q)$, $GL(2, Z)$ or maybe even $SL(2, Z)$ ($ad - bc = 1$) satisfying additional constraints. Since TGD predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2, Z)$ and by a scaling factor K.

   Could one understand the general qualitative features of color and weak coupling contant evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs + d$ and color confinement with the zero of $as + b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as + b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as+b/)(cs+d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p \simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a, b, c, d). In the sequel this vision is discussed in more detail.

13.7.7 Generalization of Riemann zeta to Dedekind zeta and adelic physics

13.7.8 Generalization of Riemann zeta to Dedekind zeta and adelic physics

A further insight to adelic physics comes from the possible physical interpretation of the L-functions appearing also in Langlands program [K43]. The most important L-function would be generalization of Riemann zeta to extension of rationals. I have proposed several roles for ζ, which would be the simplest L-function assignable to rational primes, and for its zeros.

1. Riemann zeta itself could be identifiable as an analog of partition function for a system with energies given by logarithms of prime. One can define also the fermionic counterpart of ζ as ζ_F. In ZEO this function could be regarded as complex square root of thermodynamical partition function in accordance with the interpretation of quantum theory as complex square root of thermodynamics.

2. The zeros of zeta could define the conformal weights for the generators of super-symplectic algebra so that the number of generators would be infinite. The rough idea - certainly not correct as such except at the limit of infinitely large CD - is that the scaling operator $L_0 = r_M d/dr_M$, where r_M is light-like coordinate of light-cone boundary (containing upper or lower boundary of the causal diamond (CD)), has as eigenfunctions the functions $(r_M/r_0)^{s_n}$ $s_n = 1/2 + iy_n$, where s_n is the radial conformal weight identified as complex zero of ζ. Periodic boundary conditions for CD do not allow all possible zeros as conformal weights so that for given CD only finite subset corresponds to generators of the supersymplectic algebra. Conformal confinement would hold true in the sense that the sum $\sum_n s_n$ for physical states would be integer. Roots and their conjugates should appear as pairs in physical states.

3. On basis of numerical evidence Dyson [A76] (`http://tinyurl.com/hjbfsuv`) has conjectured that the Fourier transform for the set formed by zeros of zeta consists of primes so that one could regard zeros as one-dimensional quasi-crystal. This hypothesis makes sense if the zeros of zeta decompose into disjoint sets such that each set corresponds to its own prime (and its powers) and one has $p^{iy} = U_{m/n} = exp(i2\pi m/n)$ (see the appendix of [L25]). This hypothesis is also motivated by number theoretical universality [K94, L47].

4. I have considered the possibility [K28] that the discrete values for the inverse of the electroweak $U(1)$ coupling constant for a gauge field assignable to the Kähler form of CP_2 assignable to p-adic coupling constant evolution corresponds to poles of the fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ coming from $s_n/2$ (denominator) and pole at $s = 1$ (numerator) zeros of zeta assignable to rational primes. Note that also odd negative integers at real axis would be poles.

   It is also possible to consider scaling of the argument of $\zeta_F(s)$. More general coupling constant evolutions could correspond to $\zeta_F(m(s))$, where $m(s) = (as+b)/(cs+d)$ is Möbius transformation performed for the argument mapping upper complex plane to itself so that a, b, c, d are real and also rational by number theoretical universality.

Suppose for a moment that more precise formulations of these physics inspired conjectures make sense and even that their generalization for extensions K/Q of rationals holds true. This would solve a big part of adelic physics! Not surprisingly, the generalization of zeta function was proposed already by Dedekind (see `http://tinyurl.com/yarwbo6h`).

1. The definition of Dedekind zeta function ζ_K relies on the product representation and analytic continuation allows to deduce ζ_K elsewhere. One has a product over prime ideals of K/Q of rationals with the factors $1/(1-p^{-s})$ associated with the ordinary primes in Riemann zeta replaced with the factors $X(P) = 1/(1 - N_{K/Q}(P)^{-s})$, where P is prime for the integers $O(K)$ of extension and $N_{K/Q}(P)$ is the norm of P in the extension. In the region $s > 1$ where the product converges, ζ_K is non-vanishing and $s = 1$ is a pole of ζ_K. The functional identifies of ζ hold true for ζ_K as well. Riemann hypothesis is generalized for ζ_K.

2. It is possible to understand ζ_K in terms of a physical picture. By the results of `http://tinyurl.com/yckfjgpk` one has $N_{K/Q}(P) = p^r$, $r > 0$ integer. This implies that one can arrange in ζ_K all primes P for which the norm is power or given p in the same group. The prime ideals p of ordinary integers decompose to products of prime ideals P of the extension: one has $p = \prod_{r=1}^{g} P_r^{e_r}$, where e_r is so called ramification index. One can say that each factor of ζ decomposes to a product of factors associated with corresponding primes P with norm power of p. In the language of physics, the particle state represented by p decomposes in improved resolution to a product of many-particle states consisting of e_r particles in state P_r, very much like hadron decomposes to quarks.

   The norms of $N_{K/Q}(P_r) = p^{d_r}$ satisfy the condition $\sum_{r=1}^{g} d_r e_r = n$. Mathematician would say that the prime ideals of Q modulo p decompose in n-dimensional extension K to products of prime power ideals $P_r^{e_r}$ and that P_r corresponds to a finite field $G(p, d_r)$ with algebraic dimension d_r. The formula $\sum_{r=1}^{g} d_r e_r = n$ reflects the fact the dimension n of extension is same independent of p even when one has $g < n$ and ramification occurs.

   Physicist would say that the number of degrees of freedom is n and is preserved although one has only $g < n$ different particle types with e_r particles having d_r internal degrees of freedom. The factor replacing $1/(1-p^{-s})$ for the general prime p is given by $\prod_{r=1}^{g} 1/(1 - p^{-e_r d_r s})$.

3. There are only finite number of ramified primes p having $e_r > 1$ for some r and they correspond to primes dividing the so called discriminant D of the irreducible polynomial P defining the extension. $D\ mod\ p$ obviously vanishes if D is divisible by p. For second order polynomials $P = x^2 + bx + c$ equals to the familiar $D = b^2 - 4c$ and in this case the two roots indeed co-incide. For quadratic extensions with $D = b^2 - 4c > 0$ the ramified primes divide D.

   Remark: Resultant $R(P, Q)$ and discriminant $D(P) = R(P, dP/dx)$ are elegant tools used by number theorists to study extensions of rationals defined by irreducible polynomials (see `http://tinyurl.com/oyumsnk` and `http://tinyurl.com/p67rdgb`). From Wikipedia articles one finds an elegant proof for the facts that $R(P, Q)$ is proportional to the product of differences of the roots of P and Q, and D to the product of squares for the differences of distinct roots. $R(P, Q) = 0$ tells that two polynomials have a common root. $D\ mod\ p = 0$ tells that polynomial and its derivative have a common root so that there is a degenerate root modulo p and the prime is indeed ramified. For modulo p reduction of P the vanishing of $D(P)\ mod\ p$ follows if D is divisible by p. There exists clearly only a finite number of primes of this kind.

   Most primes are unramified and one has maximum number n of factors in the decomposition and $e_r = 1$: maximum splitting of p occurs. The factor $1/(1-p^{-s})$ is replaced with its n:th power $1/(1-p^{-s})^n$. The geometric interpretation is that space-time sheet is replaced with n-fold covering and each sheet gives one factor in the power. It is also possible to have a situation in which no splitting occurs and one as $e_r = 1$ for one prime $P_r = p$. The factor is in this case equal to $1/(1-p^{-ns})$.

From Wikipedia (see `http://tinyurl.com/yckfjgpk`) one learns that for Galois extensions L/K the ratio ζ_L/ζ_K is so called Artin L-function of the regular representation (group algebra) of Galois group factorizing in terms of irreps of $Gal(L/K)$ is *holomorphic* (no poles!) so that ζ_L must have also the zeros of ζ_K. This holds in the special case $K = Q$. Therefore extension of rationals can only bring new zeros but no new poles!

1. This result is quite far reaching if one accepts the hypothesis about super-symplectic conformal weights as zeros of ζ_K and the conjecture about coupling constant evolution. In the case

of $\zeta_{F,K}$ this means new poles meaning new conformal weights due to increased complexity and a modification of the conjecture for the coupling constant evolution due to new primes in extension. The outcome looks physically sensible.

2. Quadratic field $Q(\sqrt{m})$ serves as example. Quite generally, the factorization of rational primes to the primes of extension corresponds to the factorization of the minimal polynomial for the generating element θ for the integers of extension and one has $p = P_i^{e_i}$, where e_i is ramification index. The norm of p factorizes to the produce of norms of $P_i^{e_i}$.

   Rational prime can either remain prime in which case $x^2 - m$ does not factorize mod p, split when $x^2 - m$ factorizes mod P, or ramify when it divides the discriminant of $x^2 - m = 4m$. From this it is clear that for unramfied primes the factors in ζ are replaced by either $1/(1-p^{-s})^2$ or $1/(1-p^{-2s}) = 1/(1-p^{-s})(1+p^{-s})$. For a finite number of unramified primes one can have something different.

   For Gaussian primes with $m = -1$ one has $e_r = 1$ for $p\ mod\ 4 = 3$ and $e_r = 2$ for $p =\ mod\ 4 = 1$. z_K therefore decomposes into two factors corresponding to primes $p\ mod\ 4 = 3$ and $p\ mod\ 4 = 3$. One can extract out Riemann zeta and the remaining factor

$$\prod_{p\ mod\ 4=3} \frac{1}{(1-p^{-s})} \times \prod_{p\ mod\ 4=1} \frac{1}{(1+p^{-s})}$$

   should be holomorphic and without poles but having possibly additional zeros at critical line. That ζ_K should possess also the poles of ζ as poles looks therefore highly non-trivial.

13.7.9 Other applications of NTU

NTU in the strongest form says that all numbers involved at "basic level" (whatever this means!) of adelic TGD are products of roots of unity and of power of a root of e. This is extremely powerful physics inspired conjecture with a wide range of possible mathematical applications.

1. For instance, vacuum functional defined as an exponent of action for preferred externals would be number of this kind. One could define functional integral as adelic operation in all number fields: essentially as sum of exponents of action for stationary preferred extremals since Gaussian and metric determinants potentially spoiling NTU would cancel each other leaving only the exponent.

2. The implications of NTU for the zeros of Riemann zeta [L25] will be discussed in more detail in the Appendix. Suffice it to say that the observations about Fourier transform for the distribution of loci of non-trivial zeros of zeta together with NTU leads to explicit proposal for the algebraic for of zeros of zeta. The testable proposal is that zeros decompose to disjoint classes $C(p)$ labelled by primes p and the condition that p^{iy} is root of unity in given class $C(p)$.

3. NTU generalises to all Lie groups. Exponents $exp(in_iJ_i/n)$ of lie-algebra generators define generalisations of number theoretically universal group elements and generate a discrete subgroup of compact Lie group. Also hyperbolic "phases" based on the roots $e^{m/n}$ are possible and make possible discretized NTU versions of all Lie-groups expected to play a key role in adelization of TGD.

   NTU generalises also to quaternions and octonions and allows to define them as number theoretically universal entities. Note that ordinary p-adic variants of quaternions and octonions do not give rise to a number field: inverse of quaternion can have vanishing p-adic variant of norm squared satisfying $\sum_n x_n^2 = 0$.

   NTU allows to define also the notion of Hilbert space as an adelic notion. The exponents of angles characterising unit vector of Hilbert space would correspond to roots of unity.

13.7.10 Going to the roots of p-adicity

The basic questions raised by the p-adic mass calculations concern the origin of preferred p-adic primes and of p-adic length scale hypothesis. One can also ask whether there might be a natural origin for p-adicity at the level of WCW.

Preferred primes as ramified primes for extensions of rationals?

Preferred primes as ramified primes for extensions of rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and to me the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of *ramification of primes* (`http://tinyurl.com/hddljlf`) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Ramification is completely analogous to the degeneracy of some roots of polynomial and corresponds to criticality if the polynomial corresponds to criticality (catastrophe theory of Thom is one application). Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this sloppy language of physicist): as one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called *integral closure* (`http://tinyurl.com/js6fpvr`) over integers of K decompose to products of prime ideals of L (prime ideal is a more rigorous manner to express primeness). Note that the general ideal is analog of integer.

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

   Prime ideals of K can be decomposed to products of prime ideals of L: $P = \prod P_i^{e_i}$, where e_i is the ramification index. If $e_i > 1$ is true for some i, *ramification* occurs. P_i:s in question are like co-inciding roots of polynomial, which for in thermodynamics and Thom's catastrophe theory corresponds to criticality. Ramification could therefore be a natural aspect of quantum criticality and ramified primes P are good candidates for preferred primes for a given extension of rationals. Note that the ramification make sense also for extensions of given extension of rationals.

2. A physical analogy for the decomposition of ideals to ideals of extension is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, the physical analog would be the number of elementary particles of type i in the state (`http://tinyurl.com/h9528pl`). Unramified prime P would be analogous a state with e fermions. Maximally ramified prime would be analogous to Bose-Einstein condensate of e bosons. General ramified prime would be analogous to an e-particle state containing both fermions and condensed bosons. This is of course just a formal analogy.

3. There are two further basic notions related to ramification and characterizing it. *Relative discriminant* is the ideal divided by all ramified ideals in K (integer of K having no ramified prime factors) and relative different for P is the ideal of L divided by all ramified P_i:s (product of prime factors of P in L). These ideals represent the analogs of product of preferred primes P of K and primes P_i of L dividing them. These two integers ideals would characterize the ramification.

In TGD framework the extensions of rationals (`http://tinyurl.com/h9528pl`) and p-adic number fields (`http://tinyurl.com/zq22tvb`) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would gradually proceed to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

How could ramification relate to p-adic and adelic physics and could it explain preferred primes?

1. Ramified p-adic prime $P = P_i^e$ would be replaced with its e:th root P_i in p-adicization. Same would apply to general ramified primes. Each un-ramified prime of K is replaced with $e = K : L$ primes of L and ramified primes P with $\#\{P_i\} < e$ primes of L: the increase of algebraic dimension is smaller. An interesting question relates to p-adic length scale. What happens to p-adic length scales. Is p-adic prime effectively replaced with e:th root of p-adic prime: $L_p \propto p^{1/2}L_1 \rightarrow p^{1/2e}L_1$? The only physical option is that the p-adic temperature for P would be scaled down $T_p = 1/n \rightarrow 1/ne$ for its e:th root (for fermions serving as fundamental particles in TGD one actually has $T_p = 1$). Could the lower temperature state be more stable and select the preferred primes as maximimally ramified ones? What about general ramified primes?

2. This need not be the whole story. Some algebraic extensions would be more favored than others and p-adic view about realizable imaginations could be involved. p-Adic pseudo constants are expected to allow p-adic continuations of string world sheets and partonic 2-surfaces to 4-D preferred extremals with number theoretic discretization. For real continuations the situation is more difficult. For preferred extensions - and therefore for corresponding ramified primes - the number of real continuations - realizable imaginations - would be especially large.

   The challenge would be to understand why primes near powers of 2 and possibly also of other small primes would be favored. Why for them the number of realizable imaginations would be especially large so that they would be winners in number theoretical fight for survival?

Can one make this picture more concrete? What kind of algebraic extensions could be considered?

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. It would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (`http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one

or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n:th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified.

A good illustration is provided by equations $x^2+1=0$ allowing roots $x_\pm = \pm i$ and equation $x^2+2px+p=0$ allowing roots $x_\pm = -p\pm\sqrt{p}p-1$. In the first case the ideals associated with $\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

4. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \rightarrow x+1$ transforms $(x^n-1)/(x-1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

The origin of p-adic length scale hypothesis?

p-Adic length scale hypothesis emerged from p-adic length scale hypothesis. A possible generalization of this hypothesis is that p-adic primes near powers of prime are physically favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [?] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present: this is discussed in TGD inspired theory of music harmony and genetic code [K70]. See also [L36, L31].

One explanation would be that for preferred primes the number of p-adic space-time sheets representable also as real space-time sheets is maximal. Imagined worlds would be maximally realizable. Preferred p-adic primes would correspond to ramified primes for extensions with the property that the number of realizable imaginations is especially large for them. Why primes satisfying p-adic length scale hypothesis or its generalization would appear as ramified primes for extensions, which are winners in number theoretical evolution?

Also the weak form of NMP (WNMP) applying also to the purely number theoretic form of NMP [K49] might come in rescue here.

1. Entanglement negentropy for a NE [K49] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretical entanglement negentropy (NEN). If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured, which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real PE as p-adic PE (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. WNMP [K49, K91] suggests a more feasible explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. WNMP allows "free will" so that all dimensions $n - k$, $k = 0, 1, ...n - 1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n - k$ is power of prime. For $n = 2^k = M_k + 1$, where M_k is Mersenne prime $n - 1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n = 2^k - r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p = 2$ can be replaced by any prime.

13.8 What could be the role of complexity theory in TGD?

I have many times wondered what could be the role of chaos theory or better in TGD. In fact, I would prefer to talk about complexity theory since the chaos in the sense as it is used is only apparent and very different from thermodynamical chaos.

Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a nice summary about the history of chaos theory and I repeat only some main points here. Chaos theory has roots already at the end of 18the century by the works of Poincare (non-periodic orbits in 3-body system) and Hadamard (free particle gliding frictionlessly on surface of constant negative curvature, "Hadamard billiard". In this case all trajectories are unstable diverging exponentially from each other: this is characterized by positive Lyapunov exponent.

Chaos theory got is start from ergodic theory (see `http://tinyurl.com/pfcrz4c`) studying dynamical systems with the original motivation coming from statistical physics. For instance, spin glasses are a representative example of non-ergodic system in which the trajectory of point does not go arbitrary near to every point. The study of non-linear differential equations George David Birkhoff, Andrey Nikolaevich Kolmogorov, Mary Lucy Cartwright and John Edensor Littlewood, and Stephen Smale provides was purely mathematical study of chaotic systems. Smale discovered strange attractor at which periodic orbits form a dense set. Chaos theory was formalized around 1950. At this time it was also discovered that finite-D linear systems do not allow chaos.

The emergence of computers meant breakthrough. Much of chaos theory involves repeated iteration of simple mathematical formulas. Edward Lorentz was a pioneer of chaos theory working with weather prediction and accidentally discovered initial value sensitivity. Benard Mandelbrot discovered fractality and Mitchell Feigenbaum the universality of chaos for iteration of functions of real variable.

Chaotic systems are as far from integrable systems as one could imagine: all orbits are cycles in integrable Hamiltonian dynamics. There are good reasons to suspect that TGD Universe is completely integrable classically. Chaos theory however describes also the emergence of complexity through phase transition like steps - period n-tupling and most importantly by period doubling for iteration of maps.

Chaotic (or actually extremely complex and only apparently chaotic) systems seem to be the diametrical opposite of completely integrable systems about which TGD is a possible example. There is however also something common: in completely integrable classical systems all orbits are cyclic and in chaotic systems they form a dense set in the space of orbits. Furthermore, in chaotic systems the approach to chaos occurs via steps as a control parameter is changed. Same would take place in adelic TGD fusing the descriptions of matter and cognition.

In TGD Universe the hierarchy of extensions of rationals inducing finite-dimensional extension of p-adic number fields defines a hierarchy of adelic physics and provides a natural correlate for evolution. Galois groups and ramified primes appear as characterizers of the extensions. The sequences of Galois groups could characterize an evolution by phase transitions increasing the dimension of the extension associated with the coordinates of "world of classical worlds" (WCW) in turn inducing the extension used at space-time and Hilbert space level. WCW decomposes to sectors characterized by Galois groups G_3 of extensions associated with the 3-surfaces at the ends of space-time surface at boundaries of causal diamond (CD) and G_4 characterizing the space-time surface itself. G_3 (G_4) acts on the discretization and induces a covering structure of the 3-surface (space-time surface). If the state function reduction to the opposite boundary of CD involves localization into a sector with fixed G_3, evolution is indeed mapped to a sequence of G_3s.

Also the cognitive representation defined by the intersection of real and p-adic surfaces with coordinates of points in an extension of rationals evolve. The number of points in this representation becomes increasingly complex during evolution. Fermions at partonic 2-surfaces connected by fermionic strings define a tensor network, which also evolves since the number of fermions can change.

The points of space-time surface invariant under non-trivial subgroup of Galois group define singularities of the covering, and the positions of fermions at partonic surfaces could correspond to these singularities - maybe even the maximal ones, in which case the singular points would be rational. There is a temptation to interpret the p-adic prime characterizing elementary particle as a ramified prime of extension having a decomposition similar to that of singularity so that category theoretic view suggests itself.

One also ends up to ask how the number theoretic evolution could select preferred p-adic primes satisfying the p-adic length scale hypothesis as a survivors in number theoretic evolution, and ends up to a vision bringing strongly in mind the notion of conserved genes as analogy for conservation of ramified primes in extensions of extension. $h_{eff}/h = n$ has natural interpretation as divisor of the order of Galois group of extension. The generalization of $\hbar_{gr} = GMm/v_0 = \hbar_{eff}$ hypothesis to other interactions is discussed in terms of number theoretic evolution as increase of G_3, and one ends up to surprisingly concrete vision for what might happen in the transition from prokaryotes to eukaryotes.

13.8.1 Basic notions of chaos theory

It is good to begin by summarizing the basic concepts of chaos theory. Again Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a more detailed representation and references. Citing Wikipedia freely: Within the apparent randomness of chaotic complex systems there are patterns, constant feedback loops, repetition, self-similarity, fractals, self-organization and there is sensitivity to initial conditions (butterfly effect) implying the loss of predictability although chaotic systems as such are deterministic.

Basic prerequisites for chaotic dynamics

Wikipedia article lists three basic conditions for chaotic dynamics. Dynamics must **a)** be sensitive to initial conditions, **b)** allow topological mixing, **c)** have dense set of periodic orbits.

1. Sensitivity to initial conditions.

   Mathematical formulation for the sensitivity to initial conditions can be formulated by perturbation theory for differential equations. The rate of separation of images of points initially near to each other increases exponentially as $exp(\lambda t)$ in initial value sensitive situation and the approximation fails soon. Lyapunov exponent λ characterizes the time evolution of the difference. In multi-dimensional case there are several Lyapunov exponents but the largest one is often enough to characterize the situation.

2. Topological mixing (transitivity).

This notion corresponds to everyday intuition about mixing. For instance, the flow defined by a vector field mixes the marker completely with the fluid. Iteration of simple scaling operation is initial value sensitive but does not cause topological mixing. In 1-D case all points larger than one approach to infinity and smaller than 1 to zero so that the behavior is extremely simple.

3. Dense set of periodic orbits.

   Periodic orbits should form a dense set in the space of orbits: every point of space is approached arbitrarily closely by a periodic orbit. In completely integrable system all orbits would be periodic orbits so that the difference of these systems is very delicate and one can wonder whether the conditions a) and b) follow from this delicate difference. One can also ask whether there might be a deep connection between completely integrable and chaotic systems.

   Sharkovkii's theorem states that any 1-D system with dynamics determined by iteration of a continuous function of real argument exhibits a regular cycle of period 3 exhibits all other cycles. This theorem can be generalized further (see `http://tinyurl.com/l7q3rah`). Introduce Sharkovskii ordering of integers as union of sets consisting of odd integers multiplied by powers of 2. The generalization of the theorem states that if n is a peri89od and precedes k in Sharkovskii ordering then k is prime period (it is not a multiple of smaller period).

   The theorem holds true for reals but not for periodic functions at circle which are encountered for iterations defined by powers of cyclic group elements. The discrete subgroup of hyperbolic subgroups of Lie groups do not have not cycles at all.

Strange attractors and Julia sets

Logistic map $x \rightarrow kx(1-x)$ is chaotic everywhere but many systems are chaotic only in a subset of phase space. An interesting situation arises when the chaotic behavior takes place at attractor, since all initial positions in the basic of the attractor lead to the attractor and to a chaotic behavior. Lorentz attractor is a well-known example of strange attractor (see Wikipedia article for illustration). It contains dense sets of both periodic and aperiodic orbits.

Julia set (see `http://tinyurl.com/l8jl5ne`) is the boundary of the basin of attraction in chaotic systems defined by iteration of a rational function of complex argument mapping complex plane to itself. Both Julia sets and strange attractors have a fractal structure.

Strange attractors can appear only in spaces with dimension $D \geq 3$. Poincare-Bendixon theorem states that 2-D differential equations on Euclidian plane have very regular behavior. In non-Euclidian geometry situation changes and the hyperbolic character of the geometry implying initial value sensitivity of geodesic motion is the reason for this. Also infinite-D linear systems can exhibit chaotic behavior.

13.8.2 How to assign chaos/complexity theory with TGD?

Completely integrable systems can be seen as a diametric opposite of chaotic systems. If classical TGD indeed represents a completely integrable system meaning that space-time surfaces as preferred extremals can be constructed explicitly, one might think that chaos theory need not have much to do with classical TGD. Chaos is however the end product of transitions making the system more complex, and it might well be that the understanding about the emergence of complexity in chaotic systems could help to develop the vision about emergence of complexity in TGD. Note also that periodic orbit are dense in chaotic systems so that diametrical opposites might actually meet.

The most relevant TGD based ingredients used in the sequel are following: WCW [K73]; strong form of holography (SH) [K96], quantum classical correspondence (QCC), zero energy ontology (ZEO) [K53], dark matter as hierarchy of phases with effective Planck constant $h_{eff}/h = n$ [K29, ?, K63], p-Adic physics as physics of cognition [K56, K49, K67] [L51], adelic physics [L51] fusing the physics of matter and cognition by integrating reals and extensions of various p-adic number fields induced by an extension of rationals to a larger structure, and the notions of adelic manifold and associated cognitive representation [L34] , Negentropy Maximization Principle (NMP) [K49] satisfied automatically in statistical sense in adelic physics [L51].

Complexity in TGD

Complexity is often taken to mean computational complexity for classical computations. Complexity as it is understood in the sequel relates very closely cognition. Too complex looks chaotic since our cognitive abilities do not allow to discern too complex patterns. Hence complexity theory should characterize cognitive representations whatever they are.

Number theoretic vision about TGD serves as the guideline here.

1. In adelic TGD [K44] cognitive representations correspond to the intersections of real space-time surfaces and their p-adic variants obeying same field equations and representing correlates for cognition. In these intersections the coordinates of points belong to an extension of rationals defining adele [L34].

   One ends up with a generalization of the notion of manifold to adelic manifold. Intersection defines a common discrete spine consisting of points with coordinates in the extension of rationals defining the adele. These poins are shared by the real and p-adic variants of the adelic manifold. I have called this manifold also monadic manifold since there is strong resemblance with the ideas of Leibniz. In real sector this manifold differs from ordinary manifold in that the open sets are labelled by a discrete set of points in the intersection.

   In TGD framework it is essential that the spine of the space-time surface consists of points of embedding space for which it is convenient to use preferred coordinates.

2. Complexity corresponds roughly to the dimension of extension of rationals defining the adeles. p-Adic differential equations are non-deterministic due to the existence of p-adic pseudo constants depending on finite number of p-adic digits of the p-adic number. This non-determinism is identified as a correlate for imagination. p-Adic variants of space-time surfaces are not uniquely determined this means finite cognitive resolution.

   By SH [K93] the data associated with string world sheets, partonic 2-surfaces, and discretization allow to construct space-time surfaces as preferred extremals of the action principle defining classical TGD and to find the Kähler function for WCW geometry. It is quite well possible that the data allowing to construct p-adic space-time surfaces does not allow continuation to a preferred extremal: all imaginations are not realizable!

   The algebraic dimension of the extension could be relevant for the ability of mathematical cognition to imagine spaces with dimension higher than that for the real 3-space. Besides the extensions of p-adics induced by algebraic extensions of rationals also those induced by some root of e are algebraically finite-dimensional. One can imagine also other extensions involving transcendentals in real sense but it is not clear whether there are finite dimensional extensions among them. The finiteness of cognition suggests that only these extensions can be allowed. All imaginations are not realizable!

3. Extension is characterized partially by Galois group (see `http://tinyurl.com/mrvqhz2`) acting as automorphisms meaning that Galois group permutes the roots of the n:th order polynomials defining extensions of rationals via their non-rational roots. So called ramified primes (see `http://tinyurl.com/m32nvcz` and `http://tinyurl.com/oh7tgsw`) provide additional characteristics.

   Iteration cycles appearing in complexity theory for iteration of functions and repeated action of an element Galois group defining a finite Abelian group are mathematically similar notions. Now only cycles are present whereas chaotic systems have aperiodic orbits. The cyclic subgroups of Galois group do not seem to have an natural realization as iterative dynamics except in quantum sense meaning that cyclic orbits are replaced with wave functions labelled by number theoretic integer valued "momenta" for the action of the analog of Cartan subgroup as maximal commutative subgroup for the Galois group. The maximal Abelian Galois group is analog of Cartan subgroup for Galois group of algebraic numbers and states are in its irreducible representations.

   Remark: What is interesting that for polynomials with order larger than 4, one cannot write closed analytic expressions for the roots of the polynomials. This obviously means a fundamental limitation on symbolic cognitive representations provided by explicit formulas.

The realization of was a huge step in the evolution of mathematics. Could also the emergence of Galois groups with order larger at space-time level than 5 have meant cognitive revolution - probably at much lower level in the hierarchy? Could this relate also to the fact that space-time dimension is $D = 4$ and thus imaginable using 4-D algebraic extension of rationals?

A possible measure for the cognitive complexity is the dimension of the Galois group of the extension. One can speak also about the complexity of the Galois group itself - the non-Abelianity of Galois group brings in additional complexity. The number of generators and number of relations between them serve as a measure for complexity of Galois group.

Extension of rationals is also characterized by so called ramified primes and should have a profound physical meaning. p-Adic length scale hypothesis states that physically preferred primes are near powers of 2 and perhaps also other small primes. Could they correspond to ramified primes. Why just these ramified primes would be survivors in the number theoretic evolution, is the fascinating question to be addressed later.

4. The increase of the dimension of extension or complexity of its Galois group corresponds naturally to evolution interpreted as emergence of algebraic complexity and evolutionary paths could be seen as sequences of inclusions for Galois groups. Chaos would correspond to the limit when the extension of rationals approaches to infinite sub-field of algebraic numbers - say maximal Abelian extension of rationals - so that the number of points in the cognitive representation becomes infinite.

   The Galois group of algebraic numbers - the magic Absolute Group - would characterize this limit as a kind of never achievable mathematical enlightenment. A more practical definition would be that external system is experienced as complex, when its number theoretical complexity exceeds that of the conscious observer so that it is impossible to form a faithful cognitive representation about the system. Note that these cognitive representations could be formulated as homomorphisms between Galois groups. This would suggest a rather nice category theoretical picture about cognitive representations in the self hierarchy.

5. Galois group acts on the cognitive representation associated with the space-time sheet and in general gives n-fold covering of the space-time sheet: n is naturally the dimension of the extension and thus a divisor of the order of Galois group since Galois group acts on the discretization and implies n-sheeted structure for it and therefore also for the space-time surface.

   The value of the effective Planck constant assigned with dark matter as phases of ordinary matter $h_{eff}/h = n$ was identified from very beginning as number of sheets for some kind of covering space of embedding space. n would correspond to a divisor for the order of Galois group for discretized embedding space consisting of points with coordinates in extension of rational. The increase of h_{eff} corresponds to the emergence of also cognitive complexity. Physically it is accompanied by the emergence of quantum coherence and non-locality in increasingly long scales.

General vision about evolution as emergence of complexity

Evolution would mean emergence of number theoretical complexity. Evolutionary paths would naturally correspond to sequences of inclusions (note that recent view allows also temporary "de-evolutions" but in statistical sense evolution occurs). There are infinitely many evolutionary pathways of this kind.

There is a strong resemblance with the inclusion sequences of hyper-finite factors of type II_1 (HHFs) for von Neumann algebras [K95] also playing a central role in TGD and assignable to a fractal hierarchy of isomorphic sub-algebras of super-symplectic algebra associated with the isometries of WCW and related Kac-Moody algebras. It is difficult to believe that this could be an accident.

Evolution must mean a discrete time evolution of some kind - most naturally by non-deterministic quantum version of discrete dynamics, which can be deterministic only in statistical sense. By QCC this evolution should have classical correlates at space-time level. ZEO and TGD inspired theory of consciousness, which can be regarded as a generalization of quantum measurement theory in ZEO, is essential in attempts to concretize this intuition.

1. Galois group codes for the complexity and evolution means the emergence of increasingly complex Galois groups assignable to spacetime surface in a sector of WCW for which WCW coordinates are in corresponding extension of rationals. One can say that evolution defines a path in the space of sectors of WCW characterized by Galois groups. Although the space-time dynamics is expected to be integrable, the notion of complexity still has meaning, and ultimate chaos would emerge at the limit of algebraic numbers as extension of rationals.

2. One can assign Galois group G_5 to space-time surface. Suppose that one an assign Galois groups $G_3 \subset G_4$ with the 3-surfaces at the ends of space-time surfaces at boundaries of CD. This point will be discused below in more detail.

3. At quantum level conscious entities - selves - correspond to sequences off steps consisting of unitary evolution followed by a localization in the moduli space of CD. State function reduction to the opposite boundary of CD means death of self and re-incarnation of self with opposite arrow of time: also this means localization to a definite sector of WCW [L51, L49]. The sequence of pairs of selves and their time reversals associated with the opposite boundaries of CD (, which itself increases in size) defines a candidate for the non-deterministic quantum analog of iteration in complexity theory.

4. There is a temptation to assume that for the passive boundary of CD all 3-surfaces in quantum superposition have same G_3 - the G_3 that emerged in the first state function reduction to the passive boundary when this self was born. G_3 so would be automatically measured observable and sequence of reductions would define a sequence of G_3s analogous to iteration sequence and also to evolution.

   But can one assume that G_3 is measured automatically in the re-incarnation of self as its time-reversal [K4, K44]? Could only some charateristics of G_3 - say order $n = h_{eff}/h$ - be measured? Also ramified primes characteristize extensions and their measurement is also possible and proposed to characterize elementary particles: they do not fix G_3. These uncertainties are not relevant for the general vision.

5. For the active boundary one would have a superposition of 3-surfaces with different Galois groups and the sequence of the steps consisting of unitary evolution followed by a localization in the moduli space of CDs including also a localization in clock time determined by distance between the tips of CD. Also this would give to quantal discrete dynamics. Also now one can wonder whether Galois group is measured or not. If not, one would have a dispersion like process in the space of Galois groups labelling sectors of WCW.

6. Also the evolution of the tensor net defined by fermionic strings connecting the positions of fermions at partonic 2-surfaces would define a discrete dynamics in the space of these networks both at classical and quantum level [L32]. The dynamics of many-fermion states would determine this evolution.

In the sequel this picture is discussed in more detail.

How can one assign an extension of rationals to WCW, embedding space, and a region of space-time surface?

What fixes the extension used at both WCW level, embedding space level, and space-time level? The natural assumption is that the extension used for WCW coordinates induces the extension used at embedding space level and space-time level. At the level of space-time surfaces WCW coordinates appear as moduli (parameters) characterizing preferred extremals and would have values in an extension of rationals characterizing the adele by inducing the extensions of p-adic sectors.

1. The simplest option is that the extension is dictated by WCW. Preferred WCW coordinates - made possible by maximal isometries and fixed apart from the isometries of WCW - are in the extension: this makes the space of allowed 3-surfaces discrete. This in turn induces a constraint on space-time surfaces: WCW coordinates define parameters characterizing the

space-time surface as a preferred extremal. One could use also other coordinates of WCW but these would not be optimal as cognitive representations.

This applies also at the level of embedding space. Contrary to what I first thought, it is not actually absolutely necessary to use preferred space-time coordinates (subset of embedding space coordinates) since cognitive representation depends on coordinates in finite measurement resolution: consider only spherical and Cartesian coordinates with given resolution defining different discretizations. The preferred coordinates would be preferred because they are cognitively optimal.

2. Real embedding space is replaced with a discrete set of points of H with preferred coordinates in an extension of rationals. The direct identification of the points of extension as real numbers with p-adic numbers is extremely discontinuous although it would respect algebraic symmetries. The situation is saved by the lower dimensionality of space-time surfaces for which the set of points with coordinates in extension is discrete and even finite in the generic case. The surface $x^n + y^n = z^n$ has only one rational point for $n > 2$! $D = 4 < 8$ for space-time surfaces automatically brings in finite measurement resolution and cognitive resolution induced directly from the restriction on WCW parameters.

   SH has as data the intersection plus string world sheets (SH). String world sheets are in the intersection of reality and p-adicities defined by rational functions with coefficients of polynomials in extension, and makes sense both in real and p-adic sense. To these initial data one can assign as a preferred extremal of Kähler action a smooth p-adic space-time surface such that each point is contained in an open set consisting of points with p-adic coordinates having norm smaller than some power of p. This extremal is not unique in the p-adic sectors. In real sector it might not exist at all as already discussed.

3. 3-surface is seen as pair of 3-surfaces assigned to the ends of the space-time surface at boundaries of CD. WCW coordinates parameterize this pair and correspond to extension in 4-D sense. These parameters are expected to decompose to sets of parameters characterizing the 3-D members of pair and parameters characterizing the connecting space-time surface unless it is unique. If so, one can assign to the initial and final 3-surfaces subsets of WCW coordinates.

   The extensions associated with the ends of CD would be extensions in 3-D sense and sub-extensions of the extension in 4-D sense. Hence one can say that classical space-time evolution connecting initial and final 3-surfaces can modify the extension, its Galois group, and therefore also $h_{eff}/h = n$. This would be the classical view a about number theoretic evolution and also about quantum critical fluctuation changing the value of $h_{eff}/h = n$.

4. The extension of rationals for WCW coordinates induces the cognitive representation posing constraints of p-adic space-time surfaces. Adelic sub-WCW consisting of preferred extremals inside given CD decomposes to sectors characterized by an extension of rationals and evolution should correspond number theoretically to a path in the space of WCW sectors.

   This is a restriction on p-adic space-time sheets and thus cognition: the larger the number of points in the intersection, the more precise the cognitive representation is. The increase of the dimension of extension implies that the number of points of cognitive representation increases and it becomes more precise. The cognitive abilities of the system evolve. p-Adic pseudo constants allow imagination but also make the representation imprecise in scales below that defined by the cognitive representation. The continuation to smooth p-adic surface would however explain the highly non-trivial fact that we automatically tend to associate continuous structures with discrete data.

5. The fermions at partonic 2-surfaces are at positions for which preferred space-time coordinates are in extension and can be said to actualize the cognitive representation. It turns out that these positions could naturally correspond to the singularities of the space-time surfaces as n-fold covering in the sense that the dimension of the orbit of Galois group would be reduced at these points.

Can one assign the analog of discrete dynamics to TGD at fundamental level?

Could one assign a discrete symbolic dynamics to classical and quantum TGD?

At classical level the dynamics would correspond to space-time surface connecting the boundaries of CD and 3-surfaces at them. As already explained, the WCW coordinates characterizing space-time surface as a preferred extremal correspond to what might be called Galois group in 4-D sense. These coordinates decompose to coordinates characterizing the coordinates at the 3-surfaces at the ends of of space-time at boundaries of CD in extensions characterized by Galois groups in 3-D sense - the initial and final Galois group. The classical evolutionary step would be a step leading from the initial fo final Galois group serving as classical correlate for quantum evolution.

What about quantum level?

1. One expects that zero energy state in general is a superposition of space-time surfaces with different Galois groups in 4-D sense, G_4. The Galois groups in 3-D sense - G_3 - assignable to the ends of space-time surface would be sub-groups of G_4. If the first state function reduction to the opposite boundary of CD involves a localization to a sector of WCW having same G_3 at passive boundary for all 3-surfaces in the superposition.

   Subsequent reductions at opposite boundaries would define evolutionary pathway in the space of Galois groups G_3 leading in statistical sense to the increase of complexity.

2. The original vision was that Negentropy Maximization Principle (NMP) [K49] is needed as a separate principle to guarantee evolution but adelic physics implies it in statistical sense automatically [L51]. There is infinite number of extensions more complex than given one and only finite number of them less complex.

3. At quantum level the basic notion is self. It corresponds to a discrete sequence steps consisting of unitary evolution followed by a localization in the moduli space of CDs. This would correspond to a dispersion in WCW to sectors characterized by different Galois groups G_4 and G_3 associated with the 3-surface at active boundary. As explained, the state function reduction to the opposite boundary of CD analogous to a halting of quantum computation would correspond to a localization to a sector with definite Galois group G_3.

4. These time discrete time evolutions are non-deterministic unlike the dynamical evolutions studied in chaos theory defined by differential equations or iteration of function. The sequence of unitary time evolutions involving localization in the moduli of CD would however give rise to a quantum analog of iteration and one can ask whether the quantum counterparts for the notions of cycle, super-stable cycle etc... could make sense for the quantum superpositions of 4-surfaces involved. One expects dispersion in the space of Galois groups so that this idea does not look promising. One can also wonder if the sequence of unitary transformations could lead to some kind of asymptotic self-organization pattern before the first state function reduction to the opposite boundary of CD.

It is natural to consider also the evolution of the cognitive representation itself both at the space-time level and forced by the change of the many-fermion state and at quantum level.

1. For a given preferred extremal cognitive representation defines a discrete set of points in an extension of rationals and the number of points in the extension increases as it grows. The positions of fermions at partonic 2-surfaces define the nodes of a graph with strings connecting fermions at different partonic 2-surfaces serving as edges. Evolution of fermionic state changes the topology of this network by adding vertices and changing the connection.

   One can assign a complexity theory to these graphs. A connection with tensor nets [L32] emerging in the description of quantum complexity is highly suggestive. The nodes of the tensor net would correspond to fermions at partonic 2-surfaces. As the number of fermions increases, the complexity of this network increases and also the space-time surface itself becomes more complex. The maximum number of fermions increases with the dimension of extension.

   An interesting proposal is that fermion lines are accompanied by magnetic flux tubes taking the role of wormholes in ER-EPR correspondence (see `http://tinyurl.com/hzqlo6r`), which emerged more than half decade after its TGD analog. The discrete evolution of many-fermion state in state function reductions in the fermionic sector induces the evolution of this network.

2. In the case of graphs one can speak about various kinds of cycles, in particular Hamiltonian cycles going through all points of graph and having no self-intersections. Interestingly, Hamiltonian cycles for icosahedron (here the isometry group of icosahedron is involved as an additional structure) lead to a vision about genetic code and music harmonies [L23].

3. An interesting question concerns the extensions of rationals having as Galois group the isometry groups of Platonic solids: they probably exist. One can also consider the counterparts of Galois groups as discrete subgroups of the Galois group $SO(3)$ of quaternions. They emerge naturally for algebraic discretizations of M^4 regarded as a subspace of complexified quaternions with time axis identified as the real axis for quaternions (for $M^8 - H$ correspondence [K82, K94] see `http://tinyurl.com/mdvazmr`). Platonic solids correspond to *finite* discretizations with finite isometry groups belonging to a hierarchy of finite discrete subgroups of $SO(3)$ labelling the hierarchy of inclusions of HFFs: a connection between HFFs and quaternions is suggestive. For HFFs Platonic solids are in unique role in the sense that only for them the action of $SO(3)$ is genuinely 3-D. In Mac Kay correspondence they correspond to exceptional groups.

   For this generalization evolution would correspond to evolution in the space of Galois groups for finite-dimensional extensions of rational valued quaternions. p-Adic quaternions do not however form a field since p-adic quaternion can have vanishing norm squared.

4. The wave functions in the Galois group G reduce to wave functions in its coset space G/H if they are invariant under subgroup H. One can also perform the analog of second quantization for fermions in Galois group labelling the space-time sheets (or those of 3-space). In the model of harmony based on Hamilton's cycles the notes of 12-note scale would correspond to vertices of icosahedron obtained as coset space of I/Z_5, where I is icosahedral group with 60 elements. 3-chords of the harmony for a given Hamiltonian cycle would correspond to faces, which are triangles. Single particle fermion states localized at vertices (points of coset space) would correspond to notes of the scale and 3-fermion states localized at vertices of triangle to allowed 3-chords. The observation that one can understand the degeneracies of vertebrate genetic code by introducing besides icosahedron also tetrahedron suggests that both music and genetic code could relate directly to cognition described number theoretically.

5. It is also known that graphs can be identified as representations for Boolean statements (see `http://tinyurl.com/myrkhny`). Many-fermion states represent in TGD framework quantum Boolean statements with fermion number taking the role of bit. Could it be that this graphs indeed represent entanglement many-fermion states having interpretation as quantum Boolean statements?

Can one imagine a quantum counterpart of iteration cycle? The space-time sheets can be seen as covering spaces with the number of sheets equal to the order $n = h_{eff}/h$ of Galois group. This gives additional discrete degrees of freedom and one could have wave functions in Galois group and also in its cyclic subgroup. These might serve as quantum counterparts for iteration cycles. An open question is whether n is always accompanied by $1/n$ fractionization of quantum numbers so that dark elementary particles would have same quantum numbers as ordinary ones but could be said to decompose to n pieces corresponding to sheets of covering.

One can also imagine that the cycles appear in the statistical description. At this limit one obtains deterministic kinetic equations and by their non-linearity one expects that they exhibit chaotic behavior in the usual sense.

Why would primes near powers of two (or small primes) be important?

p-Adic length scale hypothesis states that physically preferred p-adic primes come as primes near prime powers of two and possibly also other small primes. Does this have some analog to complexity theory, period doubling, and with the super-stability associated with period doublings?

Also ramified primes characterize the extension of rationals and would define naturally preferred primes for a given extension.

1. Any rational prime p can be decomposes to a product of powers P_i^k of primes P_i of extension given by $p = \prod_i P_i^{k_i}$, $\sum k_i = n$. If one has $k_i \neq 1$ for some i, one has ramified prime. Prime p is Galois invariant but ramified prime decomposes to lower-dimensional orbits of Galois group formed by a subset of $P_i^{k_i}$ with the same index k_i . One might say that ramified primes are more structured and informative than un-ramified ones. This could mean also representative capacity.

2. Ramification has as its analog criticality leading to the degenerate roots of a polynomial or the lowering of the rank of the matrix defined by the second derivatives of potential function depending on parameters. The graph of potential function in the space defined by its arguments and parameters if n-sheeted singular covering of this space since the potential has several extrema for given parameters. At boundaries of the n-sheeted structure some sheets degenerate and the dimension is reduced locally . Cusp catastrophe with 3-sheets in catastrophe region is standard example about this.

   Ramification also brings in mind super-stability of n-cycle for the iteration of functions meaning that the derivative of n:th iterate $f(f(...)(x) \equiv f^{n)}(x)$ vanishes. Superstability occurs for the iteration of function $f = ax + bx^2$ for $a = 0$.

3. I have considered the possibility that the n-sheeted coverings of the space-time surface are singular in that the sheet co-incide at the ends of space-time surface or at some partonic 2-surfaces. One can also consider the possibility that only some sheets or partonic 2-surfaces co-incide.

   The extreme option is that the singularities occur only at the points representing fermions at partonic 2-surfaces. Fermions could in this case correspond to different ramified primes. The graph of $w = z^{1/2}$ defining 2-fold covering of complex plane with singularity at origin gives an idea about what would be involved. This option looks the most attractive one and conforms with the idea that singularities of the coverings in general correspond to isolated points. It also conforms with the hypothesis that fermions are labelled by p-adic primes and the connection between ramifications and Galois singularities could justify this hypothesis.

4. Category theorists love structural similarities and might ask whether there might be a morphism mapping these singularities of the space-time surfaces as Galois coverings to the ramified primes so that sheets would correspond to primes of extension appearing in the decomposition of prime to primes of extension.

   Could the singularities of the covering correspond to the ramification of primes of extension? Could this degeneracy for given extension be coded by a ramified prime? Could quantum criticality of TGD favour ramified primes and singularities at the locations of fermions at partonic 2-surfaces?

   Could the fundamental fermions at the partonic surfaces be quite generally localize at the singularities of the covering space serving as markings for them? This also conforms with the assumption that fermions with standard value of Planck constants corresponds to 2-sheeted coverings.

5. What could the ramification for a point of cognitive representation mean algebraically? The covering orbit of point is obtained as orbit of Galois group. For maximal singularity the Galois orbit reduces to single point so that the point is rational. Maximally ramified fermions would be located at rational points of extension. For non-maximal ramifications the number of sheets would be reduced but there would be several of them and one can ask whether only maximally ramified primes are realized. Could this relate at the deeper level to the fact that only rational numbers can be represented in computers exactly.

6. Can one imagine a physical correlate for the singular points of the space-time sheets at the ends of the space-time surface? Quantum criticality as analogy of criticality associated with super-stable cycles in chaos theory could be in question. Could the fusion of the space-time

sheets correspond to a phenomenon analogous to Bose-Einstein condensation? Most naturally the condensate would correspond to a fractionization of fermion number allowing to put n fermions to point with same M^4 projection? The largest condensate would correspond to a maximal ramification $p = P_i^n$.

Why ramified primes would tend to be primes near powers of two or of small prime? The attempt to answer this question forces to ask what it means to be a survivor in number theoretical evolution. One can imagine two kinds of explanations.

1. Some extensions are winners in the number theoretic evolution, and also the ramified primes assignable to them. These extensions would be especially stable against further evolution representing analogs of evolutionary fossils. As proposed earlier, they could also allow exceptionally large cognitive representations that is large number of points of real space-time surface in extension.

2. Certain primes as ramified primes are winners in the sense the further extensions conserve the property of being ramified.

   (a) The first possibility is that further evolution could preserve these ramified primes and only add new ramified primes. The preferred primes would be like genes, which are conserved during biological evolution. What kind of extensions of existing extension preserve the already existing ramified primes. One could naïvely think that extension of an extension replaces P_i in the extension for $P_i = Q_{ik}^{k_i}$ so that the ramified primes would remain ramified primes.

   (b) Surviving ramified primes could be associated with a exceptionally large number of extensions and thus with their Galois groups. In other words, some primes would have strong tendency to ramify. They would be at criticality with respect to ramification. They would be critical in the sense that multiple roots appear.

   Can one find any support for this purely TGD inspired conjecture from literature? I am not a number theorist so that I can only go to web and search and try to understand what I found. Web search led to a thesis (see `http://tinyurl.com/mkhrssy`) studying Galois group with prescribed ramified primes.

   The thesis contained the statement that not every finite group can appear as Galois group with prescribed ramification. The second statement was that as the number and size of ramified primes increases more Galois groups are possible for given predetermined ramified primes. This would conform with the conjecture. The number and size of ramified primes would be a measure for complexity of the system, and both would increase with the size of the system.

   (c) Of course, both mechanisms could be involved.

Why ramified primes near powers of 2 would be winners? Do they correspond to ramified primes associated with especially many extension and are they conserved in evolution by subsequent extensions of Galois group. But why? This brings in mind the fact that $n = 2^k$-cycles becomes super-stable and thus critical at certain critical value of the control parameter. Note also that ramified primes are analogous to prime cycles in iteration. Analogy with the evolution of genome is also strongly suggestive.

$h_{eff}/h = n$ hypothesis and Galois groups

The natural hypothesis is that $h_{eff}/h = n$ equals to dimension of the extension of rationals in the case that it gives the number of sheets of the covering assignable to the space-time surfaces. The stronger hypothesis is that $h_{eff}/h = n$ is associated with flux tubes and is proportional to the quantum numbers associated with the ends.

1. The basic idea is that Mother Nature is theoretician friendly. As perturbation theory breaks down, the interaction strength expressible as a product of appropriate charges divided by Planck constant, is reduced in the phase transition $\hbar \to \hbar_{eff}$.

2. In the case of gravitation $GMm \rightarrow= GMm(h/h_{eff})$. Equivalence Principle is satisfied if one has $\hbar_{eff} = \hbar_{gr} = GMm/v_0$, where v_0 is parameter with dimensions of velocity and of the order of some rotation velocity associated with the system. If the masses move with relativistic velocities the interaction strength is proportional to the inner product of four-momenta and therefore to Lorentz boost factors for energies in the rest system of the entire system. In this case one must assume quantization of energies to satisfy the constraint or a compensating reduction of v_0. Interactions strength becomes equal to $\beta_0 = v_0/c$ having no dependence on the masses: this brings in mind the universality associated with quantum criticality.

3. The hypothesis applies to all interactions. For electromagnetism one would have the replacements $Z_1Z_2\alpha \rightarrow Z_1Z_2\alpha(h/h_{em})$ and $\hbar_{em} = Z_1Z_2\alpha/\beta_0$ giving Universal interaction strength. In the case of color interactions the phase transition would lead to the emergence of hadron and it could be that inside hadrons the valence quark have $h_{eff}/h = n > 1$. In this case one could consider a generalization in which the product of masses is replaced with the inner product of four-momenta. In this case quantization of energy at either or both ends is required. For astrophysical energies one would have effective energy continuum.

This hypothesis suggests the interpretation of $h_{eff}/h = n$ as either the dimension of the extension or the order of its Galois group. If the extensions have dimensions n_1 and n_2, then the composite system would be n_2-dimensional extension of n_1-dimensional extension and have dimension $n_1 \times n_2$. This could be also true for the orders of Galois groups. This would be the case if Galois group of the entire system is free group generated by the G_1 and G_2. One just takes all products of elements of G_1 and G_2 and assumes that they commute to get $G_1 \times G_2$.

Consider gravitation as example.

1. The dimension of the extension should coincide with $\hbar_{eff}/\hbar = n = \hbar_{gr}/\hbar = GMm/v_0\hbar$. The transition occurs only if the value of $\hbar_{gr}/\hbar$ is larger than one. One can say that the dimension of the extension is proportional the product of masses using as unit Planck mass. Rather large extensions are involved and the number of sheets in the Galois covering is huge.

   Note that it is difficult to say how larger Planck constants are actually involved since by gravitational binding the classical gravitational forces are additive and by Equivalence principle same potential is obtained as sum of potentials for splitting of masses into pieces. Also the gravitational Compton length $\lambda_{gr} = GM/v_0$ for m does not depend on m at all so that all particles have same $\lambda_{gr} = GM/v_0$ irrespective of mass (note that v_0 is expressed using units with $c = 1$).

   The maximally incoherent situation would correspond to ordinary Planck constant and the usual view about gravitational interaction between particles. The extreme quantum coherence would mean that both M and m behave as single quantum unit. In many-sheeted space-time this could be understood in terms of a picture based on flux tubes. The interpretation for the degree of coherence is discussed in terms of flux tube connections mediating gravitational flux is discussed in [**?**].

2. h_{gr}/h would be the dimension of the extension, and there is a temptation to associate with the product of masses the product $n = n_1n_2$ of dimensions n_i associated masses M and m at least in some situations.

   The problem is that the dimension of the extension associated with m would be smaller than 1 for masses $m < m_P/\sqrt{\beta_0}$. Planck mass is about 1.3×10^{19} proton masses and corresponds to a blob of water with size scale 10^{-4} meters - size scale of a large neuron so that only above these scale gravitational quantum coherence would be possible. For $v_0 < 1$ it would seem that even in the case of large neurons one must have more than one neurons. Maybe pyramidal neurons could satisfy the mass constraint and would represent higher level of conscious as compared to other neurons and cells. The giant neurons discovered by the group led by Christof Koch in the brain of of mouse having axonal connections distributed over the entire brain might fulfil the constraint (see `http://tinyurl.com/gvwggsc`).

3. It is difficult to avoid the idea that macroscopic quantum gravitational coherence for multicellular objects with mass at least that for the largest neurons could be involved with

biology. Multicellular systems can have mass above this threshold for some critical cell number. This might explain the dramatic evolutionary step distinguishing between prokaryotes (mono-cellulars consisting of Archaea and bacteria including also cellular organelles and cells with sub-critical size) and eukaryotes (multi-cellulars).

4. I have proposed an explanation of the fountain effect appearing in super-fluidity and apparently defying the law of gravity. In this case m was assumed to be the mass of ^4He atom in case of super-fluidity to explain fountain effect [?]. The above arguments however allow to ask whether anything changes if one allows the blobs of superfluid to have masses coming as a multiple of $m_P/\sqrt{\beta_0}$. One could check whether fountain effect is possible for super-fluid volumes with mass below $m_P/\sqrt{\beta_0}$.

What about h_{em}? In the case of super-conductivity the interpretation of h_{em}/h as product of orders of Galois groups would allow to estimate the number $N = Q/2e$ of Cooper pairs of a minimal blob of super-conducting matter from the condition that the order of its Galois group is larger than integer. The number $N = Q/2e$ is such that one has $2N\sqrt{\alpha/\beta_0} = n$. The condition is satisfied if one has $\alpha/\beta_0 = q^2$, with $q = k/2l$ such that N is divisible by l. The number of Cooper pairs would be quantized as multiples of l. What is clear that em interaction would correspond to a lower level of cognitive consciousness and that the step to gravitation dominated cognition would be huge if the dark gravitational interaction with size of astrophysical systems is involved [K63]. Many-sheeted space-time allows this in principle.

These arguments support the view that quantum information theory indeed closely relates not only to gravitation but also other interactions. Speculations revolving around blackhole, entropy, and holography, and emergence of space would be replaced with the number theoretic vision about cognition providing information theoretic interpretation of basic interactions in terms of entangled tensor networks [L32]. Negentropic entanglement would have magnetic flux tubes (and fermionic strings at them) as topological correlates. The increase of the complexity of quantum states could occur by the "fusion" of Galois groups associated with various nodes of this network as macroscopic quantum states are formed. Galois groups and their representations would define the basic information theoretic concepts. The emergence of gravitational quantum coherence identified as the emergence of multi-cellulars would mean a major step in biological evolution.

13.9 Why The Non-trivial Zeros Of Riemann Zeta Should Reside At Critical Line?

The following argument shows that the troublesome looking "1/2" in the non-trivial zeros of Riemann zeta can be understood as being necessary to allow a hermitian realization of the radial scaling generator rd/dr at light-cone boundary, which in the radial light-like radial direction corresponds to half-line R^+. Its presence allows unitary inner product and reduces the situation to that for ordinary plane waves on real axis. For preferred extremals strong form of holography poses extremely strong conditions expected to reduce the scaling momenta $s = 1/2 + iy$ to the zeros of zeta at critical line. RH could be also seen as a necessary condition for the existence of super-symplectic representations and thus for the existence of the "World of Classical Worlds" as a mathematically well-defined object. We can thank the correctness of Riemann's hypothesis for our existence!

13.9.1 What Is The Origin Of The Troublesome 1/2 In Non-trivial Zeros Of Zeta?

Riemann Hypothess (RH) states that the non-trivial (critical) zeros of zeta lie at critical line $s = 1/2$. It would be interesting to know how many physical justifications for why this should be the case has been proposed during years. Probably this number is finite, but very large it certainly is. In Zero Energy Ontology (ZEO) forming one of the cornerstones of the ontology of quantum TGD, the following justification emerges naturally.

1. The "World of Classical Worlds" (WCW) consisting of space-time surfaces having ends at the boundaries of causal diamond (CD), the intersection of future and past directed light-cones

times CP_2 (recall that CDs form a fractal hierarchy). WCW thus decomposes to sub-WCWs and conscious experience for the self associated with CD is only about space-time surfaces in the interior of CD: this is a trong restriction to epistemology, would philosopher say.

Also the light-like orbits of the partonic 2-surfaces define boundary like entities but as surfaces at which the signature of the induced metric changes from Euclidian to Minkowskian. By holography either kinds of 3-surfaces can be taken as basic objects, and if one accepts strong form of holography, partonic 2-surfaces defined by their intersections plus string world sheets become the basic entities.

2. One must construct tangent space basis for WCW if one wants to define WCW Kähler metric and gamma matrices. Tangent space consists of allowed deformations of 3-surfaces at the ends of space-time surface at boundaries of CD, and also at light-like parton orbits extended by field equations to deformations of the entire space-time surface. By strong form of holography only very few deformations are allowed since they must respect the vanishing of the elements of a sub-algebra of the classical symplectic charges isomorphic with the entire algebra. One has almost 2-dimensionality: most deformations lead outside WCW and have zero norm in WCW metric.

3. One can express the deformations of the space-like 3-surface at the ends of space-time using a suitable function basis. For CP_2 degrees of freedom color partial waves with well defined color quantum numbers are natural. For light-cone boundary $S^2 \times R^+$, where R^+ corresponds to the light-like radial coordinate, spherical harmonics with well defined spin are natural choice for S^2 and for R^+ analogs of plane waves are natural. By scaling invariance in the light-like radial direction they look like plane waves $\psi_s(r) = r^s = exp(us)$, $u = log(r/r_0)$, $s = x + iy$. Clearly, u is the natural coordinate since it replaces R^+ with R natural for ordinary plane waves.

4. One can understand why $Re[s] = 1/2$ is the only possible option by using a simple argument. One has super-symplectic symmetry and conformal invariance extended from 2-D Riemann surface to metrically 2-dimensional light-cone boundary. The natural scaling invariant integration measure defining inner product for plane waves in R^+ is $du = dr/r = dlog(r/r_0)$ with u varying from $-\infty$ to $+\infty$ so that R^+ is effectively replaced with R. The inner product must be same as for the ordinary plane waves and indeed is for $\psi_s(r)$ with $s = 1/2 + iy$ since the inner product reads as

$$\langle s_1, s_2\rangle \equiv \int_0^\infty \overline{\psi_{s_1}} \psi_{s_2} dr = \int_0^\infty exp(i(y_1 - y_2) r^{-x_1-x_2} dr \quad .$$

For $x_1 + x_2 = 1$ one obtains standard delta function normalization for ordinary plane waves:

$$\langle s_1, s_2\rangle \int \int_{-\infty}^\infty exp[i(y_1 - y_2)u]du \propto \delta(y_1 - y_2) \quad .$$

If one requires that this holds true for all pairs (s_1, s_2), one obtains $x_i = 1/2$ for all s_i. Preferred extremal condition gives extremely powerful additional constraints and leads to a quantisation of $s = -x - iy$: the first guess is that non-trivial zeros of zeta are obtained: $s = 1/2 + iy$. This identification would be natural by generalised conformal invariance. Thus RH is physically extremely well motivated but this of course does not prove it.

5. The presence of the real part $Re[s] = 1/2$ in the eigenvalues of scaling operator apparently breaks hermiticity of the scaling operator. There is however a compensating breaking of hermiticity coming from the fact that real axis is replaced with half-line and origin is pathological. What happens that real part 1/2 effectively replaces half-line with real axis and obtains standard plane wave basis. Note also that the integration measure becomes scaling invariant - something very essential for the representations of super-symplectic algebra. For $Re[s] = 1/2$ the hermicity conditions for the scaling generator rd/dr in R^+ reduce to those for the translation generator d/du in R.

13.9.2 Relation To Number Theoretical Universality And Existence Of WCW

This relates also to the number theoretical universality and mathematical existence of WCW in an interesting manner.

1. If one assumes that p-adic primes p correspond to zeros $s = 1/2+y$ of zeta in 1-1 manner in the sense that $p^{iy(p)}$ is root of unity existing in all number fields (algebraic extension of p-adics) one obtains that the plane wave exists for p at points $r = p^n$. p-Adically wave function is discretized to a delta function distribution concentrated at $(r/r_0) = p^n$- a logarithmic lattice. This can be seen as space-time correlate for p-adicity for light-like momenta to be distinguished from that for massive states where length scales come as powers of $p^{1/2}$. Something very similar is obtained from the Fourier transform of the distribution of zeros at critical line (Dyson's argument), which led to a the TGD inspired vision about number theoretical universality [L25] (see `http://tinyurl.com/y7gl4huo`).

2. My article "Strategy for Proving Riemann Hypothesis" (`http://tinyurl.com/yd7k46ar`) [L1] written for 12 years ago ((for a slightly improved version see `http://tinyurl.com/ydcfkxwr`) relies on coherent states instead of eigenstates of Hamiltonian. The above approach in turn absorbs the problematic 1/2 to the integration measure at light cone boundary and conformal invariance is also now central.

3. Quite generally, I believe that conformal invariance in the extended form applying at metrically 2-D light-cone boundary (and at light-like orbits of partonic 2-surfaces) could be central for understanding why physics requires RH and maybe even for proving RH assuming it is provable at all in existing standard axiomatic system. For instance, the number of generating elements of the extended supersymplectic algebra is infinite (rather than finite as for ordinary conformal algebras) and generators are labelled by conformal weights defined by zeros of zeta (perhaps also the trivial conformal weights). $s = 1/2 + iy$ guarantees that the real parts of conformal weights for all states are integers. By conformal confinement the sum of ys vanishes for physical states. If some weight is not at critical line the situation changes. One obtains as net conformal weights all multiples of x shifted by all half odd integer values. And of course, the realisation as plane waves at boundary of light-cone fails and the resulting loss of unitary makes things too pathological and the mathematical existence of WCW is threatened.

4. The existence of non-trivial zeros outside the critical line could thus spoil the representations of super-symplectic algebra and destroy WCW geometry. RH would be crucial for the mathematical existence of the physical world! And the physical worlds exist only as mathematical objects in TGD based ontology: there are no physical realities behind the mathematical objects (WCW spinor fields) representing the quantum states. TGD inspired theory of consciousness tells that quantum jumps between the zero energy states give rise to conscious experience, and this is in principle all that is needed to understand what we experience.

13.10 Why Mersenne primes are so special?

Mersenne primes are central in TGD based world view. p-Adic thermodynamics combined with p-adic length scale hypothesis stating that primes near powers of two are physically preferred provides a nice understanding of elementary particle mass spectrum. Mersenne primes $M_k = 2^k - 1$, where also k must be prime, seem to be preferred. Mersenne prime labels hadronic mass scale (there is now evidence from LHC for two new hadronic physics labelled by Mersenne and Gaussian Mersenne), and weak mass scale. Also electron and tau lepton are labelled by Mersenne prime. Also Gaussian Mersennes $M_{G,k} = (1 + i)^k - 1$ seem to be important. Muon is labelled by Gaussian Mersenne and the range of length scales between cell membrane thickness and size of cell nucleus contains 4 Gaussian Mersennes!

What gives Mersenne primes so special physical status? I have considered this problem many times during years. The key idea is that natural selection is realized in much more general sense than usually thought, and has chosen them and corresponding p-adic length scales. Particles characterized by p-adic length scales should be stable in some well-defined sense.

Since evolution in TGD corresponds to generation of information, the obvious guess is that Mersenne primes are information theoretically special. Could the fact that $2^k - 1$ represents almost k bits be of significance? Or could Mersenne primes characterize systems, which are information theoretically especially stable? In the following a more refined TGD inspired quantum information theoretic argument based on stability of entanglement against state function reduction, which would be fundamental process governed by Negentropy Maximization Principle (NMP) and requiring no human observer, will be discussed.

13.10.1 How to achieve stability against state function reductions?

TGD provides actually several ideas about how to achieve stability against state function reductions. This stability would be of course marvellous fact from the point of view of quantum computation since it would make possible stable quantum information storage. Also living systems could apply this kind of storage mechanism.

1. p-Adic physics leads to the notion of negentropic entanglement (NE) for which number theoretic entanglement entropy is negative and thus measures genuine, possibly conscious information assignable to entanglement (ordinary entanglement entropy measures the lack of information about the state of either entangled system). NMP favors the generation of NE. NE can be however transferred from system to another (stolen using less diplomatically correct expression!), and this kind of transfer is associated with metabolism. This kind of transfer would be the most fundamental crime: biology would be basically criminal activity! Religious thinker might talk about original sin.

   In living matter NE would make possible information storage. In fact, TGD inspired theory of consciousness constructed as a generalization of quantum measurement theory in Zero Energy Ontology (ZEO) identifies the permanent self of living system (replaced with a more negentropic one in biological death, which is also a reincarnation) as the boundary of CD, which is not affected in subsequent state function reductions and carries NE. The changing part of self - sensory input and cognition - can be assigned with opposite changing boundary of CD.

2. Also number theoretic stability can be considered. Suppose that one can assign to the system some extension of algebraic numbers characterizing the WCW coordinates ("world of classical worlds") parametrizing the space-time surface (by strong form of holography (SH) the string world sheets and partonic 2-surfaces continuable to 4-D preferred extremal) associated with it.

   This extension of rationals and corresponding algebraic extensions of p-adic numbers would define the number fields defining the coefficient fields of Hilbert spaces. Assume that you have an entangled system with entanglement coefficients in this number field. Suppose you want to diagonalize the corresponding density matrix. The eigenvalues belong in general case to a larger algebraic extension since they correspond to roots of a characteristic polynomials assignable to the density matrix. Could one say, that this kind of entanglement is stable (at least to some degree) against state function reduction since it means going to an eigenstate which does not belong to the extension used? Reader can decide!

3. Hilbert spaces are like natural numbers with respect to direct sum and tensor product. The dimension of the tensor product is product mn of the dimensions of the tensor factors. Hilbert space with dimension n can be decomposed to a tensor product of prime Hilbert spaces with dimensions which are prime factors of n. In TGD Universe state function reduction is a dynamical process, which implies that the states in state spaces with prime valued dimension are stable against state function reduction since one cannot even speak about tensor product decomposition, entanglement, or reduction of entanglement. These state spaces are quantum indecomposable and would be thus ideal for the storage of quantum information!

   Interestingly, the system consisting of k qubits have Hilbert space dimension $D = 2^k$ and is thus maximally unstable against decomposition to $D = 2$-dimensional tensor factors! In TGD

Universe NE might save the situation. Could one imagine a situation in which Hilbert space with dimension $M_k = 2^k - 1$ stores the information stably? When information is processed this state space would be mapped isometrically to 2^k-dimensional state space making possible quantum computations using qubits. The outcome of state function reduction halting the computation would be mapped isometrically back to M_k-D space. Note that isometric maps generalizing unitary transformations are an essential element in the proposal for the tensor net realization of holography and error correcting codes [L32]. Can one imagine any concrete realization for this idea? This question be considered in the sequel.

13.10.2 How to realize $M_k = 2^k - 1$-dimensional Hilbert space physically?

One can imagine at least three physical realizations of $M_k = 2^k - 1$-dimensional Hilbert space.

1. The set with k elements has 2^k subsets. One of them is empty set and cannot be physically realized. Here the reader might of course argue that if they are realized as empty boxes, one can realize them. If empty set has no physical realization, the wave functions in the set of non-empty subsets with $2^k - 1$ elements define $2^k - 1$-dimensional Hilbert space. If $2^k - 1$ is Mersenne prime, this state state space is stable against state function reductions since one cannot even speak about entanglement!

   To make quantum computation possible one must map this state space to 2^k dimensional state space by isometric embedding. This is possible by just adding a new element to the set and considering only wave functions in the set of subsets containing this new element. Now also the empty set is mapped to a set containing only this new element and thus belongs to the state space. One has 2^k dimensions and quantum computations are possible. When the computation halts, one just removes this new element from the system, and the data are stored stably!

2. Second realization relies on k bits represented as spins such that $2^k - 1$ is Mersenne prime. Suppose that the ground state is spontaneously magnetized state with $k + l$ parallel spins, with the l spins in the direction of spontaneous magnetization and stabilizing it. $l > 1$ is probably needed to stabilize the direction of magnetization: $l \leq k$ suggests itself as the first guess. Here thermodynamics and a model for spin-spin interaction would give a better estimate.

   The state with the k spins in direction opposite to that for l spins would be analogous to empty set. Spontaneous magnetization disappears, when a sufficient number of spins is in direction opposite to that of magnetization. Suppose that k corresponds to a critical number of spins in the sense that spontaneous magnetization occurs for this number of parallel spins. Quantum superpositions of $2^k - 1$ states for k spins would be stable against state function reduction also now.

   The transformation of the data to a processable form would require an addition of $m \geq 1$ spins in the direction of the magnetization to guarantee that the state with all k spins in direction opposite to the spontaneous magnetization does not induce spontaneous magnetization in opposite direction. Note that these additional stabilizing spins are classical and their direction could be kept fixed by a repeated state function reduction (Zeno effect). One would clearly have a critical system.

3. Third realization is suggested by TGD inspired view about Boolean consciousness. Boolean logic is represented by the Fock state basis of many-fermion states. Each fermion mode defines one bit: fermion in given mode is present or not. One obtains 2^k states. These states have different fermion numbers and in ordinary positive energy ontology their realization is not possible.

   In ZEO situation changes. Fermionic zero energy states are superpositions of pairs of states at opposite boundaries of CD such that the total quantum numbers are opposite. This applies to fermion number too. This allows to have time-like entanglement in which one has superposition of states for which fermion numbers at given boundary are different. This kind of states might be realized for super-conductors to which one at least formally assigns coherent state of Cooper pairs having ill-defined fermion number.

Now the non-realizable state would correspond to fermion vacuum analogous to empty set. Reader can of course argue that the bosonic degrees of freedom assignable to the space-time surface are still present. I defend this idea by saying that the purely bosonic state might be unstable or maybe even non-realizable as vacuum state and remind that also bosons in TGD framework consists of pairs of fundamental fermions.

If this state is effectively decoupled from the rest of the Universe, one has 2^k-1-dimensional state space and states are stable against state function reduction. Information processing becomes possible by adding some positive energy fermions and corresponding negative energy fermions at the opposite boundaries of CD. Note that the added fermions do not have time-like quantum entanglement and do not change spin direction during time evolution.

The proposal is that Boolean consciousness is realized in this manner and zero energy states represents quantum Boolean thoughts as superposition of pairs $(b_1 \otimes b_2)$ of positive and negative energy states and having identification as Boolean statements $b_1 \rightarrow b_2$. The mechanism would allow both storage of thoughts as memories and their processing by introducing the additional fermion.

13.10.3 Why Mersenne primes would be so special?

Returning to the original question "Why Mersenne primes are so special?". A possible explanation is that elementary particle or hadron characterized by a p-adic length scale $p = M_k = 2^k - 1$ both stores and processes information with maximal effectiveness. This would not be surprising if p-adic physics defines the physical correlates of cognition assumed to be universal rather than being restricted to human brain.

In adelic physics p-dimensional Hilbert space could be naturally associated with the p-adic adelic sector of the system. Information storage could take place in $p = M_k = 2^k - 1$ phase and information processing (cognition) would take place in 2^k-dimensional state space. This state space would be reached in a phase transition $p = 2^k - 1 \rightarrow 2$ changing effective p-adic topology in real sector and genuine p-adic topology in p-adic sector and replacing padic length scale $\propto \sqrt{p} \simeq 2^{k/2}$ with k-nary 2-adic length scale $\propto 2^{k/2}$.

Electron is characterized by the largest not completely super-astrophysical Mersenne prime M_{127} and corresponds to $k = 127$ bits. Intriguingly, the secondary p-adic time scale of electron corresponds to .1 seconds defining the fundamental biorhythm of 10 Hz.

This proposal suffers from deficiencies. It does not explain why Gaussian Mersennes are also special. Gaussian Mersennes correspond ordinary primes near power of 2 but not so near as Mersenne primes do. Neither does it explain why also more general primes $p \simeq 2^k$ seem to be preferred. Furthermore, p-adic length scale hypothesis generalizes and states that primes near powers of at least small primes q: $p \simeq q^k$ are special at least number theoretically. For instance, $q = 3$ seems to be important for music experience and also $q = 5$ might be important (Golden Mean)

Could it be that the proposed model relying on criticality generalizes. There would be $p < 2^k$-dimensional state space allowing isometric embedding to 2^k-dimensional space such that the bit configurations orthogonal to the image would be unstable in some sense. Say against a phase transition changing the direction of magnetization. One can imagine the variants of above described mechanism also now. For $q > 2$ one should consider pinary digits instead of bits but the same arguments would apply (except in the case of Boolean logic).

13.10.4 Brain and Mersenne integers

I received a link to an interesting the article "Brain Computation Is Organized via Power-of-Two-Based Permutation Logic" by Kun Xie, Grace E. Fox, Jun Liu, Cheng Lyu, Jason C. Lee, Hui Kuang, Stephanie Jacobs, Meng Li, Tianming Liu, Sen Song and Joe Z. Tsien in Frontiers in Systems Neuroscience [?]see `http://tinyurl.com/zfymqrq`).

The proposed model is about how brain classifies neuronal inputs. The following represents my attempt to understand the model of the article.

1. One can consider a situation in which one has n inputs identifiable as bits: bit could correspond to neuron firing or not. The question is however to classify various input combinations.

The obvious criterion is how many bits are equal to 1 (corresponding neuron fires). The input combinations in the same class have same number of firing neurons and the number of subsets with k elements is given by the binomial coefficient $B(n,k) = n!/k!(n-k)!$. There are clearly $n-1$ different classes in the classification since no neurons firing is not a possible observation. The conceptualization would tell how many neurons fire but would not specify which of them.

2. To represent these bit combinations one needs 2^n-1 neuron groups acting as unit representing one particular firing combination. These subsets with k elements would be mapped to neuron cliques with k firing neutrons. For given input individual firing neurons ($k=1$) would represent features, lowest level information. The n cliques with $k=2$ neurons would represent a more general classification of input. One obtains $M_n = 2^n-1$ combinations of firing neurons since the situations in which no neurons are firing is not counted as an input.

3. If all neurons are firing then all the however level cliques are also activated. Set theoretically the subsets of set partially ordered by the number of elements form an inclusion hierarchy, which in Boolean algebra corresponds to the hierarchy of implications in opposite direction. The clique with all neurons firing correspond to the most general statement implying all the lower level statements. At k:th level of hierarchy the statements are inconsistent so that one has $B(n,k)$ disjoint classes.

The $M_n = 2^n - 1$ (Mersenne number) labelling the algorithm is more than familiar to me.

1. For instance, electron's p-adic prime corresponds to Mersenne prime $M_{127} = 2^{127} - 1$, the largest not completely super-astrophysical Mersenne prime for which the mass of particle would be extremely small. Hadron physics corresponds to M_{107} and M_{89} to weak bosons and possible scaled up variant of hadron physics with mass scale scaled up by a factor 512 ($= 2^{(107-89)/2}$). Also Gaussian Mersennes seem to be physically important: for instance, muon and also nuclear physics corresponds to $M_{G,n} = (1+i)^n - 1$, $n = 113$.

2. In biology the Mersenne prime $M_7 = 2^7 - 1$ is especially interesting. The number of statements in Boolean algebra of 7 bits is 128 and the number of statements that are consistent with given atomic statement (one bit fixed) is $2^6 = 64$. This is the number of genetic codons which suggests that the letters of code represent 2 bits. As a matter of fact, the so called Combinatorial Hierarchy $M(n) = M_{M(n-1)}$ consists of Mersenne primes $n = 3, 7, 127, 2^{127}-1$ and would have an interpretation as a hierarchy of statements about statements about ... It is now known whether the hierarchy continues beyond M_{127} and what it means if it does not continue. One can ask whether M_{127} defines a higher level code - memetic code as I have called it - and realizable in terms of DNA codon sequences of 21 codons [L31] (see `http://tinyurl.com/jukyq6y`).

3. The Gaussian Mersennes $M_{G,n}$ $n = 151, 157, 163, 167$, can be regarded as a number theoretical miracles since the these primes are so near to each other. They correspond to p-adic length scales varying between cell membrane thickness 10 nm and cell nucleus size 2.5 μm and should be of fundamental importance in biology. I have proposed that p-adically scaled down variants of hadron physics and perhaps also weak interaction physics are associated with them.

I have made attempts to understand why Mersenne primes M_n and more generally primes near powers of 2 seem to be so important physically in TGD Universe.

1. The states formed from n fermions form a Boolean algebra with 2^n elements, but one of the elements is vacuum state and could be argued to be non-realizable. Hence Mersenne number $M_n = 2^n-1$. The realization as algebra of subsets contains empty set, which is also physically non-realizable. Mersenne primes are especially interesting as sine the reduction of statements to prime nearest to M_n corresponds to the number M_n-1 of physically representable Boolean statements.

2. Quantum information theory suggests itself as explanation for the importance of Mersenne primes since M_n would correspond the number of physically representable Boolean statements of a Boolean algebra with n-elements. The prime $p \leq M_n$ could represent the number of elements of Boolean algebra representable p-adically [L36] (see http://tinyurl.com/gp9mspa).

3. In TGD Fermion Fock states basis has interpretation as elements of quantum Boolean algebra and fermionic zero energy states in ZEO expressible as superpositions of pairs of states with same net fermion numbers can be interpreted as logical implications. WCW spinor structure would define quantum Boolean logic as "square root of Kähler geometry". This Boolean algebra would be infinite-dimensional and the above classification for the abstractness of concept by the number of elements in subset would correspond to similar classification by fermion number. One could say that bosonic degrees of freedom (the geometry of 3-surfaces) represent sensory world and spinor structure (many-fermion states) represent that logical thought in quantum sense.

4. Fermion number conservation would seem to represent an obstacle but in ZEO it can circumvented since zero energy states can be superpositions of pair of states with opposite fermion number F at opposite boundaries of causal diamond (CD) in such a way that F varies. In state function reduction however localization to single value of F is expected to happen usually. If superconductors carry coherent states of Cooper pairs, fermion number for them is ill defined and this makes sense in ZEO but not in standard ontology unless one gives up the super-selection rule that fermion number of quantum states is well-defined.

One can of course ask whether primes n defining Mersenne primes (see http://tinyurl.com/l3lxe2n) could define preferred numbers of inputs for subsystems of neurons. This would predict $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257, ..$ define favoured numbers of inputs. $n = 127$ would correspond to memetic code.

13.11 Number Theoretical Feats and TGD Inspired Theory of Consciousness

Number theoretical feats of some mathematicians like Ramanujan remain a mystery for those believing that brain is a classical computer. Also the ability of idiot savants - lacking even the idea about what prime is - to factorize integers to primes challenges the idea that an algorithm is involved. In this article I discuss ideas about how various arithmetical feats such as partitioning integer to a sum of integers and to a product of prime factors might take place. The ideas are inspired by the number theoretic vision about TGD suggesting that basic arithmetics might be realized as naturally occurring processes at quantum level and the outcomes might be "sensorily perceived". One can also ask whether zero energy ontology (ZEO) could allow to perform quantum computations in polynomial instead of exponential time.

The indian mathematician Srinivasa Ramanujan is perhaps the most well-known example about a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess. A possible TGD based explanation of this feat relies on the idea that in zero energy ontology (ZEO) quantum computation like activity could consist of steps consisting quantum computation and its time reversal with long-lasting part of each step performed in reverse time direction at opposite boundary of causal diamond so that the net time used would be short at second boundary.

The adelic picture about state function reduction in ZEO suggests that it might be possible to have direct sensory experience about prime factorization of integers [L35]. What about partitions of integers to sums of primes? For years ago I proposed that symplectic QFT is an essential part of TGD. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian

Wilson line. I could not specify at that time how to select these polygons. A very natural manner to fix the vertices of polygon (or polygons) is to assume that they correspond ends of fermion lines which appear as boundaries of string world sheets. The polygons would be fixed rather uniquely by requiring that fermions reside at their vertices.

The number 1 is the only prime for addition so that the analog of prime factorization for sum is not of much use. Polygons with $n = 3, 4, 5$ vertices are special in that one cannot decompose them to non-degenerate polygons. Non-degenerate polygons also represent integers $n > 2$. This inspires the idea about numbers $\{3, 4, 5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. These polygons could be associated many-fermion states with negentropic entanglement (NE) - this notion relate to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define conscious representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition.

13.11.1 How Ramanujan did it?

Lubos Motl wrote recently a blog posting (`http://tinyurl.com/zduu72p`) about $P \neq NP$ computer in the theory of computation based on Turing's work. This unproven conjecture relies on a classical model of computation developed by formulating mathematically what the women doing the hard computational work in offices at the time of Turing did. Turing's model is extremely beautiful mathematical abstraction of something very every-daily but does not involve fundamental physics in any manner so that it must be taken with caution. The basic notions include those of algorithm and recursive function, and the mathematics used in the model is mathematics of integers. Nothing is assumed about what conscious computation is and it is somewhat ironic that this model has been taken by strong AI people as a model of consciousness!

1. A canonical model for classical computation is in terms of Turing machine, which has bit sequence as inputs and transforms them to outputs and each step changes its internal state. A more concrete model is in terms of a network of gates representing basic operations for the incoming bits: from this basic functions one constructs all recursive functions. The computer and program actualize the algorithm represented as a computer program and eventually halts - at least one can hope that it does so. Assuming that the elementary operations require some minimum time, one can estimate the number of steps required and get an estimate for the dependence of the computation time as function of the size of computation.

2. If the time required by a computation, whose size is characterized by the number N of relevant bits, can be carried in time proportional to some power of N and is thus polynomial, one says that computation is in class P. Non-polynomial computation in class NP would correspond to a computation time increasing with N faster than any power of N, say exponentially. Donald Knuth, whose name is familiar for everyone using Latex to produce mathematical text, believes on $P = NP$ in the framework of classical computation. Lubos in turn thinks that the Turing model is probably too primitive and that quantum physics based model is needed and this might allow $P = NP$.

What about quantum computation as we understand it in the recent quantum physics: can it achieve $P = NP$?

1. Quantum computation is often compared to a superposition of classical computations and this might encourage to think that this could make it much more effective but this does not seem to be the case. Note however that the amount of information represents by N qubits is however exponentially larger than that represented by N classical bits since entanglement is possible. The prevailing wisdom seems to be that in some situations quantum computation can be faster than the classical one but that if $P = NP$ holds true for classical computation, it holds true also for quantum computations. Presumably because the model of quantum computation begins from the classical model and only (quantum computer scientists must experience this statement as an insult - apologies!) replaces bits with qubits.

2. In quantum computer one replaces bits with entangled qubits and gates with quantum gates and computation corresponds to a unitary time evolution with respect to a discretized time parameter constructed in terms of fundamental simple building bricks. So called tensor networks realize the idea of local unitary in a nice manner and has been proposed to defined error correcting quantum codes. State function reduction halts the computation. The outcome is non-deterministic but one can perform large number of computations and deduce from the distribution of outcomes the results of computation.

What about conscious computations? Or more generally, conscious information processing. Could it proceed faster than computation in these sense of Turing? To answer this question one must first try to understand what conscious information processing might be. TGD inspired theory of consciousnesss provides one a possible answer to the question involving not only quantum physics but also new quantum physics.

1. In TGD framework Zero energy ontology (ZEO) replaces ordinary positive energy ontology and forces to generalize the theory of quantum measurement. This brings in several new elements. In particular, state function reductions can occur at both boundaries of causal diamond (CD), which is intersection of future and past direct light-cones and defines a geometric correlate for self. Selves for a fractal hierarchy - CDs within CDs and maybe also overlapping. Negentropy Maximization Principle (NMP) is the basic variational principle of consciousness and tells that the state function reductions generate maximum amount of conscious information. The notion of negentropic entanglement (NE) involving p-adic physics as physics of cognition and hierarchy of Planck constants assigned with dark matter are also central elements.

2. NMP allows a sequence of state function reductions to occur at given boundary of diamond-like CD - call it passive boundary. The state function reduction sequence leaving everything unchanged at the passive boundary of CD defines self as a generalized Zeno effect. Each step shifts the opposite - active - boundary of CD "upwards" and increases its distance from the passive boundary. Also the states at it change and one has the counterpart of unitary time evolution. The shifting of the active boundary gives rise to the experienced time flow and sensory input generating cognitive mental images - the "Maya" aspect of conscious experienced. Passive boundary corresponds to permanent unchanging "Self".

3. Eventually NMP forces the first reduction to the opposite boundary to occur. Self dies and reincarnates as a time reversed self. The opposite boundary of CD would be now shifting "downwards" and increasing CD size further. At the next reduction to opposite boundary re-incarnation of self in the geometric future of the original self would occur. This would be re-incarnation in the sense of Eastern philosophies. It would make sense to wonder whose incarnation in geometric past I might represent!

Could this allow to perform fast quantal computations by decomposing the computation to a sequence in which one proceeds in both directions of time? Could the incredible feats of some "human computers" rely on this quantum mechanism (see `http://tinyurl.com/hk5baty`). The indian mathematician Srinivasa Ramanujan (see `http://tinyurl.com/l42q7a2`) is the most well-known example of a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess.

Might it be possible in ZEO to perform quantally computations requiring classically non-polynomial time much faster - even in polynomial time? If this were the case, one might at least try to understand how Ramanujan did it although higher levels selves might be involved also (did his Goddess do the job?).

1. Quantal computation would correspond to a state function reduction sequence at fixed boundary of CD defining a mathematical mental image as sub-self. In the first reduction to the opposite boundary of CD sub-self representing mathematical mental image would die and

quantum computation would halt. A new computation at opposite boundary proceeding to opposite direction of geometric time would begin and define a time-reversed mathematical mental image. This sequence of reincarnations of sub-self as its time reversal could give rise to a sequence of quantum computation like processes taking less time than usually since one half of computations would take place at the opposite boundary to opposite time direction (the size of CD increases as the boundary shifts).

2. If the average computation time is same at both boundaries, the computation time would be only halved. Not very impressive. However, if the mental images at second boundary - call it A - are short-lived and the selves at opposite boundary B are very long-lived and represent very long computations, the process could be very fast from the point of view of A! Could one overcome the $P \neq NP$ constraint by performing computations during time-reversed re-incarnations?! Short living mental images at this boundary and very long-lived mental images at the opposite boundary - could this be the secret of Ramanujan?

3. Was the Goddess of Ramanujan - self at higher level of self-hierarchy - nothing but a time reversal for some mathematical mental image of Ramanujan (Brahman=Atman!), representing very long quantal computations! We have night-day cycle of personal consciousness and it could correspond to a sequence of re-incarnations at some level of our personal self-hierarchy. Ramanujan tells that he met his Goddess in dreams. Was his Goddess the time reversal of that part of Ramanujan, which was unconscious when Ramanujan slept? Intriguingly, Ramanujan was rather short-lived himself - he died at the age of 32! In fact, many geniuses have been rather short-lived.

4. Why the alter ego of Ramanujan was Goddess? Jung intuited that our psyche has two aspects: anima and animus. Do they quite universally correspond to self and its time reversal? Do our mental images have gender?! Could our self-hierarchy be a hierarchical collection of anima and animi so that gender would be something much deeper than biological sex! And what about Yin-Yang duality of Chinese philosophy and the ka as the shadow of persona in the mythology of ancient Egypt?

13.11.2 Symplectic QFT, $\{3,4,5\}$ as Additive Primes, and Arithmetic Consciousness

For years ago I proposed that symplectic QFT is an essential part of TGD [K16, K82]. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian Wilson line. I could not specify at that time how to select these polygons in the case of partonic 2-surfaces.

The recent proposal of Maldacena and Arkani-Hamed [B53] (see `http://tinyurl.com/ych26gcm`) that CMB might contain signature of inflationary cosmology as triangles and polygons for which the magnitude of n-point correlation function is enhanced led to a progress in this respect. In the proposal of Maldacena and Arkani-Hamed the polygons are defined by momentum conservation. Now the polygons would be fixed rather uniquely by requiring that fermions reside at their vertices and momentum conservation is not involved.

This inspires the idea about numbers $\{3,4,5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. Geometrically one could speak of prime polygons not decomposable to lower non-degenerate polygons. These polygons are different from those of Maldacena and Arkani-Hamed and would be associated many-fermion states with negentropic entanglement (NE) - this notion relates to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition. I have already earlier considered a possible counterpart for conscious prime factorization in the adelic framework [L35].

Basic ideas of TGD inspired theory of conscious very briefly

Negentropy Maximization Principle (NMP) is the variational principle of consciousness in TGD framework. It says that negentropy gain in state function reduction (quantum jump re-creating Universe) is maximal. State function reduction is basically quantum measurement in standard QM and sensory qualia (for instance) could be perhaps understood as quantum numbers of state resulting in state function reduction. NMP poses conditions on whether this reduction can occur. In standard ontology it would occur always when the state is entangled: reduction would destroy the entanglement and minimize entanglement entropy. When cognition is brought in, the situation changes.

The first challenge is to define what negentropic entanglement (NE) and negentropy could mean.

1. In real physics without cognition one does not have any definition of negentropy: on must define negentropy as reduction of entropy resulting as conscious entity gains information. This kind of definition is circular in consciousness theory.

2. In p-adic physics one can define number theoretic entanglement entropy with same basic properties as ordinary Shannon entropy. For some p-adic number fields this entropy can be negative and this motivates an interpretation as conscious information related to entanglement - rather to the ignorance of external observer about entangled state. The prerequisite is that the entanglement probabilities belong to an an extension of rationals inducing a finite-dimensional extension of rationals. Algebraic extensions are such extensions as also those generate by a root of e (e^p is p-adic number in Q_p).

A crucial step is to fuse together sensory and cognitive worlds as different aspects of existence.

1. One must replace real universe with adelic one so that one has real space-time surfaces and their p-adic variants for various primes p satisfying identical field equations. These are related by strong form of holography (SH) in which 2-D surfaces (string world sheets and partonic 2-surfaces) serve as "space-time genes" and obey equations which make sense both p-adically in real sense so that one can identify them as points of "world of classical worlds" (WCW).

2. One can say that these 2-surfaces belong to intersection of realities and p-adicities - intersection of sensory and cognitive. This demands that the parameters appearing in the equations for 2-surface belong algebraic extension of rational numbers: the interpretation is that this hierarchy of extensions corresponds to evolutionary hierarchy. This also explains imagination in terms of the p-adic space-time surfaces which are not so unique as the real one because of inherent non-determinism of p-adic differental equations. What can be imagined cannot be necessarily realized. You can continued p-adic 2-surface to 4-D surface but not to real one.

There is also second key assumption involved.

1. Hilbert space of quantum states is *same* for real and p-adic sectors of adelic world: for instance, tensor product would lead to total nonsense since there would be both real and p-adic fermions. This means same quantum state and same entanglement but seen from sensory and various cognitive perspectives. This is the basic idea of adelicity: the p-adic norms of rational number characterize the norm of rational number. Now various p-adic conscious experiences characterize the quantum state.

2. Real perspective sees entanglement always as entropic. For some finite number number of primes p p-adic entanglement is however negentropic. For instance, for entanglement probabilities $p_i = 1/N$, the primes appearing as factors of N are such information carrying primes. The presence of these primes can make the entanglement stable. The total entropy equal to the sum of negative real negentropy + various p-adic negentropies can be positive and cannot be reduced in the reduction so that reduction does not occur at all! Entanglement is stabilized by cognition and the randomness of state function reduction tamed: matter has power over matter!

3. There is analogy with the reductionism-holism dichotomy. Real number based view is reductionistic: information is obtained when the entangled state is split into un-entangled part. p-Adic number based view is holistic: information is inthe negentropic entanglement and can be seen as abstraction or rule. The superposition of state pairs represents a rule with state pairs (a_i, b_i) representing the instance of the rule A↔B. Maximal entanglement defined by entanglement probabilities $p_i = 1/N$ makes clear the profound distinction between these views. In real sector the negentropy is negative and smallest possible. In p-adic sector the negentropy is maximum for p-adic primes appearing as factors of N and total negentropy as their sum is large. NE allows to select unique state basis if the probabilities p_i are different.

   For $p_i = 1/N$ one can choose any unitary related state basis since unit matrix is invariant under unitary transformations. From the real point of view the ignorance is maximal and entanglement entropy is indeed maximal. For instance, in case of Schrödinger cat one could choose the cat's state basis to be any superposition of dead and alive cat and a state orthogonal to it. From p-adic view information is maximal. The reports of meditators, in particular Zen buddhists, support this interpretation. In "enlightened state" all discriminations disappear: it does not make sense to speak about dead or alive cat or anything between these two options. The state contains information about entire state - not about its parts. It is not information expressible using language relying on making of distinctions but silent wisdom.

How do polygons emerge in TGD framework?

The duality defined by strong form of holography (SH) has 2 sides. Space-time side (bulk) and boundary side (string world sheets and partonic 2-surfaces). 2-D half of SH would suggest a description based on string world sheets and partonic 2-surfaces. This description should be especially simple for the quantum states realized as spinor fields in WCW ("world of classical worlds"). The spinors (as opposed to spinor fields) are now fermionic Fock states assignable to space-time surface defining a point of WCW. TGD extends ordinary 2-D conformal invariance to super-symplectic symmetry applying at the boundary of light-cone: note that given boundary of causal diamond (CD) is contained by light-cone boundary.

1. The correlation functions at embedding space level for fundamental objects, which are fermions at partonic 2-surfaces could be calculated by applying super-symplectic invariance having conformal structure. I have made rather concrete proposals in this respect. For instance, I have suggested that the conformal weights for the generators of supersymplectic algebra are given by poles of fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ and thus include zeros of zeta scaled down by factor 1/2 [K28]. A related proposal is conformal confinement guaranteeing the reality of net conformal weights.

2. The conformally invariant correlation functions are those of super-symplectic CFT at light-cone boundary or its extension to CD. There would be the analog of conformal invariance associated with the light-like radial coordinate r_M and symplectic invariance associated with CP_2 and sphere S^2 localized with respect to r_M analogous to the complex coordinate in ordinary conformal invariance and naturally continued to hypercomplex coordinate at string world sheets carrying the fermionic modes and together with partonic 2-surfaces defining the boundary part of SH.

Symplectic invariants emerge in the following manner. Positive and negative energy parts of zero energy states would also depend on zero modes defined by super-symplectic invariants and this brings in polygons. Polygons emerge also from four-momentum conservation. These of course are also now present and involve the product of Lorentz group and color group assignable to CD near its either boundary. It seems that the extension of Poincare translations to Kac-Moody type symmetry allows to have full Poincare invariance (in its interior CD looks locally like $M^4 \times CP_2$).

1. One can define the symplectic invariants as magnetic fluxes associated with S^2 and CP_2 Kähler forms. For string world sheets one would obtain non-integrable phase factors. The vertices of polygons defined by string world sheets would correspond to the intersections of the string world sheets with partonic 2-surfaces at the boundaries of CD and at partonic 2-surfaces defining generalized vertices at which 3 light-like 3-surfaces meet along their ends.

2. Any polygon at partonic 2-surface would also allow to define such invariants. A physically natural assumption is that the vertices of these polygons are realized physically by adding fermions or antifermions at them. Kähler fluxes can be expressed in terms of non-integrable phase factors associated with the edges. This assumption would give the desired connection with quantum physics and fix highly uniquely but not completely the invariants appearing in physical states.

The correlated polygons would be thus naturally associated with fundamental fermions and a better analogy would be negentropically entangled n-fermion state rather than corresponding to maximum of the modulus of n-point correlation function. Hierarchy of Planck constants makes these states possible even in cosmological scales. The point would be that negentropic entanglement assignable to the p-adic sectors of WCW would be in key role.

Symplectic invariants and Abelian non-integrable phase factors

Consider now the polygons assignable to many-fermion states at partonic 2-surfaces.

1. The polygon associated with a given set of vertices defined by the position of fermions is far from unique and different polygons correspond to different physical situations. Certainly one must require that the geodesic polygon is not self-intersecting and defines a polygon or set of polygons.

2. Geometrically the polygon is not unique unless it is convex. For instance, one can take regular n-gon and add one vertex to its interior. The polygon can be also constructed in several ways. From this one obtains a non-convex $n+1$-gon in $n+1$ ways.

3. Given polygon is analogous with Hamiltonian cycle connecting all points of given graph. Now one does not have graph structure with edges and vertices unless one defines it by nearest neighbor property. Platonic solids provide an example of this kind of situation. Hamiltonian cycles [A17, A35] are key element in the TGD inspired model for music harmony leading also to a model of genetic code [K70] [L23].

4. One should somehow fix the edges of the polygon. For string world sheets the edges would be boundaries of string world sheet. For partonic 2-surfaces the simplest option is that the edges are geodesic lines and thus have shortest possible length. This would bring in metric so that the idea about TGD as almost topological QFT would be realized.

One can distinguish between two cases: single polygon or several polygons.

1. One has maximal entanglement between fundamental fermions, when the vertices define single polygon. One can however have several polygons for a given set of vertices and in this case the coherence is reduced. Minimal correlations correspond to maximal number of 3-gons and minimal number of 4-gons and 5-gons.

2. For large $h_{eff} = n \times h$ the partonic 2-surfaces can have macroscopic and even astrophysical size and one can consider assigning many-fermion states with them. For instance, anyonic states could be interpreted in this manner. In this case it would be natural to consider various decompositions of the state to polygons representing entangled fermions.

The definition of symplectic invariant depends on whether one has single polygon or several polygons.

1. In the case that there are several polygons not containing polygons inside them (if this the case,then the complement of polygon must satisfy the condition) one can uniquely identify the interior of each polygon and assign a flux with it. Non-integrable phase factor is well-defined now. If there is only single polygon then also the complement of polygon could define the flux. Polygon and its complement define fluxes Φ and $\Phi_{tot} - \Phi$.

2. If partonic 2-surface carries monopole Kähler charge Φ_{tot} is essentially $n\pi$, where n is magnetic monopole flux through the partonic 2-surface. This is half integer - not integer: this is key feature of TGD and forces the coupling of Kähler gauge potential to the spinors leading to the quantum number spectrum of standard model. The exponent can be equal to -1 for half-odd integer.

   This problem disappears if both throats of the wormhole contact connecting the space-time sheets with Minkowski signature give their contribution so that two minus-signs give one plus sign. Elementary particles necessarily consist of wormhole contacts through which monopole flux flows and runs along second space-time sheet to another contact and returns along second space-time sheet so that closed monopole flux tube is obtained. The function of the flux must be single valued. This demands that it must reduce to the cosine of the integer multiple of the flux and identifiable as as the real part of the integer power of magnetic flux through the polygon.

The number theoretically deepest point is geometrically completely trivial.

1. Only $n > 2$-gons are non-degenerate and 3-, 4- and 5-gons are prime polygons in the sense that they cannot be sliced to lower polygons. Already 6-gon decomposes to 2 triangles.

2. One can wonder whether the appearance of 3 prime polygons might relate to family replication phenomenon for which TGD suggests an explanation in terms of genus of the partonic 2-surface [K18]. This does not seem to be the case. There is however other three special integers: namely 0,1, and 2.

   The connection with family replication phenomenon could be following. When the number of handles at the parton surface exceeds 2, the system forms entangled/bound states describable in terms of polygons with handles at vertices. This would be kind of phase transition. Fundamental fermion families with handle number 0,1,2 would be analogous to integers 0,1,2 and the anyonic many-handle states with NE would be analogous to partitions of integers $n > 2$ represented by the prime polygons. They would correspond to the emergence of p-adic cognition. One could not assign NE and cognition with elementary particles but only to more complex objects such as anyonic states associated with large partonic 2-surfaces (perhaps large because they have large Planck constant $h_{eff} = n \times h$) [K64].

Integers $(3, 4, 5)$ as "additive primes" for integers $n \geq 3$: a connection with arithmetic consciousness

The above observations encourage a more detailed study of the decomposition of polygons to smaller polygons as a geometric representation for the partition of integers to a sum of smaller integers. The idea about integers $(3, 4, 5)$ as "additive primes" represented by prime polygons is especially attractive. This leads to a conjecture about NE associated with polygons as quantum correlates of arithmetic consciousness.

1. Motivations

The key idea is to look whether the notion of divisibility and primeness could have practical value in additive arithmetics. 1 is the only prime for addition in general case. $n = 1 + 1 + ...$ is analogous to p^n and all integers are "additive powers" of 1.

What happens if one considers integers $n \geq 3$? The basic motivation is that $n \geq 3$ is represented as a non-degenerate n-gon for $n \geq 3$. Therefore geometric representation of these primes is used in the following. One cannot split triangles from 4-gon and 5-gon. But already for 6-gon one can and obtains 2 triangles. Thus $\{3, 4, 5\}$ would be the additive primes for $n \geq 3$ represented as prime polygons.

The n-gons with $n \in \{3, 4, 5\}$ appear as faces of the Platonic solids! The inclusions of von Neumann algebras known as hyperfinite factors of type II_1 central in TGDs correspond to quantum phases $exp(\pi/n)$ $n = 3, 4, 5....$ Platonic solids correspond to particular finite subgroups of 3-D rotation group, which are in one-one correspondence with simply laced Lie-groups (ADE). There is also a direct connection with the classification of $\mathcal{N} = 2$ super-conformal theories, which seem to be relevant for TGD.

I cannot resist the temptation to mention also a personal reminiscence about a long lasting altered state of consciousness about 3 decades ago. I called it Great Experience and it boosted among other things serious work in order to understand consciousness in terms of quantum physics. One of the mathematical visions was that number 3 is in some sense fundamental for physics and mathematics. I also precognized infinite primes and much later indeed discovered them. I have repeatedly returned to the precognition about number 3 but found no really convincing reason for its unique role although it pops up again and again in physics and mathematics: 3 particle families, 3 colors for quarks, 3 spatial dimensions, 3 quaternionic imaginaryunits, triality for octonions, to say nothing about the role of trinity in mystics and religions. The following provides the first argument for the special role of number 3 that I can take seriously.

2. Partition of integer to additive primes

The problem is to find a partition of an integer to additive primes $3, 4, 5$. The problem can be solved using a representation in terms of $n > 2$-gons as a geometrical visualization. Some general aspects of the representation.

1. The detailed shape of n-gons in the geometric representation of partitions does not matter: they just represent geometrically a partition of integer to a sum. The partition can be regarded as a dynamical process. n-gons splits to smaller n-gons producing a representation for a partition $n = \sum_i n_i$. What this means is easiest to grasp by imagining how polygon can be decomposed to smaller ones. Interestingly, the decompositions of polytopes to smaller ones - triangulations - appear also in Grassmannian twistor approach to $\mathcal{N} = 4$ super Yang Mills theory.

2. For a given partition the decomposition to n-gons is not unique. For instance, integer 12 can be represented by 3 4-gons or 4 3-gons. Integers $n \in \{3, 4, 5\}$ are special and partitions to these n-gons are in some sense maximal leading to a maximal decoherence as quantum physicist might say.

   The partitions are not unique and there is large number of partitions involving 3-gons,4-gons, 5-gons. The reason is that one can split from n-gons any n_1-gon with $n_1 < n$ except for $n = 3, 4, 5$.

3. The daydream of non-mathematician not knowing that everything has been very probably done for aeons ago is that one could chose n_1 to be indivisible by 4 and 5, n_2 indivisible by 3 and 5 and n_3 indivisible by 3 and 4 so that one might even hope for having a unique partition. For instance, double modding by 4 and 5 would reduce to double modding of $n_1 \times 3$ giving a non-vanishing result, and one might hope that n_1, n_2 and n_3 could be determined from the double modded values of n_i uniquely. Note that for $n_i \in \{1, 2\}$ the number $n = 24 = 2 \times 3 + 2 \times 4 + 2 \times 5$ playing key role in string model related mathematics is the largest integer having this kind of representation. One should numerically check whether any general orbit characterized by the above formulas contains a point satisfying the additional number theoretic conditions.

   Therefore the task is to find partitions satisfying these indivisibility conditions. It is however reasonable to consider first general partitions.

4. By linearity the task of finding general partitions (forgetting divisibility conditions) is analogous to that of finding of solutions of non-homogenous linear equations. Suppose that one has found a partition

$$n = n_1 \times 3 + n_2 \times 4 + n_3 \times 5 \leftrightarrow (n_1, n_2, n_3) \ . \tag{13.11.1}$$

   This serves as the analog for the special solution of non-homogenous equation. One obtains a general solutions of equation as the sum $(n_1 + k_1, n_2 + k_1, n_3 + k_3)$ of the special solution and general solution of homogenous equation

$$k_1 \times 3 + k_2 \times 4 + k_3 \times 5 = 0 \ . \tag{13.11.2}$$

This is equation of plane in N^3 - 3-D integer lattice.

Using $4 = 3 + 1$ and $5 = 3 + 2$ this gives equations

$$k_2 + 2 \times k_3 = 3 \times m \ , \quad k_1 - k_3 + 4 \times m = 0 \ , \quad m = 0, 1, 2, ... \tag{13.11.3}$$

5. There is periodicity of $3 \times 4 \times 5 = 60$. If (k_1, k_2, k_3, m) is allowed deformation, one obtains a new one with same divisibility properties as the original one as $(k_1 + 60, k_2 - 120, k_3 + 60, m)$. If one does not require divisibility properties for all solutions, one obtains much larger set of solutions. For instance $(k_1, k_2, k_3) = m \times (1, -2, 1)$ defines a line in the plane containing the solutions. Also other elementary moves than (1,-2,1) are possible.

One can identify very simple partitions deserving to be called standard partitions and involve mostly triangles and minimal number of 4- and 5-gons. The physical interpretation is that the coherence is minimal for them since mostly the quantum coherent negentropically entangled units are minimal triangles.

1. One starts from n vertices and constructs n-gon. For number theoretic purposes the shape does not matter and the polygon can be chosen to be convex. One slices from it 3-gons one by one so that eventually one is left with $k \equiv n \ mod \ 3 == 0, 1$ or 2 vertices. For $k = 0$ no further operations are needed. For $k = 1$ *resp.* $k = 2$ one combines one of the triangles and edge associated with 1 *resp.* 2 vertices to 4-gon *resp.* 5-gon and is done. The outcome is one of the partitions

$$n = n_1 \times 3 \ , \quad n = n_1 \times 3 + 4, n = n_1 \times 3 + 5 \tag{13.11.4}$$

   These partitions are very simple, and one can easily calculate similar partitions for products and powers. It is easy to write a computer program for the products and powers of integers in terms of these partitions.

2. There is however a uniqueness problem. If n_1 is divisible by 4 or 5 - $n_1 = 4 \times m_1$ or $n_1 = 5 \times m_1$ - one can interpret $n_1 \times 3$ as a collection of m_1 4-gons or 5-gons. Thus the geometric representation of the partition is not unique. Similar uniqueness condition must apply to n_2 and n_3 and is trivially true in above partitions.

   To overcome this problem one can pose a further requirement. If one wants n_1 to be indivisible by 4 and 5 one can transform 2 or or 4 triangles and existing 4-gon or 5-gon or 3 or 6 triangles to 4-gons and 5-gons.

   (a) Suppose $n = n_1 \times 3 + 4$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 2$ is not and 4-gon and 2 3-gons can be transformed to 2 5-gons: $(n_1, 1, 0) \rightarrow (n_1 - 2, 0, 2)$. If $n_1 - 2$ is divisible by 5, $n_1 - 3$ is not divisible by either 4 or 5 and 3 triangles can be transformed to 4-gon and 5-gon: $(n_1, 1, 0) \rightarrow (n_1 - 3, 2, 1)$.

   (b) Suppose $n = n_1 \times 3 + 5$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 1$ is not and triangle and 5-gon can be transformed to 2 4-gons: $(n_1, 0, 1) \rightarrow (n_1 - 1, 2, 0)$. If $n_1 - 1$ is divisible by 4 or 5, $n_1 - 3$ is not and 3 triangles and 5-gon can be transformed to 2 5-gons and 4-gon: $(n_1, 0, 1) \rightarrow (n_1 - 3, 1, 2)$.

   (c) For $n = n_1 \times 3$ divisible by 4 or 5 or both one can remove only $m \times 3$ triangles, $m \in \{1, 2\}$ since only in these case the resulting $m \times 3$ (9 or 18) vertices can partitioned to a union of 4-gon and 5-gon or of 2 4-gons and 2 5-gons: $(n_1, 0, 0) \rightarrow (n_1 - 3, 1, 1)$ or $(n_1, 0, 0) \rightarrow (n_1 - 6, 2, 2)$.

These transformations seem to be the minimal transformations allowing to achieve indivisibility by starting from the partition with maximum number of triangles and minimal coherence.

Some further remarks about the partitions satisfying the divisibility conditions are in order.

1. The multiplication of n with partition (n_1, n_2, n_3) satisfying indivisibility conditions by an integer m not divisible by $k \in \{3, 4, 5\}$ gives integer with partition $m \times (n_1, n_2, n_3)$. Note also that if n is not divisible by $k \in \{3, 4, 5\}$ the powers of n, n^k has partition $n^{k-1} \times (n_1, n_2, n_3)$ and this could help to solve Diophantine equations.

2. Concerning the uniqueness of the partition satisfying the indivisibility conditions, the answer is negative. $8 = 3 + 5 = 4 + 4$ is the simplest counter example. Also the m-multiples of 8 such that m is indivisible by 2,3,4,5 serve as counter examples. 60-periodicity implies that for sufficiently large values of n the indivisibility conditions do not fix the partition uniquely. (n_1, n_2, n_3) can be replaced with $(n_1 + 60 + n_2 - 120, n_3 + 60)$ without affecting divisibility properties.

3. Intriguing observations related to 60-periodicity

60-periodicity seems to have deep connections with both music consciousness and genetic code if the TGD inspired model of genetic code is taken seriously code [K70] [L23].

1. The TGD inspired model for musical harmony and genetic involves icosahedron with 20 triangular faces and tetrahedron with 4 triangular faces. The 12 vertices of icosahedron correspond to the 12 notes. The model leads to the number 60. One can say that there are 60 +4 DNA codons and each 20 codon group is 60=20+20+20 corresponds to a subset of aminoacids and 20 DNAs assignable to the triangles of icosahedron and representing also 3-chords of the associated harmony. The remaining 4 DNAs are associated with tetrahedron.

   Geometrically the identification of harmonies is reduced to the construction of Hamiltonian cycles - closed isometrically non-equivalent non-self-intersecting paths at icosahedron going through all 12 vertices. The symmetries of the Hamiltonian cycles defined by subgroups of the icosahedral isometry group provide a classification of harmonies and suggest that also genetic code carries additional information assignable to what I call bio-harmony perhaps related to the expression of emotions - even at the level of biomolecules - in terms of "music" defined as sequences 3-chords realized in terms of triplets of dark photons (or notes) in 1-1 correspondence with DNA codons in given harmony.

2. Also the structure of time units and angle units involves number 60. Hour consists of 60 minutes, which consists of 60 seconds. Could this accident somehow reflect fundamental aspects of cognition? Could we be performing sub-conscious additive arithmetics using partitions of n-gons? Could it be possible to "see" the partitions if they correspond to NE?

4. Could additive primes be useful in Diophantine mathematics?

The natural question is whether it could be number theoretically practical to use "additive primes" $\{3, 4, 5\}$ in the construction of natural numbers $n \geq 3$ rather than number 1 and successor axiom. This might even provide a practical tool for solving Diophantine equations (it might well be that mathematicians have long ago discovered the additive primes).

The most famous Diophantine equation is $x^n + y^n = z^n$ and Fermat's theorem - proved by Wiles - states that for $n > 2$ it has no solutions. Non-mathematician can naïvely ask whether the proposed partition to additive primes could provide an elementary proof for Fermat's theorem and continue to test the patience of a real mathematician by wondering whether the parition for a sum of powers $n > 2$ could be always different from that for single power $n > 2$ perhaps because of some other constraints on the integers involved?

5. Could one identify quantum physical correlates for arithmetic consciousness?

Even animals and idiot savants can do arithmetics. How this is possible? Could one imagine physical correlates for arithmetic consciousness for which product and addition are the fundamental aspects? Is elementary arithmetic cognition universal and analogous to direct sensory experience. Could it reduce at quantum level to a kind of quantum measurement process quite generally giving rise to mental images as outcomes of quantum measurement by repeated state function reduction lasting as long as the corresponding sub-self (mental image) lives?

Consider a partition of integer to a product of primes first. I have proposed a general model for how partition of integer to primes could be experienced directly [L35]. For negentropically entangled state with maximal possible negentropy having entanglement probabilities $p_i = 1/N$, the negentropic primes are factors of N and they could be directly "seen" as negentropic p-adic factors in the adelic decomposition (reals and extensions of various p-adic number fields defined by extension of rationals defined the factors of adele and space-time surfaces as preferred extremals of Kähler action decompose to real and p-adic sectors).

What about additive arithmetics?

1. The physical motivation for n-gons is provided symplectic QFT [K16, K82], which is one aspect of TGD forced by super symplectic conformal invariance having structure of conformal symmetry. Symplectic QFT would be analogous to conformal QFT. The key challenge is to identify symplectic invariants on which the positive and negative energy parts of zero energy states can depend. The magnetic flux through a given area of 2-surface is key invariant of this kind. String world sheet and partonic 2-surfaces are possible identifications for the surface containing the polygon.

   Both the Kähler form associated with the light-cone boundary, which is metrically sphere with constant radius r_M (defining light-like radial coordinate) and the induced Kähler form of CP_2 define these kind of fluxes.

2. One can assign fluxes with string world sheets. In this case one has analog of magnetic flux but over a surface with metric signature (1,-1). Fluxes can be also assigned as magnetic fluxes with partonic 2-surfaces at which fundamental fermions can be said to reside. n fermions defining the vertices at partonic 2-surface define naturally an n-gon or several of them. The interpretation would be as Abelian Wilson loop or equvalently non-integrable phase factor.

3. The polygons are not completely unique but this reflect the possibility of several physical states. n-gon could correspond to NE. The imaginary exponent of Kähler magnetic flux Φ through n-gon is symplectic invariant defining a non-integrable phase factor and defines a multiplicative factor of wave function. When the state decomposes to several polygons, one can uniquely identify the interior of the polygon and thus also the non-integrable phase factor.

   There is however non-uniqueness, when one has only single n-gon since also the complement of n-gon at partonic 2-surface containing now now polygons defines n-gon and the corresponding flux is $\Phi_{tot} - \Phi$. The flux Φ_{tot} is quantized and equal to the integer valued magnetic charge times 2π. The total flux disappears in the imaginary exponent and the non-integrable phase factor for the complementary polygon reduces to complex conjugate of that for polygon. Uniqueness allows only the cosine for an integer multiple of the flux.

   The non-integrable phase factor assignable to fermionic polygon would give rise to a correlation between fermions in zero modes invariant under symplectic group. The correlations defined by the n-gons at partonic 2-surfaces would be analogous to that in momentum space implied by the momentum conservation forcing the momenta to form a closed polygon but having totally different origin.

Could it be that the wave functions representing collections of n-gons representing partition of integer to a sum could be experienced directly by people capable of perplexing mathematical feats. The partition to a sum would correspond to a geometric partition of polygon representing partition of positive integer $n \geq 3$ to a sum of integers. Quantum physically it would correspond to NE as a representation of integer.

This might explain number theoretic miracles related to addition of integers in terms of direct "seeing". The arithmetic feats could be dynamical quantum processes in which polygons would decompose to smaller polygons, which would be directly "seen". This would require at least two representations: the original polygon and the decomposed polygon resulting in the state function reduction to the opposite boundary of CD. An ensemble of arithmetic sub-selves would seem to be needed. NMP does not seem to favour this kind of partition since negentropy is reduced but if its time reversal occurs in geometric time direction opposite to that of self it might look like partition for the self having sub-self as mental image.

13.12 p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework

In TGD framework p-adicization and adelization are carried out at all levels of geometry: embedding space, space-time and WCW. Adelization at the level of state spaces requires that it is common from all sectors of the adele and has as coefficient field an extension of rationals allowing both real and p-adic interpretations: the sectors of adele give only different views about the same quantum state.

In the sequel the recent view about the p-adic variants of embedding space, space-time and WCW is discussed. The notion of finite measurement resolution reducing to number theoretic existence in p-adic sense is the fundamental notion. p-Adic geometries replace discrete points of discretization with p-adic analogs of monads of Leibniz making possible to construct differential calculus and formulate p-adic variants of field equations allowing to construct p-adic cognitive representations for real space-time surfaces.

This leads to a beautiful construction for the hierarchy of p-adic variants of embedding space inducing in turn the construction of p-adic variants of space-time surfaces. Number theoretical existence reduces to conditions demanding that all ordinary (hyperbolic) phases assignable to (hyperbolic) angles are expressible in terms of roots of unity (roots of e).

For $SU(2)$ one obtains as a special case Platonic solids and regular polygons as preferred p-adic geometries assignable also to the inclusions of hyperfinite factors [K95, K30]. Platonic solids represent idealized geometric objects ofthe p-adic world serving as a correlate for cognition as contrast to the geometric objects of the sensory world relying on real continuum.

In the case of causal diamonds (CDs) - the construction leads to the discrete variants of Lorentz group $SO(1,3)$ and hyperbolic spaces $SO(1,3)/SO(3)$. The construction gives not only the p-adicizable discrete subgroups of $SU(2)$ and $SU(3)$ but applies iteratively for all classical Lie groups meaning that the counterparts of Platonic solids are countered also for their p-adic coset spaces. Even the p-adic variants of WCW might be constructed if the general recipe for the construction of finite-dimensional symplectic groups applies also to the symplectic group assignable to $\Delta CD \times CP_2$.

The emergence of Platonic solids is very remarkable also from the point of view of TGD inspired theory of consciousness and quantum biology. For a couple of years ago I developed a model of music harmony [K70] [L23] relying on the geometries of icosahedron and tetrahedron. The basic observation is that 12-note scale can be represented as a closed curve connecting nearest number points (Hamiltonian cycle) at icosahedron going through all 12 vertices without self intersections. Icosahedron has also 20 triangles as faces. The idea is that the faces represent 3-chords for a given harmony characterized by Hamiltonian cycle. Also the interpretation terms of 20 amino-acids identifiable and genetic code with 3-chords identifiable as DNA codons consisting of three letters is highly suggestive.

One ends up with a model of music harmony predicting correctly the numbers of DNA codons coding for a given amino-acid. This however requires the inclusion of also tetrahedron. Why icosahedron should relate to music experience and genetic code? Icosahedral geometry and its dodecahedral dual as well as tetrahedral geometry appear frequently in molecular biology but its appearance as a preferred p-adic geometry is what provides an intuitive justification for the model of genetic code. Music experience involves both emotion and cognition. Musical notes could code for the points of p-adic geometries of the cognitive world. The model of harmony in fact generalizes. One can assign Hamiltonian cycles to any graph in any dimension and assign chords and harmonies with them. Hence one can ask whether music experience could be a form of p-adic geometric cognition in much more general sense.

The geometries of biomolecules brings strongly in mind the geometry p-adic space-time sheets. p-Adic space-time sheets can be regarded as collections of p-adic monad like objects at algebraic space-time points common to real and p-adic space-time sheets. Monad corresponds to p-adic units with norm smaller than unit. The collections of algebraic points defining the positions of monads and also intersections with real space-time sheets are highly symmetric and determined by the discrete p-adicizable subgroups of Lorentz group and color group. When the subgroup of the rotation group is finite one obtains polygons and Platonic solids. Bio-molecules typically consists of this kind of structures - such as regular hexagons and pentagons - and could be seen as cognitive representations of these geometries often called sacred! I have proposed this idea long time ago and the discovery of the recipe for the construction of p-adic geometries gave a justification for this idea.

13.12.1 p-Adic variants of causal diamonds

To construct p-adic variants of space-time surfaces one must construct p-adic variants of the embedding space. The assumption that the p-adic geometry for the embedding space induces p-adic geometry for sub-manifolds implies a huge simplification in the definition of p-adic variants of preferred extremals. The natural guess is that real and p-adic space-time surfaces gave algebraic points as common: so that the first challenge is to pick the algebraic points of the real space-time surface. To define p-adic space-time surface one needs field equations and the notion of p-adic continuum and by assigning to each algebraic point a p-adic continuum to make it monad, one can solve p-adic field equations inside these monads.

The idea of finite measurement resolution suggests that the solutions of p-adic field equations inside monads are arbitrary. Whether this is consistent with the idea that same solutions of field equations can be interpreted either p-adically or in real sense is not quite clear. This would be guaranteed if the p-adic solution has same formal representation as the real solution in the vicinity of given discrete point - say in terms of polynomials with rational coefficients and coordinate variables which vanish for the algebraic point.

Real and p-adic space-time surfaces would intersect at points common to all number fields for given adele: cognition and sensory worlds intersect not only at the level of WCW but also at the level of space-time. I had already considered giving up the latter assumption but it seems to be necessary at least for string world sheets and partonic 2-surfaces if not for entire space-time surfaces.

General recipe

The recipe would be following.

1. One starts from a discrete variant of $CD \times CP_2$ defined by an appropriate discrete symmetry groups and their subgroups using coset space construction. This discretization consists of points in finite-dimensional extension of p-adics induced by an extension of rationals. These points are assumed to be in the intersection of reality and p-adicities at space-time level - that is common for real and p-adic space-time surfaces. Cognitive representations in the real world are thus discrete and induced by the intersection. This is the original idea which I was ready to give up as the vision about discretization at WCW level allowing to solve all problems related to symmetries emerged. At space-time level the p-adic discretization reduces symmetry groups to their discrete subgroups: cognitive representations unavoidably break the symmetries. What is important the distance between discrete p-adic points labelling monads is naturally their real distance. This fixes metrically real-p-adic/sensory-cognitive correspondence.

2. One replaces each point of this discrete variant $CD \times CP_2$ with p-adic continuum defined by an algebraic extension of p-adics for the adele considered so that differentiation and therefore also p-adic field equations make sense. The continuum for given discrete point of $CD_d \times CP_{2,d}$ defines kind of Leibnizian monad representing field equations p-adically. The solution decomposes to p-adically differentiable pieces and the global solution of field equations makes sense since it can be interpreted in terms of pseudo-constants. p-Adicization means discretization but with discrete points replaced with p-adic monads preserving also the information about local behavior. The loss of well-ordering inside p-adic monad reflects its loss due to the finiteness of measurement resolution.

3. The distances between monads correspond to their distances for real variant of $CD \times CP_2$. Are there natural restrictions on the p-adic sizes of monads? Since p-adic units are in question that size in suitable units is $p^{-N} < 1$. It would look natural that the p-adic size of the is smaller than the distance to the nearest monad. The denser the discretization is, the larger the value of N would be. The size of the monad decreases at least like $1/p$ and for large primes assignable to elementary particles ($M_{127} = 2^{127} - 1$) is rather small. The discretizations of the subgroups share the properties of the group invariant geometry of groups so that they are to form a regular lattice like structure with constant distance to nearest neighbors. At the embedding level therefore p-adic geometries are extremely symmetric. At the level of space-time geometries only a subset of algebraic points is picked and the symmetry tends to be lost.

CD degrees of freedom

Consider first CD degrees of freedom.

1. For M^4 one has 4 linear coordinates. Should one p-adicize these or should one discretize CDs defined as intersections of future and past directed light-cones and strongly suggested by ZEO. CD seems to represent the more natural option. The construction of a given CD suggests that one should replace the usual representation of manifold as a union of overlapping regions with intersection of two light-cones with coordinates related in the intersection as in the case of ordinary manifold: $\cup \rightarrow \cap$.

2. For a given light-cone one must introduce light-cone proper time a, hyperbolic angle η and two angle coordinates (θ, ϕ). Light-cone proper time a is Lorentz invariant and corresponds naturally to an ordinary p-adic number of more generally to a p-adic number in algebraic extension which does not involve phases.

   The two angle coordinates (θ, ϕ) parameterizing S^2 can be represented in terms of phases and discretized. The hyperbolic coordinate can be also discretized since e^p exists p-adically, and one obtains a finite-dimensional extension of p-adic numbers by adding roots of e and its powers. e is completely exceptional in that it is p-adically an algebraic number.

3. This procedure gives a discretization in angle coordinates. By replacing each discrete value of angle by p-adic continuum one obtains also now the monad structure. The replacement with continuum means the replacement

$$U_{m,n} \equiv exp(i2\pi m/n) \rightarrow U_{m,n} \times exp(i\phi) \quad , \tag{13.12.1}$$

   where ϕ is p-adic number with norm $p^{-N} < 1$ It can also belong to an algebraic extension of p-adic numbers. Building the monad is like replacing in finite measurement the representative point of measurement resolution interval with the entire interval. By finite measurement resolution one cannot fix the order inside the interval. Note that one obtains a hierarchy of subgroups depending on the upper bound p^{-n} for the modulus. For $p \ mod \ 4 = 1$ imaginary unit exist as ordinary p-adic number and for $p \ mod \ 4 = 3$ in an extension including $\sqrt{-1}$.

4. For the hyperbolic angle one has

$$E_{m,n} \equiv exp(m/n) \rightarrow E_{m,n} \times exp(\eta) \tag{13.12.2}$$

   with the ordinary p-adic number η having norm $p^{-N} < 1$. Lorentz symmetry is broken to a discrete subgroup: this could be interpreted in terms of finite cognitive resolution. Since e^p is p-adic number also hyperbolic angle has finite number of values and one has compactness in well-defined sense although in real context one has non-compactness.

In cosmology this discretization means quantization of redshift and thus recession velocities. A concise manner to express the discretization to say that the cosmic time constant hyperboloids are discrete variants of Lobatchevski spaces $SO(3,1)/SO(3)$. The spaces appear naturally in TGD inspired cosmology.

5. The coordinate transformation relating the coordinates in the two intersecting coordinate patches maps hyperbolic and ordinary phases to each other as such. Light-cone proper time coordinates are related in more complex manner. $a_+^2 = t^2 - r^2$ and $a_-^2 = (t-T)^2 - r^2$ are related by $a_+^2 - a_-^2 = 2tT - T^2 = 2a_+cosh(\eta)T - T^2$.

   This leads to a problem unless one allows a_+ and a_- to belong to an algebraic extension containing the roots of e making possible to define hyperbolic angle. The coordinates $a_\pm$ can also belong to a larger extension of p-adic numbers. The expectation is that one obtains an infinite hierarchy of algebraic extensions of rationals involving besides the phases also other non-Abelian extension parameters. It would seem that the Abelian extension for phases and the extension for a must factorize somehow. Note also that the expression of a_+ in terms of a_- given by

$$a_+ = -cosh(\eta)T \pm \sqrt{sinh^2(\eta)T^2 + a_-^2} \quad . \tag{13.12.3}$$

   This expression makes sense p-adically for all values of a_- if one can expand the square root as a converning power series with respect to a_-. This is true if $a_-/sinh(\eta)T$ has p-adic norm smaller than 1.

6. What about the boundary of CD which corresponds to a coordinate singularity? It seems that this must be treated separately. The boundary has topology $S^2 \times R_+$ and S^2 can be p-adicized as already explained. The light-like radial coordinate $r = asinh(\eta)$ vanishes identically for finite values of $sinh(\eta)$. Should one regard r as ordinary p-adic number? Or should one think that entire light-one boundary corresponds to single point $r = 0$? The discretization of r is in powers of a roots of e is very natural so that each power $E_{m,n}$ corresponds to a p-adic monad. If now powers $E_{m,n}$ are involved, one obtains just the monad at $r = 0$.

   The construction of quantum TGD leads to the introduction of powers $exp(log(r/r_0)s)$, where s is zero of Riemann Zeta [K28]. These make sense p-adically if $u = log(r/r_0)$ has p-adic norm smaller than unity and s makes sense p-adically. The latter condition demanding that the zeros are algebraic numbers is quite strong.

13.12.2 Construction for $SU(2)$, $SU(3)$, and classical Lie groups

In the following the detailed construction for $SU(2)$, $SU(3)$, and classical Lie groups will be sketched.

Subgroups of $SU(2)$ having p-adic counterparts

In the case $U(1)$ the subgroups defined by roots of unity reduce to a finite group Z_n. What can one say about p-adicizable discrete subgroups of $SU(2)$?

1. To see what happens in the case of $SU(2)$ one can write $SU(2)$ element explicitly in quaternionic matrix representation

$$(\theta, n) \equiv cos(\theta)Id + sin(\theta)\sum_i n_i I_i \quad . \tag{13.12.4}$$

   Here Id is quaternionic real unit and I_i are quaternionic imaginary units. $n = (n_1, n_2, n_2)$ is a unit vector representable as $(cos(\phi), sin(\phi)cos(\psi), sin(\phi)sin(\psi))$. This representation exists p-adically if the phases $exp(i\theta)$, $exp(i(\phi)$ and $exp(i\psi)$ exist p-adically so that they must be roots of unity.

The geometric interpretation is that n defines the direction of rotation axis and θ defines the rotation angle.

2. This representation is not the most general one in p-adic context. Suppose that one has two elements of this kind characterized by (θ_i, n_i)such that the rotation axes are different. From the multiplication table of quaternions one has for the product (θ_{12}, n_{12}) of these

$$cos(\theta_{12}) = cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)n_1 \cdot n_2 \ . \quad (13.12.5)$$

This makes sense p-adically if the inner product $cos(\chi) \equiv n_1 \cdot n_2$ corresponds to root of unity in the extension of rationals used. Therefore the angle between the rotation axes is number theoretically quantized in order that p-adicization works.

One can solve θ_{12} from the above equation in real context but in the general case it does not correspond to $U_{m,n}$. This is not however a problem from p-adic point of view. The reduction to a root of unity is true only in some special cases. For $n_1 = n_2$ the group generated by the products reduces a discrete $Z_n \subset U(1)$ generated by a root of unity. If n_1 and n_2 are orthogonal the angle between rotation axes corresponds trivially to a root of unity. In this case one has the isometries of cube. For other Platonic solids the angles between rotation axes associated with various $U(1)$ subgroups generating the entire sub-group are fixed by their geometries. The rotation angles correspond to $n = 3$ for tetrahedron and icosahedron and $n = 5$ dodecahedron and for $n = 3$. There is also duality between cube and octahedron and icosahedron and dodecahedron.

3. Platonic solids can be geometrically seen as discretized variants of $SU(2)$ and it seems that they correspond to finite discrete subgroups of $SU(2)$ defining $SU(2)_d$. Platonic sub-groups appear in the hierarchy of Jones inclusions. The other finite subgroups of $SU(2)$ appearing in this hierarchy act on polygons of plane and being generated by Z_n and rotations around the axes of plane and would naturally correspond to discrete U(1) sub-groups of $SU(2)$ and in a well-defined sense to a degenerate situation. By Mc-Kay correspondence all these groups correspond to ADE type Lie groups. These subgroups define finite discretizations of $SU(2)$ and S^2. p-Adicization would lead directly to the hierarchy of inclusions assigned also with the hierarchy of sub-algebras of super-symplectic algebra characterized by the hierarchy of Planck constants.

4. There are also p-adicizable discrete subgroups, which are infinite. By taking two rotations with angles which correspond to root of unity with rotation axes, whose mutual angle corresponds to root of unity one can generate an infinite discrete subgroup of $SU(2)$ existing in p-adic sense. More general discrete U(1) subgroups are obtained by taking n rotation axes with mutual angles corresponding to roots of unity and generating the subgroup from these. In case of Platonic solids this gives a finite subgroup.

Construction of p-adicizable discrete subgroups of CP_2

The construction of p-adic CP_2 proceeds along similar lines.

1. In the original ultra-naïve approach the local p-adic metric of CP_2 is obtained by a purely formal replacement of the ordinary metric of CP_2 with its p-adic counterpart and it defines the CP_2 contribution to induced metric. This makes sense since Kähler function is rational function and components of CP_2 metric and spinor connection are rational functions. This allows to formulate p-adic variants of field equations. This description is however only local. It says nothing about global aspects of CP_2 related to the introduction of algebraic extension of p-adic numbers.

One should be able to realize the angle coordinates of CP_2 in a physically acceptable manner. The coordinates of CP_2 can be expressed by compactness in terms of trigonometric functions, which suggests a realization of them as phases for the roots of unity. The number of points depends on the Abelian extension of rationals inducing that of p-adics which is chosen. This gives however only discrete version of p-adic CP_2 serving as a kind of spine. Also the flesh replacing points with monads is needed.

2. A more profound approach constructs the algebraic variants of CP_2 as discrete versions of the coset space $CP_2 = SU(3)/U(2)$. One restricts the consideration to an algebraic subgroup of $SU(3)_d$ with elements, which are 3×3 matrices with components, which are algebraic numbers in the extension of rationals. Since they are expressible in terms of phases one can express them in terms of roots of unity. In the same manner one identifies $U(2)_d \subset SU(3)_d$. $CP_{2,d}$ is the coset space $SU(3)_D/U(2)_d$ of these. The representative of a given coset is a point in the coset and expressible in terms of roots of unity.

3. The construction of the p-adicizable subgroups of $SU(3)$ suggests a generalization. Since $SU(3)$ is 8-D and Cartan algebra is 2-D the coset space is 6-dimensional flag-manifold $F = SU(3)/U(1) \times U(1)$ with coset consisting of elements related by automorphism $g \equiv hgh^{-1}$. F defines the twistor space of CP_2 characerizing the choices for the quantization axes of color quantum numbers. The points of F should be expressible in terms of phase angles analogous to the angle defining rotation axis in the case of SU(2).

   In the case of $SU(2)$ n U(1) subgroups with specified rotation axes with p-adically existing mutual angles are considered. The construction as such generates only $SU(2)_d$ subgroup which can be trivially extended to $U(2)_d$. The challenge is to proceed further.

   Cartan decomposition of the Lie algebra (see `http://tinyurl.com/y7cjbm4c`) seems to provide a solution to the problem. In the case of $SU(3)$ it corresponds to the decomposition to U(2) sub-algebra and its complement. One could use the decomposition $G = KAK$ where K is maximal compact subgroup. A is exponentiation of the maximal Abelian subalgebra, which is 3-dimensional for CP_2. By Abelianity the p-adicization of A in terms of roots of unity simple. The image of A in G/K is totally geodesic sub-manifold. In the recent case one has $G/Ki = CP_2$ so that the image of A is geodesic sphere S^2. This decomposition implies the representation using roots of unity. The construction of discrete p-adicizable subgroups of $SU(n)$ for $n > 3$ would continue iteratively.

4. Since the construction starts from $SU(2)$, $U(1)$, and Abelian groups, and proceeds iteratively it seems that Platonic solids have counterparts for all classical Lie groups containing $SU(2)$. Also level p-adicizable discrete coset spaces have analogous of Platonic solids.

The results imply that $CD \times CP_2$ is replaced by a discrete set of p-adic monads at a given level of hierarchy corresponding to the finite cognitive resolution.

Generalization to other groups

The above argument demonstrates that p-adicization works iteratively for $SU(n)$ and thus for $U(n)$. For finite-dimensional symplectic group $Sp(n, R)$ the maximal compact sub-group is $U(n)$ so that KAK construction should work also now. $SO(n)$ can be regarded as subgroup of $SU(n)$ so that the p-adiced discretrized variants of maximal compact subgroups should be constructible and KAK give the groups. The inspection of the table of the Wikipedia article (see `http://tinyurl.com/j44639q`) encourages the conjecture that the construction of $SU(n)$ and $U(n)$ generalizes to all classical Lie groups.

This construction could simplify enormously also the p-adicization of WCW and the theory would discretize even in non-compact degrees of freedom. The non-zero modes of WCW correspond to the symplectic group for $\delta M^4 \times CP_2$, and one might hope that the p-adicization works also at the limit of infinite-dimensional symplectic group with $U(\infty)$ taking the role of K.

13.13 Some layman considerations related to the fundamentals of mathematics

I am not a mathematician and therefore should refrain from consideration of anything related to fundamentals of mathematics. In the discussions with Santeri Satama I could not avoid the temptation to break this rule. I however feel that I must confess my sins and in the following I will do this.

1. Gödel's problematics is shown to have a topological analog in real topology, which however disappears in p-adic topology which raises the question whether the replacement of the arithmetics of natural numbers with that of p-adic integers could allow to avoid Gödel's problematics.

2. Number theory looks from the point of view of TGD more fundamental than set theory and inspires the question whether the notion of algebraic number could emerge naturally from TGD. There are two ways to understand the emergence of algebraic numbers: the hierarchy of infinite primes in which ordinary primes are starting point and the arithmetics of Hilbert spaces with tensor product and direct sum replacing the usual arithmetic operations. Extensions of rationals give also rise to cognitive variants of n-D spaces.

3. The notion of empty set looks artificial from the point of view of physicist and a possible cure is to take arithmetics as a model. Natural numbers would be analogous to nonempty sets and integers would correspond to pairs of sets (A, B), $A \subset B$ or $B \subset A$ with equivalence $A, B) \equiv (A \cup C, B \cup C)$. Empty set would correspond to pairs (A, A). In quantum context the generalization of the notion of being member of set $a \in A$ suggests a generalization: being an element in set would generalize to being single particle state which in general is de-localized to the set. Subsets would correspond to many-particle states. The basic operation would be addition or removal of element represented in terms of oscillator operator. The order of elements of set does not matter: this would generalize to bosonic and fermionic many particle states and even braid statistics can be considered. In bosonic case one can have multiple points - kind of Bose-Einstein condensate.

4. One can also start from finite-D Hilbert space and identify set as the collection of labels for the states. In infinite-D case there are two cases corresponding to separable and non-separable Hilbert spaces. The condition that the norm of the state is finite without infinite normalization constants forces selection of de-localized discrete basis in the case of a continuous set like reals. This inspires the question whether the axiom of choice should be given up. One possibility is that one can have only states localized to finite or at least discrete set of points which correspond points with coordinates in an extension of rationals.

13.13.1 Geometric analog for Gödel's problematics

Godel's problematics involves statements which cannot be proved to be true or false or are simultaneously true and false. This problematics has also a purely geometric analog in terms of set theoretic representation of Boolean algebras when real topology is used but not when p-adic topology is used.

The natural idea is that Boolean algebra is realized in terms of open sets such that the negation of statement corresponds to the complement of the set. In p-adic topologies open sets are simultaneously also closed and there are no boundaries: this makes them and - more generally Stone spaces - ideal for realizing Boolean algebra set theoretically. In real topology the complement of open set is closed and therefore not open and one has a problem.

Could one circumvent the problem somehow?

1. If one replaces open sets with their closures (the closure of open set includes also its boundary, which does not belong to the open set) and closed complements of open sets, the analog of Boolean algebra would consist of closed sets. Closure of an open set and the closure of its open complement - stament and its negation - share the common boundary. Statement and its negation would be simultaneously true at the boundary. This strange situation reminds of Russell's paradox but in geometric form.

2. If one replaces the closed complements of open sets with their open interiors, one has only open sets. Now the sphere would represent statement about which one cannot say whether it is true or false. This would look like Gödelian sentence but represented geometrically.

This leads to an already familiar conclusion: p-adic topology is natural for the geometric correlates of cognition, in particular Boolean cognition. Real topology is natural for the geometric correlates of sensory experience.

3. Gödelian problematics is encountered already for arithmetics of natural numbers although naturals have no boundary in the discrete topology. Discrete topology does not however allow well-ordering of natural numbers crucial for the definition of natural number. In the induced real topology one can order them and can speak of boundaries of subsets of naturals. The ordering of natural numbers by size reflects the ordering of reals: it is very difficult to think about discrete without implicitly bringing in the continuum.

   For p-adic integers the induced topology is p-adic. Is Gödelian problematics is absent in p-adic Boolean logic in which set and its complement are both open and closed. If this view is correct, p-adic integers might replace naturals in the axiomatics of arithmetics. The new element would be that most p-adic integers are of infinite size in real sense. One has a natural division of them to cognitively representable ones finite also in real sense and non-representable ones infinite in real sense. Note however that rationals have periodic pinary expansion and can be represented as pairs of finite natural numbers.

In algebraic geometry Zariski topology in which closed sets correspond to algebraic surfaces of various dimensions, is natural. Open sets correspond to their complements and are of same dimension as the embedding space. Also now one encounters asymmetry. Could one say that algebraic surfaces characterize "representable" (="geometrically provable"?) statements as elements of Boolean algebra and their complements the non-representable ones? 4-D space-time (as possibly associative/co-associative) algebraic variety in 8-D octonionic space would be example of representable statement. Finite unions and intersections of algebraic surfaces would form the set of representable statements. This new-to-me notion of representability is somehow analogous to provability or demonstrability.

13.13.2 Number theory from quantum theory

Could one define or at least represent the notion of number using the notions of quantum physics? A natural starting point is hierarchy of extensions of rationals defining hierarchy of adeles. Could one obtain rationals and their extensions from simplest possible quantum theory in which one just constructs many particle states by adding or removing particles using creation and annihilation operators?

How to obtain rationals and their extensions?

Rationals and their extensions are fundamental in TGD. Can one have quantal construction for them?

1. One should construct rationals first. Suppose one starts from the notion of finite prime as something God-given. At the first step one constructs infinite primes as analogs for many-particle states in super-symmetric arithmetic quantum field theory [K80]. Ordinary primes label states of fermions and bosons. Infinite primes as the analogs of free many-particle states correspond to rationals in a natural manner.

2. One obtains also analogs of bound states which are mappable to irreducible polynomials, whose roots define algebraic numbers. This would give hierarchy of algebraic extensions of rationals. At higher levels of the hierarchy one obtains also analogs of prime polynomials with number of variables larger than 1. One might say that algebraic geometry has quantal representation. This might be very relevant for the physical representability of basic mathematical structures.

Arithmetics of Hilbert spaces

The notions of prime and divisibility and even basic arithmetics emerge also from the tensor product and direct sum for Hilbert spaces. Hilbert spaces with prime dimension do not decompose to tensor products of lower-dimensional Hilbert spaces. One can even perform a formal generalization of the dimension of Hilbert space so that it becomes rational and even algebraic number.

For some years ago I indeed played with this thought but at that time I did not have in mind reduction of number theory to the arithemetics of Hilbert spaces. If this really makes sense, numbers could be replaced by Hilbert spaces with product and sum identified as tensor product and direct sum!

Finite-dimensional Hilbert space represent the analogs of natural numbers. The analogs of integers could be defined as pairs (m, n) of Hilbert spaces with spaces (m, n) and $(m + r, n + r)$ identified (this space would have dimension $m - n$. This identification would hold true also at the level of states. Hilbert spaces with negative dimension would correspond to pairs with $(m-n) < 0$: the canonical representives for m and $-m$ would be $(m, 0)$ and $(0, m)$. Rationals can be defined as pairs (m, n) of Hilbert spaces with pairs (m, n) and (km, kn) identified. These identifications would give rise to kind of gauge conditions and canonical representatives for m and $1/m$ are $(m, 1)$ and $(1, m)$.

What about Hilbert spaces for which the dimension is algebraic number? Algebraic numbers allow a description in terms of partial fractions and Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq` and `http://tinyurl.com/yc6hhboo`) containing given rational number once. S-B tree allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. Algebraic number could be seen as a periodic partial fraction defining an infinite path in S-B tree. Each node along this path would correspond to a rational having Hilbert space analog. Hilbert space with algebraic dimension would correspond to this kind of path in the space of Hilbert spaces with rational dimension. Transcendentals allow identification as non-pediodic partial fraction and could correspond to non-periodic paths so that also they could have Hilbert spaces counterparts.

How to obtain the analogs higher-D spaces?

Algebraic extensions of rationals allow cognitive realization of spaces with arbitrary dimension identified as algebraic dimension of extension of rationals.

1. One can obtain n-dimensional spaces (in algebraic sense) with integer valued coordinates from n-D extensions of rationals. Now the n-tuples defining numbers of extension and differing by permutations are not equivalent so that one obtains n-D space rather than n-D space divided by permutation group S_n. This is enough at the level of cognitive representations and could explain why we are able to imagine spaces of arbitrary dimension although we cannot represent them cognitively.

2. One obtains also Galois group and orbits of set A of points of extension under Galois group as $G(A)$. One obtains also discrete coset spaces G/H and alike. These do not have any direct analog in the set theory. The hierarchy of Galois groups would bring in discrete group theory automatically. The basic machinery of quantum theory emerges elegantly from number theoretic vision.

3. In octonionic approach to quantum TGD one obtains also hierarchy of extensions of rationals since space-time surface correspond zero loci for RE or IM for octonionic polynomials obtained by algebraic continuation from real polynomials with coeffficients in extension of rationals [L42].

13.13.3 Could quantum set theory make sense?

In the following my view point is that of quantum physicist fascinated by number theory and willing to reduce set theory to what could be called called quantum set theory. It would follow from physics as generalised number theory (adelic physics) and have ordinary set theory as classical correlate.

1. From the point of quantum physics set theory and the notion of number based on set theory look somewhat artificial constructs. Nonempty set is a natural concept but empty set and set having empty set as element used as basic building brick in the construction of natural numbers looks weird to me.

2. From TGD point of view it would seem that number theory plus some basic pieces of quantum theory might be more fundamental than set theory. Could set theory emerge as a classical correlate for quantum number theory already considered and could quantal set theory make sense?

Quantum set theory

What quantum set theory could mean? Suppose that number theory-quantum theory connection really works. What about set theory? Or perhaps its quantum counterpart having ordinary set theory as a classical correlate?

1. A purely quantal input to the notion of set would be replacement of points delocalized states in the set. A generic single particle quantum state as analog of element of set would not be localized to a single element of set. The condition that the state has finite norm implies in the case of continuous set like reals that one cannot have completely localized states. This would give quantal limitation to the axiom of choice. One can have any discrete basis of state functions in the set but one cannot pick up just one point since this state would have infinite norm.

   The idea about allowing only say rationals is not needed since there is infinite number of different choices of basis. Finite measurement resolution is however unvoidable. An alternative option is restriction of the domains of wave functions to a discrete set of points. This set can be chosen in very many ways and points with coordinates in extension of rationals are very natural and would define cognitive representation.

2. One can construct also the analogs of subsets as many-particle states. The basic operation would be addition/removal of a particle from quantum state represented by the action of creation/annihilation operator.

   Bosonic states would be invariant under permutations of single particle states just like set is the equivalence class for a collection of elements $(a_1, ..., a_n)$ such that any two permutations are equivalent. Quantum set theory would however bring in something new: the possibility of both bosonic and fermionic statistics. Permutation would change the state by phase factor -1. One would have fermionic and bosonic sets. For bosonic sets one could have multiplet elements ("Bose-Einstein condensation"): in the theory of surfaces this could allow multiple copies of the same surface. Even braid statistics is possible. The phase factor in permutation could be complex. Even non-commutative statistics can be considered.

   Many particle states formed from particles, which are not identical are also possible and now the different particle types can be ordered. On obtains n-ples decomposing to ordered K-ple of n_i-ples, which are consist of identical particles and are quantum sets. One could talk about K-sets as a generalization of set as analogs of classical sets with K-colored elements. Group theory would enter into the picture via permutation groups and braid groups would bring in braid statistics. Braids strands would have K colors.

How to obtain classical set theory?

How could one obtain classical set theory?

1. Many-particle states represented algebraically are detected in lab as sets: this is quantum classical correspondence. This remains to me one of the really mysterious looking aspects in the interpretation of quantum field theory. For some reason it is usually not mentioned at all in popularizations. The reason is probably that popularization deals typically with wave mechanics but not quantum field theory unless it is about Higgs mechanism, which is the weakest part of quantum field theory!

2. From the point of quantum theory empty set would correspond to vacuum. It is not observable as such. Could the situation change in the presence of second state representing the environment? Could the fundamental sets be always *non-empty* and correspond to states with non-vanishing particle number. Natural numbers would correspond to eigenvalues of an observable telling the cardinality of set. Could representable sets be like natural numbers?

3. Usually integers are identified as pairs of natural numbers (m, n) such that integer corresponds to $m - n$. Could the set theoretic analog of integer be a pair (A, B) of sets such that A is subset of B or vice versa? Note that this does not allow pairs with disjoint members. (A, A) would correspond to empty set. This would give rise to sets (A, B) and their "antisets" (B, A) as analogs of positive and negative integers.

   One can argue that antisets are not physically realizable. Sets and antisets would have as analogs two quantizations in which the roles of oscillator operators and their hermitian conjugates are changed. The operators annihilating the ground state are called annilation operators. Only either of these realization is possible but not both simultaneously.

   In ZEO one can ask whether these two options correspond to positive and negative energy parts of zero energy states or to the states with state function reduction at either boundary of CD identified as correlates for conscious entities with opposite arrows of geometric time (generalized Zeno effect).

4. The cardinality of set, the number of elements in the set, could correspond to eigenvalue of observable measuring particle number. Many-particle states consisting of bosons or fermions would be analogs for sets since the ordering does not matter. Also braid statistics would be possible.

   What about cardinality as a p-adic integer? In p-adic context one can assign to integer m, integer $-m$ as $m \times (p-1) \times (1 + p + p^2 + ...)$. This is infinite as real integer but finite as p-adic integer. Could one say that the antiset of m-element as analog of negative integer has cardinality $-m = m(p-1)(1+p+p^2+..)$. This number does not have cognitive representation since it is not finite as real number but is cognizable.

   One could argue that negative numbers are cognizable but not cognitively representable as cardinality of set? This representation must be distinguished from cognitive representations as a point of embedding space with coordinates in extension of rationals. Could one say that antisets and empty set as its own antiset can be cognized but cannot be cognitively represented?

Nasty mathematician would ask whether I can really start from Hilbert space of state functions and deduce from this the underlying set. The elements of set itself should emerge from this as analogs of completely localized single particle states labelled by points of set. In the case of finite-dimensional Hilbert space this is trivial. The number of points in the set would be equal to the dimension of Hilbert space. In the case of infinite-D Hilbert space the set would have infinite number of points.

Here one has two views about infinite set. One has both separable (infinite-D in discrete sense: particle in box with discrete momentum spectrum) and non-separable (infinite-D in real sense: free particle with continuous momentum spectrum) Hilbert spaces. In the latter case the completely localized single particle states would be represented by delta functions divided by infinite normalization factors. They are routinely used in Dirac's bra-ket formalism but problems emerge in quantum field theory.

A possible solution is that one weakens the axiom of choice and accepts that only discrete points set (possibly finite) are cognitively representable and one has wave functions localized to discrete set of points. A stronger assumption is that these points have coordinates in extension of rationals so that one obtains number theoretical universality and adeles. This is TGD view and conforms also with the identification of hyper-finite factors of type II_1 as basic algebraic objects in TGD based quantum theory as opposed to wave mechanics (type I) and quantum field theory (type III). They are infinite-D but allow excellent approximation as finite-D objects.

This picture could relate to the notion of non-commutative geometry, where set emerges as spectrum of algebra: the points of spectrum label the ideals of the integer elements of algebra.

13.14 Abelian Class Field Theory And TGD

The context leading to the discovery of adeles (`http://tinyurl.com/64pgerm`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p\ mod\ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p\ mos\ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (see `http://tinyurl.com/y8aefmg2`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

13.14.1 Adeles And Ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://tinyurl.com/y8aefmg2`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [K42, A57, A56]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where embedding space is replaced with Cartesian product of real embedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://tinyurl.com/64pgerm`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod'_p Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://tinyurl.com/yc3yrcxx`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

13.14.2 Questions About Adeles, Ideles And Quantum TGD

The intriguing general result of class field theory (`http://tinyurl.com/y8aefmg2`) is that the the maximal Abelian extension for rationals is homomorphic with the multiplicative group of ideles. This correspondence plays a key role in Langlands correspondence.

Does this mean that it is not absolutely necessary to introduce p-adic numbers? This is actually not so. The Galois group of the maximal abelian extension is rather complex objects (absolute Galois group, AGG, defines as the Galois group of algebraic numbers is even more complex!). The ring $\hat{Z}$ of adeles defining the group of ideles as its invertible elements homeomorphic to the Galois group of maximal Abelian extension is profinite group (`http://tinyurl.com/y9d8vro7`). This means that it is totally disconnected space as also p-adic integers and numbers are. What is intriguing that p-dic integers are however a continuous structure in the sense that differential calculus is possible. A concrete example is provided by 2-adic units consisting of bit sequences which can have literally infinite non-vanishing bits. This space is formally discrete but one can construct differential calculus since the situation is not democratic. The higher the pinary digit in the expansion is, the less significant it is, and p-adic norm approaching to zero expresses the reduction of the insignificance.

1. Could TGD based physics reduce to a representation theory for the Galois groups of quaternions and octonions?

Number theoretical vision about TGD raises questions about whether adeles and ideles could be helpful in the formulation of TGD. I have already earlier considered the idea that quantum TGD could reduce to a representation theory of appropriate Galois groups. I proceed to make questions.

1. Could real physics and various p-adic physics on one hand, and number theoretic physics based on maximal Abelian extension of rational octonions and quaternions on one hand, define equivalent formulations of physics?

2. Besides various p-adic physics all classical number fields (reals, complex numbers, quaternions, and octonions) are central in the number theoretical vision about TGD. The technical problem is that p-adic quaternions and octonions exist only as a ring unless one poses some additional conditions. Is it possible to pose such conditions so that one could define what might be called quaternionic and octonionic adeles and ideles?

   It will be found that this is the case: p-adic quaternions/octonions would be products of rational quaternions/octonions with a p-adic unit. This definition applies also to algebraic extensions of rationals and makes it possible to define the notion of derivative for corresponding adeles. Furthermore, the rational quaternions define non-commutative automorphisms of quaternions and rational octonions at least formally define a non-associative analog of group of octonionic automorphisms [K82, K94].

3. I have already earlier considered the idea about Galois group as the ultimate symmetry group of physics. The representations of Galois group of maximal Abelian extension (or even that for algebraic numbers) would define the quantum states. The representation space could be group algebra of the Galois group and in Abelian case equivalently the group algebra of ideles or adeles. One would have wave functions in the space of ideles.

   The Galois group of maximal Abelian extension would be the Cartan subgroup of the absolute Galois group of algebraic numbers associated with given extension of rationals and it would be natural to classify the quantum states by the corresponding quantum numbers (number theoretic observables).

   If octonionic and quaternionic (associative) adeles make sense, the associativity condition would reduce the analogs of wave functions to those at 4-dimensional associative sub-manifolds of octonionic adeles identifiable as space-time surfaces so that also space-time physics in various number fields would result as representations of Galois group in the maximal Abelian Galois group of rational octonions/quaternions. TGD would reduce to classical number theory! One can hope that WCW spinor fields assignable to the associative and co-associative space-time surfaces provide the adelic representations for super-conformal algebras replacing symmetries for point like objects.

   This of course involves huge challenges: one should find an adelic formulation for WCW in terms octonionic and quaternionic adeles, similar formulation for WCW spinor fields in terms of adelic induced spinor fields or their octonionic variants is needed. Also zero energy ontology, causal diamonds, light-like 3-surfaces at which the signature of the induced metric changes, space-like 3-surfaces and partonic 2-surfaces at the boundaries of CDs, $M^8 - H$ duality, possible representation of space-time surfaces in terms of of O_c-real analytic functions (O_c denotes for complexified octonions), etc. should be generalized to adelic framework.

4. Absolute Galois group is the Galois group of the maximal algebraic extension and as such a poorly defined concept. One can however consider the hierarchy of all finite-dimensional algebraic extensions (including non-Abelian ones) and maximal Abelian extensions associated with these and obtain in this manner a hierarchy of physics defined as representations of these Galois groups homomorphic with the corresponding idele groups.

5. In this approach the symmetries of the theory would have automatically adelic representations and one might hope about connection with Langlands program [K42], [K42, A57, A56].

2. Adelic variant of space-time dynamics and spinorial dynamics?

As an innocent novice I can continue to pose stupid questions. Now about adelic variant of the space-time dynamics based on the generalization of Kähler action discussed already earlier but without mentioning adeles ([K97]).

1. Could one think that adeles or ideles could extend reals in the formulation of the theory: note that reals are included as Cartesian factor to adeles. Could one speak about adelic space-time surfaces endowed with adelic coordinates? Could one formulate variational principle in terms of adeles so that exponent of action would be product of actions exponents associated with various factors with Neper number replaced by p for Z_p. The minimal interpretation would

be that in adelic picture one collects under the same umbrella real physics and various p-adic physics.

2. Number theoretic vision suggests that 4: th/8: th Cartesian powers of adeles have interpretation as adelic variants of quaternions/ octonions. If so, one can ask whether adelic quaternions and octonions could have some number theoretical meaning. Adelic quaternions and octonions are not number fields without additional assumptions since the moduli squared for a p-adic analog of quaternion and octonion can vanish so that the inverse fails to exist at the light-cone boundary which is 17-dimensional for complexified octonions and 7-dimensional for complexified quaternions. The reason is that norm squared is difference $N(o_1) - N(o_2)$ for $o_1 \oplus io_2$. This allows to define differential calculus for Taylor series and one can consider even rational functions. Hence the restriction is not fatal.

   If one can pose a condition guaranteeing the existence of inverse for octonionic adel, one could define the multiplicative group of ideles for quaternions. For octonions one would obtain non-associative analog of the multiplicative group. If this kind of structures exist then four-dimensional associative/co-associative sub-manifolds in the space of non-associative ideles define associative/co-associative adeles in which ideles act. It is easy to find that octonionic ideles form 1-dimensional objects so that one must accept octonions with arbitrary real or p-adic components.

3. What about equations for space-time surfaces. Do field equations reduce to separate field equations for each factor? Can one pose as an additional condition the constraint that p-adic surfaces provide in some sense cognitive representations of real space-time surfaces: this idea is formulated more precisely in terms of p-adic manifold concept [K97] (see the appendix of the book). Or is this correspondence an outcome of evolution?

   Physical intuition would suggest that in most p-adic factors space-time surface corresponds to a point, or at least to a vacuum extremal. One can consider also the possibility that same algebraic equation describes the surface in various factors of the adele. Could this hold true in the intersection of real and p-adic worlds for which rationals appear in the polynomials defining the preferred extremals.

4. To define field equations one must have the notion of derivative. Derivative is an operation involving division and can be tricky since adeles are not number field. The above argument suggests this is not actually a problem. Of course, if one can guarantee that the p-adic variants of octonions and quaternions are number fields, there are good hopes about well-defined derivative. Derivative as limiting value $df/dx = \lim(f(x + dx) - f(x))/dx$ for a function decomposing to Cartesian product of real function $f(x)$ and p-adic valued functions $f_p(x_p)$ would require that $f_p(x)$ is non-constant only for a finite number of primes: this is in accordance with the physical picture that only finite number of p-adic primes are active and define "cognitive representations" of real space-time surface. The second condition is that dx is proportional to product $dx \times \prod dx_p$ of differentials dx and dx_p, which are rational numbers. dx goes to xero as a real number but not p-adically for any of the primes involved. dx_p in turn goes to zero p-adically only for Q_p.

5. The idea about rationals as points common to all number fields is central in number theoretical vision. This vision is realized for adeles in the minimal sense that the action of rationals is well-defined in all Cartesian factors of the adeles. Number theoretical vision allows also to talk about common rational points of real and various p-adic space-time surfaces in preferred coordinate choices made possible by symmetries of the embedding space, and one ends up to the vision about life as something residing in the intersection of real and p-adic number fields. It is not clear whether and how adeles could allow to formulate this idea.

6. For adelic variants of embedding space spinors Cartesian product of real and p-adc variants of embedding spaces is mapped to their tensor product. This gives justification for the physical vision that various p-adic physics appear as tensor factors. Does this mean that the generalized induced spinors are infinite tensor products of real and various p-adic spinors and Clifford algebra generated by induced gamma matrices is obtained by tensor product construction? Does the generalization of massless Dirac equation reduce to a sum of

d'Alembertians for the factors? Does each of them annihilate the appropriate spinor? If only finite number of Cartesian factors corresponds to a space-time surface which is not vacuum extremal vanishing induced Kähler form, Kähler Dirac equation is non-trivial only in finite number of adelic factors.

3. Objections leading to the identification of octonionic adeles and ideles

The basic idea is that appropriately defined invertible quaternionic/octonionic adeles can be regarded as elements of Galois group assignable to quaternions/octonions. The best manner to proceed is to invent objections against this idea.

1. The first objection is that p-adic quaternions and octonions do not make sense since p-adic variants of quaternions and octonions do not exist in general. The reason is that the p-adic norm squared $\sum x_i^2$ for p-adic variant of quaternion, octonion, or even complex number can vanish so that its inverse does not exist.

2. Second objection is that automorphisms of the ring of quaternions (octonions) in the maximal Abelian extension are products of transformations of the subgroup of $SO(3)$ (G_2) represented by matrices with elements in the extension and in the Galois group of the extension itself. Ideles separate out as 1-dimensional Cartesian factor from this group so that one does not obtain 4-field (8-fold) Cartesian power of this Galois group.

One can define quaternionic/octonionic ideles in terms of rational quaternions/octonions multiplied by p-adic number. For adeles this condition produces non-sensical results.

1. This condition indeed allows to construct the inverse of p-adic quaternion/octonion as a product of inverses for rational quaternion/octonion and p-adic number. The reason is that the solutions to $\sum x_i^2 = 0$ involve always p-adic numbers with an infinite number of pinary digits - at least one and the identification excludes this possibility. The ideles form also a group as required.

2. One can interpret also the quaternionicity/octonionicity in terms of Galois group. The 7-dimensional non-associative counterparts for octonionic automorphisms act as transformations $x \rightarrow gxg^{-1}$. Therefore octonions represent this group like structure and the p-adic octonions would have interpretation as combination of octonionic automorphisms with those of rationals.

3. One cannot assign to ideles 4-D idelic surfaces. The reason is that the non-constant part of all 8-coordinates is proportional to the same p-adic valued function of space-time point so that space-time surface would be a disjoint union of effectively 1-dimensional structures labelled by a subset of rational points of M^8. Induced metric would be 1-dimensional and induced Kähler and spinor curvature would vanish identically.

4. One must allow p-adic octonions to have arbitrary p-adic components. The action of ideles representing Galois group on these surfaces is well-defined. Number field property is lost but this feature comes in play as poles only when one considers rational functions. Already the Minkowskian signature forces to consider complexified octonions and quaternions leading to the loss of field property. It would not be surprising if p-adic poles would be associated with the light-like orbits of partonic 2-surfaces. Both p-adic and Minkowskian poles might therefore be highly relevant physically and analogous to the poles of ordinary analytic functions. For instance, n-point functions could have poles at the light-like boundaries of causal diamonds and at light-like partonic orbits and explain their special physical role.

   The action of ideles in the quaternionic tangent space of space-time surface would be analogous to the action of of adelic linear group $Gl_n(A)$ in n-dimensional space.

5. Adelic variants of octonions would be Cartesian products of ordinary and various p-adic octonions and would define a ring. Quaternionic 4-surfaces would define associative local sub-rings of octonion-adelic ring.

Chapter 14

Knots and TGD

14.1 Introduction

Witten has highly inspiring popular lecture about knots and quantum physics [A34] mentioning also his recent work with knots related to an attempt to understand Khovanov homology. Witten manages to explain in rather comprehensible way both the construction recipe of Jones polynomial and the idea about how Jones polynomial emerges from topological quantum field theory as a vacuum expectation of so called Wilson loop defined by path integral with weighting coming from Chern-Simons action [A58]. Witten also tells that during the last year he has been working with an attempt to understand in terms of quantum theory the so called Khovanov polynomial associated with a much more abstract link invariant whose interpretation and real understanding remains still open. In particular, he mentions the approach of Gukov, Schwartz, and Vafa [A71, A71] as an attempt to understand Khovanov polynomial.

This kind of talks are extremely inspiring and lead to a series of questions unavoidably culminating to the frustrating "Why I do not have the brain of Witten making perhaps possible to answer these questions?". This one must just accept. In the following I summarize some thoughts inspired by the associations of the talk of Witten with quantum TGD and with the model of DNA as topological quantum computer. In my own childish way I dare believe that these associations are interesting and dare also hope that some more brainy individual might take them seriously.

An idea inspired by TGD approach which also main streamer might find interesting is that the Jones invariant defined as vacuum expectation for a Wilson loop in 2+1-D space-time generalizes to a vacuum expectation for a collection of Wilson loops in 2+2-D space-time and could define an invariant for 2-D knots and for cobordisms of braids analogous to Jones polynomial. As a matter fact, it turns out that a generalization of gauge field known as gerbe is needed and that in TGD framework classical color gauge fields defined the gauge potentials of this field. Also topological string theory in 4-D space-time could define this kind of invariants. Of course, it might well be that this kind of ideas have been already discussed in literature.

Khovanov homology generalizes the Jones polynomial as knot invariant. The challenge is to find a quantum physical construction of Khovanov homology analous to the topological QFT defined by Chern-Simons action allowing to interpret Jones polynomial as vacuum expectation value of Wilson loop in non-Abelian gauge theory.

Witten's approach to Khovanov homology relies on fivebranes as is natural if one tries to define 2-knot invariants in terms of their cobordisms involving violent un-knottings. Despite the difference in approaches it is very useful to try to find the counterparts of this approach in quantum TGD since this would allow to gain new insights to quantum TGD itself as almost topological QFT identified as symplectic theory for 2-knots, braids and braid cobordisms. This comparison turns out to be extremely useful from TGD point of view.

1. A highly unique identification of string world sheets and therefore also of the braids whose ends carry quantum numbers of many particle states at partonic 2-surfaces emerges if one identifies the string word sheets as singular surfaces in the same ways as is done in Witten's approach.

This identification need not of course be correct and in TGD framework the localization of the modes of the induced spinor fields at 2-D surfaces carrying vanishing induced W boson fields guaranteeing that the em charge of spinor modes is well-defined for a generic preferred extremal is natural. Besides string world sheets partonic 2-surfaces are good candidates for this kind of surfaces. It is not clear whether one can have continuous slicing of this kind by string world sheets and partonic 2-surfaces orthogonal to them or whether only discrete set of these surfaces is possible.

2. Also a physical interpretation of the operators Q, F, and P of Khovanov homology emerges. P would correspond to instanton number and F to the fermion number assignable to right handed neutrinos. The breaking of M^4 chiral invariance makes possible to realize Q physically. The finding that the generalizations of Wilson loops can be identified in terms of the gerbe fluxes $\int H_A J$ supports the conjecture that TGD as almost topological QFT corresponds essentially to a symplectic theory for braids and 2-knots.

The basic challenge of quantum TGD is to give a precise content to the notion of generalization Feynman diagram and the reduction to braids of some kind is very attractive possibility inspired by zero energy ontology. The point is that no $n > 2$-vertices at the level of braid strands are needed if bosonic emergence holds true.

1. For this purpose the notion of algebraic knot is introduced and the possibility that it could be applied to generalized Feynman diagrams is discussed. The algebraic structrures kei, quandle, rack, and biquandle and their algebraic modifications as such are not enough. The lines of Feynman graphs are replaced by braids and in vertices braid strands redistribute. This poses several challenges: the crossing associated with braiding and crossing occurring in non-planar Feynman diagrams should be integrated to a more general notion; braids are replaced with sub-manifold braids; braids of braids....of braids are possible; the redistribution of braid strands in vertices should be algebraized. In the following I try to abstract the basic operations which should be algebraized in the case of generalized Feynman diagrams.

2. One should be also able to concretely identify braids and 2-braids (string world sheets) as well as partonic 2-surfaces and I have discussed several identifications during last years. Legendrian braids turn out to be very natural candidates for braids and their duals for the partonic 2-surfaces. String world sheets in turn could correspond to the analogs of Lagrangian sub-manifolds or two minimal surfaces of space-time surface satisfying the weak form of electric-magnetic duality. The latter option turns out to be more plausible. This identification - if correct - would solve quantum TGD explicitly at string world sheet level which corresponds to finite measurement resolution.

3. Also a brief summary of generalized Feynman rules in zero energy ontology is proposed. This requires the identification of vertices, propagators, and prescription for integrating over al 3-surfaces. It turns out that the basic building blocks of generalized Feynman diagrams are well-defined.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L21].

14.2 Some TGD Background

What makes quantum TGD [L4, L5, L8, L9, L6, L3, L7, L13] interesting concerning the description of braids and braid cobordisms is that braids and braid cobordisms emerge both at the level of generalized Feynman diagrams and in the model of DNA as a topological quantum computer [K2].

14.2.1 Time-Like And Space-Like Braidings For Generalized Feynman Diagrams

1. In TGD framework space-times are 4-D surfaces in 8-D embedding space. Basic objects are partonic 2-surfaces at the two ends of causal diamonds CD (intersections of future and past directed light-cones of 4-D Minkowski space with each point replaced with CP_2). The light-like orbits of partonic 2-surfaces define 3-D light-like 3-surfaces identifiable as lines of generalized Feynman diagrams. At the vertices of generalized Feynman diagrams incoming and outgoing light-like 3-surfaces meet. These diagrams are not direct generalizations of string diagrams since they are singular as 4-D manifolds just like the ordinary Feynman diagrams.

   By strong form of holography one can assign to the partonic 2-surfaces and their tangent space data space-time surfaces as preferred extremals of Kähler action. This guarantees also general coordinate invariance and allows to interpret the extremals as generalized Bohr orbits.

2. One can assign to the partonic 2-surfaces discrete sets of points carrying quantum numbers. These sets of points emerge from the solutions of of the Kähler-Dirac equation, which are localized at 2-D surfaces - string world sheets and possibly also partonic 2-surfaces - carrying vanishing induced W fields and also Z^0 fields above weak scale. These points and their orbits identifiable as boundaries of string world sheets define braid strands at the light-like orbits of partonic 2-surfaces. In the generic case the strands get tangled in time direction and one has linking and knotting giving rise to a time-like braiding. String world sheets and also partonic surfaces define 2-braids and 2-knots at 4-D space-time surface so that knot theory generalizes.

3. Also space-like braidings are possible. One can imagine that the partonic 2-surfaces are connected by space-like curves defining TGD counterparts for strings and that in the initial state these curves define space-like braids whose ends belong to different partonic 2-surfaces. Quite generally, the basic conjecture is that the preferred extremals define orbits of string-like objects with their ends at the partonic 2-surfaces. One would have slicing of space-time surfaces by string world sheets one one hand and by partonic 2-surface on one hand. This string model is very special due to the fact that the string orbits define what could be called braid cobordisms representing which could represent unknotting of braids. String orbits in higher dimensional space-times do not allow this topological interpretation.

14.2.2 Dance Metaphor

Time like braidings induces space-like braidings and one can speak of time-like or dynamical braiding and even duality of time-like and space-like braiding. What happens can be understood in terms of dance metaphor.

1. One can imagine that the points carrying quantum numbers are like dancers at parquettes defined by partonic 2-surfaces. These parquettes are somewhat special in that it is moving and changing its shape.

2. Space-like braidings means that the feet of the dancers at different parquettes are connected by threads. As the dance continues, the threads connecting the feet of different dancers at different parquettes get tangled so that the dance is coded to the braiding of the threads. Time-like braiding induce space-like braiding. One has what might be called a cobordism for space-like braiding transforming it to a new one.

14.2.3 DNA As Topological Quantum Computer

The model for topological quantum computation is based on the idea that time-like braidings defining topological quantum computer programs. These programs are robust since the topology of braiding is not affected by small deformations.

1. The first key idea in the model of DNA as topological quantum computer is based on the observation that the lipids of cell membrane form a 2-D liquid whose flow defines the dance in which dancers are lipids which define a flow pattern defining a topological quantum computation. Lipid layers assignable to cellular and nuclear membranes are the parquettes. This 2-D flow pattern can be induced by the liquid flow near the cell membrane or in case of nerve pulse transmission by the nerve pulses flowing along the axon. This alone defines topological quantum computation.

2. In DNA as topological quantum computer model one however makes a stronger assumption motivated by the vision that DNA is the brain of cell and that information must be communicated to DNA level wherefrom it is communicated to what I call magnetic body. It is assumed that the lipids of the cell membrane are connected to DNA nucleotides by magnetic flux tubes defining a space-like braiding. It is also possible to connect lipids of cell membrane to the lipids of other cell membranes, to the tubulins at the surfaces of microtubules, and also to the aminoadics of proteins. The spectrum of possibilities is really wide.

   The space-like braid strands would correspond to magnetic flux tubes connecting DNA nucleotides to lipids of nuclear or cell membrane. The running of the topological quantum computer program defined by the time-like braiding induced by the lipid flow would be coded to a space-like braiding of the magnetic flux tubes. The braiding of the flux tubes would define a universal memory storage mechanism and combined with 4-D view about memory provides a very simple view about how memories are stored and how they are recalled.

14.3 Could Braid Cobordisms Define More General Braid Invariants?

Witten says that one should somehow generalize the notion of knot invariant. The above described framework indeed suggests a very natural generalization of braid invariants to those of braid cobordisms reducing to braid invariants when the braid at the other end is trivial. This description is especially natural in TGD but allows a generalization in which Wilson loops in 4-D sense describe invariants of braid cobordisms.

14.3.1 Difference Between Knotting And Linking

Before my modest proposal of a more general invariant some comments about knotting and linking are in order.

1. One must distinguish between internal knotting of each braid strand and linking of 2 strands. They look the same in the 3-D case but in higher dimensions knotting and linking are not the same thing. Codimension 2 surfaces get knotted in the generic case, in particular the 2-D orbits of the braid strands can get knotted so that this gives additional topological flavor to the theory of strings in 4-D space-time. Linking occurs for two surfaces whose dimension d_1 and d_2 satisfying $d_1 + d_2 = D - 1$, where D is the dimension of the embedding space.

2. 2-D orbits of strings do not link in 4-D space-time but do something more radical since the sum of their dimensions is $D = 4$ rather than only $D - 1 = 3$. They intersect and it is impossible to eliminate the intersection without a change of topology of the stringy 2-surfaces: a hole is generated in either string world sheet. With a slight deformation intersection can be made to occur generically at discrete points.

14.3.2 Topological Strings In 4-D Space-Time Define Knot Cobordisms

What makes the 4-D braid cobordisms interesting is following.

1. The opening of knot by using brute force by forcing the strands to go through each other induces this kind of intersection point for the corresponding 2-surfaces. From 3-D perspective this looks like a temporary cutting of second string, drawing the string ends to some distance and bringing them back and gluing together as one approaches the moment when the strings

would go through each other. This surgical operation for either string produces a pair of non-intersecting 2-surfaces with the price that the second string world sheet becomes topologically non-trivial carrying a hole in the region were intersection would occur. This operation relates a given crossing of braid strands to its dual crossing in the construction of Jones polynomial in given step (string 1 above string 2 is transformed to string 2 above string 1).

2. One can also cut both strings temporarily and glue them back together in such a way that end a/b of string 1 is glued to the end c/d of string 2. This gives two possibilities corresponding to two kinds of reconnections. Reconnections appears as the second operation in the construction of Jones invariant besides the operation putting the string above the second one below it or vice versa. Jones polynomial (see `http://tinyurl.com/2jctzy`) relates in a simple manner to Kauffman bracket (see `http://tinyurl.com/yc2wu47x`) allowing a recursive construction. At a given step a crossing is replaced with a weighted sum of the two reconnected terms [A2, A18]. Reconnection represents the analog of trouser vertex for closed strings replaced with braid strands.

3. These observations suggest that stringy diagrams describe the braid cobordisms and a kind of topological open string model in 4-D space-time could be used to construct invariants of braid cobordisms. The dynamics of strand ends at the partonic 2-surfaces would partially induce the dynamics of the space-like braiding. This dynamics need not induce the un-knotting of space-like braids and simple string diagrams for open strings are enough to define a cobordism leading to un-knotting. The holes needed to realize the crossover for braid strands would contribute to the Wilson loop an additional factor corresponding to the rotation of the gauge potential around the boundary of the hole (non-integrable phase factor). In abelian case this gives simple commuting phase factor.

Note that braids are actually much more closer to the real world than knots since a useful strand of knotted structure must end somewhere. The abstract closed loops of mathematician floating in empty space are not very useful in real life albeit mathematically very convenient as Witten notices. Also the braid cobordisms with ends of a collection of space-like braids at the ends of causal diamond are more practical than 2-knots in 4-D space. Mathematician would see these objects as analogous to surfaces in relative homology allowed to have boundaries if they located at fixed sub-manifolds. Homology for curves with ends fixed to be on some surfaces is a good example of this. Now these fixed sub-manifolds would correspond to space-like 3-surfaces at the ends CDs and light-like wormhole throats at which the signature of the induced metric changes and which are carriers of elementary particle quantum numbers.

14.4 Invariants 2-Knots As Vacuum Expectations Of Wilson Loops In 4-D Space-Time?

The interpretation of string world sheets in terms of Wilson loops in 4-dimensional space-time is very natural. This raises the question whether Witten's a original identification of the Jones polynomial as vacuum expectation for a Wilson loop in 2+1-D space might be replaced with a vacuum expectation for a collection of Wilson loops in 3+1-D space-time and would characterize in the general case (multi-)braid cobordism rather than braid. If the braid at the lower or upped boundary is trivial, braid invariant is obtained. The intersections of the Wilson loops would correspond to the violent un-knotting operations and the boundaries of the resulting holes give an additional Wilson loop. An alternative interpretation would be as the analog of Jones polynomial for 2-D knots in 4-D space-time generalizing Witten's theory. This description looks completely general and does not require TGD at all.

The following considerations suggest that Wilson loops are not enough for the description of general 2-knots and that Wilson loops must be replaced with 2-D fluxes. This requires a generalization of gauge field concept so that it corresponds to a 3-form instead of 2-form is needed. In TGD framework this kind of generalized gauge fields exist and their gauge potentials correspond to classical color gauge fields.

14.4.1 What 2-Knottedness Means Concretely?

It is easy to imagine what ordinary knottedness means. One has circle imbedded in 3-space. One projects it in some plane and looks for crossings. If there are no crossings one knows that un-knot is in question. One can modify a given crossing by forcing the strands to go through each other and this either generates or removes knottedness. One can also destroy crossing by reconnection and this always reduces knottedness. Since knotting reduces to linking in 3-D case, one can find a simple interpretation for knottedness in terms of linking of two circles. For 2-knots linking is not what gives rise to knotting.

One might hope to find something similar in the case of 2-knots. Can one imagine some simple local operations which either increase of reduce 2-knottedness?

1. To proceed let us consider as simple situation as possible. Put sphere in 3-D time= constant section E^3 of 4-space. Add a another sphere to the same section E^3 such that the corresponding balls do not intersect. How could one build from these two spheres a knotted 2-sphere?

2. From two spheres one can build a single sphere in topological sense by connecting them with a small cylindrical tube connecting the boundaries of disks (circles) removed from the two spheres. If this is done in E^3, a trivial 2-knot results. One can however do the gluing of the cylinder in a more exotic manner by going temporarily to "hyper-space", in other words making a time travel. Let the cylinder leave the second sphere from the outer surface, let it go to future or past and return back to recent but through the interior. This is a good candidate for a knotted sphere since the attempts to deform it to self-non-intersecting sphere in E^3 are expected to fail since the cylinder starting from interior necessarily goes through the surface of sphere if wants to the exterior of the sphere.

3. One has actually 2×2 ways to perform the connected sum of 2-spheres depending on whether the cylinders leave the spheres through exterior or interior. At least one of them (exterior-exterior) gives an un-knotted sphere and intuition suggests that all the three remaining options requiring getting out from the interior of sphere give a knotted 2-sphere. One can add to the resulting knotted sphere new spheres in the same manner and obtain an infinite number of them. As a matter fact, the proposed 1+3 possibilities correspond to different versions of connected sum and one could speak of knotting and non-knotting connected sums. If the addition of knotted spheres is performed by non-knotting connected sum, one obtains composites of already existing 2-knots. Connected sum composition is analogous to the composition of integer to a product of primes. One indeed speaks of prime knots and the number of prime knots is infinite. Of course, it is far from clear whether the connected sum operation is enough to build all knots. For instance it might well be that cobordisms of 1-braids produces knots not producible in this manner. In particular, the effects of time-like braiding induce braiding of space-like strands and this looks totally different from local knotting.

14.4.2 Are All Possible 2-Knots Possible For Stringy WorldSheets?

Whether all possible 2-knots are allowed for stringy world sheets, is not clear. In particular, if they are dynamically determined it might happen that many possibilities are not realized. For instance, the condition that the signature of the induced metric is Minkowskian could be an effective killer of 2-knottedness not reducing to braid cobordism.

1. One must start from string world sheets with Minkowskian signature of the induced metric. In other words, in the previous construction one must E^3 with 3-dimensional Minkowski space M^3 with metric signature 1+2 containing the spheres used in the construction. Time travel is replaced with a travel in space-like hyper dimension. This is not a problem as such. The spheres however have at least one two special points corresponding to extrema at which the time coordinate has a local minimum or maximum. At these points the induced metric is necessarily degenerate meaning that its determinant vanishes. If one allows this kind of singular points one can have elementary knotted spheres. This liberal attitude is encouraged by the fact that the light-like 3-surfaces defining the basic dynamical objects of

quantum TGD correspond to surfaces at which 4-D induced metric is degenerate. Otherwise 2-knotting reduces to that induced by cobordisms of 1-braids. If one allows only the 2-knots assignable to the slicings of the space-time surface by string world sheets and even restricts the consideration to those suggested by the duality of 2-D generalization of Wilson loops for string world sheets and partonic 2-surfaces, it could happen that the string world sheets reduce to braidings.

2. The time=constant intersections define a representation of 2-knots as a continuous sequence of 1-braids. For critical times the character of the 1-braids changes. In the case of braiding this corresponds to the basic operations for 1-knots having interpretation as string diagrams (reconnection and analog of trouser vertex). The possibility of genuine 2-knottedness brings in also the possibility that strings pop up from vacuum as points, expand to closed strings, are fused to stringy words sheet temporarily by the analog of trouser vertex, and eventually return to the vacuum. Essentially trouser diagram but second string open and second string closed and beginning from vacuum and ending to it is in question. Vacuum bubble interacting with open string would be in question. The believer in string model might be eager to accept this picture but one must be cautious.

14.4.3 Are Wilson Loops Enough For 2-Knots?

Suppose that the space-like braid strands connecting partonic 2-surfaces at given boundary of CD and light-like braids connecting partonic 2-surfaces belonging to opposite boundaries of CD form connected closed strands. The collection of closed loops can be identified as boundaries of Wilson loops and the expectation value is defined as the product of traces assignable to the loops. The definition is exactly the same as in 2+1-D case [A58].

Is this generalization of Wilson loops enough to describe 2-knots? In the spirit of the proposed philosophy one could ask whether there exist two-knots not reducible to cobordisms of 1-knots whose knot invariants require cobordisms of 2-knots and therefore 2-braids in 5-D space-time. Could it be that dimension $D = 4$ is somehow very special so that there is no need to go to $D = 5$? This might be the case since for ordinary knots Jones polynomial is very faithful invariant.

Innocent novice could try to answer the question in the following manner. Let us study what happens locally as the 2-D closed surface in 4-D space gets knotted.

1. In 1-D case knotting reduces to linking and means that the first homotopy group of the knot complement is changed so that the embedding of first circle implies that the there exists embedding of the second circle that cannot be transformed to each other without cutting the first circle temporarily. This phenomenon occurs also for single circle as the connected sum operation for two linked circles producing single knotted circle demonstrates.

2. In 2-D case the complement of knotted 2-sphere has a non-trivial second homotopy group so that 2-balls have homotopically non- equivalent embeddings, which cannot be transformed to each other without intersection of the 2-balls taking place during the process. Therefore the description of 2-knotting in the proposed manner would require cobordisms of 2-knots and thus 5-D space-time surfaces. However, since 3-D description for ordinary knots works so well, one could hope that the generalization the notion of Wilson loop could allow to avoid 5-D description altogether. The generalized Wilson loops would be assigned to 2-D surfaces and gauge potential A would be replaced with 2-gauge potential B defining a three-form $F = dB$ as the analog of gauge field.

3. This generalization of bundle structure known as gerbe structure has been introduced in algebraic geometry [A12, A89] and studied also in theoretical physics [A79]. 3-forms appear as analogs of gauge fields also in the QFT limit of string model. Algebraic geometer would see gerbe as a generalization of bundle structure in which gauge group is replaced with a gauge groupoid. Essentially a structure of structures seems to be in question. For instance, the principal bundles with given structure group for given space defines a gerbe. In the recent case the space of gauge fields in space-time could be seen as a gerbe. Gerbes have been also assigned to loop spaces and WCW can be seen as a generalization of loop space. Lie groups define a much more mundane example about gerbe. The 3-form F is given by

$F(X, Y, Z) = B(X, [Y, Z])$, where B is Killing form and for $U(n)$ reduces to $(g^{-1}dg)^3$. It will be found that classical color gauge fields define gerbe gauge potentials in TGD framework in a natural manner.

14.5 TGD Inspired Theory Of Braid Cobordisms And 2-Knots

In the sequel the considerations are restricted to TGD and to a comparison of Witten's ideas with those emerging in TGD framework.

14.5.1 Weak Form Of Electric-Magnetic Duality And Duality Of Space-Like And Time-Like Braidings

Witten notices that much of his work in physics relies on the assumption that magnetic charges exist and that rather frustratingly, cosmic inflation implies that all traces of them disappear. In TGD Universe the non-trivial topology of CP_2 makes possible Kähler magnetic charge and inflation is replaced with quantum criticality. The recent view about elementary particles is that they correspond to string like objects with length of order electro-weak scale with Kähler magnetically charged wormhole throats at their ends. Therefore magnetic charges would be there and LHC might be able to detect their signatures if LHC would get the idea of trying to do this.

Witten mentions also electric-magnetic duality. If I understood correctly, Witten believes that it might provide interesting new insights to the knot invariants. In TGD framework one speaks about weak form of electric magnetic duality. This duality states that Kähler electric fluxes at space-like ends of the space-time sheets inside CDs and at wormhole throats are proportional to Kähler magneic fluxes so that the quantization of Kähler electric charge quantization reduces to purely homological quantization of Kähler magnetic charge.

The weak form of electric-magnetic duality fixes the boundary conditions of field equations at the light-like and space-like 3-surfaces. Together with the conjecture that the Kähler current is proportional to the corresponding instanton current this implies that Kähler action for the preferred extremal sof Kähler action reduces to 3-D Chern-Simons term so that TGD reduces to almost topological QFT. This means an enormous mathematical simplification of the theory and gives hopes about the solvability of the theory. Since knot invariants are defined in terms of Abelian Chern-Simons action for induced Kähler gauge potential, one might hope that TGD could as a by-product define invariants of braid cobordisms in terms of the unitary U-matrix of the theory between zero energy states. The detailed construction of U-matrix is discussed in [K53].

Electric magnetic duality is 4-D phenomenon as is also the duality between space-like and time like braidings essential also for the model of topological quantum computation. Also this suggests that some kind of topological string theory for the space-time sheets inside CDs could allow to define the braid cobordism invariants.

14.5.2 Could Kähler Magnetic Fluxes Define Invariants Of Braid Cobordisms?

Can one imagine of defining knot invariants or more generally, invariants of knot cobordism in this framework? As a matter fact, also Jones polynomial describes the process of unknotting and the replacement of unknotting with a general cobordism would define a more general invariant. Whether the Khovanov invariants might be understood in this more general framework is an interesting question.

1. One can assign to the 2-dimensional stringy surfaces defined by the orbits of space-like braid strands Kähler magnetic fluxes as flux integrals over these surfaces and these integrals depend only on the end points of the space-like strands so that one deform the space-like strands in an arbitrarily manner. One can in fact assign these kind of invariants to pairs of knots and these invariants define the dancing operation transforming these knots to each other. In the special case that the second knot is un-knot one obtains a knot-invariant (or link- or braid-invariant).

2. The objection is that these invariants depend on the orbits of the end points of the space-like braid strands. Does this mean that one should perform an averaging over the ends with the condition that space-like braid is not affected topologically by the allowed deformations for the positions of the end points?

3. Under what conditions on deformation the magnetic fluxes are not affect in the deformation of the braid strands at 3-D surfaces? The change of the Kähler magnetic flux is magnetic flux over the closed 2-surface defined by initial non-deformed and deformed stringy two-surfaces minus flux over the 2-surfaces defined by the original time-like and space-like braid strands connected by a thin 2-surface to their small deformations. This is the case if the deformation corresponds to a U(1) gauge transformation for a Kähler flux. That is diffeomorphism of M^4 and symplectic transformation of CP_2 inducing the U(1) gauge transformation.

   Hence a natural equivalence for braids is defined by these transformations. This is quite not a topological equivalence but quite a general one. Symplectic transformations of CP_2 at light-like and space-like 3-surfaces define isometries of the world of classical worlds so that also in this sense the equivalence is natural. Note that the deformations of space-time surfaces correspond to this kind of transformations only at space-like 3-surfaces at the ends of CDs and at the light-like wormhole throats where the signature of the induced metric changes. In fact, in quantum TGD the sub-spaces of world of classical worlds with constant values of zero modes (non-quantum fluctuating degrees of freedom) correspond to orbits of 3-surfaces under symplectic transformations so that the symplectic restriction looks rather natural also from the point of view of quantum dynamics and the vacuum expectation defined by Kähler function be defined for physical states.

4. A further possibility is that the light-like and space-like 3-surfaces carry vanishing induced Kähler fields and represent surfaces in $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2 carrying vanishing Kähler form. The interior of space-time surface could in principle carry a non-vanishing Kähler form. In this case weak form of self-duality cannot hold true. This however implies that the Kähler magnetic fluxes vanish identically as circulations of Kähler gauge potential. The non-integrable phase factors defined by electroweak gauge potentials would however define non-trivial classical Wilson loops. Also electromagnetic field would do so. It would be therefore possible to imagine vacuum expectation value of Wilson loop for given quantum state. Exponent of Kähler action would define for non-vacuum extremals the weighting. For 4-D vacuum extremals this exponent is trivial and one might imagine of using imaginary exponent of electroweak Chern-Simons action. Whether the restriction to vacuum extremals in the definition of vacuum expectations of electroweak Wilson loops could define general enough invariants for braid cobordisms remains an open question.

5. The quantum expectation values for Wilson loops are non-Abelian generalizations of exponentials for the expectation values of Kähler magnetic fluxes. The classical color field is proportional to the induced Kähler form and its holonomy is Abelian which raises the question whether the non-Abelian Wilson loops for classical color gauge field could be expressible in terms of Kähler magnetic fluxes.

14.5.3 Classical Color Gauge Fields And Their Generalizations Define Gerbe Gauge Potentials Allowing To Replace Wilson Loops With Wilson Sheets

As already noticed, the description of 2-knots seems to necessitate the generalization of gauge field to 3-form and the introduction of a gerbe structure. This seems to be possible in TGD framework.

1. Classical color gauge fields are proportional to the products $B_A = H_A J$ of the Hamiltonians of color isometries and of Kähler form and the closed 3-form $F_A = dB_A = dH_A \wedge J$ could serve as a colored 3-form defining the analog of U(1) gauge field. What would be interesting that color would make F non-vanishing. The "circulation" $h_A = \oint H_A J$ over a closed partonic 2-surface transforms covariantly under symplectic transformations of CP_2, whose Hamiltonians can be assigned to irreps of SU(3): just the commutator of Hamiltonians defined by Poisson bracket appears in the infinitesimal transformation. One could hope that the expectation

values for the exponents of the fluxes of B_A over 2-knots could define the covariants able to catch 2-knotted-ness in TGD framework. The exponent defining Wilson loop would be replaced with $exp(iQ^A h_A)$, where Q^A denote color charges acting as operators on particles involved.

2. Since the symplectic group acting on partonic 2-surfaces at the boundary of CD replaces color group as a gauge group in TGD, one can ask whether symplectic SU(3) should be actually replaced with the entire symplectic group of $\cup_{\pm}\delta M^4_{\pm} \times CP_2$ with Hamiltonians carrying both spin and color quantum numbers. The symplectic fluxes $\oint H_A J$ are indeed used in the construction of both quantum states and of WCW geometry. This generalization is indeed possible for the gauge potentials $B_A J$ so that one would have infinite number of classical gauge fields having also interpretation as gerbe gauge potentials.

3. The objection is that symplectic transformations are not symmetries of Kähler action. Therefore the action of symplectic transformation induced on the space-time surface reduces to a symplectic transformation only at the partonic 2-surfaces. This spoils the covariant transformation law for the 2-fluxes over stringy world sheets unless there exist preferred stringy world sheets for which the action is covariant. The proposed duality between the descriptions based on partonic 2-surfaces and stringy world sheets realized in terms of slicings of space-time surface by string world sheets and partonic 2-surfaces suggests that this might be the case.

   This would mean that one can attach to a given partonic 2-surface a unique collection string world sheets. The duality suggests even stronger condition stating that the total exponents $exp(iQ^A h_A)$ of fluxes are the same irrespective whether h_A evaluated for partonic 2-surfaces or for string world sheets defining the analog of 2-knot. This would mean an immense calculational simplification! This duality would correspond very closely to the weak form of electric magnetic duality whose various forms I have pondered as a must for the geometry of WCW . Partonic 2-surfaces indeed correspond to magnetic monopoles at least for elementary particles and stringy world sheets to surfaces carrying electric flux (note that in the exponent magnetic charges do not make themselves visible so that the identity can make sense also for $H_A = 1$).

4. Quantum expectation means in TGD framework a functional integral over the symplectic orbits of partonic 2-surfaces plus 4-D tangent space data assigned to the upper and lower boundaries of CD. Suppose that holography fixes the space-like 3-surfaces at the ends of CD and light-like orbits of partonic 2-surfaces. In completely general case the braids and the stringy space-time sheets could be fixed using a representation in terms of space-time coordinates so that the representation would be always the same but the embedding varies as also the values of the exponent of Kähler function, of the Wilson loop, and of its 2-D generalization. The functional integral over symplectic transforms of 3-surfaces implies that Wilson loop and its 2-D generalization varies.

   The proposed duality however suggests that both Wilson loop and its 2-D generalization are actually fixed by the dynamics of quantum TGD. One can ask whether the presence of 2-D analog of Wilson loop has a direct physical meaning bringing into almost topological stringy dynamics associated with color quantum numbers and coding explicit information about space-time interior and topology of field lines so that color dynamics would also have interpretation as a theory of 2-knots. If the proposed duality suggested by holography holds true, only the data at partonic 2-surfaces would be needed to calculate the generalized Wilson loops.

   In TGD framework the localization of the modes of the induced spinor fields at 2-D surfaces carrying vanishing induced W boson fields guaranteeing that the em charge of spinor modes is well-defined for a generic preferred extremal is natural [K96]. Besides string world sheets partonic 2-surfaces are good candidates for this kind of surfaces. It is not clear whether one can have a continuous slicing of this kind by string world sheets and partonic 2-surfaces orthogonal to them or whether only discrete set of these surfaces is possible.

This picture is very speculative and sounds too good to be true but follows if one consistently applies holography.

14.5.4 Summing Sup The Basic Ideas

Let us summarize the ideas discussed above.

1. Instead of knots, links, and braids one could study knot and link cobordisms, that is their dynamical evolutions concretizable in terms of dance metaphor and in terms of interacting string world sheets. Each space-like braid strand can have purely internal knotting and braid strands can be linked. TGD could allow to identify uniquely both space-like and time-like braid strands and thus also the stringy world sheets more or less uniquely and it could be that the dynamics induces automatically the temporary cutting of braid strands when knot is opened violently so that a hole is generated. Gerbe gauge potentials defined by classical color gauge fields could make also possible to characterize 2-knottedness in symplectic invariant manner in terms of color gauge fluxes over 2-surfaces.

   The weak form of electric-magnetic duality would reduce the situation to almost topological QFT in general case with topological invariance replaced with symplectic one which corresponds to the fixing of the values of non-quantum fluctuating zero modes in quantum TGD. In the vacuum sector it would be possible to have the counterparts of Wilson loops weighted by 3-D electroweak Chern-Simons action defined by the induced spinor connection.

2. One could also leave TGD framework and define invariants of braid cobordisms and 2-D analogs of braids as vacuum expectations of Wilson loops using Chern-Simons action assigned to 3-surfaces at which space-like and time-like braid strands end. The presence of light-like and space-like 3-surfaces assignable to causal diamonds could be assumed also now.

I checked whether the article of Gukov, Scwhartz, and Vafa entitled "Khovanov-Rozansky Homology and Topological Strings" [A71, A71] relies on the primitive topological observations made above. This does not seem to be the case. The topological strings in this case are strings in 6-D space rather than 4-D space-time.

There is also an article by Dror Bar-Natan with title "Khovanov's homology for tangles and cobordisms" [A51]. The article states that the Khovanov homology theory for knots and links generalizes to tangles, cobordisms and 2-knots. The article does not say anything explicit about Wilson loops but talks about topological QFTs.

An article of Witten about his physical approach to Khovanov homology has appeared in arXiv [A59]. The article is more or less abracadabra for anyone not working with M-theory but the basic idea is simple. Witten reformulates 3-D Chern-Simons theory as a path integral for $\mathcal{N} = 4$ SYM in the 4-D half space $W\times; R$. This allows him to use dualities and bring in the machinery of M-theory and 6-branes. The basic structure of TGD forces a highly analogous approach: replace 3-surfaces with 4-surfaces, consider knot cobordisms and also 2-knots, introduce gerbes, and be happy with symplectic instead of topological QFT, which might more or less be synonymous with TGD as almost topological QFT. Symplectic QFT would obviously make possible much more refined description of knots.

14.6 Witten's Approach To Khovanov Homology From TGD Point Of View

Witten's approach to Khovanov comohology [A59] relies on fivebranes as is natural if one tries to define 2-knot invariants in terms of their cobordisms involving violent un-knottings. Despite the difference in approaches it is very useful to try to find the counterparts of this approach in quantum TGD since this would allow to gain new insights to quantum TGD itself as almost topological QFT identified as symplectic theory for 2-knots, braids and braid cobordisms.

An essentially unique identification of string world sheets and therefore also of the braids whose ends carry quantum numbers of many particle states at partonic 2-surfaces emerges if one identifies the string word sheets as singular surfaces in the same manner as is done in Witten's approach [A59].

Also a physical interpretation of the operators Q, F, and P of Khovanov homology emerges. P would correspond to instanton number and F to the fermion number assignable to right handed neutrinos. The breaking of M^4 chiral invariance makes possible to realize Q physically. The finding that the generalizations of Wilson loops can be identified in terms of the gerbe fluxes $\int H_A J$ supports the conjecture that TGD as almost topological QFT corresponds essentially to a symplectic theory for braids and 2-knots.

14.6.1 Intersection Form And Space-Time Topology

The violent unknotting corresponds to a sequence of steps in which braid or knot becomes trivial and this very process defines braid invariants in TGD approach in nice concordance with the basic recipe for the construction of Jones polynomial. The topological invariant characterizing this process as a dynamics of 2-D string like objects defined by braid strands becomes knot invariant or more generally, invariant depending on the initial and final braids.

The process is describable in terms of string interaction vertices and also involves crossings of braid strands identifiable as self-intersections of the string world sheet. Hence the intersection form for the 2-surfaces defining braid strand orbits becomes a braid invariant. This intersection form is also a central invariant of 4-D manifolds and Donaldson's theorem [A7] says that for this invariant characterizes simply connected smooth 4-manifold completely. Rank, signature, and parity of this form in the basis defined by the generators of 2-homology (excluding torsion elements) characterize smooth closed and orientable 4-manifold. It is possible to diagonalize this form for smoothable 4-surfaces. Although the situation in the recent case differs from that in Donaldson theory in that the 4-surfaces have boundary and even fail to be manifolds, there are reasons to believe that the theory of braid cobordisms and 2-knots becomes part of the theory of topological invariants of 4-surfaces just as knot theory becomes part of the theory of 3-manifolds. The representation of 4-manifolds as space-time surfaces might also bring in physical insights.

This picture leads to ideas about string theory in 4-D space-time as a topological QFT. The string world sheets define the generators of second relative homology group. "Relative" means that closed surfaces are replaced with surfaces with boundaries at wormhole throats and ends of CD. These string world sheets, if one can fix them uniquely, would define a natural basis for homology group defining the intersection form in terms of violent unbraiding operations (note that also reconnections are involved).

Quantum classical correspondence encourages to ask whether also physical states must be restricted in such a way that only this minimum number of strings carrying quantum numbers at their ends ending to wormhole throats should be allowed. One might hope that there exists a unique identification of the topological strings implying the same for braids and allowing to identify various symplectic invariants as Hamiltonian fluxes for the string world sheets.

14.6.2 Framing Anomaly

In 3-D approach to knot theory the framing of links and knots represents an unavoidable technical problem [A59]. Framing means a slight shift of the link so that one can define self-linking number as a linking number for the link and its shift. The problem is that this framing of the link - or trivialization of its normal bundle in more technical terms- is not topological invariant and one obtains a large number of framings. For links in S^3 the framing giving vanishing self-linking number is the unique option and Atyiah has shown that also in more general case it is possible to identify a unique framing.

For 2-D surfaces self-linking is replaced with self-intersection. This is well-defined notion even without framing and indeed a key invariant. One might hope that framing is not needed also for string world sheets. If needed, this framing would induce the framing at the space-like and light-like 3-surfaces. The restriction of the section of the normal bundle of string world sheet to the 3-surfaces must lie in the tangent space of 3-surfaces. It is not clear whether this is enough to resolve the non-uniqueness problem.

14.6.3 Khovanov Homology Briefly

Khovanov homology involves three charges Q, F, and P. Q is analogous to super charge and satisfies $Q^2 = 0$ for the elements of homology. The basic commutation relations between the charges are $[F, Q] = Q$ and $[P, Q] = 0$. One can show that the Khovanov homology $\kappa(L)$ for link can be expressed as a bi-graded direct sum of the eigen-spaces $V_{m,n}$ of F and P, which have integer valued spectra. Obviously Q increases the eigenvalue of F and maps $V_{m,n}$ to $V_{m+1,n}$ just as exterior derivative in de-Rham comology increases the degree of differential form. P acts as a symmetry allowing to label the elements of the homology by an integer valued charge n.

Jones polynomial can be expressed as an index assignable to Khovanov homology:

$$\mathcal{J}(q|L) = Tr((-1)^F q^P \quad . \tag{14.6.1}$$

Here q defining the argument of Jones polynomial is root of unity in Chern-Simons theory but can be extended to complex numbers by extending the positive integer valued Chern-Simons coupling k to a complex number. The coefficients of the resulting Laurent polynomial are integers: this result does not follow from Chern-Simons approach alone. Jones polynomial depends on the spectrum of F only modulo 2 so that a lot of information is lost as the homology is replaced with the polynomial.

Both the need to have a more detailed characterization of links and the need to understand why the Wilson loop expectation is Laurent polynomial with integer coefficients serve as motivations of Witten for searching a physical approach to Khovanov polynomial.

The replacement of $D = 2$ in braid group approach to Jones polynomial with $D = 3$ for Chern-Simons approach replaced by something new in $D = 4$ would naturally correspond to the dimensional hierarchy of TGD in which partonic 2-surfaces plus their 2-D tangent space data fix the physics. One cannot quite do with partonic 2-surfaces and the inclusion of 2-D tangent space-data leads to holography and unique space time surfaces and perhaps also unique string world sheets serving as duals for partonic 2-surfaces. This would realize the weak form of electric magnetic duality at the level of homology much like Poincare duality relates cohomology and homology.

14.6.4 Surface Operators And The Choice Of The Preferred 2-Surfaces

The choice of preferred 2-surfaces and the identification of surface operators in $\mathcal{N} = 4$ YM theory is discussed in [A53]. The intuitive picture is that preferred 2-surfaces- now string world sheets defining braid cobordisms and 2-knots- correspond to singularities of classical gauge fields. Surface operator can be said to create this singularity. In functional integral this means the presence of the exponent defining the analog of Wilson loop.

1. In [A53] the 2-D singular surfaces are identified as poles for the magnitude r of the Higgs field. One can assign to the 2-surface fractional magnetic charges defined for the Cartan algebra part A_C of the gauge connection as circulations $\oint A_C$ around a small circle around the axis of singularity at $r = \infty$. What happens that 3-D $r = constant$ surface reduces to a 2-D surface at $r = \infty$ whereas A_C and entire gauge potential behaves as $A = A_C = \alpha d\phi$ near singularity. Here ϕ is coordinate analogous to angle of cylindrical coordinates when t-z plane represents the singular 2-surface. α is a linear combination of Cartan algebra generators.

2. The phase factor assignable to the circulation is essentially $exp(i2\pi\alpha)$ and for non-fractional magnetic charges it differs from unity. One might perhaps say that string word sheets correspond to singularities for the slicing of space-time surface with 3-surfaces at which 3-surfaces reduce to 2-surfaces.

Consider now the situation in TGD framwork.

1. The gauge group is color gauge group and gauge color gauge potentials correspond to the quantities $H_A J$. One can also consider a generalization by allowing all Hamiltonians generating symplectic transformations of CP_2. Kähler gauge potential is in essential role since color gauge field is proportional to Kähler form.

2. The singularities of color gauge fields can be identified by studing the theory locally as a field theory from CP_2 to M^4. It is quite possible to have space-time surfaces for which M^4 coordinates are many-valued functions of CP_2 coordinates so that one has a covering of CP_2 locally. For singular 2-surfaces this covering becomes singular in the sense that separate sheets coincide. These coverings do not seem to correspond to those assignable to the hierarchy of Planck constants implied by the many-valuedness of the time derivatives

of the embedding space coordinates as functions of canonical momentum densities but one must be very cautious in making too strong conclusions here.

3. To proceed introduce the Eguchi-Hanson coordinates

$$(\xi^1, \xi^2) = [rcos(\theta/2)exp(i(\Psi + \Phi)/2), rsin(\theta/2)exp(i(-\Psi + \Phi)/2]$$

for CP_2 with the defining property that the coordinates transform linearly under $U(2) \subset SU(3)$. In QFT context these coordinates would be identified as Higgs fields. The choice of these coordinates is unique apart from the choice of the $U(2)$ subgroup and rotation by element of $U(2)$ once this choice has been made. In TGD framework the definition of CD involves the fixing of these coordinates and the interpretation is in terms of quantum classical correspondence realizing the choice of quantization axes of color at the level of the WCW geometry.

r has a natural identification as the magnitude of Higgs field invariant under $U(2) \subset SU(3)$. The $SU(2) \times U(1)$ invariant 3-sphere reduces to a homologically non-trivial geodesic 2-sphere at $r = \infty$ so that for this choice of coordinates this surface defines in very natural manner the counterpart of singular 2-surface in CP_2 geometry. At this sphere the second phase associated with CP_2 coordinates- Φ - becomes a redundant coordinate just like the angle Φ at the poles of sphere. There are two other similar spheres and these three spheres are completely analogous to North and South poles of 2-sphere.

4. One possibility is that the singular surfaces correspond to the inverse images for the projection of the embedding map to $r = \infty$ geodesic sphere of CP_2 for a CD corresponding to a given choice of quantization axes. Also the inverse images of all homological non-trivial geodesic spheres defining the three poles of CP_2 can be considered. The inverse images of this geodesic 2-sphere under the embedding-projection map would naturally correspond to 2-D string world sheets for the preferred extremals for a generic space-time surface. For cosmic strings and massless extremals the inverse image would be 4-dimensional but this problem can be circumvented easily. The identification turned out to be somewhat ad hoc and later a much more convincing unique identification of string world sheets emerged and will be discussed in the sequel. Despite this the general aspects of the proposal deserves a discussion.

5. The existence of dual slicings of space-time surface by 3-surfaces and lines on one hand and by string world sheets Y^2 and 2-surfaces X^2 with Euclidian signature of metric on one hand, is one of the basic conjectures about the properties of preferred extremals of Kähler action. A stronger conjecture is that partonic 2-surfaces represent particular instances of X^2. The proposed picture suggests an amazingly simple and physically attractive identification of these slicings.

   (a) The slicing induced by the slicing of CP_2 by $r = constant$ surfaces defines an excellent candidate for the slicing by 3-surfaces. Physical the slices would correspond to equivalence classes of choices of the quantization axes for color group related by $U(2)$. In gauge theory context they would correspond to different breakings of $SU(3)$ symmetry labelled by the vacuum expectation of the Higgs field r which would be dynamical for CP_2 projections and play the role of time coordinate.

   (b) The slicing by string world sheets would naturally correspond to the slicing induced by the 2-D space of homologically non-trivial geodesic spheres (or triplets of them) and could be called "CP_2/S^2". One has clearly bundle structure with S^2 as base space and "CP_2/S^2" as fiber. Partonic 2-surfaces could be seen locally as sections of this bundle like structure assigning a point of "CP_2/S^2" to each point of S^2. Globally this does not make sense for partonic 2-surfaces with genus larger than $g = 0$.

6. In TGD framework the Cartan algebra of color gauge group is the natural identification for the Cartan algebra involved and the fluxes defining surface operators would be the classical fluxes $\int H_A J$ over the 2-surfaces in question restricted to Cartan algebra. What would be new is the interpretation as gerbe gauge potentials so that flux becomes completely analogous to Abelian circulation.

If one accepts the extension of the gauge algebra to a symplectic algebra, one would have the Cartan algebra of the symplectic algebra. This algebra is defined by generators which depend on the second half P_i or Q_i of Darboux coordinates. If P_i are chosen to be functions of the coordinates (r, θ) of CP_2 coordinates whose Poisson brackets with color isospin and hyper charge generators inducing rotations of phases (Ψ, Φ) of CP_2 complex coordinates vanish, the symplectic Cartan algebra would correspond to color neutral Hamiltonians. The spherical harmonics with non-vanishing angular momentum vanish at poles and one expects that same happens for CP_2 spherical harmonics at the three poles of CP_2. Therefore Cartan algebra is selected automatically for gauge fluxes.

This subgroup leaves the ends of the points of braids at partonic 2-surfaces invariant so that symplectic transformations do not induce braiding.

If this picture -resulting as a rather straightforward translation of the picture applied in QFT context- is correct, TGD would predict uniquely the preferred 2-surfaces and therefore also the braids as inverse images of CP_2 geodesic sphere for the embedding of space-time surface to $CD \times CP_2$. Also the conjecture slicings by 3-surfaces and string world sheets could be identified. The identification of braids and slicings has been indeed one of the basic challenges in quantum TGD since in quantum theory one does not have anymore the luxury of topological invariance and I have proposed several identifications. If one accepts only these space-time sheets then the stringy content for a given space-time surface would be uniquely fixed.

The assignment of singularities to the homologically non-trivial geodesic sphere suggests that the homologically non-trivial space-time sheets could be seen as 1-dimensional idealizations of magnetic flux tubes carrying Kähler magnetic flux playing key role also in applications of TGD, in particular biological applications such as DNA as topological quantum computer and bio-control and catalysis.

14.6.5 The Identification Of Charges Q, P And F Of Khovanov Homology

The challenge is to identify physically the three operators Q, F, and P appearing in Khovanov homology. Taking seriously the proposal of Witten [A59] and looking for its direct counterpart in TGD leads to the identification and physical interpretation of these charges in TGD framework.

1. In Witten's approach P corresponds to instanton number assignable to the classical gauge field configuration in space-time. In TGD framework the instanton number would naturally correspond to that assignable to CP_2 Kähler form. One could consider the possibility of assigning this charge to the deformed CP_2 type vacuum extremals assigned to the space-like regions of space-time representing the lines of generalized Feynman diagrams having elementary particle interpretation. P would be or at least contain the sum of unit instanton numbers assignable to the lines of generalized Feynman diagrams with sign of the instanton number depending on the orientation of CP_2 type vacuum extremal and perhaps telling whether the line corresponds to positive or negative energy state. Note that only pieces of vacuum extremals defined by the wormhole contacts are in question and it is somewhat questionable whether the rest of them in Minkowskian regions is included.

2. F corresponds to $U(1)$ charge assignable to R-symmetry of $N = 4$ SUSY in Witten's theory. The proposed generalization of twistorial approach in TGD framework suggests strongly that this identification generalizes to TGD. In TGD framework all solutions of Kähler-Dirac equation at wormhole throats define super-symmetry generators but the supersymmetry is badly broken. The covariantly constant right handed neutrino defines the minimally broken supersymmetry since there are no direct couplings to gauge fields. This symmetry is however broken by the mixing of right and left handed M^4 chiralities present for both Dirac actions for induced gamma matrices and for Kähler-Dirac equations defined by Kähler action and Chern-Simons action at parton orbits. It is caused by the fact that both the induced and Kähler-Dirac gamma matrices are combinations of M^4 and CP_2 gamma matrices. F would therefore correspond to the net fermion number assignable to right handed neutrinos and antineutrinos. F is not conserved because of the chirality mixing and electroweak interactions respecting only the conservation of lepton number.

Note that the mixing of M^4 chiralities in sub-manifold geometry is a phenomenon characteristic for TGD and also a direct signature of particle massivation and SUSY breaking. It would be nice if it would allow the physical realization of Q operator of Khovanov homology.

3. Witten proposes an explicit formula for Q in terms of 5-dimensional time evolutions interpolating between two 4-D instantons and involving sum of sign factors assignable to Dirac determinants. In TGD framework the operator Q should increase the right handed neutrino number by one unit and therefore transform one right-handed neutrino to a left handed one in the minimal situation. In zero energy ontology Q should relate to a time evolution either between ends of CD or between the ends of single line of generalized Feynman diagram. If instanton number can be assigned solely to the wormhole contacts, this evolution should increase the number of CP_2 type extremals by one unit. 3-particle vertex in which right handed neutrino assignable to a partonic 2-surface transforms to a left handed one is thus a natural candidate for defining the action of Q.

4. Note that the almost topological QFT property of TGD together with the weak form of electric-magnetic duality implies that Kähler action reduces to Abelian Chern-Simons term. Ordinary Chern-Simons theory involves imaginary exponent of this term but in TGD the exponent would be real. Should one replace the real exponent of Kähler function with imaginary exponent? If so, TGD would be very near to topological QFT also in this respect. This would also force the quantization of the coupling parameter k in Chern-Simons action. On the other hand, the Chern-Simons theory makes sense also for purely imaginary k [A59].

14.6.6 What Does The Replacement Of Topological Invariance With Symplectic Invariance Mean?

One interpretation for the symplectic invariance is as an analog of diffeo-invariance. This would imply color confinement. Another interpretation would be based on the identification of symplectic group as a color group. Maybe the first interpretation is the proper restriction when one calculates invariants of braids and 2-knots.

The replacement of topological symmetry with symplectic invariance means that TGD based invariants for braids carry much more refined information than topological invariants. In TGD approach M^4 diffeomorphisms act freely on partonic 2-surfaces and 4-D tangent space data but the action in CP_2 degrees of freedom reduces to symplectic transformations. One could of course consider also the restriction to symplectic transformations of the light-cone boundary and this would give additional refinements.

It is is easy to see what symplectic invariance means by looking what it means for the ends of braids at a given partonic 2-surface.

1. Symplectic transformations respect the Kähler magnetic fluxes assignable to the triangles defined by the finite number of braid points so that these fluxes defining symplectic areas define some minimum number of coordinates parametrizing the moduli space in question. For topological invariance all n-point configurations obtained by continuous or smooth transformations are equivalent braid end configurations. These finite-dimensional moduli spaces would be contracted with point in topological QFT.

2. This picture led to a proposal of what I call symplectic QFT [K16] in which the associativity condition for symplectic fusion rules leads the hierarchy of algebras assigned with symplectic triangulations and forming a structures known as operad in category theory. The ends of braids at partonic 2-surfaces would would define unique triangulation of this kind if one accepts the identification of string like 2-surfaces as inverse images of homologically non-trivial geodesic sphere.

Note that both diffeomorphisms and symplectic transformations can in principle induce braiding of the braid strands connecting two partonic 2-surfaces. Should one consider the possibility that the allow transformations are restricted so that they do not induce braiding?

1. These transformations induce a transformation of the space-time surface which however is not a symplectic transformation in the interior in general. An attractive conjecture is that

for the preferred extremals this is the case at the inverse images of the homologically non-trivial geodesic sphere. This would conform with the proposed duality between partonic 2-surfaces and string world sheets inspired by holography and also with quantum classical correspondence suggesting that at string world sheets the transformations induced by symplectic transformations at partonic 2-surfaces act like symplectic transformations.

2. If one allows only the symplectic transformations in Cartan algebra leaving the homologically non-trivial geodesic sphere invariant, the infinitesimal symplectic transformations would affect neither the string word sheets nor braidings but would modify the partonic 2-surfaces at all points except at the intersections with string world sheets.

14.7 Algebraic Braids, Sub-Manifold Braid Theory, And Generalized Feynman Diagrams

Ulla send me a link to an article by Sam Nelson about very interesting new-to-me notion known as algebraic knots (see `http://tinyurl.com/yauy7asy`) [A95, A83], which has initiated a revolution in knot theory. This notion was introduced 1996 by Louis Kauffmann [A85] so that it is already 15 year old concept. While reading the article I realized that this notion fits perfectly the needs of TGD and leads to a progress in attempts to articulate more precisely what generalized Feynman diagrams are.

In the following I will summarize briefly the vision about generalized Feynman diagrams, introduce the notion of algebraic knot, and after than discuss in more detail how the notion of algebraic knot could be applied to generalized Feynman diagrams. The algebraic structrures kei, quandle, rack, and biquandle and their algebraic modifications as such are not enough. The lines of Feynman graphs are replaced by braids and in vertices braid strands redistribute. This poses several challenges: the crossing associated with braiding and crossing occurring in non-planar Feynman diagrams should be integrated to a more general notion; braids are replaced with sub-manifold braids; braids of braids....of braids are possible; the redistribution of braid strands in vertices should be algebraized. In the following I try to abstract the basic operations which should be algebraized in the case of generalized Feynman diagrams.

One should be also able to concretely identify braids and 2-braids (string world sheets) as well as partonic 2-surfaces and I have discussed several identifications during last years. Legendrian braids turn out to be very natural candidates for braids and their duals for the partonic 2-surfaces. String world sheets in turn could correspond to the analogs of Lagrangian sub-manifolds or to minimal surfaces of space-time surface satisfying the weak form of electric-magnetic duality. The latter option turns out to be more plausible. Finite measurement resolution would be realized as symplectic invariance with respect to the subgroup of the symplectic group leaving the end points of braid strands invariant. In accordance with the general vision TGD as almost topological QFT would mean symplectic QFT. The identification of braids, partonic 2-surfaces and string world sheets - if correct - would solve quantum TGD explicitly at string world sheet level in other words in finite measurement resolution.

Irrespective of whether the algebraic knots are needed, the natural question is what generalized Feynman diagrams are. It seems that the basic building bricks can be identified so that one can write rather explicit Feynman rules already now. Of course, the rules are still far from something to be burned into the spine of the first year graduate student.

14.7.1 Generalized Feynman Diagrams, Feynman Diagrams, And Braid Diagrams

How knots and braids a la TGD differ from standard knots and braids?

TGD approach to knots and braids differs from the knot and braid theories in given abstract 3-manifold (4-manifold in case of 2-knots and 2-braids) is that space-time is in TGD framework identified as 4-D surface in $M^4 \times CP_2$ and preferred 3-surfaces correspond to light-like 3-surfaces defined by wormhole throats and space-like 3-surfaces defined by the ends of space-time sheets at the two light-like boundaries of causal diamond CD.

The notion of finite measurement resolution effectively replaces 3-surfaces of both kinds with braids and space-time surface with string world sheets having braids strands as their ends. The 4-dimensionality of space-time implies that string world sheets can be knotted and intersect at discrete points (counterpart of linking for ordinary knots). Also space-time surface can have self-intersections consisting of discrete points.

The ordinary knot theory in E^3 involves projection to a preferred 2-plane E^2 and one assigns to the crossing points of the projection an index distinguishing between two cases which are transformed to each other by violently taking the first piece of strand through another piece of strand. In TGD one must identify some physically preferred 2-dimensional manifold in embedding space to which the braid strands are projected. There are many possibilities even when one requires maximal symmetries. An obvious requirement is however that this 2-manifold is large enough.

1. For the braids at the ends of space-time surface the 2-manifold could be large enough sphere S^2 of light-cone boundary in coordinates in which the line connecting the tips of CD defines a preferred time direction and therefore unique light-like radial coordinate. In very small knots it could be also the geodesic sphere of CP_2 (apart from the action of isometries there are two geodesic spheres in CP_2).

2. For light-like braids the preferred plane would be naturally M^2 for which time direction corresponds to the line connecting the tips of CD and spatial direction to the quantization axis of spin. Note that these axes are fixed uniquely and the choices of M^2 are labelled by the points of projective sphere P^2 telling the direction of space-like axis. Preferred plane M^2 emerges naturally also from number theoretic vision and corresponds in octonionic pictures to hyper-complex plane of hyper-octonions. It is also forced by the condition that the choice of quantization axes has a geometric correlate both at the level of embedding space geometry and the geometry of the "world of classical worlds".

The braid theory in TGD framework could be called sub-manifold braid theory and certainly differs from the standard one.

1. If the first homology group of the 3-surface is non-trivial as it when the light-like 3-surfaces represents an orbit of partonic 2-surface with genus larger than zero, the winding of the braid strand (wrapping of branes in M-theory) meaning that it represents a homologically non-trivial curve brings in new effects not described by the ordinary knot theory. A typical new situation is the one in which 3-surface is locally a product of higher genus 2-surface and line segment so that knot strand can wind around the 2-surface. This gives rise to what are called non-planar braid diagrams for which the projection to plane produces non-standard crossings.

2. In the case of 2-knots similar exotic effects could be due to the non-trivial 2-homology of space-time surface. Wormhole throats assigned with elementary particle wormhole throats are homologically non-trivial 2-surfaces and might make this kind of effects possible for 2-knots if they are possible.

The challenge is to find a generalization of the usual knot and braid theories so that they apply in the case of braids (2-braids) imbedded in 3-D (4-D) surfaces with preferred highly symmetry sub-manifold of $M^4 \times CP_2$ defining the analog of plane to which the knots are projected. A proper description of exotic crossings due to non-trivial homology of 3-surface (4-surface) is needed.

Basic questions

The questions are following.

1. How the mathematical framework of standard knot theory should be modified in order to cope with the situation encountered in TGD? To my surprise I found that this kind of mathematical framework exists: so called algebraic knots [A95, A83] define a generalization of knot theory very probably able to cope with this kind of situation.

2. Second question is whether the generalized Feynman diagrams could be regarded as braid diagrams in generalized sense. Generalized Feynman diagrams are generalizations of ordinary Feynman diagrams. The lines of generalized Feynman diagrams correspond to the orbits of wormhole throats and of wormhole contacts with throats carrying elementary particle quantum numbers.

   The lines meet at vertices which are partonic 2-surfaces. Single wormhole throat can describe fermion whereas bosons have wormhole contacts with fermion and anti-fermion at the opposite throats as building bricks. It seems however that all fermions carry Kähler magnetic charge so that physical particles are string like objects with magnetic charges at their ends.

   The short range of weak interactions results from the screening of the axial isospin by neutrinos at the other end of string like object and also color confinement could be understood in this manner. One cannot exclude the possibility that the length of magnetic flux tube is of order Compton length.

3. Vertices of the generalized Feynman diagrams correspond to the partonic 2-surfaces along which light-like 3-surfaces meet and this is certainly a challenge for the required generalization of braid theory. The basic objection against the reduction to algebraic braid diagrams is that reaction vertices for particles cannot be described by ordinary braid theory: the splitting of braid strands is needed.

   The notion of bosonic emergence however suggests that 3-vertex and possible higher vertices correspond to the splitting of braids rather than braid strands. By allowing braids which come from both past and future and identifying free fermions as wormhole throats and bosons as wormhole contacts consisting of a pair of wormhole throats carrying fermion and anti-fermion number, one can understand boson excanges as recombinations without anyneed to have splitting of braid strands. Strictly and technically speaking, one would have tangles like objects instead of braids. This would be an enormous simplification since $n > 2$-vertices which are the source of divergences in QFT: s would be absent.

4. Non-planar Feynman diagrams are the curse of the twistor approach and I have already earlier proposed that the generalized Feynman amplitudes and perhaps even twistorial amplitudes could be constructed as analogs of knot invariants by recursively transforming non-planar Feynman diagrams to planar ones for which one can write twistor amplitudes. This forces to answer two questions.

   (a) Does the non-nonplanarity of Feynman diagrams - completely combinatorial objects identified as diagrams in plane - have anything to do with the non-planarity of algebraic knot diagrams and with the non-planarity of generalized Feynman diagrams which are purely geometric objects?

   (b) Could these two kind of non-planarities be fused to together by identifying the projection 2-plane as preferred $M^2 \subset M^4$. This would mean that non-planarity in QFT sense is defined for entire braids: braid A can have virtual crossing with B. Non-planarity in the sense of knot theory would be defined for braid strands inside the braids. At vertices braid strands are redistributed between incoming lines and the analog of virtual crossing be identifiable as an exchange of braid strand between braids. Several kinds of non-planarities would be present and the idea about gradual unknotting of a non-planar diagram so that a planar diagram results as the final outcome might make sense and allow to generalize the recursion recipe for the twistorial amplitudes.

   (c) This approach could be combined with the number theoretic vision that amplitudes correspond to sequences of computations with vertices identified as product and co-product for a Yangian variant of super-symplectic algebra [A36] [B33, B24, B25]. When incoming and outgoing algebraic objects are specified there would be unique smallest diagram leading from input to output. This diagram would be tree diagram in ordinary Feynman diagrammatics. This would mean huge generalization of the duality symmetry of string models if all diagrams connecting initial and final collections of algebraic objects correspond to the same amplitude.

Non-planar diagrams of quantum field theories should have natural counterpart and linking and knotting for braids defines it naturally. This suggests that the amplitudes can be interpreted as generalizations of braid diagrams defining braid invariants: braid strands would appear as legs of 3-vertices representing product and co-product. Amplitudes could be constructed as generalized braid invariants transforming recursively braided tree diagram to an un-braided diagram using same operations as for braids. In [L26] I considered a possible breaking of associativity occurring in weak sense for conformal field theories and was led to the vision that there is a fractal hierarchy of braids such that braid strands themselves correspond to braids. This hierarchy would define an operad with subgroups of permutation group in key role. Hence it seems that various approaches to the construction of amplitudes converge.

(d) One might consider the possibility that inside orbits of wormhole throats defining the lines of Feynman diagrams the R-matrix for integrable QFT in M^2 (only permutations of momenta are allowed) describes the dynamics so that one obtains just a permutation of momenta assigned to the braid strands. Ordinary braiding would be described by existing braid theories. The core problem would be the representation of the exchange of a strand between braids algebraically.

One can consider different and much simpler general approach to the non-planarity problem. In twistor Grassmannian approach [K86] generalized Feynman diagrams correspond to TGD variants of stringy diagrams. In stringy approach one gets rid of non-planarity problem altogether.

14.7.2 Brief Summary Of Algebraic Knot Theory

Basic ideas of algebraic knot theory

In ordinary knot theory one takes as a starting point the representation of knots of E^3 by their plane plane projections to which one attach a "color" to each crossing telling whether the strand goes over or under the strand it crosses in planar projection. These numbers are fixed uniquely as one traverses through the entire knot in given direction.

The so called Reidermeister moves are the fundamental modifications of knot leaving its isotopy equivalence class unchanged and correspond to continuous deformations of the knot. Any algebraic invariant assignable to the knot must remain unaffected under these moves. Reidermeister moves as such look completely trivial and the non-trivial point is that they represent the minimum number of independent moves which are represented algebraically.

In algebraic knot theory topological knots are replaced by typographical knots resulting as planar projections. This is a mapping of topology to algebra. It turns out that the existing knot invariants generalize and ordinary knot theory can be seen as a special case of the algebraic knot theory. In a loose sense one can say that the algebraic knots are to the classical knot theory what algebraic numbers are to rational numbers.

Virtual crossing is the key notion of the algebraic knot theory. Virtual crossing and their rules of interaction were introduced 1996 by Louis Kauffman as basic notions [A2]. For instance, a strand with only virtual crossings should be replaceable by any strand with the same number of virtual crossings and same end points. Reidermeister moves generalize to virtual moves. One can say that in this case crossing is self-intersection rather than going under or above. I cannot be eliminated by a small deformation of the knot. There are actually several kinds of non-standard crossings: examples listed in figure 7 of [A95]) are virtual, flat, singular, and twist bar crossings.

Algebraic knots have a concrete geometric interpretation.

(a) Virtual knots are obtained if one replaces E^3 as embedding space with a space which has non-trivial first homology group. This implies that knot can represent a homologically non-trivial curve giving an additional flavor to the unknottedness since homologically non-trivial curve cannot be transformed to a curve which is homologically non-trivial by any continuous deformation.

(b) The violent projection to plane leads to the emergence of virtual crossings. The product $(S^1 \times S^1) \times D$, where $(S^1 \times S^1)$ is torus D is finite line segment, provides the simplest example. Torus can be identified as a rectangle with opposite sides identified and homologically non-trivial knots correspond to curves winding n_1 times around the first S^1 and n_2 times around the second S^1. These curves are not continuous in the representation where $S^1 \times S^1$ is rectangle in plane.

(c) A simple geometric visualization of virtual crossing is obtained by adding to the plane a handle along which the second strand traverses and in this manner avoids intersection. This visualization allows to understand the geometric motivation for the virtual moves.

This geometric interpretation is natural in TGD framework where the plane to which the projection occurs corresponds to $M^2 \subset M^4$ or is replaced with the sphere at the boundary of S^2 and 3-surfaces can have arbitrary topology and partonic 2-surfaces defining as their orbits light-like 3-surfaces can have arbitrary genus.

In TGD framework the situation is however more general than represented by sub-manifold braid theory. Single braid represents the line of generalized Feynman diagram. Vertices represent something new: in the vertex the lines meet and the braid strands are redistributed but do not disappear or pop up from anywhere. That the braid strands can come both from the future and past is also an important generalization. There are physical argments suggesting that there are only 3-vertices for braids but not higher ones [K18]. The challenge is to represent algebraically the vertices of generalized Feynman diagrams.

Algebraic knots

The basic idea in the algebraization of knots is rather simple. If x and y are the crossing portions of knot, the basic algebraic operation is binary operation giving "the result of x going under y", call it $x \triangleright y$ telling what happens to x. "Portion of knot" means the piece of knot between two crossings and $x \triangleright y$ denotes the portion of knot next to x. The definition is asymmetrical in x and y and the dual of the operation would be $y \triangleleft x$ would be "the result of y going above x". One can of course ask, why not to define the outcome of the operation as a pair $(x \triangleleft y, y \triangleright x)$. This operation would be bi-local in a well-defined sense. One can of course do this: in this case one has binary operation from $X \times X \rightarrow X \times X$ mapping pairs of portions to pairs of portions. In the first case one has binary operation $X \times X \rightarrow X$.

The idea is to abstract this basic idea and replace X with a set endowed with operation $\triangleright$ or $\triangleleft$ or both and formukate the Reidermeister conditions given as conditions satisfied by the algebra. One ends up to four basic algebraic structures kei, quandle, rack, and biquandle.

(a) In the case of non-oriented knots the kei is the algebraic structure. Kei - or invontary quandle-is a set X with a map $X \times X \rightarrow X$ satisfying the conditions

  i. $x \triangleright x = x$ (idenpotency, one of the Reidemeister moves)

  ii. $(x \triangleright y) \triangleright y$ =x (operation is its own right inverse having also interpretation as Reidemeister move)

  iii. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity)

  $Z([t])/(t^2)$ module with $x \triangleright y = tx + (1-t)y$ is a kei.

(b) For orientable knot diagram there is preferred direction of travel along knot and one can distinguish between $\triangleright$ and its right inverse $\triangleright^{-1}$. This gives quandle satisfying the axios

  i. $x \triangleright x = x$

  ii. $(x \triangleright y) \triangleright^{-1} y = (x \triangleright^{-1} y) \triangleright y = x$

  iii. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

  $Z[t^{\pm 1}]$ nodule with $x \triangleright y = tx + (1-t)y$ is a quandle.

(c) One can also introduce framed knots: intuitively one attaches to a knot very near to it. More precise formulation in terms of a section of normal bundle of the knot.

This makes possible to speak about self-linking. Reidermeister moves must be modified appropriately. In this case rack is the appropriate structure. It satisfied the axioms of quandle except the first axiom since corresponding operation is not a move anymore. Rack axioms are eqivalent with the requirement that functions $f_y : X \to X$ defined by $f_y(x)x \triangleright y)$ are automorphisms of the structure. Therefore the elements of rack represent its morphisms. The modules over $Z[t^{\pm 1}, s]/s(t + s - 1)$ are racks. Coxeter racks are inner product spaces with $x \triangleright y$ obtained by reflecting x across y.

(d) Biquandle consists of arcs connecting the subsequent crossings (both under- and over-) of oriented knot diagram. Biquandle operation is a map $B : X \times X \to X \times X$ of order pairs satisfying certain invertibility conditions together with set theoretic Yang-Baxter equation:

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B) \ .$$

Here $I : X \to X$ is the identity map. The three conditions to which Yang-Baxter equation decomposes gives the counterparts of the above discussed axioms. Alexander biquandle is the module $Z(t^{\pm 1}, s^{\pm} 1$ with $B(x, y) = (ty + (1 - ts)x, sx)$ where one has $s \neq 1$. If one includes virtual, flat and singular crossings one obtains virtual/singular aundles and semiquandles.

14.7.3 Generalized Feynman Diagrams As Generalized Braid Diagrams?

Zero energy ontology suggests the interpretation of the generalized Feynman diagrams as generalized braid diagrams so that there would be no need for vertices at the fundamental braid strand level. The notion of algebraic braid (or tangle) might allow to formulate this idea more precisely.

Could one fuse the notions of braid diagram and Feynman diagram?

The challenge is to fuse the notions of braid diagram and Feynman diagram having quite different origin.

(a) All generalized Feynman diagrams are reduced to sub-manifold braid diagrams at microscopic level by bosonic emergence (bosons as pairs of fermionic wormhole throats). Three-vertices appear only for entire braids and are purely topological whereas braid strands carrying quantum numbers are just re-distributed in vertices. No 3-vertices at the really microscopic level! This is an additional nail to the coffin of divergences in TGD Universe.

(b) By projecting the braid strands of generalized Feynman diagrams to preferred plane $M^2 \subset M^4$ (or rather 2-D causal diamond), one could achieve a unified description of non-planar Feynman diagrams and braid diagrams. For Feynman diagrams the intersections have a purely combinatorial origin coming from representations as 2-D diagrams.

For braid diagrams the intersections have different origin and non-planarity has different meaning. The crossings of entire braids analogous to those appearing in non-planar Feynman diagrams should define one particular exotic crossing besides virtual crossings of braid strands due to non-trivial first homology of 3-surfaces.

(c) The necessity to choose preferred plane M^2 looks strange from QFT point of view. In TGD framework it is forced by the number theoretic vision in which M^2 represents hyper-complex plane of sub-space of hyper-octonions which is subspace of complexified octonions. The choice of M^2 is also forced by the condition that the choice of quantization axes has a geometric correlate both at the level of embedding space geometry and the geometry of the “world of classical worlds”.

(d) Also 2-braid diagrams defined as projections of string world sheets are suggestive and would be defined by a projections to the 3-D boundary of CD or to $M^3 \subset M^4$. They would provide a more concrete stringy illustration about generalized Feynman diagram as analog of string diagram. Another attractive illustration is in terms of dance metaphor with the boundary of CD defining the 3-D space-like parquette. The duality between space-like and light-like braids is expected to be of importance.

The obvious conjecture is that Feynman amplitudes are a analogous to knot invariants constructible by gradually reducing non-planar Feynman diagrams to planar ones after which the already existing twistor theoretical machinery of $\mathcal{N} = 4$ SYMs would apply [K31, K86, L33].

Does 2-D integrable QFT dictate the scattering inside the lines of generalized Feynman diagrams

The preferred plane M^2 (more precisely, 2-D causal diamond having also interpretation as Penrose diagram) plays a key role as also the preferred sphere S^2 at the boundary of CD. It is perhaps not accident that a generalization of braiding was discovered in integrable quantum field theories in M^2. The S-matrix of this theory is rather trivial looking: particle moving with different velocities cross each other and suffer a phase lag and permutation of 2-momenta which has physical effects only in the case of non-identical particles. The R-matrix describing this process reduces to the R-matrix describing the basic braiding operation in braid theories at the static limit.

I have already earlier conjectured that this kind of integrable QFT is part of quantum TGD [K20]. The natural guess is that it describes what happens for the projections of 4-momenta in M^2 in scattering process inside lines of generalized Feynman diagrams. If integrable theories in M^2 control this scattering, it would cause only phase changes and permutation of the M^2 projections of the 4-momenta. The most plausible guess is that M^2 QFT characterized by R-matrix describes what happens to the braid momenta during the free propagation and the remaining challenge would be to understand what happens in the vertices defined by 2-D partonic surfaces at which re-distribution of braid strands takes place.

How quantum TGD as almost topological QFT differs from topological QFT for braids and 3-manifolds

One must distinguish between two topological QFTs. These correspond to topological QFT defining braid invariants and invariants of 3-manifolds respectively. The reason is that knots are an essential element in the procedure yielding 3-manifolds. Both 3-manifold invariants and knot invariants would be defined as Wilson loops involving path integral over gauge connections for a given 3-manifold with exponent o non-Abelkian f Chern-Simons action defining the weight.

(a) In TGD framework the topological QFT producing braid invariants for a given 3-manifold is replaced with sub-manifold braid theory. Kähler action reduces Chern-Simons terms for preferred extremals and only these contribute to the functional integral. What is the counterpart of topological invariance in this framework? Are general isotopies allowed or should one allow only sub-group of symplectic group of CD boundary leaving the end points of braids invariant? For this option Reidermeister moves are undetectable in the finite measurement resolution defined by the subgroup of the symplectic group. Symplectic transformations would not affect 3-surfaces as the analogs of abstract contact manifold since induced Kähler form would not be affected and only the embedding would be changed.

In the approach based on inclusions of HFFs gauge invariance or its generalizations would represent finite measurement resolution (the action of included algebra would generate states not distiguishable from the original one).

(b) There is also ordinary topological QFT allowing to construct topological invariants for 3-manifold. In TGD framework the analog of topological QFT is defined by Chern-Simons-Kähler action in the space of preferred 3-surfaces. Now one sums over small deformations of 3-surface instead of gauge potentials. If extremals of Chern-Simons-Kähler action are in question, symplectic invariance is the most that one can hope for and this might be the situation quite generally. If all light-like 3-surfaces are allowed so that only weak form of electric-magnetic duality at them would bring metric into the theory, it might be possible to have topological invariance at 3-D level but not at 4-D level. It however seems that symplectic invariance with respect to subgroup leaving end points of braids invariant is the realistic expectation.

Could the allowed braids define Legendrian sub-manifolds of contact manifolds?

The basic questions concern the identification of braids and 2-braids. In quantum TGD they cannot be arbitrary but determined by dynamics providing space-time correlates for quantum dynamics. The deformations of braids should mean also deformations of 3-surfaces which as topological manifolds would however remain as such. Therefore topological QFT for given 3-manifold with path integral over gauge connections would in TGD correspond to functional integral of 3-surfaces corresponding to same topology even symplectic structure. The quantum fluctuating degrees of freedom indeed correspond to symplectic group divided by its subgroup defining measurement resolution.

What is the dynamics defining the braids strands? What selects them? I have considered this problem several times. Just two examples is enough here.

(a) Could they be some special light-like curves? Could the condition that the end points of the curves correspond to rational points in some preferred coordinates allow to select these light-like curves? But what about light-like curves associated with the ends of the space-time surface?

(b) The solutions of Kähler-Dirac equation [K96] are localized to curves by using the analog of periodic boundary conditions: the length of the curve is quantized in the effective metric defined by the Kähler-Dirac gamma matrices. Here one however introcuced a coordinate along light-like 3-surface and it is not clear how one should fix this preferred coordinate.

1. Legendrian and Lagrangian sub-manifolds

A hint about what is missing comes from the observation that a non-vanishing Chern-Simons-Kähler form A defines a contact structure (see `http://tinyurl.com/yblj4hlq`) [A6] at light-like 3-surfaces if one has $A \wedge dA \neq 0$. This condition states complete non-intebrability of the distribution of 2-planes defined by the condition $A_\mu t^\mu = 0$, where t is tangent vector in the tangent bundle of light-like 3-surface. It also states that the flow lines of A do *not* define global coordinate varying along them.

(a) It is however possible to have 1-dimensional curves for which $A_\mu t^\mu = 0$ holds true at each point. These curves are known as Legendrian sub-manifolds to be distinguished from Lagrangian manifolds for which the projection of symplectic form expressible locally as $J = dA$ vanishes. The set of this curves is discrete so that one obtains braids. Legendrian knots are the simplest example of Legendrian sub-manifolds and the question is whether braid strands could be identified as Legendrian knots. For Legendrian braids symplectic invariance replaces topological invariance and Legendrian knots and braids can be trivial in topological sense. In some situations the property of being Legendrian implies un-knottedness.

(b) For Legendrian braid strands the Kähler gauge potential vanishes. Since the solutions of the Kähler-Dirac equation are localized to braid strands, this means that the coupling to Kähler gauge potential vanishes. From physics point of view a generalization of Legendre braid strand by allowing gauge transformations $A \to A + d\Phi$ looks natural since it means that the coupling of induced spinors is pure gauge terms and can be eliminated by a gauge transformation.

2. 2-D duals of Legendrian sub-manifolds

One can consider also what might be called 2-dimensional duals of Legendrian sub-manifolds.

(a) Also the one-form obtained from the dual of Kähler magnetic field defined as $B^\mu = \epsilon^{\mu\nu\gamma}J_{\nu\nu}$ defines a distribution of 2-planes. This vector field is ill-defined for light-like surfaces since contravariant metric is ill-defined. One can however multiply B with the square root of metric determining formally so that metric would disappear completely just as it disappears from Chern-Simons action. This looks however somewhat tricky mathematically. At the 3-D space-like ends of space-time sheets at boundaries of CD B^μ is however well-defined as such.

(b) The distribution of 2-planes is integrable if one has $B \wedge dB = 0$ stating that one has Beltrami field: physically the conditions states that the current dB feels no Lorentz force. The geometric content is that B defines a global coordinate varying along its flow lines. For the preferred extremals of Kähler action Beltrami condition is satisfied by isometry currents and Kähler current in the interior of space-time sheets. If this condition holds at 3-surfaces, one would have an global time coordinate and integrable distribution of 2-planes defining a slicing of the 2-surface. This would realize the conjecture that space-time surface has a slicing by partonic 2-surfaces. One could say that the 2-surfaces defined by the distribution are orthogonal to B. This need not however mean that the projection of J to these 2-surfaces vanishes. The condition $B \wedge dB = 0$ on the space-like 3-surfaces could be interpreted in terms of effective 2-dimensionality. The simplest option posing no additional conditions would allow two types of braids at space-like 3-surfaces and only Legendrian braids at light-like 3-surfaces.

These observations inspire a question. Could it be that the conjectured dual slicings of space-time sheets by space-like partonic 2-surfaces and by string world sheets are defined by A_μ and B^μ respectively associated with slicings by light-like 3-surfaces and space-like 3-surfaces? Could partonic 2-surfaces be identified as 2-D duals of 1-D Legendrian sub-manifolds?

The identification of braids as Legendrian braids for light-like 3-surfaces and with Legendrian braids or their duals for space-like 3-surfaces would in turn imply that topological braid theory is replaced with a symplectic braid theory in accordance with the view about TGD as almost topological QFT. If finite measurement resolution corresponds to the replacement of symplectic group with the coset space obtained by dividing by a subgroup, symplectic subgroup would take the role of isotopies in knot theory. This symplectic subgroup could be simply the symplectic group leaving the end points of braids invariant.

An attempt to identify the constraints on the braid algebra

The basic problems in understanding of quantum TGD are conceptual. One must proceed by trying to define various concepts precisely to remove the many possible sources of confusion. With this in mind I try collect essential points about generalized Feynman diagrams and their relation to braid diagrams and Feynman diagrams and discuss also the most obvious constraints on algebraization.

Let us first summarize what generalized Feynman diagrams are.

(a) Generalized Feynman diagrams are 3-D (or 4-D, depends on taste) objects inside $CD \times CP_2$. Ordinary Feynman diagrams are in plane. If finite measurement resolution has as a space-time correlate discretization at the level of partonic 2-surfaces, both space-like and light-like 3-surfaces reduce to braids and the lines of generalized Feynman diagrams correspond to braids. It is possible to obtain the analogs of ordinary Feynman diagrams by projection to $M^2 \subset M^4$ defined uniquely for given CD. The resulting apparent intersections would represent ne particular kind of exotic intersection.

(b) Light-like 3-surfaces define the lines of generalized Feynman diagrams and the braiding results naturally. Non-trivial first homology for the orbits of partonic 2-surfaces with genus $g > 0$ could be called homological virtual intersections.

(c) It zero energy ontology braids must be characterized by time orientation. Also it seems that one must distinguish in zero energy ontology between on mass shell braids and off mass shell braid pairs which decompose to pairs of braids with positive and negative energy massless on mass shell states. In order to avoid confusion one should perhaps speak about tangles insie CD rather than braids. The operations of the algebra are same except that the braids can end either to the upper or lower light-like boundary of CD. The projection to M^2 effectively reduces the CD to a 2-dimensional causal diamond.

(d) The vertices of generalized Feynman diagrams are partonic 2-surfaces at which the light-like 3-surfaces meet. This is a new element. If the notion of bosonic emergence is accepted no $n > 2$-vertices are needed so that braid strands are redistributed in the reaction vertices. The redistribution of braid strands in vertices must be introduced as an additional operation somewhat analogous to $\triangleright$ and the challenge is to reduce this operation to something simple. Perhaps the basic operation reduces to an exchange of braid strand between braids. The process can be seen as a decay of of braid with the conservation of braid strands with strands from future and past having opposite strand numbers. Also for this operation the analogs of Reidermeister moves should be identified. In dance metaphor this operation corresponds to a situation in which the dancer leaves the group to which it belongs and goes to a new one.

(e) A fusion of Feynman diagrammatic non-planarity and braid theoretic non-planarity is needed and the projection to M^2 could provide this fusion when at least two kinds of virtual crossings are allowed. The choice of M^2 could be global. An open question is whether the choice of M^2 could characterize separately each line of generalized Feynman diagram characterized by the four-momentum associated with it in the rest system defined by the tips of CD. Somehow the theory should be able to fuse the braiding matrix for integrable QFT in M^2 applying to entire braids with the braiding matrix for braid theory applying at the level of single braid.

Both integral QFTs in M^2 and braid theories suggest that biquandle structure is the structure that one should try to generalized.

(a) The representations of resulting bi-quandle like structure could allow abstract interesting information about generalized Feynman diagrams themselves but the dream is to construct generalized Feynman diagrams as analogs of knot invariants by a recursive procedure analogous to un-knotting of a knot.

(b) The analog of bi-quandle algebra should have a hierarchical structure containing braid strands at the lowest level, braids at next level, and braids of braids...of braids at higher levels. The notion of operad would be ideal for formulating this hierarchy and I have already proposed that this notion must be essential for the generalized Feynman diagrammatics. An essential element is the vanishing of total strand number in the vertex (completely analogous to conserved charged such as fermion number). Again a convenient visualization is in terms of dancers forming dynamical groups, forming groups of groups forming .....

I have already earlier suggested [K20] that the notion of operad [A26] relying on permutation group and its subgroups acting in tensor products of linear spaces is central for understanding generalized Feynman diagrams. $n \to n_1 + n_2$ decay vertex for n-braid would correspond to "symmetry breaking" $S_n \to S_{n_1} \times S_{n_2}$. Braid group represents the covering of permutation group so that braid group and its subgroups permuting braids would suggest itself as the basic group theoretical notion. One could assign to each strand of n-braid decaying to n_1 and n_2 braids a two-valued color telling whether it becomes a strand of n_1-braid or n_2-braid. Could also this "color" be interpreted as a particular kind of exotic crossing?

(c) What could be the analogs of Reidermaster moves for braid strands?

i. If the braid strands are dynamically determined, arbitrary deformations are not possible. If however all isotopy classes are allowed, the interpretation would be that a kind of gauge choice selecting one preferred representation of strand among all possible ones obtained by continuous deformations is in question.
ii. Second option is that braid strands are dynamically determined within finite measurement resolution so that one would have braid theory in given length scale resolution.
iii. Third option is that topological QFT is replaced with symplectic QFT: this option is suggested by the possibility to identify braid strands as Legendrian knots or their duals. Subgroup of the symplectic group leaving the end points of braids invariant would act as the analog of continous transformations and play also the role of gauge group. The new element is that symplectic transformations affect partonic 2-surfaces and space-time surfaces except at the end points of braid.

(d) Also 2-braids and perhaps also 2-knots could be useful and would provide string theory like approach to TGD. In this case the projections could be performed to the ends of CD or to M^3, which can be identified uniquely for a given CD.
(e) There are of course many additional subtleties involved. One should not forget loop corrections, which naturally correspond to sub-CDs. The hierarchy of Planck constants and number theoretical universality bring in additional complexities.

All this looks perhaps hopelessly complex but the Universe around is complex even if the basic principles could be very simple.

14.7.4 About String World Sheets, Partonic 2-Surfaces, And Two-Knots

String world sheets and partonic 2-surfaces provide a beatiful visualization of generalized Feynman diagrams as braids and also support for the duality of string world sheets and partonic 2-surfaces as duality of light-like and space-like braids. Dance metaphor is very helpful here.

(a) The projection of string world sheets and partonic 2-surfaces to 3-D space replaces knot projection. In TGD context this 3-D of space could correspond to the 3-D light-like boundary of CD and 2-knot projection would correspond to the projection of the braids associated with the lines of generalized Feynman diagram. Another identification would be as $M^1 \times E^2$, where M^1 is the line connecting the tips of CD and E^2 the orthogonal complement of M^2.
(b) Using dance metaphor for light-like braiding, braids assignable to the lines of generalized Feynman diagrams would correspond to groups of dancers. At vertices the dancing groups would exchange members and completely new groups would be formed by the dancers. The number of dancers (negative for those dancing in the reverse time direction) would be conserved. Dancers would be connected by threads representing strings having braid points at their ends. During the dance the light-like braiding would induce space-like braiding as the threads connecting the dancers would get entangled. This would suggest that the light-like braids and space-like braidings are equivalent in accordance with the conjectured duality between string-world sheets and partonic 2-surfaces. The presence of genuine 2-knottedness could spoil this equivalence unless it is completely local.

Can string world sheets and partonic 2-surfaces get knotted?

(a) Since partonic 2-surfaces (wormhole throats) are imbedded in light-cone boundary, the preferred 3-D manifolds to which one can project them is light-cone boundary (boundary of CD). Since the projection reduces to inclusion these surfaces cannot get knotted. Only if the partonic 2-surfaces contains in its interior the tip of the light-cone something non-trivial identifiable as virtual 2-knottedness is obtained.

(b) One might argue that the conjectured duality between the descriptions provided by partonic 2-surfaces and string world sheets requires that also string world sheets represent trivial 2-braids. I have shown earlier that nontrivial local knots glued to the string world sheet require that M^4 time coordinate has a local maximum. Does this mean that 2-knots are excluded? This is not obvious: TGD allows also regions of space-time surface with Euclidian signature and generalized Feynman graphs as 4-D space-time regions are indeed Euclidian. In these regions string world sheets could get knotted.

What happens for knot diagrams when the dimension of knot is increased to two? According to the articles of Nelson (see `http://tinyurl.com/yauy7asy`) [A95] and Carter (see `http://tinyurl.com/yclgj739`) [A83] the crossings for the projections of braid strands are replaced with more complex singularities for the projections of 2-knots. One can decompose the 2-knots to regions surrounded by boxes. Box can contain just single piece of 2-D surface; it can contain two intersection pieces of 2-surfaces as the counterpart of intersecting knot strands and one can tell which of them is above which; the box can contain also a discrete point in the intersection of projections of three disjoint regions of knot which consists of discrete points; and there is also a box containing so called cone point. Unfortunately, I failed to understand the meaning of the cone point.

For 2-knots Reidemeister moves are replaced with Roseman moves. The generalization would allow virtual self intersections for the projection and induced by the non-trivial second homology of 4-D embedding space. In TGD framework elementary particles have homologically non-trivial partonic 2-surfaces (magnetic monpoles) as their building bricks so that even if 2-knotting in standard sense might be not allowed, virtual 2-knotting would be possible. In TGD framework one works with a subgroup of symplectic transformations defining measurement resolution instead of isotopies and this might reduce the number of allowed mov

The dynamics of string world sheets and the expression for Kähler action

The dynamics of string world sheets is an open question. Effective 2-dimensionality suggests that Kähler action for the preferred extremal should be expressible using 2-D data but there are several guesses for what the explicit expression could be, and one can only make only guesses at this moment and apply internal consistency conditions in attempts to kill various options.

1. Could weak form of electric-magnetic duality hold true for string world sheets?

If one believes on duality between string world sheets and partonic 2-surfaces, one can argue that string world sheets are most naturally 2-surfaces at which the weak form of electric magnetic duality holds true. One can even consider the possibility that the weak form of electric-magnetic duality holds true only at the string world sheets and partonic 2-surfaces but not at the preferred 3-surfaces.

(a) The weak form of electric magnetic duality would mean that induced Kähler form is non-vanishing at them and Kähler magnetic flux over string world sheet is proportional to Kähler electric flux.

(b) The flux of the induced Kähler form of CP_2 over string world sheet would define a dimensionless "area". Could Kähler action for preferred extremals reduces to this flux apart from a proportionality constant. This "area" would have trivially extremum with respect to symplectic variations if the braid strands are Legendrian sub-manifolds since in this case the projection of Kähler gauge potential on them vanishes. This is a highly non-trivial point and favors weak form of electric-magnetic duality and the identification of Kähler action as Kähler magnetic flux. This option is also in spirit with the vision about TGD as almost topological QFT meaning that induced metric appears in the theory only via electric-magnetic duality.

(c) Kähler magnetic flux over string world sheet has a continuous spectrum so that the identification as Kähler action could make sense. For partonic 2-surfaces the magnetic flux would be quantized and give constant term to the action perhaps identifiable as the contribution of CP_2 type vacuum extremals giving this kind of contribution.

The change of space-time orientation by changing the sign of permutation symbol would change the sign in electric-magnetic duality condition and would not be a symmetry. For a given magnetic charge the sign of electric charge changes when orientation is changed. The value of Kähler action does not depend on space-time orientation but weak form of electric-magnetic duality as boundary condition implies dependence of the Kähler action on space-time orientation. The change of the sign of Kähler electric charge suggests the interpretation of orientation change as one aspect of charge conjugation. Could this orientation dependence be responsible for matter antimatter asymmetry?

2. Could string world sheets be Lagrangian sub-manifolds in generalized sense?

Legendrian sub-manifolds (see `http://tinyurl.com/yblj4hlq`) can be lifted to Lagrangian sub-manifolds [A6] Could one generalize this by replacing Lagrangian sub-manifold with 2-D sub-manifold of space-times surface for which the projection of the induced Kähler form vanishes? Could string world sheets be Lagrangian sub-manifolds?

I have also proposed that the inverse image of homologically non-trivial sphere of CP_2 under embedding map could define counterparts of string world sheets or partonic 2-surfaces. This conjecture does not work as such for cosmic strings, massless extremals having 2-D projection since the inverse image is in this case 4-dimensional. The option based on homologically non-trivial geodesic sphere is not consistent with the identification as analog of Lagrangian manifold but the identification as the inverse image of homologically trivial geodesic sphere is.

The most general option suggested is that string world sheet is mapped to 2-D Lagrangian sub-manifold of CP_2 in the embedding map. This would mean that theory is exactly solvable at string world sheet level. Vacuum extremals with a vanishing induced Kähler form would be exceptional in this framework since they would be mapped as a whole to Lagrangian sub-manifolds of CP_2. The boundary condition would be that the boundaries of string world sheets defined by braids at preferred 3-surfaces are Legendrian sub-manifolds. The generalization would mean that Legendrian braid strands could be continued to Lagrangian string world sheets for which induced Kähler form vanishes. The physical interpretation would be that if particle moves along this kind of string world sheet, it feels no covariant Lorentz-Kähler force and contra variant Lorentz forces is orthogonal to the string world sheet.

There are however serious objections.

(a) This proposal does not respect the proposed duality between string world sheets and partonic 2-surfaces which as carries of Kähler magnetic charges cannot be Lagrangian 2-manifolds.

(b) One loses the elegant identification of Kähler action as Kähler magnetic flux since Kähler magnetic flux vanishes. Apart from proportionality constant Kähler electric flux

$$\int_{Y^2} *J$$

is as a dimensionless scaling invariant a natural candidate for Kähler action but need not be extremum if braids are Legendrian sub-manifolds whereas for Kähler magnetic flux this is the case. There is however an explicit dependence on metric which does not conform with the idea that almost topological QFT is symplectic QFT.

(c) The sign factor of the dual flux which depends on the orientation of the string world sheet and thus changes sign when the orientation of space-time sheet is changed by changing that of the string world sheet. This is in conflict with the independence of Kähler action on orientation. One can however argue that the orientation makes itself actually physically visible via the weak form of electric-magnetic duality. If the above discussed duality holds true, the net contribution to Kähler action would vanish as the total Kähler magnetic flux for partonic 2-surfaces. Therefore the duality cannot hold true if Kähler action reduces to dual flux.

(d) There is also a purely formal counter argument. The inverse images of Lagrangian sub-manifolds of CP_2 can be 4-dimensional (cosmic strings and massless extremals) whereas string world sheets are 2-dimensonal.

String world sheets as minimal surfaces

Effective 2-dimensionality suggests a reduction of Kähler action to Chern-Simons terms to thearea of minimal surfaces defined by string world sheets holds true [K39]. Skeptic could argue that the expressibility of Kähler action involving no dimensional parameters except CP_2 scaled does not favor this proposal. The connection of minimal surface property with holomorphy and conformal invariance however forces to take the proposal seriously and it is easy to imagine how string tension emerges since the size scale of CP_2 appears in the induced metric [K39].

One can ask whether the mimimal surface property conforms with the proposal that string worlds sheets obey the weak form of electric-magnetic duality and with the proposal that they are generalized Lagrangian sub-manifolds.

(a) The basic answer is simple: minimal surface property and possible additional conditions (Lagrangian sub-manifold property or the weak form of electric magnetic duality) poses only additional conditions forcing the space-time sheet to be such that the imbedded string world sheet is a minimal surface of space-time surface: minimal surface property is a condition on space-time sheet rather than string world sheet. The weak form of electric-magnetic duality is favored because it poses conditions on the first derivatives in the normal direction unlike Lagrangian sub-manifold property.

(b) Any proposal for 2-D expression of Kähler action should be consistent with the proposed real-octonion analytic solution ansatz for the preferred extremals [K10]. The ansatz is based on real-octonion analytic map of embedding space to itself obtained by algebraically continuing real-complex analytic map of 2-D sub-manifold of embedding space to another such 2-D sub-manifold. Space-time surface is obtained by requiring that the "imaginary" part of the map vanishes so that image point is hyper-quaternion valued. Wick rotation allows to formulate the conditions using octonions and quaternions. Minimal surfaces (of space-time surface) are indeed objects for which the embedding maps are holomorphic and the real-octonion analyticity could be perhaps seen as algebraic continuation of this property.

(c) Does Kähler action for the preferred exremals reduce to the area of the string world sheet or to Kähler magnetic flux or are the representations equivalent so that the induced Kähler form would effectively define area form? If the Kähler form form associated with the induced metric on string world sheet is proportional to the induced Kähler form the Kähler magnetic flux is proportional to the area and Kähler action reduces to genuine area. Could one pose this condition as an additional constraint on string world sheets? For Lagrangian sub-manifolds Kähler electric field should be proportional to the area form and the condition involves information about space-time surface and is therefore more complex and does not look plausible.

Explicit conditions expressing the minimal surface property of the string world sheet

It is instructive to write explicitly the condition for the minimal surface property of the string world sheet and for the reduction of the area Kähler form to the induced Kähler form. For string world sheets with Minkowskian signature of the induced metric Kähler structure must be replaced by its hyper-complex analog involving hyper-complex unit e satisfying $e^2 = 1$ but replaced with real unit at the level hyper-complex coordinates. e can be represented as antisymmetric Kähler form J_g associated with the induced metric but now one has $J_g^2 = g$

instead of $J_g^2 = -g$. The condition that the signed area reduces to Kähler electric flux means that J_g must be proportional to the induced Kähler form: $J_g = kJ$, $k = constant$ in a given space-time region.

One should make an educated guess for the embedding of the string world sheet into a preferred extremal of Kähler action. To achieve this it is natural to interpret the minimal surface property as a condition for the preferred Kähler extremal in the vicinity of the string world sheet guaranteeing that the sheet is a minimal surface satisfying $J_g = kJ$. By the weak form of electric-magnetic duality partonic 2-surfaces represent both electric and magnetic monopoles. The weak form of electric-magnetic duality requires for string world sheets that the Kähler magnetic field at string world sheet is proportional to the component of the Kähler electric field parallel to the string world sheet. Kähler electric field is assumed to have component only in the direction of string world sheet.

1. Minkowskian string world sheets

Let us try to formulate explicitly the conditions for the reduction of the signed area to Kähler electric flux in the case of Minkowskian string world sheets.

(a) Let us assume that the space-time surface in Minkowskian regions has coordinates coordinates $(u, v, w, \overline{w})$ [K10]. The pair (u, v) defines light-like coordinates at the string world sheet having identification as hyper-complex coordinates with hyper-complex unit satisfying $e = 1$. u and v need not - nor cannot as it turns out - be light-like with respect to the metric of the space-time surface. One can use (u, v) as coordinates for string world sheet and assume that $w = x^1 + ix^2$ and $\overline{w}$ are constant for the string world sheet. Without a loss of generality one can assume $w = \overline{w} = 0$ at string world sheet.

(b) The induced Kähler structure must be consistent with the metric. This implies that the induced metric satisfies the conditions

$$g_{uu} = g_{vv} = 0 \ . \tag{14.7.1}$$

The analogs of these conditions in regions with Euclidian signature would be $g_{zz} = g_{\overline{z}\overline{z}} = 0$.

(c) Assume that the embedding map for space-time surface has the form

$$s^m = s^m(u, v) + f^m(u, v, x^m)_{kl} x^k x^l \ , \tag{14.7.2}$$

so that the conditions

$$\partial_l k s^m = 0 \ , \ \partial_k \partial_u s^m = 0, \ \partial_k \partial_v s^m = 0 \tag{14.7.3}$$

are satisfies at string world sheet. These conditions imply that the only non-vanishing components of the induced CP_2 Kähler form at string world sheet are J_{uv} and $J_{w\overline{w}}$. Same applies to the induced metric if the metric of M^4 satisfies these conditions (no non-vanishing components of form m_{uk} or m_{vk}).

(d) Also the following conditions hold true for the induced metric of the space-time surface

$$\partial_k g_{uv} = 0 \ , \ \partial_u g_{kv} = 0 \ , \ \partial_v g_{ku} = 0 \ . \tag{14.7.4}$$

at string world sheet as is easy to see by using the ansatz.

Consider now the minimal surface conditions stating that the trace of the four components of the second fundamental form whose components are labelled by the coordinates $\{x^\alpha\} \equiv (u, v, w, \overline{w})$ vanish for string world sheet.

(a) Since only g_{uv} is non-vanishing, only the components H^k_{uv} of the second fundamental form appear in the minimal surface equations. They are given by the general formula

$$\begin{aligned} H^{\alpha}_{uv} &= H^{\gamma}P^{\alpha}_{\gamma} \ , \\ H^{\alpha} &= (\partial_u\partial_v x^{\alpha} + \left({}_{\beta}{}^{\alpha}{}_{\gamma}\right)\partial_u x^{\beta}\partial_v x^{\gamma}) \ . \end{aligned} \tag{14.7.5}$$

Here P^{α}_{γ} is the projector to the normal space of the string world sheet. Formula contains also Christoffel symbols $\left({}_{\beta}{}^{\alpha}{}_{\gamma}\right)$.

(b) Since the embedding map is simply $(u,v) \to (u,v,0,0)$ all second derivatives in the formula vanish. Also $H^k = 0$, $k \in \{w,\overline{w}\}$ holds true. One has also $\partial_u x^{\alpha} = \delta^{\alpha}_u$ and $\partial_v x^{\beta} = \delta^{\beta}_v$. This gives

$$H^{\alpha} = \left({}_{u}{}^{\alpha}{}_{v}\right) \ . \tag{14.7.6}$$

All these Christoffel symbols however vanish if the assumption $g_{uu} = g_{vv} = 0$ and the assumptions about embedding ansatz hold true. Hence a minimal surface is in question.

Consider now the conditions on the induced metric of the string world sheet

(a) The conditions reduce to

$$g_{uu} = g_{vv} = 0 \ . \tag{14.7.7}$$

The conditions on the diagonal components of the metric are the analogs of Virasoro conditions fixing the coordinate choices in string models. The conditions state that the coordinate lines for u and v are light-like curves in the induced metric.

(b) The conditions can be expressed directly in terms of the induced metric and read

$$\begin{aligned} m_{uu} + s_{kl}\partial_u s^k\partial_u s^l &= 0 \ , \\ m_{vv} + s_{kl}\partial_v s^k\partial_v s^l &= 0 \ . \end{aligned} \tag{14.7.8}$$

The CP_2 contribution is negative for both equations. The conditions make sense only for $(m_{uu} > 0, m_{vv} > 0)$. Note that the determinant condition $m_{uu}m_{vv} - m_{uv}m_{vu} < 0$ expresses the Minkowskian signature of the (u,v) coordinate plane in M^4.

The additional condition states

$$J^g_{uv} = kJ_{uv} \ . \tag{14.7.9}$$

It reduces signed area to Kähler electric flux. If the weak form of electric-magnetic duality holds true one can interpret the area as magnetic flux defined as the flux of the dual of induced Kähler form over space-like surface and defining electric charge. A further condition is that the boundary of string world sheet is Legendrean manifold so that the flux and thus area is extremized also at the boundaries.

2.Conditions for the Euclidian string world sheets

One can do the same calculation for string world sheet with Euclidian signature. The only difference is that (u,v) is replaced with $(z,\overline{z})$. The embedding map has the same form assuming that space-time sheet with Euclidian signature allows coordinates $(z,\overline{z},w,\overline{w})$ and the local conditions on the embedding are a direct generalization of the above described conditions. In this case the vanishing for the diagonal components of the string world sheet metric reads as

$$
\begin{aligned}
h_{kl}\partial_z s^k \partial_z s^l &= 0 \ , \\
h_{kl}\partial_{\overline{z}} s^k \partial_{\overline{z}} s^l &= 0 \ .
\end{aligned}
\tag{14.7.10}
$$

The natural ansatz is that complex CP_2 coordinates are holomorphic functions of the complex coordinates of the space-time sheet.

3. Wick rotation for Minkowskian string world sheets leads to a more detailed solution ansatz

Wick rotation is a standard trick used in string models to map Minkowskian string world sheets to Euclidian ones. Wick rotation indeed allows to define what one means with real-octonion analyticity. Could one identify string world sheets in Minkowskian regions by using Wick rotation and does this give the same result as the direct approach?

Wick rotation transforms space-time surfaces in $M^4 \times CP_2$ to those in $E^4 \times CP_2$. In $E^4 \times CP_2$ octonion real-analyticity is a well-defined notion and one can identify the space-time surfaces surfaces at which the imaginary part of of octonion real-analytic function vanishes: imaginary part is defined via the decomposition of octonion to two quaternions as $o = q_1 + Iq_2$ where I is a preferred octonion unit. The reverse of the Wick rotation maps the quaternionic surfaces to what might be called hyper-quaternionic surfaces in $M^4 \times CP_2$.

In this picture string world sheets would be hyper-complex surfaces defined as inverse imagines of complex surfaces of quaternionic space-time surface obtained by the inverse of Wick rotation. For this approach to be equivalent with the above one it seems necessary to require that the treatment of the conditions on metric should be equivalent to that for which hyper-complex unit e is not put equal to 1. This would mean that the conditions reduce to independent conditions for the real and imaginary parts of the real number formally represented as hyper-complex number with $e = 1$.

Wick rotation allows to guess the form of the ansatz for CP_2 coordinates as functions of space-time coordinates In Euclidian context holomorphich functions of space-time coordinates are the natural ansatz. Therefore the natural guess is that one can map the hypercomplex number $t \pm ez$ to complex coordinate $t \pm iz$ by the analog of Wick rotation and assume that CP_2 complex coordinates are analytic functions of the complex space-time coordinates obtained in this manner.

The resulting induced metric could be obtained directly using real coordinates (t, z) for string world sheet or by calculating the induced metric in complex coordinates $t \pm iz$ and by mapping the expressions to hyper-complex numbers by Wick rotation (by replacing i with $e = 1$). If the diagonal components of the induced metric vanish for $t \pm iz$ they vanish also for hyper-complex coordinates so that this approach seem to make sense.

Electric-magnetic duality for flux Hamiltonians and the existence of Wilson sheets

One must distinguish between two conjectured dualities. The weak form of electric-magnetic duality and the duality between string world sheets and partonic 2-surfaces. Could the first duality imply equivalence of not only electric and magnetic flux Hamiltonians but also electric and magnetic Wilson sheets? Could the latter duality allow two different representations of flux Hamiltonians?

(a) For electric-magnetic duality holding true at string world sheets one would have non-vanishing Kähler form and the fluxes would be non-vanishing. The Hamiltonian fluxes

$$Q_{m,A} = \int_{X^2} JH_A dx^1 dx^2 = \int_{X^2} H_A J_{\alpha\beta} dx^\alpha \wedge dx^\beta \quad (14.7.11)$$

for partonic 2-surfaces X^2 define WCW Hamiltonians playing a key role in the definition of WCW Kähler geometry. They have also interpretation as a generalization of Wilson loops to Wilson 2-surfaces.

(b) Weak form of electric magnetic duality would imply both at partonic 2-surfaces and string world sheets the proportionality

$$Q_{m,A} = \int_{X^2} JH_A dx^1 \wedge dx^2 \propto Q^*_{m,A} = \int_{X^2} H_A * J_{\alpha\beta} dx^\alpha \wedge dx^\beta \ . \quad (14.7.12)$$

Therefore the electric-magnetic duality would have a concrete meaning also at the level of WCW geometry.

(c) If string world sheets are Lagrangian sub-manifolds Hamiltonian fluxes would vanish identically so that the identification as Wilson sheets does not make sense. One would lose electric-magnetic duality for flux sheets. The dual fluxes

$$*Q_A = \int_{Y^2} *JH_A dx^1 \wedge dx^2 = \int_{Y^2} \epsilon_{\alpha\beta}{}^{\gamma\delta} J_{\gamma\delta} = \int_{Y^2} \frac{\sqrt{det(g_4)}}{det(g_2^\perp)} J_{34}^\perp dx^1 \wedge dx^2$$

for string world sheets Y^2 are however non-vanishing. Unlike fluxes, the dual fluxes depend on the induced metric although they are scaling invariant.

Under what conditions the conjectured duality between partonic 2-surface and string world sheets hold true at the level of WCW Hamiltonians?

(a) For the weak form of electric-magnetic duality at string world sheets the duality would mean that the sum of the fluxes for partonic 2-surfaces and sum of the fluxes for string world sheets are identical apart from a proportionality constant:

$$\sum_i Q_A(X_i^2) \propto \sum_i Q_A(Y_i^2) \ . \quad (14.7.13)$$

Note that in zero ontology it seems necessary to sum over all the partonic surfaces (at both ends of the space-time sheet) and over all string world sheets.

(b) For Lagrangian sub-manifold option the duality can hold true only in the form

$$\sum_i Q_A(X_i^2) \propto \sum_i Q_A^*(Y_i^2) \ . \quad (14.7.14)$$

Obviously this option is less symmetric and elegant.

Summary

There are several arguments favoring weak form of electric-magnetic duality for both string world sheets and partonic 2-surfaces. Legendrian sub-manifold property for braid strands follows from the assumption that Kähler action for preferred extremals is proportional to the Kähler magnetic flux associated with preferred 2-surfaces and is stationary with respect to the variations of the boundary. What is especially nice is that Legendrian sub-manifold property implies automatically unique braids. The minimal option favored by the idea that 3-surfaces are basic dynamical objects is the one for which weak form of electric-magnetic duality holds true only at partonic 2-surfaces and string world sheets. A stronger option assumes it at preferred 3-surfaces. Duality between string world sheets and partonic 2-surfaces suggests that WCW Hamiltonians can be defined as sums of Kähler magnetic fluxes for either partonic 2-surfaces or string world sheets.

14.7.5 What Generalized Feynman Rules Could Be?

After all these explanations the skeptic reader might ask whether this lengthy discussion gives any idea about what the generalized Feynman rules might look like. The attempt to answer this question is a good manner to make a map about what is understood and what is not understood. The basic questions are simple. What constraints does zero energy ontology (ZEO) pose? What does the necessity to projecti the four-momenta to a preferred plane M^2 mean? What mathematical expressions one should assign to the propagator lines and vertices? How does one perform the functional integral over 3-surfaces in finite measurement resolution? The following represents tentatative answers to these questions but does not say much about exact role of algebraic knots.

Zero energy ontology

Zero energy ontology (ZEO) poses very powerful constraints on generalized Feynman diagrams and gives hopes that both UV and IR divergences cancel.

(a) ZEO predicts that the fermions assigned with braid strands associated with the virtual particles are on mass shell massless particles for which the sign of energy can be also negative: in the case of wormhole throats this can give rise to a tachyonic exchange.

(b) The on mass shell conditions for each wormhole throat in the diagram involving loops are very stringent and expected to eliminate very large classes of diagrams. If however given diagonal diagram leading from n-particle state to the same n-particle state -completely analogous to self energy diagram- is possible then the ladders form by these diagrams are also possible and one one obtains infinite of this kind of diagrams as generalized self energy correction and is excellent hopes that geometric series gives a closed algebraic function.

(c) IR divergences plaguing massless theories are cancelled if the incoming and outgoing particles are massive bound states of massless on mass shell particles. In the simplest manner this is achieved when the 3-momenta are in opposite direction. For internal lines the massive on-mass shell-condition is not needed at all. Therefore there is an almost complete separation of the problem how bound state masses are determined from the problem of constructing the scattering amplitudes.

(d) What looks like a problematic aspect ZEO is that the massless on-mass-shell propagators would diverge for wormhole throats. The solution comes from the projection of 4-momenta to M^2. In the generic the projection is time-like and one avoids the singularity. The study of solutions of the Kähler-Dirac equation [K96] and number theoretic vision [K80] indeed suggests that the four-momenta are obtained by rotating massless M^2 momenta and their projections to M^2 are in general integer multiples of hyper-complex primes or light-like. The light-like momenta would be treated like in the case of ordinary Feynman diagrams using $i\epsilon$-prescription of the propagator and would also give a finite contributions corresponding to integral over physical on mass shell states. This guarantees also the vanishing of the possible IR divergences coming from the summation over different M^2 momenta.

There is a strong temptation to identify - or at least relate - the M^2 momenta labeling the solutions of the Kähler-Dirac equation with the region momenta of twistor approach [K86]. The reduction of the region momenta to M^2 momenta could dramatically simplify the twistorial description. It does not seem however plausible that $\mathcal{N} = 4$ supersymmetric gauge theory could allow the identification of M^2 projections of 4-momenta as region momenta. On the other hand, there is no reason to expect the reduction of TGD certainly to a gauge theory containing QCD as part. For instance, color magnetic flux tubes in many-sheeted space-time are central for understanding jets, quark gluon plasma, hadronization and fragmentation [L19] but cannot be deduced from QCD. Note also that the splitting of parton momenta to their M^2 projections and transversal parts is an ad hoc assumption motivated by parton model rather than first principle implication of QCD: in TGD framework this splitting would emerge from first principles.

(e) ZEO strongly suggests that all particles (including photons, gluons, and gravitons) have mass which can be arbitrarily small and could be perhaps seen as being due to the fact that particle "eats" Higgs like states giving it the otherwise lacking polarization states. This would mean a generalization of the notion of Higgs particle to a Higgs like particle with spin. It would also mean rearrangmenet of massless states at wormhole throat level to massives physical states. The slight massication of photon by p-adic thermodynamics does not however mean disappearance of Higgs from spectrum, and one can indeed construct a model for Higgs like states [K35].

The projection of the momenta to M^2 is consistent with this vision. The natural generalization of the gauge condition $p \cdot \epsilon = 0$ is obtained by replacing p with the projection of the total momentum of the boson to M^2 and ϵ with its polarization so that one has $p_{||} \cdot \epsilon$. If the projection to M^2 is light-like, three polarization states are possible in the generic case, so that massivation is required by internal consistency. Note that if intermediate states in the unitary condition were states with light-like M^2-momentum one could have a problematic situation.

(f) A further assumption vulnerable to criticism is that the M^2 projections of all momenta assignable to braid strands are parallel. Only the projections of the momenta to the orthogonal complement E^2 of M^2 can be non-parallel and for massive wormhole throats they must be non-parallel. This assumption does not break Lorentz invariance since in the full amplitude one must integrate over possible choices of M^2. It also interpret the gauge conditions either at the level of braid strands or of partons. Quantum classical correspondence in strong form would actually suggests that quantum 4-momenta should co-incide with the classical ones. The restriction to M^2 projections is however necessary and seems also natural. For instance, for massless extremals only M^2 projection of wave-vector can be well-defined: in transversal degrees of freedom there is a superposition over Fourier components with diffrent transversal wave-vectors. Also the partonic description of hadrons gives for the M^2 projections of the parton momenta a preferred role. It is highly encouraging that this picture emerged first from the Kähler-Dirac equation and purely number theoretic vision based on the identification of M^2 momenta in terms of hyper-complex primes.

The number theoretical approach also suggests a number theoretical quantization of the transversal parts of the momenta [K80]: four-momenta would be obtained by rotating massless M^2 momenta in M^4 in such a way that the components of the resulting 3-momenta are integer valued. This leads to a classical problem of number theory which is to deduce the number of 3-vectors of fixed length with integer valued components. One encounters the n-dimensional generalization of this problem in the construction of discrete analogs of quantum groups (these "classical" groups are analogous to Bohr orbits) and emerge in quantum arithmetics [K61], which is a deformation of ordinary arithmetics characterized by p-adic prime and giving rigorous justification for the notion of canonical identification mapping p-adic numbers to reals.

(g) The real beauty of Feynman rules is that they guarantee unitarity automatically. In fact, unitarity reduces to Cutkosky rules which can be formulated in terms of cut obtained by putting certain subset of interal lines on mass shell so that it represents on mass shell state. Cut analyticity implies the usual $iDisc(T) = TT^{\dagger}$. In the recent context the cutting of the internal lines by putting them on-mass-shell requires a generalization.

i. The first guess is that on mass shell property means that M^2 projection for the momenta is light-like. This would mean that also these momenta contribute to the amplitude but the contribution is finite just like in the usual case. In this formulation the real particles would be the massless wormhole throats.

ii. Second possibility is that the internal lines on on mass shell states corresponding to massive on mass-shell-particles. This would correspond to the experimental meaning of the unitary conditions if real particles are the massive on mass shell particles. Mathematically it seems possible to pick up from the amplitude the states which correspond to massive on mass shell states but one should understand why the discontinuity should be associated with physical net masses for wormhole contacts or many-particle states formed by them. General connection with unitarity and analyticity might allow to understand this.

(h) CDs are labelled by various moduli and one must integrate over them. Once the tips of the CD and therefore a preferred M^1 is selected, the choice of angular momentum quantization axis orthogonal to M^1 remains: this choice means fixing M^2. These choices are parameterized by sphere S^2. It seems that an integration over different choices of M^2 is needed to achieve Poincare invariance.

How the propagators are determined?

In accordance with previous sections it will be assumed that the braid are Legendrian braids and therefore completely well-defined. One should assign propagator to the braid. A good guess is that the propagator reduces to a product of three terms.

(a) A multi-particle propagator which is a product of collinear massless propagators for braid strands with fermionin number $F = 0, 1-1$. The constraint on the momenta is $p_i = \lambda_i p$ with $\sum_i \lambda_i = 1$. So that the fermionic propagator is $\frac{1}{\prod_i \lambda_i} p^k \gamma_k$. If one gas $p = nP$, where P is hyper-complex prime, one must sum over combinations of $\lambda_i = n_i$ satisfying $\sum_i n_i = n$.

(b) A unitary S-matrix for integrable QFT in M^2 in which the velocities of particles assignable to braid strands appear for which fixed by R-matrix defines the basic 2-vertex representing the process in which a particle passes through another one. For this S-matrix braids are the basic units. To each crossing appearing in non-planar Feynman diagram one would have an R-matrix representing the effect of a reconnection the ends of the lines coming to the crossing point. In this manner one could gradually transform the non-planar diagram to a planar diagram. One can ask whether a formulation in terms of a suitable R-matrix could allow to generalize twistor program to apply in the case of non-planar diagrams.

(c) An S-matrix predicted by topological QFT for a given braid. This S-matrix should be constructible in terms of Chern-Simons term defining a sympletic QFT.

There are several questions about quantum numbers assignable to the braid strands.

(a) Can braid strands be only fermionic or can they also carry purely bosonic quantum numbers corresponding to WCW Hamiltonians and therefore to Hamiltonians of $\delta M^4_\pm \times CP_2$? Nothing is lost if one assumes that both purely bosonic and purely fermionic lines are possible and looks whether this leads to inconsistencies. If virtual fermions correspond to single wormhole throat they can have only time-like M^2-momenta. If virtual fermions correspond to pairs of wormhole throats with second throat carrying purely bosonic quantum numbers, also fermionic can have space-like net momenta. The interpretation would be in terms of topological condensation. This is however not possible if all strands are fermionic. Situation changes if one identifies physical fermions wormhole throats at the ends of Kähler magnetic flux tube as one indeed does: in this case virtual net momentum can be space-like if the sign of energy is opposite for the ends of the flux tube.

(b) Are the 3-momenta associated with the wormholes of wormhole contact parallel so that only the sign of energy could distinguish between them for space-like total momentum and M^2 mass squared would be the same? This assumption simplifies the situation but is not absolutely necessary.

(c) What about the momentum components orthogonal to M^2? Are they restricted only by the massless mass shell conditions on internal lines and quantization of the M^2 projection of 4-momentum?

(d) What kind of braids do elementary particles correspond? The braids assigned to the wormhole throat lines can have arbitrary number n of strands and for $n = 1, 2$ the treatment of braiding is almost trivial. A natural assumption is that propagator is simply a product of massless collinear propagators for M^2 projection of momentum [?]. Collinearity means that propagator is product of a multifermion propagator $\frac{1}{\lambda_i p_k \gamma_k}$, znd multiboson propagator $\frac{1}{\mu_i p_k \gamma_k}$, $\sum \lambda_i + \sum_i \mu_i = 1$. There are also quantization conditions on M^2 projections of momenta from Kähler-Dirac equation implying that multiplies of hyper-complex prime are in question in suitable units. Note however that it is not clear whether purely bosonic strands are present.

(e) For ordinary elementary particles with propagators behaving like $\prod_i \lambda_i^{-1} 1 p^{-n}$, only $n \leq 2$ is possible. The topologically really interesting states with more than two braid strands are something else than what we have used to call elementary particles. The proposed interpretation is in terms of anyonic states [K64]. One important implication is that $\mathcal{N} = 1$ SUSY generated by right-handed neutrino or its antineutrino is SUSY for which all members of the multiplet assigned to a wormhole throat have braid number smaller than 3. For $\mathcal{N} = 2$ SUSY generated by right-handed neutrino and its antiparticle the states containing fermion and neutrino-antineutrino pair have three braid strands and SUSY breaking is expected to be strong.

Vertices

Conformal invariance raises the hope that vertices can be deduced from super-conformal invariance as n-point functions. Therefore lines would come from integrable QFT in M^2 and topological braid theory and vertices from confofmal field theory: both theories are integrable.

The basic questions is how the vertices are defined by the 2-D partonic surfaces at which the ends of lines meet. Finite measurement resolution reduces the lines to braids so that the vertices reduces to the intersection of braid strands with the partonic 2-surface.

(a) Conformal invariance is the basic symmetry of quantum TGD. Does this mean that the vertices can be identified as n-point functions for points of the partonic 2-surface defined by the incoming and outgoing braid strands? How strong constraints can one pose on this conformal field theory? Is this field theory free and fixed by anti-commutation relations of induced spinor fields so that correlation function would reduce to product of fermionic two points functions with standard operator in the vertices represented by strand ends. If purely bosonic vertices are present, their correlation functions must result from the functional integral over WCW .

(b) For the fermionic fields associated with each incoming braid the anti-commutators of fermions and anti-fermions are trivial just as the usual equal time anti-commutation relations. This means that the vertex reduces to sum of products of fermionic correlation functions with arguments belonging to different incoming and outgoing lines. How can one calculate the correlators?

   i. Should one perform standard second quantization of fermions at light-like 3-surface allowing infinite number of spinor modes, apply a finite measurement resolution to obtain braids, for each partonic 2-surface, and use the full fermion fields to calculate the correlators? In this case braid strands would be discontinuous in vertices. A possible problem might be that the cutoff in spinor modes seems to come from the theory itself: finite measurement resolution is a property of quantum state itself.

   ii. Could finite measurement resolution allow to approximate the braid strands with continuous ones so that the correlators between strands belonging to different lines are given by anti-commutation relations? This would simplify enormously the situation and would conform with the idea of finite measurement resolution and the vision that interaction vertices reduce to braids. This vision is encouraged by the previous considerations and would mean that replication of braid strands analogous to replication of DNA strands can be seen as a fundamental process of Nature. This of course represents an important deviation from the standard picture.

(c) Suppose that one accepts the latter option. What can happen in the vertex, where line goes from one braid to another one?

i. Can the direction of momentum changed as visual intuition suggests? Is the total braid momentum conservation the only constraint so that the velocities assignable braid strands in each line would be constrained by the total momentum of the line.

ii. What kind of operators appear in the vertex? To get some idea about this one can look for the simplest possible vertex, namely FFB vertex which could in fact be the only fundamental vertex as the arguments of [K18] suggest. The propagator of spin one boson decomposes to product of a projection operator to the polarization states divited by p^2 factor. The projection operator sum over products $\epsilon_i^k\gamma_k$ at both ends where γ_k acts in the spinor space defined by fermions. Also fermion lines have spinor and its conjugate at their ends. This gives rise to $p^k\gamma_k/p^2$. $p^k\gamma_k$ is the analog of the bosonic polarization tensor factorizing into a sum over products of fermionic spinors and their conjugates. This gives the BFF vertex $\epsilon_i^k\gamma_k$ slashed between the fermionic propagators which are effectively 2-dimensional.

iii. Note that if H-chiralities are same at the throats of the wormhole contact, only spin one states are possible. Scalars would be leptoquarks in accordance with general view about lepton and quark number conservation. One particular implication is that Higgs in the standard sense is not possible in TGD framework. It can appear only as a state with a polarization which is in CP_2 direction. In any case, Higgs like states would be eaten by massless state so that all particles would have at least a small mass.

Functional integral over 3-surfaces

The basic question is how one can functionally integrate over light-like 3-surfaces or space-like 3-surfaces.

(a) Does effective 2-dimensionality allow to reduce the functional integration to that over partonic 2-surfaces assigned with space-time sheet inside CD plus radiative corrections from the hierarchy of sub-CDs?

(b) Does finite measurement resolution reduce the functional integral to a ordinary integral over the positions of the end points of braids and could this integral reduce to a sum? Symplectic group of $\delta M^4_\pm \times CP_2$ basically parametrizes the quantum fluctuating degrees of freedom in WCW . Could finite measurement resolution reduce the symplectic group of $\delta M^4_\pm \times CP_2$ to a coset space obtained by dividing with symplectic transformations leaving the end points invariant and could the outcome be a discrete group as proposed? Functional integral would reduce to sum.

(c) If Kähler action reduces to Chern-Simons-Kähler terms to surface area terms in the proposed manner, the integration over WCW would be very much analogous to a functional integral over string world sheets and the wisdom gained in string models might be of considerable help.

Summary

What can one conclude from these argument? To my view the situation gives rise to a considerable optimism. I believe that on basis of the proposed picture it should be possible to build a concrete mathematical models for the generalized Feynman graphics and the idea about reduction to generalized braid diagrams having algebraic representations could pose additional powerful constraints on the construction. Braid invariants could also be building bricks of the generalized Feynman diagrams. In particular, the treatment of the non-planarity of Feynman diagrams in terms of M^2 braiding matrix would be something new and therefore can be questioned.

Few years after writing these lines a view about generalized Feynman diagrams as a stringy generalization of twistor Grassmannian diagrams has emerged [K86]. This approach relies heavily on the localization of spinor modes on 2-D string world sheets (covariantly constant right-handed neutrino is an exception) [K96]. This approach can be regarded as an effective

QFT (or rather, effective string theory) approach: all information about the microscopic character of the fundamental particle like entities has been integrated out so that a string model type description at the level of imbdding space emerges. The presence of gigantic symmetries, in particular, the Yangian generalization of super-conformal symmetries, raises hopes that this approach could work. The approach to generalized Feynman diagrams considered above is obviously microscopic.

14.8 Electron As A Trefoil Or Something More General?

The possibility that electron, and also other elementary particles could correspond to knot is very interesting. The video model (see `http://tinyurl.com/ycz4jm48`) [B57] was so fascinating (I admire the skills of the programmers) that I started to question my belief that all related to knots and braids represents new physics (say anyons, see `http://tinyurl.com/y89xp4bu`) [K64] and that it is hopeless to try to reduce standard model quantum numbers with purely group theoretical explanation (except family replication) to topological quantum numbers.

Electroweak and color quantum numbers should by quantum classical correspondence have geometric correlates in space-time geometry. Could these correlates be topological? As a matter of fact, the correlates existing if the present understanding of the situation is correct but they are not topological.

Despite this, I played with various options and found that in TGD Universe knot invariants do not provide plausible space-time correlates for electroweak quantum numbers. The knot invariants and many other topological invariants are however present and mean new physics. As following arguments try to show, elementary particles in TGD Universe are characterized by extremely rich spectrum of topological quantum numbers, in particular those associated with knotting and linking: this is basically due to the 3-dimensionality of 3-space.

For a representation of trefoil knot by R.W. Gray see `http://tinyurl.com/ycz4jm48`. The homepage of Louis Kauffman (see `http://tinyurl.com/y7r3w5jq`) [A13] is a treasure trove for anyone interested in ideas related to possible applications of knots to physics. One particular knotty idea is discussed in the article "Emergent Braided Matter of Quantum Geometry" (see `http://tinyurl.com/y7lnn3wa`) by Bilson-Thompson, Hackett, and Kauffman [B17].

14.8.1 Space-Time As 4-Surface And The Basic Argument

Space-time as a 4-surface in $M^4 \times CP_2$ is the key postulate. The dynamics of space-time surfaces is determined by so called Kähler action - essentially Maxwell action for the Kähler form of CP_2 induced to X^4 in induced metric. Only so called preferred extremals are accepted and one can in very loose sense say that general coordinate invariance is realized by assigning to a given 3-surface a unique 4-surface as a preferred extremal analogous to Bohr orbit for a particle identified as 3-D surface rather than point-like object.

One ends up with a radical generalization of space-time concept to what I call many-sheeted space-time. The sheets of many-sheeted space-time are at distance of CP_2 size scale (10^4 Planck lengths as it turns out) and can touch each other which means formation of wormhole contact with wormhole throats as its ends. At throats the signature of the induced metric changes from Minkowskian to Euclidian. Euclidian regions are identified as 4-D analogs of lines of generalized Feynman diagrams and the M^4 projection of wormhole contact can be arbitrarily large: macroscopic, even astrophysical. Macroscopic object as particle like entity means that it is accompanied by Euclidian region of its size.

Elementary particles are identified as wormhole contacts. The wormhole contacts born in mere touching are not expected to be stable. The situation changes if there is a monopole magnetic flux (CP_2 carries self dual purely homological monopole Kähler form defining Maxwell field, this is not Dirac monopole) since one cannot split the contact. The lines of the Kähler magnetic field must be closed, and this requires that there is another wormhole contact nearby. The magnetic flux from the upper throat of contact A travels to the upper throat of contact B along "upper" space-time sheet, goes to "lower" space-time sheet along contact B and returns back to the wormhole contact A so that closed loop results.

In principle, wormhole throat can have arbitrary orientable topology characterized by the number g of handles attached to sphere and known as genus. The closed flux tube corresponds to topology $X^2_g \times S^1$, g=0, 1, 2, ... Genus-generation correspondence (see `http://tinyurl.com/ybowqm5v`) [K18] states that electron, muon, and tau lepton and similarly quark generations correspond to $g = 0, 1, 2$ in TGD Universe and CKM mixing is induced by topological mixing.

Suppose that one can assign to this flux tube a closed string: this is indeed possible but I will not bother reader with details yet. What one can say about the topology of this string?

(a) X^2_g has homology Z^{2g} and S^1 homology S^1. The entire homology is Z^{2g+1} so that there are $2g+1$ additional integer valued topological quantum numbers besides genus. Z^{2g+1} obviously breaks topologically universality stating that fermion generations are exact copies of each other apart from mass. This would be new physics. If the size of the flux loop is of order Compton length, the topological excitations need not be too heavy. One should however know how to excite them.

(b) The circle S^1 is imbedded in 3-surface and can get knotted. This means that all possible knots characterize the topological states of the fermion. Also this means extremely rich spectrum of new physics.

14.8.2 What Is The Origin Of Strings Going Around The Magnetic Flux Tube?

What is then the origin of these knotted strings? The study of the Kähler-Dirac equation [K96] determining the dynamics of induced spinor fields at space-time surface led to a considerable insight here. This requires however additional notions such as zero energy ontology (ZEO), and causal diamond (CD) defined as intersection of future and past directed light-cones (double 4-pyramid is the M^4 projection. Note that CD has CP_2 as Cartesian factor and is analogous to Penrose diagram.

(a) ZEO means the assumption that space-time surfaces for a particular sub- WCW ("world of classical worlds") are contained inside given CD identifiable as a the correlate for the "spotlight of consciousness" in TGD inspired theory of consciousness. The space-time surface has ends at the upper and lower light-like boundaries of CD. The 3-surfaces at the ends define space-time correlates for the initial and final states in positive energy ordinary ontology. In ZEO they carry opposite total quantum numbers.

(b) General coordinate invariance (GCI) requires that once the 3-D ends are known, space-time surface connecting the ends is fixed (there is not path integral since it simply fails). This reduces ordinary holography to GCI and makes classical physics defined by preferred extremals an exact part of quantum theory, actually a key element in the definition of Kähler geometry of WCW .

    Strong form of GCI is also possible. One can require that 3-D light-like orbits of wormhole throats at which the induced metric changes its signature, and space-like 3-surfaces at the ends of CD give equivalent descriptions. This implies that quantum physics is coded by the their intersections which I call partonic 2-surfaces - wormhole throats - plus the 4-D tangent spaces of X^4 associated with them. One has strong form of holography. Physics is almost 2-D but not quite: 4-D tangent space data is needed.

(c) The study of the Kähler-Dirac equation [K96] leads to further results. The mere conservation of electromagnetic charge defined group theoretically for the induced spinors of $M^4 \times CP_2$ carrying spin and electroweak quantum numbers implies that for all other fermion states except right handed neutrino (, which does not couple at all all to electroweak fields), are localized at 2-D string world sheets and partonic 2-surfaces.

String world sheets intersect the light-like orbits of wormhole throats along 1-D curves having interpretation as time-like braid strands (a convenient metaphor: braiding in time direction si created by dancers in the parquette).

One can say that dynamics automatically implies effective discretization: the ends of time like braid strands at partonic 2-surfaces at the ends of CD define a collection of discrete points to each of which one can assign fermionic quantum numbers.

(d) Both throats of the wormhole contact can carry many fermion state and known fermions correspond to states for which either throat carries single braid strand. Known bosons correspond to states for which throats carry fermion and anti-fermion number.

(e) Partonic 2-surface is replaced with discrete set of points effectively. The interpretation is in terms of a space-time correlate for finite measurement resolution. Quantum correlate would be the inclusion of hyperfinite factors of type II_1.

This interpretation brings in even more topology!

(a) String world sheets - present both in Euclidian and Minkowskian regions - intersect the 3-surfaces at the ends of CD along curves - one could speak of strings. These strings give rise to the closed curves that I discussed above. These strings can be homologically non-trivial - in string models this corresponds to wrapping of branes.

(b) For known bosons one has two closed loop but these loops could fuse to single. Space-like 2-braiding (including linking) becomes possible besides knotting.

(c) When the partonic 2-surface contains several fermionic braid ends one obtains even more complex situation than above when one has only single braid end. The loops associated with the braid ends and going around the monopole flux tube can form space-like N-braids. The states containing several braid ends at either throat correspond to exotic particles not identifiable as ordinary elementary particles.

14.8.3 How Elementary Particles Interact As Knots?

Elementary particles could reveal their knotted and even braided character via the topological interactions of knots. There are two basic interactions.

(a) The basic interaction for single string is by self-touching and this can give to a local connected sum or a reconnection. In both cases the knot invariants can change and it is possible to achieve knotting or unknotting of the string by this mechanism. String can also split into two pieces but this might well be excluded in the recent case.

The space-time dynamics for these interactions is that of closed string model with 4-D target space. The first guess would be topological string model describing only the dynamics of knots. Note that string world sheets define 2-knots and braids.

(b) The basic interaction vertex for generalized Feynman diagrams (lines are 4-D space-time regions with Euclidian signature) is join along 3-D boundaries for the three particles involved: this is just like ordinary 3-vertex for Feynman diagrams and is not encountered in string models. The ends of lines must have same genus g. In this interaction vertex the homology charges in Z^{2g+1} is conserved so that these charges are analogous to U(1) gauge charges. The strings associated with the two particles can touch each other and connected sum or reconnection is the outcome.

Consider now in more detail connected sum and reconnection vertices responsible for knotting and un-knotting.

(a) The first interaction is connected sum (see `http://tinyurl.com/lye7pvp`) of knots [A5]. A little mental exercise demonstrates that a local connected sum for the pieces of knot for which planar projections cross, can lead to a change in knotted-ness. Local connected sum is actually used to un-knot the knot in the construction of knot invariants.

In dimension 3 knots form a module with respect to the connected sum. One can identify unique prime knots and construct all knots as products of prime knots with product defined as a connected sum of knots. In particular, one cannot have a situation on which a product of two non-trivial knots is un-knot so that one could speak about the inverse of a knot (indeed, the inverse of ordinary prime is not an integer!). For higher-dimensional knots the situation changes (string world sheets at space-time surface could form 2-knots but instead of linking they intersect at discrete points).

Connected sum in the vertex of generalized Feynman graph (as described above) can lead to a decay of particle to two particles, which correspond to the summands in the connected sum as knots. Could one consider a situation in which un-knotted particle decomposes via the time inverse of the connected sum to a pair of knotted particles such that the knots are inverses of each other? This is not possible since knots do not have inverse.

(b) Touching knots can also reconnect. For braids the strands $A \rightarrow B$ and $C \rightarrow D$ touch and one obtains strands $A \rightarrow D$ and $C \rightarrow B$. If this reaction takes place for strands whose planar projections cross, it can also change the character of the knot. One one can transform knot to un-knot by repeatedly applying connected sum and reconnection for crossing strands (the Alexandrian way).

(c) In the evolution of knots as string world sheets these two vertices corresponds to closed string vertices. These vertices can lead to topological mixing of knots leading to a quantum superposition of different knots for a given elementary particle. This mixing would be analogous to CKM mixing understood to result from the topological mixing of fermion genera in TGD framework. It could also imply that knotted particles decay rapidly to un-knots and make the un-knot the only long-lived state.

A naïve application of Uncertainty Principle suggests that the size scale of string determines the life time of particular knot configuration. The dependence on the length scale would however suggest that purely topological string theory cannot be in question. Zero energy ontology suggests that the size scale of the causal diamond assignable to elementary particle determines the time scale for the rates as secondary p-adic time scale: in the case of electron the time scale would be.1 seconds corresponding to Mersenne prime $M_{127} = 2^{127} - 1$ so that knotting and unknotting would be very slow processes. For electron the estimate for the scale of mass differences between different knotted states would be about $10^{-19} m_e$: electron mass is known for certain for 9 decimals so that there is no hope of detecting these mass differences. The pessimistic estimate generalizes to all other elementary particles: for weak bosons characterized by M_{89} the mass difference would be of order $10^{-13} m_W$.

(d) A natural guess is that p-adic thermodynamics can be applied to the knotting. In p-adic thermodynamics Boltzmann weights in are of form $p^{H/T}$ (p-adic number) and the allowed values of the Hamiltonian H are non-negative integer powers of p. Clearly, H representing a contribution to p-adic valued mass squared must be a non-negative integer valued invariant additive under connected sum. This guarantees extremely rapid convergence of the partition function and mass squared expectation value as the number of prime knots in the decomposition increases.

An example of an knot invariant (see `http://tinyurl.com/ya6pdykc`) [A21] additive under connected sum is knot genus (see `http://tinyurl.com/y8nfykh3`) [A20] defined as the minimal genus of 2-surface having the knot as boundary (Seifert surface). For trefoil and figure eight knot one has $g = 1$. For torus knot $(p, q) \equiv (q, p)$ one has $g = (p-1)(q-1)/2$. Genus vanishes for un-knot so that it gives the dominating contribution to the partition function but a vanishing contribution to the p-adic mass squared.

p-Adic mass scale could be assumed to correspond to the primary p-adic mass scale just as in the ordinary p-adic mass calculations. If the p-adic temperature is $T = 1$ in natural units (highest possible), and if one has $H = 2g$, the lowest order contribution corresponds to the value $H = 2$ of the knot Hamiltonian, and is obtained for trefoil and figure eight knot so that the lowest order contribution to the mass would indeed be about $10^{-19} m_e$ for electron. An equivalent interpretation is that $H = g$ and $T = 1/2$ as assumed for gauge bosons in p-adic mass calculations.

There is a slight technical complication involved. When the string has a non-trivial homology in $X_g^2 \times S^1$ (it always has by construction), it does not allow Seifert surface in the ordinary sense. One can however modify the definition of Seifert surface so that it isolates knottedness from homology. One can express the string as connected sum of homologically non-trivial un-knot carrying all the homology and of homologically trivial knot carrying all knottedness and in accordance with the additivity of genus define the genus of the original knot as that for the homologically trivial knot.

(e) If the knots assigned with the elementary particles have large enough size, both connected sum and reconnection could take place for the knots associated with different elementary particles and make the many particle system a single connected structure. TGD based model for quantum biology is indeed based on this kind of picture. In this case the braid strands are magnetic flux tubes and connect bio-molecules to single coherent whole. Could electrons form this kind of stable connected structures in condensed matter systems? Could this relate to super-conductivity and Cooper pairs somehow? If one takes p-adic thermodynamics for knots seriously then knotted and braided magnetic flux tubes are more attractive alternative in this respect.

What if the thermalization of knot degrees of freedom does not take place? One can also consider the possibility that knotting contributes only to the vacuum conformal weight and thus to the mass squared but that no thermalization of ground states takes place. If the increment Δm of inertial mass squared associated with knotting is of from kgp^2, where k is positive integer and g the above described knot genus, one would have $\Delta m/m \simeq 1/p$. This is of order $M_{127}^{-1} \simeq 10^{-38}$ for electron.

Could the knotting and linking of elementary particles allow topological quantum computation at elementary particle level? The huge number of different knottings would give electron a huge ground state degeneracy making possible negentropic entanglement. For negentropic entanglement probabilities must belong to an algebraic extension of rationals: this would be the case in the intersection of p-adic and real worlds and there is a temptation to assign living matter to this intersection. Negentropy Maximization Principle could stabilize negentropic entanglement and therefore allow to circumvent the problems due to the fact that the energies involved are extremely tiny and far below thus thermal energy. In this situation bit would generalize to "nit" corresponding to N different ground states of particle differing by knotting.

A very naïve dimensional analysis using Uncertainty Principle would suggest that the number changes of electron state identifiable as quantum computation acting on q-nits is of order $1/\Delta t = \Delta m/\hbar$. More concretely, the minimum duration of the quantum computation would be of order $\Delta t = \hbar/\Delta m$. Single quantum computation would take an immense amount time: for electron single operation would take time of order 10^{17} s, which is of the order of the recent age of the Universe. Therefore this quantum computation would be of rather limited practical value!

Chapter i

Appendix

A-1 Introduction

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of embedding space and related spaces are discussed and the relationship of CP_2 to the standard model is summarized. The basic vision is simple: the geometry of the embedding space $H = M^4 \times CP_2$ geometrizes standard model symmetries and quantum numbers. The assumption that space-time surfaces are basic objects, brings in dynamics as dynamics of 3-D surfaces based on the induced geometry. Second quantization of free spinor fields of H induces quantization at the level of H, which means a dramatic simplification.

The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Also the relationship to string models is discussed briefly.

Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure analogous to adele [L48, L47]. In the recent view of quantum TGD [L109], both notions reduce to physics as number theory vision, which relies on $M^8 - H$ duality [L80, L81] and is complementary to the physics as geometry vision.

Zero energy ontology (ZEO) [L68] [K99] has become a central part of quantum TGD and leads to a TGD inspired theory of consciousness as a generalization of quantum measurement theory having quantum biology as an application. Also these aspects of TGD are briefly discussed.

A-2 Embedding space $M^4 \times CP_2$

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that embedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Embedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) [L68, L98] [K99] causal diamond (CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure [A74] so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2.1 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrate that it codes for the symmetries of standard models via its isometries and holonomies.

CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A61] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.2}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.3}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://tgdtheory.fi/appfigures/cp2.jpg`

Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i \bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-2.5}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.6}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r, \phi) = (tan(\theta/2), \phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-2.7}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \qquad \text{(A-2.8)}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \qquad \text{(A-2.9)}$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , & e^1 &= \frac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} \ , & e^3 &= \frac{r\sigma_3}{F} \ . \end{aligned} \qquad \text{(A-2.10)}$$

The explicit representations of vierbein vectors are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , & e^1 &= \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 &= \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 &= \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{aligned} \qquad \text{(A-2.11)}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \qquad \text{(A-2.12)}$$

From this expression one finds that at coordinate infinity $r = \infty$ line element reduces to $\frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2)$ of S^2 meaning that 3-sphere degenerates metrically to 2-sphere and one can say that CP_2 is obtained by adding to R^4 a 2-sphere at infinity.

The vierbein connection satisfying the defining relation

$$de^A = -V_B^A \wedge e^B \ , \qquad \text{(A-2.13)}$$

is given by

$$\begin{aligned} V_{01} &= -\frac{e^1}{r} \ , & V_{23} &= \frac{e^1}{r} \ , \\ V_{02} &= -\frac{e^2}{r} \ , & V_{31} &= \frac{e^2}{r} \ , \\ V_{03} &= (r - \frac{1}{r})e^3 \ , & V_{12} &= (2r + \frac{1}{r})e^3 \ . \end{aligned} \qquad \text{(A-2.14)}$$

The representation of the covariantly constant curvature tensor is given by

$$
\begin{array}{lllllll}
R_{01} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , & R_{23} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\
R_{02} & = & e^0 \wedge e^2 - e^3 \wedge e^1 \ , & R_{31} & = & -e^0 \wedge e^2 + e^3 \wedge e^1 \ , \\
R_{03} & = & 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , & R_{12} & = & 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ .
\end{array} \quad \text{(A-2.15)}
$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$
J = -is_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \quad \text{(A-2.16)}
$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$
J^k_{\ r} J^{rl} = -s^{kl} \ . \quad \text{(A-2.17)}
$$

The condition states that J and g give representations of real unit and imaginary units related by the formula $i^2 = -1$.

Kähler form is expressible locally in terms of Kähler gauge potential

$$
J = dB \ , \quad \text{(A-2.18)}
$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

$dJ = ddB = 0$ gives the topological half of Maxwell equations (vanishing of magnetic charges and Faraday's induction law) and self-duality $*J = J$ reduces the remaining equations to $dJ = 0$. Hence the Kähler form can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling).

The magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$
\begin{aligned}
B &= 2re^3 \ , \\
J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2} dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F} sin\Theta d\Theta \wedge d\Phi \ .
\end{aligned} \quad \text{(A-2.19)}
$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1, 1).

Useful coordinates for CP_2 are the so called canonical (or symplectic or Darboux) coordinates in which the Kähler potential and Kähler form have very simple expressions

$$
\begin{aligned}
B &= \sum_{k=1,2} P_k dQ_k \ , \\
J &= \sum_{k=1,2} dP_k \wedge dQ_k \ .
\end{aligned} \quad \text{(A-2.20)}
$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned}
P_1 &= -\frac{1}{1+r^2} \ , \\
P_2 &= -\frac{r^2 cos\Theta}{2(1+r^2)} \ , \\
Q_1 &= \Psi \ , \\
Q_2 &= \Phi \ .
\end{aligned} \tag{A-2.21}$$

Spinors In CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A50]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1- factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential $exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the embedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A94] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \text{ for } X,Y,Z \in t \ . \tag{A-2.22}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S_I^2 \ : \ \xi^1 = \bar{\xi}^2 \text{ or equivalently } (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S_{II}^2 \ : \ \xi^1 = \xi^2 \text{ or equivalently } (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S_I^2. S_{II}^2 is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2.2 CP_2 geometry and Standard Model symmetries

Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B47] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-2.23}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \otimes \gamma_5$, $1 \otimes \gamma_5$ and $\gamma_5 \otimes 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4)$ having as its covering group $SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \tag{A-2.24}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_{\pm}$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{llllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ , \end{array} \qquad \text{(A-2.25)}$$

and

$$B = 2re^3 \ , \qquad \text{(A-2.26)}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \qquad \text{(A-2.27)}$$

where one have defined

$$\begin{array}{lll} I_L^1 & = & \frac{(\Sigma_{01}-\Sigma_{23})}{2} \ , \\ I_L^2 & = & \frac{(\Sigma_{02}-\Sigma_{13})}{2} \ . \end{array} \qquad \text{(A-2.28)}$$

A_{ch} is clearly left handed so that one can perform the identification of the gauge potential as

$$W^{\pm} = \frac{2(e^1 \pm ie^2)}{r} \ , \qquad \text{(A-2.29)}$$

where $W^{\pm}$ denotes the charged intermediate vector boson.

The covariantly constant curvature tensor is given by

$$\begin{array}{l} R_{01} = -R_{23} = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} = -R_{31} = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \\ R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , \\ R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \qquad \text{(A-2.30)}$$

The charged part of the curvature tensor is left handed.

This is to be compared with the Weyl tensor, which defines a representation of quaternionic imaginary units.

$$
\begin{array}{l}
W_{03} = W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\
W_{01} = W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\
W_{02} = W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ .
\end{array} \tag{A-2.31}
$$

The charged part of the Weyl tensor is right-handed and that the relative sign of the two terms in the curvature tensor and Weyl tensor are opposite.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$
\begin{array}{rcl}
X & = & re^3 \ , \\
Y & = & \frac{e^3}{r} \ ,
\end{array} \tag{A-2.32}
$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$
\begin{array}{rcl}
\bar{\gamma} & = & aX + bY \ , \\
\bar{Z}^0 & = & cX + dY \ ,
\end{array} \tag{A-2.33}
$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$
\begin{array}{rcl}
A_{nc} & = & [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\
& + & [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ .
\end{array} \tag{A-2.34}
$$

Identifying Σ_{12} and $\Sigma_{03} = 1\times\gamma_5\Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$
c = -d \ . \tag{A-2.35}
$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$
A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.36}
$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$
\begin{aligned}
Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ , \\
I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ .
\end{aligned}
\tag{A-2.37}
$$

The fields γ and Z^0 are defined via the relations

$$
\begin{aligned}
\gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ , \\
Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \ .
\end{aligned}
\tag{A-2.38}
$$

The value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{3b}{2(a+b)} \ , \tag{A-2.39}
$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of the Weinberg angle is a dynamical problem. The original approach was based on the assumption that it makes sense to talk about electroweak action defined at fundamental level and introduce a symmetry breaking by adding an additional term proportional to Kähler action. The recent view is that Kähler action plus volume term defines the fundamental action.

The Weinberg angle is completely fixed if one requires that the electroweak action contains no cross term of type γZ^0. This leads to a definite value for the Weinberg angle.

One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle. As a matter fact, color gauge action identifying color gauge field as proportional to $H^A J_{\alpha\beta}$ is proportional to Kähler action. A possible interpretation would be as a sum of electroweak and color gauge interactions.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$
F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \tag{A-2.40}
$$

where one has

$$
\begin{aligned}
R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\
R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) \ , \\
J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ ,
\end{aligned}
\tag{A-2.41}
$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$
F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.42}
$$

Evaluating the expressions above, one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{A-2.43}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \tag{A-2.44}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \tag{A-2.45}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$\begin{aligned} X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\ K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \end{aligned} \tag{A-2.46}$$

This parameter can be calculated by substituting the values of quark and lepton charges and weak isospins.

In the general case the value of the coefficient K is given by

$$K = \sum_i \left[-\frac{(18 + 2n_i^2)sin^2\theta_W}{9}\right] \ , \tag{A-2.47}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18 + n_i^2))} \ . \tag{A-2.48}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \tag{A-2.49}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is not far from the typical value 9/24 of GUTs at high energies [B10]. The experimental value at the scale length scale of the electron can be deduced from the ratio of W and Z boson masses as $sin^2\theta_W = 1 - (m_W/m_Z)^2 \simeq .22290$. This ratio and also the weak boson masses depend on the length scale.

If one interprets the additional term proportional to J as color action, one could perhaps interpret the value of Weinberg angle as expressing a connection between strong and weak coupling constant evolution. The limit $f \to 0$ should correspond to an infinite value of color coupling strength and at this limit one would have $sin^2\theta_W = \frac{9}{28}$ for $f/g^2 \to 0$. This does not make sense since the Weinberg angle is in the standard model much smaller in QCD scale Λ corresponding roughly to pion mass scale. The Weinberg angle is in principle predicted by the p-adic coupling constant evolution fixed by the number theoretical vision of TGD.

One could however have a sum of electroweak action, correction terms changing the value of Weinberg angle, and color action and coupling constant evolution could be understood in terms of the coupling parameters involved.

Electroweak symmetry breaking

One of the hardest challenges in the development of the TGD based view of weak symmetry breaking was the fact that classical field equations allow space-time surfaces with finite but arbitrarily large size. For a fixed space-time surface, the induced gauge fields, including classical weak fields, are long ranged. On the other hand, the large mass for weak bosons would require a short correlation length. How can one understand this together with the fact that a photon has a long correlation length?

In zero energy ontology quantum states are superpositions of space-time surfaces as analogs of almost unique Bohr orbits of particles identified as 3-D surfaces. For some reason the superposition should be such that the quantum averages of weak gauge boson fields vanish below the weak scale whereas the quantum average of electromagnetic fields is non-vanishing.

This is indeed the case.

(a) The supersymplectic symmetries form isometries of the world of classical worlds (WCW) and they act in CP_2 degrees of freedom as symplectic transformations leaving the CP_2 symplectic form J invariant and therefore also its contribution to the electromagnetic field since this part is the same for all space-time surfaces in the superposition of space-time surfaces as a representation of supersymplectic isometry group (as a special case a representation of color group).

(b) In TGD, color and electroweak symmetries acting as holonomies are not independent and for the $SU(2)_L$ part of induced spinor connection the symplectic transformations induces $SU(2)_L \times U(1)_R$ gauge transformation. This suggests that the quantum expectations of the induced weak fields over the space-time surfaces vanish above the quantum coherence scale. The averages of W and of the left handed part of Z^0 should therefore vanish.

(c) $\langle Z^0 \rangle$ should vanish. For $U(1)_R$ part of Z^0, the action of gauge transformation is trivial in gauge theory. Now however the space-time surface changes under symplectic transformations and this could make the average of the right-handed part of Z^0 vanishing. The vanishing of the average of the axial part of the Z^0 is suggested by the partially conserved axial current hypothesis.

One can formulate this picture quantitatively.

(a) The electromagnetic field [L119] contains, besides the induced Kähler form, also the induced curvature form R_{12}, which couples vectorially. Conserved vector current hypothesis suggests that the average of R_{12} is non-vanishing. One can express the neutral part of the induced gauge field in terms of induced spinor curvature and Kähler form J as

$$
\begin{array}{rcl}
R_{03} & = & 2(2e^0 \wedge e^3 + e^1 \wedge e^2) = J + 2e^0 \wedge e^3 \ , \\
J & = & 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\
R_{12} & = & 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) = 3J - 2e^0 \wedge e^3 \ ,
\end{array} \qquad \text{(A-2.50)}
$$

(b) The induced fields γ and Z^0 (photon and Z- boson) can be expressed as

$$
\begin{array}{rcl}
\gamma & = & 3J - sin^2\theta_W R_{12} \ , \\
Z^0 & = & 2R_{03} = 2(J + 2e^0 \wedge e^3) \qquad \text{(A-2.51)} \\
per. & & \qquad \text{(A-2.52)}
\end{array}
$$

The condition $\langle Z^0 \rangle = 0$ gives $2\langle e^0 \wedge e^3 \rangle = -2J$ and this in turn gives $\langle R_{12} \rangle = 4J$. The average over γ would be

$$\langle \gamma \rangle = (3 - 4sin^2\theta_W)J \ .$$

For $sin^2\theta_W = 3/4\ langle\gamma\rangle$ would vanish.

The quantum averages of classical weak fields quite generally vanish. What about correlation functions?

(a) One expects that the correlators of classical weak fields as color invariants, and perhaps even symplectic invariants, are non-vanishing below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as hadron.

(b) The intuitive picture is that in longer length scales one has disjoint 3-surfaces with a size scale of Compton length. If the states associated with two disjoint 3-surfaces are separately color invariant there are no correlations in color degrees of freedom and correlators reduce to the products of expectations of classical weak fields and vanish. This could also hold when the 3-surfaces are connected by flux tube bonds.

Below the Compton length weak bosons would thus behave as correlated massless fields. The Compton lengths of weak bosons are proportional to the value of effective Planck constant h_{eff} and in living systems the Compton lengths are proposed to be even of the order of cell size. This would explain the mysterious chiral selection in living systems requiring large parity violation.

(c) What about the averages and correlators of color gauge fields? Classical color gauge fields are proportional to the products of Hamiltonians of color isometries induced Kähler form and the expectations of color Hamiltonians give vanishing average above Compton length and therefore vanishing average. Correlators are non-vanishing below the hadron scale. Gluons do not propagate in long scales for the same reason as weak bosons. This is implied by color confinement, which has also classical description in the sense that 3-surfaces have necessarily a finite size.

A large value of h_{eff} allows colored states even in biological scales below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as dark hadron.

Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

(a) Symmetries must be realized as purely geometric transformations.

(b) Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B18] .

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.53}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.54}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.55}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

A-3.1 Induction procedure for gauge fields and spinor connection

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has embedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as a base space the imbedded manifold, whose points have as fiber the fiber if embedding space at their image points. In the recent case the embedding of space-time surface to embedding space defines the induction procedure. The induced gauge potentials and gauge fields are projections of the spinor connection of the embedding space to the space-time surface (see `http://tgdtheory.fi/appfigures/induct.jpg`).

Induction procedure makes sense also for the spinor fields of embedding space and one obtains geometrization of both electroweak gauge potentials and of spinors. The new element is induction of gamma matrices which gives their projections at space-time surface.

As a matter fact, the induced gamma matrices cannot appear in the counterpart of massless Dirac equation. To achieve super-symmetry, Dirac action must be replaced with Kähler-Dirac action for which gamma matrices are contractions of the canonical momentum currents of Kähler action with embedding space gamma matrices. Induced gamma matrices in Dirac action would correspond to 4-volume as action.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://tgdtheory.fi/appfigures/induct.jpg`.

A-3.2 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP$_2$ projection, only vacuum extremals and space-time surfaces for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

A-3.3 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four embedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. `http://tgdtheory.fi/appfigures/manysheeted.jpg`

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. `http://tgdtheory.fi/appfigures/wormholecontact.jpg`

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg`

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of embedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)") as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-3.4 Embedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

(a) Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

(b) Spinor harmonics of embedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of embedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

(a) Although the embedding space spinor connection carries W gauge potentials one can say that the embedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

(b) The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

(c) Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

(d) One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

(e) This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-3.5 About induced gauge fields

In the following the induced gauge fields are studied for general space-time surface without assuming the preferred extremal property (Bohr orbit property). Therefore the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$
\begin{aligned}
J &= \frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta)d\Phi) + \frac{r^2}{2F} sin(\Theta)d\Theta \wedge d\Phi \ , \\
F &= 1 + r^2 \ .
\end{aligned}
\tag{A-3.1}
$$

The general expression of electromagnetic field reads as

$$
\begin{aligned}
F_{em} &= (3+2p)\frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta)d\Phi) + (3+p)\frac{r^2}{2F} sin(\Theta)d\Theta \wedge d\Phi \ , \\
p &= sin^2(\Theta_W) \ ,
\end{aligned}
\tag{A-3.2}
$$

where Θ_W denotes Weinberg angle.

(a) The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$
\begin{aligned}
\Psi &= k\Phi \ , \\
(3+2p)\frac{1}{r^2 F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p)sin(\Theta) = 0 \ ,
\end{aligned}
\tag{A-3.3}
$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$
\begin{aligned}
r &= \sqrt{\frac{X}{1-X}} \ , \\
X &= D\left[|\frac{k+u}{C}|\right]^{\epsilon} \ , \\
u &\equiv cos(\Theta) \ , \ C = k + cos(\Theta_0) \ , \ D = \frac{r_0^2}{1+r_0^2} \ , \ \epsilon = \frac{3+p}{3+2p} \ ,
\end{aligned}
\tag{A-3.4}
$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1 + r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k+1 \ ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \tag{A-3.5}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

(b) The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

(c) The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \quad X = D|k+u| \ , \\ \gamma &= -\frac{p}{2} Z^0 \ . \end{aligned} \tag{A-3.6}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta}) d\Theta^2 + (s_{\Phi\Phi} + 2k s_{\Phi\Psi}) d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta} d\Theta^2 + s^{eff}_{\Phi\Phi} d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \tag{A-3.7}$$

and is useful in the construction of vacuum embedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$
\begin{aligned}
\Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\
\Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ . \qquad \text{(A-3.8)}
\end{aligned}
$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global embedding for, say a constant magnetic field. Although global embedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$
\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \qquad \text{(A-3.9)}
$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-4 The relationship of TGD to QFT and string models

The recent view of the relationship of TGD to QFT and string models has developed slowly during years and it seems that in a certain sense TGD means a return to roots: instead of QFT like description involving path integral one would have wave mechanics for 3-surfaces.

A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces

The first vision of TGD was as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

The later work has revealed that TGD could be seen as a generalization of the wave mechanism based on the replacement of a point-like particle with 3-D surface. This is due to holography implied by general coordinate invariance. The definition of the metric of the "world of classical worlds" (WCW) must assign a unique or at least almost unique space-time surface to a given 3-surface. This 4-surface is analogous to Bohr orbit so that also Bohr orbitology becomes an exact part of quantum physics. The failure of strict determinism forces to replace 3-surfaces with 4-surfaces and this leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K39, K21, K73] [L99, L109].

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://tgdtheory.fi/appfigures/particletgd.jpg`

A-4.2 Extension of superconformal invariance

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in a completely unique position as far as symmetries are considered.

In fact, this leads to a generalization of the Kac-Moody type symmetries of string models. $\delta M^4_+ \times CP_2$ allows huge supersymplectic symmetries for which the radial light-like coordinate of δM^4_+ plays the role of complex string coordinate in string models. These symmetries are assumed to act as isometries of WCW.

A-4.3 String-like objects and strings

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of the TGD Universe and the inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string-like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situations in Minkowskian regions of space-time surface. `http://tgdtheory.fi/appfigures/fermistring.jpg`

A-4.4 TGD view of elementary particles

The TGD based view about elementary particles has two key aspects.

(a) The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidean signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

(b) At the level of H Fermion number is carried by the modes of the induced spinor field. In space-time regions with Minkowski signature the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle orbit corresponds to a 4-D generalization of a world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidean signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacts carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompanied by fermionic strings connecting the throats at the same sheet: the strings do not extend inside the wormhole contacts. `http://tgdtheory.fi/appfigures/elparticletgd.jpg`

Particle interactions involve both stringy and QFT aspects.

(a) The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having a hadronic string as a physical counterpart. Long strings should be distinguished from wormhole contacts which due to their super-conformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

(b) Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats behave like virtual bosons so that one has BFF type vertices typically.

(c) In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particles along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. `http://tgdtheory.fi/appfigures/tgdgraphs.jpg`

A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K39, K73].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

A-5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [L50] [L99, L102, L103] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and $T(CP_2)$ of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A74] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

(a) Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

(b) Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

(c) The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

    For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

(a) The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

(b) Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

A-5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

(a) If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

(b) WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L109]. There are also other important algebras but these will not be discussed now.

(a) The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

The super symplectic algebra A has an infinite hierarchy of sub-algebras [L109] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

(b) The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on $M^8 - H$ duality [L109] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

(a) The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by $M^8 - H$ duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

(b) This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L109] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P) = n(braid) = n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

(c) The actions related to different values of $n(P) = n(braid) = n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

(d) Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L106] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i = 2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r = 1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

(e) A subset of p-adic primes allowed by the p-adic length scale hypothesis $p \simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K50, K51]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

(a) Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L106], and the forward scattering amplitude corresponds to the propagator.

This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

(b) According to [L106], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L106].

(c) p-Adic length scale hypothesis [L110] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

(a) The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

 In [L109] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

(b) The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K39, K21]. As isometries they would naturally permute the maxima with each other.

A-6 Number theoretic vision of TGD

Physics as number theory vision is complementary to the physics as geometry vision and has developed gradually since 1993. Langlands program is the counterpart of this vision in mathematics [L108].

The notion of p-adic number fields emerged with the motivation coming from the observation that elementary particle mass scales and mass ratios could be understood in terms of the so-called p-adic length scale hypothesis [K54, K46, K18]. The fusion of the various p-adic physics leads to what I call adelic physics [L48, L47]. Later the hypothesis about hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter emerged [K24, K25, K26, K26].

Eventually this led to that the values of effective Planck constant could be identified as the dimension of an algebraic extension of rationals assignable to polynomials with rational coefficients. This led to the number theoretic vision in which so-called $M^8 - H$ duality [L80, L81] plays a key role. M^8 (actually a complexification of real M^8) is analogous to momentum space so that the duality generalizes momentum position duality for point-like particles. M^8 has an interpretation as complexified octonions.

The dynamics of 4-surfaces in M^8 is coded by polynomials with rational coefficients, whose roots define mass shells H^3 of $M^4 \subset M^8$. It has turned out that the polynomials satisfy stringent additional conditions and one can speak of number theoretic holography [L106, L108]. Also the ordinary 3→ 4 holography is needed to assign 4-surfaces with these 3-D mass shells. The number theoretic dynamics is based on the condition that the normal space of the 4-surface in M^8 is associative (quaternionic) and contains a commutative complex sub-space. This makes it possible to assign to this surface space-time surface in $H = M^4 \times CP_2$.

At the level of H the space-time surfaces are by holography preferred extremals and are assumed to be determined by the twistor lift of TGD [L50] giving rise to an action which is sum of the Kähler action and volume term. The preferred extremals would be minimal surfaces analogous to soap films spanned by frames. Outside frames they would be simultaneous extremals of the Kähler action, which requires a generalization of the holomorphy characterizing string world sheets.

In the following only p-adic numbers and hierarchy of Planck constants will be discussed.

A-6.1 p-Adic numbers and TGD

p-Adic number fields

p-Adic numbers (p is prime: 2, 3, 5, ...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A48]. p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k , \; x(k) = 0, ..., p-1 \; . \tag{A-6.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \; . \tag{A-6.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \; , \tag{A-6.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \; . \tag{A-6.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \leq D \; . \tag{A-6.5}$$

This division of the metric space into classes has following properties:

(a) Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

(b) Distances of points x and y inside single class are smaller than distances between different classes.

(c) Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B40]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

1. Basic form of the canonical identification

There exists a natural continuous map $I : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0, 1, .., p-1\} \ . \end{aligned} \tag{A-6.6}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{A-6.7}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{A-6.8}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

2. The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p |y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{A-6.9}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{A-6.10}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.11}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3. Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.12}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4. Generalization of number concept and notion of embedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic embedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R^n_p of R^n each of which projects to a copy of R^n_+ (four quadrants in the case of plane). The common points of p-adic and real embedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \; mod \; p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real embedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles".

Fig. 16. The basic idea between p-adic manifold. `http://tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

(a) Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

(b) Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

(c) Canonical identification violates general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic embedding space with chart maps to real embedding space and assuming preferred coordinates made possible by isometries of embedding space: one however obtains several inequivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-6.2 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the embedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of embedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of embedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-6.3 $M^8 - H$ duality as it is towards the end of 2021

The view of $M^8 - H$ duality (see Appendix **??**) has changed considerably towards the end 2021 [L99] after the realization that this duality is the TGD counterpart of momentum position duality of wave mechanics, which is lost in QFTs. Therefore M^8 and also space-time surface is analogous to momentum space. This forced us to give up the original simple identification of the points $M^4 \subset M^4 \times E^4 = M^8$ and of $M^4 \times CP_2$ so that it respects Uncertainty Principle (UP).

The first improved guess for the duality map was the replacement with the inversion $p^k \rightarrow m^k = \hbar_{eff} p^k/p^2$ conforming in spirit with UP but turned out to be too naive.

The improved form [L99] of the $M^8 - H$ duality map takes mass shells $p^2 = m^2$ of $M^4 \subset M^8$ to cds with size $L(m) = \hbar_{eff}/m$ with a common center. The slicing by mass shells is mapped to a Russian doll like slicing by cds. Therefore would be no CDs in M^8 contrary to what I believed first.

Quantum classical correspondence (QCC) inspires the proposal that the point $p^k \in M^8$ is mapped to a geodesic line corresponding to momentum p^k starting from the common center of cds. Its intersection with the opposite boundary of cd with size $L(m)$ defines the image point. This is not yet quite enough to satisfy UP but the additional details [L99] are not needed in the sequel.

The 6-D brane-like special solutions in M^8 are of special interest in the TGD inspired theory of consciousness. They have an M^4 projection which is $E = E_n$ 3-ball. Here E_n is a root of the real polynomial P defining $X^4 \subset M^8_c$ (M^8 is complexified to M^8_c) as a "root" of its octonionic continuation [L80, L81]. E_n has an interpretation as energy, which can be complex. The original interpretation was as moment of time. For this interpretation, $M^8 - H$ duality would be a linear identification and these hyper planes would be mapped to hyperplanes in $M^4 \subset H$. This motivated the term "very special moment in the life of self" for the image of the $E = E_n$ section of $X^4 \subset M^8$ [L63]. This notion does not make sense at the level M^8 anymore.

The modified $M^8 - H$ duality forces us to modify the original interpretation [L99]. The point $(E_n, p = 0)$ is mapped $(t_n = \hbar_{eff}/E_n, 0)$. The momenta (E_n, p) in $E = E_n$ plane are mapped to the boundary of cd and correspond to a continuous time interval at the boundary of CD: "very special moment" becomes a "very special time interval".

The quantum state however corresponds to a set of points corresponding to quark momenta, which belong to a cognitive representation and are therefore algebraic integers in the extension determined by the polynomial. These active points in E_n are mapped to a discrete set at the boundary of cd(m). A "very special moment" is replaced with a sequence of "very special moments".

So called Galois confinement [L93] forces the total momenta for bound states of quarks and antiquarks to be rational integers invariant under Galois group of extension of rationals determined by the polynomial P [L99]. These states correspond to states at boundaries of sub-CDs so that one obtains a hierarchy. Galois confinement provides a universal number theoretic mechanism for the formation of bound states.

A-7 Zero energy ontology (ZEO)

ZEO is implied by the holography forced in the TGD framework by general coordinate invariance.

A-7.1 Basic motivations and ideas of ZEO

The following gives a brief summary of ZEO [L68] [K99].

(a) In ZEO quantum states are not 3-dimensional but superpositions of 4-dimensional deterministic time evolutions connecting ordinary initial 3-dimensional states. By holography they are equivalent to pairs of ordinary 3-D states identified as initial and final states of time evolution. One can say that in the TGD framework general coordinate invariance implies holography and the slight failure of its determinism in turn forces ZEO.

Quantum jumps replace this state with a new one: a superposition of deterministic time evolutions is replaced with a new superposition. Classical determinism of individual time evolution is not violated and this solves the basic paradox of quantum measurement theory. There are two kinds of quantum jumps: ordinary ("big") state function reductions (BSFRs) changing the arrow of time and "small" state function reductions (SSFRs) (weak measurements) preserving it and giving rise to the analog of Zeno effect [L68].

(b) To avoid getting totally confused it is good to emphasize some aspects of ZEO.

i. ZEO does not mean that physical states in the usual 3-D sense as snapshots of time evolution would have zero energy state pairs defining zero energy states as initial and final states have same conserved quantities such as energy. Conservation implies that one can adopt the conventions that the values of conserved quantities are opposite for these states so that their sum vanishes: one can think that incoming and outgoing particles come from geometric past and future is the picture used in quantum field theories.

ii. ZEO means two times: subjective time as sequence of quantum jumps and geometric time as space-time coordinate. These times are identifiable but are strongly correlated.

(c) In BSFRs the arrow of time is changed and the time evolution in the final state occurs backwards with respect to the time of the external observer. BSFRs can occur in all scales since TGD predicts a hierarchy of effective Planck constants with arbitrarily large values. There is empirical support for BSFRs.

i. The findings of Minev et al [L60] in atomic scale can be explained by the same mechanism [L60]. In BSFR a final zero energy state as a superposition of classical deterministic time evolutions emerges and for an observer with a standard arrow of time looks like a superposition of deterministic smooth time evolutions leading to the final state. Interestingly, once this evolution has started, it cannot be stopped unless one changes the stimulus signal inducing the evolution in which case the process does not lead to anywhere: the interpretation would be that BSFR back to the initial state occurs!

ii. Libets' experiments about active aspects of consciousness [?] can be understood. Subject person raises his finger and neural activity starts before the conscious decision to do so. In the physicalistic framework it is thought to lead to raising of the finger. The problem with the explanation is that the activity beginning .5 seconds earlier seems to be dissipation with a reversed arrow of time: from chaotic and disordered to ordered at around .15 seconds. ZEO explanation is that macroscopic quantum jump occurred and generated a signal proceeding backwards in time and generated neural activity and dissipated to randomness.

iii. Earthquakes involve a strange anomaly: they are preceded by ELF radiation. One would expect that they generate ELF radiation. The identification as BSFR would explain the anomaly [L62]. In biology the reversal of the arrow of time would occur routinely and be a central element of biological self-organization, in particular self-organized quantum criticality (see [L66, L122].

A-7.2 Some implications of ZEO

ZEO has profound implications for understanding self-organization and self-organized quantum criticality in terms of dissipation with non-standard arrow of time looking like generation of structures [L66, L122]. ZEO could also allow understanding of what planned actions - like realizing the experiment under consideration - could be.

(a) Second law in the standard sense does not favor - perhaps even not allow - realization of planned actions. ZEO forces a generalization of thermodynamics: dissipation with a non-standard arrow of time for a subsystem would look like self-organization and planned action and its realization.

Could most if not all planned action be like this - induced by BSFR in the geometric future and only apparently planned? There would be however the experience of planning and realizing induced by the signals from geometric future by a higher level in the hierarchy of conscious entities predicted by TGD! In long time scales we would be realizing our fates or wishes of higher level conscious entities rather than agents with completely free will.

(b) The notion of magnetic body (MB) serving as a boss of ordinary matter would be central. MB carries dark matter as $h_{eff} = nh_0$ phases of ordinary matter with n serving as a measure for algebraic complexity of extension of rationals as its dimension and defining a kind of universal IQ. There is a hierarchy of these phases and MBs labelled by extension of rationals and the value of n.

MBs would form a hierarchy of bosses - a realization for master slave hierarchy. Ordinary matter would be at the bottom and its coherent behavior would be induced from quantum coherence at higher levels. BSFR for higher level MB would give rise to what looks like planned actions and experienced as planned action at the lower levels of hierarchy. One could speak of planned actions inducing a cascade of planned actions in shorter time scales and eventually proceeding to atomic level.

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred embedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes.

There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. `http://tgdtheory.fi/appfigures/padictoreal.jpg`

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. `http://tgdtheory.fi/appfigures/sharing.jpg`

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see **Fig.** `http://tgdtheory.fi/appfigures/timemirror.jpg` or **Fig.** 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. `http://tgdtheory.fi/appfigures/timemirror.jpg`

Acknowledgements: I am grateful for Dainis Zeps for enlightnening discussions concerning CP_2 geometry.

A-9 Could $\mathcal{N}=4$ Super-Conformal Symmetry Be Realized In TGD?

Both $\mathcal{N}=4$ and possible $\mathcal{N}=2$ super-conformal symmetry would be symmetries generated by the solutions of the Kähler-Dirac equation for the second quantized induced spinor fields at string world sheets. $\mathcal{N}=2$ SUSY at space-time level would follow from corresponding super-conformal algebra and would be naturally realized in terms of right handed neutrino and antineurino. It is however far from obvious whether large $\mathcal{N}=4$ super-conformal symmetry makes sense.

(a) One has two conserved fermionic numbers (quarks and leptons) and this allows 4-super generators but they SUSY generated by right-handed neutrino does not have any counterpart in quark sector so that one can hope only $\mathcal{N}=4$ SCA broken down to $\mathcal{N}=2$ realized by adding to quark or lepton state right-handed neutrino or antineutrino.

(b) In the case of $\mathcal{N}=2$ one has inherent $SU(2)_-\times U(1)$ symmetry assignable to CP_2 naturally. For $\mathcal{N}=4$ one has inherent $SU(2)_+\times SU(2)_-\times U(1)$ Kac-Moody symmetry, which should correspond to a fundamental partonic super-conformal symmetry in TGD framework.

    The assignment of both $SU(2)$ with CP_2 degrees of freedom is highly questionable since the holonomy group in these degrees of freedom reduces to electro-weak group. The assignment of the second $SU(2)$ with M^4 spin is questionable since M^4 has trivial holonomy group. In zero energy ontology (ZEO) positive and negative energy parts of zero energy states are assigned to the light-like boundaries of causal diamond (CD) and having SU(2) as holonomy group. Could one assign the second $SU(2)$ with it? One does not however have induced spinor connection in M^4 degrees of freedom that this identification is questionable.

The conservative conclusion would be that one has $\mathcal{N}=\in$ SCA with quarks and leptons defining separate irreducible representations of SCA. Despite this the $\mathcal{N}=\triangle$ alternative deserves a separate study.

Needless to say, a lot remains to be understood. One of the problems is that my understanding of $\mathcal{N}=4$ super-conformal symmetry at technical level is rather modest. There are also profound differences between these two kinds of super conformal symmetries. In TGD framework super generators carry quark or lepton number, super-symplectic and super Kac-Moody generators are identified as Hamiltonians rather than vector fields, and symplectic group is infinite-dimensional whereas the Lie groups associated with Kac-Moody algebras are finite-dimensional. On the other hand, finite measurement resolution implies discretization and cutoff in conformal weight. Therefore the naïve attempt to re-interpret results of standard super-conformal symmetry to TGD framework might lead to erratic conclusions.

$N>0$ super-conformal algebras contain besides super Virasoro generators also other types of generators and this raises the question whether it might be possible to find an algebra coding the basic quantum numbers of the induced spinor fields.

There are several variants of $\mathcal{N}=4$ SCAs and they correspond to the Kac-Moody algebras $SU(2)$ (small SCA), $SU(2)\times SU(2)\times U(1)$ (large SCA) and $SU(2)\times U(1)^4$. Rasmussen has found also a fourth variant based on $SU(2)\times U(1)$ Kac-Moody algebra [A81]. It seems that only minimal and maximal $\mathcal{N}=4$ SCAs can represent realistic options. The reduction to almost topological string theory in critical phase is probably lost for other than minimal SCA but could result as an appropriate limit for other variants.

A-9.1 Large $\mathcal{N}=4$ SCA

Large $\mathcal{N}=4$ SCA is described in the following in detail since it might be a natural algebra in TGD framework.

The structure of large $\mathcal{N}=4$ SCA algebra

A concise discussion of this symmetry with explicit expressions of commutation and anti-commutation relations can be found in [A81]. The representations of SCA are characterized by three central extension parameters for Kac-Moody algebras but only two of them are independent and given by

$$\begin{aligned} k_\pm &\equiv k(SU(2)_\pm) \ , \\ k_1 &\equiv k(U(1)) = k_+ + k_- \ . \end{aligned} \tag{A-9.1}$$

The central extension parameter c is given as

$$c = \frac{6k_+k_-}{k_+ + k_-} \ . \tag{A-9.2}$$

and is rational valued as required.

A much studied $\mathcal{N}=4$ SCA corresponds to the special case

$$\begin{aligned} k_- &= 1 \ , \ k_+ = k+1 \ , \ k_1 = k+2 \ , \\ c &= \frac{6(k+1)}{k+2} \ . \end{aligned} \tag{A-9.3}$$

$c=0$ would correspond to $k_+ = 0, k_- = 1, k_1 = 1$. For $k_+ > 0$ one has $k_1 = k_+ + k_- \neq k_+$.

About unitary representations of large $\mathcal{N}=4$ SCA

The unitary representations of large $\mathcal{N}=4$ SCA are briefly discussed in [A63]. The representations are labeled by the ground state conformal weigh h, SU(2) spins l_+, l_-, and U(1) charge u. Besides the inherent Kac-Moody algebra there is also “external” Kac-Moody group G involved and could correspond in TGD framework to the symplectic algebra associated with $\delta H_\pm = \delta M^4_\pm \times CP_2$ or to Kac-Moody group respecting light-likeness of light-like 3-surfaces. External Kac-Moody algebra can be also assigned with color degrees of freedom.

Unitarity constraints apply completely generally irrespective of G so that one can apply them also in TGD framework. There are two kinds of unitary representations.

(a) Generic/long/massive representations which are generated from vacuum state as usual. In this case there are no null vectors.

(b) Short or massless representations have a null vector. The expression for the conformal weigt h_{short} of the null vector reads in terms of l_+, l_- and k_+, k_- as

$$h_{short} = \frac{1}{k_+ + k_-}(k_-l_+ + k_+l_- + (l_+ - l_-)^2 + u^2) \ . \tag{A-9.4}$$

Unitarity demands that both short and long representations lie at or above $h \geq h_{short}$ and that spins lie in the range $l_\pm = 0, 1/2, ..., (k_\pm - 1)/2$.

(c) Interesting examples of $\mathcal{N}=4$ SCA are provided by WZW coset models $\mathcal{W} \times U(1)$, where $\mathcal{W}$ is WZW model associated with a quaternionic (Wolf) space. Examples based on classical groups are $\mathcal{W} = G/H = SU(n)/SU(n-1)\times U(1)$, $SO(n)/SO(n-4)\times SU(2)$, and $Sp(2n)/Sp(2n-2)$. For $n=3$ first series gives CP_2 whereas second series gives for $\mathcal{N}=4$ $SO(4)/SU(2) = SU(2)$. In this case one has $k_+ = \kappa + 1$, and $k_- = \hat{c}_G$, where κ is the level of the bosonic current algebra for G and $\hat{c}_G$ is its dual Coxeter number.

WZW coset model $\mathcal{W} = G/H = CP_2$ is of special interest in TGD framework and could allow to bring in the color Kac-Moody algebra. The $U(1)$ algebra might be however problematic since the standard model U(1) is already contained in the SCA.

A-9.2 Overall View About How Different $\mathcal{N} = 4$ SCAs Could Emerge In TGD Framework

The basic idea is simple $\mathcal{N} = 4$ fermion states obtained as different combinations of spin and isospin for given H-chirality of embedding space spinor correspond to $\mathcal{N} = 4$ multiplet. In the case of leptons the holonomy group of $S^2 \times CP_2$ for given spinor chirality is $SU(2)_R \times SU(2)_R$ or $SU(2)_L \times SU(2)_R$ depending on M^4 chirality of the spinor. In case of quark one has $SU(2)_L \times SU(2)_L$ or $SU(2)_R \times SU(2)_R$. The coupling to Kähler gauge potential adds to the group $U(1)$ factor so that large $\mathcal{N} = 4$ SCA is obtained. For covariantly constant right handed neutrino electro-weak part of holonomy group drops away as also $U(1)$ factor so that one obtains $SU(2)_L$ or $SU(2)_R$ and small $\mathcal{N} = 4$ SCA.

How maximal $\mathcal{N} = 4$ SCA could emerge in TGD framework?

Consider the Kac-Moody algebra $SU(2) \times SU(2) \times U(1)$ associated with the maximal $\mathcal{N} = 4$ SCA. Besides Kac-Moody currents it contains 4 spin 1/2 fermionic generators having an identification as quantum counterparts of leptonic spinor fields. The interpretation of the first $SU(2)$ is as rotations as rotations leaving invariant the sphere $S^2 \subset \delta M^4_\pm$.

Here it is essential to notice that the holonomy of light-cone boundary is non-trivial unlike the holonomy of M^4. In zero energy ontology (ZEO) assigning positive and negative energy parts of zero energy states to the boundaries of causal diamond (CD) this holonomy group would emerge naturally.

$U(2)$ has interpretation as electro-weak gauge group and as maximal linearly realized subgroup of $SU(3)$. This algebra acts naturally as symmetries of the 8-component spinors representing super partners of quaternions.

The algebra involves the integer value central extension parameters k_+ and k_- associated with the two SU(2) algebras as parameters. The value of $U(1)$ central extension parameter k is given by $k = k_+ + k_-$. The value of central extension parameter c is given by

$$c = 6k_- \frac{x}{1+x} < 6k_+ \quad , \quad x = \frac{k_+}{k_-} \quad .$$

c can have all non-negative rational values m/n for positive values of $k_\pm$ given by $k_+ = rm, k_- = (6nr-1)m$. Unitarity might pose further restrictions on the values of c. At the limit $k_- = k$, $k_+ \to \infty$ the algebra reduces to the minimal $\mathcal{N} = 4$ SCA with $c = 6k$ since the contributions from the second $SU(2)$ and $U(1)$ to super Virasoro currents vanish at this limit.

How small $\mathcal{N} = 4$ SCA could emerge in TGD framework?

Consider the TGD based interpretation of the small $\mathcal{N} = 4$ SCA.

(a) The group $SU(2)$ associated with the small $\mathcal{N} = 4$ SCA and acting as rotations of covariantly constant right-handed neutrino spinors allows also an interpretation as a group $SO(3)$ leaving invariant the sphere S^2 of the light-cone boundary identified as $r_M = m^0$=constant surface defining generalized Kähler and symplectic structures in $\delta M^4_\pm$. Electro-weak degrees of freedom are obviously completely frozen so that $SU(2)_- \times U1$ factor indeed drops out.

(b) The choice of the preferred coordinate system should have a physical justification. The interpretation of $SO(3)$ as the isotropy group of the rest system defined by the total four-momentum assignable to the 3-surface containing partonic 2-surfaces is supported by the quantum classical correspondence. The subgroup $U(1)$ of $SU(2)$ acts naturally as rotations around the axis defined by the light ray from the tip of $M^4_\pm$ orthogonal to S^2. For $c = 0, k = 0$ case these groups define local gauge symmetries. In the more general case local gauge invariance is broken whereas global invariance remains as it should.

In $M^2 \times E^2$ decomposition E^2 corresponds to the tangent space of S^2 at a given point and M^2 to the plane orthogonal to it. The natural assumption is that the right handed neutrino spinor is annihilated by the momentum space Dirac operator corresponding to the light-like momentum defining $M^2 \times E^2$ decomposition.

(c) For covariantly constant right handed neutrinos the dynamics would be essentially that defined by a topological quantum field theory and this kind of almost trivial dynamics is indeed associated with small $\mathcal{N}=4$ SCA.

1. Why $\mathcal{N}=4$ SUSY

$\mathcal{N}=2$ super-conformal invariance has been claimed to imply the vanishing of all amplitudes with more than 3 external legs for closed critical $\mathcal{N}=2$ strings having $c=6, k=1$ which is proposed to correspond to $n \rightarrow \infty$ limit [A47, A84]. Only the partition function and $2 \leq N \leq 3$ scattering amplitudes would be non-vanishing. The argument of [A47] relies on the embedding of $\mathcal{N}=2$ super-conformal field theory to $\mathcal{N}=4$ topological string theory whereas in [A84] the Ward identities for additional unbroken symmetries associated with the chiral ring accompanying $\mathcal{N}=2$ super-symmetry [A70] are utilized. In fact, $\mathcal{N}=4$ topological string theory allows also embeddings of $N=1$ super strings [A47].

The properties of $c=6$ critical theory allowing only integral valued $U(1)$ charges and fermion numbers would conform nicely with what we know about the perturbative electro-weak physics of leptons and gauge bosons. $c=1, k=1$ sector with $\mathcal{N}=2$ super-conformal symmetry would involve genuinely stringy physics since all N-point functions would be non-vanishing and the earlier hypothesis that strong interactions can be identified as electro-weak interactions which have become strong inspired by HO-H duality [K82] could find a concrete realization.

In $c=6$ phase $\mathcal{N}=2$-vertices the loop corrections coming from the presence of higher lepton genera in amplitude could be interpreted as topological mixing forced by unitarity implying in turn leptonic CKM mixing for leptons. The non-triviality of 3-point amplitudes would in turn be enough to have a stringy description of particle number changing reactions, such as single photon brehmstrahlung. The amplitude for the emission of more than one brehmstrahlung photons from a given lepton would vanish. Obviously the connection with quantum field theory picture would be extremely tight and imbeddability to a topological $\mathcal{N}=4$ quantum field theory could make the theory to a high degree exactly solvable.

2. Objections

There are also several reasons for why one must take the idea about the usefulness of $c=6$ super-conformal strings from the point of view of TGD with an extreme caution.

(a) Stringy diagrams have quite different interpretation in TGD framework. The target space for these theories has dimension four and metric signature (2, 2) or (0, 4) and the vanishing theorems hold only for $(2,2)$ signature. In lepton sector one might regard the covariantly constant complex right-handed neutrino spinors as generators of $\mathcal{N}=2$ super-symmetries but in quark sector there are no super-symmetries.

(b) The spectrum looks unrealistic: all degrees of freedom are eliminated by symmetries except single massless scalar field so that one can wonder what is achieved by introducing the extremely heavy computational machinery of string theories. This argument relies on the assumption that time-like modes correspond to negative norm so that the target space reduces effectively to a 2-dimensional Euclidian sub-space E^2 so that only the vibrations in directions orthogonal to the string in E^2 remain. The situation changes if one assigns negative conformal weights and negative energies to the time like excitations. In the generalized coset representation used to construct physical states this is indeed assumed.

(c) The central charge has only values $c = 6k$, where k is the central extension parameter of SU(2) algebra [A38] so that it seems impossible to realize the genuinely rational values of c which should correspond to the series of Jones inclusions. One manner to circumvent the problem would be the reduction to $\mathcal{N} = 2$ super-conformal symmetry.

(d) SU(2) Kac-Moody algebra allows to introduce only 2-component spinors naturally whereas super-quaternions allow quantum counterparts of 8-component spinors.

The $\mathcal{N} = 2$ super-conformal algebra automatically extends to the so called small $\mathcal{N} = 4$ algebra with four super-generators $G_\pm$ and their conjugates [A47]. In TGD framework $G_\pm$ degeneracy corresponds to the two spin directions of the covariantly constant right handed neutrinos and the conjugate of $G_\pm$ is obtained by charge conjugation of right handed neutrino. From these generators one can build up a right-handed $SU(2)$ algebra.

Hence the $SU(2)$ Kac-Moody of the small $\mathcal{N} = 4$ algebra corresponds to the three imaginary quaternionic units and the $U(1)$ of $\mathcal{N} = 2$ algebra to ordinary imaginary unit. Energy momentum tensor T and SU(2) generators would correspond to quaternionic units. $G_\pm$ to their super counterparts and their conjugates would define their "square roots".

What about $\mathcal{N} = 4$ SCA with $SU(2) \times U(1)$ Kac-Moody algebra?

Rasmussen [A81] has discovered an $\mathcal{N} = 4$ super-conformal algebra containing besides Virasoro generators and 4 Super-Virasoro generators $SU(2) \times U(1)$ Kac-Moody algebra and two spin 1/2 fermions and a scalar.

The first identification of $SU(2) \times U(1)$ is as electro-weak algebra for a given spin state. Second identification is as the algebra defined by rotation group and electromagnetic or Kähler charge acting on given charge state of fermion and naturally resulting in electro-weak symmetry breaking. Scalar might relate to Higgs field which is M^4 scalar but CP_2 vector.

There are actually two versions about Rasmussen's article [A81]: in the first version the author talks about $SU(2) \times U(1)$ Kac-Moody algebra and in the second one about $SL(2) \times U(1)$ Kac-Moody algebra.

.9.3 How Large $\mathcal{N} = 4$ SCA Could Emerge In Quantum TGD?

The formulation of TGD as an almost topological super-conformal QFT with light-like partonic 3-surfaces identified as basic dynamical objects has increased considerably the understanding of super-conformal symmetries and their breaking in TGD framework. $\mathcal{N} = 4$ super-conformal algebra would correspond to the maximal algebra with $SU(2) \times U(2)$ Kac-Moody algebra as inherent fermionic Kac-Moody algebra.

Concerning the interpretation the first guess would be that $SU(2)_+$ and $SU(2)_-$ correspond to vectorial spinor rotations in M^4 and CP_2 and $U(1)$ to Kähler charge or electromagnetic charge. For given embedding space chirality (lepton/quark) and M^4 chirality $SU(2)$ groups are completely fixed.

There are many kinds of fermionic super generators and the role of these algebras is not yet well-understood.

Well-definedness of electromagnetic charge implies stringiness

There is also a new element not present in the original speculations. The condition that em charge is well defined for spinor modes implies that the space-time region in which spinor mode is non-vanishing has 2-D CP_2 projection such that the induced W boson fields are vanishing. The vanishing of classical Z^0 field can be poses as additional condition - at least in scales above weak scale. In the generic case this requires that the spinor mode is restricted to 2-D surface: string world sheet or possibly also partonic 2-surface. This implies that TGD reduces to string model in fermionic sector. Even for preferred extremals with 2-D projecting the modes are expected to allow restriction to 2-surfaces. This localization is possible only for Kähler-Dirac action.

Identification of super generators associated with WCW metric

The definition of the metric of "world of classical worlds" (WCW) is as anticommutators of WCW gamma matrices carrying fermion number and in one-one correspondence with the infinitesimal isometries of WCW . WCW gamma matrices can be interpreted as supergenerators but do not seem to be identifiable as super counterparts of Noether charges. Fermionic generators can be divided into those associated with symplectic transformations, isometries, or symplectic isometries.

1. Generators of the symplectic algebra of $\delta M^4_\pm \times CP_2$ defined in terms of covariantly constant right-handed neutrino and second quantized induced spinor field. The form of current is $\overline{\nu}_R j^k_A \gamma_k \Psi$ and only leptonic Ψ contributes.

2. Fermionic generators defined in terms of all spinor modes for the symplectic isometries by the same formulas as in the case of symplectic algebra. This algebra is Kac-Moody type algebra with radial light-like coordinate r_M of $\delta M^4_\pm$ playing the role of complex coordinate. There is conformal weight associated with r_M but also with the fermionic modes since the fermions are localized to 2-D string world sheets and labelle by integer valued conformal weight. The form of the fermionic current is $\overline{\Psi}_n j^k_A \gamma_k \Psi$ and both quark-like anbd leptonic Ψ contribute.

3. One can also consider fermionic generators assignable as a Noether super charges to the isometries of $\delta M^4_\pm = S^2 \times R_+$, which are in 1-1 correspondence with the conformal transformations of S^2. The conformal scaling of S^2 is compensated by the S^2 dependent scaling of the light-like radial coordinate r_M. It is not completely clear whether these should be included. If not, it would be a slight dis-appointment since the metric 2-dimensionality of the $\delta M^4_\pm$ makes 4-D Minkowski space unique. Same applies to 4-D space-time since light-like 3-surfaces representing partonic 2-surfaces allow also 2-D conformal symmetries as isometries.

Supercharges accompanying conserved fermion numbers

There are also fermionic super-charges defined as super-currents serving as super counter-parts of conserved fermion number in quark-like and leptonic sector.

1. Assume that the Kähler-Dirac operator decomposition $D = D(Y^2) + D(X^2)$ reflecting the dual slicings of space-time surfaces to string world sheets Y^2 and partonic 2-surfaces X^2. If the conditions guaranteing well-defined em charge hold true, when can forget the presence of X^2 and the parameters λ_k labelling spinor modes in these degrees of freedom. The highly non-trivial consistency condition possible for Kähler-Dirac action is that $D(X^2)$ vanishes at string world sheets and thus allows the localization.

2. Y^1 represents light-like direction and also string connecting braid strands at same component of X^3_l or at two different components of X^3_l. Kähler-Dirac equation implies that the charges

$$\int_{X^3_l} \overline{\Psi}_n \hat{\Gamma}^v \Psi \tag{.9.5}$$

define conserved super charges in time direction associated with Y^1 and carrying quark or lepton number. Here Ψ_n corresponds to n: th conformal excitation of Ψ and has conformal weight n (plus possible ground state conformal weight). In the case of ordinary Dirac equation essentially fermionic oscillator operators would be in question.

3. The zero modes of $D(X^2)$ define a sub-algebra which is a good candidate for representing super gauge symmetries. If localizations to 2-D string world sheets takes place, only these transformations are present.

   In particular, covariantly constant right handed neutrinos define this kind of super gauge super-symmetries. $\mathcal{N} = 2$ super-conformal symmetry would correspond in TGD framework to covariantly constant complex right handed neutrino spinors with two spin directions forming a right handed doublet and would be exact and act only in the leptonic sector relating WCW

Hamiltonians and super-Hamiltonians. This algebra extends to the so called small $\mathcal{N} = 4$ algebra if one introduces the conjugates of the right handed neutrino spinors. This symmetry is exact if only leptonic chirality is present in theory or if free quarks carry leptonic charges.

A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K41] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of CDs and wormhole throats would define space-like and time-like braids uniquely.

The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A59] to TGD framework. It leads to the identification of slicing by 3-surfaces as that induced by the inverse images of $r = constant$ surfaces of CP_2, where r is $U(2)$ invariant radial coordinate of CP_2 playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing CD and would have the choice of braids and string world sheets as a space-time correlate.

Identification of Kac-Moody generators

Consider next the generators of inherent Kac-Moody algebras for $SU(2) \times SU(2) \times U(1)$ and freely chosen group G.

1. Generators of Kac-Moody algebra associated with isometries correspond Noether currents associated with the infinitesimal action of Kac-Moody algebra to the induced spinor fields. Local $SO(3) \times SU(3)$ algebra is in question and excitations should have dependence on the coordinate u in direction of Y^1. The most natural guess is that this algebra corresponds to the Kac-Moody algebra for group G.

2. The natural candidate for the inherent Kac-Moody algebra is the holonomy algebra associated with $S^2 \times CP_2$. This algebra should correspond to a broken symmetry.

   The generalized eigen modes of $D(X^2)$ labeled by λ_k should from the representation space in this case: if localization to 2-D string world sheets occurs, this space is 1-D. If Kac-Moody symmetry were not broken these representations would correspond a degeneracy associated with given value of λ_k. Electro-weak symmetry breaking is however present and coded already into the geometry of CP_2. Also $SO(3)$ symmetry is broken due to the presence of classical electro-weak magnetic fields. The broken symmetries could be formulated in terms of initial values of generalized eigen modes at X^2 defining either end of X^3_l. One can rotate these initial values by spinor rotations. Symmetry breaking would mean that the modes obtained by a rotation by angle $\phi = \pi$ from a mode with fixed eigenvalue λ_k have different eigenvalues. Four states would be obtained for a given embedding space chirality (quark or lepton). One expects that an analog of cyclotron spectrum with cutoff results with each cyclotron state split to four states with different eigenvalues λ_k. Kac-Moody generators could be expressed as matrices acting in the space spanned by the eigen modes.

Consistency with p-adic mass calculations

The consistency with p-adic mass calculations provides a strong guide line in attempts to interpret $\mathcal{N} = 4$ SCA. The basis ideas of p-adic mass calculations are following.

1. Fermionic partons move in color partial waves in their cm degrees of freedom. This gives to conformal weight a vacuum contribution equal to the CP_2 contribution to mass squared. The contribution depends on electro-weak isospin and equals $(h_c(U), h_c(D)) = (2, 3)$ for quarks and one has $(h_c(\nu), h_c(L)) = (1, 2)$.

2. The ground state can correspond also to non-negative value of L_0 for SKMV algebra, which gives rise to a thermal degeneracy of massless states. p-Adic mass calculations require $(h_{gr}(U), h_{gr}(D)) = (1, 0)$ and $(h_{gr}(\nu), h_{gr}(L)) = (2, 1)$ so that the super-symplectic operator O_c screening the anomalous color charge has conformal weight $h_c = -3$ for all fermions.

The simplest interpretation is that the free parameter h appearing in the representations of the SCA corresponds to the conformal weight due to the color partial wave so that the correlation with electromagnetic charge would indeed emerge but from the correlation of color partial waves and electro-weak quantum numbers.

The requirement that ground states are null states with respect to the SCV associated with the radial light-like coordinate of $\delta M^4_\pm$ gives an additional consistency condition and $h_c = -3$ should satisfy this condition. p-Adic mass calculations do not pose non-trivial conditions on h for option 1) if one makes the identification $u = Q_{em}$ since one has $h_{short} < 1$ for all values of $k_+ + k_-$. Therefore both options 1) and 2) can be considered.

About symmetry breaking for large $\mathcal{N} = 4$ SCA

Partonic formulation predicts that large $\mathcal{N} = 4$ SCA is a broken symmetry, and the first guess is that breaking occurs via several steps. First a "small" $\mathcal{N} = 4$ SCA with Kac-Moody group $SU(2)_+ \times U(1)$, where $SU(2)_+$ corresponds to ordinary rotations on spinor with fixed helicity, would result in electro-weak symmetry breaking. The next step in breaking of the spin symmetry would lead to $\mathcal{N} = 2$ SCA and the final step to $N = 0$ SCA. Several symmetry breaking scenarios are possible.

1. The interpretation of $SU(2)_+$ in terms of right- or left- handed spin rotations and $U(1)$ as electromagnetic gauge group conforms with the general vision about electro-weak symmetry breaking in non-stringy phase. The interpretation certainly makes sense for covariantly constant right handed neutrinos for which spin direction is free. For left handed charged electro-weak bosons the action of right-handed spinor rotations is trivial so that the interpretation would make sense also now.

2. The next step in the symmetry breaking sequence would be $\mathcal{N} = 2$ SCA with electromagnetic Kac-Moody algebra as inherent Kac-Moody algebra $U(1)$.

.9.4 Relationship To Super String Models, M-theory And WZW Model

In hope of achieving more precise understanding one can try to understand the relationship of $\mathcal{N} = 4$ super conformal symmetry as it might appear in TGD to super strings, M theory and WZW model.

Relationship to super-strings and M-theory

The (4, 4) signature characterizing $\mathcal{N} = 4$ SCA topological field theory is not a problem since in TGD framework the target space becomes a fictive concept defined by the Cartan algebra. Both $M^4 \times CP_2$ decomposition of the embedding space and space-time dimension are crucial for the $2 + 2 + 2 + 2$ structure of the Cartan algebra, which together with the notions of WCW and generalized coset representation formed from super Kac-Moody and super-symplectic algebras guarantees $\mathcal{N} = 4$ super-conformal invariance.

Including the 2 gauge degrees of freedom associated with M^2 factor of $M^4 = M^2 \times E^2$ the critical dimension becomes $D = 10$ andincluding the radial degree of light-cone boundary the critical dimension becomes $D = 11$ of M-theory. Hence the fictive target space associated with the vertex operator construction corresponds to a flat background of super-string theory and flat background of M-theory with one light-like direction. From TGD point view the difficulties of these approaches are due to the un-necessary assumption that the fictive target space defined by the Cartan algebra corresponds to the physical embedding space. The flatness of the fictive target space forces to introduce the notion of spontaneous compactification and dynamical embedding space and this in turn leads to the notion of landscape.

Consistency with critical dimension of super-string models and M-theory

Mass squared is identified as the conformal weight of the positive energy component of the state rather than as a contribution to the conformal weight canceling the total conformal weight. Also the Lorentz invariance of the p-adic thermodynamics requires this. As a consequence, the pseudo 4-momentum p assignable to M^4 super Kac-Moody algebra could be always light-like or even tachyonic.

Super-symplectic algebra would generate the negative conformal weight of the ground state required by the p-adic mass calculations and super-Kac Moody algebra would generate the non-negative net conformal weight identified as mass squared. In this interpretation SKM and SC degrees of freedom are independent and correspond to opposite signs for conformal weights.

The construction is consistent with p-adic mass calculations [K46, K46] and the critical dimension of super-string models.

1. Five Super Virasoro sectors are predicted as required by the p-adic mass calculations (the predicted mass spectrum depends only on the number of tensor factors). Super-symplectic algebra gives $Can(CP_2)$ and $Can(S^2)$. In SKM sector one has $SU(2)_L$, $U(1)$, local SU(3), SO(2) and E^2 orthogonal to strong world sheets so that 5 sectors indeed result.

2. The Cartan algebras involved of SC is 2-dimensional and that of SKM is 7-dimensional so that 10-dimensional Cartan algebra results. This means that vertex operator construction implies generation of 10-dimensional target space which in super-string framework would be identified as embedding space. Note however that these dimensions have Euclidian signature unlike in superstring models. SKM algebra allows also the option $SO(3) \times E(3)$ in M^4 degrees of freedom: this would mean that SKM Cartan algebra is 10-dimensional and the whole algebra 11-dimensional.

$\mathcal{N} = 4$ super-conformal symmetry and WZW models

One can question the naïve idea that the basic structure $G_{int} = SU(2) \times U(2)$ structure of $\mathcal{N} = 4$ SCA generalizes as such to the recent framework.

1. $\mathcal{N} = 4$ SCA is originally associated with Majorana spinors. $\mathcal{N} = 4$ algebra can be transformed from a real form to complex form with 2 complex fermions and their conjugates corresponding to complex H-spinors of definite chirality having spin and weak isospin. At least at formal level the complexification of $\mathcal{N} = 4$ SCA algebra seems to make sense and might be interpreted as a direct sum of two $\mathcal{N} = 4$ SCAs and complexified quaternions. Central charge would remain $c = 6k_+k_-/(k_+ + k_-)$ if naïve complexification works. The fact that Kac-Moody algebra of spinor rotations is $G_{int} = SO(4) \times SO(4) \times U(1)$ is naturally assignable naturally to spinors of H suggests that it represents a natural generalization of $SO(4) \times U(1)$ algebra to inherent Kac-Moody algebra.

2. One might wonder whether the complex form of $\mathcal{N} = 4$ algebra could result from $N = 8$ SCA by posing the associativity condition.

3. The article of Gunaydin [A87] about the representations of $\mathcal{N} = 4$ super-conformal algebras realized in terms of Goddard-Kent-Olive construction and using gauged Wess-Zumino-Witten models forces however to question the straightforward translation of results about $\mathcal{N} = 4$ SCA to TGD framework and it must be admitted that the situation is something confusing. Of course, there is no deep reason to believe that WZW models are appropriate in TGD framework.

   (a) Gauged WZW models are constructed using super-space formalism which is not natural in TGD framework. The coset space $CP_2 \times U(2)$ where $U(2)$, could be identified as sub-algebra of color algebra or possibly as electro-weak algebra provides one such realization. Also the complexifixation of the $\mathcal{N} = 4$ algebra is something new.

   (b) The representation involves 5-grading by the values of color isospin for $SU(3)$ and makes sense as a coset space realization for $G/H \times U(1)$ if H is chosen in such a way that $G/H \times SU(2)$ is quaternionic space. For SU(3) one has $H = U(1)$ identifiable in terms of color hyper charge CP_2 is indeed quaternionic space. For $SU(2)$ 5-grading degenerates since spin 1/2 Lie-algebra generators are absent and H is trivial group. In M^4 degrees of gauged WZW model would be trivial.

(c) $\mathcal{N} = 4$ SCA results as an extension of $\mathcal{N} = 2$ SCA using so called Freudenthal triple system. $\mathcal{N} = 2$ SCA has realization in terms of $G/H \times U(1)$ gauged WZW theory whereas the extension to $\mathcal{N} = 4$ SCA gives $G \times U(1)/H$ gauged WZW model: note that $SU(3) \times U(1)/H$ does not have an obvious interpretation in TGD framework. The Kac-Moody central extension parameters satisfy the constraint $k_+ = k + 1$ and $k_- = \hat{g} - 1$, where k is the central extension parameter for G. For $G = SU(3)$ one obtains $k_- = 1$ and $c = 6(k + 1)/(k + 2)$. $H = U(1)$ corresponding to color hyper-charge and $U(1)$ for $\mathcal{N} = 2$ algebra corresponds to color isospin. The group $U(1)$ appearing in $SU(3) \times U(1)$ might be interpreted in terms of fermion number or Kähler charge.

(d) What looks somewhat puzzling is that the generators of second SU(2) algebra carry fermion number $F = 4I_3$. Note however that the sigma matrices of WCW with fermion number ± 2 are non-vanishing since corresponding gamma matrices anti-commute. Second strange feature is that fermionic generators correspond to 3+3 super-coordinates of the flag-manifold $SU(3)/U(1) \times U(1)$ plus 2 fermions and their conjugates. Perhaps the coset realization in CP_2 degrees of freedom is not appropriate in TGD framework and that one should work directly with the realization based on second quantized induced spinor fields.

.9.5 The Interpretation Of The Critical Dimension $D = 4$ And The Objection Related To The Signature Of The Space-Time Metric

The first task is to show that $D = 4$ ($D = 8$) as critical dimension of target space for $\mathcal{N} = 2$ ($\mathcal{N} = 4$) super-conformal symmetry makes sense in TGD framework and that the signature $(2, 2)$ ($(4, 4)$ of the metric of the target space is not a fatal flaw. The lifting of TGD to twistor space seems the most promising manner to bring in $(2, 2)$ signature. One must of course remember that super-conformal symmetry in TGD sense differs from that in the standard sense so that one must be very cautious with comparisons at this level.

Space-time as a target space for partonic string world sheets?

Since partonic 2-surfaces are sub-manifolds of 4-D space-time surface, it would be natural to interpret space-time surface as the target space for $\mathcal{N} = 2$ super-conformal string theory so that space-time dimension would find a natural explanation. Different Bohr orbit like solutions of the classical field equations could be the TGD counterpart for the dynamic target space metric of M-theory. Since partonic two-surfaces belong to 3-surface X^3_V, the correlations caused by the vacuum functional would imply non-trivial scattering amplitudes with CP_2 type extremals as pieces of X^3_V providing the correlate for virtual particles. Hence the theory could be physically realistic in TGD framework and would conform with perturbative character for the interactions of leptons. $\mathcal{N} = 2$ super-conformal theory would of course not describe everything. This algebra seems to be still too small and the question remains how the functional integral over the configuration space degrees of freedom is carried out. It will be found that $\mathcal{N} = 4$ super-conformal algebra results neatly when super Kac-Moody and super-symplectic degrees of freedom are combined.

The interpretation of the critical signature

The basic problem with this interpretation is that the signature of the induced metric cannot be (2, 2) which is essential for obtaining the cancelation for $\mathcal{N} = 2$ SCA imbedded to $\mathcal{N} = 4$ SCA with critical dimension $D = 8$ and signature (4, 4). When super-generators carry fermion number and do not reduce to ordinary gamma matrices for vanishing conformal weights, there is no need to pose the condition of the metric signature. The (4, 4) signature of the target space metric is not so serious limitation as it looks if one is ready to consider the target space appearing in the calculation of N-point functions as a fictive notion.

The resolution of the problems relies on two observations.

1. The super Kac-Moody and super-symplectic Cartan algebras have dimension $D = 2$ in both M^4 and CP_2 degrees of freedom giving total effective dimension $D = 8$.

2. The generalized coset construction to be discussed in the sequel allows to assign opposite signatures of metric to super Kac-Moody Cartan algebra and corresponding super-symplectic Cartan algebra so that the desired signature (4, 4) results. Altogether one has 8-D effective target space with signature (4, 4) characterizing $\mathcal{N} = 4$ super-conformal topological strings. Hence the number of physical degrees of freedom is $D_{phys} = 8$ as in super-string theory. Including the non-physical M^2 degrees of freedom, one has critical dimension $D = 10$. If also the radial degree of freedom associated with $\delta M^4_\pm$ is taken into account, one obtains $D = 11$ as in M-theory.

Small $\mathcal{N} = 4$ SCA as sub-algebra of $N = 8$ SCA in TGD framework?

A possible interpretation of the small $\mathcal{N} = 4$ super-conformal algebra would be quaternionic sub-SCA of the non-associative octonionic SCA. The $\mathcal{N} = 4$ algebra associated with a fixed fermionic chirality would represent the fermionic counterpart for the restriction to the hyper-quaternionic sub-manifold of HO and $\mathcal{N} = 2$ algebra in the further restriction to commutative sub-manifold of HO so that this algebra would naturally appear at the parton level. Super-affine version of the quaternion algebra can be constructed straightforwardly as a special case of corresponding octonionic algebra [A39]. The construction implies 4 fermion spin doublets corresponding and unit quaternion naturally corresponds to right handed neutrino spin doublet. The interpretation is as leptonic spinor fields appearing in Sugawara representation of Super Virasoro algebra.

A possible octonionic generalization of Super Virasoro algebra would involve 4 doublets $G^{i)}_\pm$, $i = 1, ..., 4$ of super-generators and their conjugates having interpretation as SO(8) spinor and its its conjugate. $G^{i)}_\pm$ and their conjugates $\overline{G}^{i)}_\pm$ would anti-commute to SO(8) vector octet having an interpretation as a super-affine algebra defined by the octonionic units: this would conform nicely with SO(8) triality.

One could say that the energy momentum tensor T extends to an octonionic energy momentum tensor T as real component and affine generators as imaginary components: the real part would have conformal weight $h = 2$ and imaginary parts conformal weight $h = 1$ in the proposed constructions reflecting the special role of real numbers. The ordinary gamma matrices appearing in the expression of G in Sugawara construction should be represented by units of complexified octonions to achieve non-associativity. This construction would differ from that of [A39] in that G fields would define an SO(8) octet in the proposed construction: HO-H duality would however suggest that these constructions are equivalent.

One can consider two possible interpretations for $G^{i)}_\pm$ and corresponding analogs of super Kac-Moody generators in TGD framework.

1. Leptonic right handed neutrino spinors correspond to $G^{i)}_\pm$ generating quaternionic units and quark like left-handed neutrino spinors with leptonic charges to the remaining non-associative octonionic units. The interpretation in terms of so called mirror symmetry would be natural. What is is clear the direct sum of $\mathcal{N} = 4$ SCAs corresponding to the Kac-Moody group $SU(2) \times SU(2)$ would be exact symmetry if free quarks and leptons carry integer charges. One might however hope of getting also $N = 8$ super-conformal algebra. The problem with this interpretation is that SO(8) transformations would in general mix states with different fermion numbers. The only way out would be the allowance of mixtures of right-handed neutrinos of both chiralities and also of their conjugates which looks an ugly option.

   In any case, the well-definedness of the fermion number would require the restriction to $\mathcal{N} = 4$ algebra. Obviously this restriction would be a super-symmetric version for the restriction to 4-D quaternionic- or co-quaternionic sub-manifold of H.

2. One can ask whether $G^{i)}_\pm$ and their conjugates could be interpreted as components of leptonic H-spinor field. This would give 4 doublets plus their conjugates and mean $N = 16$ super-symmetry by generalizing the interpretation of $\mathcal{N} = 4$ super-symmetry. In this case fermion number conservation would not forbid the realization of $SO(8)$ rotations. Super-conformal variant of complexified octonionic algebra obtained by adding a commuting imaginary unit

would result. This option cannot be excluded since in TGD framework complexified octonions and quaternions play a key role. The fact that only right handed neutrinos generate associative super-symmetries would mean that the remaining components $G^{i)}_{\pm}$ and their conjugates could be used to construct physical states. $N = 8$ super-symmetry would thus break down to small $\mathcal{N} = 4$ symmetry for purely number theoretic reasons and the geometry of CP_2 would reflect this breaking.

The objection is that the remaining fermion doublets do not allow covariantly constant modes at the level of embedding space. They could however allow these modes as induced H-spinors in some special cases which is however not enough and this option can be considered only if one accepts breaking of the super-conformal symmetry from beginning. The conclusion is that the $N = 8$ or even $N = 16$ algebra might appear as a spectrum generating algebra allowing elegant coding of the primary fermionic fields of the theory.

.9.6 How Could Exotic Kac-Moody Algebras Emerge From Jones Inclusions?

Also other Kac-Moody algebras than those associated with the basic symmetries of quantum TGD could emerge from Jones inclusions. The interpretation would be the TGD is able to mimic various conformal field theories. The discussion is restricted to Jones inclusions defined by discrete groups acting in CP_2 degrees of freedom in TGD framework but the generalization to the case of M^4 degrees of freedom is straightforward.

$\mathcal{M} : \mathcal{N} = \beta < 4$ case

The first situation corresponds to $\mathcal{M} : \mathcal{N} = \beta < 4$ for which a finite subgroup $G \subset SU(2)_L$ defines Jones inclusion $\mathcal{N}^G \subset \mathcal{M}^G$, with G commuting with the Clifford algebra elements creating physical states. $\mathcal{N}$ corresponds to a subalgebra of the entire infinite-dimensional Clifford algebra Cl for which one 8-D Clifford algebra factor identifiable as Clifford algebra of the embedding space is replaced with Clifford algebra of M^4.

Each M^4 point corresponds to G orbit in CP_2 and the order of maximal cyclic subgroup of G defines the integer n defining the quantum phase $q = exp(i\pi/n)$. In this case the points in the covering give rise to a representation of G defining multiplets for Kac-Moody group $\hat{G}$ assignable to G via the ADE diagram characterizing G using McKay correspondence. Partonic boundary component defines the Riemann surface in which the conformal field theory with Kac Moody symmetry is defined. The formula $n = k + h_{\hat{G}}$ would determine the value of Kac-Moody central extension parameter k. The singletness of fermionic oscillator operators with respect to G would be compensated by the emergence of representations of G realized in the covering of M^4.

$\mathcal{M} : \mathcal{N} = \beta = 4$ case

Second situation corresponds to $\beta = 4$. In this case the inclusions are classified by extended ADE diagrams assignable to Kac Moody algebras. The interpretation $n = k + h_G$ assigning the quantum phase to $SU(2)$ Kac Moody algebra corresponds to the Jones inclusion $\mathcal{N}^{\hat{G}} \subset \mathcal{M}^{\hat{G}}$ of WCW spinor s for $\hat{G} = SU(2)_L$ with index $\mathcal{M} : \mathcal{N} = 4$ and trivial quantum phase $q = 1$. The Clifford algebra elements in question would be products of fermionic oscillator operators having vanishing $SU(2)_L$ quantum numbers but arbitrary $U(1)_R$ quantum numbers if the identification $\hat{G} = SU(2)_L$ is correct. Thus only right handed fermions carrying homological magnetic charge would be allowed and obviously these fermions must behave like massless particles so that $\beta < 4$ could be interpreted in terms of massivation. The ends of cosmic strings $X^2 \times S^2 \subset M^4 \times CP_2$ would represent an example of this phase having only Abelian electro-weak interactions.

According to the proposal of [K95] the finite subgroup $G \subset SU(2)$ defining the quantum phase emerges from the effective decomposition of the geodesic sphere $S^2 \subset CP_2$ to a lattice having S^2/G as the unit cell. The discrete wave functions in the lattice would give rise to $SU(2)_L \supset G$-multiplets defining the Kac Moody representations and S^2/G would represent the 2-dimensional Riemann surface in which the conformal theory in question would be defined. Quantum phases would correspond to the holonomy of S^2/G. Therefore the singletness in fermionic degrees of freedom would be compensated by the emergence of G- multiplets in lattice degrees of freedom.

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