TGD AS A GENERALIZED NUMBER THEORY: PART II

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April 29, 2024

0.1 PREFACE

This book belongs to a series of online books summarizing the recent state Topological Geometrodynamics (TGD) and its applications. TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students at seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 37 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of imbedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, main stream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to same multiplet of the gauge group implying instability of proton.

There have been also longstanding problems.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. Furthermore, the imbeddings of Robertson-Walker cosmologies turned out to be vacuum extremals with respect to the inertial energy. About 25 years was needed to realize that the sign of the inertial energy can be also negative and in cosmological scales the density of inertial energy vanishes: physically acceptable universes are creatable from vacuum. Eventually this led to the notion of zero energy ontology (ZEO) which deviates dramatically from the standard ontology being however consistent with the crossing symmetry of quantum field theories. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of so called causal diamonds defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle as it is expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields.

  It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to dark matter. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem, and 32 years was needed to discover that what I call weak form of electric-magnetic duality gives a satisfactory solution of the problem and provides also surprisingly powerful insights to the mathematical structure of quantum TGD.

  The latest development was the realization that the well- definedness of electromagnetic charge as quantum number for the modes of the induced spinors field requires that the CP_2 projection of the region in which they are non-vanishing carries vanishing W boson field and is 2-D. This implies in the generic case their localization to 2-D surfaces: string world sheets and possibly also partonic 2-surfaces. This localization applies to all modes except covariantly constant right handed neutrino generating supersymmetry and mplies that string model in 4-D space-time is part of TGD. Localization is possible only for Kähler-Dirac assigned with Kähler action defining the dynamics of space-time surfaces. One must however leave open the question whether W field might vanish for the space-time of GRT if related to many-sheeted space-time in the proposed manner even when they do not vanish for space-time sheets.

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but the first discovery made already during first year of TGD was that these formalisms might be useless due to the extreme non-linearity and enormous vacuum degeneracy of the theory. This turned out to be the case.

- It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as 3-dimensional surfaces. Still few years had to pass before I understood that general coordinate invariance leads to a more or less unique solution of the problem and in positive energyontology implies that space-time surfaces are analogous to Bohr orbits. This in positive energy ontology in which space-like 3-surface is basic object. It is not clear whether Bohr orbitology is necessary also in ZEO in which space-time surfaces connect space-like 3-surfaces at the light-like boundaries of causal diamond CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). The reason is that the pair of 3-surfaces replaces the boundary conditions at single 3-surface involving also time derivatives. If one assumes Bohr orbitology then strong correlations between the 3-surfaces at the ends of CD follow. Still a couple of years and I discovered that quantum states of the Universe can be identified as classical spinor fields in WCW. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

- TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Pronciple with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

- One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

  During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

  Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

  The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. It has indeed turned out that the dream about explicit formula is unrealistic before one has understood what happens in quantum jump. Symmetries and general physical principles have turned out to be the proper guide line here. To give some impressions about what is required some highlights are in order.

- With the emergence of ZEO the notion of S-matrix was replaced with M-matrix defined between positive and negative energy parts of zero energy states. M-matrix can be interpreted as a complex square root of density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally. M-matrices in turn bombine to form the rows of unitary U-matrix defined between zero energy states.

- A decisive step was the strengthening of the General Coordinate Invariance to the requirement that the formulations of the theory in terms of light-like 3-surfaces identified as 3-surfaces at which the induced metric of space-time surfaces changes its signature and in terms of space-like 3-surfaces are equivalent. This means effective 2-dimensionality in the sense that partonic 2-surfaces defined as intersections of these two kinds of surfaces plus 4-D tangent space data at partonic 2-surfaces code for the physics. Quantum classical correspondence requires the coding of the quantum numbers characterizing quantum states assigned to the partonic 2-surfaces to the geometry of space-time surface. This is achieved by adding to the modified Dirac action a measurement interaction term assigned with light-like 3-surfaces.

- The replacement of strings with light-like 3-surfaces equivalent to space-like 3-surfaces means enormous generalization of the super conformal symmetries of string models. A further generalization of these symmetries to non-local Yangian symmetries generalizing the recently discovered Yangian symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is highly suggestive. Here the replacement of point like particles with partonic 2-surfaces means the replacement of conformal symmetry of Minkowski space with infinite-dimensional super-conformal algebras. Yangian symmetry provides also a further refinement to the notion of conserved quantum numbers allowing to define them for bound states using non-local energy conserved currents.

- A further attractive idea is that quantum TGD reduces to almost topological quantum field theory. This is possible if the Kähler action for the preferred extremals defining WCW Kähler function reduces to a 3-D boundary term. This takes place if the conserved currents are so called Beltrami fields with the defining property that the coordinates associated with flow lines extend to single global coordinate variable. This ansatz together with the weak form of electric-magnetic duality reduces the Kähler action to Chern-Simons term with the condition that the 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric magnetic duality. It is the latter constraint which prevents the trivialization of the theory to a topological quantum field theory. Also the identification of the Kähler function of WCW as Dirac determinant finds support as well as the description of the scattering amplitudes in terms of braids with interpretation in terms of finite measurement resolution coded to the basic structure of the solutions of field equations.

- In standard QFT Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

  In TGD framework this generalization of Feynman diagrams indeed emerges unavoidably. Light-like 3-surfaces replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. The approximate localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In TGD framework fermionic variant of twistor Grassmann formalism leads to a stringy variant of twistor diagrammatics in which basic fermions can be said to be on mass-shell but carry non-physical helicities in the internal lines. This suggests the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras.

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 32 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks.

Karkkila, October 30, 2010, Finland

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ACKNOWLEDGEMENTS

Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family and Kalevi and Ritva Tikkanen and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 45 lonely years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During the last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss my work. Pertti Kärkkäinen is my old physicist friend and has provided continued economic support for a long time. I have also had stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Tommi Ullgren has provided both economic support and encouragement during years. Pekka Rapinoja has offered his help in this respect and I am especially grateful to him for my Python skills.

During the last five years I have had inspiring discussions with many people in Finland interested in TGD. We have had video discussions with Sini Kunnas and had podcast discussions with Marko Manninen related to the TGD based view of physics and consciousness. Marko has also helped in the practical issues related to computers and quite recently he has done a lot of testing of chatGPT helping me to get an overall view of what it is. The discussions in a Zoom group involving Marko Manninen, Tuomas Sorakivi and Rode Majakka have given me the valuable opportunity to clarify my thoughts.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation in CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. I am grateful to Mark McWilliams, Paul Kirsch, Gary Ehlenberg, and Ulla Matfolk and many others for providing links to possibly interesting websites and articles. We have collaborated with Peter Gariaev and Reza Rastmanesh. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the public through the iron wall of academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as an individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without direct support from power holders- even in archives like arXiv.org.

Situation changed as Andrew Adamatsky proposed the writing of a book about TGD when I had already gotten used to the thought that my work would not be published during my lifetime. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loophole. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy.

And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his 73th birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During the period when the right wing held political power this trend was steadily strengthening and the situation is the same as I am writing this. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

Karkkila, August 30, 2023, Finland

Matti Pitkänen

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Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K2].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A150]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

   M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of

electromagnetic fields are nonvanishing. The correlations functions for weak fields are nonvanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question: super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A97] [B28, B18, B19]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B15]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4 \times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n \times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A134, A149, A114, A145].

The identification of the space-time as a sub-manifold [A135, A171] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. ??** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

[1] There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.1.5 Construction of scattering amplitudes

Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A159, A178, A193]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K58]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, M^8-H duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and M^8-H duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and M^8-H duality would map the algebraic physics in M^8 to the ordinary physics in $M^4\times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their $M^8 - H$ duals in M_c^8 are defined for both reals and various p-adic number fields. This is true if they are defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p = 3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired "Universe as Computer" vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.
2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since $M^8 - H$ duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K53].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to “mind stuff”, the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \rightarrow f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{(Re(E)^2 - Im(E)^2)}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{(Re(E)^2 - Im(E)^2)}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} . \tag{1.1.1}$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \rightarrow Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \rightarrow 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \rightarrow SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \to CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same.

The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of M^8-H duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4\times CP_2$?

1. The selection of $SU(3)\subset G_2$ for ordinary M^8-H duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3\subset M^8_c$ and its image $H^3\subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?
2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3)\subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.
3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of M^8-H duality as something more fundamental than the ordinary M^8-H duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3\times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1)\times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1\oplus 3\oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3\oplus\overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8\oplus 3\oplus\overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3)\times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x)\times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X^3_i.

1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that

Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [?] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K80].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K72, K73, K71]) support the view that dark matter might be a key player in living matter.

Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [K91]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A150]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   $M^8 - H$ duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L54].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L42]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t - m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

1.2 Bird's Eye of View about the Topics of the "Quantum Physics as Number theory"

The focus of the book "Quantum Physics as Number theory" is the number theoretic vision about physics. This vision involves three loosely related parts and the chapters represent the evolution of ideas rather than just the final outcome.

The three chapters of the first part of "Quantum Physics as Number theory: Part I" introduce the general ideas of number theoretic vision that is p-adic physics and their fusion to adelic physics, algebraic physics realized in terms of complexified octionons, and infinite primes.

1. The first chapter discusses the fusion of real physics and various p-adic physics to a single larger whole by generalizing the number concept by fusing real numbers and various p-adic number fields along common rationals. Extensions of p-adic number fields can be introduced by gluing them along common algebraic numbers to reals.

   Algebraic continuation of the physics from rationals and their extensions to various number fields (completion of rational physics to physics in various number fields) is the key idea and the challenge is to understand whether one could achieve this dream. A very profound implication is that purely local p-adic physics codes for the p-adic fractality of long length scale real physics and vice versa. As a consequence, one can understand the origins of the p-adic length scale hypothesis and one ends up with a very concrete view about space-time correlates of cognition. The fusion of various p-adic physics to a single coherent whole leads to what I call adelic physics [L42, L43].

   Infinite primes is a physically motivated notion and their construction corresponds formally to a hierarchy of second quantizations of an arithmetic number theory.

2. Second part of the vision involves what the classical number fields defined as subspaces of their complexifications with Minkowskian signature of the metric. The hypothesis is that allowed space-time surfaces correspond to quaternionic sub-manifolds of complexified octonionic space. The proposed interpretation of quaternionicity would in terms of being zero for the real or imaginary part of octonionic polynomial with rational or perhaps even algebraic coefficients.

Real/imaginary part refers to a composition of octonion to quaternion and imaginary unit multiplying second quaternion analogous to the decomposition of ordinary complex number to real and imaginary parts. Space-time surface would correspond to imaginary roots (in the sense that they are proportional to the imaginary unit i commuting with the octonionic units). It is argued that this notion of quaternionicity is equivalent with the assumption that the tangent space or normal of space-time surface in M^8 at each point is quaternionic.

Besides this one assumes that one can assign to each point of space-time surface a complex plane M_c^2 as subspace of the quaternionic plane M_c^4. These planes could even depend on point of space-time surface and define an integrable distribution - kind of string world sheet.

Quaternionicity of the tangent plane in this sense allows to map the space-time surface in M^8 to a space-time surface in $H = M^4 \times CP_2$. This involves a projection to M^4 in the decomposition $M^8 = M^4 \times C_2$ and the assignment to the point of space-time surface point of CP_2 labelling its tangent space.

It is not clear whether one can assign also to each point of space-time surface in H a quaternionic tangent or normal in the tangent space M^8 of H. In the case in H this plane could be the tangent/normal plane defined by the modified gamma matrices or induced gamma matrices. These two planes co-incide with each other only for action defined by the metric determinant. Hence the basic variational principle of TGD would have deep number theoretic content. Reduction to a closed form would also mean that classical TGD would define a generalized topological field theory with Noether charges defining topological invariants.

3. The third part of the vision involves infinite primes, which can be identified in terms of an infinite hierarchy of second quantized arithmetic quantum fields theories on one hand, and as having representations as space-time surfaces analogous to zero surfaces of polynomials on the other hand. In this framework space-time surface would represent an infinite number. This vision leads also the conclusion that single point of space-time has an infinitely complex structure since real unity can be represented as a ratio of infinite numbers in infinitely many ways each having its own number theoretic anatomy. Thus single space-time point is in principle able to represent in its structure the quantum state of the entire universe. This number theoretic variant of Brahman=Atman identity also means that Universe is an algebraic hologram.

1.2.1 Organization of "Physics as Generalized Number Theory: Part II"

The book consists of 3 parts.

1. In the 1st part $M^8 - H$ duality is discussed. It states that the purely algebraic physics (no variational principle nor partial differential equations) based on algebraic surfaces in complexified M^8 regarded as complexified octonions is dual to the physics defined by preferred extremals (presumably minimal surfaces) in $H = M^4 \times CP_2$. Quantum criticality would bring in infinite number of constraints analogous to gauge conditions implying that space-time surfaces in H are analogs of Bohr orbits. The dynamics based on variational principle in H would be equivalent with purely algebraic physics in M^8.
2. The 2nd part includes various TGD inspired considerations related to Riemann hypothesis - in particular, a strategy for proving Riemann hypothesis using a modification of Hilbert-Polya conjecture replacing quantum states with coherent states of a unique conformally invariant physical system. The proposal that zeros of Riemann zeta could correspond to complex values of coupling constant is also discussed. Although the values of the coupling parameter fit rather nicely with those of U(1) coupling strength for electro-weak interactions, I have more or less given up this conjecture in favor of much more convincing conjecture justifiable from a model of coupling constant evolution reducing to that for the length scale dependent cosmological constant taking the role of cutoff parameter and emerging from the twistor lift of TGD. For this option the values of coupling constant are labelled by the zeros of zeta but are not so directly related to them.
3. In lack of better title I have have referred the contents of the 3rd part as "miscellaneous topics". These topics touch the boundaries of my mathematical understanding and skills, and I do

not regard these chapters as core TGD. The first chapter represents the first serious attempt to define the notion of p-adic manifold. It started from the question what p-adic variant of icosahedron could mean. Later I realized that it is better to approach the problems from the perspective of TGD inspired physics rather than trying to mimick what mathematicians have done. Much simpler and physically more attractve approach emerges from the notion of cognitive representation based on extensions of rationals defining a hierarchy if adeles. There are also chapters about TGD and non-standard numbers and infinite primes and motives. The last chapter is about Langlands program and TGD.

1.3 Sources

The eight online books about TGD [K97, K92, K75, K64, K19, K59, K39, K83] and nine online books about TGD inspired theory of consciousness and quantum biology [K88, K16, K70, K15, K36, K49, K51, K82, K87] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://tinyurl.com/ybv8dt4n`) contains a lot of material about TGD. In particular, a TGD glossary at `http://tinyurl.com/yd6jf3o7`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://tinyurl.com/ycyrxj4o` founded by Lian Sidorov and in *Prespacetime Journal* (`http://tinyurl.com/ycvktjhn`), *Journal of Consciousness Research and Exploration* (`http://tinyurl.com/yba4f672`), and *DNA Decipher Journal* (`http://tinyurl.com/y9z52khg`), all of them founded by Huping Hu. One can find the list about the articles published at `http://tinyurl.com/ybv8dt4n`. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.4 The contents of the book

1.4.1 PART I: $M^8 - H$ DUALITY

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part I

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

2. It will be shown how $M^8 - H$ duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in M^8 would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o_c decomposing as $o_c = q_c^1 + q_c^2 I^4$ and projected to a Minkowskian subspace M^8 of complexified O. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

   One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c\overline{q_c}$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and M^8-H correspondence could generalize.

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case M^8-H duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. M^8-H duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

   One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

   One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by M^8-H duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. M^8-H duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. M^8-H duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound

states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

Does M^8-H duality reduce classical TGD to octonionic algebraic geometry?: Part II

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called M^8-H duality is one of these approaches. The beauty of M^8-H duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

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2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case M^8-H duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. M^8-H duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

   One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by M^8-H duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. M^8-H duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. M^8-H duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

   I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

Also a sketchy proposal for the description of interactions is discussed.

1. The surprise that $RE(P)=0$ and $IM(P)=0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety X^4. The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices.

   The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

2. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

   The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i)=0$ reduces to $IM(P_i)=0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i)=0$ does not reduce to $RE(\prod P_i)=0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

3. The possibility of super octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N}=4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

Scattering diagrams would be determined by points of space-time variety, which are in extension of rationals. In adelic physics the interpretation is as cognitive representations.

1. Cognitive representations are identified as sets of rational points for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [?]
2. Some aspects related to homology charge (Kähler magnetic charge) and genus-generation correspondence are discussed. Both topological quantum numbers are central in the proposed model of elementary particles and it is interesting to see whether the picture is internally consistent and how algebraic variety property affects the situation. Also possible problems related to $h_{eff}/h = n$ hierarchy []adelicphysics realized in terms of n-fold coverings of space-time surfaces are discussed from this perspective.

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part III

Cognitive representations are the basic topic of the third chapter related to $M^8 - H$ duality. Cognitive representations are identified as sets of points in extension of rationals for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces.

The notion is applied in various cases and the connection with $M^8 - H$ duality is rather loose.

1. Extensions of rationals are essentially coders of information. There the possible analogy of extensions of rationals with genes deserves discussion. Extensions, which are not extensions of extensions would be analogous to genes. The notion of conserved gene as number theoretical analogy for Galois extensions as the Galois group of extension which is normal subgroup of Galois extension.
2. The possible physical meaning of the notion of perfectoid introduced by Peter Scholze is discussed in the framework of p-adic physics. Extensions of p-adic numbers involving roots of the prime defining the extension are involved and are not considered previously in TGD framework. There there possible physical meaning deserves discussion.
3. The construction of cognitive representation reduces to a well-known mathematical problem of finding the points of space-time surface with embedding space coordinates in given extension of rationals. The work of Kim and Coates represents new ideas in this respect and there is a natural connection with TGD.
4. One expects that large cognitive representations are winners in the number theoretical fight for survival. Strong form of holography suggests that it is enough to consider cognitive representations restricted to string world sheets and partonic 2-surfaces. If the 2-surface possesses large group of symmetries acting in extension of rationals, one can have large cognitive representations as orbit of point in extension. Examples of highly symmetric 2-D surfaces are geodesic spheres assignable to partonic 2-surfaces and cosmic strings and elliptic curves assignable with string world sheets and cosmic strings.
5. Rationals and their extensions give rise to a unique discretizations of space-time surface (for instance) - cognitive representation - having interpretation in terms of finite measurement resolution. There are however many open questions. Should one allow only octonionic polynomials defined as algebraic continuations of real polynomials or should one allow also analytic functions and regard polynomials as approximations. Zeta functions are especially interesting analytic functions and Defekind zetas characterize extensions of rationals and one can pose physically motivated questions about them.

Breakthrough in understanding of $M^8 - H$ duality

A critical re-examination of $M^8 - H$ duality is discussed. $M^8 - H$ duality is one of the cornerstones of Topological Geometrodynamics (TGD). The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent or normal space and contains a complex or co-complex surface, it can be mapped to a 4-surface in $H = M^4 \times CP_2$.

Later emerged the idea that octonionic analyticity realized in terms of real polynomials P algebraically continued to polynomials of complexified octonion could fulfill the dream. The vanishing of the real part $Re_Q(P)$ (imaginary part $Im_Q(P)$) in the quaternionic sense would give rise to an associative (co-associative) space-time surface.

The realization of the general coordinate invariance motivated the notion of strong form of holography (SH) in H allowing realization of a weaker form of $M^8 - H$ duality by assuming that associativity/co-associativity conditions are needed only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits.

The outcome of the re-examination yielded both positive and negative surprises.

1. Although no interesting associative space-time surfaces are possible, every distribution of normal associative planes (co-associativity) is integrable.

2. Another positive surprise was that Minkowski signature is the only possible option. Equivalently, the image of M^4 as real co-associative subspace of O_c (complex valued octonion norm squared is real valued for them) by an element of local G_2 or rather, its subgroup $SU(3)$, gives a real co-associative space-time surface.

3. The conjecture based on naive dimensional counting, which was not correct, was that the polynomials P determine these 4-D surfaces as roots of $Re_Q(P)$. The normal spaces of these surfaces possess a fixed 2-D commuting sub-manifold or possibly their distribution allowing the mapping to H by $M^8 - H$ duality as a whole.

   If this conjecture were correct, strong form of holography (SH) would not be needed and would be replaced with extremely powerful number theoretic holography determining space-time surface from its roots and selection of real subspace of O_c characterizing the state of motion of a particle. erate

4. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [L37, L38, L39] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [L63, L72].

5. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X = 0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.

   The expectation was wrong! The equations $X = 0$ and $Y = 0$ involve the same(!) complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X = 0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.

   There are solutions $X = 0$ and $Y = 0$ only if the two polynomials considered have a common a_c^2 as a root! Also now the solutions are complex mass shells H^7_c.

How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .

1. The key observation is that G_2 acts as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space M^4 gives a co-associative four-surface as a real projection. Octonion analyticity would correspond to G_2 gauge transformation: this would realize the original idea about octonion analyticity.

2. A co-associative X^4_c satisfying also the conditions posed by the existence of M^8-H duality is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines M^8-H duality even if the co-associativity in geometric sense were not satisfied.

   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

   The group $SU(3)$ has interpretation as a Kac-Moody type analog of color group and the map defining space-time surface. This picture conforms with the H-picture in which gluon gauge potentials are identified as color gauge potentials. Note that at QFT limit the gauge potentials are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

3. Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \rightarrow M^4 \subset H$ from an identification to an almost inversion. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.

This picture combined with zero energy ontology leads also to a view about quantum TGD at the level of M^8. A local $SU(3)$ element defining 4-surface in M^8, which suggests a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

New findings related to the number theoretical view of TGD

The geometric vision of TGD is rather well-understood but there is still a lot of fog in the number theoretic vision.

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?
2. The detailed realization of M^8-H duality involves uncertainties: in particular, how the complexification of M^8 to M^8_c can be consistent with the reality of $M^4 \subset H$.
3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?
4. How unique is the choice of 3-D surfaces at the mass shells $H^3_m \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

These and many other questions motivated this article and led to the observation that the model geometries used in the classification of 3-manifolds seem to be rather closely related to the known space-time surfaces extremizing practically any general coordinate invariant action constructible in terms of the induced geometry.

The 4-surfaces in M^8 would define coupling constant evolutions for quantum states as analogs of and mappable to time evolutions at the level of H and obeying conservation laws associated with the dual conformal invariance analogous to that in twistor approach.

The momenta of fundamental fermions in the quantum state would be coded by the cusp singularities of 3-surfaces at the mass shells of M^8 and also its image in H provided by M^8-H duality. One can consider the possibility of $2 \rightarrow 3$ holography in which the boundaries of fundamental region of H^3/Γ is 2-D hyperbolic space H^2/Γ so that TGD could to high degree reduce to algebraic geometry.

Could quantum randomness have something to do with classical chaos?

Tim Palmer has proposed that classical chaos and quantum randomness might be related. It came as a surprise to me that these to notions could a have deep relationship in TGD framework.

1. Strong form of Palmer's idea stating that quantum randomness reduces to classical chaos certainly fails but one can consider weaker forms of the idea. Even these variants fail in Copenhagen interpretation since strictly speaking there is no classical reality, only wave function coding for the knowledge about the system. Bohr orbits should be more than approximation and in TGD framework space-time surface as preferred extremal of action is analogous to Bohr orbit and classical physics defined by Bohr orbits is an exact part of quantum theory.
2. In the zero energy ontology (ZEO) of TGD the idea works in weaker form and has very strong implications for the more detailed understanding of ZEO and $M^8-M^4\times CP_2$ duality. Ordinary ("big") state functions (BSFRs) meaning the death of the system in a universal sense and re-incarnation with opposite arrow of time would involve quantum criticality accompanied by classical chaos assignable to the correspondence between geometric time and subjective time identified as sequence of "small" state function reductions (SSFRs) as analogs of weak measurements. The findings of Minev et al give strong support for this view and Libet's findings about active aspects of consciousness can be understood if the act of free will corresponds to BSFR.

M^8 picture identifies 4-D space-time surfaces X^4 as roots for "imaginary" or "real" part of octonionic polynomial P_2P_1 obtained as a continuation of real polynomial $P_2(L-r)P_1(r)$, whose arguments have origin at the the tips of B and A and roots a the light-cone boundaries associated with tips. Causal diamond (CD) is identified intersection of future and past directed light-cones light-cones A and B. In the sequences of SSFRs $P_2(L-r)$ assigned to B varies and $P_1(r)$ assigned to A is unaffected. L defines the size of CD as distance $\tau=2L$ between its tips.

Besides 4-D space-time surfaces there are also brane-like 6-surfaces corresponding to roots $r_{i,k}$ of $P_i(r)$ and defining "special moments in the life of self" having $t_i=r_{i,k}$ ball as M^4_+ projection. The number of roots and their density increases rapidly in the sequence of SSFRs. The condition that the largest root belongs to CD gives a lower bound to it size L as largest root. Note that L increases.

Concerning the approach to chaos, one can consider three options.

Option I: The sequence of steps consisting of unitary evolutions followed by SSFR corresponds to a functional factorization at the level of polynomials as sequence $P_2=Q_1\circ Q_2\circ ...Q_n$. If the size of CD is assumed to increase, also the tip of active boundary of CD must shift so that the argument of P_2 $r-L$ is replaced in each iteration step to with updated argument with larger value of L.

Option II: A completely unexpected connection with the iteration of analytic functions and Julia sets, which are fractals assigned also with chaos interpreted as complexity emerges. In a reasonable approximation quantum time evolution by SSFRs could be induced by an iteration of a polynomial or even an analytic function: $P_2=P_2\rightarrow P_2^{\circ 2}\rightarrow ....$ For $P_2(0)=0$ the roots of the iterate consists of inverse images of roots of P_2 by $P_2^{\circ -k}$ for $k=0,...,N-1$.

Suppose that M^8 and X^4 are complexified and thus also $t=r$ and "real" X^4 is the projection of X^4_c to real M^8. Complexify also the coefficients of polynomials P. If so, the Mandelbrot and Julia sets (`http://tinyurl.com/cplj9pe` and `http://tinyurl.com/cvmr83g`) characterizing fractals would have a physical interpretation in ZEO.

One approaches chaos in the sense that the $N-1$:th inverse images of the roots of P_2 belonging to filled Julia set approach to points of Julia set of P_2 as the number N of iterations increases. Minimal L would increase with N if CD is assumed to contain all roots. The density of the roots in Julia set increases near L since the size of CD is bounded by the size Julia set. One could perhaps say that near the $t = L$ in the middle of CD the life of self when the size of CD has become almost stationary, is the most intensive.

Option III: A conservative option is to consider also real polynomials $P_2(r)$ with real argument r. Only non-negative real roots r_n are of interest whereas in the general case one considers all values of r. For a large N the new roots with possibly one exception would approach to the real Julia set obtained as a real projection of Julia set for complex iteration.

How the size L of CD is determined and when can BSFR occur?

Option I: If L is minimal and thus given by the largest (non-exceptional) root of iterate of P_2 in Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). L should smaller than the sizes of Julia sets of both$A$ and B since the iteration gives no roots outside Julia sets.

Could BSFR become probable when L as the largest allowed root for iterate P_2 is larger than the size of Julia set of A? There would be no more new "special moments in the life of self" and this would make death (in universal sense) and re-incarnation with opposite arrow of time probable. The size of CD could decrease dramatically in the first iteration for P_1 if it is determined as the largest allowed root of P_1: the re-incarnated self would have childhood.

Option II: The size of CD could be determined in SSFR statistically as an allowed root of P_2. Since the density of roots increases, one would have a lot of choices and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time. BSFR would occur for the same reason as for the first option.

1.4.2 PART II: RIEMANN ZETA AND PHYSICS

Riemann hypothesis and physics

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

1. Super-conformal invariance and generalization of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

2. Zero energy ontology and RH

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

3. Miscellaneous ideas

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

Does Riemann Zeta Code for Generic Coupling Constant Evolution?

A general model for the coupling constant evolution is proposed. The analogy of Riemann zeta and fermionic zeta $\zeta_F(s)/\zeta_F(2s)$ with complex square root of a partition function natural in Zero Energy Ontology suggests that the the poles of $\zeta_F(ks)$, $k = 1/2$, correspond to complexified critical temperatures identifiable as inverse of Kähler coupling strength itself having interpretation as inverse of critical temperature. One can actually replace the argument s of ζ_F with Möbius transformed argument $w = (as+b)/(cs+d)$ with a, b, c, d real numbers, rationals, or even integers. For α_K $w = (s + b)/2$ is proper choices and gives zeros of $\zeta(s)$ and $s = 2 - b$ as poles. The identification $\alpha_K = \alpha_{U(1)}$ leads to a prediction for α_{em}, which deviates by .7 per cent from the experimental value at low energies (atomic scale) if the experimental value of the Weinberg angle is used. The conjecture generalizes also to weak, color and gravitational interactions when general Möbius transformation leaving upper half-plane invariant is allowed. One ends up with a general model predicting successfully the entire electroweak coupling constant evolution successfully from the values of fine structure constant at atomic or electron scale and in weak scale.

TGD View about Coupling Constant Evolution

New results related to the TGD view about coupling constant evolution are discussed. The results emerge from the discussion of the recent claim of Atyiah that fine structure constant could be understood purely mathematically. The new view allows to understand the recently introduced TGD based construction of scattering amplitudes based on the analog of micro-canonical ensemble as a cognitive representation for the much more complex construction of full scattering amplitudes using real numbers rather than p-adic number fields. This construction utilizes number theoretic discretization of space-time surface inducing that of “world of classical worlds” (WCW) and makes possible adelization of quantum TGD.

The understanding of coupling constant evolution has been one of most longstanding problems of TGD and I have made several proposals during years.

Could number theoretical constraints fix the evolution? Adelization suffers from serious number theoretical problem due to the fact that the action exponentials do not in general exist p-adically for given adele. The solution of the problem turned out to be trivial. The exponentials disappear from the scattering amplitudes! Contrary to the first beliefs, adelization does not therefore seem to determine coupling constant evolution.

TGD view about cosmological constant turned out to be the solution of the problem. The formulation of the twistor lift of Kähler action led to a rather detailed view about the interpretation of cosmological constant as an approximate parameterization of the dimensionally reduced 6-D Kähler action (or energy) allowing also to understand how it can decrease so fast as a function of p-adic length scale. In particular, a dynamical mechanism for the dimensional reduction of 6-D Kähler action giving rise to the induction of the twistor structure and predicting this evolution emerges.

In standard QFT view about coupling constant evolution ultraviolet cutoff length serves as the evolution parameter. TGD is however free of infinities and there is no cutoff parameter. It turned out cosmological constant replaces this parameter and coupling constant evolution is induced by that for cosmological constant from the condition that the twistor lift of the action is not affected by small enough modifications of the moduli of the induced twistor structure. The moduli space for them corresponds to rotation group $SO(3)$. This leads to explicit evolution equations for α_K, which can be studied numerically.

The approach is also related to the view about coupling constant evolution based on the inclusions of hyper-finite factors of type II_1, and it is proposed that Galois group replaces discrete subgroup of $SU(2)$ leaving invariant the algebras of observables of the factors appearing in the inclusion.

About the role of Galois groups in TGD framework

This article was inspired by the inverse problem of Galois theory. Galois groups are realized as number theoretic symmetry groups realized physically in TGD a symmetries of space-time surfaces. Galois confinement as an analog of color confinement is proposed in TGD inspired quantum biology.

Two instances of the inverse Galois problem, which are especially interesting in TGD, are following:

Q1: Can a given finite group appear as Galois group over Q? The answer is not known.

Q2: Can a given finite group G appear as a Galois group over some EQ? Answer to Q2 is positive as will be found and the extensions for a given G can be explicitly constructed.

The TGD based formulation based on M^8-H duality in which space-time surface in complexified M^8 are coded by polynomials with rational coefficients involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The idea allowing to answer this question is the requirement that TGD adelic physics is able to represent all finite groups as Galois groups of Q or some EQ acting physical symmetry group.

If the answer to **Q1** is positive, it is enough to have polynomials with coefficients in Q. It not, then also EQs are needed as coefficient fields for polynomials to get all Galois groups. The first option would be the more elegant one.

In the sequel the inverse problem is considered from the perspective of TGD. Galois groups, in particular simple Galois groups, play a fundamental role in the TGD view of cognition. The TGD based model of the genetic code involves in an essential manner the groups A_5 (icosahedron), which is the smallest simple and non-commutative group, and A_4 (tetrahedron). The identification of these groups as Galois groups leads to a more precise view about genetic code.

Some Questions about Coupling Constant Evolution

In this chapter questions related to the hierarchy of Planck constants and p-adic coupling constant evolution (CCE) in the TGD framework are considered.

1. Is p-adic length scale hypothesis (PLS) correct in this recent form and can one deduce this hypothesis or its generalization from the basic physics of TGD defined by Kähler function of the "world of classical worlds" (WCW)? The fact, that the scaling of the roots of polynomial does not affect the algebraic properties of the extension strongly suggests that p-adic prime does not depend on purely algebraic properties of EQ. In particular, the proposed identification of p as a ramified prime of EQ could be wrong.

Number theoretical universality suggests the formula $exp(\Delta K) = p^n$, where ΔK is the contribution to Kähler function of WCW for a given space-time surface inside causal diamond (CD).

2. The understanding of p-adic length scale evolution is also a problem. The "dark" CCE would be $\alpha_K = g_K^2/2h_{eff} = g_K^2/2nh_0$, and the PLS evolution $g_K^2(k) = g_K^2(max)/k$ should define independent evolutions since scalings commute with number theory. The total evolution $\alpha_K = \alpha_K(max)/nk$ would induce also the evolution of other coupling strengths if the coupling strenghts are related to α_K by Möbius transformation as suggested.

3. The formula $h_{eff} = nh_0$ involves the minimal value h_0. How could one determine it? p-Adic mass calculations for $h_{eff} = h$ lead to the conclusion that the CP_2 scale R is roughly $10^{7.5}$ times longer than Planck length l_P. Classical argument however suggests $R \simeq l_P$. If one assumes $h_{eff} = h_0$ in the p-adic mass calculations, this is indeed the case for $h/h_0 = (R(CP_2)/l_P)^2$. This ratio follows from number theoretic arguments as $h/h_0 = n_0 = (7!)^2$. This gives $\alpha_K = n_0/kn$, and perturbation theory can converge even for $n = 1$ for sufficiently long p-adic length scales. Gauge coupling strengths are predicted to be practically zero at gravitational flux tubes so that only gravitational interaction is effectively present. This conforms with the view about dark matter.

4. Nottale hypothesis predicts gravitational Planck constant $\hbar_{gr} = GMm/\beta_0$ ($\beta_0 = v_0/c$ is velocity parameter), which has gigantic values. Gravitational fine structure constant is given by $\alpha_{gr} = \beta_0/4\pi$. Kepler's law $\beta^2 = GM/r = r_S/2r$ suggests length scale evolution $\beta^2 = xr_S/2L_N = \beta_{0,max}^2/N^2$, where x is proportionality constant, which can be fixed.

   Phase transitions changing β_0 are possible at $L_N/a_{gr} = N^2$ and these scales correspond to radii for the gravitational analogs of the Bohr orbits of hydrogen. p-Adic length scale hierarchy is replaced by that for the radii of Bohr orbits. The simplest option is that β_0 obeys a CCE induced by α_K.

   This picture conforms with the existing applications and makes it possible to understand the value of β_0 for the solar system, and is consistent with the application to the superfluid fountain effect.

1.4.3 PART III: MISCELLANEOUS TOPICS

What p-adic icosahedron could mean? And what about p-adic manifold?

The original focus of this chapter was p-adic icosahedron. The discussion of attempt to define this notion however leads to the challenge of defining the concept of p-adic sphere, and more generally, that of p-adic manifold, and this problem soon became the main target of attention since it is one of the key challenges of also TGD.

There exists two basic philosophies concerning the construction of both real and p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying the common rationals. Finite pinary cutoff is however required to avoid totally wild fluctuations and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous manner but is not consistent with p-adic analyticity nor field equations unless one poses a pinary cutoff. It seems that pinary cutoff reflecting the notion of finite measurement resolution is necessary in both approaches. This represents a new notion from the point of view of mathematics.

1. One can try to generalize the theory of real manifolds to p-adic context. The basic problem is that p-adic balls are either disjoint or nested so that the usual construction by gluing partially overlapping spheres fails. One attempt to solve the problem relies on the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultrametric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold path connected. The notion makes sense for the p-adic counterparts of projective spaces, which suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.

2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach looks very natural in TGD framework - at least for embedding space. Preferred extremals of Kähler action can be characterized purely algebraically - even in a manner independent of the action principle - so that they might make sense also p-adically.

Number theoretical universality is central element of TGD. Physical considerations force to generalize the number concept by gluing reals and various p-adic number fields along rationals and possible common algebraic numbers. This idea makes sense also at the level of space-time and of "world of classical worlds" (WCW).

Algebraic continuation between different number fields is the key notion. Algebraic continuation between real and p-adic sectors takes place along their intersection , which at the level of WCW ("world of classical worlds") correspond to surfaces allowing interpretation both as real and p-adic surfaces for some value(s) of prime p. The algebraic continuation from the intersection of real and p-adic WCWs is not possible for all p-adic number fields. For instance, real integrals as functions of parameters need not make sense for all p-adic number fields. This apparent mathematical weakness can be however turned to physical strength: real space-time surfaces assignable to elementary particles can correspond only some particular p-adic primes. This would explain why elementary particles are characterized by preferred p-adic primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!).

Although Berkovich construction of p-adic disk does not look promising in TGD framework, it suggests that the difficulty posed by the total disconnectedness of p-adic topology is real. TGD in turn suggests that the difficulty could be overcome without the completion to a non-ultrametric topology. Two approaches emerge, which ought to be equivalent.

The TGD inspired solution to the construction of path connected effective p-adic topology is based on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous manner. The trivial but striking observation was that canonical identification satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals and vice versa and preferred extremal property allows to complete the discrete image to hopefully space-time surface unique within finite measurement resolution so that topological and algebraic approach are combined. Finite resolution would become part of the manifold theory. p-Adic manifold theory would also have interpretation in terms of cognitive representations as maps between realities and p-adicities.

TGD and Non-Standard Numbers

The chapter represents a comparison of ultrapower fields (loosely surreals, hyper-reals, long line) and number fields generated by infinite primes having a physical interpretation in Topological Geometrodynamics.

Ultrapower fields are discussed in very physicist friendly manner in the articles of Elemer Rosinger and these articles are taken as a convenient starting point. The physical interpretations and principles proposed by Rosinger are considered against the background provided by TGD. The construction of ultrapower fields is associated with physics using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields.

Non-standard numbers are compared with the numbers generated by infinite primes and it is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1 represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra.

The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

Infinite Primes and Motives

In this chapter the goal is to find whether the general mathematical structures associated with twistor approach, superstring models and M-theory could have a generalization or a modification in TGD framework. The contents of the chapter is an outcome of a rather spontaneous process, and represents rather unexpected new insights about TGD resulting as outcome of the comparisons.

1. Infinite primes, Galois groups, algebraic geometry, and TGD

In algebraic geometry the notion of variety defined by algebraic equation is very general: all number fields are allowed. One of the challenges is to define the counterparts of homology and cohomology groups for them. The notion of cohomology giving rise also to homology if Poincare duality holds true is central. The number of various cohomology theories has inflated and one of the basic challenges to find a sufficiently general approach allowing to interpret various cohomology theories as variations of the same motive as Grothendieck, who is the pioneer of the field responsible for many of the basic notions and visions, expressed it.

Cohomology requires a definition of integral for forms for all number fields. In p-adic context the lack of well-ordering of p-adic numbers implies difficulties both in homology and cohomology since the notion of boundary does not exist in topological sense. The notion of definite integral is problematic for the same reason. This has led to a proposal of reducing integration to Fourier analysis working for symmetric spaces but requiring algebraic extensions of p-adic numbers and an appropriate definition of the p-adic symmetric space. The definition is not unique and the interpretation is in terms of the varying measurement resolution.

The notion of infinite has gradually turned out to be more and more important for quantum TGD. Infinite primes, integers, and rationals form a hierarchy completely analogous to a hierarchy of second quantization for a super-symmetric arithmetic quantum field theory. The simplest infinite primes representing elementary particles at given level are in one-one correspondence with many-particle states of the previous level. More complex infinite primes have interpretation in terms of bound states.

1. What makes infinite primes interesting from the point of view of algebraic geometry is that infinite primes, integers and rationals at the n:th level of the hierarchy are in 1-1 correspondence with rational functions of n arguments. One can solve the roots of associated polynomials and perform a root decomposition of infinite primes at various levels of the hierarchy and assign to them Galois groups acting as automorphisms of the field extensions of polynomials defined by the roots coming as restrictions of the basic polynomial to planes $x_n = 0$, $x_n = x_{n-1} = 0$, etc...

2. These Galois groups are suggested to define non-commutative generalization of homotopy and homology theories and non-linear boundary operation for which a geometric interpretation in terms of the restriction to lower-dimensional plane is proposed. The Galois group G_k would be analogous to the relative homology group relative to the plane $x_{k-1} = 0$ representing boundary and makes sense for all number fields also geometrically. One can ask whether the invariance of the complex of groups under the permutations of the orders of variables in the reduction process is necessary. Physical interpretation suggests that this is not the case and that all the groups obtained by the permutations are needed for a full description.

3. The algebraic counterpart of boundary map would map the elements of G_k identified as analog of homotopy group to the commutator group $[G_{k-2}, G_{k-2}]$ and therefore to the unit element of the abelianized group defining cohomology group. In order to obtains something analogous to the ordinary homology and cohomology groups one must however replaces Galois groups by their group algebras with values in some field or ring. This allows to define the analogs of homotopy and homology groups as their abelianizations. Cohomotopy, and cohomology would emerge as duals of homotopy and homology in the dual of the group algebra.

4. That the algebraic representation of the boundary operation is not expected to be unique turns into blessing when on keeps the TGD as almost topological QFT vision as the guide line. One can include all boundary homomorphisms subject to the condition that the anticommutator $\delta^i_k\delta^j_{k-1}+\delta^j_k\delta^i_{k-1}$ maps to the group algebra of the commutator group $[G_{k-2}, G_{k-2}]$. By adding dual generators one obtains what looks like a generalization of anticommutative fermionic algebra and what comes in mind is the spectrum of quantum states of a SUSY algebra

spanned by bosonic states realized as group algebra elements and fermionic states realized in terms of homotopy and cohomotopy and in abelianized version in terms of homology and cohomology. Galois group action allows to organize quantum states into multiplets of Galois groups acting as symmetry groups of physics. Poincare duality would map the analogs of fermionic creation operators to annihilation operators and vice versa and the counterpart of pairing of k:th and $n-k$:th homology groups would be inner product analogous to that given by Grassmann integration. The interpretation in terms of fermions turns however to be wrong and the more appropriate interpretation is in terms of Dolbeault cohomology applying to forms with homomorphic and antiholomorphic indices.

5. The intuitive idea that the Galois group is analogous to 1-D homotopy group which is the only non-commutative homotopy group, the structure of infinite primes analogous to the braids of braids of braids of ... structure, the fact that Galois group is a subgroup of permutation group, and the possibility to lift permutation group to a braid group suggests a representation as flows of 2-D plane with punctures giving a direct connection with topological quantum field theories for braids, knots and links. The natural assumption is that the flows are induced from transformations of the symplectic group acting on $\delta M^2_\pm \times CP_2$ representing quantum fluctuating degrees of freedom associated with WCW ("world of classical worlds"). Discretization of WCW and cutoff in the number of modes would be due to the finite measurement resolution. The outcome would be rather far reaching: finite measurement resolution would allow to construct WCW spinor fields explicitly using the machinery of number theory and algebraic geometry.

6. A connection with operads is highly suggestive. What is nice from TGD perspective is that the non-commutative generalization homology and homotopy has direct connection to the basic structure of quantum TGD almost topological quantum theory where braids are basic objects and also to hyper-finite factors of type II_1. This notion of Galois group makes sense only for the algebraic varieties for which coefficient field is algebraic extension of some number field. Braid group approach however allows to generalize the approach to completely general polynomials since the braid group make sense also when the ends points for the braid are not algebraic points (roots of the polynomial).

This construction would realize the number theoretical, algebraic geometrical, and topological content in the construction of quantum states in TGD framework in accordance with TGD as almost TQFT philosophy, TGD as infinite-D geometry, and TGD as generalized number theory visions.

2. p-Adic integration and cohomology

This picture leads also to a proposal how p-adic integrals could be defined in TGD framework.

1. The calculation of twistorial amplitudes reduces to multi-dimensional residue calculus. Motivic integration gives excellent hopes for the p-adic existence of this calculus and braid representation would give space-time representation for the residue integrals in terms of the braid points representing poles of the integrand: this would conform with quantum classical correspondence. The power of 2π appearing in multiple residue integral is problematic unless it disappears from scattering amplitudes. Otherwise one must allow an extension of p-adic numbers to a ring containing powers of 2π.

2. Weak form of electric-magnetic duality and the general solution ansatz for preferred extremals reduce the Kähler action defining the Kähler function for WCW to the integral of Chern-Simons 3-form. Hence the reduction to cohomology takes places at space-time level and since p-adic cohomology exists there are excellent hopes about the existence of p-adic variant of Kähler action. The existence of the exponent of Kähler gives additional powerful constraints on the value of the Kähler fuction in the intersection of real and p-adic worlds consisting of algebraic partonic 2-surfaces and allows to guess the general form of the Kähler action in p-adic context.

3. One also should define p-adic integration for vacuum functional at the level of WCW. p-Adic thermodynamics serves as a guideline leading to the condition that in p-adic sector exponent of Kähler action is of form $(m/n)^r$, where m/n is divisible by a positive power of p-adic

prime p. This implies that one has sum over contributions coming as powers of p and the challenge is to calculate the integral for K= constant surfaces using the integration measure defined by an infinite power of Kähler form of WCW reducing the integral to cohomology which should make sense also p-adically. The p-adicization of the WCW integrals has been discussed already earlier using an approach based on harmonic analysis in symmetric spaces and these two approaches should be equivalent. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $n + p - 2$ powers of $e^{1/n}$ can be considered.

4. If the Galois group algebras indeed define a representation for WCW spinor fields in finite measurement resolution, also WCW integration would reduce to summations over the Galois groups involved so that integrals would be well-defined in all number fields.

3. Floer homology, Gromov-Witten invariants, and TGD

Floer homology defines a generalization of Morse theory allowing to deduce symplectic homology groups by studying Morse theory in loop space of the symplectic manifold. Since the symplectic transformations of the boundary of $\delta M^4_\pm \times CP_2$ define isometry group of WCW, it is very natural to expect that Kähler action defines a generalization of the Floer homology allowing to understand the symplectic aspects of quantum TGD. The hierarchy of Planck constants implied by the one-to-many correspondence between canonical momentum densities and time derivatives of the embedding space coordinates leads naturally to singular coverings of the embedding space and the resulting symplectic Morse theory could characterize the homology of these coverings.

One ends up to a more precise definition of vacuum functional: Kähler action reduces Chern-Simons terms (imaginary in Minkowskian regions and real in Euclidian regions) so that it has both phase and real exponent which makes the functional integral well-defined. Both the phase factor and its conjugate must be allowed and the resulting degeneracy of ground state could allow to understand qualitatively the delicacies of CP breaking and its sensitivity to the parameters of the system. The critical points with respect to zero modes correspond to those for Kähler function. The critical points with respect to complex coordinates associated with quantum fluctuating degrees of freedom are not allowed by the positive definiteness of Kähler metric of WCW. One can say that Kähler and Morse functions define the real and imaginary parts of the exponent of vacuum functional.

The generalization of Floer homology inspires several new insights. In particular, space-time surface as hyper-quaternionic surface could define the 4-D counterpart for pseudo-holomorphic 2-surfaces in Floer homology. Holomorphic partonic 2-surfaces could in turn correspond to the extrema of Kähler function with respect to zero modes and holomorphy would be accompanied by super-symmetry.

Gromov-Witten invariants appear in Floer homology and topological string theories and this inspires the attempt to build an overall view about their role in TGD. Generalization of topological string theories of type A and B to TGD framework is proposed. The TGD counterpart of the mirror symmetry would be the equivalence of formulations of TGD in $H = M^4 \times CP_2$ and in $CP_3 \times CP_3$ with space-time surfaces replaced with 6-D sphere bundles.

4. K-theory, branes, and TGD

K-theory and its generalizations play a fundamental role in super-string models and M-theory since they allow a topological classification of branes. After representing some physical objections against the notion of brane more technical problems of this approach are discussed briefly and it is proposed how TGD allows to overcome these problems. A more precise formulation of the weak form of electric-magnetic duality emerges: the original formulation was not quite correct for space-time regions with Euclidian signature of the induced metric. The question about possible TGD counterparts of R-R and NS-NS fields and S, T, and U dualities is discussed.

5. p-Adic space-time sheets as correlates for Boolean cognition

p-Adic physics is interpreted as physical correlate for cognition. The so called Stone spaces are in one-one correspondence with Boolean algebras and have typically 2-adic topologies. A generalization to p-adic case with the interpretation of p pinary digits as physically representable Boolean statements of a Boolean algebra with $2^n > p > p^{n-1}$ statements is encouraged by p-adic length scale hypothesis. Stone spaces are synonymous with profinite spaces about which both finite and infinite Galois groups represent basic examples. This provides a strong support for the connection between Boolean cognition and p-adic space-time physics. The Stone space character of Galois groups suggests also a deep connection between number theory and cognition and some arguments providing support for this vision are discussed.

Langlands Program and TGD

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on one hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). If TGD can be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful. I of course hasten to confess that I am not number theorists nor group theorists and that the following considerations are just speculations inspired by TGD.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also a generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW in terms of hyper-finite factors of type II_1 in turn inspires further generalization of the notion of embedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of embedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

1. Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.
2. The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in

the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..

3. TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

4. Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g,R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

1. Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

2. There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

   (a) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

   (b) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

   (c) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \to H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

Langlands Program and TGD: Years Later

Langlands correspondence is for mathematics what unified theories are for physics. The number theoretic vision about TGD has intriguing resemblances with number theoretic Langlands program. There is also geometric variant of Langlands program. I am of course amateur and do not have grasp about the mathematical technicalities and can only try to understand the general ideas and related them to those behind TGD. Physics as geometry of WCW ("world of classical worlds") and physics as generalized number theory are the two visions about quantum TGD: this division brings in mind geometric and number theoretic Langlands programs. This motivates re-consideration of Langlands program from TGD point of view. I have written years ago a chapter about this earlier but TGD has evolved considerably since then so that it is time for a second attempt to understand what Langlands is about.

By Langlands correspondence the representations of $G \rtimes Gal$ and G should correspond to each other. The analogy with the representations of Lorentz group suggests that the representations of G should have “spin” for some compact subgroup acting from left or right such that the dimension of this representation is same as the representation of non-commutative Galois group.

Automorphic functions are indeed typically functions in G, which reduce to a function invariant under left and/or right action of a compact or even discrete subgroups H_1 and H_2 or more generally, belong to a finite-dimensional unitary representation of $H_1 \times H_2$ in $H_1 \backslash G / H_2$. Therefore they can be said to have $H_1 \times H_2$ quantum numbers analogous to spin if interpreted as “field modes” in the space of double cosets $H_1 g H_2$. This would conform with the vision about physics as generalized number theory. If I have understood correctly, the question is whether a finite-dimensional representation of H_1 or H_2 could correspond to a finite-dimensional representation of Galois group at the number theory side.

Some New Ideas Related to Langlands Program *viz.* TGD

Langlands' program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Langlands program is described by Edward Frenkel as a kind of grand unified theory of mathematics.

In the TGD framework, $M^8 - M^4 \times CP_2$ duality assigns to a rational polynomial a set of mass shells H^3 in $M^4 \subset M^8$ and by associativity condition a 4-D surface in M^8, and its it to $H = M^4 \times CP_2$. $M^8 - M^4 \times CP_2$ means that number theoretic vision and geometric vision of physics are dual or at least complementary. This vision could extend to a trinity of number theoretic, geometric and topological views since geometric invariants defined by the space-time surfaces as Bohr orbit-like preferred extremals could serve as topological invariants.

Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC), which is formulated as a correspondence between the quantum states in the "world of classical worlds" (WCW) characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via Galois group representation partition function. QCC would define a kind of closed loop giving rise to a hierarchy.

If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.

One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework. Elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters. Multiple residue integral is proposed as a way to obtain L-functions defining space-time surfaces.

One challenge is to construct Riemann zeta and the associated ξ function and the Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

This construction leads to the question whether also finite fields could play a fundamental role in the number theoretic vision. Prime polynomial with prime order $n = p$ and integer coefficients smaller than $n = p$ can be regarded as a polynomial in a finite field. If it is irreducible, it defines an infinite prime. The proposal is that all physically allowed polynomials are constructible as functional composites of these.

Finite Fields and TGD

TGD involves geometric and number theoretic physics as complementary views of physics. Almost all basic number fields: rationals and their algebraic extensions, p-adic number fields and their extensions, reals, complex number fields, quaternions, and octonions play a fundamental role in the number theoretical vision of TGD.

Even a hierarchy of infinite primes and corresponding number fields appears. At the first level of the hierarchy of infinite primes, the integer coefficients of a polynomial Q defining infinite prime have no common prime factors. $P = Q$ hypothesis states that the polynomial P defining space-time surface is identical with a polynomial Q defining infinite prime at the first level of hierarchy.

However, finite fields, which appear naturally as approximations of p-dic number fields, have not yet gained the expected preferred status as atoms of the number theoretic Universe. Also additional constraints on polynomials P are suggested by physical intuition.

Here the notions of prime polynomial and concept of infinite prime come to rescue. Prime polynomial P with prime order $n = p$ and integer coefficients smaller than p can be regarded as a polynomial in a finite field. The proposal is that all physically allowed polynomials are constructible as functional composites of prime polynomials satisfying $P = Q$ condition.

One of the long standing mysteries of TGD is why preferred p-adic primes, characterizing elementary particles and even more general systems, satisfy the p-adic length scale hypothesis. The proposal is that p-adic primes correspond to ramified primes as factors of discriminant D of polynomial $P(x)$. $D = P$ condition reducing discriminant to a single prime is an attractive hypothesis for preferred ramified primes. $M^8 - H$ duality suggests that the exponent $exp(K)$ of Kähler function corresponds to a negative power D^{-k}. Spin glass character of WCW suggests that the preferred ramified primes for, say prime polynomials of a given degree, and satisfying $D = P$, have an especially large degeneracy for certain ramified primes P, which are therefore of a special physical importance.

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

This article deals with two questions.

1. The ideas related to topological quantum computation suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of state space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum.

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Or could one replace the roots labelling states with representations? Or could even the coefficient field for state space be replaced with the representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums in quantum-classical correspondence, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k,C)$, $k = 2,3,4$ to those of $SL(n,C)$ at least. Is there a deep connection between finite subgroups of $SL(n,C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

In the TGD framework $M^8 - H$ duality relates number theoretic and differential geometric views about physics: could it provide some understanding of this mystery? The proposal is that for cognitive representations associated with extended Dynkin diagrams (EEDs), Galois group Gal acts as Weyl group on McKay diagrams defined by irreps of the isotropy group Gal_I of given root of a polynomial which is monic polynomial but with roots replaced with direct sums of irreps of Gal_I. This could work for p-adic number fields and finite fields. One also ends up with a more detailed view about the connection between the hierarchies of inclusion of Galois groups associated with functional composites of polynomials and hierarchies of inclusions of hyperfinite factors of type II_1 assignable to the representation of super-symplectic algebra.

Quantum Arithmetics and the Relationship between Real and p-Adic Physics

This chapter considers possible answers to the basic questions of the p-adicization program, which are following.

Some of the basic questions of the p-adicization program are following.

1. Is there some kind of duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this)?

   This chapter studies some ideas but does not provide a clearcut answer to these questions. The notion of quantum arithmetics obtained is central in this approach.

The starting point of quantum arithmetics is the map $n \rightarrow n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \rightarrow p_q$. The map takes prime first to finite field $G(p, 1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.
2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.
3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p, 1)$ there are no primes.

One can distinguish between two basic options concerning the definition of quantum integers.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.
2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a+b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n)p^n$. For p-adic numbers this is not the case.
3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

1. The quantum counterparts of special linear groups $SL(n,F)$ exists always. For the covering group $SL(2,C)$ of $SO(3,1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.
2. For the quantum counterparts of $SO(3)$ ($SU(2)/\ SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k+7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \rightarrow 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.
2. An alternative idea is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

Quantum Adeles

Quantum arithmetics provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations. p-Adic numbers have p-adic pinary expansions $\sum a_n p^n$ satisfying $a_n < p$. of powers p^n to be products of primes $p_1 < p$ satisfying $a_n < p$ for ordinary p-adic numbers. One could map this expansion to its quantum counterpart by replacing a_n with their counterpart and by canonical identification map $p \rightarrow 1/p$ the expansion to real number. This definition might be criticized as being essentially equivalent with ordinary p-adic numbers since one can argue that the map of coefficients a_n to their quantum counterparts takes place only in the canonical identification map to reals.

One could however modify this recipe. Represent integer n as a product of primes l and allow for l all expansions for which the coefficients a_n consist of primes $p_1 < p$ but give up the condition $a_n < p$. This would give 1-to-many correspondence between ordinary p-adic numbers and their quantum counterparts.

It took time to realize that $l < p$ condition might be necessary in which case the quantization in this sense - if present at all - could be associated with the canonical identification map to reals. It would correspond only to the process taking into account finite measurement resolution rather than replacement of p-adic number field with something new, hopefully a field. At this step one might perhaps allow $l > p$ so that one would obtain several real images under canonical identification.

One can however imagine a third generalization of number concept. One can replace integer n with n-dimensional Hilbert space and sum $+$ and product $\times$ with direct sum $\oplus$ and tensor product $\otimes$ and introduce their co-operations, the definition of which is highly non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. The construction would also generalize the notion of algebraic holography and provide self-referential cognitive representation of mathematics.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the embedding space. Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. One could interpret $\times_q$ and $+_q$ and their co-algebra operations as 3-vertices for number theoretical Feynman diagrams describing algebraic identities $X = Y$ having natural interpretation in zero energy ontology. The two vertices have direct counterparts as two kinds of basic topological vertices in quantum TGD (stringy vertices and vertices of Feynman diagrams). The definition of co-operations would characterize quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers. One prediction is that all loops can be eliminated from generalized Feynman diagrams and diagrams are in projective sense invariant under permutations of incoming (outgoing legs).

About Absolute Galois Group

Absolute Galois Group defined as Galois group of algebraic numbers regarded as extension of rationals is very difficult concept to define. The goal of classical Langlands program is to understand the Galois group of algebraic numbers as algebraic extension of rationals - Absolute Galois Group (AGG) - through its representations. Invertible adeles -ideles - define Gl_1 which can be shown to be isomorphic with the Galois group of maximal Abelian extension of rationals (MAGG) and the Langlands conjecture is that the representations for algebraic groups with matrix elements replaced with adeles provide information about AGG and algebraic geometry.

I have asked already earlier whether AGG could act is symmetries of quantum TGD. The basis idea was that AGG could be identified as a permutation group for a braid having infinite number of strands. The notion of quantum adele leads to the interpretation of the analog of Galois group for quantum adeles in terms of permutation groups assignable to finite l braids. One can also assign to infinite primes braid structures and Galois groups have lift to braid groups.

Objects known as dessins d'enfant provide a geometric representation for AGG in terms of action on algebraic Riemann surfaces allowing interpretation also as algebraic surfaces in finite fields. This representation would make sense for algebraic partonic 2-surfaces, and could be important in the intersection of real and p-adic worlds assigned with living matter in TGD inspired quantum biology, and would allow to regard the quantum states of living matter as representations of AGG. Adeles would make these representations very concrete by bringing in cognition represented in terms of p-adics and there is also a generalization to Hilbert adeles.

Part I

$M^8 - H$ DUALITY

Chapter 2

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part I

2.1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

2.1.1 Various approaches to classical TGD

World of classical worlds

The first approach is based on the geometry of the "world of classical worlds" (WCW) [K42, K24, K76].

1. The study of classical field equations led rather early to the realization that preferred extremals for the twistor lift of Kähler action with Minkowskian signature of induced metric define a slicing of space-time surfaces defined by 2-D string world sheets and partonic two-surfaces locally orthogonal to them. The interpretation is in terms of position dependent light-like momentum vector and polarization vector defining the local decompositions $M^2(x) \times E^2(x)$ of tangent space integrating to a foliation by partonic 2-surfaces and string world sheets. I christened this structure Hamilton-Jacobi structure. Its Euclidian counterpart is complex structure in Euclidian regions of space-time surface.
2. The formulation of quantum TGD in terms of spinor fields in WCW [K100] leads to the conclusion that WCW must have Kähler geometry [K42, K24] and has it only if it has maximal group of isometries identified as symplectic transformations of $\delta M^4_\pm \times CP_2$, where $\delta M^4_\pm$ denotes light cone boundary two which upper/lower boundary of causal diamond (CD) belongs. Symplectic Lie algebra extends naturally to supersymplectic algebra (SSA).
3. Space-time surfaces would be preferred extremals of twistor lift of Kähler action [K79] and the conditions realizing strong form of holography (SH) would state that sub-algebra of SSA isomorphic with it and its commutator with SSA give rise to vanishing Noether charges and these charges annihilate physical states or create zero norm states from them. One should solve these conditions.
4. The dynamics involves also fermions. Induced spinor fields are located inside space-time surface but for some yet not completely understood reason only the information about spinor at 2-D string world sheets is needed in the construction of scattering amplitudes. This dynamics would be 2-dimensional. The construction of twistor amplitudes even suggests that

it is 1-dimensional in the sense that 1-D light-like curves at light-like partonic orbits defining boundaries of Minkowskian and Euclidian regions determines the scattering amplitudes. String world sheets are however needed only as correlates for entanglement between fermions at different partonic orbits.

The 2-D character of fermionic dynamics conforms with the strong form of holography (SH) but how the string world sheets and partonic 2-surfaces are selected from Hamilton-Jacobi slicing? Electromagnetic neutrality could select string worlds sheets but one can actually always find a gauge in which the induced classical electroweak field at these surfaces is purely electromagnetic.

Twistor lift of TGD

Second approach to preferred extremals is based on TGD version [K91, K35, K11, K79] of twistor Grassmann approach [B22, B48, B29].

1. The twistor lift of TGD leads to a proposal that space-time surfaces can be represented as sections in their 6-D twistor spaces identified as twistor bundles in the product $T(H) = T(M^4) \times T(CP_2)$ of 6-D twistor spaces of M^4 and CP_2. Twistor structure would be induced from $T(H)$. Kähler action can be lifted to the level of twistor spaces only for $M^4 \times CP_2$ since only for these spaces twistor space allows Kähler structure [A150]. Twistors were originally introduced by Penrose with the motivation that one could apply algebraic geometry in Minkowskian signature. The bundle property is extremely powerful and should be consistent with the algebraic geometrization at the level of M^8_c. The challenge is to formulate the twistor lift at the level of M^8.
2. The twistor lift of Kähler action contains also volume term. Field equations have two kinds of solutions. For the solutions of first kind the dynamics of volume term and Käction are coupled and the interpretation is in terms of interaction regions. Solutions of second kind are minimal surfaces and extremals of both Kähler action and volume term, whose dynamics decouple completely and all coupling constants disappear from the dynamics. These extremals are natural candidates for external particles. For these solutions at least the field equations reduce to the existence of Hamilton-Jacobi structure. The completely universal dynamics of these regions suggests interpretation in terms of maximal quantum criticality characterized by the extension of the usual conformal invariance to its quaternionic analog.
3. A connection with zero energy ontology (ZEO) emerges. Causal diamond (CD, intersection of future and past directed light-cones of M^4 with points replaced by CP_2) would naturally determine the interaction region to which external particles enter through its 2 future and past boundaries. But where does ZEO emerge?

$M^8 - H$ duality

The third approach is based on number theoretic vision [K85, K86, K84, K98].

1. M^8-H duality [K86, K98, K8] means that one can see space-times as 4-surfaces in either M^8 or $H = M^4 \times CP_2$. One could speak "number theoretical compactification" having however nothing to do with stringy version of compactification, which is dynamical. $M^8 - H$ duality suggests that space-time surfaces in $H = M^4 \times CP_2$ are images of space-time surfaces in M^8 or actually of M^8 projections of complexified space-time surfaces in M^8_c identified as space of complexified octonions. These space-time surfaces could contain the integrated distributions of string world sheets and partonic 2-surfaces mentioned in the previous item. Space-time surfaces must have associative tangent or normal space for $M^8 - H$ correspondence to exist.
2. The fascinating possibility mentioned already earlier is that in M^8 these surfaces could correspond to zero loci for real or imaginary parts of real analytic octonionic polynomials $P(o) = RE(P) + IM(P)I_4$, I_4 an octonionic imaginary unit orthogonal to quaternionic ones. The condition $IM(P) = 0$ ($RE(P) = 0$) would give associative (co-associative) space-time surface. In the simplest case these functions would be polynomials so that one would have algebraic geometry for algebraically 4-D complex surfaces in 8-D complex space.

Remark: The naive guess that space-time surfaces reduce to quaternionic curves in quaternionic plane fails due to the non-commutativity of quaternions meaning that one has $P(o) = P(q_1, q_2, \overline{q}_1, \overline{q}_2)$ rather than $P(o) = P(q_1, q_2)$.

Remark: Why not rational functions expressible as ratios $R = P_1/P_2$ of octonionic polynomials? It has become clear that one can develop physical arguments in favor of this option. The zero loci for $IM(P_i)$ would represent space-time varieties. Zero loci for $RE(P_1/P_2) = 0$ and $RE(P_1/P_2) = \infty$ would represent their interaction presumably realized as wormhole contacts connecting these varieties. In the sequel most considerations are for polynomials: the replacement of polynomials with rational functions does not introduce big differences and its discussed in the section "Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view" of [L38].

3. The objection against this proposal is obvious. $M^8 - H$ correspondence cannot hold true since the dynamics defined by octonionic polynomials in M^8 contains no coupling constants whereas the dynamics of twistor lift of Kähler action depends on coupling constants in the generic space-time region. However, for space-time surfaces representing external particles entering inside CD at its boundaries this is however not the case! They could satisfy $M^8 - H$ correspondence!

   This suggests that inside CDs the space-time surfaces are not associative/co-associative in M^8 so that $M^8 - H$ correspondence cannot map them to H and the twistor lifted Kähler action and SH take care of the dynamics. External particles are associative and quantum critical and $M^8 - H$ correspondence makes sense. The quantum criticality and associativity at the boundaries of CD poses extremely powerful conditions and allows to satisfy infinite number of vanishing conditions for SSA charges.

   It has later turned out [L56] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

4. This picture is consistent with the the explicit formulation of the associativity conditions $Re(P) = 0$ and $IM(P) = 0$ for varieties. The calculations shows that associativity can be realized either by posing a condition making them 3-dimensional except, when the situation is critical in the sense that the 4-D variety is analogous to a double root of polynomial: now however the polynomial corresponds to prime polynomial decomposing to product of polynomials in the extension of rationals such that the product contains higher powers of the factors. One has ramification at the level of polynomial primes so that the criticality condition does not bring anything new but need not make the situation associative. At most 3 conditions need to be applied to guarantee associativity and they might leave the space-time surface 4-D.

5. The coordinates of M^4 as octonionic roots $x + iy$ of the 4 real polynomials need not be real. Space-time surface must have M_c^4 projection, which reduces to M^4. There are two options.

   (a) The original proposal was that the *projection* from M_c^8 to real M^4 (for which M^1 coordinate is real and E^3 coordinates are imaginary with respect to i!) defines the real space-time surface mappable by $M^8 - H$ duality to CP_2. One can howeerver critize the allowance of a nonvanishing imaginary part of space-time surface in M_c^4.

   (b) A more stringent condition is that the roots of the 4 vanishing polynomials as coordinates of M_c^4 belong automatically to M^4 so that m^0 would be real root and m^k, $k = 1, ..., 3$ imaginary with respect to $i \to -i$. M_c^8 coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality.*

   This could allow to identify CDs in very elegant way: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

6. This octonionic view as also lower-dimensional quaternionic counterpart. In this case one considers 2-D commutative/co-commutative surfaces tentatively identifiable as string world

sheets and partonic 2-surfaces. Why not all 2-surfaces appearing in the Hamilton-Jacobi slicing are not selected? The above mechanism would work also now. The commutativity conditions reduce in the generic case give 1-D curve as a solution. The interpretation would be as orbit of point like particle at 3-D partonic orbit appearing in the construction of twistorial amplitudes. In critical situation one would obtains string world sheet serving as a correlate for entanglement between point like particles at its ends: one would have quantum critical bound state.

I have considered also other attempts to define what quaternion structure could mean.

1. One could also consider the possibility that the tangent spaces of space-time surfaces in H are associative or co-associative [K98]. This is not necessary although it seems that this might be the case for the known extremals. If this holds true, one can construct further preferred extremals by functional composition by generalization of M^8-H correspondence to $H-H$ correspondence.
2. I have considered also the possibility of quaternion analyticity in the sense of generalization of Cauchy-Riemann equations, which tell that left- or right quaternionic differentiation makes sense [L28]. It however seems that this approach is not promising. The conditions are quite too restrictive and bring nothing essentially new. Octonion/quaternion analyticity in the above mentioned sense does not require the analogs of Cauchy-Riemann conditions.

2.1.2 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.
2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by CP_2 points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.
3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L19]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P)=0$ can transform to $IM(P)=0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.
4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adele for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.

5. Also a connection with infinite primes is suggestive [K86]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to $M^8 - H$ duality. The strategy is simple: try to remember all previous objections against $M^8 - H$ duality and invent new ones since this is the best way to make real progress.

2.1.3 Topics to be discussed

Key notions and ideas of algebraic geometry

Before going of octonionic algebraic geometry, I will discuss basic notions of algebraic geometry such as algebraic variety (see `http://tinyurl.com/hl6sjmz`), - surface (see `http://tinyurl.com/y8d5wsmj`), and - curve (see `http://tinyurl.com/nt6tkey`), rational point of variety central for TGD view about cognitive representation, elliptic curves (see `http://tinyurl.com/lovksny`) and - surfaces (see `http://tinyurl.com/yc33a6dg`), and rational points (see `http://tinyurl.com/ybbnnysu`) and potentially rational varieties (see `http://tinyurl.com/yablk4xt`). Also the notion of Zariski topology (see `http://tinyurl.com/h5pv4vk`) and Kodaira dimension (see `http://tinyurl.com/yadoj2ut`) are discussed briefly. I am not a mathematician. What hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

Much of algebraic geometry is counting numbers of say rational points or of varieties satisfying some conditions. One can also count dimensions of moduli spaces. Hence the basic notions and methods of enumerative geometry are discussed. There is also a discussion of Gromow-Witten invariants and Riemann-Roch theorem having Atyiah-Singer index theorem as a generalization. These notions will be applied in the second part of the article [L38].

$M^8 - H$ duality

$M^8 - H$ duality [K8, K86, K98] would reduce classical TGD to the algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. Space-time surfaces in M^8 would be algebraic varieties identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o decomposing as $o = q_c^1 + q_c^2 I_4$ and projected to a Minkowskian sub-space M^8 of o. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces in M^8 would form commutative and associative algebra with addition, product and functional composition.

As already noticed, the associativity conditions do not allow 4-D solutions except for criticality so that $M^8 - H$ correspondence can hold true only in these space-time regions and one has these nice features at the level of M^8. In critical regions $M^8 - H$ correspondence is true and these features have H counterparts

The basic problem is to understand the map mediating $M^8 - H$ duality mapping the point (m, e) of $M^8 = M_0^4 \times E^4$ to a point (m, s) of $M_0^4 \times CP_2$, where M_0^4 point is obtained as a projection to a suitably chosen $M_0^4 \subset M^8$ and CP_2 point parameterizes the tangent space as quaternionic sub-space containing preferred $M_0^2(x) \subset M^4(x)$. This map involves slightly non-local information and could allow to understand why the preferred extremals at the level of H are determined by partial differential equations rather than algebraic equations. Also the generalization to the level of twistor lift is briefly touched.

Challenges of the octonionic algebraic geometry

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a $M^8 - H$ correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

   One can have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A129] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

   One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commu-

tative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

The easiest way to kill $M^8 - H$ duality in the form it is represented here is to prove that 4-D zero loci for imaginary/real parts of octonionic polynomials with real coefficients can never be associative/co-associative being always 3-D. I hope that some professional mathematician would bother to check this.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

2.2 Some basic notions, ideas, results, and conjectures of algebraic geometry

In this section I will summarize very briefly the basic notions of algebraic geometry needed in the sequel.

2.2.1 Algebraic varieties, curves and surfaces

The basic notion of algebraic geometry is algebraic variety.

1. One considers affine space A^n with n coordinates $x^1, ..., x^n$ having values in a number field K usually assumed to be algebraically closed (note that affine space has no preferred origin like linear space). Algebraic variety is defined as a solution of one or more algebraic equations stating the vanishing of polynomials of n variables: $P^i(x^1, ..., x^n) = 0$, $i = 1, ..., r \leq n$. One can restrict the coefficients of polynomials to p-adic number field or or its extension to an extension of rationals. One talks about polynomials on $k \subset K$.

2. The basic condition is that the variety is not a union of disjoint varieties. This for instance happens, when the polynomial $P(x^1, .., x^n)$ defining co-dimension 1 manifold is product of polynomials $P = \prod_r P_r$. Algebraic variety need not be a manifold meaning that it can have singular points. For instance, for co-dimension 1 variety the Jacobian matrix $\partial P/\partial x^i$ of the polynomial can vanish at singularity.

3. One can define projective varieties (see `http://tinyurl.com/ybsqvy3r`) in projective space P^n having coordinatization in terms of $n+1$ homogenous coordinates $(x^1, ..., x^{n+1})$ in K with points differing by an overall scaling identified. Projective variety is defined as zero locus of homogenous polynomials of $n + 1$ coordinates so that solutions remain solutions under the overall scaling of all coordinates. By identifying the points related by scaling one obtains a surface in P^n. Grassmannian of linear space V^n (not affine space!) is a projective spaces defined as space of k-planes of V^n. These spaces are encountered in twistor Grassmannian approach to scattering amplitudes.

For polynomials of single variable one obtains just the roots of $P_n(x) = 0$ in an algebraic extension assignable to the polynomial. For several variables one can in principle proceed step by step by solving variable x^1 as algebraic function of others from $P_1(x^1, ..., x^n) = 0$, proceed to solve x^2 from $P_2(x^1(x^2, ...), x^2, ...) = 0$ as as algebraic function of the remaining variables, and so one. The algebraic functions involved get increasingly complex but in some exceptional situations the solution has parametric representation in terms of *rational* rather than algebraic functions of parameters t^k. For co-dimension $d_c > 1$ case the intersection of surfaces $P^i = 0$ need not be complete and the tangent spaces of the hyper-surfaces $P^i = 0$ need not intersect transversally in the generic case. Therefore $d_c > 1$ case is not gained so much attention as $d_c = 1$ case.

A more advanced treatment relies on ring theory by assigning to polynomials a ring as the ring of polynomials in the space involved divided by the ring of polynomials vanishing at zero loci of polynomials P^i.

1. The notion of ideal is central and determined as a subring invariant under the multiplication by elements of ring. Prime ideal generalizes the notion of prime and one can say that the notion of integer generalizes to that of ideal. One can also define the notion of fractional ideal.
2. Zariski topology (see `http://tinyurl.com/h5pv4vk`) replacing the topology based on real norm is second highly advanced notion. The closed sets in this topology are algebraic varieties of various dimensions. Since the complement of any algebraic variety is open set this topology and open also in the ordinary real topology, this topology is considerable rougher than the ordinary than the ordinary topology.

Some remarks from the point of view of TGD are in order.

1. In the scenario inspired by M^8-H duality one has co-dimension 4 surfaces in 8-D complex space. Octonionicity of polynomials however implies huge symmetries since the polynomial is determined by single real polynomial of real variable, whose values at finite number of points determined the polynomial.
2. In TGD the extension of rationals can be assumed to contain also powers for some root of e since in p-adic context this gives rise to a finite-dimensional extensions due to the fact that e^p is ordinary p-adic number. Also a restriction to a finite field are possible and restriction of rational coefficients to their modulo p counterparts reduces the polynomial to polynomial in finite field. This reduction is used as a technical tool. In the case of Diophantine equations (see `http://tinyurl.com/nt6tkey` and `http://tinyurl.com/y8hm4zce`) the coefficients are restricted to be integers.
3. In adelic TGD [L42] [L41] the number fields involved are reals and extensions of p-adic numbers. The coefficient field for the coefficients of polynomials would be naturally extension of rationals or extension of p-adics induced by it. The coefficients of polynomials serve as coordinates of adelic WCW. p-Adic numbers are not algebraically closed and one must assume an extension of p-adic numbers from that for the coefficients one to allow maximal number of roots.

   This suggests an evolutionary process [L44] extending the number field for the coefficients of polynomials. Arbitrary root of polynomial for given extension can be realized only if the original extension is extended further. But this allows polynomial coefficients in this new extension: WCW is now larger. Now one has however roots in even larger extension so that the unavoidable outcome is number theoretic evolution as increase of complexity.
4. What is so remarkable is that octonionic polynomials with rational coefficients could be determined by their values at finite set of points for a polynomial of real argument once the order of polynomial is fixed. Real coordinate corresponds to preferred time axis naturally. A cognitive representation consisting of finite number of rational points could fix the entire space-time surface! This would extend ordinary holography to its discrete variant!
5. Algebraic variety is rather simple object as compared to the solutions of partial differential equations encountered in physics - say those for minimal surfaces. Now one must fix boundary values or initial values at $n-1$-dimensional surface to fix the solution. For integrable theories the situation can change. In TGD SH suggests that the classical solutions are determined by data at 2-surfaces, which together with conformal invariance could reduce the data to one-dimensional data specified by a polynomial. M^8-H correspondence allows to consider this option seriously.
6. M^8-H duality suggests that space-time surfaces are co-dimension $d_c=4$ algebraic curves in M^8. Could space-time surfaces define closed sets for the analog of Zariski topology? Could string world sheets and partonic 2-surfaces do the same inside space-time surfaces? An interesting question is whether this generalizes also to the level of embedding space H and could perhaps define a topology rougher than real topology in better accord with the notion of finite measurement resolution.

2.2.2 About algebraic curves and surfaces

The realization M^8-H correspondence to be considered allows to understand space-time surfaces as 4-D complex algebraic surfaces X^4_c in the space o of complexified octonions projected to real sub-space of O^c with Minkowskian signature. Due to the non-commutativity of quaternions, the reduction of space-time surfaces to curves in quaternionic plane is not possible. Despite this it is instructive to start from the algebraic geometry of curves and surfaces.

Degree and genus of the algebraic curve

Algebraic curve is defined as zero locus of a polynomial $P(x^1, x^2, ..., x^n)$ with x^n in some - preferably algebraically closed - number field K and coefficients in some number field $k \subset K$. In adelic physics K corresponds to real or complex numbers and k to the extension of rationals defining adeles. In p-adic sectors k corresponds to tje extension of p-adic numbers induced by k. In general roots belong to an extension of k.

Degree, genus, and Euler characteristic are the basic characterizers of algebraic curve.

1. The degree d of algebraic curve corresponds to the highest power for the variables appearing in the polynomial. One can also define multi-degree in an obvious manner. A useful geometric interpretation for the degree is that line intersects curve (also complex) of degree d in at most d points as is clear from the fact that the equation of curve reduces the equation for curve to an equation for the roots of d:th order polynomial of single variable.

2. Also the genus g of the curve (see `http://tinyurl.com/ybm3wfue`) is important characteristic. One can distinguish between topological genus, geometric genus and arithmetic genus. For curves these notions are equivalent. The connection between genus and degree d of non-singular algebraic curve is very useful:

$$g = \frac{(d-1)(d-2)}{2} \ . \tag{2.2.1}$$

Spherical topology for complex curves corresponds to $n = 1$ and $n = 2$.

A more general formula reads as:

$$g = \frac{(d-1)(d-2)}{2} + \frac{n_s}{2} \ . \tag{2.2.2}$$

Here n_s is the number of holes of the curve behaving like holes and increasing the genus.

3. Euler characteristic (for Euler characteristic see `http://tinyurl.com/pp52zd4`) is a homological invariant making sense in arbitrary dimension and also for manifolds. Homological definition based on simplicial homology relies on counting of simplexes of various dimension. The definition in terms of dimensions of homology groups H_n is given by

$$\chi = b_0 - b_1 + b_2 ... + (-1)^n b_n \ , \tag{2.2.3}$$

where b_k is the dimension of k:th homology group (see `http://tinyurl.com/j48ojys`).

The following gives the engineering rules for obtaining Euler characteristic of the surface obtained from simpler building blocks. Note that algebraic variety property is not essential here.

1. Euler characteristic is homotopy invariant so that it does not change one adds homologically trivial space such as E^n as a Cartesian factor.

2. χ is additive under disjoint union. Inclusion-exclusion principle states that if M and N intersect, one has $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$.

3. Euler characteristic for the connected sum $A\#B$ of n-dimensional manifolds obtained by drilling balls B^n from summands, giving opposite orientation to the boundaries of the hole, and connecting with cylinder $D \times S^{n-1}$ is given by $\chi(A) + \chi(B) - \chi(S^{n-1})$. One has $\chi(S^2) = 2$ and $\chi(D^2) = 1$.

4. The Euler characteristic for product $M \times N$ is $\chi(M) \times \chi(N)$.

5. The Euler characteristic for N-fold covering space M_n is $N \times \chi(M)$ with a correction term coming from the singularities of the covering (ramified covering space).

6. For a fibration $M \to B$ with fiber S, which differs from fiber bundle in that the fibers are only homeomorphic, one has $\chi(M) = \chi(B) \times \chi(S)$.

   Euler characteristic and the genus of 2-surface (or complex) curve are related by the equation

$$\chi = 2(1-g) \ . \tag{2.2.4}$$

having values $2, 0, -2, ...$. If the 2-surface has n_s holes (punctures), one has

$$\chi = 2(1-g) - n_s \ . \tag{2.2.5}$$

Punctures must be distinguished from singularities at which some sheets of covering meet at single point.

A formal generalization of the definition of genus for varieties in terms of Euler characteristic makes sense.

$$g = -\frac{\chi}{2} + 1 - \frac{n_s}{2} \ . \tag{2.2.6}$$

Disk has genus 1/2 and drilling of n holes increases genus by $n/2$. Pair of holes gives same contribution to g and the cylinder connecting the holes. Note that for complex curves the definition of puncture is obvious. For real curves the puncture would mean missing point of the curve.

The latter definitions of genus can be identified in terms of Euler characteristic also for higher-dimensional varieties. For curves these notions are equivalent if there are no singularities. For algebraic curves g is same for the real and complex variants of the curve in RP_1 and CP_1 respectively.

Elliptic curves and elliptic surfaces

Elliptic curves (see `http://tinyurl.com/lovksny`) are cubic curves with no singularities (cusps or self-intersections) having representation of form $y^2 - x^3 - ax - b = 0$. These singularities can occur only at special values of parameters ($(a = 0, b = 0)$. Since the degree equals to $d = 3$, elliptic curve has genus $g = 1$.

Elliptic curves allow a group of Abelian symmetries generated by a finite number of generators. The emergence of abelian group structure can be intuitively understood as follows.

1. Given line intersects the curve of degree 3 in at most 3 points. Let P and Q be two of these points. Then there can be also a third intersection point R and by the Z^2 symmetry changing the sign of y also the reflection of R - identify it as $-R$ - belongs to the curve. Define the sum of $P + Q$ to be $-R$.

   The actual proof is slightly more complicated since the number of intersection points for the line with curve can be also 2 or 1. By writing explicit expressions for the coordinates x_R and y_R, one can also find that they are indeed rational if the points P and Q are rational. If the elliptic curve as single rational point it has infinite number of them.

2. The generators with finite order give rise to torsion. The rank of generators of infinite order is called rank and conjectured to be arbitrarily large (see `http://tinyurl.com/lovksny`) . Therefore elliptic curve is an Abelian group and one talks about Abelian variety. If elliptic curve contains a rational point it contains entire lattice of rational points obtained as shifts of this point.

Remark: Complex elliptic curves are 2-surfaces in complex projective plane CP_2 and therefore highly interesting from TGD point of view. $g = 1$ partonic 2-surfaces would in TGD framework correspond to second generation fermions [K21]. Abelian varieties define a generalization of elliptic curves to higher dimensions and simplest space-time surfaces allowing also large cognitive representations could correspond to such.

Elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) are fibrations with an algebraic curve as base space and elliptic curve as fiber (fibration is more general notion than fiber space since the fibers are only homeomorphic). The singular fibers failing to be elliptic curves have been classified by Kodaira.

2.2.3 The notion of rational point and its generalization

The notion of algebraic integer (see http://tinyurl.com/y8z389a7) makes sense for any number field as a root of a monic polynomial (polynomial with integer coefficients with coefficient of highest power equal to unity). The field of fractions for given number field consists of ratios of algebraic integers. The same is true for the notion of prime. The more precise definition forces to replace integers and primes with ideals.

Rational varieties are expressible as maps defined by rational functions with rational coefficients in some extension of Q and contain infinite number of rational points. If the variety is not rational, one can ask whether it could allow a dense set of rational points with rational number replaced with the ratio of algebraic integers for some extension of Q. This leads to the idea of potentially rational point, and one can classify algebraic varieties according to whether they are potentially rational or not. The variety is potentially rational if it allows a parameteric representation using rational functions. Otherwise the parametric representation involves algebraic functions such as roots of rational functions.

The interpretation in terms of cognition would be that large enough extension makes the situation "cognitively easy" since cognitive representations involving fermions at the rational points and defining discretizations of the algebraic variety could be arbitrary large. The simpler the surface is cognitively, the large the number of rational points or potentially rational points is.

Complexity of algebraic varieties is measured by Kodaira dimension d_K (see http://tinyurl.com/yadoj2ut). The spectrum for this dimension varies in the range $(-\infty, 0, 1, 2, ...d)$, where d is the algebraic dimension of the variety. Maximal value equals to the ordinary topological dimension d and corresponds to maximal complexity: in this case the set of rational points is finite. Minimal Kodaira dimension is $d_K = -\infty$: in this case the set of rational points is infinite. Rational surfaces are maximally simple and this corresponds to the existence of parametric representations using only rational functions.

Rational points for algebraic curves

The sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see http://tinyurl.com/y9oq37ce) states that a curve over Q with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere CP_1 in CP_2 has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A158, A169]).

   $g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in CP_2 with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in CP_2 (see http://tinyurl.com/lovksny) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

   $g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and CP_2 as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in CP_2 (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially dense set of rational points.

Remark: In TGD framework algebraic varieties could be zero loci of octonionic polynomials and have algebraic dimension 4 so that the classification for algebraic curves does not help. Octonion analyticity must bring in symmetries which simplify the situation.

Enriques-Kodaira classification

The tables of (see http://tinyurl.com/ydelr4np) give an overall view about the Enriques-Kodaira classification of algebraic curves, surfaces, and varieties in terms of Kodaira dimension (see http://tinyurl.com/yadoj2ut).

1. For instance, general curves ($g \geq 2$) have $d_K = 1$, elliptic curves ($g = 1$) have $d_K = 0$ and projective line ($g = 0$) has $d_K = -\infty$. $CP_1 \subset CP_2$ is a rational curve so that rational points are dense. Elliptic curves allow infinite number or rational points forming an Abelian group if they containing single rational point and are therefore cognitively easy.
2. Algebraic varieties (with real dimension $d_R = 4$ in complex case) with $d_K = 2$ are surfaces of general type, elliptic surfaces (see http://tinyurl.com/yc33a6dg) have $d_K = 1$, surfaces with attribute abelian, hyper-elliptic, K3, and Enriques, have $d_K = 0$.

   Remark: All real 2-surfaces are hyper-elliptic for $g \leq 2$, in other words allow Z_2 as global conformal symmetry. Genus-generation correspondence [K21] for fermions allows to assign to the 3 lowest generations of fermions hyper-elliptic partonic 2-surfaces with genus $g = 0, 1, 2$. These surfaces would have $d_K = 0$ and be rather simple as real surfaces in Kodaira classification. Could one regard them as M^4 projection of complex hyper-elliptic surfaces of real dimension $d_R = 4$? $d_K = -\infty$ holds true for rational surfaces and ruled surfaces, which allow straight line through any point are maximally simple. In complex case the line would be CP_1.
3. The Wikipedia article gives also a table about the classification of algebraic 3-folds. Real algebraic 3-surfaces might well occur in TGD framework. The twistor space for space-time surface might allow realization as complex 3-fold and since it has S^2 has fiber, it would naturally correspond to an uni-ruled surface with $d_K = -\infty$. The table shows that one can build higher dimensional algebraic varieties with $d_K < d$ from lower-dimensional ones as fiber-space like structures, which based or fiber having $d_K < d$. 3-D Abelian varieties and Calabi-Yau 3-folds are complex manifolds with $d_K = 0$, which cannot be engineered in this manner.
4. Space-time surfaces would be surfaces of algebraic dimension 4. Wikipedia tables do not give direct information about this case but one can make guesses on basis of the three tables. Octonionic polynomials are analytic continuations of real polynomials of real variable, which must mean a huge simplification, which also favor cognitive representability. The best that one might have infinite sets of rational points. The examples about extremals of Kähler action does not however favor this wish.

Bombieri-Lang conjecture (see http://tinyurl.com/y887yn5b) states that, for any variety X of general type over a number field k, the set of k-rational points of X fails to be Zariski dense (see http://tinyurl.com/jm9fh74) in X. This means that , the k-rational points are contained in a *finite* union of lower-dimensional sub-varieties of X. In dimension 1, this is exactly Faltings theorem, since a curve is of general type if and only if it has $g \geq 2$. The conjecture of Vojta (see http://tinyurl.com/y9sttuu4) states that varieties of general type cannot be potentially dense. As will be found, these conjectures might be highly relevant for TGD.

2.3 About enumerative algebraic geometry

Algebraic geometry is something very different from Riemann geometry, Kähler geometry, or sub-manifold geometry based on local notions. Sub-manifolds are replaced with sub-varieties defined as zero loci for polynomials with coefficients in the field of rationals or extension of rationals. Partial differential equations are replaced with algebraic ones. One can generalize algebraic geometry to any number field.

String theorists have worked with algebraic geometry with motivation coming from various moduli spaces emerging in string theory. The moduli spaces for closed and open strings possibly in presence of branes are involved. Also Calabi-Yau compacticication leads to algebraic geometry, and topological string theories of type A and B involve also moduli spaces and enumerative algebraic geometry.

In TGD the motivation for enumerative algebraic geometry comes from several sources.

1. Twistor lift of TGD lifts space-time surfaces to their 6-D twistor spaces representable as surfaces in the product of 6-D twistor spaces of M^4 and CP_2 and possessing Kähler structure - this makes these spaces completely unique and strongly suggests the role of algebraic geometry, in particular in the generalization of twistor Grassmannian approach [L38].
2. There are three threads in number theoretic vision: p-adic numbers and adelics, classical number fields, and infinite primes. Adelic physics [L42] as physics of sensory experience and cognition unifies real physics and various p-adic physics in the adele characterized by an extension of rationals inducing those of p-adic number fields. This leads to algebraic geometry and counting of points with embedding space coordinates in the extension of rationals and defining a discrete cognitive representation. The core of the scattering amplitude would be defined by this cognitive representation identifiable in terms of points appearing as arguments of n-point function in QFT picture [L36].
3. $M^8 - M^4 \times CP_2$ duality is the analog of the rather adhoc spontaneous compactification in string models but would be non-dynamical and thus allow to avoid landscape catastrophe. Classical physics would reduce to octonionic algebraic geometry at the level of complexified octonions with several special features due to non-commutativity and non-associativity: space-time could be seen as 4-surface in the complexification of of octonions. The commuting imaginary unit would make possible the realization of algebraic extensions of rationals.

   The moduli space for the varieties is discrete if the coefficients of the polynomials are in the extension of rationals. If one poses additional conditions such as associativity of 4-surfaces, the moduli space is further reduced by the resulting criticality conditions realizing quantum criticality at the fundamental level raising hopes about extremely simple formulation of scattering amplitudes at the level of M^8 [L38].

   Also complex and co-complex sub-manifolds of associative space-time surface are important and would realize strong form of holography (SH). For non-associative regions of space-time surface it might not be possible to define complex and co-complex surfaces in unique manner since the basic $M^2 \subset M^4$ local flag structure is missing. String world sheets and partonic 2-surfaces and their moduli spaces are indeed in key role and the topology of partonic surfaces plays a key role in understanding of family replication phenomenon in TGD [L36].

In this framework one cannot avoid enumerative algebraic geometry.

1. One might want to know the number of points of sub-variety belonging to the number field defining the coefficients of the polynomials. This problem is very relevant in M^8 formulation of TGD, where these points are carriers of sparticles. In TGD based vision about cognition [L42] they define cognitive representations as points of space-time surface, whose M^8 coordinates can be thought of as belonging to both real number field and to extensions of various p-adic number fields induced by the extension of rationals. If these cognitive representations define the vertices of analogs of twistor Grassmann diagrams in which sparticle lines meet, one would have number theoretically universal adelic formulation of scattering amplitudes and a deep connection between fundamental physics and cognition.
2. Second kind of problem involves a set algebraic surfaces represented as zero loci for polynomials - lines and circles in the simplest situations. One must find the number of algebraic surfaces intersecting or touching the surfaces in this set. Here the notion of incidence is central. Point can be incident on line or two lines (being their intersection), line on plane, etc.. This leads to the notion of Grassmannians and flag-manifolds.

Moduli spaces parameterizing sub-varieties of given kind - lines, circles, algebraic curves of given degree, are central for the more advanced formulation of algebraic geometry. These moduli spaces emerge also in the formulation of TGD. The moduli space of conformal equivalence classes of partonic 2-surfaces is one example involved with the explanation of family replication phenomenon [K21]. One can assign moduli spaces also to octonion and quaternion structures in M^8 (or equivalently with the complexification of E^8). One can identify CP_2 as a moduli space of quaternionic sub-spaces of octonions containing preferred complex sub-space.

One cannot avoid these moduli spaces in the formulation of the scattering amplitudes and this leads to $M^8 - H$ duality. The hard core of the calculation should however reduce to the understanding of the algebraic geometry of 4-surfaces in octonionic space. Clearly, M^8 picture seems to provide the simplest formulation of the number theoretic vision.

2.3.1 Some examples about enumerative algebraic geometry

Some examples give an idea about what enumerative algebraic geometry (see http://tinyurl.com/y7yzt67b) is.

1. Consider 4 lines in 3-D space. What is the number of lines intersecting these 4 lines [A187] (see http://tinyurl.com/ycrbr5aj). One could deduce the number of lines and lines by writing the explicit equations for the lines with each line characterized by 2+3=5 parameters specifying direction t vector and arbitarily chosen point x_0 on the line. 2+3=5 parameters characterize each sought-for line.

   For intersection points x_i of sought for line with i:th one has $x_i = x_0 + k_i t_0$, $i = 1, ..., 4$ for the sought for line with direction t_0. At the intersection points at the 4 lines one has $x_i = x_{0i} + K_i t_i$ with fixed directions t_i. Combining the two equations for each line one has $4 \times 3 = 12$ equations and 3+4+2 parameters for the sought for line plus 4 parameters K_i for the four lines. This gives 13 unknown parameters corresponding to x_0, k_i, K_i. One would have one parameter set of solutions: something goes wrong.

   One has however projective invariance: one can shift x_0 along the line by $x_0 \rightarrow x_0 - at$, $k_i \rightarrow k_i + a$ and using this freedom assume for instance $k_1 = 0$. This reduces the number of parameters to 12 and one has finite number of solutions in the generic case. Actually the number is 2 in the generic case but can be infinite in some special cases. The challenge is to deduce the number of the solutions by geometric arguments.Below Schubert's argument proving that the number of solutions is 2 will be discussed.

   The idea of enumerative geometry is to do this using general geometric arguments allowing to deform the problem topologically to a simpler one in which case the number of solutions is obvious which in the most abstract formulation become topological.

2. Apollonius can be seen as founder of enumerative algebraic geometry. Apollonian circles (see http://tinyurl.com/ycvxe688) represent second example. One has 3 circles in plane. What is the number of circles tangential to all these 3 circles. Wikipedia link represents the geometric solution of the problem. The number of circles is 8 in the generic case but there are exceptional cases.

3. In Steiner's conic problem (see http://tinyurl.com/yahshsjo) one have 5 conical sections (circles, cones, ellipsoids, hyperbole) in plane. How many different conics tangential to the conics there exist? This problem is rather difficult and the thumb rules of enumerative geometry (dimension counting, Bezout's rule, Schubert calculus) fail. This is a problem in projective geometry where one is forced to introduce moduli space for conics tangential to given conic. This space is algebraic sub-variety of all conics in plane which is 5-D projective space. One must be able to deduce the number of points in the intersection of these sub-varieties so that the original problem in 2-D plane is replaced with a problem in moduli space.

2.3.2 About methods of algebraic enumerative geometry

A brief summary about methods of algebraic geometry is in order to give some idea about what is involved (see http://tinyurl.com/y7yzt67b).

1. Dimension counting is the simplest method. If two geometric objects of n-D space have dimensions k and l, there intersection is $n-k-l$-dimensional for $n-k-l \geq 0$ or empty in the generic case. For $k+l=n$ one obtains discrete set of intersection points.

2. Bezout's theorem is a more advanced method. Consider for instance, curves in plane defined by the curves polynomials $x = P^m(y)$ and $x = P^n(y)$ of degrees $k = m$ and $k = n$. The number N of intersection points in the generic case is bounded above by $N = m \times n$ (in this case all roots are real). One can understand this by noticing that one has m roots y_k or given x giving rise to a m-branched graph of function $y = f(x)$. The number of intersections for the graphs of the two polynomials is at most $m \times n$. If one has curve in plane represented by polynomial equation $P^{m,n}(x, y) = 0$, one can also estimate immediately the minimal multi-degree (m, n) for this polynomials.

3. Schubert calculus http://tinyurl.com/y766ddw2) is a more advanced but not completely rigorous method of enumerative geometry [A187] (see http://tinyurl.com/ycrbr5aj).

   Schubert's vision was that the number of intersection points is stable against deformations in the generic case. This is not quite true always but in exceptional cases one can say that two separate solutions degenerate to single one, just like roots of polynomial can do for suitable values of coefficients.

   For instance, Schubert's solution to the already mentioned problem of finding a line intersecting 4 lines in generic position relies on this assumption. The idea is to deform the situation so that one has two intersecting pairs of lines. One solution to the problem is a line going through the intersection points for line pairs. Second solution is obtained as intersection of the planes. It can happen that planes are parallel in which case this does not work.

   Schubert calculus it applies to linear sub-varieties but can be generalized also to non-linear varieties. The notion of incidence allowing a general formulation for intersection and tangentiality (touching) is central. This leads to the notions of flag, flag manifold, and Schubert variety as sub-variety of Grassmannian.

   Flag is a hierarchy of incident subspaces $A_0 \subset A_1 \subset A_2 ... \subset A_n$ with the property that the dimension $d_i \leq n$ of A_i satisfies $d_i \geq i$. As a special case this notion leads to the notion of Grassmannian $G(k,n)$ consisting of k-planes in n-dimensional space: in this case A_0 corresponds to k-planes and A_2 to space A_n. More general flag manifolds are moduli spaces and sub-varieties of Grassmannian providing a solution to some conditions. Flag varieties as sub-varieties of Grassmannians are Schubert varieties (see http://tinyurl.com/y7ehcrzg). They are also examples of singular varieties. More general Grassmannians are obtained as coset spaces of G/P, where G is algebraic group and P is parabolic sub-group of G.

   Remark: CP_2 corresponds to the space of complex lines in C^3. CP_2 can be also understood as the space of quaternionic planes in octonionic 8-space containing fixed 2-plane so that also now one has flag. String world sheets inside space-time surfaces define curved flags with 2-D and 4-D tangent spaces defining an integrable distribution of local flags.

4. Cohomology combined with Poincare duality allows a rigorous formulation of Schubert calculus. Schubert's idea about possibility to deform the generic position corresponds to homotopy invariance, when the degeneracies of the solutions are taken into account. Homology and cohomology become basic tools and the so called cup product for cohomology together with Poincare duality and Künneth formula for the cohomology of Cartesian product in terms of cohomologies of factors allows to deduce intersection numbers algebraically. Schubert cells define a basis for the homology of Grassmannian containing only even-dimensional generators.

   Grassmannians play a key role in twistor Grassmannian approach as auxiliary manifolds. In particular, the singularities of the integrand of the scattering amplitude defined as a multiple residue integral over $G(k,n)$ define a hierarchy of Schubert cells. The so called positive Grassmannian [B25] defines a subset of singularities appearing in the scattering amplitudes of $\mathcal{N} = 4$ SUSY. This hierarchy and its CP_2 counterpart are expected also in TGD framework.

   Remark: Schubert's vision might be relevant for the notion of conscious intelligence. Could problem solving involve the transformation of a problem to a simple critical problem, which is easy but for which some solutions can become degenerate? The transformation of general position for 4 lines to a pair of intersecting lines would be example of this. One can wonder whether quantum criticality could help problem solving by finding critical cases.

5. Moduli spaces of curves and varieties provide the most refined methods. Flag manifolds define basic examples of moduli spaces. Quantum cohomology represents even more refined conceptualization: the varieties (branes in M-theory terminology) are said to be connected or intersect if each of them has a common point with the same pseudo-holomorphic variety ("string world sheet"). Pseudo-holomorphy - which could have minimal surface property as counterpart - implies that the connecting 2-surface is not arbitrary.

   Quantum intersection for the "string world sheet" and "brane" is possible also when it is not stable classically (the co-dimension of brane is smaller than 2). Even in the case that it possible classically quantum intersection makes possible kind of "telepathic" quantum contact mediated by the "string world sheet" naturally involved with the description of quantum entanglement in TGD framework.

2.3.3 Gromow-Witten invariants

Gromow-Witten invariants repreent example of so called quantum invariants natural in string models and M-theory. They provide new invariants in algebraic and symplectic geometry.

Formal definition

Consider first the definition of Gromow-Witten (G-W) invariants (see `http://tinyurl.com/y9b5vbcw`). G-W invariants are rational number valued topological invariants useful in algebraic and symplectic geometry. These quantum invariants give information about these geometries not provided by classical invariants. Despite being rational numbers in the general case G-W invariants in some sense give the number of string world sheets connecting given branes.

1. One considers collection of n surfaces ("branes") with even dimensions in some symplectic manifold X of dimension $D = 2k$ (say Kähler manifold) and pseudo-holomorphic curves ("string world sheets") X^2, which have the property that they connect these n surfaces in the sense that they intersect the "branes" in the marked points x_i, $i = 1, .., n$.

   "Connect" does not reduce to intersection in topologically stable sense since connecting is possible also for branes with dimension smaller than $D-2$. One allows all surfaces that X^2 that intersects the n surfaces at marked points if they are pseudo-holomorphic even if the basic dimension rule is not satisfied. In 4-dimensional case this does not seem to have implications since partonic 2-surfaces satisfy automatically the dimension rule. The n branes intersect or touch in quantum sense: there is no concrete intersection but intersection with the mediation of "string world sheet".

2. Pseudo-holomorphy means that the Jacobian df of the embedding map $f : X^2 \rightarrow X$ commutes with the symplectic structures j *resp.* J of X^2 *resp.* X: i.e. one has $df(jT) = Jdf(T)$ for any tangent vector T at given point of X^2. For $X^2 = X = C$ this gives Cauchy-Riemann conditions.

   In the symplectic case X^2 is characterized topologically by its genus g and homology class A as surface of X. In algebraic geometry context the degree d of the polynomial defining X^2 replaces A. In TGD X^2 corresponds to string world sheet having also boundary. X^2 has also n marked points $x_1, ..., x_n$ corresponding to intersections with the n surfaces.

3. G-W invariant $GW_{g,n}^{X,A}$ gives the number of pseudo-holomorphic 2-surfaces X^2 connecting n given surfaces in X - each at single marked point. In TGD these surfaces would be partonic 2-surfaces and marked points would be carriers of sparticles.

The explicit definition of G-W invariant is rather hard to understand by a layman like me. I however try to express the basic idea on basis of Wikipedia definition (see `http://tinyurl.com/y9b5vbcw`). I apologize for my primitive understanding of higher algebraic geometry. The article of Vakil [L25] (see `http://tinyurl.com/ybobccub`) discusses the notion of G-W invariant in detail.

1. The situation is conformally invariant meaning that one considers only the conformal equivalence classes for the marked pseudo-holomorphic curves X^2 parameterized by the points of so called Deligne-Mumford moduli space $\overline{M}_{g,n}$ of curves of genus g with n marked points (see `http://tinyurl.com/yaq8n6dp`): note that these curves are just abstract objects without no embedding as surface to X assumed. $\overline{M}_{g,n}$ has *complex* dimension

   $$d_0 = 3(g-1) + n \ .$$

   n corresponds complex dimensions assignable to the marked points and $3(g-1)$ correspond to the complex moduli in absence of marked points. This space appears in TGD framework in the construction of elementary particle vacuum functionals [K21].

2. Since these curves must be represented as surfaces in X one must introduces the moduli space $\overline{M}_{g,n}(X, A)$ of their maps f to X with given homology equivalence class. The elements in this space are of form $(C, x_1, .., x_n, f)$ where C is one particular representative of A.

3. The complex dimension d of $\overline{M}_{g,n}(X, A)$ can be calculated. One has

   $$d = d_0 + c_1^X(A) + (g-1)k \ .$$

Here $c_1^X(A)$ is the first Chern class defining element of second cohomology of X evaluated for A. For Calabi-Yau manifolds one has $c_1 = 0$. The contribution $(g-1)k$ to the dimension vanishing for torus topology should have some simple explanation.

4. One defines so called evaluation map ev from $\overline{M}_{g,n}(X,A) \rightarrow Y$, $Y = \overline{M}_{g,n} \times X^n$ in terms of stabilization $st(C, x_1, ..., x_n) \in \overline{M}_{g,n}(X,A)$ of C (I understand that stabilization means that the automophism group of the stabilized surface defined by f is finite [A182] (see http://tinyurl.com/y8r44uhl). I am not quite sure what the finiteness of the automorphism group means. One might however think that conformal transformations must be in question. One has

$$ev(C, x_1, .., x_n, f) = (st(C, x_1, .., x_n), f(x_1), ...f(x_n)) \ .$$

Evaluation map assigns to the concrete realization of string world sheet with marked points the abstract curve $st(C, x_1, .., x_n)$ and points $(f(x_i), ..., f(x_n)) \in X^n$ possibly interpretable as positions $f(x_i)$ of n particles. One could say that one has many particle system with particles represented by surfaces of X_i of X connected by X^2 - string world sheet - mediating interaction between X_i via the intersection points.

5. Evaluation map takes the fundamental class of $\overline{M}_{g,n}(X,A)$ in $H_d(\overline{M}_{g,n}(X,A))$ to an element of homology group $H_d(Y)$. This homology equivalence class defines G-W invariant, which is rational valued in the general case.

6. One can make this more concrete by considering homology equivalence class β in $\overline{M}_{g,n}$ and homology equivalence classes α_i, $i = 1, ..., n$ represented by the surfaces X_i. The co-dimensions of these $n+1$ homology equivalence classes must sum up to d. The homologies of $\overline{M}_{g,n}$ and $Y = \overline{M}_{g,n} \times X^n$ induce homology of Y by Künneth formula (see http://tinyurl.com/yd9ttlfr) implying that Y has class of $H_d(Y)$ given by the product $\beta \cdot \alpha_1 ... \cdot \alpha_n$.

   One can identify the value of $GW_{g,n}^{X,A}$ for a given class $\beta \cdot \alpha_1 ... \cdot \alpha_n$ as the coefficients in its expansion as sum of all elements in $H_d(Y)$. This coefficient is the value of its intersection product of $GW_{g,n}^{X,A}$ with the product $\beta \cdot \alpha 1 ... \cdot \alpha_n$ and gives element of $H_0(Q)$, which is rational number.

7. There are two non-classical features. Classically intersection must be topologically stable. This would require α_i to have codimension 2 but all even co-dimensions are allowed. That the value for the number of connecting string world sheets is rational number does not conform with the classical geometric intuition. The Wikipedia explanation is that the orbifold singularities for the space $\overline{M}_{g,n}(X,A)$ of stable maps are responsible for rational number.

Application to string theory

Topological string theories give a physical realization of this picture. Here the review article *Instantons, Topological Strings, and Enumerative Geometry* of Szabo [A182] (see http://tinyurl.com/y8r44uhl) is very helpful.

1. In M-theory framework and for topological string models of type A and B the physical interpretation for the varieties associated with α_i would be as branes of various dimensions needed to satisfy Dirichlet boundary conditions for strings.

2. In topological string theories one considers sigma model with target space X, which can be rather general. The symplectic or complex structure of X is however essential. X is forced to be 3-D (in complex sense) Calabi-Yau manifold by consistency of quantum theory. Interestingly, the super twistor space $CP(3|4)$ is super Calabi-Yau manifold although CP_3 is not and must therefore have trivial first Chern class c_1 appearing in the formula for the dimension d above. I must admit that I do not understand why this is the case.

   Closed topological strings have no marked points and one has $n = 0$. Open topological strings world sheets meet n branes at points x_i, where they satisfy Dirichlet boundary conditions. Branes an be identified as even-dimensional Lagrangian sub-manifolds with vanishing induced symplectic form.

3. For topological closed string theories of type A one considers holomorphically imbedded curves in X characterized by genus g and homology class A: one speaks of world sheet instantons. $A = \sum n_i S_i$ is sum over the generating classes S_i with integer coefficients. For given g and A one has analog of product of non-interacting systems at temperatures $1/t_i$ assignable to the homology classes S_i with energies identifiable as n_i. One can assign Boltzmann weight labelled by (g, A) as $Q^\beta = \prod_i Q_i^{n_i}$, $Q_i = exp(-t_i)$.

   One can construct partition function for the entire system as sum over Boltzmann weights with degeneracy factors telling the number of world sheet instantons with given (g, A). One can calculate free energy as sum $\sum N_{g,\beta} Q^\beta$ over contributions labelled by (g, A). The coefficients $N_{g,\beta}$ count the rational valued degeneracies of the world sheet instantons of given type and reduce to G-W invariants $GW_{g,0}^{X,A}$.

   Remark: If one allows powers of a root $e^{-1/n}$, $t = n$, in the extension of rationals or replace e^{-t} with p^n, partition functions make sense also in the p-adic context.

4. For topological open string theories of type A one has also branes. Homology equivalence classes are relative to the brane configuration. The coefficients $N_{g,\beta}$ are given by $GW_{g,n}^{X,A}$ for a given configuration of branes: the above described general formulas correspond to these.

5. For topological string theories of type B, string world sheets reduce to single point and thus correspond to constant solutions to the field equations of sigma model. Quantum intersection reduces to ordinary intersection and one has $x_1 = x_2 ... = x_n$. G-W invariants involve only classical cohomology and give for $n = 2$ the number of common points for two surfaces in X with dimension d_1 and $d_2 = n - d$. The duality between topological string theories of type A and B related to the mirror symmetry supports the idea that one could generalize the calculation of these invariants in theories B to theories A. It is not clear whether this option as any analog in TGD.

The so called Witten conjecture (see `http://tinyurl.com/yccahv3q`) proved by Kontsevich states that the partition function in one formulation of stringy quantum gravity and having as coefficients of free energy G-W invariants of the target space is same as the partition function in second formulation and expressible in terms of so called tau function associated with KdV hierarchy. This leads to non-trivial identities. Witten conjecture actually follows from the invariance of partition function with respect to half Virasoro algebra and Virasoro conjecture (see `http://tinyurl.com/y7xcc9hm`) stating just this generalizes Witten's conjecture.

2.3.4 Riemann-Roch theorem

Riemann-Roch theorem (RR) is also part of enumerative geometry albeit more abstract. Instead of counting of numbers of points, one counts dimensions of various function spaces associated with Riemann surfaces. RR provides information about the dimensions for the spaces of meromorphic functions and 1-forms with prescribed zeros and poles.

Basic notions

Riemann surface is the basic notion. Riemann surface is orientable is characterized by its genus g and number of holes/punctures in it. Riemann surface can also possess marked points, which seem to be equivalent with punctures. The moduli space of these complex curves is parameterized by a moduli space $\overline{M}_{g,n}$ of curves of genus g with n marked points (see `http://tinyurl.com/yaq8n6dp`) (see `http://tinyurl.com/yaq8n6dp`).

Dolbeault cohomology (see `http://tinyurl.com/y7cvs5sx`) generalizes the notion of differential form so that it has has well-defined degrees with respect to complex coordinates and their conjugates: one can write in general complex manifold this kind of form as

$$\omega = \omega_{i_1 i_2 .. i_n, j_1 j_2 ... j_n} dz^{i_1} \wedge dz^{i_2} ... dz^{i_n} d\overline{z}^{j_1} \wedge d\overline{z}^{j_2} ... dz^{j_n} \quad .$$

The ordinary exterior derivative d is replaced with its holomorphic counterpart ∂ and its conjugate. One can construct the counterparts of cohomology groups (Hodge theory) $H^{p,q} = H^{q,p}$. Betti numbers as numbers $h_{i,j}$ defining the dimensions of the cohomology groups forms of degrees i and

j with respect to dz^i and $d\bar{z}^j$. One can define the holomorphic Euler's characteristic as $\chi_C = h_{0,0} - h_{01} = 1 - g$ whereas orinary Euler characteristic is $\chi_R = h_{0,0} - (h_{01} + h_{10}) + h_{1,1} = 2(1 - g)$.

One considers meromorphic functions having poles and zeros as the only singularities (points at which the map does not preserve angles): rational functions provide the basic example. Riemann zeta provides example of meromorphic function not reducing to rational function. Holomorphic functions have only zeros and entire functions have neither zeros nor poles. If analytic functions on Riemann surfaces can be interpreted as maps of compact Riemann surface to itself rather than to complex plane, meromorphy reduces to holomorphy since the point ∞ belongs to the Riemann surface.

The elements of free group of divisors are defined as formal sums of integers associated with the points P of Riemann surface. Divisors $D = \sum_P n(P)$, where (P) is integer, are analogous to integer valued "wave functions" on Riemann surface. The number of points with $n(P) \neq 0$ is countable. The degree of divisor is obtained as the ordinary sum $deg(D)$ of the integers defining the divisor.

Although divisors can be seen as purely formal objects, they are in practice associated to both meromorphic functions and 1-forms. The divisor of a meromorphic function is known as principal divisor. Meromorphic functions and 1-forms differing by a multiplication with meromorphic function are regarded as linearly equivalent - in other words, one can add to a given divisor a divisor of a meromorphic function without changing its equivalence class. It can be shown that all divisors associated with meromorphic 1-forms linearly equivalent and one can talk about canonical divisor. Note that $deg(D)$ is linear invariant since the degree of globally meromorphic function is zero.

The motivation for the divisors is following. Consider the space of meromorphic functions h with the property that the degrees of poles associated with the poles of these functions are not higher than given integers $n(P)$. In other words, one has $\langle h(P)\rangle + D(P) \geq 0$ for all points P ($\langle h\rangle$ is the divisor of h). For $D(P) > 0$ the pole has degree not higher than $D(P)$. For non-positive $D(P)$ the function has zero of order $D(P)$ at least.

Formulation of RR theorem

With these prerequisites it is possibly to formulate RR (for Wikipedia article see `http://tinyurl.com/mdmbcx6`). The Wikipedia article is somewhat confusing and a more precise description of RR can be found in the article "Riemann-Roch theorem" by Vera Talovikova [A191] (see `http://tinyurl.com/ktww7ks`).

Let $l(D)$ be the dimension of the space of meromorphic functions with principal divisor D or 1-forms linearly equivalent with canonical divisor K. Then the equality

$$l(D) - l(K - D) = deg(D) - g + 1 \tag{2.3.1}$$

is true for both meromorphic functions and canonical divisors. For $D = K$ one obtains using $l(0) = 1$

$$l(K) = deg(K) - g + 2 \tag{2.3.2}$$

giving the dimension of the space of canonical divisors. $l(K) > 0$ in general is not in conflict with the fact that canonical divisors are linearly equivalent. $deg(K) = 2g - 2$ in the above formula gives $l(K) = g$.

$l(K) = 0$ for $g = 0$ case looks strange: one should actually make notational distinction between dimensions of spaces of meromorphic functions and one-forms (this is done in the article of Talivakova). The explanation is that $l(K)$ here is not the dimension of the space of canonical 1-forms but that of the holomorphic functions with the divisor of K. The canonical form K for the sphere has second order pole at ∞ so that one cannot have meromorphic forms holomorphic outside P.

Riemann's inequality

$$l(D) \geq deg(D) - g + 1 \tag{2.3.3}$$

follows from $l(K-D) \geq 0$, which can be seen as a correction term to the formula

$$l(D) = deg(D) - g + 1 \ . \tag{2.3.4}$$

In what sense this is true, becomes clear from what follows.

The dimension of the space meromorphic functions corresponding to given divisor

The simplest divisor associated with meromorphic function involves only one point. Multiplying a function, which is non-vanishing and finite at P by $(z - z(P))^{-n}$ gives a pole of order n (zero has negative order in this sense). One is interested on the dimension $l(nP)$ of the space nP of meromorphic functions and RR allows to deduce information about $l(nP)$. One is interested on the behavior of $l(nP)$ as function of genus g of Riemann surface (more general situation would allow also punctures). For $n = 0$ one has entire function without poles and zeros. Only constant function is possible: $l(0) = 1$.

First some general observations. K has degree $deg(K) = 2g - 2$, which gives $l(K) = g$. For $n = deg(D) > deg(K) = 2g - 2$ the correction term vanishes since $deg(K - D)$ becomes negative, and one has $l(D) = deg(D) - g + 1$. This gives $l(n = 2g - 1) = g$. Therefore $n \in \{2g - 1, 2g, ...\}$ corresponds to $l(nP) \in \{g, g + 1, ...\}$. $n < 2g - 2$ corresponds to $l(nP) = 1$. What about the range $n \in \{2, ..., 2g - 2\}$? Note that $2g - 2$ is the negative of the Euler character of Riemann surface.

1. $g = 0$ case. K on sphere. dz canonical 1-form on Riemann sphere covered by two complex coordinate patches. $z \rightarrow w = 1/z$ relates the coordinates. There is second order pole at infinity ($dw = -dz/z^2$). One has therefore $deg(K) = -2$ for sphere in accordance with the general formula $deg(K) = 2g - 2$. The formula $l(nP) = deg(D) + 1$ holds for all n and there is no correction term now. One as $l(nP) = n + 1$.

2. $g = 1$ case.

   One has $deg(K) = 2g - 2 = 0$ for torus reflecting the fact that the canonical form $\omega = dz$ has no poles or zeros (torus is obtained by identifying the cells of a periodic lattice in complex plane). Correction term vanishes since it would have negative degree for all n and one has $l(nP) \in \{1, 1, 2, 3, ...\}$.

3. $g = 2$ case.

   For $n = deg(D) \geq 2 \times 2 - 1 = 3$ gives $l(D) = n - 1$ giving for $n \geq 3$ $l(nP) \in \{2, 3, ...\}$. What about $n = g = 2$? For generic points one has $l(2) = 1$. There are 6 points at which one has $l(D) = 2$ so that there is additional meromorphic function having pole of order 2 at this kind of point. These points are fixed points under Z_2 defining hyper-ellipticity. Note that $g \leq 2$ Riemann surfaces are always hyper-elliptic in the sense that it allows Z_2 as conformal symmetry (see `http://tinyurl.com/y9sdu4o3`).

   One has therefore $l(nP) \in \{1, 1, 1, 2, ..\}$ for a generic point and $l(nP) \in \{1, 1, 2, 2......\}$ for 6 points fixed under Z_2. An interesting question is whether this phenomenon could have physical interpretation in TGD framework.

4. $g > 2$ case.

   For $g > 2$. $l(nP)$ in the range $\{2, 2g - 2\}$ can depend on point and even on the conformal moduli. There are more special points in which correction term differs from that in the generic case. $g = 3$ illustrates the situation. $n \in \{1, 1, 1, 1, 1, 2, ...\}$ is obtained for a generic point. At special points and for $n < 3$ there are also other options for $l(nP)$. Also the dependence of $l(nP)$ on moduli emerges for $g \geq 3$. The natural guess layman is that these points are fixed points of conformal symmetries. Also now hyper-elliptic surfaces allowing projective Z_2 covering are special. In the general case hyper-ellipticity is not possible.

In TGD framework Weierstrass points(see `http://tinyurl.com/y9wehsml`) are of special interest physically.

1. My layman guess is that special points known as Weierstrass points (see `http://tinyurl.com/y9wehsml`) correspond to singularities for projective coverings for which conformal symmetries permute the sheets of the covering. Several points coincide for the covering since a sub-group of conformal symmetries would act trivially on the Weierstrass point.

Note that for $g > 2$ Z_2 covering is not possible except for hyper-elliptic surfaces, and one can wonder whether this relates to the experimental absence fo $g > 2$ fermion families [K21]. Second interesting point is that elementary particles indeed correspond to double sheeted structures from the condition that monopole fluxes flow along closed flux tubes (there are no free magnetic monopoles).

2. There is an obvious analogy with the coverings associated with the cognitive representation defined by the points of space-time surface with coordinates in an extension of rationals [L42, L36] [L41]. Fixed points for a sub-group of Galois group generate singularities at which sheets touch each other. These singular points are physically the most interesting and could carry sparticles. The action of discrete conformal groups restricted to cognitive representation could be represented as the action of Galois group on points of cognitive representation. Cognitive representation would indeed represent!

   Remarkably, if the tangent spaces are not parallel for the touching sheets, these points are mapped to several points in H in M^8-H correspondence. If this picture is correct, the hyper-elliptic symmetry Z_2 of genera $g \leq 2$ could give rise to this kind of exceptional singularities for $g \geq 2$.

   What is worrying that there are two views about twistorial amplitudes. One view relying on the notion of octonionic super-space M^8 [L36] is analogous to that of SUSYs: sparticles can be seen as completely local composites of fermions. Second view relies on embedding space $M^4 \times CP_2$ [K79] and on the identification sparticles as non-local many-fermion states at partonic 2-surfaces. These two views could be actually equivalent by $M^8 - H$ duality.

3. When these singular points are present at partonic 2-surfaces at boundaries of CD and at vertices, the topology of partonic 2-surface is in well-defined sense between g and $g + 1$ external particles: one has criticality. The polynomials representing external particles indeed satisfy criticality conditions guaranteeing associativity or co-associativity (quantum criticality of TGD Universe is the basic postulate of quantum TGD). At partonic orbits the touching pieces of partonic 2-surface could separate (g) or fuse ($g + 1$). Could this topological mixing give rise to CKM mixing of fermions [K21, K50, K61]?

RR for algebraic varieties and bundles

RR can be generalized to algebraic varieties (see `http://tinyurl.com/y9asz4qg`). In complex case the real dimension is four so that this generalization is interesting from TGD point of view and will be considered later. The generalization involves rather advanced mathematics such as the notion of sheaf (see `http://tinyurl.com/nudhxo6`). Zeros and poles appearing in the divisor are for complex surfaces replaced with 2-D varieties so that the generalization is far from trivial.

The following is brief summary based on Wikipedia article.

1. Genus g is replaced with algebraic genus and $deg(D)$ plus correction term is replaced with the intersection number (see `http://tinyurl.com/y7dcffb6`) for D and $D - K$, where K is the canonical divisor associated with 2-forms, which is also unique apart from linear equivalence Points of divisor are replaced with 2-varieties.

2. The generalization to complex surfaces (with real dimension equal to 4) reads as

$$\chi(D) = \chi(0) + \frac{1}{2}D \cdot (D - K) \quad . \tag{2.3.5}$$

   $\chi(D)$ is holomorphic Euler characteristic associated with the divisor. $\chi(0)$ is defined as $\chi(0) = h_{0,0} - h_{0,1} + h_{0,2}$, where $h_{i,j}$ are Betti numbers for holomorphic forms. '$\cdot$' denotes intersection product in cohomology made possibly by Poincare duality. K is canonical two-form which is a section of determinant bundle having unique divisor (their is linear equivalence due to the possibility to multiply with meromorphic function.

   One has $\chi(0) = 1 + p_a$, where p_a is arithmetic genus. Noether's formula gives

$$\chi(0) = \frac{c_1^2 + c_2}{12} = \frac{K \cdot K + e}{12} \quad . \tag{2.3.6}$$

c_1^2 is Chern number and $e = c_2$ is topological Euler characteristic.

Clearly the information given by $\chi(D)$ is about Dolbeault homology. For comparison note that RR for curves states $l(D) - l(K - D) = \chi(D) = \chi(0) + deg(D)$.

RR can be also generalized so that it applies to vector bundles. Ordinary RR can be interpreted as applying to a bundle for which the fiber is point. This requires the notion of the inverse bundle defined as a bundle with the property that its direct sum (Whitney sum) with the bundle itself is trivial bundle. One ends up with various characteristic classes, which represent homologically non-trivial forms in the base spaces characterizing the bundle. For instance, the generalizations of RR give information about the dimensions of the spaces of sections of the vector bundle.

Atyiah-Singer index theorem (see `http://tinyurl.com/k6daqco`) deals with so called elliptic operators in compact manifolds and represents a generalization important in recent theoretical physics, in particular gauge theories and string models. The theorem relates analytical index - typically characterizing the dimension for the spectrum of solutions of elliptic operator to a topological index. Elliptic operator is assigned with small perturbations for a given solution of field equations. Perturbations of a given solution of say Yang-Mills equations is a representative example.

2.4 Does $M^8 - H$ duality allow to use the machinery of algebraic geometry?

The machinery of algebraic geometry is extremely powerful. In particular, the number theoretical universality of algebraic geometry implies that same equations make sense for all number fields: this is just what adelic physics [L42] [L41] demands. Therefore it would be extremely nice if one could somehow use this machinery also in TGD framework as it is used in string models. How this could be achieved? There are several guide lines.

1. Twistor lift of TGD [K91, K35, K11, K79] is now a rather well-established idea although a lot of work remains to be done with the details. Twistors were originally introduced in order to be able to use this machinery and involves complexification of Minkowski space M^4 to M^4_c as an auxiliary tool. Complexification in sufficiently general sense seems to be a necessary auxiliary tool but it cannot be a trick (like Wick rotation) but something fundamental and here complexification at the level of M^8 is suggestive. In the sequel I will used M^4 for M^4_c and M^8 for M^8_c unless it is necessary to emphasize that M^8_c is in question. The essential point is that the Euclidian metric is complexified and it reduces to a real metric in various sub-spaces defining besides Euclidian space also Minkowski spaces with varying signature when the complex coordinates are real or imaginary.

2. If $M^8 - H$ duality holds true, one can solve field equations in $M^8 = M^4 \times E^8$ by assuming that either the tangent space or normal space of the space-time surface X^4 is associative (quaternionic) at each point and contains preferred M^2 in its tangent space. M^2 could depend on x but $M^2(x)$:s should integrate to a 2-surface. This allows to map space-time surface M^8 to a surface in $M^4 \times CP_2$ since tangent spaces are parameterized by points of CP_2 and CP_2 takes the role of moduli space. The image of tangent space as point of CP_2 is same irrespective of whether one has quaternions or complexified quaternions (H_c).

   It came a surprise that associativity/co-associativity is possible only if the space-time surface is critical in the sense that some gradients of 8 complex components of the octonionic polynomial P vanish without posing them as additional conditions reducing thus the dimension of the space-time surface. This occurs when the coefficients of P satisfy additional conditions. One obtains associative/co-associative space-time regions and regions without either property and they correspond nicely to two solution types for the twistor lift of Kähler action.

3. Contrary to the original expectations, $M^4 \subset M^8_c$ must be identified as co-associative (co-quaternionic) subspace so that E^4 is the associative/quaternionic sub-space. This allows to have light-cone boundary as the counterpart of point-like singularity in ordinary algebraic geometry and also allows to understand the emergence of CDs and ZEO.

Remark: A useful convention to be used in the sequel. $RE(o)$ and $IM(o)$ denote the real and imaginary parts of the octionion in the decomposition $o = RE(o) + IM(o)I_4$ and $Re(o)$ and $Im(o)$ its real number valued and purely imaginary parts in the usual decomposition.

The problems related to the signature of M^4 have been a longstanding head-ache of M^8 duality.

1. The intuitive vision has been that the problems can be solved by replacing M^8 with its complexification M^8_c identifiable as complexified octonions o. This requires introduction of imaginary unit i commuting with the octonionic units $E^k \leftrightarrow (1, I_1, ..., I_7)$. The real octonionic components are thus replaced with ordinary complex numbers $z_i = x_i + iy_i$.
2. Importantly, complex conjugation $o \to \overline{o}$ changes only the sign of I_i but *not!* that of i so that the octonionic inner product $(o_1, o_2) = o_1\overline{o}_2 = o_1^k o_2^l \delta_{k,l}$ becomes complex valued. Norm is equal to $O\overline{O} = \sum_i z_i^2$. Both norm and inner product are in general complex valued and real valued only in sub-spaces in which octonionic coordinates are real or imaginary. Sub-spaces have all possible signatures of metric. These sub-spaces are not closed under multiplication and this has been an obstacle in the earlier attempts based on the notion of octonion analyticity. This argument applies also to quaternions and one obtains signatures $(1, 1, 1, 1)$, $(1, 1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, -1, -1, -1)$. Why just the usual Minkowskian signature $(1, -1, -1, -1)$ is physical, should be understood.

   The convention consistent with that used in TGD corresponds to a negative length squared for space-like vectors and positive for time-like vectors. This gives $m = (o^0, io^1, ..., io^7)$ with real o^k. The projection $M^8_c \to M^8$ defines the projection of $X^4_c \subset M^8_c$ to $X^4 \subset M^8$ serving as the pre-image of $X^4 \subset M^8$ in $M^8 - H$ correspondence.
3. o is not field anymore as is clear from the fact that $1/o = \overline{o}/o\overline{o}$ is formally infinite in Minkowskian sub-spaces, when octonion defines a light-like vector. o (and H_c) remains however a ring so that sum and products are well-defined but division can lead to problems unless one stays inside 7+7-dimensional light-cone with $Re(o\overline{o}) > 0$ ($Re(q\overline{q}) > 0$).

   Although the number field structure is lost, one can still define polynomials needed to define algebraic varieties by requiring their simultaneous vanishing and rational functions make sense inside the light-cone. Also rational functions can be defined but poles are replaced with light-cones in Minkowskian section. Algebraic geometry would thus be forced by the complexification of octonions. This looks to me highly non-trivial! The extension of zeros and poles to light-cones making propagation possible could be a good reason for why Minkowskian signature is physical. Interestingly, the allowed octonionic momenta are light-like quaternions [K79].
4. An interesting question is whether ZEO and the emergence of CDs relates to the failure of field property. It seems now clear that CDs must be assigned even with elementary particles. I have asked whether they could form an analog for the covering of manifold by coordinate patches (in TGD inspired theory of consciousness CDs would be correlates for perceptive fields for conscious entities assignable to CDs [L44]). These observations encourage to ask whether the tips of CD should correspond to a pair formed by two poles/two zeros or by pole and zero assignable to positive and negative energy states.

   It turns out that the space-time surfaces in the interior of CD would naturally correspond to non-associative surfaces and only their 3-D boundaries would have associative 4-D tangent spaces allowing mapping to H by M^8-duality, which is enough by holography.
5. The relationship between light-like 3-surface bounding Minkowskian and Euclidian space-time regions and light-like boundaries of CDs is interesting. Could also the partonic orbits be understood a singularities of octonionic polynomials with $IM(P) = RE(P) = 0$?

2.4.1 What does one really mean with $M^8 - H$ duality?

The original proposal was that M^8 duality should map the associative tangent/normal planes of associative/co-associative space-time surface containing preferred M^2, call it M^2_0, to CP_2: the map read as $(m, e) \in M^4 \times E^4 \to (m, s) \in M^4 \times CP_2$. Eventually it became clear that the choice of M^2 can depend on position with $M^2(x)$ forming an integrable distribution to CP_2: this would define what I have called Hamilton-Jacobi structures [K8]. String like objects have minimal surface as

M^4 projection for almost any general coordinate invariant action, and internal consistency requires that $M^2(x)$ integrate to a minimal surface. The details are however not understood well enough.

1. M^4 coordinate would correspond simply to projection to a fixed M_0^4 in the decomposition $M^8 = M_0^4 \times E_0^4$. One can however challenge this interpretation. How M_0^4 is chosen? Is it possible to choose it uniquely? Could also M^4 coordinates represent moduli analogous to CP_2 coordinates? What about ZEO?

   There is an elegant general manner to formulate the choice of M_0^4 at the level of M^8. The complexified quaternionic sub-spaces of M_c^8 (M^8) are parameterized by moduli space defining the quaternionic moduli. The moduli space in question is CP_2. The choice of M_0^4 corresponds to fixing of the quaternionic moduli by fixing a point of CP_2.

   Warning: Note that one should be very careful in distinguishing between quaternionic as sub-spaces of M^8 and as the tangent space M^8 of given point of M^8.

2. One can ask whether there could be a connection with ZEO, where CDs play a key role. Indeed, the complexified Minkowski inner product means that the complexified octonions (quaternions) inside M_c^8 (M_c^4) have inverse only inside 7-D (4-D) complexified light-cone and this would motivate the restriction of space-time surfaces inside future or past light-cone or both but not yet force CD.

   If one allows rational functions and even meromorphic functions of octonionic or quaternionic variable, one could consider the possibility of restricting the space-time surface defined as their zeros to a maximally sized region containing no poles.

3. Consider complexified quaternions H_c. Poles $(q\overline{q})^{-n}$, $n \geq 1$ would correspond M^4 light-cone boundaries since $q\overline{q} = 0$ at them. Also zeros $q\overline{q} = 0$, for $n \geq 1$ correspond to light-like boundaries. Could one have two poles with with time-like distance defining CD or a pair of pole and zero?

   There is also a possible connection with the notion of infinite primes [K84]. The notion of infinite prime leads to the proposal that rationals defined as ratios of infinite integers but having unit real norm (and also p-adic norms) could correspond pairs of positive and negative energy states with identical total quantum numbers and located at opposite boundaries of CD. Infinite rationals can be mapped to rational functions. Could positive energy states correspond to the numerators with zeros at second boundary of CD and negative energy states to denominators with zeros at opposite boundary of CD?

Is the choice of the pair (M_0^2, M_0^4) consistent with the properties of known extremals in H

It should be made clear that the notion of associativity/co-associativity (quaternionicity/co-quaternionicity) of the tangent/normal space need not make sense at the level of H. I shall however study this working hypothesis in the sequel.

The choice of the pair (M_0^2, M_0^4) means choosing preferred co-commutative (commutative) sub-space M_0^2 of M^8 defining a subspace of fixed co-quaternionic (quaternionic) sub-space $M_0^4 \subset M^8$.

Remark: M^4 should indeed be the co-associative/co-quaternionic subspace of M^8 if the argument about emergence of CDs is accepted and if $M^8 - H$ correspondence maps associative to associative and co-associative to co-associative.

M_0^4 in turn contains preferred M_0^2 defining co-commutative (hyper-complex) structure. Both M_0^2 and M_0^4 are needed in order to label the choice by CP_2 point (that is as a point of Grassmannian).

Is the projection to a fixed factor $M_0^4 \subset M_0^4 \times E^4$ as a choice of co-quaternionic moduli consistent with what we know about the extremals of twistor lift of Kähler action in H? How could one fix M_0^4 from the data about the extremal in H? One can make similar equations about the choice of M_0^2 as a fixed co-complex moduli characterized by a unit quaternion. Note that this choice is expected to relate closely to the twistor structure and Kähler structure.

It is best to check the proposal for the known extremals in H [K8]. Consider first CP_2 type extremals for which M^4 projection is a piece of light-like geodesic.

1. The CP_2 projection for the image of $X^4 \subset M^8$ differs from single point only if the tangent space isomorphic to M^4 and parameterized by CP_2 point varies. Consider CP_2 type extremals for the twistor lift of Kähler action [**?**]n H having light-like geodesic as M^4 projection as an example. The light-like geodesic defines a light-like vector in the tangent space of CP_2 type extremal. This light-like vector together with its dual spans fixed M^2, which however does not belong to the tangent space so that associative surface would not be in question.

   What about co-associativity or associativity (the latter is favored by above argument)? This property should hold true for the pre-image of CP_2 type extremal in M^8 but I am not able to say anything about this. It is questionable to require this property at the level H but one can of course look what it would give.

   What about associativity for CP_2 tangent space? The normal space of CP_2 type extremal is 3-D (!) since the only the light-like tangent vector of the geodesic and 2 vectors orthogonal to it are orthogonal to CP_2 tangent vectors. For Euclidian signature this would mean that tangent space is 5-D and cannot be associative but now the tangent space is 4-D. Can one still say that tangent space is associative. The co-associativity of the tangent space makes sense trivially. Can one conclude that CP_2 is co-associative.

   The associativity for CP_2 tangent space might make sense since the tangent space is 4-D. The light-like vector k defines M_0^2. The associativity conditions involving two tangent space vectors of CP_2 and the light-like vector k contracted with the corresponding octonion components. The contributions from the components of k to the associator should cancel each other. Since one can change the relative sign of the components of k, this mechanism does not seem to work for all components. Hence associativity cannot hold true. Neither does M_0^2 belong to the normal space since k and its dual are not orthogonal.

   Could one conclude that CP_2 type extremal is co-associative in accordance with the original belief thanks to the light-like projection to M^4? This does not conform with what the singularity considerations for the octonionic polynomials would suggest. Or is it simply not correct to try to apply associativity at the level of H. Or does $M^8 - H$ correspondence map associative tangent spaces to co-associative ones?

2. The normal space M^4 of CP_2 type extremal have all orientations characterized by its CP_2 projection. The normal space must contain the M_0^2 determined by the tangent of the light-like geodesic and this is indeed the case. Note that CP_2 type extremals cannot have entire CP_2 as CP_2 projection: they necessarily have hole at either end, which would be naturally be at the boundary of CD.

   CP_2 type extremals seem to be consistent with $M^8 - H$ correspondence. It however seems that one cannot fix the choice of M_0^4 uniquely in terms of the properties of the extremal. There is a moduli space for M_0^4:s defined by CP_2 and obviously codes for moduli for quaternion structures in octonionic space. The distributions of $M^2(x)$ (minimal surfaces) would code for quaternion structures (decomposition of octonionic coordinates to quaternionic coordinates in turn decomposing to pairs of complex coordinates).

Consider next the associativity condition for cosmic strings in $X^2 \times Y^2 \subset M^4 \times CP_2$. Now CP_2 projection is 2-D complex surfaces and M^4 projection is minimal surface. Situation is clearly associative. How unique the choice of M_0^4 is now?

1. Now $M^2(x)$ depends on position but $M^2(x)$:s define an integrable distribution defining string orbit X^2 as a minimal surface. M_0^4 must contain all surfaces $M^2(x)$, which would fix M_0^4 to a high degree for complex enough cosmic strings.

2. Each point of X^2 corresponds to the same partonic surface $Y^2 \subset CP_2$ labelling the tangent spaces for its pre-image in M^8. All the tangent surfaces $M^2(x) \times E^2(y)$ for $X^2 \times Y^2 \subset M^8$ share only $M^2(x) \subset M_0^4$. M_0^4 must contain all tangent spaces $M^2(x)$ and the inverse image of $Y^2 \subset CP_2$ must belong to the orthogonal complement E^4 of M_0^4. This is completely analogous with the condition $X^2 = X^2 \times Y^2 \subset M^4 \times CP_2$.

   Consider a decomposition $M^8 = M_0^4 \times E^4$, $M_0^4 = M_0^2 \times E_0^2$. If the inverse image of Y^2 at point x belongs to E^4, the M_0^4 projection belongs to M_0^4 also in M^8. If this does not pose any condition on the tangent spaces assignable to the points of Y^2 defining points of CP_2, there are no problems. What could happen that the tangent spaces assignable to Y^2 could force the projection of the inverse image of Y^2 to intersect M_0^4.

One should also understand massless extremals (MEs). How to choose M_0^4 in this case?

1. MEs are given as zeros of arbitrary functions of CP_2 coordinates and 2 M^4 coordinates u and v representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant and define $M_0^4 = M_0^2 \times E_0^2$ decomposition everywhere so that M_0^4 is uniquely defined. Same applies also when the directions are not constant. In the general case light-like direction would define the local tangent plane of string world sheet and local polarization plane. Since the dimension of M^4 projection is 4 there seems to be no problems involved.
2. Tangent plane of X^4 is parameterized by CP_2 coordinates depending on 2 coordinates u and v. The surface $X^4 \subset M^8$ must be graph for a map $M_0^4 \to E^4$ so that a 2-parameter deformation of M_0^4 as tangent plane is in question. The distribution of tangent planes of $X^4 \subset M^8$ is 2-D as is also the CP_2 projection in H.

To sum up, the original vision about the associativity properties of the known extremals at level of H survives. On the other hand, CDs emerge if M^4 corresponds to the co-associative part of O. Does this mean that $M^8 - H$ correspondence maps associative to co-associative by multiplying the quaternionic tangent space in M^8 by I_4 to get that in H and vice versa or that the notions of associative and co-associative do not make sense at the level of H? This does not affect the correspondence since the same CP_2 point parametrizes both associative tangent space and its complement.

Space-time surfaces as co-dimension 4 algebraic varieties defined by the vanishing of real or imaginary part of octonionic polynomial?

If the theory intended to be a theory of everything, the solution ansatz for the field equations defining space-time surfaces should be ambitious enough: nothing less than a general solution of field equations should be in question.

1. One cannot exclude the possibility that all analytic functions of complexified octonionic variable with real Taylor or even Laurent coefficients. These would would a commutative and associative algebra. Space-time surfaces would be identified as their zero loci. This option is however number theoretically attractive and can also leads to problems with adelic physics. Since Taylor series at rational point need not anymore give a rational value.
2. Polynomials of complexified octonion variable o with real coefficients define the simplest option but also rational functions formed as ratios of this kind of polynomials must be considered. Polynomials form a non-associative ring allowing sum, product, and functional decomposition as basic operations. If the coefficients o_n of polynomials are complex numbers $o_n = a_n + ib_n$, a_n, b_n real, where i refers to the commutative imaginary unit complexifying the octonions, the ring is associative. It is essential to allow only powers o^n (or $(o - o_0))^n$ with $o_0 = a_0 + ib_0$, a_0, b_0 real numbers). Physically this means that a preferred time axis is fixed. This time axis could connect the tips of CD in ZEO.

   One can write

$$P(o) = \sum_k p_k o^k \equiv RE(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) + IM(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) \times I_4 \quad , p_k \ \text{real} \quad , \tag{2.4.1}$$

   where the notations

$$o = q_1 + q_2 I_4 \quad , \quad q_i = z_i^1 + z_i^2 I_2 \quad , \quad \overline{q}_i = z_i^1 - z_i^2 I_2 \quad , z_i^j = x_i^j + i y_i^j \tag{2.4.2}$$

   Note that the conjugation does *not* change the sign of i. Due to the non-commutativity of octonions P^i as functions of quaternions are in general *not* analytic in the sense that they would be polynomials of q_i with real coefficients! They are however analytic functions of z_i. The real and imaginary parts of x_i^j correspond to Minkowskian and Euclidian signatures.

In adelic physics coefficients o_n of the octonionic polynomials define WCW coordinates and should be rational numbers or rationals in the extension of rationals defining the adele. The polynomials form an associative algebra since associativity holds for powers o^n multiplied by real number. Thus complex analyticity crucial in algebraic geometry would be a key element of adelic physics.

3. If the preferred extremals correspond to the associative algebra formed by these polynomials, one could construct a completely general solution of the field equations as zero loci of their real or imaginary parts and build up of new solutions using algebra operation sum, product, and functional decomposition. One could identify space-time regions as associative or co-associative algebraic varieties in terms of these polynomials and they would form an algebra.

The motivation for this dream comes from 2-D electrostatics, where conducting surfaces correspond to curves at which the real part u or imaginary part v of analytic function $w = f(z) = u + iv$ vanishes. In electrostatics curves form families with curves orthogonal to each other locally and the map $w = u + iv \rightarrow v - iu$ defines a duality in which curves of constant potential and the curves defining their normal vectors are mapped to each other.

1. The generalization to the recent situation would be vanishing of the imaginary part $IM(P)$ or real part $RE(P)$ of the octonionic polynomial, where real and imaginary parts are defined via $o = q_c^1 + q_c^2 I_4$. One can consider also the possibility that imaginary or real part has constant value c are restricted to be rational so that one can regard the constant value set also as zero set for a polynomial with constant shift. Note that the rationals could be also complexified by addition of i. One would have

$$RE(P)(z_i^k) \quad \text{or} \quad IM(P)(z_i^k) = c \ , \quad c = c_0 \ \text{rational} \ . \tag{2.4.3}$$

c_0 must be real. These two options should correspond to the situations in which tangent space or normal space is associative (associativity/co-associativity). Complexified space-time surfaces X_c^4 corresponding to different constant values c of imaginary or real part (with respect to i) would define foliations of M_c^8 by locally orthogonal 4-dimensional surfaces in M_c^8 such that normal space for surface X_c^4 would be tangent space for its co-surface.

CDs and ZEO emerges naturally if the $IM(o)$ corresponds to co-quaternionic part of octonion.

2. It must be noticed that one has moduli space for the quaternionic structures even when M_0^4 is fixed. The simplest choices of complexified quaternionic space $H_c = M_{c,0}^4$ containing preferred complex plane $M_{c,0}^2$ and its orthogonal complement are parameterized by CP_2. More complex choices are characterized by the choice of distribution of $M^2(x)$ integrable to (presumably minimal) 2-surface in M^4. Also the choice of the origin matters as found and one has preferred coordinates. Also the 8-D Lorentz boosts give rise to further quaternionic moduli. The physically interesting question concerns the interpretation of space-time surfaces with different moduli. For instance, under which conditions they can interact?

The proposal has several extremely nice features.

1. Single real valued polynomial of real coordinate extended to octonionic polynomial and fixed by real coefficients in extension of rationals would determine space-time surfaces.

2. The notion of analyticity needed in concrete equations is just the ordinary complex analyticity forced by the octonionic complexification: there is no need for the application to have left- or right quaternion analyticity since quaternionic derivatives are not needed. Algebraically one has the most obvious guess for the counterpart of real analyticity for polynomials generalized to octonionic framework and there is no need for the quaternionic generalization of Cauchy-Riemann equations [A195, A164] [A195, A164] (`http://tinyurl.com/yb8l34b5`) plagued by the problems with the definition of differentiation in non-commutative and non-associative context. There would be no problems with non-associativity and non-commutativity thanks to commutativity of complex coordinates with octonionic units.

3. The vanishing of the real or imaginary part gives rise to 4 conditions for 8 complex coordinates z_1^k and z_2^k allowing to solve z_2^k as algebraic functions $z_2^k = f^k(z_1^l)$ or vice versa. The conditions would reduce to algebraic geometry in complex co-dimension $d_c = 4$ and all methods and concepts of algebraic geometry can be used! Algebraic geometry would become part of TGD as it is part of M-theory too.

2.4.2 Is the associativity of tangent-/normal spaces really achieved?

The non-trivial challenge is to prove that the tangent/normal spaces are indeed associative for the two options. The surfaces X_c^4 are indeed associative/co-associative if one considers the *internal* geometry since points are in M_c^4 or its orthogonal complement.

One should however prove that X_c^4 are also associative *as sub-manifolds* of O and therefore have quaternionic tangent space or normal space at each point parameterized by a point of CP_2 in the case that tangent space containing position dependent M_c^2, which integrate to what might be called a 2-D complexified string world sheet inside M_c^4.

1. The first thing to notice that associativity and quaternionicity need not be identical concepts. Any surface with complex dimension $d < 4$ in O is associative and any surface with dimension $d > 4$ co-associative. Quaternionic and co-quaternionic surfaces are 4-D by definition. One can of course ask whether one should consider a generalization of brane hierarchy of M-theory also in TGD context and allow associativity in its most general sense. In fact, the study of singularity of o^2 shows that 6 and 5-dimensional surfaces are allowed for which the only interpretation would be as co-associative spaces. This exceptional situation is due to the additional symmetries increasing the dimension of the zero locus.
2. One has clearly quaternionicity at the level of o obtained by putting $Y = 0$ and at the level of the tangent space for the resulting surface. The tangent space should be quaternionic. The Jacobian of the map defined by P is such that it takes fixed quaternionic subspace $H_c \to M_{0,c}^4$ of O to a quaternionic tangent space of X^4. The Jacobian applied to the vectors of H_c gives the octonionic tangent vectors and they should span a quaternionic sub-space.
3. The notion of quaternionic surface is rigorous. $M^8 - H$ correspondence could be actually interpreted in terms of the construction of quaternionic surface in M^8. One has 4-D integrable distribution of quaternionic planes in O with given quaternion structure labelled by points of CP_2 and has representation at the level of H as space-time surface and should be preferred extremals. These quaternion planes should integrate to a slicing by 4-surfaces and their duals. One obtains this slicing by fixing the values 4 of the suitably defined octonionic coordinates P^i, $i = 1,..,8$, to a real constants depending on the surface of the slicing. This gives a space-time surfaces for which tangent space-spaces or normal spaces are quaternionic.

   The first guess for these coordinates P^i be as real or imaginary parts of real polynomials $P(o)$. But how to prove and understand this?

   Could the following argument be more than wishful thinking?

1. In complex case an analytic function $w(z) = u + iv$ of $z = x + iy$ mediates a map between complex planes Z and W. One can interpret the imaginary unit appearing in w locally as a tangent vector along $u = constant$ coordinate line.
2. One can interpret also octonionic polynomials with real coefficients as mediating a map from octonionic plane O to second octonionic plane, call if W. The decomposition $P = P^{1)} + P^{2)} I_4$ would have interpretation in terms of coordinates of W with coordinate lines representing quaternions and co-quaternions.
3. This would suggests that the quaternionic coordinate lines in W can be identified as coordinate curves in O - that space-time surfaces - which are quaternionic/co-quaternionic surfaces for $P^1 = constant/P^2 = constant$ lines. One would have a representation of the same thing in two spaces, and if sameness includes also quaternionicity/co-quaternionicity as attributes, then also associativity and co-associativity should hold true.

The most reasonable approach is based on generality. Associativity/quaternionicity means a slicing of octonion space by orthogonal quaternionic and co-quaternionic 4-D surfaces defined by constant value surfaces of octonionic polynomial with real coefficients. This slicing should make sense also for quaternions: one should have a slicing by complex and co-complex (commutative/co-commutative) surfaces and in TGD string world sheets and partonic 2-surfaces assignable to Hamilton-Jacobi structure would define this kind of slicing. In the case of complex numbers one has a slicing in terms of constant value curves for real and imaginary parts of analytic function and Cauchy-Riemann equations should define the property and co-property. The first guess that the tangent space of the curve is real or imaginary is wrong.

Could associativity and commutativity conditions be seen as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "Whatever it is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of division algebras by assuming only algebra property by using Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the structure already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

Complex curves in real plane cannot have real tangent space

Going from octonions to quaternions to complex numbers, could constant value curves of real and imaginary parts of ordinary analytic function in complex plane make sense? The curves $u = 0$ and $v = 0$ of functions $f(z) = u + iv$, $z = x + iy$ define a slicing of plane by orthogonal curves completely analogous to its octonionic and quaternionic variants. Can one say that the tangent vectors for these curves are real/imaginary? For $u = 0$ these curves have tangent $\partial_x u + i\partial_y u$, which is not real unless one has $f(z) = k(x + iy)$, k real.

Reality condition is clearly too strong. In fact, it is the well-ordering of the points of the 1-dimensional curve, which is the property in question and lost for complex numbers and regained at $u = 0$ and $v = 0$ curves. The reasonable interpretation is in terms of hierarchy of conditions multilinear in the gradients of coordinates proposed above and linear Cauchy-Riemann conditions is the only option in the case of complex plane. What is special in this curves that the tangent vectors define flows which by Cauchy-Riemann conditions are divergenceless and irrotational locally.

Pessimistic would conclude that since the conjecture fails except for linear polynomials in complex case, it fails also in the case of quaternions and octonions. For quaternionic polynomial q^2 the conditions are however satisfied and it turns out that the resulting conditions make sense also in the general case. Optimistic would argue that reality condition is not analogous to commutativity and associativity so that this example tells nothing. Less enthusiastic optimist might admit that the reality condition is a natural generalization to complex case but that the conjecture might be true only for a restricted set of polynomials - in complex case of for $f(z) = kz$, k real. In quaternionic and octonionic case but hopefully for a larger set of polynomials with real coefficients, maybe even all polynomials with real coefficients.

Associativity and commmutativity conditions as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial

maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "whatever-it-is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of number fields by assuming only algebra property by adding additional imaginary units as done in Cayley-Hamilton construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the algebra already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions? Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

One would have also a nice physical interpretation: in the case of quaternions one would have "quaternionic conformal invariance" as conformal invariances inside string world sheets and partonic 2-surfaces in a nice agreement with basic vision about TGD. At the level of octonions would have "quaternionic conformal invariance" inside space-time surfaces and their duals. What selects the preferred commutative or co-commutative surfaces is of course an interesting problem. Is a gauge choice in question? Are these surfaces selected by some special property such as singular character? Or does one have wave function in the set of these surfaces for a given space-time surface?

Could quaternionic polynomials define complex and co-complex surfaces in H_c?

What about complex and co-complex (commutative/co-commutative) surfaces in the space of quaternions? One would have a slicing of the quaternionic space by pairs of complex and co-complex surfaces and would have natural identification as quaternion/Hamilton-Jacobi structure and relate to the decomposition of space-time to string world sheets and partonic 2-surfaces. Now the condition of associativity would be replaced with commutativity.

1. In the quaternionic case the tangent vectors of the 2-D complex sub-variety would be commuting. Can this be the case for the zero loci real polynomials $P(q)$ with $IM(P) = 0$ or $RE(P) = 0$? In this case the commutativity condition is that the tangent vectors have imaginary parts (as quaternions) proportional to each other but can have different real parts. The vanishing of cross product is the condition now and involves only two vectors whereas associativity condition involves 3 vectors and is more difficult.

2. The tangent vectors of a commutative 2-surface commute: $[t^1, t^2] = 0$. The commutator reduces to the vanishing of the cross product for the imaginary parts:

$$Im(t^1) \times Im(t^2) = 0 \quad . \tag{2.4.4}$$

3. Expressing z_1^i as a function of z_2^k and using (z_1^i, z_2^k) as coordinates in quaternionic space, the tangent vectors in quaternionic spaces can be written in terms of partial derivatives $\partial z_1^{1)}/\partial z_2^{k)}$ as

$$t_k^i = (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta_k^i) \quad , \tag{2.4.5}$$

Here the first part corresponds to $RE(t^i)$ as quaternion and second part to $IM(t^i)$ as quaternion.

The condition that the vectors are parallel implies

$$\frac{\partial z_1^{1)}}{\partial z_2^{k)}} = 0 \ . \tag{2.4.6}$$

At the commutative 2-surface X^2 $z_1^{1)}$ is constant and $z_1^{2)}$ is a function of $z_2^{1)}$ and $z_2^{2)}$. One would have a graph of a function $z_1^{2)} = f_2(z_2^{k)})$ at X^2 but not elsewhere. One could regard $z_1^{1)}$ as an extremum of a function $z_1^{1)} = f_1(z_2^{k)})$.

How to interpret this result?

1. In the generic case this condition eliminates 1 dimension so that 2-D surface would reduce to a 1-D curve.
2. If one poses constraints on the coefficients of $P(q)$ analogous to the conditions forcing the potential function for say cusp catastrophe to have degenerate extrema at the boundaries of the catastrophe one can get 2-D solution. For these values of parameters the conditions would be equivalent with $RE(P) = 0$ or $IM(P) = 0$ conditions.

   The vanishing of the gradient of z_1^1 would indeed correspond in the case of cups catastrophe to the condition for the co-incidence of two roots of the behavior variable x as extremum of potential function $V(x, a, b)$ fixing the control variable a as function of b.

   This would pose constraints on the coefficients of P not all polynomials would be allowed. This kind of conditions would realize the idea of quantum criticality of TGD at the level of quaternion polynomials. This option is attractive if realizable also at the level of octonion polynomials. This turns out to be the case.
3. One would thus have two kinds of commutative/co-commutative surfaces. The generic 1-D surfaces and 2-D ones which are commutative/commutative and critical and assignable to string world sheets and partonic 2-surfaces. 1-D surfaces would correspond to fermion lines at the orbits of partonic 2-surfaces appearing in the twistor amplitudes in the interaction regions defined by CDS. 2-D surfaces would correspond to the orbits of fermionic strings connecting point-like fermions at their ends and serving as correlates for bound state entanglement for external fermions arriving into CD. This picture would allow also to understand why just some string world sheets and partonic 2-surfaces are selected.

The simplest manner to kill the proposal is to look for $P = q^2$ and $RE(P(q^2)) = 0$ surface. In this case this condition is indeed satisfied. One has

$$\begin{aligned}
&RE(P) = X^{1)} + X^{2)} I_1 \ , \\
&X^{1)} = (z_1^{1)})^2 - (z_1^{2)})^2 + (z_2^{1)})^2 - (z_2^{2)})^2 \ , \quad X^{2)} = 2 z_1^{1)} z_1^{2)} I_1 \ , \\
&IM(P) = Y^{1)} + Y^{2)} I_1 \ , \\
&Y^{1)} = (z_2^{1)} + \overline{z_2^{1)}}) z_1^{1)} \ , \qquad Y^{2)} = (z_2^{2)} + \overline{z_2^{2)}}) z_1^{2)}
\end{aligned} \tag{2.4.7}$$

$X^{2)} = 0$ gives $z_1^{1)} z_1^{2)} = 0$ so that one has either $z_1^{1)} = 0$ or $z_1^{2)} = 0$. $X^{1)} = 0$ gives for $z_1^{1)} = 0$ $z_1^{2)} = \pm\sqrt{(z_2^{1)})^2 + (z_2^{2)})^2}$.

The partial derivative $\partial z_1^{1)}/\partial z_2^{k)}$ is from implicit function theorem - following from the vanishing of the differential $d(RE(P))$ along the surface - proportional $\partial X^{1)}/\partial z_2^{k)}$, but vanishes as required.

Clearly, the quaternionic variant of the proposal survives in the simplest case its simplest test. 2-D character of the surface would be due to the criticality of q^2 making it possible to satisfy the conditions without the reduction of dimension.

Explicit form of associativity/quaternionicity conditions

Consider now the explicit conditions for associativity in the octonionic case.

1. One should calculate the octonionic tangent (normal) vectors t^i for $X = 0$ in associative ($Y = 0$ in co-associative case) and show that there associators $Ass(t^i, t^j, t^k)$ vanish for all possible or all possible combinations i, j, k. In other words, one that one has

$$Ass(t^i, t^j, t^k) = 0 \ , \quad i, j, k \in \{1, .., 4\} \ , \quad Ass(a, b, c) \equiv (ab)c - a(bc) \ . \tag{2.4.8}$$

   One can cast the condition to simpler from by expressing t^i as octonionic vectors $t^i_k E^k$:

$$Ass(E^a, E^b, E^b) = \equiv f^{abcd} E_d \ , \quad a, b, c, d \in \{1, .., 7\} \ ,$$
$$f^{abcd} = \epsilon^{abe} \epsilon_e^{\ cd} - \epsilon^{aed} \epsilon^{bc}_{\ \ e} = 2\epsilon^{abe} \epsilon_e^{\ cd} \ . \tag{2.4.9}$$

   The permutation symbols for a given triplet i, j, k are structures constants for quaternionic inner product and completely antisymmetric (see `http://tinyurl.com/p42tqsq`).. $\epsilon_{ijk} = 1$ is true for the seven triplets $123, 145, 176, 246, 257, 347, 365$ defining quaternionic sub-spaces with 1-D intersections. The anti-associativity condition $(E_i E_j) E_k = -(E_i E_j) E_k$ holds true so that one has obtains the simpler expression for f^{ijks} having values ± 2.
   Using this representation $Ass(t^i, t^j, t^k)$ reduces to 7 conditions for each triplet:

$$t^i_r t^j_s t^k_t f^{rstu} = 0 \ , \quad i, j, k \in \{1, .., 4\} \ , \quad r, s, t, u \in \{1, .., 7\} \ . \tag{2.4.10}$$

2. If the vanishing condition $X = 0$ or $Y = 0$ is crucial for associativity then every polynomial is its own case to be studied separately and a general principle behind associativity should be identified: the proposal is as a non-linear generalization of Cauchy-Riemann conditions. As the following little calculation shows, the vanishing condition indeed appears as one calculates partial derivatives $\partial z_1^{k)} / \partial z_2^{l)}$ in the expression for the tangent vectors of the surface deduced from the vanishing gradient of X or Y.

3. I have proposed the octonionic polynomial ansatz already earlier but failed to prove that it gives associative tangent or normal spaces. Besides the intuitive geometric argument I failed to notice that the complex 8-D tangent vectors in coordinates $z_1^{k)}$ or $z_2^{k)}$ for complexified space-time surface and coordinates $(z_1^{k)}, z_2^{k)})$ for o have components

$$\frac{\partial o^i}{\partial z^1_k} \leftrightarrow (\delta^i_k, \frac{\partial z_2^{i)}}{\partial z_1^{k)}})$$

$$\text{or}$$

$$(\frac{\partial o^i}{\partial z^2_k}) \leftrightarrow (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta^i_k) \ . \tag{2.4.11}$$

   These vectors correspond to complexified octonions O_i given by

$$\delta^i_k E^k + \frac{\partial z_2^{i)}}{\partial z_1^{k)}} E^k E_4 \ , \tag{2.4.12}$$

   where the unit octonions are given by $(E_0, E_1, E_2, E_3) = (1, I_1, I_2, I_3)$ and $(E_5, E_5, E_7, E_8) = (1, I_1, I_2, I_3) E_4$. The vanishing of the associators stating that one has

4. One can calculate the partial derivatives $\frac{\partial z_i^k}{\partial z_j^k}$ explicitly without solving the equations or the complex valued quaternionic components of $RE(P) \equiv X = 0$ or $IM(P) \equiv Y = 0$ (note that X and Y have for complex components labelled by X^i and Y^i respectively.

$$Y^i(z_1^{k)}, z_2^{l)}) = c \in R \ , \quad i = 1, ..., 4 \ , \quad \text{associativity} \ ,$$

or

$$X^i(z_1^{k)}, z_2^{l)}) = c \in R \ , \quad i = 1, ..., 4 \ , \quad \text{co-associativity} \ . \tag{2.4.13}$$

explicitly and check whether associativity holds true. The derivatives can be deduced from the constancy of Y or X.

5. For instance, if one has $z_2^{k)}$ as function of $z_1^{k)}$, one obtains in the associative case

$$RE(Y)^i{}_k + IM(Y)^i{}_k \frac{\partial z_2^{r)}}{\partial z_1^{k)}} = 0$$

$$RE(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_1^{k)}} \ , \qquad IM(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_2^{k)}} \ . \tag{2.4.14}$$

In co-associative case one must consider normal vectors expressible in terms of Y^i so that X is replaced with Y in these equations.
This allows to solve the partial derivatives needed in associator conditions

$$\frac{\partial z_2^{i)}}{\partial z_1^{k)}} = \left[Im(Y)^{-1}\right]^i{}_r Re(Y)^r{}_k \ . \tag{2.4.15}$$

6. The vanishing conditions for the associators are however multilinear and one can multiply each factor by the matrix $IM(P)$ without affecting the condition so that $IM(P)^{-1}$ disappears and one obtains the conditions for vectors

$$T_r^i T_s^j T_t^k f^{rstu} = 0 \ , \qquad i,j,k \in \{1,..,4\} \ , \quad r,s,t,u \in \{1,..,7\} \ ,$$

$$T^i = IM(Y)^i{}_k E^k - RE(Y)^i{}_k E^k E_4 \ . \tag{2.4.16}$$

This form of conditions is computationally much more convenient.

How to solve these equations?

1. The antisymmetry of f^{rstu} with respect to the first two indices r,s leads one to ask whether one could have

$$T_r^i T_s^j T_t^k = 0 \tag{2.4.17}$$

for the 7 quaternionic triplets. This is guaranteed if one has either $RE(Y)^i{}_k = \partial Y^i/\partial z_1^k = 0$ (coquaternionic part of T^i) or $IM(Y)^i{}_k = \partial Y^i/\partial z_2^k = 0$ (co-quaternionic part of T^i) for *one* member in each triplet.
The study of the structure constants listed above shows that indices 1,2,3 are contained in all 7 triplets. Same holds for the indices appearing in any quaternionic triplet. Hence it is enough to require that three gradients $RE(Y)^i k = 0$ or $IM(Y)^i{}_k = 0$ $k \in \{1,2,3\}$ vanish. This condition is obviously too strong since already single vanishing condition reduces the dimension of space-time variety to 3 in the generic case and it becomes trivially associative.

Octonionic automorphism group G_2 gives additional basis with their own quaternion triplets and the general condition would be that 3 partial derivatives vanish for a triplet obtained from the basic triplet $\{1,2,3\}$ by G_2 transformation. It is not quite clear to me whether the G_2 transformation can depend on position on space-time surface.

2. As noticed, the vanishing of all triplets is an un-necessarily strong condition. Already the vanishing of single gradient $RE(Y)^i{}_k$ or $IM(Y)^i{}_k$ reduces the dimension of the surface from 4 to 3 in the generic case. If one accepts that the dimension of associative surface is lower than 4 then single criticality condition is enough to obtain 3-D surface.

   In the generic case associativity holds true only at the 4-D tangent spaces of 3-surfaces at the ends of CD (at light-like partonic orbits it holds true trivially in 4-D) and that the twistor lift of Kähler action determines the space-time surfaces in their interior.

   In this case one can map only the boundaries of space-time surface by $M^8 - H$ duality to H. The criticality at these 3-surfaces dictates the boundary conditions and provides a solution to infinite number of conditions stating the vanishing of SSA Noether charges at space-like boundaries. These space-time regions would correspond to the regions of space-time surfaces inside CDs identifiable as interaction regions, where Kähler action and volume term couple and dynamics depends on coupling constants.

   The mappability of M^8 dynamics to H dynamics in all space-time regions does not look feasible: the dynamics of octonionic polynomials involves no coupling constants whereas twistor lift of Kähler action involves couplings parameters. The dynamics would be non-associative in the geometric sense in the interior of CDs. Notice that also conformal field theories involve slight breaking of associativity and that octonions break associativity only slightly ($a(bc) = -(ab)c$ for octonionic imaginary units). I have discussed the breaking of associativity from TGD viewpoint in [K44] .

3. Twistor lift of Kähler action allows also space-time regions, which are minimal surfaces [L19] and for which the coupling between Kähler action and volume term vanishes. Preferred extremal property reduces to the existence of Hamilton-Jacobi structure as image of the quaternionic structure at the level of M^8. The dynamics is universal as also critical dynamics and independent of coupling constants so that $M^8 - H$ duality makes sense for it. External particles arriving into CD via its boundaries would correspond to critical 4-surfaces: I have discussed their interpretation from the perspective of physics and biology in [L20].

4. One should be able to produce associativity without the reduction of dimension. One can indeed hope of obtaining 4-D associative surfaces by posing conditions on the coefficients of the polynomial P by requiring that one $RE(Y)^i_k$ or $IM(Y)^i_k$, $i = i_1$ -call it just X_1 - should vanish so that Y^i would be critical as function of z_1^k or z_2^k.

   At $X_1 = 0$ would have degenerate zero at the 4-surface. The decomposition of X_1 to a product of monomial factors with root in extension of rationals would have one or more factors appearing at least twice. The associative 4-surfaces would be ramified. Also the physically interesting p-adic primes are conjectured to be ramified in the sense that their decomposition to primes of extension of rationals contains powers of primes of extension. The ramification of the monomial factors is nothing but ramification for polynomials primes in field of rationals in terms of polynomial primes in its extension.

   This could lead to vanishing of say one triplet while keeping $D = 4$. This need not however give rise to associativity in which case also second $RE(Y)^i_i$ or $IM(Y)^i_k$, $i = i_2$, call it X_2, should vanish. The maximal number of X_i would be $n_{max} = 3$. The natural condition consistent with quantum criticality of TGD Universe would be that the variety is associative but maximally quantum critical and has therefore dimension $D = 3$ or $D = 4$. Stronger condition allows only $D = 4$.

   These $n \leq 3$ additional conditions make the space-time surface analogous to a catastrophe with $n \leq 3$ behavior variables in Thom's classification of 7 elementary catastrophes with less than 11 control variables [A129]. Thom's theory does not apply now since it has only one potential function $V(x)$ (now $n \leq 3$ corresponding to the critical coordinates Y^i!) as a function of behaviour variables and control variables). Also the number of non-vanishing coefficients in the polynomial having values in an extension of rationals and acting as control variables is unlimited. In quaternionic case the number of potential functions is indeed 1 but the number of control variables unlimited.

5. One should be able to understand the $D = 3$ associative objects - say light-like 3-surfaces or 3-surfaces at the boundaries of CD - as 3-surfaces along which 4-D associative (co-associative) and non-associative (non-co-associative) surfaces are glued together.

   Consider a product P of polynomials allowing 3-D surfaces as necessarily associative zero loci to which a small interaction polynomial vanishing at the boundaries of CD (proportional to o^n, $n > 1$) is added. Could P allow 4-D surface as a zero locus of real or imaginary part and containing the light-like 3-surfaces thanks to the presence of additional parameters coming from the interaction polynomial. Can one say that this small interaction polynomial would generate 4-D space-time in some sense? 4-D associative space-time regions would naturally emerge from the increasing algebraic complexity both via the increase of the degree of the polynomial and the increase of the dimension of the extension of rationals making it easier to satisfy the criticality conditions!

   There are two regions to be considered: the interior and exterior of CD. Could associativity/co-associativity be possible outside CD but not inside CD so that one would indeed have free external particles entering to the non-associative interaction region. Why associativity conditions would be more difficult to satisfy inside CD? Certainly the space-likeness of M^4 points with respect to the preferred origin of M^8 in this region should be crucial since Minkowski norm appears in the expressions of $RE(P)$ and $IM(P)$.

Do the calculations for the associative case generalize to the co-associative case?

1. Suppose that one has possibly associative surface having $RE(P) = 0$. One would have $IM(P) = 0$ for dual space-time surface defining locally normal space of $RE(P) = 0$ surface. This would transform the co-associativity conditions to associativity conditions and the preceding arguments should go through essentially as such.

   Associative and co-associative surfaces would meet at singularity $RE(P) = IM(P) = 0$, which need not be point in Minkowskian signature (see $P = o^2$ example in the Appendix) and can be even 4-D! This raises the possibility that the associative and co-associative surfaces defined by $RE(P) = 0$ and $IM(P) = 0$ meet along 3-D light-like orbits partonic surfaces or 3-D ends of space-time surfaces at the ends of CD.

2. If $D = 3$ for associative surfaces are allowed besides $D = 4$ as boundaries of 4-surfaces, one can ask why not allow $D = 5$ for co-associative surfaces. It seems that they do not have a reasonable interpretation as a surface at which co-associative and non-co-associative 4-D space-time regions would meet. Or could they in some sense be geometric "co-boundaries" of 4-surfaces like branes in M-theory serve as co-boundaries of strings? Could this mean that 4-D space-time-surface is boundary of 5-D co-associative surface defining a TGD variant of brane with strings world sheets replaced with 4-D space-time surfaces?

   What came as a surprise that $P = o^2$ allows 5-D and 6-D surfaces as zero loci of $RE(P)$ or $IM(P)$ as shown in Appendix. The vanishing of the entire o^2 gives 4-D interior or exterior of CD forced also by associativity/co-associativity and thus maximal quantum criticality. This is very probably due to the special properties of o^2 as polynomial: in the generic case the zero loci should be 4-D.

This discussion can be repeated for complex/co-complex surfaces inside space-time surfaces associated with fermionic dynamics.

1. Associativity condition does not force string world sheets and partonic 2-surfaces but they could naturally correspond to commutative or co-commutative varieties inside associative/co-associative varieties.

   In the generic case commutativity/co-commutativity allows only 1-D curves - naturally light-like fermionic world lines at the boundaries of partonic orbits and representing interacting point-like fermions inside CDs and used in the construction of twistor amplitudes [K35, K79]. There is coupling between Kähler part and volume parts of modified Dirac action inside CDs so that coupling constants are visible in the spinor dynamics and in dynamics of string world sheet.

2. At criticality one obtains 2-D commutative/co-commutative surfaces necessarily associated with external particles quantum critical in 4-D sense and allowing quaternionic structure.

String world sheets would serve as correlates for bound state entanglement between fermions at their ends. Criticality condition would select string world sheets and partonic 2-surfacs from the slicing of space-time surface provided by quaternionic structure (having Hamilton-Jacobi structure as H-counterpart).

If associativity holds true and fixed M^2_c is contained in the tangent space of space-time surface, one can map the M^4 projection of the space-time surface to a surface in $M^4 \times CP_2$ so that the quaternionic tangent space at given point is mapped to CP_2 point. One obtains 4-D surface in $H = M^4 \times CP_2$.

1. The condition that fixed M^2_c belongs to the tangent space of X^4_c is true in the sense that the coordinates $z_2^{k)}$ do not depend on $z_1^{1)}$ and $z_1^{2)}$ defining the coordinates of M^2_c. It is not clear whether this condition can be satisfied in the general case: octonionic polynomials are expected to imply this dependence un-avoidably.

   The more general condition allows M^2_c to depend on position but assumes that M^2_c:s associated with different points integrate to a family 2-D surfaces defining a family of complexified string world sheets. In the similar manner the orthogonal complements E^2_c of M^2_c would integrate to a family of partonic 2-surfaces. At each point these two tangent spaces and their real projections would define a decomposition analogous to that define by light-like momentum vector and polarization vector orthogonal to it. This decomposition would define decomposition of quaternionic sub-spaces to complexified complex subspace and its co-complex normal space. The decomposition would correspond to Hamilton-Jacobi structure proposed to be central aspect of extremals [K8].

2. What is nice that this decomposition of M^4_c (M^4) would (and of course should!) follow automatically from the octonionic decomposition. This decomposition is lower-dimensional analog to that of the complexified octonionic space induced by level sets of real octonionic polymials but at lower level and extremely natural due to the inclusion hierarchy of classical number fields. Also M^2_c could have similar decomposition orthogonal complex curves by the value sets of polynomials. The hierarchy of grids means the realization of the coordinate grid consisting of quaternionic, complex, and real curves for complexified coordinates o^k and their quaternionic and complex variants and is accompanied by corresponding real grids obtained by projecting to M^4 and mapping to CP_2.

   Thus these decompositions would be obtained from the octonionic polynomial decomposing it to real quaternionic and imaginary quaternionic parts first to get a grid of space-time surfaces as constant value sets of either real or imaginary part, doing the same for the non-constant quaternionic part of the octonionic polynomial to get similar grid of complexified 2-surfaces, and repeating this for the complexified complex octonionic part.

Unfortunately, I do not have computer power to check the associativity directly by symbolic calculation. I hope that the reader could perform the numerical calculations in non-trivial cases to this!

General view about solutions to $RE(P) = 0$ and $IM(P) = 0$ conditions

The first challenge is to understand at general level the nature of $RE(P) = 0$ and $IM(P) = 0$ conditions. Appendix shows explicitly for $P(o) = o^2$ that Minkowski signature gives rise to unexpected phenomena. In the following these phenomena are shown to be completely general but not quite what one obtains for $P(o) = o^2$ having double root at origin.

1. Consider first the octonionic polynomials $P(o)$ satisfying $P(0) = 0$ restricted to the light-like boundary δM^8_+ assignable to 8-D CD, where the octonionic norm of o vanishes.

   (a) $P(o)$ reduces along each light-ray of δM^8_+ to the same real valued polynomial $P(t)$ of a real variable t apart from a multiplicative unit $E = (1 + in)/2$ satisfying $E^2 = E$. Here n is purely octonion-imaginary unit vector defining the direction of the light-ray.
   $IM(P) = 0$ corresponds to quaterniocity. If the E^4 ($M^8 = M^4 \times E^4$) projection is vanishing, there is no additional condition. 4-D light-cones $M^4_\pm$ are obtained as solutions of $IM(P) = 0$. Note that $M^4_\pm$ can correspond to any quaternionic subspace.

If the light-like ray has a non-vanishing projection to E^4, one must have $P(t) = 0$. The solutions form a collection of 6-spheres labelled by the roots t_n of $P(t) = 0$. 6-spheres are not associative.

(b) $RE(PE) = 0$ corresponding to co-quaternionicity leads to $P(t) = 0$ always and gives a collection of 6-spheres.

2. Suppose now that $P(t)$ is shifted to $P_1(t) = P(t) - c$, c a real number. Also now $M^4_\pm$ is obtained as solutions to $IM(P) = 0$. For $RE(P) = 0$ one obtains two conditions $P(t) = 0$ and $P(t-c) = 0$. The common roots define a subset of 6-spheres which for special values of c is not empty.

The above discussion was limited to δM^8_+ and light-likeness of its points played a central role. What about the interior of 8-D CD?

1. The natural expectation is that in the interior of CD one obtains a 4-D variety X^4. For $IM(P) = 0$ the outcome would be union of X^4 with M^4_+ and the set of 6-spheres for $IM(P) = 0$. 4-D variety would intersect M^4_+ in a discrete set of points and the 6-spheres along 2-D varieties X^2. The higher the degree of P, the larger the number of 6-spheres and these 2-varieties.
2. For $RE(P) = 0$ X^4 would intersect the union of 6-spheres along 2-D varieties. What comes in mind that these 2-varieties correspond in H to partonic 2-surfaces defining light-like 3-surfaces at which the induced metric is degenerate.
3. One can consider also the situation in the complement of 8-D CD which corresponds to the complement of 4-D CD. One expects that $RE(P) = 0$ condition is replaced with $IM(P) = 0$ condition in the complement and $RE(P) = IM(P) = 0$ holds true at the boundary of 4-D CD.

6-spheres and 4-D empty light-cones are special solutions of the conditions and clearly analogs of branes. Should one make the (reluctant-to-me) conclusion that they might be relevant for TGD at the level of M^8.

1. Could M^4_+ (or CDs as 4-D objects) and 6-spheres integrate the space-time varieties inside different 4-D CDs to single connected structure with space-time varieties glued to the 6-spheres along 2-surfaces X^2 perhaps identifiable as pre-images of partonic 2-surfaces and maybe string world sheets? Could the interactions between space-time varieties X^4_i assignable with different CDs be describable by regarding 6-spheres as bridges between X^4_i having only a discrete set of common points. Could one say that X^2_i interact via the 6-sphere somehow. Note however that 6-spheres are not dynamical.
2. One can also have Poincare transforms of 8-D CDs. Could the description of their interactions involve 4-D intersections of corresponding 6-spheres?
3. 6-spheres in $IM(P) = 0$ case do not have image under $M^8 - H$ correspondence. This does not seem to be possible for $RE(P) = 0$ either: it is not possible to map the 2-D normal space to a unique CP_2 point since there is 2-D continuum of quaternionic sub-spaces containing it.

2.4.3 $M^8 - H$ duality: objections and challenges

In the following I try to recall all objections against the reduction of classical physics to octonionic algebraic geometry and against the notion of $M^8 - H$ duality and also invent some new counter arguments and challenges.

Can on really assume distribution of $M^2(x)$?

Hamilton-Jacobi structure means that $M^2(x)$ depends on position and $M^2(x)$ should define an integrable distribution integrating to a 2-D surface. For cosmic string extremals this surface would be minimal surface so that the term "string world sheet" is appropriate. There are objections.

1. It seems that the coefficients of octonionic polynomials cannot contain information about string world sheet, and the only possible choice seems to be that string world sheets and partonic 2-surfaces parallel to it assigned with integrable distribution of orthogonal complements

$E^2(x)$ should be coded by quaternionic moduli. It should be possible to define quaternionic coordinates q_i decomposing to pairs of complex coordinates to each choice of $M^2(x) \times E^2(x)$ decomposition of given M_0^4. Octonionic coordinates would be given by $o = q_1 + q_2 I_4$ where q_i are associated with the same quaternionic moduli. The choice of Hamilton-Jacobi structure would not be ad hoc procedure anymore but part of the definition of solutions of field equations at the level of M^8.

2. It would be very nice if the quaternionic structure could be induced from a fixed structure defined for M_c^8 once the choice of curvilinear M^4 coordinates is made. Since Hamiltoni-Jacobi structure [K8] involves a choice of generalized Kähler form for M^4 and since quaternionic structure means that there is full S^2 of Kähler structures determined by quaternionic imaginary units (ordinary Kähler form for sub-space $E^8 \subset M_c^8$) the natural proposal is that Hamilton-Jacobi structures is determined by a particular local choice of the Kähler form for M^4 involving fixing of quaternionic imaginary unit fixing $M^2(x) \subset M_0^4$ identifiable as point of S^2. This might relate closely also to the fixing of twistor structure, which indeed involves also self-dual Kähler form and a similar choice.

3. One can argue that it is not completely clear whether massless extremals (MEs) [K8] allow a general Hamilton-Jacobi structure. It is certainly true that if the light-like direction and orthogonal polarization direction are constant, MEs exist. It is clear that if the form of field equations is preserved and thus reduces to contractions of various tensors with second fundamental form one obtains only contractions of light-like vector with itself or polarization vector and these contractions vanish. For spatially varying directions one could argue that light-like direction codes for a direction of light-like momentum and that problems with local conservation laws expressed by field equations might emerge.

Can one assign to the tangent plane of $X^4 \subset M^8$ a unique CP_2 point when M^2 depends on position

One should show that the choice $s(x) \in CP_2$ for a given distribution of $M^2(x) \subset M^4(x)$ is unique in order to realize the $M^8 - H$ correspondence as a map $M^8 \to H$. It would be even better if one had an analytic formula for $s(x)$ using tangent space-data for $X^4 \subset H$.

1. If $M^2(x) = M_0^2$ holds true but the tangent space $M^4(x)$ depends on position, the assignment of CP_2 point $s(x)$ to the tangent space of $X^4 \subset M^8$ is trivial. When $M^4(x)$ is not constant, the situation is not so easy.

2. The space $M^2(x) \subset M^4(x)$ satisfies also the constraint $M^2(x) \subset M_0^4$ since quaternionic moduli are fixed. To avoid confusion notice that $M^4(x)$ denotes tangent space of X^4 and is different from M_0^4 fixing the quaternionic moduli.

3. $M^2(x)$ determines the local complex subspace and its completion to quaternionic tangent space $M^4(x)$ determines a point $s(x)$ of CP_2. The idea is that M_0^2 defines a standard reference and that one should be able to map $M^2(x)$ to M_0^2 by G_2 automorphism mapping also the $s(x)$ to a unique point $s_0(x) \in CP_2$ defining the CP_2 point assignable to the point of $X^4 \subset M^8$.

4. One can assign to the point x quaternionic unit vector $n(x)$ determining $M^2(x)$ as the direction of the preferred imaginary unit. The G_2 transformation must rotate $n(x)$ to n_0 defining M_0^2 and acts on s. G_2 transformation is not unique since $u_1 g u_2$ has the same effect for $u_i \subset U(2)$ leaving invariant the point of CP_2 for initial and final situation. Hence the equivalence classes of transformations should correspond to a point of 6-dimensional double coset space $U(2)\backslash G_2/U(2)$. Intuitively it seems obvious that the $s_0(x)$ is unique but proof is required.

What about the inverse of $M^8 - H$ duality?

$M^8 - H$ duality should have inverse in the critical regions of $X^4 \subset M^8$, where associativity conditions are satisfied. How could one construct the inverse of $M^8 - H$ duality in these regions? One should map space-time points $(m, s) \in M^4 \times CP_2$ to points $(m, e) = (m, f(m, s)) \in M^8$. $M_0^4 \supset M_0^2$ parameterized by CP_2 point can be chosen arbitrarily and one can require that it corresponds to some space-time point $(m_0, s_0) \in H$. CP_2 point $s(x)$ characterizes the quaternionic tangent space containing $M^2(x)$ and can choose M_0^2 to be $M^2(x_0)$ for conveniently chosen x_0. Coordinates x can be used also for $X^4 \subset M^8$.

One obtains set of points $(m, e) = (m(x), f(m(x), s(x)) \in M^8$ and a distribution of 4-D spaces of labelled by $s(x)$. This requires that the 4-D tangent space spanned by the gradients of $m(x)$ and $f(m(x), s(x))$ and characterized by $s_1 \subset CP_2$ for given $M^2(x)$ by using the above procedure mapping the situation to that for M_0^2 is same as the tangent space determined by $s(x)$: $s(x) = s_1(x)$. Also the associativity conditions should hold true. One should have a formula for s_1 as function of tangent vectors of space-time surface in M^8. The ansatz based on algebraic geometry in M_c^8 should be equivalent with this ansatz. The problem is that the ansatz leads to algebraic functions which cannot be found explicitly. It might be that in practice the correspondence is easy only in the direction $M^8 \to H$.

What one can say about twistor lift of $M^8 - H$ duality?

One can argue that the twistor spaces CP_1 associated with M^4 and E^4 are in no way visible in the dynamics of octonion polynomials and in $M^8 - H$ duality. Hence one could argue that they are not needed for any reasonable purpose. I cannot decide whether this is indeed the case. There I will consider the existence of twistor lift of the M^8 and also the twistor lift $M^8 - H$ duality in the space-time regions, where the tangent spaces satisfy the conditions for the existence of the duality as a map $(m, e) \in M^8 \to (m, s) \in M^4 \times CP_2$ must be considered. This involves some non-trivial delicacies.

1. The twistor bundles of M_c^4 and E_c^4 would be simply $M_c^4 \times CP_1$ and $E_c^4 \times CP_1$. $CP_1 = S^2$ parameterizes direction vectors in 3-D Euclidian space having interpretation as unit quaternions so that this interpretation might make sense. The definition of twistor structure means a selection of a preferred quaternion unit and its representation as Kähler form so that these twistor bundles would have thus Kähler structure. Twistor lift replaces complex quaternionic surfaces with their twistor spaces with induced twistor structure.
2. In M^8 the radii of the spheres CP_1 associated with M^4 and E^4 would be most naturally identical whereas in $M^4 \times CP_2$ they can be different since CP_2 is moduli space. Is the value of the CP_2 radius visible at all in the classical dynamics in the critical associative/co-associative space-time regions, where one has minimal surfaces. Criticality would suggest that besides coupling constants also parameters with dimension of length should disappear from the field equations. At least for the known extremals such as massless extremals, CP_2 type extremals, and cosmic strings CP_2 radius plays no role in the equations. CP_2 radius comes however into play only in interaction regions defined by CDs since $M^8 - H$ duality works only at the 3-D ends of space-time surface and at the partonic orbits. Therefore the different radii for the CP_1 associated with CP_2 and E^4 cause no obvious problems.

Consider now the idea about twistor space as real part of octonionic twistor space regarded as quaternion-complex space.

1. One can regard $CP_1 = S^2$ as the space of unit quaternions and it is natural to replace it with the 6-sphere S^6 of octonionic imaginary units at the level of complexified octonions. The sphere of complexified (by i) unit octonions is non-compact space since the norm is complex valued and this generalization looks neither attractive nor necessary since the projection to real numbers would eliminate the complex part.

   The equations determining the twistor bundle of space-time surface can be indeed formulated as vanishing of the quaternionic imaginary part of S^6 coordinates, and one obtains a reduction to quaternionic sphere S^2 at space-time level.

   If S^2 is identified as sub-manifold $S^2 \subset S^6$, it can be chosen in very many ways (this is of course not necessary). The choices are parameterized by $SO(7)/SO(3) \times SO(4)$ having dimension $D = 12$. This choice has no physical content visible at the level of H. Note that the Kähler structure determining Hamilton-Jaboci structure is fixed by the choice of preferred direction $(M^2(x))$. If all these moduli are allowed, it seems that one has something resembling multiverse, the description at the level of M^8 is deeper one and one must ask whether the space-time surfaces with different twistorial, octonionic, and quaternionic moduli can interact.
2. The resulting octonionic analog of twistor space should be mapped by $M^8 - H$ corresponds to twistor space of space-time surface $T(M^4) \times T(CP_2)$. The radii of twistor spheres of $T(M^4)$ and $T(CP_2)$ are different and this should be also understood. It would seem that the radius

of $T(M^4)$ at $H = M^4 \times CP_2$ side should correspond to that of $T(M^4)$ at M^8 side and thus to that of S^6 as its geodesic sphere: Planck length is the natural proposal inspired by the physical interpretation of the twistor lift. The radius of $T(CP_2)$ twistor sphere should correspond to that of CP_2 and is about 2^{12} Planck lengths.

Therefore the scale of CP_2 would emerge as a scale of moduli space and does not seem to be present at the level of M^8 as a separate scale. M^8 level would correspond to what might be called Planckian realm analogous to that associated with strings before dynamical compactification which is now replaced with number theoretic compactification. The key question is what determines the ratio of the radii of CP_2 scale to Planck for which favored value is 2^{12} [K11]. Could quantum criticality determine this ratio?

2.5 Appendix: o^2 as a simple test case

Octonionic polynomial o^2 serves as a simple testing case. o^2 is not irreducible so that its properties might not be generic and it might be better to study polynomial of form $P(o) = o + po^2$ instead.

Before continuing, some conventions are needed.

1. The convention is that in $M^8 = M^1 \times E^7$ E^7 corresponds to purely imaginary complexified octonions in both octonionic sense and in the sense that they are proportional to i. M^1 corresponds to octonions real in both senses. This corresponds to the signature $(1, -1, -1, -1, ...)$ for M^8 metric obtained as restriction of complexified metric. For $M^4 = M^1 \times E^3$ analogous conventions hold true.
2. Conjugation $o = o_0 + o_k I_k \rightarrow \overline{o} \equiv o_0 - I_k o_k$ does not change the sign of i. Quaternions can be decomposed to real and imaginary parts and some notation is needed. The notation $q = Re(q) + Im(q)$ seems to be the least clumsy one: here $Im(q)$ is 3-vector.

The explicit expression in terms of quaternionic decomposition $o = q_1 + q_2 I_4$ reads as

$$P(o) = o^2 = q_1^2 - q_2\overline{q}_2 + (q_1 q_2 + q_2\overline{q}_1)I_4 \quad . \tag{2.5.1}$$

o corresponds to complexified octonion and there are two options concerning the interpretation of M^4 and E^4. M^4 could correspond to quaternionic or co-quaternionic sub-space. I have assumed the first interpretation hitherto but actually the identification is not obvious. This two cases are different and must be treated both.

With these notations quaternionic inner product reads as

$$\begin{aligned} &q_1 q_2 = Re(q_1 q_2) + Im(q_1 q_2) \quad , \\ &Re(q_1 q_2) = Re(q_1)Re(q_2) - Im(q_1) \cdot Im(q_2) \quad , \\ &Im(q_1 q_2) = Re(q_1)Im(q_2) + Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) \quad . \end{aligned} \tag{2.5.2}$$

Here $a \cdot b$ denotes the inner product of 3-vectors and $a \times b$ their cross product.

Note that one has real and imaginary parts of octonions as two quaternions and real and imaginary parts of quaternions. To avoid confusion, I will use RE and IM to denote the decomposition of octonions to quaterions and Re and Im for the decomposition of quaternions to real and imaginary parts.

One can express the $RE(o^2)$ as

$$\begin{aligned} &RE(o^2) \equiv X \equiv q_1^2 - q_2\overline{q}_2 \quad , \\ &Re(X) = Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \quad , \\ &Im(X) = Im(q_1^2) = 2Re(q_1)Im(q_1) \quad . \end{aligned} \tag{2.5.3}$$

For $IM(o^2)$ one has

$$
\begin{array}{l}
IM(o^2) \equiv Y = q_1 q_2 + q_2 \overline{q}_1 \\
Re(Y) = 2Re(q_1)Re(q_2) \ , \\
Im(Y) = Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) \ .
\end{array}
\tag{2.5.4}
$$

The essential point is that only $RE(o^2)$ contains the complexified Euclidian norm $q_2\overline{q_2}$ which becomes Minkowskian of Euclidian norm depending on whether one identifies M^4 as associative or co-associative surface in o_c^8.

2.5.1 Option I: M^4 is quaternionic

Consider first the condition $RE(o^2) = 0$. The condition decomposes to two conditions stating the vanishing of quaternionic real and imaginary parts:

$$
\begin{array}{l}
Re(X) = Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{E^4}(q_2) = 0 \ , \\
Im(X) = Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ .
\end{array}
\tag{2.5.5}
$$

$Im(X) = 0$ is satisfied for $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has two options. This gives 1-D line in time direction of 3-D hyperplane as a solution for M^4 factor.

$Re(X) = 0$ states $N_{M^4}(q_1) = N_{E^4}(q_2)$. q_2 coordinate itself is free. $N_{E^4}(q_2)$ is negative so that q_1 must be space-like with respect to the N_{M^4} so that only the solution $Re(q_1) = 0$ is possible. Therefore one has $Re(q_1) = 0$ and $N_{M^4}(q_1) = N_{E^4}(q_2)$.

One can assign to each E^4 point a section of hyperboloid with $t = 0$ hyper-plane giving a sphere and the surface is 6-dimensional sphere bundle like variety! This is completely unexpected result and presumably is due to the additional accidental symmetries due to the octonionicity. Also the fact that o^2 is not irreducible polynomial is a probably reason since for o the surface is 4-D. The addition of linear term is expected to remove the degeneracy.

Consider next the case $IM(o^2) = 0$. The conditions read now as

$$
\begin{array}{l}
Re(Y) = 2Re(q_1)Re(q_2) = 0 \ , \\
Im(Y) = Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) = 0 \ .
\end{array}
\tag{2.5.6}
$$

Since cross product is orthogonal to the factors $Im(Y) = 0$ condition requires that $Im(q_1)$ and $Im(q_2)$ are parallel vectors: $Im(q_1) = \lambda Im(q_2)$ and one has the condition $Re(q_1) = \lambda Re(q_2)$ implying $q_1 = \Lambda q_2$. Therefore to each point of E^4 is associated a line of M^4. The surface is 5-dimensional.

It is interesting to look what the situation is if both conditions are true so that one would have a singularity. In this case $Re(q_1) = 0$ and $Re(q_1) = \lambda Re(q_2)$ imply $\lambda = 0$ so that $q_1 = 0$ is obtained and the solution reduces to 4-D E^4, which would be co-associative.

2.5.2 Option II: M^4 is co-quaternionic

This case is obtained by the inspection of the previous calculation by looking what changes the identification of M^4 as co-quaternionic factor means. Now q_1 is Euclidian and q_2 Minkowskian coordinate and $q_2\overline{q}_2$ gives Minkowskian rather than Euclidian norm.

Consider first $RE(o^2) = 0$ case.

$$
\begin{array}{l}
Re(X) = Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{M^4}(q_2) = 0 \ , \\
Im(X) = Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ .
\end{array}
\tag{2.5.7}
$$

$N_{M^4}(q_1) - N_{M^4}(q_2) = 0$ condition holds true now besides the condition $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has also now two options.

1. For $Re(q_1) = 0$ $N_{M^4}(q_1)$ is non-positive and this must be the case for $N_{M^4}(q_2))$ so that the *exterior* of the light-cone is selected. In this case the points of M^4 with fixed N_{M^4} give rise to a 2-D intersection with $Re(q_1) = 0$ hyper-plane that is sphere so that one has 6-D surface, kind of sphere bundle.
2. For $Im(q_1) = 0$ Minkowski norm is positive and so must be corresponding norm in E^4 so that in E^4 surface has future ligt-cone as projection. This surface is 4-D. The emergence of future light-cone might provide justification for the emergence of CDs and zero energy ontology.

For $IM(o^2)$ the discussion is same as in quaternionic case since norm does not appear in the equations.

At singularity both $RE(o^2)$ and $IM(o^2) = 0$ vanish. The condition $q_1 = \Lambda q_2$ reduces to $\Lambda = 0$ so that $q_1 = 0$ is only allowed. This leaves only light-cone boundary under consideration.

The appearance of surfaces with dimension higher than 4 raises the question whether something is wrong. One could of course argue that associativity allows also lower than 4-D surfaces as associative surfaces and higher than 4-D surfaces as co-associative surfaces. At H-level one can say that one has 4-D surfaces. A good guess is that this behavior disappears when the linear term is absent and origin ceases to be a singularity.

Chapter 3

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part II

3.1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

3.1.1 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.

   Remark: Why not rational functions expressible as ratios $R = P_1/P_2$ of octonionic polynomials? It has become clear that one can develop physical arguments in favor of this option. The zero loci for $IM(P_i)$ would represent space-time varieties. Zero loci for $RE(P_1/P_2) = 0$ and $RE(P_1/P_2) = \infty$ would represent their interaction presumably realized as wormhole contacts connecting these varieties. In the sequel most considerations are for polynomials: the replacement of polynomials with rational functions does not introduce big differences and its discussed in the section "Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view".

2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by CP_2 points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.

3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries

of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L19]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P) = 0$ can transform to $IM(P) = 0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.

4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adele for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.

5. Also a connection with infinite primes is suggestive [K86]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to $M^8 - H$ duality. The strategy is simple: try to remember all previous objections against $M^8 - H$ duality and invent new ones since this is the best way to make real progress.

3.1.2 Topics to be discussed

Challenges of the octonionic algebraic geometry

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a $M^8 - H$ correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

   One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root.

   Various components of octonion polynomial P of degree n are polynomials of same degree. Could criticality reduces to the degeneracy of roots for some component polynomials? Could P as a polynomial of real variable have degenerate roots?

   The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A129] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

   One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

Description of interactions

Also a sketchy proposal for the description of interactions is discussed.

1. $IM(P_1P_2) = 0$ is satisfied for $IM(P_1) = 0$ and $IM(P_2) = 0$ since $IM(o_1o_2)$ is linear in $IM(o_i)$ and one obtains union of space-time varieties. $RE(P_1P_2) = 0$ cannot be satisfied in this way since $RE(o_1o_2)$ is not linear in $RE(o_i)$ so that the two varieties interact and this interaction could give rise to a wormhole contact connecting the two space-time varieties.

2. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety X^4. The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices. S^6 and therefore also X^2 are doubly critical, S^6 is also singular surface.

   The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

3. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

   The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(\prod P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

4. The possibility of super-octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward way to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

   If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic. Indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N} = 4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the appropriate extension of rationals.

   Twistor Grassmannian construction of scattering amplitudes at the level of M^8 looks feasible. The amplitudes decompose to M^4 and CP_2 parts with similar structure with E^4 spin (electroweak isospin) replacing ordinary spin. The residue integrals over Grassmannians emerging from the conservation of M^4 and E^4 4-momenta would have same form and guarantee Yangian supersymmetry in both sectors. The counterpart for the product of delta functions associated with the "negative helicities" (weak isospins with negative sign) would be expressible as a delta function in the complement of $SU(3)$ Cartan algebra $U(1) \times U(1)$ by using exponential map.

About the analogs of Gromow-Witten invariants and branes in TGD

Gromov-Witten (G-W) invariants belong to the realm of quantum enumerative geometry briefly discussed in [L37]. They count numbers of points in the intersection of varieties ("branes") with quantum intersection identified as the existence of "string world sheet(s)" intersecting the branes. Also octonionic geometry gives rise to brane like objects. G-W invariants are rational numbers but it is proposed that they could be integers in TGD framework.

Riemann-Roch theorem (RR) and its generalization Atyiah-Singer index theorem (AS) relate dimensions of various kinds of moduli spaces to topological invariants. The possible generalizations of RR and AS to octonionic framework and the implications of $M^8 - H$ duality for the possible generalizations are discussed. The adelic hierarchy of extensions of rationals and criticality conditions make the moduli spaces discrete so that one expects kind of particle in box type quantization selecting discrete points of moduli spaces about the dimension.

The discussion of RR as also the notion of infinite primes and infinite rationals as counterparts of zero energy states suggests that rational functions $R = P_1/P_2$ could be more appropriate than mere polynomials. The construction of space-time varieties would not be modified in essential way: one would have zero loci of $IM(P_i)$ identifiable as space-time sheets and zero- and ∞-loci of $RE(P_1/P_2)$ naturally identifiable as wormhole contacts connecting the space-time sheets.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

3.2 Some challenges of octonionic algebraic geometry

Space-time surfaces in $H = M^4 \times CP_2$ identified as preferred extremals of twistor lift of Kähler action leads to rather detailed view about space-time surfaces as counterparts of particles. Does this picture follow from $X^4 \subset M^8$ picture and does this description bring in something genuinely new?

3.2.1 Could free many-particle states as zero loci for real or imaginary parts for products of octonionic polynomials

In algebraic geometry zeros for the products of polynomials give rise to disjoint varieties, which are disjoint unions of surfaces assignable to the individual surfaces and possibly having lower-dimensional intersections. For instance, for complex curves these intersections consist of points. For complex surfaces they are complex curves.

In the case of octonionic polynomial $P = RE(P) + IM(P)I_4$ (Re and Im are defined in quaternionic sense) one considers zeros of quaternionic polynomial $RE(P)$ or $IM(P)$.

1. Product polynomial $P = P_1P_2$ decomposes to

$$P = RE(P_1)RE(P_2) - IM(P_1)IM(P_2) + (RE(P_1)IM(P_1) + IM(P_1)RE(P_2)I_4 \ .$$

   One can require vanishing of $RE(P)$ or $IM(P)$.

   (a) $IM(P)$ vanishes for

$$(RE(P_1) = 0, RE(P_2) = 0)$$

   or

$$I(m(P_1) = 0, IM(P_2) = 0) \ .$$

   (b) $RE(P)$ vanishes for

$$(RE(P_1) = 0, IM(P_2) = 0)$$

   or

$$IM(P_1) = 0, RE(P_2) = 0) \ .$$

   One could reduce the condition $RE(P) = 0$ to $IM(P) = 0$ by replacing $P = P_1 + P_2I_4$ with $P_2 - P_1I_4$. If this condition is satisfied for the factors, it is satisfied also for the product. The set of surfaces is a commutative and associative algebra for the condition $IM(P) = 0$. Note that the quaternionic moduli must be same for the members of product. If one has quantum superposition of quaternionic moduli, the many-particle state involves a superposition of products with same moduli.

   As found, the condition $IM(P) = 0$ can transform to $RE(P) = 0$ at singularities having $RE(P) = 0, IM(P) = 0$.

2. The commutativity of the product means that the products are analogous to many-boson states. P^n would define an algebraic analog of Bose-Einstein condensate. Does this surface correspond to a state consisting of n identical particles or is this artefact of representation? As a limiting case of product of different polynomials it might have interpretation as genuine n-boson states.
3. The product of two polynomials defines a union of disjoint surfaces having discrete intersection in Euclidian signature. In Minkowskian signature the vanishing of $q\overline{q}$ (conjugation does not affect the sign of i and changes only the sign of I_k!) can give rise to 3-D light-cone. The non-commutativity of quaternions indeed can give rise to combinations of type $q\overline{q}$ in $RE(P)$ and $IM(P)$.

What about interactions?

1. Could one introduce interaction by simply adding a polynomial P_{int} to the product? This polynomial should be small outside interaction region. CD would would define naturally interaction regions and the interaction terms should vanish at the boundaries of CD. This might be possible in Minkowskian signature, where $f(q^2)$ multiplying the interaction term might vanish at the boundary of CD: in Euclidian sector $q\overline{q} = 0$ would imply $q = 0$ but in Minkowskian sector it would give light-cone as solution. One should arrange $IM(P_{int})$ to be proportional to $q\overline{q}$ vanishing at the boundary of CD. Minkowskian signature could be crucial for the possibility to "turning interactions on".
2. If the imaginary part of the interaction term is proportional $f_1(q^2)f_2((q-T)^2)$ (T is real and corresponds to the temporal distance between the tips of CD) with $f_i(0) = 0$, one could obtain asymptotic states reducing to disjoint unions of zero loci of P^i at the boundaries of CD. If the order of of the perturbation terms is higher than the total order of polynomials P^i, one would obtain new roots and particle emission. Non-perturbative situation would correspond to a dramatic modification of the space-time surface as a zero locus of $IM(P)$. This picture would be M^8 counterpart for the reduction of preferred extremals to minimal surfaces analogous to geodesic lines near the boundaries of CD: preferred extremals reduce to extremals of both Kähler action and volume term in these regions [L19].

The singularities of scattering amplitudes at algebraic varieties of Grassmann manifolds are central in the twistor Grassmann program [B22, B48, B29]. Since twistor lift of TGD seems to be the correct manner to formulate classical TGD in H, one can wonder about the connection between space-time surfaces in M^8_c and scattering amplitudes. Witten's formulation of twistor amplitudes in terms of algebraic curves in CP_3 suggests a formulation of scattering amplitudes in terms of the 4-D algebraic varieties in M^8_c as of course, also TGD itself [K35, K79]! Could the huge multi-local Yangian symmetries of twistor Grassmann amplitudes reduce to octonion analyticity.

3.2.2 Two alternative interpretations for the restriction to M^4 subspace of M^8_c

One must complexify M^8 so that one has complexified octonions M^8_c. This means the addition of imaginary unit i commuting with octonionic imaginary units. The vanishing of real or imaginary part of octonionic polynomial in quaternionic sense ($o = q_1 + Jq_2$) defines the space-time surface. Octonionic polynomial itself is obtained from a real polynomial by algebraic continuation so that in information theoretic sense space-time is 1-D. The roots of this real polynomial fix the polynomial and therefore also space-time surface uniquely. 1-D line degenerates to a discrete set of points of an extension in information theoretic sense. In p-adic case one can allow p-adic pseudo constants and this gives a model for imagination.

The octonionic roots $x + iy$ of the real polynomial need not however be real. There are two options.

1. The original proposal in [L36, L38] was that the *projection* from M^8_c to real M^4 (for which M^1 coordinate is real and E^3 coordinates are imaginary with respect to i!) defines the real space-time surface mappable by $M^8 - H$ duality to CP_2.
2. An alternative option is that only the roots of the 4 vanishing polynomials as coordinates of M^4_c belong to M^4 so that m^0 would be real root and m^k, $k = 1, ..., 3$ imaginary with

respect to $i \to -i$. M_c^8 coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality*. This could make sense.

What is remarkable that this could allow to identify CDs in very elegant manner: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

Consider now this in detail.

1. One can think of starting from one of the 4 vanishing conditions for the components of octonionic polynomial guaranteeing associativity. Assuming real roots and continuing one by one through all 4 conditions to obtain 4-D Minkowskian real regions. The time coordinate of M^4 coordinates is real and others purely imaginary with respect to $i \to -i$. If this region does not connect 3-D surface at the boundaries of real CD, one must make a new trial.

   Cusp catastrophe determined as the zero locus of third order polynomial provides an example. There are regions with single real root, regions with two real roots (complex roots become real and identical) defining V-shaped boundary of cusp and regions with 3 real roots (the interior of the cusp).

2. The restriction of the octonionic polynomial to time axis m^0 identifiable as octonionic real axes is a real polynomial with algebraic coefficients. In this case the root and its conjugate with respect to i would define the same surface. One could say that the Galois group of the real polynomial characterizes the space-time surface although at points other than those at real axis (time axis) the Galois group can be different.

   One could consider the local Galois group of the fourth quaternionic valued polynomial, say the part of quaternionic polynomial corresponding to real unit 1 when other components are required to vanish and give rise to coordinates in $M^8 \subset M_c^8$ - Minkowskian reality. The extension and its Galois group would depend on the point of space-time surface.

   An interesting question is how strong conditions Minkowskian reality poses on the extension. Minkowskian reality seems to imply that E^3 roots are purely real so that for an octonionic polynomial obtained as a continuation of a *real* polynomial one expects that both root and complex conjugate should be allow and that Galois group should contain Z_2 reflection $i \to -i$. Space-time surface would be at least 2-sheeted. Also the model for elementary particles forces this conclusion on physical grounds. Real as opposite to imagined would mean Minkowskian reality in mathematical sense. In the case of polynomials this description would make sense in p-adic case by allowing the coefficients of the polynomial be pseudo constants.

3. What data one could use to fix the space-time surface? Can one start directly from the real polynomial and regard its coefficients as WCW coordinates? This would be easy and elegant. Space-time surface could be determined as Minkowskian real roots of the octonionic polynomial. The condition that the space-time surface has ends at boundaries of given CD and the roots are not Minkowskian real outside it would pose conditions on the polynomial. If the coefficients of the polynomial are p-adic pseudo constants, this condition might be easy to satisfy.

The situation depends also on the coordinates used. For linear coordinates such as Minkowski coordinates Minkowskian reality looks natural. One can however consider also angle like coordinates representable only in terms of complex phases p-adically and coming as roots of unity and requiring complex extension: at H-side they are very natural. For instance, for CP_2 all coordinates would be naturally represented in this manner. For future light-cone one would have hyperbolic angle and 2 ordinary angles plus light-cone proper time which would be real and positive coordinate.

This picture conforms with the proposed picture. The point is that the time coordinate m^k can be real in the sense that they are linear combinations of complex roots, say powers for the roots of unity. $E_c^4 \subset M_c^8$ could be complex and contain also complex roots since $M^8 - H$ duality does not depend on whether tangent space is complex or not. Therefore would could have complex extensions.

3.2.3 Questions related to ZEO and CDs

Octonionic polynomials provide a promising approach to the understanding of ZEO and CDs. Light-like boundary of CD as also light-cone emerge naturally as zeros of octonionic polynomials. This does not yet give CDs and ZEO: one should have intersection of future and past directed light-cones. The intuitive picture is that one has a hierarchy of CDs and that also the space-time surfaces inside different CDs an interact.

Some general observations about CDs

It is good to list some basic features of CDS, which appear as both 4-D and 8-D variants.

1. There are both 4-D and 8-D CDs defined as intersections of future and past directed light-cones with tips at say origin 0 at real point T at quaternionic or octonionic time axis. CDs can be contained inside each other. CDs form a fractal hierarchy with CDs within CDs: one can add smaller CDs with given CD in all possible ways and repeat the process for the sub-CDs. One can also allow overlapping CDs and one can ask whether CDs define the analog of covering of O so that one would have something analogous to a manifold.
2. The boundaries of two CDs (both 4-D and 8-D) can intersect along light-like ray. For 4-D CD the image of this ray in H is light-like ray in M^4 at boundary of CD. For 8-D CD the image is in general curved line and the question is whether the light-like curves representing fermion orbits at the orbits of partonic 2-surfaces could be images of these lines.
3. The 3-surfaces at the boundaries of the two 4-D CDs are expected to have a discrete intersection since 4 + 4 conditions must be satisfied (say $RE(P_i^k)) = 0$ for $i = 1, 2$, $k = 1, 4$. Along line octonionic coordinate reduces effectively to real coordinate since one has $E^2 = E$ for $E = (1 + in)/2$, n octonionic unit. The origins of CDs are shifted by a light-like vector kE so that the light-like coordinates differ by a shift: $t_2 = t_1 - k$. Therefore one has common zero for real polynomials $RE(P_1^k(t))$ and $RE(P_2^k(t - k))$.

   Are these intersection points somehow special physically? Could they correspond to the ends of fermionic lines? Could it happen that the intersection is 1-D in some special cases? The example of o^2 suggest that this might be the case. Does 1-D intersection of 3-surfaces at boundaries of 8-D CDs make possible interaction between space-time surfaces assignable to separate CDs as suggested by the proposed TGD based twistorial construction of scattering amplitudes?
4. Both tips of CD define naturally an origin of quaternionic coordinates for $D = 4$ and the origin of octonionic coordinates for $D = 8$. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be along the real line (time axis) connecting the tips of CD. Only the translations in this specified direction are symmetries preserving the commutativity and associativity of the polynomial algebra.
5. One expects that also Lorentz boosts of 4-D CDs are relevant. Lorentz boosts leave second boundary of CD invariant and Lorentz transforms the other one. Same applies to 8-D CDs. Lorentz boosts define non-equivalent octonionic and quaternionic structures and it seems that one assume moduli spaces of them.

One can of course ask whether the still somewhat ad hoc notion of CD general enough. Should one generalize it to the analog of the polygonal diagram with light-like geodesic lines as its edges appearing in the twistor Grassmannian approach to scattering diagrams? Octonionic approach gives naturally the light-like boundaries assignable to CDs but leaves open the question whether more complex structures with light-like boundaries are possible. How do the space-time surfaces associated with different quaternionic structures of M^8 and with different positions of tips of CD interact?

The emergence of causal diamonds (CDs)

CDs are a key notion of zero energy ontology (ZEO). They should emerge from the number-theoretic dynamics somehow. How? In the following this question is approached from two different directions.

1. One can ask whether the emergence of CDs could be understood in terms of singularities of octonion polynomials located at the light-like boundaries of CDs. In Minkowskian case the complex norm $q\overline{q}_i$ is present in P. Could this allow to blow up the singular point to a 3-D boundary of light-cone and allow to understand the emergence of causal diamonds (CDs) crucial in ZEO. This question will be considered below.
2. These arguments were developed before the realization that the Minkowskian reality condition discussed in the previous section is natural for the space-time surfaces as roots of the 4 polynomials defining real or imaginary part of octonionic polynomial in quaternionic sense and giving M^4 point as a solution. Minkowskian reality can hold only in some regions of M^4 and an attractive conjecture is that it fails outside CD. CD would be a prediction of number theoretical dynamics and have counterpart also at the level of H.

Consider now the second approach in more detail. The study of the special properties for zero loci of general polynomial $P(o)$ at light-rays of O indeed demonstrated that both 8-D land 4-D light-cones and their complements emerge naturally, and that the M^4 projections of these light-cones and even of their boundaries are 4-D future - or past directed light-cones. What one should understand is how CDs as their intersections, and therefore ZEO, emerge.

1. One manner to obtain CDs naturally is that the polynomials are sums $P(t) = \sum_k P_k(o)$ of products of form $P_k(o) = P_{1,k}(o)P_{2,k}(o-T)$, where T is real octonion defining the time coordinate. Single product of this kind gives two disjoint 4-varieties inside future and past directed light-cones $M^4_+(0)$ and $M^4_-(T)$ for either $RE(P) = 0$ (or $IM(P) = 0$) condition. The complements of these cones correspond to $IM(P) = 0$ (or $RE(P) = 0$) condition.
2. If one has nontrivial sum over the products, one obtains a connected 4-variety due the interaction terms. One has also as special solutions $M^4_\pm$ and the 6-spheres associated with the zeros $P(t)$ or equivalently $P_1(t_1) \equiv P(t)$, $t_1 = T - t$ vanishing at the upper tip of CD. The causal diamond $M^4_+(0) \cap M^4_-(T)$ belongs to the intersection.

   Remark: Also the union $M^4_-(0) \cup M^4_+(T)$ past and future directed light-cones belongs to the intersection but the latter is not considered in the proposed physical interpretation.
3. The time values defined by the roots t_n of $P(t)$ define a sequence of 6-spheres intersecting 4-D CD along 3-balls at times t_n. These time slices of CD must be physically somehow special. Space-time variety intersects 6-spheres along 2-varieties X^2_n at times t_n. The varieties X^2_n are perhaps identifiable as 2-D interaction vertices, pre-images of corresponding vertices in H at which the light-like orbits of partonic 2-surfaces arriving from the opposite boundaries of CD meet.

   The expectation is that in H one as generalized Feynman diagram with interaction vertices at times t_n. The higher the evolutionary level in algebraic sense is, the higher the degree of the polynomial $P(t)$, the number of t_n, and more complex the algebraic numbers t_n. $P(t)$ would be coded by the values of interaction times t_n. If their number is measurable, it would provide important information about the extension of rationals defining the evolutionary level. One can also hope of measuring t_n with some accuracy! Octonionic dynamics would solve the roots of a polynomial! This would give a direct connection with adelic physics [L42] [L43].

   Remark: Could corresponding construction for higher algebras obtained by Cayley-Dickson construction solve the "roots" of polynomials with larger number of variables? Or could Cartesian product of octonionic spaces perhaps needed to describe interactions of CDs with arbitrary positions of tips lead to this?
4. Above I have considered only the interiors of light-cones. Also their complements are possible. The natural possibility is that varieties with $RE(P) = 0$ and $IM(P) = 0$ are glued at the boundary of CD, where $RE(P) = IM(P) = 0$ is satisfied. The complement should contain the external (free) particles, and the natural expectation is that in this region the associativity/co-associativity conditions can be satisfied.
5. The 4-varieties representing external particles would be glued at boundaries of CD to the interacting non-associative solution in the complement of CD. The interaction terms should be non-vanishing only inside CD so that in the exterior one would have just product $P(o) = P_{1,k_0}(o)P_{2,k_0}(o-T)$ giving rise to a disjoint union of associative varieties representing external particles. In the interior one could have interaction terms proportional to say $t^2(T - t)^2$

vanishing at the boundaries of CD in accordance with the idea that the interactions are switched one slowly. These terms would spoil the associativity.

Remark: One can also consider sums of the products $\prod_k P_k(o-T_k)$ of n polynomials and this gives a sequence CDs intersecting at their tips. It seems that something else is required to make the picture physical.

3.2.4 About singularities of octonionic algebraic varieties

In Minkowskian signature the notion of singularity for octonionic polynomials involves new aspects as the study of o^2 singular at origin shows (see Appendix). The region in which $RE(o^2) = 0, IM(o^2) = 0$ holds true is 4-D rather than a discrete set of points as one would naïvely expect.

1. At singularity the local dimension of the algebraic variety is reduced. For instance, double cone of 3-space has origin as singular point where it becomes 0-dimensional. A more general example is local pinch in which cylinder becomes infinitely thin at some point. This kind of pinching could occur for fibrations as the fiber contracts to a lower-dimensional space along a sub-variety of the base space.

   A very simple analogy for this kind of singularity is the singularity of $P(x,y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x,y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

   The signature of the singularity of algebraic variety determined by the conditions $P^i(z^j) = 0$ is the reduction of the maximal rank r for the matrix formed by the partial derivatives $P^i_j \equiv \partial IM(P)^i/\partial z^j$ ("RE" could replace "IM"). Rank corresponds to the largest dimension of the minor of P^i_j with non-vanishing determinant. Determinant vanishes when two rows of the minor are proportional to each other meaning that two tangent vectors become linearly dependent. When the rank is reduced by Δr, one has $r = r_{max} - \Delta r$ and the local dimension is locally reduced by Δr. One has hierarchy of singularities within singularities.

   The conditions that all independent minors of the P^i_j have reduced rank gives additional constraints and define a sub-variety of the algebraic variety. Note that the dimension of the singularity corresponds to $d_s = \Delta r$ in the sense that the dimension of tangent space at singularity is effectively d_s.

2. In the recent case there are 4 polynomials and 4 complex variables so that $IM(P)^i_j$ is 4×4-matrix. Its rank r can have values in $r = 1, 2, 3, 2, 4$. One can use Thom's catastrophe theory as a guideline. Catastrophe decomposes to pieces of various dimensions characterized by the reduction of the rank of the matrix defined by the second derivatives $V_{ij} = \partial_i\partial_j V$ of the potential function defining the catastrophe. For instance, for cusp catastrophe with $V(x,a,b) = x^4 + ax^2 + bx$ one has V-shaped region in (a,b) plane with maximal reduction of rank to $r = 0$ ($\partial_x^2 V = 0$) at the tip $(a,b) = 0$ at reduction to $r = 1$ at the sides of V, where two roots of $\partial_x V = 4x^3 + 2ax + b = 0$ co-incide requiring that the discriminant of this equation vanishes.

3. In the recent case $IM(P)$ takes the role of complex quaternion valued potential function and the 4 coordinates $z_1^{k)}$ that of behavior variable x for cusp and $z_2^{k)}$ that of control parameters (a,b). The reduction of the rank of $n\times n$ matrix by Δr means that there are r linearly independent rows in the matrix. These give Δr additional conditions besides $IM(P) = 0$ so that the sub-variety along which the singularity takes places as dimension r. One can say that the r-dimensional tangent spaces integrate to the singular variety of dimension r.

   The analogy with branes would be realized as a hierarchical structure of singularities of the spacetime surfaces. This hierarchy of singularities would realize space-time correlates for quantum criticality, which is basic principle of quantum TGD. For instance, the reduction by 3-units would correspond to strings - say at the ends of CD and along the partonic orbits (fermion lines), and maximal reduction might correspond to discrete points - say the ends of fermion lines at partonic 2-surfaces. Also isolated intersection points can be regarded as singularities and are stably present but it does not make sense to add fermions to these points so that cognitive representations are not possible.

4. Note that also the associativity - and commutativity conditions already discuss involved the gradients of $IM(P)^i$ and $RE(P)^i$, which would suggests that these regions can be interpreted as singularities for which the dimension is not lowered by on unit since the vanishing conditions hold true identically by criticality.

There are two cases to be considered. The usual Euclidian case in which pinch reducing the dimension and the Minkowskian case in which metric dimension is reduced locally.

Consider first the Euclidian case.

1. In Euclidian case it is difficult to tell whether all values of Δr are possible since octonion analyticity poses strong conditions on the singularities. The pinch could correspond to the singularity of the covering associated with the space-time surface defined by Galois group for the covering associated with $h_{eff}/h = n$ identifiable as the dimension of the extension [L33]. Therefore there would be very close connection between the extensions of rationals defining the Galois group and the extension of polynomial ring of 8 complex variables z_i^k, $i = 1, 2$, $k = 1, .., 4$ by algebraic functions. At the pinch, which would be algebraic point, the Galois group would have subgroup leaving the coordinates of the point invariant and some sheets of the covering defining roots would co-incide.
2. A very simple analogy for this kind of singularity is the singularity of $P(x, y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x, y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.
3. Quaternion structure predicts the slicing of M^4 by string world sheets inducing that of space-time surfaces. One must ask whether singular space-time sheets emerge already for the slicing of M^4 by string world sheets. String world sheets could be considered as candidates for $\Delta r = 2$ singularities of this kind. The physical intuition strongly suggests that there indeed physically preferred string world sheets and identification as $\Delta r = 2$ singularities of Euclidian type is attractive. Partonic 2-surfaces are also candidates in this respect. Could some sheets of the $h_{eff}/h = n$ covering co-incide at string world sheets?

Consider next the Minkowskian case. At the level of H the rank of the induced metric is reduced. This reduction need not be same as that for the matrix P^i_j and it is of course not obvious that the partonic orbit allows description as a singularity of algebraic variety.

1. Could the matrix P^i_j take a role analogous to the dual of induced metric and one might hope that the change of the sign for P^i_j for a fixed polynomial at singular surface could be analogous to the change of the sign of $\sqrt{g_4}$ so that the idea about algebraization of this singularity at level of M^8 might make sense. The information about metric could come from the fact that $IM(P)$ depends on complex valued quaternion norm reducing to Minkowskian metric in Minkowskian sub-space.
2. The condition for the reduction of rank from its maximal value of $r = 4$ to $r = 3$ occurs if one has $det(P) = 0$, which defines co-dimension 1 surface as a sub-variety of space-time surface. The interpretation as co-incidence of two roots should make sense if $IM(P) = 0$. Root pairs would now correspond now to the points at different sides of the singular 3-surface.

   Minkowskian singularity cannot be identified as the 3-D space-like boundary of many-sheeted space-time surface located at the boundary of CD (induced metric is space-like).

   Could this sub-variety be identified as partonic orbit, the common boundary of the Euclidian and Minkowskian regions? This would require that associative region transforms to co-associative one here. $IM(P) = 0$ condition can transform to $RE(P) = 0$ condition if one has $P = 0$ at this surface. Minkowskian variant of point singularity (P^i_j vanishes) would explode it to a light-like partonic orbit.

   What does this imply about the rank of singularity? The condition $IM(P) = RE(P) = 0$ does not reduce the rank if P is linear polynomial and one could consider a hierarchy of reductions of rank. Since $q\overline{q}$ vanishes in Minkowskian sub-space at light-cone boundary rather than at point $q = 0$ only, there are reasons to expect that it appears in P and reduces the rank by $\Delta r = 4$ (see Appendix for the discussion of o^2 case). The rank of the induced 4-metric is however reduced only by $\Delta r = 1$ at partonic orbit. If the complexified complex norm $z\overline{z}$, $z = z_1 + z_2 I_2$ can take the role of $q\overline{q}$, one has $\Delta r = 2$.

3. The reduction of rank to $r = 2$ would give rise to 2-surfaces, which are at the boundaries of 3-D singularities. If partonic orbits correspond to $\Delta r = 1$ singularities one could identify them as partonic 2-surfaces at the ends partonic orbits.

   Could the singularity at partonic 2-surface correspond to the reduction of the rank of the induced metric by 2 units? This is impossible in strict sense since there is only one light-like direction in signature $(1, -1, -1, -1)$. Partonic 2-surface singularity would however correspond to a corner for both Euclidian and Minkowskian regions at which the metrically 2-D but topologically 3-D partonic orbit meets the the space-like 3-surface along the light-like boundary of CD. Also the radial direction for space-like 3-surface could become light-like at partonic 2-surface if the CP_2 coordinates have vanishing gradient with respect to the light-like radial coordinate r_M at the partonic 2-surface. In this sense the rank could be reduced by 2 units. The situation is analogous to that for fold singularity $y^2 - x = 0$.

   String world sheets cannot be subsets of $r = 3$ singularities, which suggests different interpretation for partonic 2-surfaces and string world sheets.

What could this different interpretation be?

1. Perhaps the most convincing interpretation of string world sheets/partonic 2-surfaces has been already discussed (this interpretation would generalize to associative space-time surfaces). They could be commutative/co-commutative (here permutation might be allowed!) submanifolds of associative regions of the space-time surface allowing quaternionic tangent spaces so that the notions of commutative and co-commutative make sense. The criticality conditions are satisfied without the reduction of dimension from $d = 2$ to $d = 1$. In non-associative regions string world sheets would reduce to 1-D curves. This would happen at the boundaries of partonic orbits and 3-surfaces at the ends of space-time surface and only the ends of strings at partonic orbits carrying fermion number would be needed to determine twistorial scattering amplitudes [K35, K79].

2. I have also considered an interpretation in terms of singularities of space-time surfaces represented as a sections of their own twistor bundle. Self-intersections of the space-time surface would correspond to 2-D surfaces in this case [L33] and perhaps identifiable as string world sheets. The interpretation mentioned above would be in terms of Euclidian singularities. If this is true, the question is only about whether these two interpretations are consistent with each other.

If I were forced to draw conclusion on basis of these notices, it would be that only $r = 4$ Minkowskian singularities could be interesting and at them $RE(P) = 0$ regions could be transformed to $IM(P) = 0$ regions. Furthermore, the reduction of rank for the induced metric cannot be equal to the reduction of the rank for P^i_j.

3.2.5 The decomposition of space-time surface to Euclidian and Minkowskian regions in octonionic description

The unavoidable outcome of H picture is the decomposition of space-time surface to regions with Minkowskian or Euclidian signature of the induced metric. These regions are bounded by 3-D regions at which the signature of the induced metric is $(0, -1, -1, -1)$ due to the vanishing of the determinant of the induced metric. The boundary is naturally the light-like orbit of partonic 2-surface although one can consider also the possibility that these regions have boundaries intersecting along light-like curves defining boundaries of string world sheets. A more detailed view inspired by the study of extremals is following.

1. Let us assume that the above picture about decomposition of space-time surfaces in H to two kinds regions takes place. The regions where the dynamicis universal minimal surface dynamics have associative pre-image in M^8. The regions where Kähler action and volume term couple the associative pre-image in M^8 exists only at the 3-D boundary regions and M^8 dynamics determines the boundary conditions for H dynamics, which by holography is enough.

2. In the space-time regions having associative pre-image in M^8 one has a fibration of X^4 with with partonic surface as a local base and string world sheet as local fiber. In the interior of space-time region there are no singularities but at the boundary 2-D string world sheets becomes metrically 1-D as 1-D string boundary reduces metrically to 0-D structure analogous to a point. This reduction of dimension would be metric, but not topological.

   The singularity for plane curve $P(x, y) = y^2 - x^3 = 0$ at origin illustrates the difference between Minkowskian and Euclidian singularity. One has $(\partial_x P, \partial_y P) = (-3x^2, 2y)$ vanishing at origin so that $\Delta r = 1$ singularity is in question and the dimension of singular manifold is indeed $r = 0$. From $y = \pm x^{3/2}$, $x \geq 0$. The induced metric $g_{xx} = 1 + (dy/dx)^2 = 1 + (9/4)x$, $x \geq 0$ is however non-singular at origin.

3. If the Euclidian region with pre-image corresponds to a deformation of wormhole contact, the identification as image of a co-associative space-time region in M^8 is natural so that normal space is associative and contains also the preferred $M^2(x)$. In Minkowskian regions the identification as image of associative space-time region in M^8 is natural.

What can one say about the relationship of the M^8 counterparts of neighboring Minkowskian and Euclidian regions?

1. Do these regions intersect along light-like 3-surfaces, 1-D light-like curve (orbit of fermion) or is the intersection disrete set of points possibly assignable to the partonic 2-surface at the boundaries of CD? The M^4 projections of the inverse image of the light-like partonic orbit should co-incide but E^4 projections need not do so. They could be however mappable to the same partonic two surface in $M^8 - H$ correspondence or the images could have at least have light-like curve as common.

2. Is seems impossible for the space-time surfaces determined as zeros of octonionic polynomials to have boundaries. Rather, it seems that the boundary must be between Minkowskian and Euclidian regions of the space-time surface determined by the same octonionic polynomial. At the boundary also associate region would transform to co-associative region suggesting that $IM(P) = RE(P) = 0$ holds allowing to change the condition from $IM(P) = 0$ to $RE(P) = 0$.

Consider now in more detail whether this view can be realized.

1. In $H = M^4 \times CP_2$ the boundary between the Minkowskian and Euclidian space-time regions - light-like partonic 3-surface - is a singularity possible only in Minkowskian signature. Space-time surface X^4 at the boundary is effectively 3-D since one has $\sqrt{g_4} = 0$ meaning that tangent space is effectively 3-D. The 3-D boundary itself is metrically 2-D and this gives rise to the extended conformal invariance defining crucial distinction between TGD and super string models.

2. The singularities of $P(o)$ for o identified as linear coordinate of M_c^8 were already considered. The singularities correspond to the boundaries of light-cone and the emergence of CDs can be understood. Could also the light-like orbits of partonic 2-surfaces be understood in the same manner? Does the pre-image of this singularity in M^8 emerge as a singularity of an algebraic variety determined by the vanishing of $IM(P)$ for the octonionic polynomial?

   What is common is that the rank of the induced metric by one unit also now. Now one has however also $det(g_4) = 0$. The singularities correspond to curved light-like 3-surfaces inside space-time surfaces rather than light-like surfaces in M^8: induced metric matters rather than M^4 metric.

3. Could also these regions correspond to singularities of octonionic polynomials at which $P(o) = 0$ is satisfied and associative region transforms to a co-associative region? For $M^2(x) = M_0^2$ this is impossible. Partonic 2-surfaces are planes E^2 now. One should have closed partonic 2-surfaces.

   Could the allowance of quaternionic structures with slicing of X^4 by string world sheets and partonic 2-surfaces help? If one has slicing of string world sheets by dual light-like curves corresponding to light-like coordinates u and v, this slicing gives also rise to a slicing of light-like 3-surfaces and dual light-like coordinate. The pair (u, v) in fact defines the analog of z and $\overline{z}$ in hypercomplex case. Could the singularity of $P(o)$ using the quaternionic coordinates defined by (u, v) and coordinates of partonic 2-surface allow to identify light-like partonic orbits with $det(g_4) = 0$ as a generalization of light-cone boundaries in M^4?

The decomposition $M_0^4 = M_x^2 \times E^2(x)$ associated with quaternionic structure is independent of E^4. In the other hand, tangent space of space-time surface at point decomposes $M^2(x) \times E_T^2(x)$, where $E_T^2(x)$ is in general different from $E^2(x)$. Is this enough to obtain partonic 2-surfaces as singularities with $RE(P) = IM(P) = 0$?

The question whether the boundaries between Minkowskian and Euclidian can correspond to singular regions at which $P(o)$ vanishes and the surface $RE(P) = 0$ transforms to $IM(P) = 0$ surface remains open. What remains poorly understood is the role of the induced metric. My hope is that with a further work the picture could be made more detailed.

3.2.6 About rational points of space-time surface

What one can say about rational points of space-time surface?

1. An important special case corresponds to a generalization of so called rational surfaces for which a parametric representation in terms of 4 complex coordinates t^k exists such that o_1^k are *rational* functions of t^k. The singularities for 2-complex dimensional surfaces in C^3 or equivalently CP_3 are classified by Du Val [A158, A169] (see `http://tinyurl.com/ydz93hle`).
2. In [L33] [L27] I considered possible singularities of the twistor bundle. These would correspond typically 2-D self-intersections of the embedding of space-time surfaces as 4-D base space of 6-D twistor bundle with sphere as a fiber. They could relate to string world sheets and partonic 2-surfaces and - as already found - are different from singularities at the level of M_c^8. The singularities of string world sheets and partonic 2-surfaces as hyper-complex and co-complex surfaces consist of points and could relate to the singularities at octonionic level.

As already mentioned, Bombieri-Lang conjecture (see `http://tinyurl.com/y887yn5b`) states that, for any variety X of general type over a number field k, the set of k-rational points of X is not Zariski dense (see `http://tinyurl.com/jm9fh74`) in X. Even more, the k-rational points are contained in a *finite* union of lower-dimensional sub-varieties of X.

This conjecture is highly interesting from TGD point of view if one believes in $M^8 - H$ duality. Space-time surfaces $X^4 \subset M_c^8$ can be seen as $M^8 = M^4 \times E^4$ projections of zero loci for real or imaginary parts of octonionic polynomials in o. In complex sense they reduce to $M^4 \times E^4$ projections of algebraic co-dimension 4 surfaces in C^8. If Bombieri-Lang conjectures makes sense in this context, it would state that for a space-time surface $X^4 \subset M^8$ of general type the rational points are contained in a *finite* union of lower-dimensional sub-varieties. Also the conjecture of Vojta (see `http://tinyurl.com/y9sttuu4`) stating that varieties of general type cannot be potentially dense is known to be true for curves and support this general vision.

Could the finite union of sub-varieties correspond to string world sheets, partonic 2-surfaces, and their light-like orbits define singularities? But why just singular sub-varieties would be cognitively simple and have small Kodaira dimension d_K allowing large number of rational points? In the case of partonic orbits one might understand this as a reduction of metric dimension. The orbit is effectively 2-dimensional partonic surface metrically and for the genera $g = 0, 1$ rational points are dense. For string world sheets with handle number smaller than 2 the situation is same.

The proposed realizations of associativity and commutativity provide additional support for this picture. Criticality guaranteeing associativity/commutativity would select preferred space-time surfaces as also string world sheets and partonic 2-surfaces.

Concluding, the general wisdom of algebraic geometry conforms with SH and with the vision about the localization of cognitive representations at 2-surfaces. There are of many possible options for detailed interpretation and certainly the above sketch cannot be correct at the level of details.

3.2.7 About $h_{eff}/h = n$ as the number of sheets of Galois covering

The following considerations were motivated by the observation of a very stupid mistake that I have made repeatedly in some articles about TGD. Planck constant $h_{eff}/h = n$ corresponds naturally to the number of sheets of the covering space defined by the space-time surface.

I have however claimed that one has $n = ord(G)$, where $ord(G)$ is the order of the Galois group G associated with the extension of rationals assignable to the sector of "world of classical worlds" (WCW) and the dynamics of the space-time surface (what this means will be considered below).

This claim of course cannot be true since the generic point of extension G has some subgroup H leaving it invariant and one has $n = ord(G)/ord(H)$ dividing $ord(G)$. Equality holds true only for Abelian extensions with cyclic G. For singular points isotropy group is $H_1 \sup H$ so that $ord(H_1)/ord(H)$ sheets of the covering touch each other. I do not know how I have ended up to a conclusion, which is so obviously wrong, and how I have managed for so long to not notice my blunder.

This observation forced me to consider more precisely what the idea about Galois group acting as a number theoretic symmetry group really means at space-time level and it turned out that $M^8 - H$ correspondence [L36] (see `http://tinyurl.com/yd43o2n2`) gives a precise meaning for this idea.

Consider first the action of Galois group (see `http://tinyurl.com/y8grabt2` and `http://tinyurl.com/ydze5psx`).

1. The action of Galois group leaves invariant the number theoretic norm characterizing the extension. The generic orbit of Galois group can be regarded as a discrete coset space G/H, $H \subset G$. The action of Galois group is transitive for irreducible polynomials so that any two points at the orbit are G-related. For the singular points the isotropy group is larger than for generic points and the orbit is G/H_1, $H_1 \sup H$ so that the number of points of the orbit divides n. Since rationals remain invariant under G, the orbit of any rational point contains only single point. The orbit of a point in the complement of rationals under G is analogous to an orbit of a point of sphere under discrete subgroup of $SO(3)$.

   $n = ord(G)/ord(H)$ divides the order $ord(G)$ of Galois group G. The largest possible Galois group for n-D algebraic extension is permutation group S_n. A theorem of Frobenius states that this can be achieved for $n = p$, p prime if there is only single pair of complex roots (see `http://tinyurl.com/y8grabt2`). Prime-dimensional extensions with $h_{eff}/h = p$ would have maximal number theoretical symmetries and could be very special physically: p-adic physics again!

2. The action of G on a point of space-time surface with embedding space coordinates in n-D extension of rationals gives rise to an orbit containing n points except when the isotropy group leaving the point is larger than for a generic point. One therefore obtains singular covering with the sheets of the covering touching each other at singular points. Rational points are maximally singular points at which all sheets of the covering touch each other.

3. At QFT limit of TGD the n dynamically identical sheets of covering are effectively replaced with single one and this effectively replaces h with $h_{eff} = n \times h$ in the exponent of action (Planck constant is still the familiar h at the fundamental level). n is naturally the dimension of the extension and thus satisfies $n \leq ord(G)$. $n = ord(G)$ is satisfied only if G is cyclic group.

The challenge is to define what space-time surface as Galois covering does really mean!

1. The surface considered can be partonic 2-surface, string world sheet, space-like 3-surface at the boundary of CD, light-like orbit of partonic 2-surface, or space-time surface. What one actually has is only the data given by these discrete points having embedding space coordinates in a given extension of rationals. One considers an extension of rationals determined by irreducible polynomial P but in p-adic context also roots of P determine finite-D extensions since e^p is ordinary p-adic number.

2. Somehow this data should give rise to possibly unique continuous surface. At the level of $H = M^4 \times CP_2$ this is impossible unless the dynamics satisfies besides the action principle also a huge number of additional conditions reducing the initial value data ans/or boundary data to a condition that the surface contains a discrete set of algebraic points.

   This condition is horribly strong, much more stringent than holography and even strong holography (SH) implied by the general coordinate invariance (GCI) in TGD framework. However, preferred extremal property at level of $M^4 \times CP_2$ following basically from GCI in TGD context might be equivalent with the reduction of boundary data to discrete data if $M^8 - H$ correspondence [L36] (see `http://tinyurl.com/yd43o2n2`) is accepted. These data would be analogous to discrete data characterizing computer program so that an analog of computationalism would emerge [L29] (see `http://tinyurl.com/y75246rk`).

One can argue that somehow the action of discrete Galois group must have a lift to a continuous flow.

1. The linear superposition of the extension in the field of rationals does not extend uniquely to a linear superposition in the field reals since the expression of real number as sum of units of extension with real coefficients is highly non-unique. Therefore the naïve extension of the extension of Galois group to all points of space-time surface fails.
2. The old idea already due to Riemann is that Galois group is represented as the first homotopy group of the space. The space with homotopy group π_1 has coverings for which points remain invariant under subgroup H of the homotopy group. For the universal covering the number of sheets equals to the order of π_1. For the other coverings there is subgroup $H \subset \pi_1$ leaving the points invariant. For instance, for homotopy group $\pi_1(S^1) = Z$ the subgroup is nZ and one has $Z/nZ = Z_p$ as the group of n-sheeted covering. For physical reasons its seems reasonable to restrict to finite-D Galois extensions and thus to finite homotopy groups.

   $\pi_1 - G$ correspondence would allow to lift the action of Galois group to a flow determined only up to homotopy so that this condition is far from being sufficient.
3. A stronger condition would be that π_1 and therefore also G can be realized as a discrete subgroup of the isometry group of $H = M^4 \times CP_2$ or of M^8 ($M^8 - H$ correspondence) and can be lifted to continuous flow. Also this condition looks too weak to realize the required miracle. This lift is however strongly suggested by Langlands correspondence [K46, K47] (see `http://tinyurl.com/y9x5vkeo`).

The physically natural condition is that the preferred extremal property fixes the surface or at least space-time surface from a very small amount of data. The discrete set of algebraic points in given extension should serve as an analog of boundary data or initial value data.

1. $M^8 - H$ correspondence [L36] (see `http://tinyurl.com/yd43o2n2`) could indeed realize this idea. At the level of M^8 space-time surfaces would be algebraic varieties whereas at the level of H they would be preferred extremals of an action principle which is sum of Kähler action and minimal surface term.

   They would thus satisfy partial differential equations implied by the variational principle and infinite number of gauge conditions stating that classical Noether charges vanish for a subgroup of symplectic group of $\delta M^4_\pm \times CP_2$. For twistor lift the condition that the induced twistor structure for the 6-D surface represented as a surface in the 12-D Cartesian product of twistor spaces of M^4 and CP_2 reduces to twistor space of the space-time surface and is thus S^2 bundle over 4-D space-time surface.

   The direct map $M^8 \to H$ is possible in the associative space-time regions of $X^4 \subset M^8$ with quaternionic tangent or normal space. These regions correspond to external particles arriving into causal diamond (CD). As surfaces in H they are minimal surfaces and also extremals of Kähler action and do not depend at all on coupling parameters (universality of quantum criticality realized as associativity). In non-associative regions identified as interaction regions inside CDs the dynamics depends on coupling parameters and the direct map $M^8 \to CP_2$ is not possible but preferred extremal property would fix the image in the interior of CD from the boundary data at the boundaries of CD.
2. At the level of M^8 the situation is very simple since space-time surfaces would correspond to zero loci for $RE(P)$ or $IM(P)$ (RE and IM are defined in quaternionic sense) of an octonionic polynomial P obtained from a real polynomial with coefficients having values in the field of rationals or in an extension of rationals. The extension of rationals would correspond to the extension defined by the roots of the polynomial P.

   If the coefficients are not rational but belong to an extension of rationals with Galois group G_0, the Galois group of the extension defined by the polynomial has G_0 as normal subgroup and one can argue that the relative Galois group $G_{rel} = G/G_0$ takes the role of Galois group.

   It seems that $M^8 - H$ correspondence could allow to realize the lift of discrete data to obtain continuous space-time surfaces. The data fixing the real polynomial P and therefore also its octonionic variant are indeed discrete and correspond essentially to the roots of P.

3. One of the elegant features of this picture is that the at the level of M^8 there are highly unique linear coordinates of M^8 consistent with the octonionic structure so that the notion of a M^8 point belonging to extension of rationals does not lead to conflict with GCI. Linear coordinate changes of M^8 coordinates not respecting the property of being a number in extension of rationals would define moduli space so that GCI would be achieved.

Does this option imply the lift of G to π_1 or to even a discrete subgroup of isometries is not clear. Galois group should have a representation as a discrete subgroup of isometry group in order to realize the latter condition and Langlands correspondence supports this as already noticed. Note that only a rather restricted set of Galois groups can be lifted to subgroups of $SU(2)$ appearing in McKay correspondence and hierarchy of inclusions of hyper-finite factors of type II_1 labelled by these subgroups forming so called ADE hierarchy in 1-1 correspondence with ADE type Lie groups [K99, K33] (see `http://tinyurl.com/ybavqvvr`). One must notice that there are additional complexities due to the possibility of quaternionic structure which bring in the Galois group SO(3) of quaternions.

Remark: After writing this article a considerable progress in understanding of $h_{eff}/h = n$ as number of sheets of Galois covering emerged. By M^8-duality space-time surface can be seen as zero locus for real or imaginary part (regarding octonions as sums of quaternionic real and imaginary parts) allows a nice understanding of space-time surface as an $h_{eff}/h = n$-fold Galois covering. M^8 is complexified by adding an imaginary unit i commuting with octonionic imaginary units. Also space-time surface is complexified to 8-D surface in complexified M^8. One can say that ordinary space-time surface is the "real part" of this complexified space-time surface just like x is the real part of a complex number $x + iy$. Space-time surface can be also seen as a root of n:th order polynomial with n complex branches and the projections of complex roots to "real part" of M^8 define space-time surface as an n-fold covering space in which Galois group acts.

3.2.8 Connection with infinite primes

The idea about space-time surfaces as zero loci of polynomials emerged for the first time as I tried to understand the physical interpretation of infinite primes [K84], which were motivated by TGD inspired theory of consciousness. Infinite primes form an infinite hierarchy. At the lowest level the basic entity is the product $X = \prod_p p$ of all finite primes. The physical interpretation could be as an analog of fermionic sea with fermion states labelled by finite primes p.

1. The simplest infinite primes are of form $P = X \pm 1$ as is easy to see. One can construct more complex infinite primes as infinite integers of form $nX/r + mr$. Here r is square free integer, n is integer having no common factors with r, and m can have only factors possessed also by r.

   The interpretation is that r defines fermionic state obtained by kicking from Dirac sea the fermions labelled by the prime factors of r. The integers n and m define bosonic excitations in which k:th power of p corresponds to k bosons in state labelled by p. One can also construct more complex infinite primes as polynomials of X and having no rational factors. In fact, X becomes coordinate variable in the correspondence with polynomials.

2. This process can be repeated at the next level. Now one introduces product $Y = \prod_P P$ of all primes at the previous level and repeats the same construction. These infinite correspond to polynomials of Y with coefficients given by rational functions of X. Primality means irreducibility in the field of rational functions so that solving Y in terms of X would give algebraic function.

3. At the lowest level are ordinary primes. At the next level the infinite primes are indeed infinite in real sense but have p-adic norms equal to unity. They can be mapped to polynomials $P(x_1)$ with rational coefficients and the simplest polynomials are monomials with rational root. Higher polynomials are irreducible polynomials with algebraic roots. At the third level of hierarchy one has polynomials $P(x_2|x_1)$ of two variables. They are polynomials of x_1 with coefficients with are rational functions of x_1. This hierarchy can be continued.

   One can define also infinite integers as products of infinite primes at various levels of hierarchy and even infinite rationals.

4. This hierarchy can be interpreted in terms of a repeated quantization of an arithmetic supersymmetric quantum field theory with elementary particles labelled by primes at given level of hierarchy. Physical picture suggests that the hierarchy of second quantizations is realized also in Nature and corresponds to the hierarchy of space-time sheets.

5. One could consider a mapping $P(x_n|x_{n-1}|..|x_1)$ by a diagonal projection $x_i = x$ to polynomials of single variable x. One could replace x with complexified octonic coordinate o_c. Could this correspondence give rise to octonionic polynomials and could the connection with second quantization give classical space-time correlates of real quantum states assignable to infinite primes and integers? Even quantum states defining counterparts of infinite rationals could be considered. One could require that the real norm of these infinite rationals equals to one. They would define infinite number of real units with arbitrarily complex number theoretical anatomy. The extension of real numbers by these units would mean huge extension of the notion of real number and one could say that each real point corresponds to platonic defined by these units closed under multiplication.

   In ZEO zero energy states formed by pairs of positive and negative energy could correspond to these states physically. The condition that the ratio is unit would have also a physical interpretation in terms of particle content.

6. As already noticed, the notions of analyticity, quaternionicity, and octonionicity could be seen as a manifestation of polynomials in algebras defined by adding repeatedly a new non-commuting imaginary unit to already existing algebra. The dimension of the algebra is doubled in each step so that dimension comes as a power of 2. The algebra of polynomials with real coefficients is commutative and associative. This encourages the crazy idea that the spaces are indeed realized and the generalization of $M^8 - H$ duality holds true at each level. At level k the counterpart for CP_2 (for $k = 3$) would be as moduli space for sub-spaces of dimension 2^{k-1} for which tangent space reduces to the algebra at level $k - 1$. For $k = 2$ CP_1 is the moduli space and could correspond to twistor sphere. Essentially Grassmannian $Gl(2^k, 2^{k-1})$ would be in question. This brings in mind twistor Grassmann approach involving hierarchy of Grassmannians too, which however allows all dimensions. What is interesting that the spinor bundle for space of even dimension d has fiber with dimension $2^{d/2}$.

   The number of arguments for the hierarchy of polynomials assignable to the hierarchy of infinite primes increases by one at each step. Hence these two hierarchies are different.

The vanishing of the octonionic polynomials indeed allow a decomposition to products of prime polynomials with roots which in general are algebraic numbers and an exciting possibility is that the prime polynomials have interpretation as counterparts of elementary particles in very general sense.

Infinite primes can be mapped to polynomials and the most natural counterpart for the infinite rational would be as a complexified octonionic rational function $P_1(t)/P_2(t-T)$, where T is real octonion, with coefficients in extension of rationals. This would naturally give the geometry CD. The assignment of opposite boundaries of CD to $P_1(t)$ and $P_2(t-T)$ is suggestive and identification of zero loci of $IM(P_1)$ and $IM(P_2)$ as incoming and outgoing particles would be natural. The zero and ∞ loci for $RE(P_1/P_2)$ would define interaction between these space-time varieties and should give rise to wormhole contacts connecting them. Note that the linearity of $IM(o_1o_2)$ in $IM(o_i)$ and non-linearity of $RE(o_1o_2)$ in $RE(o_i)$ would be a key element behind this identification. This idea will be discussed in more detail in the section "Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view".

3.3 Super variant of octonionic algebraic geometry and space-time surfaces as correlates for fermionic states

Could the octonionic level provide an elegant description of fermions in terms of super variant of octonionic algebraic geometry? Could one even construct scattering amplitudes at the level of M^8 using the variant of the twistor approach discussed in [K35, K79]?

The idea about super-geometry is of course very different from the idea that fermionic statistics is realized in terms of the spinor structure of "world of classical worlds" (WCW) but

$M^8 - H$ duality could however map these ideas and also number theoretic and geometric vision to each other. The angel of geometry and the devil of algebra could be dual to each other.

In the following I start from the notion of emergence generalized to the vision that entire physics emerges from the notion of number. This naturally leads to an identification of super-variants of various number fields, in particular of complexified octonions. After that super variants of $RE(P) = 0$ and $IM(P) = 0$ conditions are discussed, and the surprising finding is that the conditions might allow only single fermion states localized at strings. This would allow only single particle in the super-multiplet and would mean breaking of SUSY. This picture would be consistent with the earlier H picture about construction of scattering amplitudes [K35, K79]. Finally the problems related to the detailed physical interpretation are discussed.

3.3.1 About emergence

The notion of emergence is fashionable in the recent day physics, in particular, he belief is that 3-space emerges in some manner. In the sequel I consider briefly the standard view about emergence idea from TGD point of view, then suggest that the emergence in the deepest sense requires emergence of physics from the notion of number and that complexified octonions [L36] [L37, L38, L25, L35] are the most plausible candidate in this respect. After that I will show that number theory generalizes to super-number theory: super-number fields make sense and one can define the notion of super-prime. Every new step of progress creates worry about consistency with the earlier work, now the work done during last months with physics as octonionic algebraic geometry and also this aspect is discussed.

1. The notion of holography is behind the emergence of 3-space and implies that the notion of 2-space is taken as input. This could be justified by conformal invariance.
2. The key idea is that 3-space emerges somehow from entanglement. There is something that must entangle and this something must be labelled by points of space: one must introduce a discretised space. Then one must do some handwaving to make it 3-D - perhaps by arguing that holography based on 2-D holograms is unique by conformal invariance. The next hand-wave would replace this as a 3-D continuous space at infrared limit.
3. How to get space-time and how to get general coordinate invariance? How to get the symmetries of standard model and special relativity? Somehow all this must be smuggled into the theory when the audience is cheated to direct its attention elsewhere. This Münchausen trick requires a professional magician!
4. One attempt could take as starting point what I call strong form of holography (SH) in which 2-D data determine 4-D physics. Just like 2-D real analytic function determines analytic function of two complex variables in spacetime of 2 complex dimensions by analytic continuation (this hints strongly to quaternions). This is possible if conformal invariance is generalized to that for light-like 3-surfaces such as light-cone boundary. But the emergence magician should do the same without these.

   In TGD one could make this even simpler. Octonionic polynomials and rational functions are obtained from real polynomials of real variable by octonion-analytic continuation. And since polynomials and rational functions P_1/P_2 are in question their values at finite number of discrete points determined them if the orders of P_1 and P_2 are known!

   If one accepts adelic hierarchy based on extensions of rationals the coefficients of polynomials are in extensions of rationals and the situation simplifies further. The criticality conditions guaranteeing associativity for external particles is one more simplification: everything b becomes discrete. The physics at fundamental level could be incredibly simple: discrete number of points determines space-time surfaces as zero loci for $RE(P)$ or $IM(P)$ (octonions are decomposed to two quaternions gives $RE(o)$ and $IM(o)$).

   How this is mapped to physics leading to standard model emerging from the formulation in $M \times CP_2$ This map exists - I call it $M^8 - H$ duality - and takes space-time varieties in Minkowskian sector of complexified octonions to a space-time surface in $M^4 \times CP_2$ coding for standard model quantum numbers and classical fields.

How to get all this without bringing in octonionic embedding space: this is the challenge for the emergence-magician! I am afraid this this trick is impossible. I will however propose a deeper for what emergence is. It would not be emergence of space-time and all physics from entanglement but from the notion of number, which is at the base of all mathematics. This view led to a discovery of the notion of super-number field, a completely new mathematical concept, which should show how deep the idea is.

3.3.2 Does physics emerge from the notion of number field?

Concerning emergence one can start from a totally different point of view. Even if one gets rid of space as something fundamental from Hilbert sapce and entanglement, one has not reached the most fundamental level. Structures like Hilbert space, manifold, etc. are not fundamental mathematical structures: they require the notion of number field. Number field is the fundamental notion.

Could entire physics emerge from the notion of number field alone: space-time, fermions, standard model interactions, gravitation? There are good hopes about this in TGD framework if one accepts M^8-H duality and physics as octonionic algebraic geometry! One could however argue that fermions do not follow from the notion of number field alone. The real surprise was that formalizing this more precisely led to a realization that the very notion of number field generalizes to what one could call super-number field!

Emergence of physics from complexified octonionic algebraic geometry

Consider first the situation for number fields postponing the addition of attribute "super" later.

1. Number field endowed with basic arithmetic operations $+, -, \cdot, /$ is the basic notion for anyone wanting to make theoretical physics. There is a rich repertoire of number fields. Finite fields, rationals and their extensions, real numbers, complex numbers, quaternions, and octonions. There also p-adic numbers and their extensions induced by extensions of rationals and fusing into adele forming basic structure of adelic physics. Even the complex, quaternionic, and octonionic rationals and their extensions make sense. p-Adic variants of say octonions must be however restricted to have coefficients belonging to an extension of rationals unless one is willing to give up field property (the p-adic analog of norm squared can vanish in higher p-adic dimensions so that inverse need not exist).

   There are also function fields consisting of functions with local arithmetic operations. Analytic functions of complex variable provides the basic example. If function vanishes at some point its inverse element diverges at the same point. Function fields are derived objects rather than fundamental.

2. Octonions are the largest classical number field and are therefore the natural choice if one wants to reduce physics to the notion of number. Since one wants also algebraic extensions of rationals, it is natural to introduce the notion of complexified octonion by introducing an additional imaginary unit - call it i, commuting with the 7 octonionic imaginary units I_k. One obtains complexified octonions.

   That this is not a global number field anymore turns out to be a blessing physically. Complexified octonion z_kE^k has $z_k = z_k + iy_k$. The complex valued norm of octonion is given by $z_0^2 + ...z_7^2$ (there is no conjugation involved. The norm vanishes at the complex surface $z_0^2 + ...z_7^2 = 0$ defining a 7-D surface in 7-D O_c (the dimension is defined in complex sense). At this surface - complexified light-cone boundary - number field theory property fails but is preserved elsewhere since one can construct the inverse of octonion.

   At the real section M^8 (8-D Minkowski space with one real (imaginary) coordinate and 7 imaginary (real) coordinates the vanishing takes place also. This surface corresponds to the 7-D light-cone boundary of 8-D Minkowskian light-cone. This suggests that light-like propagation is basically due to the complexification of octonions implying local failure of the number field property. Same happens also in other real sections with $0 < n < 8$ real coordinates and $0 < m = 8 - n < 8$ imaginary coordinates and one obtains variant of light-cone with different signatures. Euclidian signature corresponding to $m = 0$ or $m = 8$ is an exception: light-cone boundary reduces to single point in this case and one has genuine number field - no propagation is possible in Euclidian signature.

Similar argument applies in the case of complexified quaternions Q_c and complexified complex numbers $z_1 + z_2 I \in C_c$, where I is octonionic imaginary unit. For Q_c one obtains ordinary 3-D light-cone boundary in real section and 1-D light-cone boundary in the case of C_c. It seems that physics demands complexification! The restriction to real sector follows from the requirement that norm squared reduces to a real number. All real sectors are possible and I have already considered the question whether this should be taken as a prediction of TGD and whether it is testable.

Super-octonionic algebraic geometry

There is also a natural generalization of octonionic TGD to super-octonionic TGD based on octonionic triality. $SO(1,7)$ allows besides 8-D vector representations also spinor representations 8_c and $\overline{8}_c$. This suggests that super variant of number field of octonions might make sense. One would have $o = o_8 + o_{c,8} + \overline{0}_{c,8}$.

1. Should one combine o_8, $o_{c,8}$ and $\overline{o}_{c,8}$ to a coordinate triplet $(o_8, o_{c,8}, \overline{o}_{c,8})$ as done in super-symmetric theories to construct super-fields? The introduction of super-fields as primary dynamical variables is a good idea now since the very idea is to reduce physics to algebraic geometry at the level of M^8. Polynomials of super-octonions defining space-time varieties as zero loci for their real or imaginary part in quaternionic sense could however take the role of super fields. Space-time surface would correspond to zero loci for $RE(P)$ or $IM(P)$.
2. The idea about super-octonions should be consistent with the idea that we live in a complexified number field. How to define the notion of super-octonion? The tensor product $8 \otimes 8_c$ contains 8_c and $8 \otimes 8_{\overline{c}}$ contains $8_{\overline{c}}$ and one can use Glebsch-Gordan coefficients to contract o and θ_c and o and $\overline{\theta}_{c,n}$. The tensor product of 8_c and $8_{\overline{c}}$ defined using structure constants defining octonion product gives 8. Therefore one must have

$$o_s = o + \Psi_c \times \theta_{\overline{c}} + \Psi_{\overline{c}} \times \theta_c \ , \tag{3.3.1}$$

   where the products are octonion products. Super parts of super-coordinates would not be just Grassmann numbers but octonionic products of Grassmann numbers with octonionic spinors in 8_c and $\overline{8}_c$. This would bring in the octonionic analogs of spinor fields into the octonionic geometry.

   This seems to be consistent with super field theories since octonionic polynomials and even rational functions would give the analogs of super-fields. What TGD would provide would be an algebraic geometrization of super-fields.
3. What is the meaning of the conditions $RE(P) = 0$ and $IM(P) = 0$ for super-octonions? Does this condition hold true for all $d_G = 2^{16}$ super components of $P(o_s)$ or is it enough to pose the condition only for the octonionic part of $P(o)$? In the latter case Ψ_c and $\Psi_{\overline{c}}$ would be free and this does not seem sensical and does not conform with octonionic super-symmetry. Therefore the first option will be studied in the sequel.

If super-octonions for a super variant of number field so that also inverse of super-octonion is well-defined, then even rational functions of complexified super-octonions makes sense and poles have interpretation in terms of 8-D light-fronts (partonic orbits at level of H). The notion must make sense also for other classical number fields, finite fields, rationals and their extensions, and p-adic numbers and their extensions. Does this structure form a generalization of number field to a super counter part of number field? The easiest manner to kill the idea is to check what happens in the case of reals.

1. The super-real would be of form $s = x + y\theta$, $\theta^2 = 0$. Sum and product are obviously well-defined. The inverse is also well-defined and given by $1/s = (x - y\theta))/x^2$. Note that for complex number $x + iy$ the inverse would be $\overline{z}/z\overline{z} = (x - yi)/(x^2 + y^2)$. The formula for super-inverse follows from the same formula as the inverse of complex number by defining conjugate of super-real s as $\overline{s} = x - y\theta$ and the norm squared of s as $|s|^2 = s\overline{s} = x^2$.

   One can identify super-integers as $N = m + n\theta$. One can also identify super-real units as number of unit norm. Any number $1_n = 1 + n\theta$ has unit norm and the norms form an Abelian group under multiplication: $1_m 1_n = 1_{m+n}$. Similar non-uniqueness of units occurs also for algebraic extensions of rationals.

2. Could one have super variant of number theory? Can one identify super-primes? Super-norm satisfies the usual defining property $|xy| = |x||y|$. Super-prime is defined only apart from the multiplicative factor 1_m giving not contribution to the norm. This is not a problem but a more rigorous formulation leads to the replacement of primes with prime ideals labelled by primes already in the ordinary number theory.

   If the norm of super-prime is ordinary prime it cannot decompose to a product of super-primes. Not all super-primes having given ordinary prime as norm are however independent. If super-primes $p + n\theta$ and $p + m\theta$ differ by a multiplication with unit $1_r = 1 + r\theta$, one has $n - m = pr$. Hence there are only p super-primes with norm p and they can be taken $p_s = p + k\theta$, $k \in \{0, p-1\}$. A structure analogous to a cyclic group Z_p emerges.

   Note that also θ is somewhat analogous to prime although its norm is vanishing.

3. Just for fun, one an ask what is the super counterpart of Riemann Zeta. Riemann zeta can be regarded as an analog of thermodynamical partition function reducing to a product for partition functions for bosonic systems labelled by primes p. The contribution from prime p is factor $1/(1-p^{-s})$. p^{-s} is analogous to Boltzmann weight $N(E)exp(-E/T)$, where $N(E)$ is number of states with energy E. The degeneracy of states labelled by prime p is for ordinary primes $N(p) = 1$. For super-primes the degeneracy is $N(p) = p$ and the weight becomes $1/(1-N(p)p^{-s}) = 1/(1-p^{-s+1})$. Super Riemann zeta is therefore $zeta(s-1)$ having critical line at $s = 3/2$ rather than at $s = 1/2$ and trivial zeros at real points $s = -1, -3, -5$, rather than at $s = -2, -4, -6, ...$

There are good reasons to expect that the above arguments work also for algebraic extensions of super-rationals and in fact for all number fields, even for super-variants of complex numbers, quaternions and octonions. This because the conditions for invertibility reduce to that for real numbers. One would have a generalization of number theory to super-number theory! Net search gives no references to anything like this. Perhaps the generalization has not been noticed because the physical motivation has been lacking. $M^8 - H$ duality would imply that entire physics, including fermion statistics, standard model interactions and gravitation reduces to the notion of number in accordance with number theoretical view about emergence.

Is it possible to satisfy super-variants of $IM(P) = 0$ and $RE(P) = 0$ conditions?

Instead of super-fields one would have a super variant of octonionic algebraic geometry.

1. Super variants of the polynomials and even rational functions make sense and reduce to a sum of octonionic polynomials $P_{kl}\theta_1^k\theta_2^l$, where the integers k and l would be tentatively identified as fermion numbers and θ_k is a shorthand for a monomial of k different thetas. The coefficients in $P_{kl} = P_{kl,n}o^n$ would be given by $P_{kl,n} = P_{n+k+l}B(n+k+l, k+l)$, where $B(r,s) = r!/(r-s)!s!$ is binomial coefficient. The space-time surfaces associated with P_{kl} would be different and they need not be simultaneously critical, which could give rise to a breaking of supersymmetry.

   One would clearly have an upper bound for k and l for given CD. Therefore these many-fermion states must correspond to fundamental particles rather than many-fermion Fock states. One would obtain bosons with non-vanishing fermion numbers if the proposed identification is correct. Octonionic algebraic geometry for single CD would describe only fundamental particles or states with bounded fermion numbers. Fundamental particles would be indeed fundamental also geometrically.

2. One can also now define space-time varieties as zero loci via the conditions $RE(P_s)(o_s) = 0$ or $IM(P_s)(o_s) = 0$. One obtains a collection of 4-surfaces as zero loci of P_{kl}. One would have a correlation with between fermion content and algebraic geometry of the space-time surface unlike in the ordinary super-space approach, where the notion of the geometry remains rather formal and there is no natural coupling between fermionic content and classical geometry. At the level of H this comes from quantum classical correspondence (QCC) stating that the classical Noether charges are equal to eigenvalues of fermionic Noether charges.

In the definition of the first variant of super-octonions I followed the standard idea about what super-coordinates assuming that the super-part of super-octonion is just an anti-commuting Grassmann number without any structure: I just replaced o with $o + \theta_k E^k + \overline{\theta}_k E^k$ regarding θ_k as anticommiting coordinates. Now θ_k receives octonionic coefficient: $\theta_k \rightarrow o_k \theta_k$. θ_k is now analogous to unit vector.

For the super-number field inspired formulation the situation is different since one assigns independent octonionic coordinates to anticommuting degrees of freedom. One has linear space with partially anti-commutative basis. O_c is effectively replaced with O_c^3 so that one has 8+8+8=24-dimensional Cartesian product (it is amusing that the magic dimension 24 for physical polarizations of bosonic string models emerges).

What is the number of equations in the new picture? For N super-coordinates one has 2^N separate monomials analogous to many-fermion states. Now one has $N = 8 + 8 = 16$ and this gives 2^{16} monomials! In the general case $RE = 0$ or $IM = 0$ gives 4 equations for each of the $d_G = 2^{16}$ monomials: the number of equations $RE = 0$ or $IM = 0$ is 4×2^{16} and exceeds the number $d_O = 24$ of octonion valued coordinates. In the original interpretation these equations were regarded as independent and gave different space-time variety for each many-fermion state.

In the new framework these equations cannot be treated independently. One has 24 octonionic coordinates and 2^{16} equations. In the generic case there are no solutions. This is actually what one hopes since otherwise one would have a state involving superposition of many-fermion states with several fermion numbers.

The freedom to pose constraints on the coefficients of Grassmann parameters however allows to reduce degrees of freedom. All coefficients must be however expressible as products of $3 \times 8 = 24$ components of super-octonion.

1. One can have solutions for which both 8_c part and $\overline{8}_c$ parts vanish. This gives the familiar 4 equations for 8 variables and 4-surfaces.
2. Consider first options, which fail. If 8_c- or $\overline{8}_c$ part vanishes one has $d_G = 2^8$ and $4 \times d_G = 4 \times 64$ equations for $dO_= 8 + 8 = 16$ variables having no solutions in the generic case. The restriction of 8_c to its 4-D quaternionic sub-space would give $d_O = 4$ and $4d_G = 4 \times 2^4 = 64$ conditions and 16 variables. The reduction to complex sub-space $z_1 + z_2 I$ of super-octonions would give $d_O = 2^2$ and $4 \times 2^2 = 16$ conditions for $8 + 2 = 10$ variables.
3. The restriction to 1-D sub-space of super-octonions would give $4 \times 2^1 = 8$ conditions and $8 + 1 = 9$ variables. Could the solution be interpreted as 1-D fermionic string assignable to the space-like boundary of space-time surface at the boundary of CD? Skeptic inside me asks whether this could mean the analog of $\mathcal{N} = 1$ SUSY, which is not consistent with H picture.

   Second possibility is restriction to light-like subspace for which powers of light-like octonion reduce effectively to powers of real coordinate. Fermions would be along light-lines in M^8 and along light-like curves in H. The powers of super-octonion have super-part, which belongs to the 1-D super-space in question: only single fermion state is present besides scalar state.
4. There are probably other solutions to the conditions but the presence of fermions certainly forces a localization of fermionic states to lower-dimensional varieties. This is what happens also in H picture. During years the localization of fermion to string worlds sheets and their boundaries has popped up again and again from various arguments. Could one hope that super-number theory provides the eventual argument.

   But how could one understand string world sheets in this framework? If they do not carry fermions at H-level, do they appear naturally as 2-D structures in the ordinary sense?

To sum up, although many details must be checked and up-dated, super-number theory provides and extremely attractive approach promising ultimate emergence as a reduction of physics to the notion of number. When physical theory leads to a discovery of new mathematics, one must take it seriously.

3.3.3 About physical interpretation

Super-octonionic algebraic geometry should be consistent with the H picture in which baryon and lepton numbers as well as other standard model quantum numbers can be understood. There are still many details, which are not properly understood.

The interpretation of theta parameters

The interpretation of theta parameters is not completely straightforward.

1. The first interpretation is that θ_c and $\theta_{\overline{c}}$ correspond to objects with opposite fermion numbers. If this is not the case, one could perhaps define the conjugate of super-coordinate as octonionic conjugate $\overline{o}_s = \overline{o} + \overline{\theta}_1 + \overline{\theta}_2$. This looks ugly but cannot be excluded.

   There is also the question about spinor property. Octonionic spinors are 2-spinors with octonion valued components. Could one say that the coefficients of octonion units have been replaced with Grassmann numbers and the entire 2-component spinor is represented as a pair of θ_c and $\theta_{\overline{c}}$? The two components of spinor in massless theories indeed correspond to massless particle and its antiparticle.

2. One should obtain particles and antiparticles naturally as also separately conserved baryon and lepton numbers (I have also considered the identification of hadrons in terms of anyonic bound states of leptons with fractional charges).

   Quarks and leptons have different coupling to the induced Kähler form at the level of H. It seems impossible to understand this at the level of M^8, where the dynamics is purely algebraic and contains no gauge couplings.

   The difference between quarks and leptons is that they allow color partial waves with triality $t = \pm 1$ and triality $t = 0$. Color partial waves correspond to wave functions in the moduli space CP_2 for $M^4_0 \supset M^2_0$. Could the distinction between quarks and leptons emerge at the level of this moduli space rather than at the fundamental octonionic level? There would be no need for gauge couplings to distinguish between quarks and leptons at the level of M^8. All couplings would follow from the criticality conditions guaranteeing 4-D associativity for external particles (on mass shell states would be critical).

   If so, one would have only the super octonions and θ_c and $\theta_{\overline{c}}$ would correspond to fermions and antifermions with no differentiation to quarks or leptons. Fermion number conservation would be coded by the Grassmann algebra. Quantum classical correspondence (QCC) however suggests that it should be possible to distinguish between quarks and leptons already at M^8 level. Is it really enough that the distinction comes at the level of moduli space for CDs?

One can imagine also other options but they have their problems. Therefore this option will be considered in the sequel.

Questions about quantum numbers

The first questions relate to fermionic statistics.

1. Do super-octonions really realize fermionic statistics and how? The polynomials of super-octonions can have only finite degree in θ and θ_c. One an say that only finite number of fermions are possible at given space-time point. As found, the conditions $IM(P) = 0$ and $RE(P) = 0$ might allow only single fermion strings as solutions perhaps assignable to partonic 2-surfaces.

   Can one allow for given CD arbitrary number of this kind of points as the idea that identical fermions can reside at different points suggests? Or is the number of fermions finite for given CD or correspond to the highest degree monomial of θ and θ_c in P?

   Finite fermion number of CD looks somewhat disappointing at first. The states with high fermion numbers would be described in terms of Cartesian products just like in condensed matter physics. Note however that space-time varieties with different octonionic time axes must be in any case described in this manner. It seems possible to describe the interactions using super-space delta functions stating that the interaction occur only in the intersection points of the space-time surfaces. The delta function would have also super-part as in SUSYs.

2. As found, the theta degree effectively reduces to $d = 1$ for the pointlike solutions, which by above argument are the only possible solutions besides purely bosonic solutions. Only single fermion would be allowed at given point. I have already earlier considered the question whether the partonic 2-surfaces can carry also many-fermion states or not [K35, K79], and adopted the working hypothesis that fermion numbers are not larger than 1 for given wormhole throat, possibly for purely dynamical reasons. This picture however looks too limited. The

many fermion states might not however propagate as ordinary particles (the proposal has been that their propagator pole corresponds to higher power of p^2).

The M^8 description of particle quantum numbers should be consistent with H description.

1. Can octonionic super geometry code for the quantum numbers of the particle states? It seems that super-octonionic polynomials multiplied by octonionic multi-spinors inside single CD can code only for the electroweak quantum numbers of fundamental particles besides their fermion and anti-fermion numbers. What about color?

   As already suggested, color corresponds to partial waves in CP_2 serving as moduli space for $M_0^4 \supset M_0^2$. Also four-momentum and angular momentum are naturally assigned with the translational degrees for the tip of CD assignable with the fundamental particle.

2. Quarks and leptons have different trialities at H level. How can one understand this at M^8 level. Could the color triality of fermion be determined by the color representation assignable to the color decomposition of octonion as $8 = 1 + 1 + 3 + \overline{3}$. This decomposition occurs for all 3 terms in the super-octonion. Could the octet in question correspond to the term $D(8 \otimes 8_c; 8_c)_k^{mn} o_{c,m} \theta_{c,n} E^k$ and analogous $\theta_{\overline{c}}$ term in super octonion. Only this kind of term survives from the entire super-octonion polynomial at fermionic string for the solutions found.

3. There is however a problem: $8 = 1 + 1 + 3 + \overline{3}$ decomposition is not consistent with the idea that θ_c and $\theta_{\overline{c}}$ have definite fermion numbers. Quarks appear only as 3, not $\overline{3}$. Why $\overline{3}$ from θ term and 3 from $\theta_{\overline{c}}$ term should drop out as allowed single fermion state?

There are also other questions.

1. What about twistors in this framework? $M^4 \times CP_1$ as twistor space with CP_1 coding for the choice of $M_0^2 \subset M_0^4$ allows projection to the usual twistor space CP_3. Twistor wave functions describing spin elegantly would correspond to wave functions in the twistor space and one expects that the notion of super-twistor is well-defined also now. The 6-D twistor space $SU(3)/U(2) \times U(1)$ of CP_2 would code besides the choice of $M_0^4 \supset M_0^2$ also quantization axis for color hypercharge and isospin.

2. The intersection of space-time surfaces with S^6 giving analogs of partonic 2-surfaces might make possible for two sparticle lines to fuse to form a third one at these surfaces. This would define sparticle 3-vertex in very much the same manner as in twistor Grassmann approach to $\mathcal{N} = 4$ SUSY.

   H-picture however supports the alternative option that sparticles just scatter but there is no contact interaction defining analog of 3-vertex. If the lines can carry only single fermion, the H picture about twistor diagrams [K35, K79] would be realized also at the level of M^8! This means breaking of SUSY since only single fermion states from the octonionic SUSY multiplet are realized. This would provide and easy - perhaps too easy - explanation for the failure to find SUSY at LHC.

3. What about the sphere S^6 serving as the moduli space for the choices of M_+^8? Should one have wave functions in S^6 or can one restrict the consideration to single M_+^8? As found, one obtains S^6 also as the zero locus of $Im(P) = 0$ for some radii identifiable as values t_n of time coordinates given as roots of $P(t)$: as matter of fact, $S^6(t_n)$ is a solution of both $RE(P) = 0$ and $IM(P) = 0$. Can one identify the intersections $X^4 \cap S^6$ are 2-D as partonic 2-surfaces serving as topological vertices?

3.4 Could scattering amplitudes be computed in the octonionic framework?

Octonionic algebraic geometry might provide incredibly simple framework for constructing scattering amplitudes since now variational principle is involved and WCW reduces to a discrete set of points in extension of rationals.

3.4.1 Could scattering amplitudes be computed at the level of M^8?

It would be extremely nice if the scattering amplitudes could be computed at the octonionic level by using a generalization of twistor approach in ZEO finding a nice justification at the level of M^8. Something rather similar to $\mathcal{N} = 4$ twistor Grassmann approach suggests itself.

1. In ZEO picture one would consider the situation in which the passive boundary of CD and members of state pairs at it appearing in zero energy state remain fixed during the sequence of state function reductions inducing stepwise drift of the active boundary of CD and change of states at it by unitary U-matrix at each step following by a localization in the moduli space for the positions of the active boundary.
2. At the active boundary one would obtain quantum superposition of states corresponding to different octonionic geometries for the outgoing particles. Instead of functional integral one would have sum over discrete points of WCW. WCW coordinates would be the coefficients of polynomial P in the extension of rationals. This would give undefined result without additional constraints since rationals are a dense set of reals.

   Criticality however serves as a constraint on the coefficients of the polynomials and is expected to realize finite measurement resolution, and hopefully give a well defined finite result in the summation. Criticality for the outgoing states would realize purely number theoretically the cutoff due to finite measurement resolution and would be absolutely essential for the finiteness and well-definedness of the theory.

3.4.2 Interaction vertices for space-time surfaces with the same CD

Consider interaction vertices for space-time surfaces associated with given CD. At the level of H the fundamental interactions vertices are partonic 2-surfaces at which 3 light-like partonic orbits meet. The incoming light-like sparticle lines scatter at this surface and they are not assumed to meet at single vertex. This assumption is motivated because it allows to avoid infinities but one must be ready to challenge it. It is essential that wormhole throats appear in pairs assignable to wormhole contacts and also contacts form pairs by the conservation of Kähler magnetic flux.

What could be the counterpart of this picture at level of M^8?

1. The simplest interaction could be associated with the common stable intersection points of the space-time regions. By dimensional consideration these intersections are stable and form a discrete set. This would however allow only 2-vertices involved in processes like mixing of states. In the generic case the intersection would consist of discrete points.
2. A stronger condition would be that these points belong to the extension of rationals defining adeles or is extension defined by the polynomial P. This would conform with the idea that scattering amplitudes involve only data associated with the points in the extension. The interaction points could be ramified points at which the action of a subgroup H of Galois group G would leave sheets of the Galois covering invariant so that some number of sheets would touch each other. I have discussed this proposal in [L33]. These points could be seen as analogs of interaction points in QFT description in terms of n-point functions and the sum over polynomials would give rise to the analog over integral over different n-point configurations.
3. A possible interpretation is that if the subgroup $H \subset G$ has k-elements the vertex represents meeting of k sparticle lines and thus k-vertex would be in question. This picture is not what the H view about twistor diagrams [K79] suggests: in these diagrams sparticle lines at the light-like orbits of partonic 2-surfaces do not meet at single point but only scatter at partonic 2-surface, where three light-like orbits of partonic 2-surfaces meet.
4. An alternative interpretation is that k-vertex describes the decay of particle to k fractional particles at partonic 2-surfaces and has nothing do with the usual interaction vertex.

This proposal need not describe usual particle scattering. Could the intersection of space-time varieties defined as zero loci for $RE(P_i)$ and $IM(P_i)$ with the special solutions $S^6(t_n)$ and $CD = M^4_+ \cap M^4_-$ define the loci of interaction? It is difficult to believe that these special solutions could be only a beauty spot of the theory. $X^2 = X^4 \cap S^6(t_n)$ is 2-D and $X^0 = X^4 \cap CD$ consists of discrete points.

Consider now the possible role of the singular ($RE(P) = IM(P) = 0$) maximally critical surface $S^6(t_n)$ in the scattering.

1. As already found, the 6-D spheres S^6 with radii t_n given by the zeros of $P(t)$ are universal and have interpretation as $t = t_n$ snapshots of 7-D spherical light front projection to $t = t_n$ 3-balls as cross sections of 4-D CD. Could the 2-D intersection $X^2 = X^4 \cap S^6(t_n)$ play a fundamental role in the description of interaction vertices?
2. Suppose that 3-vertices realize the dynamical realization of octonionic SUSY predicting large number of sparticles. Could one understand in this framework the 3-vertex for the orbits X_i^3 of partonic 2-surfaces meeting each other along their 2-D end defining partonic 2-surface and undersand how 3 fermions lines meet at single point in this picture?
3. Assume that 3 partonic orbits X_i^3, $i = 1, 2, 3$ meet at $X^2 = X^4 \cap S^6(t_n)$. That this occurs could be part of boundary conditions, which should follow from interaction consistency. If fermions just *go through* the X_i^2 in time direction they cannot meet at single point in the generic case. If the sparticle lines however can *move along* X^2 - maybe due the fact that an intersection $X^2 = X^4 \cap S^6(t_n)$ is in question - they intersect in the generic case and fuse to a third fermion line. Note that this portion of fermion line would be space-like whereas outside X^2 the line would be light-like. This can be used as an objection against the idea.

The picture allowing 3-vertices would be different from H picture in which fermion lines only scatter and only 2+2 fermion vertex assignable to topological 3-vertex is fundamental.

1. One would have 2 wormhole contacts carrying fermion and third one carrying fermion anti-fermion pair at its opposite throats and analogous to boson. Of course, one can reproduce the earlier picture by giving up the condition about supersymmetric fermionic 3-vertex. On the other hand, the idea that interactions occur only at discrete points in extension of rationals is extremely attractive.
2. The surprising outcome from the construction of solutions of super-variants of $RE(P) = 0$ and $IM(P) = 0$ conditions was that if the superpart of super-octonion is non-vanishing, the variety can be only 1-D string like entity carrying one-fermion state. This does allow strings with higher fermion number so that the 3-vertex would not be possible! This suggests that fermionic lines appear as sub-varieties of space-time variety.

   If so the original picture [K79] applying at the level of H applies also at the level of M^8. SUSY is broken dynamically allowing only single fermion states localized at strings and scattering of these occurs by classical interactions at the partonic 2-surfaces defining the topological vertices.
3. The only manner to have a point/line containing sparticle with higher fermion number would be as a singularity along which several branches of super-variety degenerate to single point/line: each variety would carry one fermion line. Unbroken octonionic SUSY would characterize singularities of the space-time varieties, which would be unstable so that SUSY would break. Singularities are indeed critical and thus unstable and also tend to possess enhanced symmetries.

What could be the interpretation of $X^0 = X^4 \cap CD$? For instance, could it be that these points code for 4-momenta classically so that quantum classical correspondence (QCC) would be realized also at the level of M^8 although there are no Noether charges now. But what about angular momenta? Could twistorialization realized in terms of the quaternionic structure of M_0^4 help here. What is the role of the intersections of 6-D twistor bundle of X^4 with 6-D twistor bundle of M_0^4 consisting of discrete points?

The interaction vertex would involve delta function telling that the interacting space-time varieties or their regions touch at same point of M^8. Delta function in theta parameter degrees of freedom and Grassmann integral over them would be also involved and guarantee fermion number conservation. Vertex factor should be determined by arguments used in Grassmannian twistor approach. I have developed a proposal in [K79] but this proposal allows only fermion number ± 1 at fermion lines. Now all members of the multiplet would be allowed.

3.4.3 How could the space-time varieties associated with different CDs interact?

The interaction of space-time surfaces inside given CD is well-defined in the octonionic algebraic geometry. The situation is not so clear for different CDs for which the choice of the origin of octonionic coordinates is in general different and polynomial bases for different CDs do not commute nor associate.

The intuitive expectation is that 4-D/8-D CDs can be located everywhere in M^4/M^8. The polynomials with different origins neither commute nor are associative. Their sum is a polynomial whose coefficients are not real. How could one avoid losing the extremely beautiful associative and commutative algebra of polynomials?

1. Should one assume that the physics observable by single conscious observer corresponds to single CD defining the perceptive field of this observer [L44].
2. Or should one give up associativity and allow products (but not sums since one should give up the assumption that the coefficients of polynomials are real) of polynomials associated with different CDs as an analog for the formation of free many-particle states.

Consider first what happens for the single particle solutions defined as solutions of either $RE(P_i) = 0$ or $IM(P_i) = 0$.

1. The polynomials associated with different 8-D CDs do not commute nor associate. Should one allow their products so that one would still *effectively* have a Cartesian product of commutative and associative algebras? This would realize non-commutative and non-associative physics emerging in conformal field theories also at the level of space-time geometry.
2. If the CDs differ by a *real* (time) translation $o_2 = o_1 + T$ one still obtains $IM(P_1) = 0$ and $IM(P_2) = 0$ as solutions to $IM(P_1P_2) = 0$. This applies also to states with more particles. The identification would be in terms of external particles. For $RE(P_1P_2) = 0$ this is not the case. If the interior of CD corresponds to $RE(P_1P_2) = 0$, the dynamics in the interior is not only non-trivial but also non-commutative and non-associative. Non-trivial interaction would be obtained even without interaction terms in the polynomial vanishing at the boundaries of CD!

   Could one consider allowing only CDs with tips at the same real axis but having all sizes scales? This hierarchy of CD would characterize a particular hierarchy of conscious observers - selves having sub-selves (sub-CDs) [L44]. The allowance of only these CD would be analogous to a fixing of quantization axes.
3. What happens if one allows CDs differing by arbitrary octonion translation? Consider external particles. For P_1 and P_2 RE and IM are defined for different decompositions $o_i = RE(o_i) + n_i Im(o_i)$, where n_i, $i = 1, 2$ is a unit octonion.

   What decomposition should one use for P_1P_2? The decomposition for P_1 or P_2 or some other decomposition? One can express $P_2(o_2)$ using o_1 as coordinate but the coefficients multiplying powers of o_1 from *right* would not be real numbers anymore implying $IM(P_2)_1 \neq IM(P_2)_2$. $IM(P_2)_1 = 0$ makes sense but the presence of particle 1 would have affected particle 2 or vice versa.

   Could one argue that the coordinate systems satisfying the condition that some external particles described by P_i have real coefficients and perhaps serving in the role of observers are preferred? Or could one imagine that o_{12} is a kind of center of mass coordinate? In this case the 4-varieties associated with both particles would be affected. What is clear that the choice of the octonionic coordinate origin would affect the space-time varieties of external particles even if they could remain associative/critical.
4. Are there preferred coordinates in which criticality is preserved? For instance, can one achiever criticality for P_2 on coordinates of o_1 if P_1 is critical. Could one see this as a kind of number theoretic observer effect at the level of space-time geometry?

   Remark: $P_i(o)$ would reduce to a real polynomial at light-like rays with origin for o_i irrespective of the octonionic coordinate used so that the spheres S^6_i with origin at the origin of o_i as solutions of $P_i(o) = 0$ would not be lost.

If one does not give up associativity and commutativity for polynomials, how can one describe the interactions between space-time surfaces inside different CDs at the level of M^8? The following proposal is the simplest one that one can imagine by assuming that interactions take place at discrete points of space-time surfaces with coordinates belonging an extension of rationals.

1. The most straightforward manner would be to introduce Cartesian powers of O and CD:s inside these powers to describe the interaction between CDs with different origin. This would be analogous to what one does in condensed matter physics. What seems clear is that $M^8 - H$ correspondence should map all the factors of $(M^8)^n$ to the same $M^4 \times CP_2$ by a kind of diagonal projection.

   In topological 3-parton vertex X^2 three light-like partonic orbits along X^4 would meet. X^2 would be the contact of X^4 with S^6 associated with second 8-D CD. Together with SH this gives hopes about an elegant description of interactions in terms of connected space-time varieties.

2. The intersection $X_i^4 \cap X_j^4$ consists of discrete set of points. This would suggest that the interaction means transfer of fermion between X_1^4 and X_2^4. The intersection of $X = S_1^6(t_m) \cap S_2^6(t_n)$ is 4-D and space-like. The intersection $X_i^4 \cap X$ consists of discrete points could these discrete points allow to construct interaction vertices.

To make this more concrete, assume that the external particles outside the interaction CD (CD_{int}) defining the interaction region correspond to associative (or co-associative) space-time varieties with different CDs.

Remark: CDs are now 8-dimensional.

1. One can assign the external particles to the Cartesian factors of $(M^8)^n$ giving $(P_1, ..., P_n)$ just like one does in condensed matter physics for particles in 3-space E^3. Inside CD_{int} the Cartesian factors would fuse to single factor and instead of Cartesian product one would have the octonionic product $P = \prod P_i$ plus the condition $RE(P) = 0$ (or $IM(P) = 0$: one should avoid too strong assumptions at this stage) would give to the space-time surface defining the interaction region.

2. $RE(P) = 0$ and $IM(P) = 0$ conditions make sense even, when the polynomials do not have origin at common real axis and give rise to 4 conditions for 8 polynomials of 8 complexified octonion components P^i. It is not possible to reduce the situation at the light-like boundaries of 8-D light-cone to a vanishing of polynomial $P(t)$ of real coordinate t anymore, and one loses the the surfaces S_i^6 as special solutions and therefore also the partonic 2-surfaces $X_i^2 = X^4 \cap S_i^6$. Should one assign all X_i^2 with the intersections of external particles with the two boundaries $\delta_\pm$ CD of CD defining the interaction region. They would intersect $\delta_\pm$CD at highly unique discrete points defining the sparticle interaction vertices. By 7-dimensionality of $\delta_\pm$CD the intersection points would be at the boundaries of 4-D CD and presumably at light-like partonic orbits at which the induced metric is singular at H side at least just as required by H picture.

   The most general external single-sparticle state would be defined by a product P of mutually commuting and associating polynomials with tips of CD along common real axis and satisfying $IM(P_i) = 0$ or $RE(P_i) = 0$. This could give both free and bound states of constituents.

3. Different orders and associations for $P = \prod P_i$ give rise to different interaction regions. This requires a sum over the scattering amplitudes $\sum_p T(\prod_i P_{p(i)})$ associated with the permutations p: $(1, ..., n) \to (p(1), .., p(n))$ and $T = \sum_p U(p) T(P_{p(1)}...P_{p(n)})$ ($T(AB) + T(BA)$ in the simplest case) with suitable phase factors $U(p)$. Note that one does *not* have a sum over the polynomials $P_{p(1)}...P_{p(n)}$ but over the scattering amplitudes associated with them.

4. Depending on the monomial of theta parameters in super-octonion part of P_i, one has plus or minus signs under the exchange of P_i and P_j. One can also have braid group as a lift of the permutation group. In this case given contribution to the scattering amplitude has a phase factor depending on the permutation (say $T = T(AB) + exp(i\theta)T(BA)$.

   One must also form the sum $T = \sum_{Ass} U(Ass) T(Ass(P))$ over all associations for a given permutation with phase factors $U(Ass)$. Here $T = T((AB)C) + UT(A(BC))$, U phase factor, is the simplest case. One has "association statistics" as the analog of braid statistics. Permutations and associations have now a concrete geometric meaning at the level of space-time geometry - also at the level of H.

5. The geometric realization of permutations and associations could relate to the basic problem encountered in the twistorial construction of the scattering amplitudes. One has essentially sum over the cyclic permutations of the external particles but does not know how to construct the amplitudes for general permutations, which correspond to non-planar Feynman diagrams. The geometric realization of the permutations and associations would solve this problem in TGD framework.

3.4.4 Twistor Grassmannians and algebraic geometry

Twistor Grassmannians provide an application of algebraic geometry involving the above described notions [B25] (see `http://tinyurl.com/yd9tf2ya`). This approach allows extremely elegant expressions for planar amplitudes of $\mathcal{N} = 4$ SYM theory in terms of amplitudes formulated in Grassmannians $G(k,n)$.

It seems that this approach generalizes to TGD in such a way that CP_2 degrees of freedom give rise to additional factors in the amplitudes having form very similar to the M^4 part of amplitudes and involving also $G(k,n)$ with ordinary twistor space CP_3 being replaced with the flag manifold $SU(3)/U(1) \times U(1)$: k would now correspond to the number sparticles with negative weak isospin. Therefore the understanding of the algebraic geometry of twistor amplitudes could be helpful also in TGD framework.

Twistor Grassmannian approach very concisely

I try to compress my non-professional understanding of twistor Grassmann approach to some key points.

1. Twistor Grassmannian approach constructs the scattering amplitudes by fusing 3-vertices (+,-,-) (one positive helicity) and (-,+,+) (one negative helicity) to a more complex diagrams. All particles are on mass shell and massless but complex. If only real massless momenta are allowed the scattering amplitudes would allow only collinear gluons. Incoming particles have real momenta.

   Remark: Remarkably, $M^4 \times CP_2$ twistor lift of TGD predicts also complex Noether charges, in particular momenta, already at classical level. Quantal Noether charges should be hermitian operators with real eigenvalues, which suggests that total Noether charges are real. For conformal weights this condition corresponds to conformal confinement. Also $M^8 - H$ duality requires a complexification of octonions by adding commuting imaginary unit and allows to circumvent problems related to the Minkowski signature since the metric tensor can be regarded as Euclidian metric tensor defining complex value norm as bilinear $m^k m_{kl} m^l$ in complexified M^8 so that real metric is obtained only in sub-spaces with real or purely imaginary coordinates. The additional imaginary unit allows also to define what complex algebraic numbers mean.

   The unique property of 3-vertex is that the twistorial formulation for the conservation of four-momentum implies that in the vertex one has either $\lambda_1 \propto \lambda_2 \propto \lambda_3$ or $\overline{\lambda}_1 \propto \overline{\lambda}_2 \propto \overline{\lambda}_3$. These cases correspond to the 2 3-vertices distinguished notationally by the color of the vertex taken to be white or black [B25].

   Remark: One must allow octonionic super-space in M^8 formulation so that octonionic SUSY broken by CP_2 geometry reducing to the quaternionicity of 8-momenta in given scattering diagram is obtained.

2. The conservation condition for the total four-momentum is quadratic in twistor variables for incoming particles. One can linearize this condition by introducing auxiliary Grassmannian $G(k,n)$ over which the tree amplitude can be expressed as a residue integral. The number theoretical beauty of the multiple residue integral is that it can make sense also p-adically unlike ordinary integral.

   The outcome of residue integral is a sum of residues at discrete set of points. One can construct general planar diagrams containing loops from tree diagrams with loops by BCFW recursion. I have considered the possibility that BCFW recursion is trivial in TGD since coupling constants should be invariant under the addition of loops: the proposed scattering diagrammatics however assumed that scattering vertices reduce to scattering vertices for 2 fermions. The justification for renormalization group invariance would be number theoretical: there is no guarantee that infinite sum of diagrams gives simple function defined in all number fields with parameters in extension of rationals (say rational function).

3. The general form of the Grassmannian integrand in $G(k,n)$ can be deduced and follows from Yangian invariance meaning that one has conformal symmetries and their duals which expand to full infinite-dimensional Yangian symmetry. The denominator of the integrand of planar tree diagram is the product of determinants of $k \times k$ minors for the $k \times n$ matrix providing representation of a point of $G(k,n)$ unique apart from $SL(k,k)$ transformations. Only minors consisting of k consecutive columns are assumed in the product. The residue integral is determined by the poles of the denominator. There are also dynamical singularities allowing the amplitude to be non-vanishing only for some special configurations of the external momenta.

4. On mass-shell diagrams obtained by fusing 3-vertices are highly redundant. One can describe the general diagram by using a disk such that its boundary contains the external particles with positive or negative helicity. The diagram has certain number n_F of faces. There are moves, which do not affect the amplitude and it is possible to reduce the number of faces to minimal one: this gives what is called reduced diagram. Reduced diagrams with n_F faces define a unique $n_F - 1$-dimensional sub-manifold of $G(k,n)$ over which the residue integral can be defined. Since the dimension of $G(k,n)$ is finite, also n_F is finite so that the number of diagrams is finite.

5. On mass shell diagrams can be labelled by the permutations of the external lines. This gives a connection with 1+1-dimensional QFTs and with braid group. In 1+1-D integral QFTs however scattering matrix induces only particle exchanges.

   The permutation has simple geometric description: one starts from the boundary point of the diagram and moves always from left or right depending on the color of the point from which one started. One arrives some other point at the boundary and the final points are different for different starting points so that the process assigns a unique perturbation for a given diagram. Diagrams which are obtained by moves from each other define the same permutation. BFCW bridge which is a way to obtain new Yangian invariant corresponds to a permutation of consecutive external particles in the diagram.

6. The poles of the denominator determine the value of the multiple residue integrals. If one allowed all minors, one would have extremely complex structure of singularities. The allowance only cyclically taken minors simplifies the situation dramatically. Singularities correspond to n subgroups of more than 2 collinear k-vectors implying vanishing of some of the minors.

7. Algebraic geometry comes in rescue in the understanding of singularities. Since residue integral is in question, the choice is rather free and only the homology equivalence class of the cell decomposition matters. The poles for a hierarchy with poles inside poles since given singularity contains sub-singularities. This hierarchy gives rise to a what is known as cell composition - stratification - of Grassmannian consisting of varieties with various dimensions. These sub-varieties define representatives for the homology group of Grassmannian. Schubert cells already mentioned define this kind of stratification.

   Remark: The stratification has very strong analogy of the decomposition of catastrophe in Thom's catastrophe theory to pieces of various dimensions. The smaller the dimension, the higher the criticality involved. A connection with quantum criticality of TGD is therefore highly suggestive.

   Cyclicity implies a reduction of the stratification to that for positive Grassmannians for which the points are representable as $k \times n$ matrices with non-negative $k \times k$ determinants. This simplifies the situation even further.

   Yangian symmetries have a geometric interpretation as symmetries of the stratification: level 1 Yangian symmetries are diffeomorphisms preserving the cell decomposition.

Problems of twistor approach

Twistor approach is extremely beautiful and elegant but has some problems.

1. The notion of twistor structure is problematic in curved space-times. In TGD framework the twistor structures of M^4 and CP_2 (E^4) induce twistor structure of space-time surface and the problem disappears just like the problems related to classical conservation laws are circumvented. Complexification of octonions allows to solve the problems related to the metric signature in twistorialization.
2. The description of massive particles is a problem. In TGD framework M^8 approach allows to replace massive particles with particles with octonionic momenta light-like in 8-D sense belonging to quaternionic subspace for a given diagram. The situation reduces to that for ordinary twistors in this quaternionic sub-space but since quaternionic sub-space can vary, additional degrees of freedom bringing in CP_2 emerge and manifest themselves as transversal 8-D mass giving real mass in 4-D sense.
3. Non-planar diagrams are also a problem. In TGD framework a natural guess is that they correspond to various permutations of free particle octonionic polynomials. Their product defines interaction region in the interior of CD to which free particles satisfying associativity conditions (quantum criticality) arrive. If the origins of polynomials are not along same time axis, the polynomials do not commute nor associate. One must sum over their permutations and for each permutation over its associations.

3.4.5 About the concrete construction of twistor amplitudes

At H-side the ground states of super-conformal representations are given by the anti-symmetrized products of the modes of H-spinor fields labelled by four-momentum, color quantum numbers, and electroweak (ew) quantum numbers. At partonic 2-surface one has finite number of many fermion states. Single fermion states are assigned with H-spinor basis and the fermion states form a representation of a finite-D Clifford algebra.

M^8 picture should reproduce the physical equivalent of H picture: in particular, one should understand four-momentum, color quantum numbers, ew quantum numbers, and B and L. M^8-H correspondence requires that the super-twistorial description of scattering amplitudes in M^8 is equivalent with that in H.

The M^8 picture is roughly following.

1. The ground states of super-conformal representations expressible in terms of spinor modes of H correspond at level of M^8 wave functions in super variant of the product $T(M^4)\times T(CP_2)$ of twistor spaces of M^4 and CP_2. This twistor space emerges naturally in M^8-H correspondence from the quaternionicity condition for 8-momenta.
2. Bosonic M^8 degrees of freedom translate to wave functions in the product $T(M^4)\times T(CP_2)$ labelled by four-momentum and color. Super parts of the M^4 and CP_2 twistors code for spin and ew degrees of freedom and fermion numbers. Only a finite number of spin-ew spin states is possible for a given fundamental particle since one has finite-D Grassmann algebra.
3. Contrary to the earlier expectations [K79], the view about scattering diagrams is very similar to that in $\mathcal{N}=4$ SUSY. The analog of 3-gluon vertex is fundamental and emerges naturally from number theoretic vision in which scattering diagrams defines a cognitive representation and vertices of the diagram correspond to fusion of sparticle lines.

Identification of H quantum numbers in terms of M^8 quantum numbers

The first challenge is to understand how M^8-H correspondence maps M^8 quantum numbers to H quantum numbers. At the level of M^8 one does not have action principle and conservation laws must follow from the properties of wave functions in various moduli spaces assignable to 4-D and 8-D CDs that is quaternion and octonion structures. The symmetries of the moduli spaces would dictate the properties of wave functions.

There are three types of symmetries and quantum numbers.

1. WCW quantum numbers

At level of H the quantum numbers in WCW "vibrational" degrees of freedom are associated with the representations of super-symplectic group acting as isometries of WCW. Super-symplectic generators correspond to Hamiltonians labelled by color and angular momentum quantum numbers for $SU(3) \times SO(3)$. In $M^4_\pm$ there are also super-symplectic conformal weights assignable to the radial light-coordinate in $\delta M^4_\pm$. These conformal weights could be complex and might relate closely to the zeros of Riemann zeta [L18]. Physical states should however have integer valued conformal weights (conformal confinement).

At the level of M^8 WCW "vibrational"degrees of freedom are discrete and correspond to the degree of the octonionic polynomial P and its coefficients in the extension of rationals considered. WCW integration reduces to a discrete sum, which should be well-defined by the criticality conditions on the coefficients of the polynomials. $M^8 - H$ correspondence guarantees that 4-varieties in M^8 are mappable to space-time surfaces in H. Therefore also quantum numbers should be mappable to each other.

There are also spinorial degrees of freedom associated with WCW spinors with spin-like quantum numbers assignable to fermionic oscillator operators labelled by spin, ew quantum numbers, fermion numbers, and by super-symplectic conformal weights.

2. Quantum numbers assignable to isometries of H.

These quantum numbers are special assignable to the ground states of the representations of Kac-Moody algebras associated with light-like partonic orbits.

1. The isometry group of H consists of Poincare group and color group for CP_2. M^8 isometries correspond to $8-D$ Poincare group. Only G_2 respects given octonion structure and 8-D Lorentz transformations transform to each other different octonion structures. Quantum numbers consist of 8-momentum and analogs of spin and ew spin. $M^8 - H$ correspondence is non-trivial since one must map light-like quaternionic 8-momenta to 4-momenta and color quantum numbers.

2. There are quantum numbers assignable to cm spinor degrees of freedom. They correspond for both M^8 and H to 8-D spinors and give rise to spin and ew quantum numbers. For these quantum numbers $M^8 - H$ correspondence is trivial. At the level of H baryon and lepton numbers are assignable to the conserved chiralities of H-spinors.

Quantum classical correspondence (QCC) is a key piece of TGD.

1. At the level of H QCC states that the eigenvalues of the fermionic Noether charges are equal to the classical bosonic Noether charges in Cartan algebra implies that fermionic quantum number as also ew quantum numbers and spin have correlates at the level of space-time geometry.

2. A the level of M^8 QCC is very concrete. Both bosonic and superpart of octonions have the decomposition $1+\overline{1}+3+\overline{3}$ under color rotations. Each monomial of theta parameters characterizes one particular many-fermion state containing leptons/antileptons and quarks/antiquarks. Leptons/antileptons are assignable to complexified octonionic units $(1 \pm iI_1)/\sqrt{2}$ defining preferred octonion plane M_2 and quarks/antiquarks are assignable to triplet and antitriplet, which also involve complexified octonion units. One obtains breaking of SUSY in the sense that space-time varieties assignable to different theta monomials are different (one can argue that the sum $8_s + \overline{8}_s$ can be regarded as real).

   Purely leptonic and antileptonic varieties correspond to 1 and $\overline{1}$ and quark and antiquark varieties to 3 and $\overline{3}$ and the monomial transforms as a tensor product of thetas. The monomial has well defined quark and lepton numbers and the interpretation is that it characterizes fundamental sparticle. At the level of H this kind of correspondence follows form QCC.

3. Also super-momentum leads to a characterization of spin and fermion numbers of the state since delta function expressing conservation of super-momentum codes the supersymmetry for scattering amplitudes and gives rise to vertices conserving fermion numbers. Does this mean QCC in the sense that the super parts of super-momentum and super twistor should be associated with space-time varieties with same fermion and spin content?

How the light-like quaternionic 8-momenta are mapped to H quantum numbers?

The key challenge is to understand how the light-like quaternionic 8-momenta are mapped to massive M^4 momenta and color quantum numbers.

1. One has wave function in the space of CP_2 quaternionic four-momenta. M_0^4 momentum can be identified as M_0^2 projection and in general massive unless M_0^2 and M_0^4 are chosen so that the light-like M^8 momentum belongs to M_0^2. The situation is analogous to that in the partonic description of hadron scattering.

   The space of quaternionic sub-spaces $M_0^4 \supset M_0^2$ with this property is parameterized by CP_2, and one obtains color partial waves. The inclusion of the choice of quantization axis extends this space to $T(CP_2) = SU(3)/U(1) \times U(1)$. Without quaternionicity/associativity condition the space of momenta would correspond to M^8.

   The wave functions in the moduli space for the position of the tip of CD and for the choice $M_0^2 \supset M_0^4$ specifying M_0^4 twistor structure and choice of quantization axis of spin correspond to wave functions in the twistor space CP_3 of $M_\pm^4$ coding for momentum and spin.

   Remark: The inclusion of M^4 spin quantization axis characterized by the choice of M_0^2 extends M_0^4 to geometric twistor space $T(M^4) = M_0^4 \times S^2 \supset M_0^2$ having bundle projection to CP_3. Twistorialization means essentially the inclusion of the choice of various quantization axis as degrees of freedom. This space is for symmetry group G the space G/H, where H is the Cartan sub-group of G. This description might make sense also at the level of super-symplectic and super-Kac-Moody symmetries.

2. Ordinary octonionic degrees of freedom for super-octonions in M^8 must be mapped to $M^4 \times CP_2$ cm degrees of freedom. Super octonionic parts should correspond to fermionic and spin and electroweak degrees of freedom. The space of super-twistorial states should same as the space of the super-symplectic grounds states describable in terms H-spinor modes.

3. One has wave function in the moduli space of CDs. The states in M^8 are labelled by quaternionic super-momenta. Bosonic part must correspond to four-momentum and color and super-part to spin and ew quantum numbers of CP_2. This part of the moduli space wave function is characterized by the spin and ew spin quantum numbers of the fundamental particle. Wave functions in the super counterpart of $T(M^4) \times T(CP_2)$ allow to characterize these degrees of freedom without the introduction of spinors and should correspond to the ground states of super-conformal representations in H.

It seems that H-description is an abstract description at the level moduli spaces and M^8 description for single space-time variety represents reduction to the primary level, where number theory dictates the dynamics.

Octonionic twistors and super-twistors

How to define octonionic twistors? Or is it enough to identify quaternionic/associative twistors as sub-spaces of octonionic twistors?

1. Ordinary twistors and super-twistors

Consider first how ordinary twistors and their super counterparts could be defined, and how they could allow an elegant description of spin and ew quantum numbers as quantum numbers analogous to angular momenta.

1. Ordinary twistors are defined as pairs of 2-spinors giving rise to a representation of four-momentum. The spinors are complex spinors transforming as a doublet representation of SL(2,C) and its conjugate.

   The 2-spinors are related by incidence relation, a linear condition in which M^4 coordinates represented as 2×2 matrix appears linearly [K79]. The expression of four-momentum is bilinear in the spinors and invariant under complex scalings of the 2-spinors compensating each other so that instead of 8-D space one has actually 6-D space, which reduces to CP_3 to which the geometric twistor space $M^4 \times S^2$ has a projection.

2. For light-like four-momenta p the determinant of the matrix having the two 2-spinors as rows and representing p as a point of M^4 vanishes. Wave functions in CP_3 allow to describe spin in terms of bosonic wave function. What is so beautiful is that this puts particles with different spin in a democratic position.

Super-twistors allow to integrate the states constructible as many-fermion states of $\mathcal{N}$ elementary fermions in the same representations involving several spins. The many-fermion states - sparticles - are in 1-1-correspondence with Grassmann algebra basis.

3. The description of massless particles in terms of M^4 (super-)twistors is elegant but one encounters problems in the case of massive particles [K91, K35, K79].

2. Octonionic twistors at the level of M^8?

How to define octonionic twistors at the level of M^8?

1. At the level of M^8 one has light-like 8-momenta. The M^4 momentum identified as M_0^4 projection can there be massive. This solves the basic problem of the standard twistor approach.
2. The additional assumption is that the 8-momenta in given vertex of scattering diagram belong to the same quaternionic sub-space $M_0^4 \subset M^8$ satisfying $M_0^4 \supset M_0^2$. This effectively transforms momentum space $M^4 \times E^4$ to $M^4 \times CP_2$. A stronger condition is that all momenta in a given diagram belong to the same sub-space $M_0^4 \supset M_0^2$.

   Remark: Quaternionicity implies that the 8-momentum is time-like or light-like if one requires that quaternionicity for an arbitrary choice of the octonionic structure (the action of 8-D Poincare group gives rise transforms octonionic structures to each other).
3. Complex 2-spinors are replaced with complexified octonionic spinors which must be consistent quaternionicity condition for 8-momenta. A good guess is that the spinors belong to a quaternionic sub-space of octonions too. This is expected to transform them effectively to quaternionic spinors. Without effective quaternionicity the number of 2-spinor components would be 8 rather than 4 times larger than for ordinary 2-spinors.

   Remark: One has complexified octonions (i commutes with the octonionic imaginary units E_k).
4. Octonionic/quaternionic twistors should be pairs of octonionic/quaternionic 2-spinors determined only modulo octonionic/quaternionic scaling. If quaternionicity holds true, the number of 2-spinor components is 4 times larger than usually. Does this mean that one has basically quaternionic twistors plus moduli space CP_2 for $M_0^4 \supset M_0^2$. One should be able to express octonionic twistors as bi-linears formed from 2 octonionic/quaternionic 2-spinors. Octonionic option should give the octonionic counterpart OP_3 of Grassmannian CP_3, which does not however exist.

   Remark: Octonions allow only projective plane OP_2 as the octonionic counterpart of CP_2 (see `http://tinyurl.com/ybwaeu2s`) but do not allow higher-D projective spaces nor Grassmannians (see `http://tinyurl.com/ybm8ubef`, whereas reals, complex numbers, and quaternions do so. The non-existence of Grassmannians for rings obtained by Cayley-Dickson construction could mean that $M^8 - H$ correspondence and TGD do not generalize beyond octonions.

   Does the restriction to quaternionic 8-momenta the Grassmannians to be quaternionic (sub-spaces of octonions). This would give quaternionic counterpart HP_3 of CP_3. Quaternions indeed allow projective spaces and Grassmannians and (see `http://tinyurl.com/y9htjstc` and `http://tinyurl.com/y87gpq8l`).

   Remark: One can wonder whether non-commutativity forces to distinguish between left- and right Grassmannians (points as lines $\{c(q_1,..,q_n)|c \in H\}$ or as lines as lines $\{(q_1,..,q_n)c|c \in H\}$.
5. Concerning the generalization to octonionic case, it is crucial to realize that the 2×2-matrix representing four-momentum as a pair 2-spinor can be regarded as an element in the sub-space of complexified quaternions. The representation of four-momentum would be as sum of $p_8 = p_1^k \sigma_k + I_4 p_2^k \sigma_k$, where I_4 octonionic imaginary unit orthogonal to σ_k representing quaternionic units.

   No! The twistorial representation of the 4-momentum is already quaternionic! Choosing the decomposition of M^8 to quaternionic sub-space and its complement suitably, one has $IM(p_8) = 0$ for quaternionic 8-momenta and one obtains standard representation of 4-momentum in this sub-space! The only new element is that one has now moduli specifying the quaternionic sub-space. If the sub-space contains a fixed M_0^2 one obtains just CP_2 and ordinary twistor codes for the choices of M_0^2. If the choice of color quantization axes matters as it indeed does, one has twistor space $SU(3)/U(1)\times U(1)$ instead of CP_2. This would suggest that ordinary representation of scattering amplitudes reduces apart from the presence of CP_2 twistor to the usual representation.

One can hope for a reduction to ordinary twistors and projective spaces, moduli space CP_2 for quaternion structures, and moduli space for the choices of real axis of octonion structures. One can even consider the possibility [K79] of using standard M_0^2 with the property that M^8 momentum reduces to M_0^2 momentum and coding the information about real M_0^2 to moduli. This could reduce the twistor space to $RP(3)$ associated with M_0^2 is considered and solve the problems related to the signature of M^4. Note however that the complexification of octonions in any case allow to regard the metric as Euclidian albeit complexified so that these problems should disappear.

3. Octonionic super-twistors at the level of M^8?

Should one generalize the notion of super-twistor to octonionic context or can one do by using only the moduli space and the fact that octonionic geometry codes for various components of octonion as analog of super-field? It seems that super-twistors are needed.

1. It seems that super-twistors are needed. Octonionic super-momentum would appear in the super variant of momentum conserving delta function resulting in the integration over translational moduli. In twistor Grassmann approach this delta function is super-twistorialized and this leads to the amazingly simple expressions for the scattering amplitudes.
2. At the level of M^8 one should generalize ordinary momentum to super-momentum and perform super-twistorialization. Different monomials of theta parameters emerging from super part of momentum conserving delta function (for $\mathcal{N}=1$ one has $\delta(\theta-\theta_0)=exp(i\theta-\theta_0)/i)$ correspond to different spin states of the super multiplet and anti-commutativity guarantees correct statistics. At the level of H the finite-D Clifford algebra of 8-spinors at fixed point of H gives states obtained as monomials or polynomials for the components of super-momentum in M^8.
3. Octonionic super-momentum satisfying quaternionicity condition can be defined as a combination of ordinary octonionic 8-momentum and super-parts transforming like 8_s and $\overline{8}_s$. One can express the octonionic super-momentum as a bilinear of the super-spinors defining quaternionic super-twistor. Quaternionicity is assumed at least for the octonionic super-momenta in the same vertex. Hence the M^4 part of the super-twistorialization reduces to that in SUSYs and one obtains standard formulas. The new elements is the super-twistorialization of $T(CP_2)$.

   Remark: Octonionic SUSY involving $8+8_s+\overline{8}_s$ would be an analog of $\mathcal{N}=8$ SUSY associated with maximal supergravity (see http://tinyurl.com/nv3aajy) and in M^4 degrees of freedom twistorialization should be straightforward.

   The octonionic super-momentum belongs to a quaternionic sub-space labelled by CP_2 point and corresponds to a particular sub-space M_0^2 in which it is light-like (has no other octonionic components). M_0^2 is characterized by point of S^2 point of twistor space $M^4\times S^2$ having bundle projection to CP_3.
4. That the twistor space $T(CP_2)=SU(3)/U(1)\times U(1)$ coding for the color quantization axes rather than only CP_2 emerges must relate to the presence of electroweak quantum numbers related to the super part of octonionic momentum. Why the rotations of $SU(2)\times U(1)\subset SU(3)$ have indeed interpretation also as tangent space-rotations interpreted as electroweak rotations. The transformations having an effect on the choice of quantization axies are parameterized by S^2 relating naturally to the choice of $SO(4)$ quantization axis in E^4 and coded by the geometric twistor space $T(E^4)=E^4\times S^2$.
5. Since the super-structure is very closely related to the construction of the exterior algebra in the tangent space, super-twistorialization of $T(CP_2)$ should be possible. Octonionic triality could be also in a key role and octonionic structure in the tangent space of $SU(3)$ is highly suggestive. $SU(3)$ triality could relate to the octonionic triality.

$SU(3)/U(1) \times U(1)$ is analogous with the ordinary twistor space CP_3 obtained from C^4 as a projective space. Now however $U(1) \times U(1)$ instead of group of complex scalings would define the equivalence classes. Generalization of projective space would be in question. The super-part of twistor would be obtained as $U(1) \times U(1)$ equivalence class and gauge choice should be possible to get manifestly 6-D representation. One can ask whether the CP_2 counterparts of higher- D Grassmannians appear at the level of generalized twistor diagrams: could the spaces $SU(n)/G$, H Cartan group correspond to these spaces?

4. How the wave functions in super-counterpart of $T(CP_2)$ correspond to quantum states in CP_2 degrees of freedom?

In CP_2 spinor partial waves have vanishing triality $t = 0$ for leptonic chirality and $t = \pm 1$ for quarks and antiquarks. One can say that the triality $t \neq 1$ states are possible thanks to the anomalous hypercharge equal to fractional electromagnetic charge $Y_A = Q_{em}$ of quarks: this gives also correlation between color quantum numbers and electroweak quantum numbers which is wrong for spinor partial waves. The super-symplectic and super Kac-Moody algebras however bring in vibrationals degrees of freedom and one obtains correct quantum number assignments [K50].

This mechanism should have a counterpart at the level of the super variant of the twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$. The group algebra of $SU(3)$ gives the scalar wave functions for all irreps of $SU(3)$ as matrix elements. Allowing only matrix elements that are left- or right invariant under $U(1) \times U(1)$ one obtains all irreps realized in $T(CP_2)$ as scalar wave functions. These representations have $t = 0$. The situation would be analogous for scalar functions in CP_2. One must however obtain also electroweak quantum numbers and $t \neq 0$ colored states. Here the octonionic algebraic geometry and superpart of the $T(CP_2)$ should come in rescue. The electroweak degrees of freedom in CP_2 should correspond to the super-parts of twistors.

The $SU(3)$ triplets assignable to the triplets 3 and $\overline{3}$ of space-time surfaces would make possible also the $t = \pm 1$ states. Color would be associated with the octonionic geometry. The simplest possibility would be that one has just tensor products of the triplets with $SU(3)/U(1) \times U(1)$ partial waves. In the case of CP_2 there is however a correlation between color partial waves and electroweak quantum numbers and the same is expected also now between super-part of the twistor and geometric color wave function: minimum correlation is via $Y_A = Q_{em}$. The minimal option is that the number theoretic color for the octonionic variety modifies the transformation properties of $T(CP_2)$ wave function only by a phase factor due to $Y_A = Q_{em}$ as in the case of CP_2.

The most elegant outcome would be that super-twistorial state basis in $T(M^4) times T(CP_2)$ is equivalent with the state basis defined by super-symplectic and super Kac-Moody representations in H.

About the analogs of twistor diagrams

There seems to be a strong analogy with the construction of twistor amplitudes in $\mathcal{N} = 4$ SUSY [B22, B48, B29] and one can hope of obtaining a purely geometric analog of SUSY with dynamics of fields replaced by the dynamics of algebraic super-octonionic surfaces.

1. Number theoretical vision leads to the proposal that the scattering amplitudes involve only data at discrete points of the space-time variety belonging to extension of rationals defining cognitive representation. The identification of these points has been already considered in the case of partonic orbits entering to the partonic 2-vertex and for the regions of space-time surfaces intersecting at discrete set of points. Scattering diagrams should therefore correspond to polygons with vertices of polygons defining cognitive representation and lines assignable to the external fundamental particles with given quark and lepton numbers having correlates at the level of space-time geometry. This occurs also in twistor Grassmannian approach [B22, B48, B29].

   Since polynomials determine space-time surfaces, this data is enough to determine the space-time variety completely. Indeed, the zeros of $P(t)$ determining the space-time variety give also rise to a set of spheres $S^6(t_n)$ and partonic 2-surfaces $X^2(t_n) = X^4 \cap S^6(t_n)$, where t_n is root of $P(t)$. The discretization need not mean a loss of information. The scattering amplitudes would be expressible as an analog of n-point function with points having coordinates in the extension of rationals.

2. (Super) octonion as"field"in X^4 is dynamically analogous to (super) gauge potentials and super-octonion to its super variant. (Super) gauge potentials are replaced with M^8 (super-) octonion coordinate and gauge interactions are geometrized. Here I encounter a problem with terminology. Neither sparticle nor sboson sounds good. Hence I will talk about sparticles.

3. The amplitude for a given space-time variety contains no information M^8-momentum. M^8-momentum emerges as a label for a wave function in the moduli space of 4-D and 8-D CDs involving both translational and orientational degrees of freedom. For fixed time axis the orientational degrees of freedom reduce to rotational degrees of freedom identifiable in terms of the twistor sphere S^2. The delta functions expressing conservation of 8-D quaternionic super-momentum in M^8 coming from the integration over the moduli space of 8-D translations.

   As found, quaternionicity of 8-momenta implies that standard M^4 twistor description of momenta applies but one obtains CP_2 twistors as additional contribution. This is of course what one would intuitively expect.

   8-D momentum conservation in turn translates to the conservation of momentum and color quantum numbers in the manner described. The amplitudes in momentum and color degrees of freedom reduce to kinematics as in SUSYs. It is however not clear whether one should also perform number theoretical discretization of various moduli spaces.

   In any case, it seems that all the details of the scattering amplitudes related to moduli spaces reduce to symmetries and the core of calculations reduces to the construction of space-time varieties as zero loci of octonionic polynomials and identification of the points of the 4-varieties in extension of rationals. Classical theory would indeed be an exact part of the quantum theory.

4. Quaternionic 8-D light-likeness reduces the situation to the level of ordinary complex and thus even positive (real) Grassmannians. This is crucial from the p-adic point of view. CP_2 twistors characterizes the moduli related to the choice of quaternionic sub-space, where 8-momentum reduces to ordinary 4-momentum. M^4 parts of the scattering amplitudes in twistor Grassmann approach should be essentially the same as in $\mathcal{N}=4$ SUSY apart from the replacement of super degrees of freedom with super-octonionic ones. The challenge is to generalize the formalism so that it applies also to CP_2 twistors. The challenge would be to generalize the formalism so that it applies also to CP_2 twistors. The M^4 and CP_2 degrees of freedom are expected to factorize in twistorial amplitudes. A good guess is that the scattering amplitudes are obtained as residue integrals in the analogs of Grassmannians associated with $T(CP_2)$. Could one have Grassmannians also now?

   Consider the formula of tree amplitude for n gluons with k negative helicities conjectured Arkani-Hamed *et al* in the twistor Grassmannian approach [B29]. The amplitude follows from the twistorial representation for momentum conservation and is equal to an $k\times n$-fold multiple residue integral over the complex variables $C_{\alpha a}$ defining coordinates for Grassmannian $Gl(n,k)$ and reduces to a sum over residues. The integrand is the inverse for the product of all $k\times k$ minors of the matrix $C_{\alpha a}$ in cyclic order and the resides corresponds to zeros for one or more minors. This part does not depend on twistor variables. The dependence on n twistor variables comes from the product $\prod_{\alpha=1}^{k}\delta(C_{\alpha a}W^a)$ of k delta functions related to momentum conservation. W^a denotes super-twistors in the 8-D representation, which is linear. One has projective invariance and therefore a reduction to $T(M^4)=CP_3=SU(4)/SU(3)\times U(1)$.

   Could this formula generalize almost as such to $T(CP_2)$ and come from the conservation of E^4 momentum? One has n sparticles to which super-twistors in $T(CP_2)$ are assigned. The first guess is that the sign of helicity are replaced by the sign of electroweak isospin - essentially E^4 spin at the level of M^8. For electromagnetic charge identified as the analog of helicity one would have problems in the case of neutrinos. $T(M^4)=CP_3=SU(4)/SU(3)\times U(1)$ is replaced with $T(CP_2)=SU(3)/U(1)\times U(1)$. $T(CP_2)$ does not have a representation as a projective space but there is a close analogy since the group of complex scalings is replaced with $U(1)\times U(1)$. The (apparent) linearity is lost but one represent the points of $T(CP_2)$ as exponentials of $su(3)$ Lie-algebra elements with vanishing $u(1)\times u(1)$ part. The resulting 3 complex coordinates are analogous to two complex CP_2 coordinates. The basic difference between M^4 and CP_2 degrees of freedom would come from the exponential representation of twistors.

5. By Yangian invariance one should obtain very similar formulas for the amplitudes except that one has instead of $\mathcal{N} = 4$ SUSY $\mathcal{N} = 8$ octonionic SUSY analogous to $\mathcal{N} = 8$ SUGRA.

Trying to understand the fundamental 3-vertex

Due to its unique twistorial properties as far as realization of four-momentum conservation is considered 3-vertex is fundamental in the construction of scattering diagrams in twistor Grassmannian approach to $\mathcal{N} = 4$ SYM [B25] (see `http://tinyurl.com/yd9tf2ya`). Twistor Grassmann approach suggests that 3-vertex with complexified light-like 8-momenta represents the basic building brick representing from which more complex diagrams can be constructed using the BCFW recursion formula [B25]. In TGD 3-vertex generalized to 8-D light-like quaternionic momenta should be highly analogous to the 4-D 3-vertex and in a well-defined sense reduce to it if all momenta of the diagram belong to the same quaternionic sub-space M_0^4. It is however not completely clear how 3-vertex emerges in TGD framework.

1. A possible identification of the 3-vertex at the level of M^8 would be as a vertex at which 3 sparticle lines with light-like complexified quaternionic 8-momenta meet. This vertex would be associated with the partonic vertex $X^2(t_n) = X^4 \cap S^6(t_n)$. Incoming sparticle lines at the light-like partonic orbits identified as boundaries of string world sheets (for entangled states at least) would be light-like.

   Does the fusion of two sparticle lines to third one require that either or both fusing lines become space-like - say pieces of geodesic line inside the Euclidian space-time region- bounded by the partonic orbit? The identification of the lines of twistor diagrams as carriers of light-like complexified quaternionic momenta in 8-D sense does not encourage this interpretation (also classical momenta are complex). Should one pose the fusion of the light-like lines as a boundary condition? Or should one give up the idea that sparticle lines make sense inside interaction region?

2. As found, one can challenge the assumption about the existence of string world sheets as commutative regions in the non-associative interaction region. Could one have just fermion lines as light-like curves at partonic orbits inside CD? Or cannot one have even them?

   Even if the polynomial $\prod_i P_i$ defining the interaction region is product of polynomials with origins of octonionic coordinates not along the same real line, the 7-D light-cones of M^8 associated with the particles still make sense in the sense that $P_i(o_i) = 0$ reduces at it to $P_i(t_i) = 0$, t_i real number, giving spheres $S^6(t_i(n))$ and partonic 2-surfaces and vertices $X_2(t_i(n))$. The light-like curves as geodesics the boundary of 7-D light-cones mapped to light-like curves along partonic orbits in H would not be lost inside interaction regions.

3. At the level of H this relates to a long standing interpretational problem related to the notion of induced spinor fields. SH suggests strongly the localization of the induced spinor fields at string world sheets and even at sparton lines in absence of entanglement. Super-conformal symmetry however requires that induced spinor fields are 4-D and thus seems to favor delocalization. The information theoretic interpretation is that the induced spinor fields at string world sheets or even at sparton lines contain all information needed to construct the scattering amplitudes. One can also say that string world sheets and sparton lines correspond to a description in terms of an effective action.

Could the M^8 view about twistorial scattering amplitudes be consistent with the earlier H picture?

The proposed M^8 picture involving super coordinates of M^8 and super-twistors does not conform with the earlier proposal for the construction of scattering amplitudes at the level of H [K79]. In H picture the introduction of super-space does not look natural, and one can say that fundamental fermions are the only fundamental particles [K35, K79]. The H view about super-symmetry is as broken supersymmetry in which many fermion states at partonic 2-surfaces give rise to supermultiplets such that fermions are at different points. Fermion 4-vertex would be the fundamental vertex and involve classical scattering without fusion of fermion lines. Only a redistribution of fermion and anti-fermion lines among the orbits of partonic 2-surfaces would take place in scattering and one would have kind of OZI rule.

Could this H view conform with the recent M^8 view much closer to the SUSY picture. The intuitive idea without a rigorous justification has been that the fermion lines at partonic 2-surfaces correspond to singularities of many-sheeted space-time surface at which some sheets co-incide. M^8 sparticle consists effectively of n fermions at the same point in M^8. Could it be mapped by M^8-H duality to n fermions at distinct locations of partonic 2-surface in H?

$M^8 - H$ correspondence maps the points of $M^4 \subset M^4 \times E^4$ to points of $M^4 \subset M^4 \times CP_2$. The tangent plane of space-time surface containing a preferred M^2 is mapped to a point of CP_2. If the effective n-fermion state M^8 is at point at which n sheets of space-time surface co-incide and if the tangent spaces of different sheets are not identical, which is quite possible and even plausible, the point is indeed mapped to n points of H with same M^4 coordinates but different CP_2 coordinates and sparticle would be mapped to a genuine many-fermion state. But what happens to scalar sparticle. Should one regard it as a pure gauge degree of freedom in accordance with the chiral symmetry at the level of M^8 and H?

3.5 From amplituhedron to associahedron

Lubos has a nice blog posting (see `http://tinyurl.com/y7ywhxew`) explaining the proposal represented in the newest article by Nima Arkani-Hamed, Yuntao Bai, Song He, Gongwang Yan [**?**]see `http://tinyurl.com/ya8zstll`). Amplituhedron is generalized to a purely combinatorial notion of associahedron and shown to make sense also in string theory context (particular bracketing). The hope is that the generalization of amplituhedron to associahedron allows to compute also the contributions of non-planar diagrams to the scattering amplitudes - at least in $\mathcal{N} = 4$ SYM. Also the proposal is made that color corresponds to something less trivial than Chan-Paton factors.

The remaining problem is that 4-D conformal invariance requires massless particles and TGD allows to overcome this problem by using a generalization of the notion of twistor: masslessness is realized in 8-D sense and particles massless in 8-D sense can be massive in 4-D sense.

In TGD non-associativity at the level of arguments of scattering amplitude corresponds to that for octonions: one can assign to space-time surfaces octonionic polynomials and induce arithmetic operations for space-time surface from those for polymials (or even rational or analytic functions). I have already earlier [L36] demonstrated that associahedron and construction of scattering amplitudes by summing over different permutations and associations of external particles (space-time surfaces). Therefore the notion of associahedron makes sense also in TGD framework and summation reduces to "integration" over the faces of associahedron. TGD thus provides a concrete interpretation for the associations and permutations at the level of space-time geometry.

In TGD framework the description of color and four-momentum is unified at the level and the notion of twistor generalizes: one has twistors in 8-D space-time instead of twistors in 4-D space-time so Chan-Paton factors are replaced with something non-trivial.

3.5.1 Associahedrons and scattering amplitudes

The following describes briefly the basic idea between associahedrons.

Permutations and associations

One starts from a non-commutative and non-associative algebra with product (in TGD framework this algebra is formed by octonionic polynomials with real coefficients defining space-time surfaces as the zero loci of their real or imaginary parts in quaternionic sense. One can indeed multiply space-time surface by multiplying corresponding polynomials! Also sum is possible. If one allows rational functions also division becomes possible.

All permutations of the product of n elements are in principle different. This is due to non-commutativity. All associations for a given ordering obtained by scattering bracket pairs in the product are also different in general. In the simplest case one has either $a(bc)$ or $(ab)c$ and these 2 give different outcomes. These primitive associations are building bricks of general associations: for instance, abc does not have well-defined meaning in non-associative case.

If the product contains n factors, one can proceed recursively to build all associations allowed by it. Decompose the n factors to groups of m and $n-m$ factors. Continue by decomposing these two groups to two groups and repeat until you have have groups consisting of 1 or two elements.

You get a large number of associations and you can write a computer code computing recursively the number $N(n)$ of associations for n letters.

Two examples help to understand. For $n = 3$ letters one obviously has $N(n = 3) = 2$. For $n = 4$ one has $N(4) = 5$: decompose first $abcd$ to $(abc)d$, $a(bcd)$ and $(ab)(cd)$ and then the two 3 letter groups to two groups: this gives $N(4) = 2 + 2 + 1 = 5$associations and associahedron in 3-D space has therefore 5 faces.

Geometric representation of association as face of associahedron

Associations of n letters can be represented geometrically as so called Stasheff polytope (see http://tinyurl.com/q9ga785). The idea is that each association of n letters corresponds to a face of polytope in $n - 2$-dimensional space with faces represented by the associations.

Associahedron is constructed by using the condition that adjacent faces (now 2-D polygons) intersecting along common face (now 1-D edges). The number of edges of the face codes for the structure particular association. Neighboring faces are obtained by doing minimal change which means replacement of some $(ab)c$ with $a(bc)$ appearing in the association as a building bricks or vice versa. This means that the changes are carried out at the root level.

How does this relate to particle physics?

In scattering amplitude letters correspond to external particles. Scattering amplitude must be invariant under permutations and associations of the external particles. In particular, this means that one sums over all associations by assigning an amplitude to each association. Geometrically this means that one "integrates" over the boundary of associahedron by assigning to each face an amplitude. This leads to the notion of associahedron generalizing that of amplituhedron.

Personally I find it difficult to believe that the mere combinatorial structure leading to associahedron would fix the theory completely. It is however clear that it poses very strong conditions on the structure of scattering amplitudes. Especially so if the scattering amplitudes are defined in terms of "volumes" of the polyhedrons involved so that the scattering amplitude has singularities at the faces of associahedron.

An important constraint on the scattering amplitudes is the realization of the Yangian generalization of conformal symmetries of Minkowski space. The representation of the scattering amplitudes utilizing moduli spaces (projective spaces of various dimensions) and associahedron indeed allows Yangian symmetries as diffeomorphisms of associahedron respecting the positivity constraint. The hope is that the generalization of amplituhedron to associahedron allows to generalize the construction of scattering amplitudes to include also the contribution of non-planar diagrams of at $\mathcal{N} = 4$ SYM in QFT framework.

3.5.2 Associations and permutations in TGD framework

Also in the number theoretical vision about quantum TGD one encounters associativity constraings leading to the notion of associahedron. This is closely related to the generalization of twistor approach to TGD forcing to introduce 8-D analogs of twistors [L36] (see http://tinyurl.com/yd43o2n2).

Non-associativity is induced by octonic non-associativity

As found in [L36], non-associativity at the level of space-time geometry and at the level of scattering amplitudes is induced from octonionic non-associativity in M^8.

1. By $M^8 - H$ duality ($H = M^4 \times CP_2$) the scattering are assignable to complexified 4-surfaces in complexified M^8. Complexified M^8 is obtained by adding imaginary unit i commutating with octonionic units I_k, $k = 1, , .., 7$. Real space-time surfaces are obtained as restrictions to a Minkowskian subspace complexified M^8 in which the complexified metric reduces to real valued 8-D Minkowski metric. This allows to define notions like Kähler structure in Minkowskian signature and the notion of Wick rotations ceases to be ad hoc concept. Without complexification one does not obtain algebraic geometry allowing to reduces the dynamics defined by partial differential equations for preferred extremals in H to purely algebraic

conditions in M^8. This means huge simplications but the simplicity is lost at the QFT-GRT limit when many-sheeted space-time is replaced with slightly curved piece of M^4.

2. The real 4-surface is determined by a vanishing condition for the real or imaginary part of octonionic polynomial with $RE(P)$ and $IM(P)$ defined by the composition of octonion to two quaternions: $o = RE(o) + I_4 IM(o)$, where I_4 is octonionic unit orthogonal to a quaternionic sub-space and $RE(o)$ and $IM(o)$ are quaternions. The coefficients of the polynomials are assumed to be real. The products of octonionic polynomials are also octonionic polynomials (this holds for also for general power series with real coefficients (no dependence on I_k. The product is not however neither commutative nor associative without additional conditions. Permutations and their associations define different space-time surfaces. The exchange of particles changes space-time surface. Even associations do it. Both non-commutativity and non-associativity have a geometric meaning at the level of space-time geometry!

3. For space-time surfaces representing external particles associativity is assumed to hold true: this in fact guarantees $M^8 - H$ correspondence for them! For interaction regions associativity does not hold true but the field equations and preferred extremal property allow to construct the counterpart of space-time surface in H from the boundary data at the boundaries of CD fixing the ends of space-time surface.

   Associativity poses quantization conditions on the coefficients of the polynomial determining it. The conditions are interpreted in terms of quantum criticality. In the interaction region identified naturally as causal diamond (CD), associativity does not hold true. For instance, if external particles as space-time surfaces correspond to vanishing of $RE(P_i)$ for polynomials representing particles labelled by i, the interaction region (CD) could correspond to the vanishing of $IM(P_i)$ and associativity would fail. At the level of H associativity and criticality corresponds to minimal surface property so that quantum criticality corresponds to universal free particle dynamics having no dependence on coupling constants.

4. Scattering amplitudes must be commutative and associative with respect to their arguments which are now external particles represented by polynomials P_i This requires that scattering amplitude is sum over amplitudes assignable to 4-surfaces obtained by allowing all permutations and all associations of a given permutation. Associations can be described combinatorially by the associahedron!

   Remark:. In quantum theory associative statistics allowing associations to be represented by phase factors can be considered (this would be associative analog of Fermi statistics). Even a generalization of braid statistics can be considered.

Yangian variants of various symmetries are a central piece also in TGD although supersymmetries are realized in different manner and generalized to super-conformal symmetries: these include generalization of super-conformal symmetries by replacing 2-D surfaces with light-like 3-surfaces, supersymplectic symmetries and dynamical Kac-Moody symmetries serving as remnants of these symmetries after supersymplectic gauge conditions characterizing preferred extremals are applied, and Kac-Moody symmetries associated with the isometries of H . The representation of Yangian symmetries as diffeomorphisms of the associahedron respecting positivity constraint encourages to think that associahedron is a useful auxiliary tool also in TGD.

Is color something more than Chan-Paton factors?

Nima *et al* talk also about color structure of the scattering amplitudes usually regarded as trivial. It is claimed that this is actually not the case and that there is non-trivial dynamics involved. This is indeed the case in TGD framework. Also color quantum numbers are twistorialized in terms of the twistor space of CP_2, and one performs a twistorialization at the level of M^8 and $M^4 \times CP_2$. At the level of M^8 momenta and color quantum numbers correspond to associative 8-momenta. Massless particles are now massless in 8-D sense but can be massive in 4-D sense. This solves one of the basic difficulty of the ordinary twistor approach. A further bonus is that the choice of the embedding space H becomes unique: only the twistor spaces of S^4 (and generalized twistor space of M^4 and CP_2 have Kähler structure playing a crucial role in the twistorialization of TGD. To sum up, all roads lead to Rome. Everyone is well-come to Rome!

3.5.3 Questions inspired by quantum associations

Associations have (or seem to have) different meaning depending on whether one is talking about cognition or mathematics. In mathematics the associations correspond to different bracketings of mathematical expressions involving symbols denoting mathematical objects and operations between them. The meaning of the expression - in the case that it has meaning - depends on the bracketing of the expression. For instance, one has $a(b+c) \neq (ab)+c$, that is $ab+ac \neq ab+c$). Note that one can change the order of bracket and operation but not that of bracket and object.

For ordinary product and sum of real numbers one has associativity: $a(bc) = (ab)c$ and $a+(b+c) = (a+b)+c$. Most algebraic operations such as group product are associative. Associativity of product holds true for reals, complex numbers, and quaternions but not for octonions and this would be fundamental in both classical and quantum TGD.

The building of different associations means different groupings of n objects. This can be done recursively. Divide first the objects to two groups, divide these tow groups to two groups each, and continue until you jave division of 3 objects to two groups - that is abc divided into $(ab)c$ or $a(bc)$. Numbers 3 and 2 are clearly the magic numbers.

This inspire several speculative quetions related to the twistorial construction of scattering amplitudes as associative singlets, the general structure of quantum entanglement, quantum measurement cascade as formation of association, the associative structure of many-sheeted space-time as a kind of linguistic structure, spin glass as a strongly associative system, and even the tendency of social structures to form associations leading from a fully democratic paradise to cliques of cliques of ... .

1. In standard twistor approach 3-gluon amplitude is the fundamental building brick of twistor amplitudes constructed from on-shell-amplitudes with complex momenta recursively. Also in TGD proposal this holds true. This would naturally follow from the fact that associations can be reduced recursively to those of 3 objects. 2- and 3-vertex would correspond to a fundamental associations. The association defined 2-particle pairing (both associated particles having either positive or negative helicities for twistor amplitudes) and 3-vertex would have universal structure although the states would be in general decompose to associations.

2. Consider first the space-time picture about scattering [L36]. CD defines interaction region for scattering amplitudes. External particles entering or leaving CD correspond to associative space-time surfaces in the sense that the tangent space or normal space for these space-time surfaces is associative. This gives rise to $M^8 - H$ correspondence.

   These surfaces correspond to zero loci for the imaginary parts (in quaternionic sense) for octonionic polynomial with coefficients, which are real in octonionic sense. The product of $\prod_i P_i)$ of polynomials with same octonion structure satisfying $IM(P_i) = 0$ has also vanishing imaginary part and space-time surface corresponds to a disjoint union of surfaces associated with factors so that these states can be said to be non-interacting.

   Neither the choice of quaternion structure nor the choice of the direction of time axis assignable to the octonionic real unit need be same for external particles: if it is the particles correspond to same external particle. This requires that one treats the space of external particles (4-surfaces) as a Cartesian product of of single particle 4-surfaces as in ordinary scattering theory.

   Space-time surfaces inside CD are non-associative in the sense that the neither normal nor tangent space is associative: $M^8 - M^4 \times CP_2$ correspondence fails and space-time surfaces inside CD must be constructed by applying boundary conditions defining preferred extremals. Now the real part of $RE(\prod_i P_i)$ in quaternionic sense vanishes: there is genuine interaction even when the incoming particles correspond to the same octonion structure since one does not have union of surfaces with vanishing $RE(P_i)$. This follows from s rather trivial observation holding true already for complex numbers: imaginary part of zw vanishes if it vanishes for z and w but this does not hold true for the real part. If octonionic structures are different, the interaction is present irrespective of whether one assumes $RE(\prod_i P_i) = 0$ or $IM(\prod_i P_i) = 0$. $RE(\prod_i P_i) = 0$ is favoured since for $IM(\prod_i P_i) = 0$ one would obtain solutions for which $IM(P_i) = 0$ would vanish for the i:th particle: the scattering dynamics would select i:th particle as non-interacting one.

3. The proposal is that the entire scattering amplitude defined by the zero energy state - is associative, perhaps in the projective sense meaning that the amplitudes related to different associations relate by a phase factor (recall that complexified octonions are considered), which could be even octonionic. This would be achieved by summing over all possible associations.
4. Quantum classical correspondence (QCC) suggests that in ZEO the zero energy states - that is scattering amplitudes determined by the classically non-associative dynamics inside CD - form a representation for the non-associative product of space-time surfaces defined by the condition $RE(\prod_i P_i) = 0$. Could the scattering amplitude be constructed from products of octonion valued single particle amplitudes. This kind of condition would pose strong constraints on the theory. Could the scattering amplitudes associated with different associations be octonionic - may be differing by octonion-valued phase factors - and could only their sum be real in octonionic sense (recall that complexified octonions involving imaginary unit i commuting with the octonionic imaginary units are considered)?

One can look the situation also from the point of view of positive and negative energy states defining zero energy states as they pairs.

1. The formation of association as subset is like formation of bound state of bound states of ... . Could each external line of zero energy state have the structure of association? Could also the internal entanglement associated with a given external line be characterized in terms of association.

   Could the so called monogamy theorem stating that only two-particle entanglement can be maximal correspond to the decomposing of $n = 3$ association to one- and two-particle associations? If quantum entanglement is behind associations in cognitive sense, the cognitive meaning of association could reduce to its mathematical meaning.

   An interesting question relates to the notion of identical particle: are the many-particle states of identical particles invariant under associations or do they transform by phase factor under association. Does a generalization of braid statistics make sense?
2. In ZEO based quantum measurement theory the cascade of quantum measurements proceeds from long to short scales and at each step decomposes a given system to two subsystems. The cascade stops when the reduction of entanglement is impossible: this is the case if the entanglement probabilities belong to an extension of extension of rationals characterizing the extension in question. This cascade is nothing but a formation of an association! Since only the state at the second boundary of CD changes, the natural interpretation is that state function reduction mean a selection of association in 3-D sense.
3. The division of n objects to groups has also social meaning: all social groups tend to divide into cliques spoiling the dream about full democracy. Only a group with 2 members - Romeo and Julia or Adam and Eve - can be a full democracy in practice. Already in a group of 3 members 2 members tend to form a clique leaving the third member outside. Jules and Catherine, Jim and Catherine, or maybe Jules and Jim! Only a paradise allows a full democracy in which non-associativity holds true. In ZEO it would be realized only at the quantum critical external lines of scattering diagram and quantum criticality means instability. Quantum superposition of all associations could realize this democracy in 4-D sense.

A further perspective is provided by many-sheeted space-time providing classical correlate for quantum dynamics.

1. Many-sheeted space-time means that physical states have a hierarchical structure - just like associations do. Could the formation of association (AB) correspond basically to a formation of flux tube bond between A and B to give AB and serve as space-time correlate for (negentropic) entanglement. Could ((AB)C) would correspond to (AB) and (C) "topologically condensed" to a larger surface. If so, the hierarchical structure of many-sheeted space-time would represent associations and also the basic structures of language.
2. Spin glass (see `http://tinyurl.com/y9yyq8ga`) is a system characterized by so called frustrations. Spin glass as a thermodynamical system has a very large number of minima of free energy and one has fractal energy landscape with valleys inside valleys. Typically there is a competition between different pairings (associations) of the basic building bricks of the system.

Could spin glass be describable in terms of associations? The modelling of spin glass leads to the introduction of ultrametric topology characterizing the natural distance function for the free energy landscape. Interestingly, p-adic topologies are ultrametric. In TGD framework I have considered the possibility that space-time is like 4-D spin glass: this idea was originally inspired by the huge vacuum degeneracy of Kähler action. The twistor lift of TGD breaks this degeneracy but 4-D spin glass idea could still be relevant.

3.6 Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view

Gromov-Witten (G-W) invariants, Riemann-Roch theorem (RR), and Atyiah-Singer index theorem (AS) are applied in advanced algebraic geometry, and it is interesting to see whether they could have counterparts in TGD framework. The basic difference between TGD and conventional algebraic geometry is due to the adelic hierarchy demanding that the coefficients of polynomials involved are in given extension of rationals. Continuous moduli spaces are replaced with discrete ones by number theoretical quantization due to the criticality guaranteeing associativity of tangent or normal space. $M^8 - H$ duality brings in powerful consistency conditions: counting of allowed combinations of coefficients of polynomials on M^8 side and counting of dimensions on H side using AS should give same results. $M^8 - H$ duality might be in fact analogous to the mirror symmetry of M-theory.

3.6.1 About the analogs of Gromow-Witten invariants and branes in TGD

Gromow-Witten invariants, whose definition was discussed in [L37], play a central role in superstring theories and M-theory and are closely related to branes. For instance, partition functions can be expressed in terms of these invariants giving additional invariants of symplectic and algebraic geometries. Hence it is interesting to look whether they could be important also in TGD framework.

1. As such the definition of G-W invariants discussed in [L37] do not make sense in TGD framework. For instance, space-time surface is not a closed symplectic manifold whereas M^8 and H are analogs of symplectic spaces. Minkowskian regions of space-time surface have Hamilton-Jacobi structure at the level of both M^8 and H and this might replace the symplectic structure. Space-time surfaces are not closed manifolds.

   Physical intuition however suggests that the generalization exists. The fact that Minkowskian metric and Euclidian metric for complexified octonions are obtained in various sectors for which complex valued length squared is real suggests that signature is not a problem. Kähler form for complexified z gives as special case analog of Kähler form for E^4 and M^4.

2. The quantum intersection defines a description of interactions in terms of string world sheets. If I have understood G-W invariant correctly, one could have for $D > 4$-dimensional symplectic spaces besides partonic $2k - 2$-D surfaces also surfaces with smaller but even dimension identifiable as branes of various dimensions. Branes would correspond to a generalization of relative cohomology. In TGD framework one has $2k = 4$ and the partonic 2-surfaces have dimension 2 so that classical intersections consisting of discrete points are possible and stable for string world sheets and partonic 2-surfaces. This is a unique feature of 4-D space-time.

   One might think a generalization of G-W invariant allowing to see string world sheets as connecting the spaced-like 3-surfaces at the boundaries of CDs and light-like orbits of partonic 2-surfaces. The intersection is not discrete now and marked points would naturally correspond to the ends points of strings at partonic 2-surfaces associated with the boundaries of CD and with the vertices of topological scattering diagrams.

3. The idea about 2-D string world sheet as interaction region could generalize in TGD to space-time surface inside CD defining 4-D interaction region. In [L38] one indeed ends up with amazingly similar description of interactions for n external particles entering CD and represented as zero loci for quaternion valued "real" part $RE(P)$ or "imaginary" part $IM(P)$ for the complexified octonionic polynomial.

Associativity forces quantum criticality posing conditions on the coefficients of the polynomials. Polynomials with the origin of octonion coordinate along the same real axis commute and associate. Since the origins are different for external particles in the general case, the polynomials representing particles neither commute nor associate inside the interaction region defined by CD but one can also now define zero loci for both $RE(\prod P_i)$ and $IM(\prod P_i)$ giving $P_i = 0$ for some i. Now different permutations and different associations give rise to different interaction regions and amplitude must be sum over all these.

3-vertices would correspond to conditions $P_i = 0$ for 3 indices i simultaneously. The strongest condition is that 3 partonic 2-surfaces X_i^2 co-incide: this condition does not satisfy classical dimension rule and should be posed as essentially 4-D boundary condition. Two partonic 2-surfaces $X_i^2(t_i(n))$ intersect at discrete set of points: could one assume that the sparticle lines intersect and there fusion is forced by boundary condition? Or could one imagine that partonic 2-surfaces turns back in time and second partonic 2-surface intersects it at the turning point?

4. In 4-D context string world sheets are associated with magnetic flux tubes connecting partonic orbits and together with strings serve as correlates for negentropic entanglement assignable to the p-adic sectors of the adele considered, to attention in consciousness theory, and to remote mental interactions in general and occurring routinely between magnetic body and biological body also in ordinary biology. This raises the question whether "quantum touch" generalizes from 2-D string world sheets to 4-D space-time surface (magnetic flux tubes) connecting 3-surfaces at the orbits and partonic orbits.

5. The above formulation applies to closed symplectic manifolds X. One can however generalize the formulation to algebraic geometry. Now the algebraic curve X^2 is characterized by genus g and order of polynomial n defining it. This formulation looks very natural in M^8 picture.

An interesting question is whether the notion of brane makes sense in TGD framework.

1. In TGD branes inside space-time variety are replaced by partonic 2-surfaces and possibly by their light-like orbits at which the induced metric changes signature. These surfaces are metrically 2-D. String world sheets inside space-time surfaces have discrete intersection with the partonic 2-surfaces. The intersection of strings as space-like *resp.* light-like boundaries of string world sheet with partonic orbit sheet *resp.* space-like 3-D ends of space-time surface at boundaries of CD is also discrete classically.

2. An interesting question concerns the role of 6-spheres $S^6(t_n)$ appearing as special solutions to the octonionic zero locus conditions solving both $RE(P_n) = 0$ and $IM(P_n) = 0$ requiring $P_n(o) = 0$. This can be true at 7-D light cone $o = et$, e light-like vector and t a real parameter. The roots t_n of $P(t) = 0$ give 6-spheres $S^6(t_n)$ with radius t_n as solutions to the singularity condition. As found, one can assign to each factor P_i in the product of polynomials defining many-particle state in interaction region its own partonic 2-surfaces $X^2(t_n)$ related to the solution of $P_i(t) = 0$

Could one interpret 6-spheres as brane like objects, which can be connected by 2-D "free" string world sheets as 2-varieties in M^8 and having discrete intersection with them implied by the classical dimension condition for the intersection. Free string world sheets would be something new and could be seen as trivially associative surfaces whereas 6-spheres would represent trivially co-associative surfaces in M^8.

The 2-D intersections of $S^6(t_n)$ with space-time surfaces define partonic 2-surfaces X^2 appearing at then ends of space-time and as vertices of topological diagrams. Light-like sparticle lines along parton orbits would fuse at the partonic 2-surfaces and give rise to the analog of 3-vertex in $\mathcal{N} = 4$ SUSY.

Some further TGD inspired remarks are in order.

1. Virasoro conjecture generalizing Witten conjecture involves half Virasoro algebra. Super-Virasoro algebra algebra and its super-symplectic counterpart (SSA) play a key role in the formulation of TGD at level of H. Also these algebras are half algebras. The analogs of super-conformal conformal gauge conditions state that sub-algebra of SSA with conformal weights coming as n-ples of those for entire algebra and its commutator with entire SSA give rise to vanishing Noether charges and annihilate physical states.

These conditions are conjecture to fix the preferred extremals and serve as boundary conditions allowing the formulation of M^8-H correspondence inside space-time regions (interaction regions), where the associativity conditions fail to be true and direct $M^8 - H$ correspondence does not make sense. Non-trivial solutions to these conditions are possible only if one assumes half super-conformal and half super-symplectic algebras. Otherwise the generators of the entire SSA annihilate the physical states and all SSA Noether charges vanish. The invariance of partition function for string world sheets in this sense could be interpreted in terms of emergent dynamical symmetries.

2. Just for fun one can consider the conjecture that the reduction of quantum intersections to classical intersections mediated by string world sheets implies that the numbers of string world sheets as given by the analog of G-W invariants are integers.

3.6.2 Does Riemann-Roch theorem have applications to TGD?

Riemann-Roch theorem (RR) (see http://tinyurl.com/mdmbcx6) is a central piece of algebraic geometry. Atyiah-Singer index theorem is one of its generalizations relating the solution spectrum of partial differential equations and topological data. For instance, characteristic classes classifying bundles associated with Yang-Mills theories (see http://tinyurl.com/y9xvkhyy) have applications in gauge theories and string models.

The advent of octonionic approach to the dynamics of space-time surfaces inspired by M^8-H duality [L36] [L37, L38] gives hopes that dynamics at the level of complexified octonionic M^8 could reduce to algebraic equations plus criticality conditions guaranteeing associativity for space-time surfaces representing external particles, in interaction region commutativity and associativity would be broken. The complexification of octonionic M^8 replacing norm in flat space metric with its complexification would unify various signatures for flat space metric and allow to overcome the problems due to Minkowskian signature. Wick rotation would not be a mere calculational trick.

For these reasons time might be ripe for applications of possibly existing generalization of RR to TGD framework. In the following I summarize my admittedly unprofessional understanding of RR discussing the generalization of RR for complex algebraic surfaces having real dimension 4: this is obviously interesting from TGD point of view.

I will also consider the possible interpretation of RR in TGD framework. One interesting idea is possible identification of light-like 3-surfaces and curves (string boundaries) as generalized poles and zeros with topological (but not metric) dimension one unit higher than in Euclidian signature.

Could a generalization of Riemann-Roch theorem be useful in TGD framework?

The generalization of RR for algebraic varieties, in particular for complex surfaces (real dimension equal to 4) exists. In M^8 picture the complexified metric Minkowskian signature need not cause any problems since the situation can be reduced to Euclidian sector. Clearly, this picture would provide a realization of Wick rotation as more than a trick to calculate scattering amplitudes.

Consider first the motivations for the desire of having analog of Riemann-Roch theorem (RR) at the level of space-time surfaces in M^8.

1. It would be very nice if partonic 2-surfaces would have interpretation as analogs of zeros or poles of a meromorphic function. RR applies to the divisors characterizing meromorphic functions and 2-forms, and one could hope of obtaining information about the dimensions of these function spaces giving rise to octonionic space-time varieties. Note however that the reduction to real polynomials or even rational functions might be already enough to give the needed information. Rational functions are required by the simplest generalization whereas the earlier approach assumed only polynomials. This generalization does not however change the construction of space-time varieties as zero loci of polynomials in an essential manner as will be found.

2. One would like to count the degeneracies for the intersections of 2-surfaces of space-time surface and here RR might help since its generalization to complex surfaces involves intersection form as was found in the brief summary of RR for complex surfaces with real dimension 4 (see Eq. 2.3.5).

In particular, one would like to know about the intersections of partonic 2-surfaces and string world sheets defining the points at which fermions reside. The intersection form reduces the problem via Poincare duality to 2-cohomology of space-time surfaces. More generally, it is known that the intersection form for 2-surfaces tells a lot about the topology of 4-D manifolds (see `http://tinyurl.com/y8tmqtef`). This conforms with SH. Gromow-Witten invariants [L25] (see `http://tinyurl.com/ybobccub`) are more advanced rational valued invariants but might reduce to integer valued in variants in TGD framework [L38].

There are also other challenges to which RR might relate.

1. One would like to know whether the intersection points for string world sheets and partonic 2-surfaces can belong in an extension of rationals used for adele. If the points belong to cognitive representations and subgroup of Galois group acts trivially then the number of points is reduces as the points at its orbit fuse together. The sheets of the Galois covering would intersect at point. The images of the fused points in H could be disjoint points since tangent spaces need not be parallel.
2. One would also like to have idea about what makes partonic 2-surfaces and string world sheets so special. In 2-D space-time one would have points instead of 2-surfaces. The obvious idea is that at the level of M^8 these 2-surfaces are in some sense analogous to poles and zeros of meromorphic functions. At the level of H the non-local character of M^8-H would imply that preferred extremals are solutions of an action principle giving partial differential equations.

What could be the analogs of zeros and poles of meromorphic function?

The basic challenge is to define what notions like pole, zero, meromorphic function, and divisor could mean in TGD context. The most natural approach based on a simple observation that rational functions need not define map of space-time surface to itself. Even though rational function can have pole inside CD, the point ∞ need not belong to the space-time variety defined the rational functions. Hence one can try the modification of the original hypothesis by replacing the octonionic polynomials with rational functions. One cannot exclude the possibility that although the interior of CD contains only finite points, the external particles outside CD could extend to infinity.

1. For octonionic analytic polynomials the notion of zero is well-defined. The notion of pole is well-defined only if one allows rational functions $R = P_1(o)/P_2(o)$ so that poles would correspond to zeros for the denominator of rational function. 0 and ∞ are both unaffected by multiplication and ∞ also by addition so that they are algebraically special. There are several variants of this picture. The most general option is that for a given variety zeros of both P_i are allowed.
2. The zeros of $IM(P_1) = 0$ and $IM(P_2) = 0$ would give solutions as unions of surfaces associated with P_i. This is because $IM(o_1o_2) = IM(o_1)RE(o_2) + IM(o_2)RE(o_1)$. There is no need to emphasize how important this property of IM for product is. One might say that one has two surfaces which behave like free non-interacting particles.
3. These surfaces should however interact somehow. The intuitive expectation is that the two solutions are glued by wormhole contacts connecting partonic 2-surfaces corresponding to $IM(P_1) = 0$ and $IM(P_2) = 0 = \infty$. For $RE(P_i) = 0$ and $RE(P_i) = \infty$ the solutions do not reduce to separate solutions $RE(P_1) = 0$ and $RE(P_2) = 0$. The reason is that the real part of o_1o_2 satisfies $Re(o_1o_2) = Re(o_1)Re(o_2) - Im(o_1)Im(o_2)$. There is a genuine interaction, which should generate the wormhole contact. Only at points for which $P_1 = 0$ and $P_2 = 0$ holds true, $RE(P_1) = 0$ and $RE(P_2) = 0$ are satisfied simultaneously. This happens in the discrete intersection of partonic 2-surfaces.
4. Elementary particles correspond even for $h_{eff} = h$ to two-sheeted structures with partonic surfaces defining wormhole throats. The model for elementary particles requires that particles are minimally 2-sheeted structures since otherwise the conservation of monopole Kähler magnetic flux cannot be satisfied: the flux is transferred between space-time sheets through wormhole contacts with Euclidian signature of induced metric and one obtains closed flux loop. Euclidian wormhole contact would connect the two Minkowskian sheets. Could the Minkowskian sheets corresponds to zeros $IM(P_i)$ for P_1 and P_2 and could wormhole contacts emerge as zeros of $RE(P_1/P_2)$?

One can however wonder whether this picture could allow more detailed specification. The simplest possibility would be following. The basic condition is that CD emerges automatically from this picture.

1. The simplest possibility is that one has $P_1(o)$ and $P_2(T-o)$ with the origin of octions at the "lower" tip of CD. One would have $P_1(0) = 0$ and $P_2(0) = 0$. $P_1(o)$ would give rise to the "lower" boundary of CD and $P_2(T-o)$ to the "upper" boundary of CD.

   ZEO combined with the ideas inspired by infinite rationals as counterparts of space-time surfaces connecting 3-surfaces at opposite boundaries of CD [K84] would suggest that the opposite boundaries of CD could correspond zeros and poles respectively and the ratio $P_1(o)/P_2(T-o)$ and to zeros of P_1 resp. P_2 assignable to different boundaries of CD. Both light-like parton orbits and string world sheets would interpolate between the two boundaries of CD at which partonic 2-surface would correspond to zeros and poles.

   The notion divisor would be a straightforward generalization of this notion in the case of complex plane. What would matter would be the rational function $P_1(t)/P_2(T-t)$ extended from the real (time) axis of octonions to the entire space of complexified octonions. Positive degree of divisor would multiply $P_1(t)$ with $(t-t_1)^m$ inducing a new zero at or increasing the order of existing zero at t_1. Negative orders n would multiply the denominator by $(t-t_1)^n$.

2. One can also consider the possibility that both boundaries of CD emerge for both P_1 and P_2 and without assigning either boundary of CD with P_i. In this case P_i would be sum over terms $P_{ik} = P_{ia_k}(o)P_{ib_k}(T-o)$ of this kind of products satisfying $P_{ia_k}(0) = 0$ and $P_{ib_k}(0) = 0$.

One can imagine also an alternative approach in which 0 and ∞ correspond to opposite tips of CD and have geometric meaning. Now zeros and poles would correspond to 2-surfaces, which need not be partonic. Note that in the case of Riemann surfaces ∞ can represent any point. This approach does not however look attractive.

Could one generalize RR to octonionic algebraic varieties?

RR is associated with complex structure, which in TGD framework seems to make sense independent of signature thanks to complexification of octonions. Divisors are the key notion and characterize what might be called local winding numbers. De-Rham cohomology is replaced with much richer Dolbeault cohomology (see `http://tinyurl.com/y7cvs5sx`) since the notion of continuity is replaced with that of meromorphy. Symplectic approach about which G-W invariants for symplectic manifolds provide an example define a different approach and now one has ordinary cohomology.

An interesting question is whether $M^8 - H$-duality corresponds to the mirror symmetry of string models (see `http://tinyurl.com/yc2m2e5m`) relating complex structures and symplectic structures. If this were the case, M^8 would correspond to complex structure and H to symplectic structure.

RR for curves gives information about dimensions for the spaces of meromorphic functions having poles with order not higher than specified by divisor. This kind of interpretation would be very attractive now since the poles and zeros represented as partonic 2-surfaces would have direct physical interpretation in terms of external particles and interaction vertices. RR for curves involves poles with orders not higher than specified by the divisor and gives a formula for the dimension of the space of meromorphic functions fora given divisor. As a special case give the dimension $l(nD)$ for a given divisor.

Could something similar be true in TGD framework?

1. Arithmetic genus makes sense for polynomials $P(t)$ since t can be naturally complexified giving a complex curve with well-defined arithmetic genus. What could correspond to the intersection form for 2-surfaces representing D and $K-D$? The most straightforward possibility is that partonic 2-surfaces correspond to poles and zeros.

   Divisor $-D$ would correspond to the inverse of P_2/P_1 representing it. $D-K$ would also a well-defined meaning provided the canonical divisor associated with holomorphic 2-form has well-defined meaning in the Dolbeault cohomology of the space-time surface with complex structure. RR would give direct information about the space of space-time varieties defined by $RE(P) = 0$ or $IM(P) = 0$ condition.

One could hope of obtaining information about intersection form for string world sheets and partonic 2-surfaces. Whether the divisor $D-K$ has anything to do string world sheets, is of course far from clear.

2. Complexification means that field property fails in the sense that complexified Euclidian norm vanishes and the inverse of complexified octonion/quaternion/complex number is infinite formally. For Euclidian sector with real coordinates this does not happen but does take place when some coordinates are real and some imaginary so that signature is effectively Minkowskian signature.

   At 7-D light-cone of M^8 the condition $P(o)=0$ reduces to a condition for real polynomial $P(t)=0$ giving roots t_n. Partonic 2-varieties are intersections of 4-D space-time varieties with 6-spheres with radii t_n. There are good reasons to expect that the 3-D light-like orbits of partonic 3-surfaces are intersections of space-time variety with 7-D light-cone boundary and their H counterparts are obtained as images under M^8-H duality.

   For light-like complefixied octonionic points the inverse of octonion does not exist since the complexified norm vanishes. Could the light-like 3-surfaces as partonic orbits correspond to images under M^8-H duality for zeros and/or poles as 3-D light-like surfaces? Could also the light-like boundaries of strings correspond to this kind of generalized poles or zeros? This could give a dynamical realization for the notions of zero and pole and increase the topological dimension of pole and zero for both 2-varieties and 4-varieties by one unit. The metric dimension would be unaffected and this implies huge extension of conformal symmetries central in TGD since the light-like coordinate appears as additional parameter in the infinitesimal generators of symmetries.

Could one formulate the counterpart of RR at the level of H? The interpretation of M^8-H duality as analog of mirror symmetry (see `http://tinyurl.com/yc2m2e5m`) suggests this. In this case the first guess for the identification of the counterpart of canonical divisor could be as Kähler form of CP_2. This description would provide symplectic dual for the description based on divisors at the level of M^8. G-W invariants and their possible generalization are natural candidates in this respect.

3.6.3 Could the TGD variant of Atyiah-Singer index theorem be useful in TGD?

Atyiah-Singer index theorem (AS) is one of the generalizations of RR and has shown its power in gauge field theories and string models as a method to deduce the dimensions of various moduli spaces for the solutions of field equations. A natural question is whether AS could be useful in TGD and whether the predictions of AS at H side could be consistent with M^8-H duality suggesting very simple counting for the numbers of solutions at M^8 side as coefficient combinations of polynomials in given extension of rationals satisfying criticality conditions. One can also ask whether the hierarchy of degrees n for octonion polynomials could correspond to the fractal hierarchy of generalized conformal sub-algebras with conformal weights coming as n-multiples for those for the entire algebras.

Atyiah-Singer index theorem (AS) and other generalizations of RR involve extremely abstract concepts. The best manner to get some idea about AS is to learn the motivations for it. The article `http://tinyurl.com/yc49lljp` gives a very nice general view about the motivations of Atyiah-Singer index theorem and also avoids killing the reader with details.

Solving problems of algebraic geometry is very demanding. The spectrum of solutions can be discrete (say number of points of space-time surface having linear M^8 coordinates in an extension of rationals) or continuous such as the space of roots for n:th order polynomials with real coefficients.

An even more difficult challenge is solving of partial differential equations in some space, call it X, of say Yang-Mills gauge field coupled to matter fields. In this case the set of solutions is typically continuous moduli space.

One can however pose easier questions. What is the number of solutions in counting problem? What is the dimension of the moduli space of solutions? Atiyiah-Singer index theorem relates this number - analytic index - to topological index expressible in terms of topological invariants assignable to complexified tangent bundle of X and to the bundle structure - call it field bundle - accompanying the fields for which field equations are formulated.

AS very briefly

Consider first the assumptions of AS.

1. The idea is to study perturbations of a given solution and linearize the equations in some manifold X often assumed to be compact. This leads to a linear partial differential equations defined by linear operator P. One can deduce the dimension of the solution space of P. This number defines the dimension of the tangent space of solution space of full partial differential equations, call it moduli space.

2. The idea is to assign to the partial differential operator P its symbol $\sigma(P)$ obtained by replacing derivatives with what might be called momentum components. The reversal of this operaion is familiar from elementary wave mechanics: $p_i \rightarrow id/dx^i$. This operation can be formulated in terms of co-tangent bundle. The resulting object is purely algebraic. If this matrix is reversible for all momentum values and points of X, one says that the operator is elliptic.

   Note that for field equations in Minkowski space M^4 the invertibility constraint is not satisfied and this produces problems. For instance, for massive M^4 d'Alembertian for scalar field the symbol is four-momentum squared, which vanishes, when on-mass shell condition is satisfied. Wick rotation is somewhat questionable manner to escape this problem. One replaces Minkowski space with its Euclidian counterpart or by 4-sphere. If all goes well the dimension of the solution space does not depend on the signature of the metric.

3. In the general case one studies linear equation of form $DP = f$, where f is homogenuity term representing external perturbation. f can also vanish. Quite generally, one can write the dimension of the solution space as

$$Ind_{anal}(P) = dim(ker(P)) - dim(coker(P)) \ . \tag{3.6.1}$$

   $ker(P)$ denotes the solution space for $DP = 0$ without taking into account the possible restrictions coming from the fact that f can involve part f_0 satisfying $Df_0 = 0$ (for instance, f_0 corresponds to resonance frequency of oscillator system) nor boundary conditions guaranteing hermiticity. Indeed, the hermitian conjugate $D^\dagger$ of D is not automatically identical with D. $D^\dagger$ is defined in terms of the inner product for small perturbations as

$$\langle D^\dagger P_1^* | DP_2 \rangle = \langle P_1 | DP_2 \rangle \ . \tag{3.6.2}$$

   The inner product involves integration over X and partial integrations transfer the action of partial derivatives from P_2 to P_1^*. This however gives boundary terms given by surface integral and hermiticity requires that they vanish. This poses additional conditions on P and contributes to $dim(coker(P))$.

The challenge is to calculate $Ind_{anal}(P)$ and here AS is of enormous help. AS relates analytical index $Ind_{anal}(P)$ for P to topological index $Ind_{top}(\sigma(P))$ for its symbol $\sigma(P)$.

1. $Ind_{top}(\sigma(P))$ involves only data associated with the topology X and with the bundles associated with field variables. In the case of Yang-Mills fields coupled to matter the bundle is the bundle associated with the matter fields with a connection determined by Yang-Mills gauge potentials. So called Todd class $Td(X)$ brings in information about the topology of complexified tangent bundle.

2. $Ind_{top}(\sigma(P))$ is not at all easy to define but is rather easily calculable as integrals of various invariants assignable to the bundle structure involved. Say instanton density for YM fields and various topological invariants expressing the topological invariants associated with the metric of the space. What is so nice and so non-trivial is that the dimension of the moduli space for non-linear partial differential equations is determined by topological invariants. Much of the dynamics reduces to topology.

The expression for $Ind_{top}(\sigma(P))$ involves besides σ_P topological data related to the field bundle and to the complexified tangent bundle. The expression Ind_{top} as a function of the symbol $\sigma(P)$ is given by

$$Ind_{top}(\sigma(P)) = (-1)^n \langle ch(\sigma(P)) \cdot Td(T_C(X), [X]) \rangle \ . \tag{3.6.3}$$

The expression involves various topological data.

1. Dimension of X.
2. The quantity $\langle x.y \rangle$ involving cup product $x.y$ of cohomology classes, which contains a contribution in the highest homology group $H^n(X)$ of X corresponding to the dimension of X and is contracted with this fundamental class $[X]$. $\langle x.y \rangle$ denotes matrix trace for the operator $ch(\sigma(P))$ formed as polynomial of $\sigma(P)$. $[X]$ denotes so called fundamental class fr X belonging to H^n and defines the orientation of X.
3. Chern character $ch_E(t)$ (see `http://tinyurl.com/ybavu66h`). I must admit that I ended up to a garden of branching paths while trying to understand the definition of ch_E is. In any case, $ch_E(t)$ characterizes complex vector bundle E expressible in terms of Chern classes (see `http://tinyurl.com/y8jlaznc`) of E. E is the bundle assignable to field variables, say Yang Mills fields and various matter fields.

   Both direct sums and tensor products of fiber spaces of bundles are possible and the nice feature of Chern class is that it is additive under tensor product and multiplicative under direct sum. The fiber space of the entire bundle is now direct sum of the tangent space of X and field space, which suggests that $Ind(top)$ is actually the analog of Chern character for the entire bundle.

   $t = \sigma P$ has interpretation as an argument appearing in the definition of Chern class generalized to Chern character. $t = \sigma(P)$ would naturally correspond to a matrix valued argument of the polynomial defining Chern class as cohomology element. $ch(\sigma(P))$ is a polynomial of the linear operator defined by symbol $\sigma(P)$. ch_E for given complex vector bundle is a polynomial, whose coefficients are relatively easily calculable as topological invariants assignable to bundle E. E must be the field bundle now.
4. Todd class $Td(T_C(X))$ for the complexified tangent bundle (see `http://tinyurl.com/yckv4w84`) appears also in the expression. Note that also now the complexification occurs. The cup product gives element in $H^n(X)$, which is contracted with fundamental class $[X]$ and integrated over X.

AS and TGD

The dynamics of TGD involves two levels: the level of complexified M^8 (or equivalently E^8) and the level of H related to $M^8 - H$ correspondence.

1. At the level of M^8 one has algebraic equations rather than partial differential equations and the situation is extremely simple as compared to the situation for a general action principle. At the level of H one has action principle and partial differential equations plus infinite number of gauge conditions selecting preferred extremals and making dynamics for partial differential equations dual to the dynamics determined by purely number theoretic conditions.

   The space-time varieties representing external particles outside CDs in M^8 satisfy associativity conditions for tangent space or normal space and reducing to criticality conditions for the real coefficients of the polynomials defining the space-time variety. In the interior of CDs associativity conditions are not satisfied but the boundary conditions fix the values of the coefficients to be those determined by criticality conditions guaranteing associativity outside the CD.

   In the interiors space-time surfaces of CDs M^8-duality does not apply but associativity of tangent spaces or normal spaces at the boundary of CD fixes boundary values and minimal surface dynamics and strong form of holography (SH) fixes the space-time surfaces in the interior of CD.

2. For the H-images of space-time varieties in H under M^8-H duality the dynamics is universal coupling constant independent critical dynamics of minimal surfaces reducing to holomorphy in appropriate sense. For minimal surfaces the 4-D Kähler current density vanishes so that the solutions are 4-D analogs of geodesic lines outside CD. Inside CD interactions are coupled on and this current is non-vanishing. Infinite number of gauge conditions for various half conformal algebras in generalized sense code at H side for the number theoretical critical conditions at M^8 side. The sub-algebra with conformal weights coming as n-ples of the entire algebra and its commutator with entire algebra gives rise to vanishing classical Noether charges. An attractive assumption is that the value of n at H side corresponds to the order n of the polynomials at M^8 side.
3. The coefficients of polynomials $P(o)$ determining space-time varieties are real numbers (also complexified reals can be considered without losing associativity) restricted to be numbers in extension of rationals. This makes it possible to speak about p-adic variants of the space-time surfaces at the level of M^8 at least.

Could Atyiah-Singer theorem have relevance for TGD?

1. For real polynomials it is easy to calculate the dimension of the moduli space by counting the number of independent real (in octonionic sense) coefficients of the polynomials of real variable (one cannot exclude that the coefficients are in complex extension of rationals). Criticality conditions reduce this number and the condition that coefficients are in extension of rationals reduces it further. One has quite nice overall view about the number of solutions and one can see them as subset of continuous moduli space. If $M^8 - H$ duality really works then this gives also the number of preferred extremals at H side.
2. This picture is not quite complete. It assumes fixing of 8-D CD in M^8 as well as fixing of the decomposition $M^2 \subset M^4 \subset M^4 \times E^4$. This brings in moduli space for different choices of octonion structures (8-D Lorentz group is involved). Also moduli spaces for partonic 2-surfaces are involved. Number theoretical universality seems to require that also these moduli spaces have only points with coordinates in extension of rationals involved.
3. In principle one can try to formulate the counterpart of AS at H side for the linearization of minimal surface equations, which are nothing but the counterpart of massless field equations in a fixed background metric. Note that additional conditions come from the requirement that the term from Kähler action reduces to minimal surface term.

   Discrete sets of solutions for the extensions of rationals should correspond to each other at the two sides. One can also ask whether the dimensions for the effective continuous moduli spaces labelled by n characterizing the sub-algebras of various conformal algebras isomorphic to the entire algebra and those for the polynomials of order n satisfying criticality conditions. One would have a number theoretic analog for a particle in box leading to the quantization of momenta.

All this is of course very speculative and motivated only by the general physical vision. If the speculations were true, they would mean huge amount of new mathematics.

3.7 Intersection form for 4-manifolds, knots and 2-knots, smooth exotics, and TGD

Gary Ehlenberger sent a highly interesting commentary related to smooth structures in R^4 discussed in the article of Gompf [A181] (`https://cutt.ly/eMracmf`) and more generally to exotics smoothness discussed from the point of view of mathematical physics in the book of Asselman-Maluga and Brans [A188] (`https://cutt.ly/DMuOdYr`). I am grateful for these links for Gary.

3.7.1 Basic ideas

The role of intersection forms in TGD

The intersection form of 4-manifold (`https://cutt.ly/jMriNdI`) characterizing partially its 2-homology is a central notion in the study of the smooth structures. I am not a topologist but have two good reasons to get interested on intersection forms.

1. In the TGD framework [L110], the intersection form describes the intersections of string world sheets and partonic 2-surfaces and therefore is of direct physical interest [K45, L38].
2. Knots have an important role in TGD. The 1-homology of the knot complement characterizes the knot. Time evolution defines a knot cobordism as a 2-surface consisting of knotted string world sheets and partonic 2-surfaces. A natural guess is that the 2-homology for the 4-D complement of this cobordism characterizes the knot cobordism. Also 2-knots are possible in 4-D space-time and a natural guess is that knot cobordism defines a 2-knot.

   The intersection form for the complement for cobordism as a way to classify these two-knots is therefore highly interesting in the TGD framework. One can also ask what the counterpart for the opening of a 1-knot by repeatedly modifying the knot diagram could mean in the case of 2-knots and what its physical meaning could be in the TGD Universe. Could this opening or more general knot-cobordism of 2-knot take place in zero energy ontology (ZEO) [L72, L108, L117] as a sequence of discrete quantum jumps leading from the initial 2-knot to the final one.

Why exotic smooth structures are not possible in TGD?

The existence of exotic 4-manifolds [A181, A188, A143] could be an anomaly in the TGD framework. In the articles [A181, A143] the term anomaly is indeed used. Could these anomalies cancel in the TGD framework?

The first naive guess was that the exotic smooth structures are not possible in TGD but it turned out that this is not trivially true. The reason is that the smooth structure of the space-time surface is not induced from that of H unlike topology. One could induce smooth structure by assuming it given for the space-time surface so that exotics would be possible. This would however bring an ad hoc element to TGD. This raises the question of how it is induced.

1. This led to the idea of a holography of smoothness, which means that the smooth structure at the boundary of the manifold determines the smooth structure in the interior. Suppose that the holography of smoothness holds true. In ZEO, space-time surfaces indeed have 3-D ends with a unique smooth structure at the light-like boundaries of the causal diamond $CD = cd \times CP_2 \subset H = M^4 \times CP_2$, where cd is defined in terms of the intersection of future and past directed light-cones of M^4. One could say that the absence of exotics implies that $D = 4$ is the maximal dimension of space-time.
2. The differentiable structure for $X^4 \subset M^8$, obtained by the smooth holography, could be induced to $X^4 \subset H$ by $M^8 - H$-duality. Second possibility is based on the map of mass shell hyperboloids to light-cone proper time $a = constant$ hyperboloids of H belonging to the space-time surfaces and to a holography applied to these.
3. There is however an objection against holography of smoothness (`https://cutt.ly/3MewYOt`). In the last section of the article, I develop a counter argument against the objection. It states that the exotic smooth structures reduce to the ordinary one in a complement of a set consisting of arbitrarily small balls so that local defects are the condensed matter analogy for an exotic smooth structure.

3.7.2 Intersection form in the case of 4-surfaces

Intersection form (`https://cutt.ly/jMriNdI`) for homologically trivial 2-surfaces of the space-time surface and 2-homology for the complement of these surfaces can be physically important in tGD framework.

Intersection forms in 2-D case

It is good to explain the notion of intersection form by starting from 1-homology. The intersection form for 1-homology is encountered for a cylinder with ends fixed. In this case, one has relative homology and homologically trivial curves are curves connecting the ends of string and characterized by a winding number.

In the case of torus obtained by identifying the ends of cylinder, one obtains two winding numbers (m, n) corresponding to to homologically non-trivial circles at torus. The intersection number for curves (m,n) and (p,q) at torus is $N = mq - np$ and for curves at cylinder one as $(m,n) = (1,n)$ giving $N = n - q$.

The antisymmetric intersection form is defined as 2×2 matrix defining intersections for the basis of the homology with $(m,n) = (1,0)$ and $(n,m) = (0,1)$ and is given by $(0,1;-1,0)$.

Intersection for 4-surfaces in TGD context

In TGD, the intersection form for a 4-surface identified as space-time surface could have a rather concrete physical interpretation and the stringy part of TGD physics would actually realize it concretely.

1. $M^8 - H$ duality requires that the 4-surface in M^8 has quaternionic/associative normal space: this distribution of normal spaces is integrable and integrates to the 4-surface in M^8.

   The normal must also contain a commutative (complex) sub-space at each point. Only this allows us to parametrize normal spaces by points of CP_2 and map them to space-time surfaces in $H = M^4 \times CP_2$. The integral distribution of these commutative sub-spaces defines a 2-D surface. Physically, these surfaces would correspond to string world sheets and partonic 2-surfaces.

2. String world sheets and partonic 2-surfaces, regarded as objects in relative homology (modulo ends of the space-time surfaces at the boundaries of causal diamond (CD)), can intersect as 2-D objects inside the space-time surface and the intersection form characterizes them.

   There is an analogy with the cylinder: time-like direction corresponds to the cylinder axis and a homologically non-trivial 2-surface of CP_2 corresponds to the circle at the cylinder.

3. If the second homology of the space-time surface is trivial, the naive expectation is that the intersections of string world sheets are not stable under large enough deformations of the string world sheets. Same applies to intersecting plane curves. At the cylinder, the situation is different since the relative first homology is non-trivial and spanned by two generators: the circle and a line connecting the ends of the cylinder.

   The intersection form is however non-trivial as in the case of the cylinder for 2-surfaces having 2-D homologically non-trivial CP_2 projection. They would represent M^4 deformations of 2-D homologically trivial surfaces of CP_2 just like a helical orbit along a cylinder surface. A 2-D generalization of CP_2 type extremal would have a light-like curve or light-like geodesic as M^4 projection and could define light-partonic orbit.

4. The intersection of string world sheet and partonic 2-surface can be stable however. Partonic 2-surface is a boundary of a wormhole contact connecting two space-time sheets.

   Consider a string arriving along space-time sheet A, going through the wormhole contact, and continuing along sheet B. The string has an intersection point with both wormhole throats. This intersection is stable against deformations. The orbit of this string intersects the light-like orbit of the partonic 2-surface along the light-like curve.

   One has a non-trivial intersection form with the number of intersections with partonic 2-surfaces equal to 1. In analogy with cylinder, also the intersections of 2-surfaces with 2-D homologically trivial CP_2 projection are unavoidable and reflect the non-trivial intersection form of CP_2.

3.7.3 About ordinary knots

Ordinary knots and 3-topologies are related and the natural expectation is that also 2-knots and 4-topologies are related.

About knot invariants

Consider first knot invariants (`https://cutt.ly/DMrgs14`)at the general level.

1. One important knot invariant of ordinary knots is the 1-homology of the complement and the associated first homotopy group whose abelianization gives the homology group.

2. The complement of the knot can be given a metric of a hyperbolic 3-manifold, which corresponds to a unit cell for a tessellation of the mass shell. M^8-H duality suggests that the intersection X^3 of 4-surface of M^8 with mass shell $H^3_m \subset M^4 \subset M^8$ is a hyperbolic manifold and identical with the hyperbolic manifold associated with the complement of a knot of H^3_a realized as light-cone proper time $a = constant$ hyperboloid of $M^4 \subset H$ and closed knotted and linked strings as ends of string world sheets at H^3_a.

   The evolution of the strings defined by the string world sheets would define a 1-knot cobordism. The 2-homology of the knot complement should characterize the topological evolution of the 1-homology of the knot.

Opening of knots and links by knot cobordisms

The procedure leading to the trivialization of knot or link can be used to define knot invariants and the procedure itself characterizes knot.

1. Ordinary knot is described by a knot diagram obtained as a projection of the knot to the plane. It contains intersections of lines and the intersection contains information telling which line is above and which line is below.
2. The opening of the knot or link to give a trivial knot or link, which is used in the construction of knot invariants, is a sequence of violent operations. In the basic step strings portions go through each other and therefore suffer a reconnection. This operation can therefore change the 1-homology of the 3-D knot complement.

   Knot or link can be modified by forcing two intersecting strands of the plane projection to go through each other. Locally the basic operation for two links is the same as for the pieces of knot. The transformation of the knot or link to a trivial knot or link corresponds to some sequence of these operations and can be used to define a knot invariants. This operation is not unique since there are moves which do not affect the knot.

   The basic opening operation can be also seen as a time evolution, knot cobordism, in which the first portion, call it A, remains unchanged and the second portion, call it B, draws a 2-D surface in E^3. A intersects the 2-D orbit at a single point.
3. The 2-homology for the string world sheets and partonic 2-surfaces as 2-surfaces in space-time serves as an invariant of knot cobordism and represents the topological dynamics of ordinary 1-knots of 3-surface and links formed by strings or flux tubes in 3-surface as cobordism defining the time evolution of a knot to another knot.

   In particular, the intersection form for the 2-homology of the complement of the cobordism defines an invariant of cobordism. This intersection form must be distinguished from the intersection form for the second homology of the space-time surface rather than the 2-knot complement.
4. One can also consider more general sequences of basic operations transforming two knots or links to each other as knot-/link cobordisms, which involve self intersections of the knots. Does this mean that the intersection form characterizes the knot cobordism. Could a string diagram involving reconnections describe the cobordism process.

Stringy description of knot cobordisms

M^8-H duality [L82, L83, L127, L125] requires string word sheets and partonic 2-surfaces. This implies that TGD physics represents the 2-homology of both space-time surfaces and the homology of the complement of the knotted links defined by them.

Although the "non-homological" intersections of string world sheets can be eliminated by a suitable deformation of the string world sheet, they should have a physical meaning. This comes from the observation that they affect nontrivially the 1-homology of the knot complement as 3-D time=constant slice.

The first thing that I am able to imagine is that strings reconnect. This is nothing but the trouser vertex for strings so that intersection form would define topological string dynamics in some sense. These reconnections play a key role in TGD, also in TGD inspired quantum biology.

The dynamics of partonic 2-surfaces and string world sheets could relate to knot cobordisms, possibly leading to the opening of ordinary knot,

3.7.4 What about 2-knots and their cobordisms?

2-D closed surfaces in 4-D space give rise to 2-knots. What is the physical meaning of 2-knots of string world sheets? What could 2-knots for orbits of linear molecules or associated magnetic flux tubes mean physically and from the point of view of quantum information theory? One can try to understand 2-knots by generalizing the ideas related to the ordinary knots.

1. Intuitively it seems that the cobordism of a 1-knot defines a 2-knot. It is not clear to me whether all 2-knots for space-time surfaces connecting the boundaries of CD can be regarded as this kind of cobordisms of 1-knots.
2. The 2-homology of the complement of 2-knot should define a 2-knot invariant. In particular, the intersection form should define a 2-knot invariant.
3. The opening of 1-knot by repeating the above described basic operation is central in the construction of knot invariants and the sequence of the operations can be said to be knot invariant modulo moves leaving the knot unaffected.

   The opening or a more general cobordism of a 2-knot could be seen as a time evolution with respect to a time parameter t_5 parametrizing the isotopy of space-time surface. The local cobordism can keep the first portion of 2-knot, call it A, unchanged and deform another portion, call it B, so that a 3-D orbit at the space-time surface is obtained. For each value of t_5, the portions A and B of 2-knot have in the generic case only points as intersections.

   This would suggest that an intersection point of A and B is generated in the operation and moves during the t_5 time evolution along A along 1-D curve during the process. This process would be the basic operation used repeatedly to open 2-knot or to transform it to another 2-knot.
4. In quantum TGD, a sequence of quantum jumps, quantum cobordism, would have the same effect as t_5 time evolution. This brings in mind DNA transcription and replication as a process proceeding along a DNA strand parallel to the monopole flux tube as a sequence of SFRs involving direct contact between DNA strand and enzymes catalyzing the process and also of corresponding flux tubes. An interesting possibility is that these quantum cobordisms appear routinely in biochemistry of the fundamental linear bio-molecules such as DNA, RNA, tRNA, and amino-acids [K36, K3, K95, K1, K106, L16] [L57].

   The quantum cobordism of 2-knot is possible only in ZEO, where the quantum state as a time= constant snapshot is replaced with a superposition of space-time surfaces.

3.7.5 Could the existence of exotic smooth structures pose problems for TGD?

The article of Gabor Etesi [A143] (`https://cutt.ly/2Md7JWP`) gives a good idea about the physical significance of the existence of exotic smooth structures and how they destroy the cosmic censorship hypothesis (CCH of GRT stating that spacetimes of GRT are globally hyperbolic so that there are no time-like loops.

Smooth anomaly

No compact smoothable topological 4-manifold is known, which would allow only a single smooth structure. Even worse, the number of exotics is infinite in every known case! In the case of non-compact smoothable manifolds, which are physically of special interest, there is no obstruction against smoothness and they typically carry an uncountable family of exotic smooth structures.

One can argue that this is a catastrophe for classical general relativity since smoothness is an essential prerequisite for tensory analysis and partial differential equations. This also destroys hopes that the path integral formulation of quantum gravitation, involving path integral over all possible space-time geometries, could make sense. The term anomaly is certainly well-deserved.

Note however that for 3-geometries appearing as basic objects in Wheeler's superspace approach, the situation is different since for $D < 3$ there is only a single smooth structure. If one has holography, meaning that 3-geometry dictates 4-geometry, it might be possible to avoid the catastrophe.

The failure of the CCH is the basic message of Etesi's article. Any exotic R^4 fails to be globally hyperbolic and Etesi shows that it is possible to construct exact vacuum solutions representing curved space-times which violate the CCH. In other words, GRT is plagued by causal anomalies.

Etesi constructs a vacuum solution of Einstein's equations with a vanishing cosmological constant which is non-flat and could be interpreted as a pure gravitational radiation. This also represents one particular aspect of the energy problem of GRT: solutions with gravitational radiation should not be vacua.

1. Etesi takes any exotic R^4 which has the topology of $S^3 \times R$ and has an exotic smooth structure, which is not a Cartesian product. Etesi maps maps R^4 to CP_2, which is obtained from C^2 by gluing CP_1 to it as a maximal ball B^3_r for which the radial Eguchi-Hanson coordinate approaches infinity: $r \to \infty$. The exotic smooth structure is induced by this map. The image of the exotic atlas defines atlas. The metric is that of CP_2 but $SU(3)$ does not act as smooth isometries anymore.
2. After this Etesi performs Wick rotation to Minkowskian signature and obtains a vacuum solution of Einstein's equations for any exotic smooth structure of R^4.

In TGD, the question of exotic smoothness is encountered both at the level of embedding space and associated fixed spaces and at the level of space-time surfaces and their 6-D twistor space analogies. Could TGD solve the smooth anomaly?

Can embedding space and related spaces have exotic smooth structure?

One can first worry about the exotic smooth structures possibly associated with the M^4, CP_2, $H = M^4 \times CP_2$, causal diamond CD= $cd \times CP_2$, where cd is the intersection of the future and past directed light-cones of M^4, and with M^8. One can also worry about the twistor spaces CP_3 *resp.* $SU(3)/U(1) \times U(1)$ associated with M^4 *resp.* CP_2.

The key assumption of TGD is that all these structures have maximal isometry groups so that they relate very closely to Lie groups, whose unique smooth structures are expected to determine their smooth structures.

1. The first sigh of relief is that all Lie groups have the standard smooth structure. In particular, exotic R^4 does not allow translations and Lorentz transformations as isometries. I dare to conclude that also the symmetric spaces like CP_2 and hyperbolic spaces such as $H^n = SO(1,n)/SO(n)$ are non-exotic since they provide a representation of a Lie group as isometries and the smoothness of the Lie group is inherited. This would mean that the charts for the coset space G/H would be obtained from the charts for G by an identification of the points of charts related by action of subgroup H.

   Note that the mass shell H^3, as any 3-surface, has a unique smooth structure by its dimension.
2. Second sigh of relief is that twistor spaces CP_3 and $SU(3)/U(1) \times U(1)$ have by their isometries and their coset space structure a standard smooth structure.

   In accordance with the vision that the dynamics of fields is geometrized to that of surfaces, the space-time surface is replaced by the analog of twistor space represented by a 6-surface with a structure of S^2 bundle with space-time surface X^4 as a base-space in the 12-D product of twistor spaces of M^4 and CP_2 and by its dimension $D = 6$ can have only the standard smooth structure unless it somehow decomposes to $(S^3 \times R) \times R^2$. Holography of smoothness would prevent this since it has boundaries because X^4 as base space has boundaries at the boundaries of CD.

   If exotic smoothness is allowed at the space-time level in the proposed sense ordinary smooth structure could be possible at the level of twistor space in the complement of a Cartesian product of the fiber space S^2 with a discrete set of points associated with partonic 2-surfaces.
3. cd is an intersection of future and past directed light-cones of M^4. Future/past directed light-cone could be seen as a subset of M^4 and implies standard smooth structure is possible. Coordinate atlas of M^4 is restricted to cd and one can use Minkowski coordinates also inside the cd. cd could be also seen as a pile of light-cone boundaries $S^2 \times R_+$ and by its dimension $S^2 \times R$ allows only one smooth structure.

4. M^8 is a subspace of complexified octonions and has the structure of 8-D translation group, which implies standard smooth structure.

The conclusion is that continuous symmetries of the geometry dictate standard smoothness at the level of embedding space and related structures.

Could TGD eliminate the smoothness anomaly or provide a physical interpretation for it?

The question of exotic smoothness is encountered both at the level of embedding space and associated fixed spaces and at the level of space-time surfaces and their 6-D twistor space analogies.

What does the induction of a differentiable structure really mean? Here my naive expectations turn out to be wrong. If a sub-manifold $S \subset H$ can be regarded as an embedding of smooth manifold N to $S \subset H$, the embedding $N \rightarrow S \subset H$ induces a smooth structure in S (`https://cutt.ly/tMtvG79`). The problem is that the smooth structure would not be induced from H but from N and for a given 4-D manifold embedded to H one could also have exotic smooth structures. This induction of smooth structure is of course physically adhoc.

It is not possible to induce the smooth structure from H to sub-manifold. The atlas defining the smooth structure in H cannot define the charts for a sub-manifold (surface). For standard R^4 one has only one atlas.

1. Could holography of smoothness make sense in the general case?

The first trial to get rid of exotics [A188] was based on the holography of smoothness and did not involve TGD. Could a smooth structure at the boundary of a 4-manifold could dictate that of the manifold uniquely. Could one speak of holography for smoothness? Manifolds with boundaries would have the standard smooth structure.

1. The obvious objection is that the coordinate atlas for 3-D boundary cannot determine 4-D atlas in any way because the boundary cannot have information of the topology of the interior.
2. The holography for smoothness is also argued to fail (`https://cutt.ly/3MewYOt`). Assume a 4-manifold W with 2 different smooth structures. Remove a ball B^4 belonging to an open set U and construct a smooth structure at its boundary S^3. Assume that this smooth structure can be continued to W. If the continuation is unique, the restrictions of the 2 smooth structures in the complement of B^4 would be equivalent but it is argued that they are not.
3. The first layman objection is that the two smooth structures of W are equivalent in the complement $W - B^3$ of an arbitrary small ball $B^3 \subset W$ but not in the entire W. This would be analogous to coordinate singularity. For instance, a single coordinate chart is enough for a sphere in the complement of an arbitrarily small disk.

   An exotic smooth structure would be like a local defect in condensed matter physics. In fact it turned out that this intuitive idea is correct: it can be shown that the exotic smooth structures are equivalent with standard smooth structure in a complement of a set having co-dimension zero (`https://cutt.ly/7MbGqx2`). This does not save the holography of smoothness in the general case but gives valuable hints for how exotic smoothness might be realized in TGD framework.

2. Could holography of smoothness make sense in the TGD framework?

Could $M^8 - H$ duality and holography make holography of smoothness possible in the TGD framework?

1. In the TGD framework space-time is 4-surface rather than abstract 4-manifold. 4-D general coordinate invariance, assuming that 3-surfaces as generalization of point-like particles are the basic objects, suggests a fully deterministic holography. A small failure of determinism is however possible and expected, and means that space-time surfaces analogous to Bohr orbits become fundamental objects. Could one avoid the smooth anomaly in this framework?

   The 8-D embedding space topology induces 4-D topology. My first naive intuition was that the 4-D smooth structure, which I believed to be somehow inducible from that of $H = M^4 \times CP_2$, cannot be exotic so that in TGD physics the exotics could not be realized. But can one really exclude the possibility that the induced smooth structure could be exotic as a 4-D smooth structure?

2. In the TGD framework and at the level of $H = M^4 \times cP_2$, one can argue that the holography implied by the general coordinate invariance somehow determines the smooth structure in the interior of space-time surface from the coordinate atlas at the boundary. One would have a holography of smoothness. It is however not obvious why this unique structure should be the standard one.
3. One has also holography in M^8 and this induces holography in H by $M^8 - H$ duality. The 3-surfaces X^3 inducing the holography in M^8 are parts of mass shells, which are hyperbolic spaces $H^3 \subset M^4 \subset M^8$. 3-surfaces X^3 could be even hyperbolic 3-manifolds as unit cells of tessellations of H^3. These hyperbolic manifolds have unique smooth structures as manifolds with dimension $D < 4$.

   The hypothesis is that one can assign to these 3-surfaces a 4-surface by a number theoretic dynamics requiring that the normal space is associative, that is quaternionic [L82, L83]. The additional condition is that the normal space contains commutative subspace makes it possible to parametrize normal spaces by points of CP_2. $M^8 - H$ duality would map a given normal space to a point of CP_2. $M^8 - H$ duality makes sense also for the twistor lift.
4. A more general statement would be as follows. A set of 3-surfaces as sub-manifolds of mass shells H^3_m determined by the roots of polynomial P having interpretation as mass square values defining the 4-surface in M^8 take the role of the boundaries. Mass-shells H^3_m or partonic 2-surfaces associated with them having particle interpretation could correspond to discontinuities of derivatives and even correspond to failure of manifold property analogous to that occurring for Feybman diagrams so that the holography of smoothness would decompose to a piece-wise holography.

   The regions of $X^4 \subset M^8$ connecting two sub-sequent mass shells would have a unique smooth structure induced by the hyperbolic manifolds H^3 at the ends.

It is important to notice that the holography of smoothness does not force the smooth 4-D structure to be the standard one.

3. Could the exotic smooth structures have a physical interpretation in the TGD framework?

In the TGD framework, exotic smooth structures could also have a physical interpretation. As noticed, the failure of the standard smooth structure can be thought to occur at a point set of dimension zero and correspond to a set of point defects in condensed matter physics. This could have a deep physical meaning.

1. The space-time surfaces in $H = M^4 \times CP_2$ are images of 4-D surfaces of M^8 by $M^8 - H$-duality. The proposal is that they reduce to minimal surfaces analogous to soap films spanned by frames. Regions of both Minkowskian and Euclidean signature are predicted and the latter correspond to wormhole contacts represented by CP_2 type extremals. The boundary between the Minkowskian and Euclidean region is a light-like 3-surface representing the orbit of partonic 2-surface identified as wormhole throat carrying fermionic lines as boundaries of string world sheets connecting orbits of partonic 2-surfaces.
2. These fermionic lines are counterparts of the lines of ordinary Feynman graphs, and have ends at the partonic 2-surfaces located at the light-like boundaries of CD and in the interior of the space-time surface. The partonic surfaces, actually a pair of them as opposite throats of wormhole contact, in the interior define topological vertices, at which light-like partonic orbits meet along their ends.
3. These points should be somehow special. Number theoretically they should correspond points with coordinates in an extension of rationals for a polynomial P defining 4-surface in H and space-time surface in H by $M^8 - H$ duality. What comes first in mind is that the throats touch each other at these points so that the distance between Minkowskian space-time sheets vanishes. This is analogous to singularities of Fermi surface encountered in topological condensed matter physics: the energy bands touch each other. In TGD, the partonic 2-surfaces at the mass shells of M^4 defined by the roots of P are indeed analogs of Fermi surfaces at the level of $M^4 \subset M^8$, having interpretation as analog of momentum space.

Could these points correspond to the defects of the standard smooth structure in X^4? Note that the branching at the partonic 2-surface defining a topological vertex implies the local failure of the manifold property. Note that the vertices of an ordinary Feynman diagram imply that it is not a smooth 1-manifold.

4. Could the interpretation be that the 4-manifold obtained by removing the partonic 2-surface has exotic smooth structure with the defect of ordinary smooth structure assignable to the partonic 2-surface at its end. The situation would be rather similar to that for the representation of exotic R^4 as a surface in CP_2 with the sphere at infinity removed [A143].

5. The failure of the cosmic censorship would make possible a pair creation. As explained, the fermionic lines can indeed turn backwards in time by going through the wormhole throat and turn backwards in time. The above picture suggests that this turning occurs only at the singularities at which the partonic throats touch each other. The QFT analog would be as a local vertex for pair creation.

6. If all fermions at a given boundary of CD have the same sign of energy, fermions which have returned back to the boundary of CD, should correspond to antifermions without a change in the sign of energy. This would make pair creation without fermionic 4-vertices possible.

   If only the total energy has a fixed sign at a given boundary of CD, the returned fermion could have a negative energy and correspond to an annihilation operator. This view is nearer to the QFT picture and the idea that physical states are Galois confined states of virtual fundamental fermions with momentum components, which are algebraic integers. One can also ask whether the reversal of the arrow of time for the fermionic lines could give rise to gravitational quantum computation as proposed in [A188].

4. A more detailed model for the exotic smooth structure associated with a topological 3-vertex

One can ask what happens to the 4-surface near the topological 3-particle vertex and what is the geometric interpretation of the point defect. The first is whether the description of the situation is possible both in M^8 and H. Here one must consider momentum conservation.

1. By Uncertainty Principle and momentum conservation at the level of M^8, the incoming real momenta of the particle reaction are integers in the scale defined by CD. In the standard QFT picture, the momenta at the vertex of physical particles are at different mass shells.

   In M^8 picture, the mass squared values of virtual fermions are in general algebraic and also complex roots of a polynomial defining the 3-D mass shells H^3_m of $M^4 \subset M^8$, determining 4-surface by associative holography.

   In the standard wave mechanical picture assumed also in TGD, a given topological vertex, describable in terms of partonic 2-surfaces, would correspond to a multi-local vertex in M^8 in accordance with the representation of a local n-vertex in M^4 as convolution of n-local vertices in momentum space realizing momentum conservation.

2. M^8-H duality maps M^4 momenta by inversion to positions in $M^4 \subset H$. This encourages the question whether the topological vertex could be described also in M^8 as a partonic surface at single algebraic mass shell in M^8, mapped by $M^8 - H$ duality to a single $a = constant$ hyperboloid in $M^4 \subset H$.

   The virtual momenta at the level of M^8 are algebraic, in general complex, integers. The algebraic mass squared values at the mass shell of M^8 would be the same for all particles of the vertex. This kind of correspondence does not make sense if $M^8 - H$ duality applies to the full algebraic momenta. The assumption has been that it applies to the rational parts of the momenta.

3. The rational parts of the algebraic integer valued 4-momenta of virtual fermions are in general not at the same mass shell. Could this make possible a description in terms of partonic 2-surfaces at fixed mass *resp.* $a = consant$ shell at the level of M^8 *resp.* H?

   The classical space-time surface in H, partonic 2-surfaces and fermion lines at them are characterized by classical momenta by Noether's theorem. Quantum classical correspondence, realized in ZEO as Bohr orbitology, suggests that the classical 4-momenta assignable to these objects correspond to the rational parts of the momenta at M^8 mass shell. Could the rational projections of M^8 momenta at H^3_n correspond to different mass squared values at given H^3?

4. Note that this additional symmetry for complexified momentum space and position space descriptions would be analogous to the duality of twistor amplitudes position space and the space of area momenta.

How to describe the topological vertex in H? The goal is to understand how exotic smooth structure and its point defects could emerge from this picture. The physical picture applied hitherto is as follows.

1. 3 partonic orbits meet at a vertex described by a partonic 2-surface. Assume that they are located to single $a = constant$ $H^3 \subset M^4 \subset H$.

2. The partonic wormhole throats appear as pairs at the opposite Minkowskian space-time sheets. There are three pairs corresponding to 3 external particle lines and one line which must be a bosonic line describing fermion-antifermion bound state disappears: this corresponds to a boson absorption (or emission).

   The opposite throats carry opposite magnetic monopole charges. The only possibility, not noticed before, is that the opposite wormhole throats for the partoni orbit, which ends at the vertex, must coincide at the vertex. The minimal option is that the exotic smooth structure is associated with this partonic orbit turning back in time. The two partonic orbits, which bind 4-D Euclidean regions as wormhole throats, would fuse to a larger 4-D surface with an exotic smooth structure.

   Fermion-antifermion annihilation occurs at a point at which fermion and antifermion lines meet. The first guess is that this point corresponds to the defect of the smooth structure.

3. There is an analogy with the construction of Etesi [A143]in which a homologically non-trivial ball CP_1 glued to the C^2 at infinity to construct an exotic smooth structure. One dimension disappears for the glued 3-surface at infinity.

   In the partonic vertex, one has actually two homologically non-trivial 2-surfaces with opposite homology charges as boundaries between wormhole contact and Minkowskian regions and they fuse together in the partonic vertex. Also now, one dimension disappears as the partonic 2-surfaces become identical so that 3-D wormhole contact contracts to single 2-D partonic 2-surface.

4. The defect for the smooth structure associated with the fusion of the pair of wormhole orbits should correspond to a point at which fermion and antifermion lines meet.

   This suggests that the throats do not fuse instantaneously but gradually. The fusion would start from a single touching point identifiable asd the fermion-antifermion vertex, serving as a seed of a phase transition, and would proceed to the entire wormhole contact so that it reduces to a partonic 2-surface.

   One can argue that one has a problem if this surface is homologically non-trivial. Could the process make the closed partonic 2-surface homologically trivial. A simplified example is the fusion of two circles with opposite winding numbers ± 1 on a cylinder. The outcome is two homologically non-trivial circles of opposite orientations on top of each other. The phase transition starting from a point would correspond to a touching of the circles.

A couple of further comments are in order.

1. The connection of the pair of wormhole throats to the associative holography is an interesting question. The 4-D tangent planes of $X^4 \subset M^8$ mass shell correspond to points of CP_2. They would be different at the two parallel sheets.

   At the mass shell H^3_m the branches would coincide. The presence of two tangent planes could give rise to two different holographic orbits, which coincide at the initial mass shell and gradually diverge from each other just as in the above model for the fusion of partonic 2-surfaces. The failure of the strict determinism for the associative holography at the partonic 2-surface would make in TGD the analogy of fermion-antifermion annihilation vertex possible.

2. There is also an analogy with the cusp catastrophe in which the projection of the cusp catastrophe as a 2-surface in 3-D space with behavior variable x and two control parameters

(a, b) has a boundary at which two real roots of a polynomial of degree 3 coincide. The projection to the (a, b) plane gives a sharp shape, whose boundary is a V-shaped curve in which the sides of V become parallel at the vertex. The vertex corresponds to maximal criticality. The particle vertex would be a critical phenomenon in accordance with the interpretation as a phase transition.

3.7.6 Is a master formula for the scattering amplitudes possible?

Marko Manninen asked whether TGD can in some sense be reduced to a single equation or principle is very interesting. My basic answer is that one could reduce TGD to a handful of basic principles but formula analogous to $F = ma$ is not possible. However, at the level of classical physics, one could perhaps say that general coordinate invariance $\rightarrow$ holography $\leftarrow$ 4-D generalization of holomorphy [**?**]educe the representations of preferred extremals as analogs of Bohr orbits for particles as 3-surfaces to a representation analogous to that of a holomorphic function.

Can one hope something analogous to happen at the level of scattering amplitudes? Is some kind of a master formula possible? I have considered many options, even replacing the S-matrix with the Kähler metric in the fermionic degrees of freedom [L98]. The motivation was that the rows of the matrix defining Kähler metric define unit vectors allowing interpretation in terms of probability conservation. However, it seems that the concept of zero energy state alone makes the definition unambiguous and unitarity is possible without additional assumptions.

1. In standard quantum field theory, correlation functions for quantum fields give rise to scattering amplitudes. In TGD, the fields are replaced by the spinor fields of the "world of classical worlds" (WCW) which can regarded as superpositions of pairs of multi-fermion states restricted at the 3-D surfaces at the ends of the 4-D Bohr orbits at the boundaries of CD.

   These 3-surfaces are extremely strongly but not completely correlated by holography implied by 4-D general coordinate invariance. The modes of WCW spinor fields at the 3-D surfaces correspond to irreducible unitary representations of various symmetries, which include supersymplectic symmetries of WCW and Kac-Moody type symmetries [K24, K76] [L110, L127, L136]. Hence the inner product is unitary.

2. Whatever the detailed form of the 3-D parts of the modes of WCW spinor fields at the boundaries of CD is, they can be constructed from ordinary many fermion states. These many-fermion state correspond in the number theoretic vision of TGD to Galois singlets realizing Galois confinement [L136, L130, L134]. They are states constructed at the level of M^8 from fermion with momenta whose components are possibly complex algebraic integers in the algebraic extension of rationals defining the 4-D region of M^8 mapped to H by $M^8 - H$ duality. Complex momentum means that the corresponding state decomposes to plane waves with a continuum of momenta. The presence of Euclidian wormhole contact makes already the classical momenta complex.

   Galois confined states have momenta, whose components are integers in the momentum scale defined by the causal diamond (CD). Galois confinement defines a universal mechanism for the formation of bound states. The induced spinor fields are second quantized free spinor fields in H and their Dirac propagators are therefore fixed. This means an enormou calculational simplification.

3. The inner products of these WCW spinor fields restricted to 3-surfaces determine the scattering amplitudes. They are non-trivial since the modes of WCW spinor fields are located at opposite boundaries of CD. These inner products define the zero energy state identifiable as such as scattering amplitudes. This is the case also in wave mechanics and quantum TGD is indeed wave mechanics for particles identified as 3-surfaces.

4. There is also a functional integral of these amplitudes over the WCW, i.e. over the 4-D Bohr orbits. This defines a unitary inner product. The functional integral replaces the path integral of field theory and is mathematically well-defined since the Kähler function, appearing in the exponent defining vacuum functional, is a non-local function of the 3-surface so that standard local divergences due to the point-like nature of particles disappear. Also the standard problems due to the presence of a Hessian coming from a Gaussian determinant is canceled by the square foot of the determinant of the Kähler metric appearing in the integration measure [K42, K76].

5. The restriction of the second quantized spinor fields to 4-surfaces and zero-energy ontology are absolutely essential. Induction turns free fermion fields into interacting ones. The spinor fields of H are free and define a trivial field theory in H. The restriction to space-time surfaces changes the situation. Non-trivial scattering amplitudes are obtained since the fermionic propagators restricted to the space-time surface are not anymore free propagators in H. Therefore the restriction of WCW spinors to the boundaries of CD makes the fermions interact in exactly the same way as it makes the induced spinor connection and the metric dynamical.

There are a lot of details involved that I don't understand, but it would seem that a simple "master formula" is possible. Nothing essentially new seems to be needed. There is however one more important "but".

Are pair production and boson emission possible?

The question that I have pondered a lot is whether the pair production and emission of bosons are possible in the TGD Universe. In this process the fermion number is conserved, but fermion and antifermion numbers are not conserved separately. In free field theories they are, and in the interacting quantum field theories, the introduction of boson fermion interaction vertices is necessary. This brings infinities into the theory.

1. In TGD, the second quantized fermions in H are free and the boson fields are not included as primary fields but are bound states of fermions and antifermions. Is it possible to produce pairs at all and therefore also bosons? For example, is the emission of a photon from an electron possible? If a photon is a fermion-antifermion pair, then the fermion and antifermion numbers cannot be preserved separately. How to achieve this?

2. If fundamental fermions correspond to light-like curves at light-like orbit of partonic 2-surfaces, pair creation requires that that fermion trajectory turns in time direction. At this point velocity is infinite and this looks like a causal anomaly. There are two options: the fermion changes the sign of its energy or transforms to antiferion with the same sign of energy.

   Different signs of energy is not possible since the annihilation operator creating the fermion with opposite energy would annihilate either the final state or some fermion in the final state so that both fermion and antifermion numbers of the final state would be the same as those of the initial state.

   On the other hand, it can be said that positive energy antifermions propagate backwards in time because in the free fermion field since the terms proportional to fermion creation operators and antifermion annihilation operators appear in the expression of the field as sum of spinor modes.

   Therefore a fermion-antifermion pair with positive energies can be created and corresponds to a pair of creation operators. It could also correspond to a boson emission and to a field theory vertex, in which the fermion, antifermion and boson occur. In TGD, however, the boson fields are not included as primary fields. Is such a "vertex without a vertex" possible at all?

3. Can one find an interpretation for this creation of a pair that is in harmony with the standard view. Space-time surfaces are associated with induced classical gauge potentials. In standard field theory, they couple to fermion-antifermion pairs, and pairs can be created in classical fields. The modified Dirac equation [K100] and the Dirac equation in H also have such a coupling. Now the modified Dirac equation holds true at the fermion lines at the light-like orbits of the partonic 2-surface. Does the creation of pairs happen in this way? It might do so: also in the path integral formalism of field theories, bosons basically correspond to classical fields and the vertex is just this except that in TGD fermions are restricted to 1-D lines.

Fundamental fermion pair creation vertices as local defects of the standard smooth structure of the space-time surface?

Here comes the possible connection with a very general mathematical problem of general relativity that I have already discussed.

1. Causal anomalies as time loops that break causality are more the rule than an exception in general relativity the essence of the causal anomaly is the reversal of the arrow of time. Causal anomalies correspond to exotic diffeo-structures that are possible only in dimension $D = 4$! Their number is infinite.

2. Quite generally, the exotic smooth structures reduce to defects of the usual differentiable structure and have measure zero. Assume that they are point like defects. Exotic differentiable structures are also possible in TGD, and the proposal is that the associated defects correspond to a creation of fermion-fermion pairs for emission of fermion pairs of of gauge bosons and Higgs particle identified in TGD as bound states of fermion-antifermion pairs. This picture generalizes also to the case of gravitons, which would involve a pair of vertices of this kind. The presence of 2 vertices might relate to the weakness of the gravitational interaction.

   The reversal of the fermion line in time direction would correspond to a creation of a fermion-antifermion pair: fermion and antiferion would have the same sign of energy. This would be a causal anomaly in the sense that the time direction of the fermion line is reversed so that it becomes an antifermion.

   I have proposed that this causal anomaly is identifiable as an anomaly of differentiable structure so that emission of bosons and fermion pairs would only be possible in dimension 4: the space-time dimension would be unique!

3. But why would a point-like local defect of the differentiable structure correspond to a fermion pair creation vertex. In TGD, the point-like fermions correspond to 1-D light-like curves at the light-like orbit of the partonic 2-surface.

   In the pair creation vertex in presence of classical induced gauge potentials, one would have a V-shaped world line of fermion turning backwards in time meaning that antifermion is transformed to fermion. The antifermion and fermion numbers are not separately conserved although the total fermion number is. If one assumes that the modified Dirac equation holds true along the entire fermion worldline, there would be no pair creation.

   If it holds true only outside the V-shaped vertex the modified Dirac action for the V-shaped fermion libe can be transformed to a difference of antifermion number equal to the discontinuity of the antifermion part of the fermion current identified as an operator at the vertex. This would give rise to a non-trivial vertex and the modified gamma matrices would code information about classical bosonic action.

4. The 1-D curve formed by fermion and antifermion trajectories with opposite time direction turns backwards in time at the vertex. At the vertex, the curve is not differentiable and this is what the local defect of the standard smooth differentiable structure would mean physically!

Master formula for the scattering amplitudes: finally?

Most pieces that have been identified over the years in order to develop a master formula for the scattering amplitudes are as such more or less correct but always partially misunderstood. Maybe the time is finally ripe for the fusion of these pieces to a single coherent whole. I will try to list the pieces into a story in the following.

1. The vacuum functional, which is the exponential Kähler function defined by the classical bosonic action defining the preferred extremal a an analog of Bohr orbit, is the starting point. Physically, the Kähler function corresponds to the bosonic action (e.g. EYM) in field theories.

   Because holography is almost unique, it replaces the path integral by a sum over 4-D Bohr trajectories as a functional integral over 3-surfaces plus discrete sum.

2. However, the fermionic part of the action is missing. I have proposed a long time ago a super symmetrization of the WCW Kähler function by adding to it what I call modified Dirac action. It relies on modified gamma matrices modified gamma matrices Γ^α, which are contractions $\Gamma_k T^{\alpha k}$ of H gamma matrices Γ_k with the canonical momentum currents $T^{\alpha k} = \partial L/\partial_{\partial_\alpha h^k}$ defined by the Lagrangian L. Modified Dirac action is therefore determined by the bosonic action from the requirement of supersymmetry. This supersymmetry is however quite different from the SUSY associated with the standard model and it assigns to fermonic Noether currents their super counterparts.

Bosonic field equations for the space-time surface actually follow as hermiticity conditions for the modified Dirac equation. These equations also guarantee the conservation of fermion number(s). The overall super symmetrized action that defines super symmetrized Kähler function in WCW would be unambiguous. One would get exactly the same master formula as in quantum field theories, but without the path integral.

3. The overall super symmetrized action is sum of contributions assignable to the space-time surface itself, its 3-D light-like parton orbits as boundaries between Minkowskian regions and Euclidian wormhole contact, 2-D string world sheets and their 1-D boundaries as orbits of point-like fermions. These 1-D boundaries are the most important and analogous to the lines of ordinary Feynman diagrams. One obtains a dimensional hierarchy.
4. One can assign to these objects of varying dimension actions defined in terms of the induced geometry and spinor structure. The supersymmetric actions for the preferred extremals analogous to Bohr orbit in turn give contributions to the super symmetrized Kähler function as an analogue of the YM action so that, apart from the reduction of path integral to a sum over 4-D Bohr orbits, there is a very close analogy with the standard quantum field theory.

However, some problems are encountered.

1. It seems natural to assume that a modified Dirac equation holds true. I have presented an argument for how it indeed emerges from the induction for the second quantized spinor field in H restricted to the space-time surface assuming modified Dirac action.

   The problem is, however, that the fermionic action, which should define vertex for fermion pair creation, disappears completely if Dirac's equation holds everywhere! One would not obtain interaction vertices in which pairs of fermions arise from classical induced fields. Something goes wrong. In this vertex total fermion number is conserved but fermion and antifermion numbers are changed since antifermion transforms to fermion at the V-shaped vertex: this condition should be essential.
2. If one gives up the modified Dirac equation, the fermionic action does not disappear. In this case, one should construct a Dirac propagator for the modified Dirac operator. This is an impossible task in practice.

   Moreover, the construction of the propagator is not even necessary and in conflict with the fact that the induced spinor fields are second quantized spinors of H restricted to the space-time surface and the propagators are therefore well-defined and calculable and define the propagation at the space-time surface.
3. Should we conclude that the modified Dirac equation cannot hold everywhere? What these, presumably lower-dimensional regions of space-time surface, are and could they give the interaction vertices as topological vertices?

The key question is how to understand geometrically the emission of fermion pairs and bosons as their bound states?

1. I have previously derived a topological description for reaction vertices. The fundamental 1 → 2 vertex (for example e→e+ gamma) generalizes the basic vertex of Feynman diagrams, where a fermion emits a boson or a boson decays into a pair of fermions. Three lines meet at the ends.

   In TGD, this vertex can topologically correspond to the decomposition of a 3-surface into two 3-surfaces, to the decomposition of a partonic 2-surface into two, to the decomposition of a string into two, and finally, to the turning of the fermion line backwards from time. One can say that the n-surfaces are glued together along their $n-1$-dimensional ends, just like the 1-surfaces are glued at the vertex in the Feynman diagram.
2. In the previous section, I already discussed how to identify vertex for fermion-antifermion pair creation as a V-shaped turning point of a 1-D fermion line. The fermion line turns back in time and fermion becomes an antifermion. In TGD, the quantized boson field at the vertex is replaced by a classical boson field. This description is basically the same as in the ordinary path integral where the gauge potentials are classical.

   The problem was that if the modified Dirac equation holds everywhere, there are no pair creation vertices. The solution of the problem is that the modified Dirac equation at the V-shaped vertex cannot hold true.

What this means physically is that fermion and antifermion numbers are not separately conserved in the vertex. The modified Dirac action for the fermion line can be transformed to the change of antifermion number as operator (or fermion number at the vertex) expressible as the change of the antifermion part of the fermion number. This is expressible as the discontinuity of a corresponding part of the conserved current at the vertex. This picture conforms with the appearance of gauge currents in gauge theory vertices. Notice that modified gamma matrices determined by the bosonic action appear in the current.

3. This argument was limited to 1-D objects but can be generalized to higher-dimensional defects by assuming that the modified Dirac equation holds true everywhere except at defects represented as vertices, which become surfaces. The modified Dirac action reduces to an integral of the discontinuity of say antifermion current at the vertex, i.e. the change of the antifermion charge as an operator.

What remains more precisely understood and generalized, is the connection with the irreducible exotic smooth structures possible only in 4-D space-time.

1. TGD strongly suggests that 0-dimensional vertices generalize to topological vertices representable as surfaces of dimension $n = 0, 1, 2, 3$ assignable to objects carrying induced spinor field. In the 1$\rightarrow$ 2 vertex, the orbit of an $n < 4$- dimensional surface would turn back in the direction of time and would define a V-shaped structure in time direction. These would be the various topological vertices that I have previously arrived at, but guided by a physical intuition. Also now the vertex would boild down to the discontinuity of say antifermion current instead of the current itself at the vertex.

2. It is known that exotic smooth structures reduce to standard ones except in a set of defects having measure zero. Also non-point-like defects might be possible in contrast to what I assumed at first. If the defects are surfaces, their dimension is less than 4. If not, then only the direction of fermion lines could change.

   If the generalization is possible, also 1-D, 2-D, and 3-D defects, defining an entire hierarchy of particles of different dimensions, is possible. As a matter of fact, a longstanding issue has been whether this prediction should be taken seriously. Note that in topological condensed matter physics, defects with various dimensions are commonplace. One talks about bulk states, boundary states, edge states and point-like singularities. In this would predict hierarchy of fermionic object of various dimensions.

To summarize, exotic smooth structures would give vertices without vertices assuming only free fermions fields and no primary boson fields! And this is possible only in space-time dimension 4!

3.8 A possible connection with family replication phenomenon?

In TGD framework the genus g of the partonic 2-surfaces is proposed to label fermion families [K21, K50, K54]. One can characterize by genus g the topology of light-like partonic orbits and identify the three fermion generators as 2-surfaces with genus $g = 0, 1, 2$ with the special property that they are always hyper-elliptic. Quantum mechanically also topological mixing giving rise to CKM mixing is possible. The view is that given connected 3-surface can contain several light-like 3-surface with different genera. For instance, hadrons would be such surfaces.

There are however questions to be answered.

1. The genera $g = 0, 1, 2$ assigned with the free fermion families correspond to Riemann surfaces, which are always hyper-elliptic allowing therefore Z_2 as a global conformal symmetry. These complex curves correspond to degrees $n = 2, 3, 4$ for the corresponding polynomials. For $n \leq 4$ can write explicit solutions for the roots of the polynomials. Could there be a deep connection between particle physics and mathematical cognition?
2. The homology and genus for 2-surfaces of CP_2 correlate with each other [A176]: is this consistent with the proposed topologicization of color hypercharge implying color confinement?
3. $h_{eff}/h = n$ hypothesis means that dark variant of particle particle characterized by genus g is n-fold covering of this surface. In the general case the genus of covering is different. Is this consistent with the genus-generation correspondence?

4. The degree of complex curve correlates with the genus of the curve. Is generation-genus correspondence consistent with the assumption that partonic 2-surfaces have algebraic curve as CP_2 projection (this need not be the case)?

3.8.1 How the homology charge and genus correlate?

Complex surfaces in CP_2 are highly interesting from TGD point of view.

1. The model for elementary particles assumes that the partonic 2-surfaces carrying fermion number are homologically non-trivial, in other words they carry Kähler magnetic monopole flux having values $q = \pm 1$ and $q = \pm 2$. The idea is that color hyper charge $Y = \{\pm 2/3, \pm 1/3\}$ is proportional to n for quarks and color confinement topologizes to the vanishing of total homology charge [K54].
2. The explanation of the family replication phenomenon [K21] in terms of genus-generation correspondence states that the three quarks and lepton generations correspond to the three lowest genera $g = 0, 1, 2$ for partonic 2-surfaces. Only these genera are always hyper-elliptic allowing thus a global Z_2 conformal symmetry. The physical vision is that for higher genera the handles behave like free particles. Is this proposal consistent with the proposal for the topologization of color confinement?

There is a result [A176] (page 124) stating that if the homology charge q is divisible by 2 then one must have $g \geq q^2/4 - 1$. If q is divisible by h, which is odd power of prime, one has $g \geq (q^2/4 - 1) - (q^2/4h^2)$. For $q = 2$ the theorem allows $g \geq 0$ so that all genera with color hyper charge $Y = \pm 2/3$ are realized.

The theorem says however nothing about $q = 0, 1$. These charges can be assigned to the two different geodesic spheres of CP_2 with $g = 0$ remaining invariant under SO(3) and U(2) subgroups of SU(3) respectively. Is $g > 0$ possible for $q = 1$ as the universality of topological color confinement would require? For $q = 3$ one would have $g \geq 1$. For $q = 4$ $h = 2$ divides q and one has $g \geq 2$. It would seem $g \geq 5$. The conditions become more restrictive for higher q, which suggests that for $q = 0, 1$ one has $g \geq 0$ so that the topologization of color hypercharge would make sense.

3.8.2 Euler characteristic and genus for the covering of partonic 2-surface

Hierarchy of Planck constants $h_{eff}/h = n$ means a hierarchy of space-time surfaces identifiable as n-fold coverings. The proposal is that the number of sheets in absence of singularities is maximal possible and equals to the dimension of the extension dividing the order of its Galois group.

The Euler characteristic of n-fold covering in absence of singular points is $\chi_n = n\chi$. If there are singular (ramified) points these give a correction term given by Riemann-Hurwitz formula (see `http://tinyurl.com/y7n2acub`.)

In absence of singularities one has from $\chi = -2(g - 1)$ and $\chi_n = n\chi$

$$g_n = n(g - 1) + 1 \ . \tag{3.8.1}$$

For $n = 1$ this indeed gives $g_1 = g$ independent of g. One can also combine this with the formula $g = (d - 1)(d - 2)/2$ holding for non-singular algebraic curves of degree d.

Singularities are unavoidable at algebraic points of cognitive representations at which some subgroup of Galois group leaves the point invariant (say rational point in ordinary sense). One can consider the possibility that fermions are located at the singular points at which several sheets of covering touch each other. This would give a correction factor to the formula. If the projection map from the covering to based is of form $\Pi(z) = z^n$ at the singular point P, one says that singularity has ramimifaction index $e_P = n$ and the algebraic genus would increase to

$$g_n = n(g - 1) + 1 + \frac{1}{2}\sum_P (e_P - 1) \ . \tag{3.8.2}$$

Indeed, singularities mean that sheets touch each other at singular points and this increases connectivity.

Under what conditions the genus of dark partonic surface with $n > 1$ can be same as that of the ordinary partonic surface representing visible matter? For the genera $g = 0$ and $g = 1$ this is possible so that these genera would be in an exceptional role also from the point of view of dark matter.

1. For $g = 1$ one has $g_n = g = 1$ independent of n in absence of singular point. Torus topology (assignable to muon and (c,s) quarks) is exceptional. In presence of singularities the genus would increase by the $\sum_P (e_P - 1)/2$ independent of the value of n. The lattice of points for elliptic surfaces would suggest existence of infinite number of singular points if the abelian group operations preserve the singular character of the points so that the genus would become infinite.

2. For $g = 0$ one would have $g_n = -n + 1$ in absence of singularities. Only $n = 1$ - ordinary matter - is possible without singularities. Dark matter is however possible if singularities are allowed. For sphere one would obtain $g_n = -n + 1 + \sum_P (e_P - 1)/2 \geq 0$. The condition $n \leq \sum_P (e_P - 1)/2 + 1$ must therefore hold true for $g \geq 0$.

   The condition $g_n = -n + 1 + \sum_P (e_P - 1)/2 = g = 0$ gives $\sum_P (e_P - 1) = 2(n - 1)$. For spherical topology it is possible to have dense set of rational points so that it is possible create cognitive representations with arbitrary number of points which can be also singular. One might argue that this kind of situation corresponds to a non-perturbative phase.

3. For $g = 2$ one would have $g_n = n + 1 + \sum_P (e_P - 1)/2$ and genus would grow with n even in absence of singularities and would be very large for large values of h_{eff}. $g_n = 2$ is obtained with $n = 1$ (ordinary matter) and no singular points not even allowed for $n = 1$. $g_n = g = 2$ is not possible for $n > 1$.

   Note that dark $g \geq 2$ fermions cannot correspond to lower generation fermions with singular points of covering. More generally, one could say that $g \geq 2$ fermions can exists only with standard value of Planck constant unless they are singular coverings of $g < 2$ fermions.

What is clear that the model of dark matter predicts breaking of universality. This breaking is not seen in the standard model couplings but makes it visible in amore delicate manner and might allow to understand why the masses of fermions increase with generation index.

3.8.3 All genera are not representable as non-singular algebraic curves

Suppose for a moment that partonic 2-surfaces correspond to rational maps of algebraic curves in CP_2 to M^4 that is deformations of these curves in M^4 direction. This assumption is of course questionable but deserves to be sttudied.

The formula (for algebraic curve see `http://tinyurl.com/nt6tkey`)

$$g = \frac{(d-1)(d-2)}{2} + \frac{\sum \delta_s}{2} \quad ,$$

where $\delta_s > 0$ characterizes the singularity, does not allow all genera for algebraic curves for $\sum \delta_s = 0$: one has $g = 0, 0, 1, 3, 6, 10, ..$ for $d = 1, 2, ...$

For instance, $g = 2$, which would correspond in TGD to third quark or lepton generation is not possible without singularities for $d = 3$ curve having $g = 1$ without singularities!

This raises questions. Could the third fermion generation actually correspond to $g = 3$? Or does it correspond to $g = 2$ 2-surface of CP_2, which is more general surface than algebraic curve meaning that it is not representable as complex surface? Or could third generation fermions correspond to $g = 0$ or $g = 1$ curves with singular point of covering by Galois group so that several sheets touch each other?

To sum up, if the results for algebraic varieties generalize to TGD framework, they suggest notable differences between different fermion families. Universality of standard model interactions says that the only differences between fermion families are due to the differ masses. It is not clear whether the different masses could be due to the differences at number theoretical level and dark matter sectors.

1. All genera can appear as as ordinary matter ($d = 1$). Dark variants of $g = 1$ states have $g_d = 1$ automatically in absence of singular points. Dark variants of $g = 0$ states must have singular point in order to give $g_n = 0$. Dark variants of $g = 2$ states with $g_d = 2$ are obtained from $g = 1$ states with singularities. The special role of the two lowest is analogous to their special role for algebraic curves.
2. If one assumes that partonic 2-surfacs correspond to algebraic curves, one obtains again that $g = 2$ surfaces must correspond to singular $g = 0$ and $g = 1$ which could be dark in TGD sense.

3.9 Summary and future prospects

In the following I give a brief summary about what has been done. I concentrate on $M^8 - H$ duality since the most significant results are achieved here.

It is fair to say that the new view answers the following a long list of open questions.

1. When $M^8 - H$ correspondence is true (to be honest, this question emerged during this work!)? What are the explicit formulas expressing associativity of the tangent space or normal space of the 4-surface?

   The key element is the formulation in terms of complexified M^8 - M^8_c - identified in terms of octonions and restriction $M^8_c \to M^8$. One loses the number field property but for polynomials ring property is enough. The level surfaces for real and imaginary parts of octonionic polynomials with real coefficients define 4-D surfaces in the generic case.

   Associativity condition is an additional condition reducing the dimension of the space-time surface unless some components of $RE(P)$ or $IM(P)$ are critical meaning that also their gradients vanish. This conforms with the quantum criticality of TGD and provides a concrete first principle realization for it.

   An important property of $IM(P_1P_2)$ is its linearity with respect to $IM(P_i)$ implying that this condition gives the surfaces $IM(P_i) = 0$ as solutions. This generalizes by induction to $IM(P_1P_2...P_n)$. For $RE(P_1P_2) = 0$ linearity does not hold true and there is a genuine interaction. A physically attractive idea idea is that $RE(P_1P_2) = 0$ holds true inside CDs and for wormhole contacts between space-time sheets with Minkoskian signature. One can generalizes this also to $IM(P_1/P_2)$ and $RE(P_1/P_2)$ if rational functions are allowed. Note however that the origins of octonionic coordinates in P_i must be on the octonionic real line.
2. How this picture corresponds to twistor lift? The twistor lift of Kähler action (dimensionally reduced Kähler action in twistor space of space-time surface) one obtains two kinds of space-time regions. The regions, which are minimal surfaces and obey dynamics having no dependence on coupling constants, correspond naturally to the critical regions in M^8 and H.

   There are also regions in which one does not have extremal property for both Kähler action and volume term and the dynamics depends on coupling constant at the level of H. These regions are associative only at their 3-D ends at boundaries of CD and at partonic orbits, and the associativity conditions at these 3-surfaces force the initial values to satisfy the conditions guaranteeing preferred extremal property. The non-associative space-time regions are assigned with the interiors of CDs. . The particle orbit like space-time surfaces entering to CD are critical and correspond to external particles.

   It has later turned out [L56] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.
3. The surprise was that $M^4 \subset M^8$ is naturally co-associative. If associativity holds true also at the level of H, $M^4 \subset H$ must be associative. This is possible if $M^8 - H$ duality maps tangent space in M^8 to normal space in H and vice versa.
4. The connection to the realization of the preferred extremal property in terms of gauge conditions of subalgebra of SSA is highly suggestive. Octonionic polynomials critical at the

boundaries of space-time surfaces would determine by $M^8 - H$ correspondence the solution to the gauge conditions and thus initial values and by holography the space-time surfaces in H.

5. A beautiful connection between algebraic geometry and particle physics emerges. Free many-particle states as disjoint critical 4-surfaces can be described by products of corresponding polynomials satisfying criticality conditions. These particles enter into CD , and the non-associative and non-critical portions of the space-time surface inside CD describe the interactions. One can define the notion of interaction polynomial as a term added to the product of polynomials. It can vanish at the boundary of CD and forces the 4-surface to be connected inside CD. It also spoils associativity: interactions are switched on. For bound states the coefficients of interaction polynomial are such that one obtains a bound state as associative space-time surface.

6. This picture generalizes to the level of quaternions. One can speak about 2-surfaces of space-time surface with commutative or co-commutative tangent space. Also these 2-surfaces would be critical. In the generic case commutativity/co-commutativity allows only 1-D curves.

   At partonic orbits defining boundaries between Minkowskian and Euclidian space-time regions inside CD the string world sheets degenerate to the 1-D orbits of point like particles at their boundaries. This conforms with the twistorial description of scattering amplitudes in terms of point like fermions.

   For critical space-time surfaces representing incoming states string world sheets are possible as commutative/co-commutative surfaces (as also partonic 2-surfaces) and serve as correlates for (long range) entaglement) assignable also to macroscopically quantum coherent system ($h_{eff}/h = n$ hierarchy implied by adelic physics).

7. The octonionic polynomials with real coefficients form a commutative and associative algebra allowing besides algebraic operations function composition. Space-time surfaces therefore form an algebra and WCW has algebra structure. This could be true for the entire hierarchy of Cayley-Dickson algebras, and one would have a highly non-trivial generalization of the conformal invariance and Cauchy-Riemann conditions to their n-linear counterparts at the n:th level of hierarchy with $n = 1, 2, 3, ..$ for complex numbers, quaternions, octonions,... One can even wonder whether TGD generalizes to this entire hierarchy!

8. In the original version of this article I did not realize that there are two options for realizing the idea that the M^4_c projection of space-time surface in M^8_c must belong to M^4.

   (a) I proposed that the *projection* from M^8_c to real M^4 (for which M^1 coordinate is real and E^3 coordinates are imaginary with respect to i!) defines the real space-time surface mappable by $M^8 - H$ duality to CP_2 [L36].

   (b) An alternative option, which I have not considered in the original versions of [L36, L38] is that only the roots of the 4 vanishing polynomials as coordinates of M^4_c belong to M^4 so that m^0 would be real root and m^k, $k = 1, ..., 3$ imaginary with respect to $i \to -i$. M^8_c coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality.* This could make sense. Outside CD these conditions would not hold true. This option looks more attractive than the first one. Why these condition can be true just inside CD, should be understood.

9. The use of polynomials or rational functions could be also an approximation. Analytic functions of real variable extended to octonionic functions would define the most general space-time surfaces but the limitations of cognition would force to use polynomial approximation. The degree n of the polynomial determining also $h_{eff} = nh_0$ would determine the quality of the approximation and at the same time the "IQ" of the system.

All big pieces of quantum TGD are now tightly interlinked.

1. The notion of causal diamond (CD) and therefore also ZEO can be now regarded as a consequence of the number theoretic vision and $M^8 - H$ correspondence, which is also understood physically.

2. The hierarchy of algebraic extensions of rationals defining evolutionary hierarchy corresponds to the hierarchy of octonionic polynomials.

3. Associative varieties for which the dynamics is critical are mapped to minimal surfaces with universal dynamics without any dependence on coupling constants as predicted by twistor lift of TGD. The 3-D associative boundaries of non-associative 4-varieties are mapped to initial values of space-time surfaces inside CDs for which there is coupling between Kähler action and volume term.
4. Free many particle states as algebraic 4-varieties correspond to product polynomials in the complement of CD and are associative. Inside CD the addition of interaction terms vanishing at its boundaries spoils associativity and makes these varieties connected.
5. The super variant of the octonionic algebraic geometry makes sense, and one obtains a beautiful correlation between the fermion content of the state and corresponding space-time variety. This suggests that twistorial construction indeed generalizes. Criticality for the external particles giving rise to additional constraints on the coefficients of polynomials could make possible to have well-define summation over corresponding varieties.

What mathematical challenges one must meet?

1. One should prove more rigorously that criticality is possible without the reduction of dimension of the space-time surface.
2. One must demonstrate that SSA conditions can be true for the images of the associative regions (with 3-D or 4-D). This would obviously pose strong conditions on the values of coupling constants at the level of H.

Concerning the description of interactions there are several challenges.

1. Do associative space-time regions have minimal surface extremals as images in H and indeed obeying universal critical dynamics? As found, the study of the known extremals supports this view.
2. Could one construct the scattering amplitudes at the level of M^8? Here the possible problems are caused by the exponents of action (Kähler action and volume term) at H side. Twistorial construction [K79] however leads to a proposal that the exponents actually cancel. This happens if the scattering amplitude can be thought as an analog of Gaussian path integral around single extremum of action and conforms with the integrability of the theory. In fact, nothing prevents from defining zero energy states in this manner! If this holds true then it might be possible to construct scattering amplitudes at the level of M^8.
3. What about coupling constants? Coupling constants make themselves visible at H side both via the vanishing conditions for Noether charges in sub-algebra of SSA and via the values of the non-vanishing Noether charges. $M^8 - H$ correspondence determining the 3-D boundaries of interaction regions within CDs suggests that these couplings must emerge from the level M^8 via the criticality conditions posing conditions on the coefficients of the octonionic polynomials coding for interactions.

   Could all coupling constant emerge from the criticality conditions at the level of M^8? The ratio of R^2/l_P^2 of CP_2 scale and Planck length appears at H level. Also this parameter should emerge from $M^8 - H$ correspondence and thus from criticality at M^8 level. Physics would reduce to a generalization of the catastrophe theory of Rene Thom!
4. The description of interactions at the space-time surface associated with single CD should be M^8 counterpart of the H picture in which 3 light-like partonic orbits meet at common end topological vertex - defined by a partonic 2-surface and fermions scatter without touching. Now one has octonionic sparticle lines and interaction vertex becomes possible. This conforms with the idea that interactions take place at discrete points belonging to the extension of rationals. The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

   The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$, which satisfy $RE(P) = IM(P) = 0$ and are singular and doubly critical. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

5. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be at real line (time axis) in the octonionic sense, and guarantees the associativity and commutativity of the polynomials. Arbitrary CDs cannot be located along this line. Can one assume that all CDs involved with *observable* processes satisfy this condition?

   If not, how do the 4-varieties associated with octonionic polynomials with different origins interact? How could one avoid losing the extremely beautiful associative and commutative algebra? It seems that one cannot form their products and sums and must form the Cartesian product of M^8:s with different tips for CDS and formulate the interaction in this framework. In the case of space-time surfaces associated with different CDs the discrete intersections of space-time surfaces would define the interaction vertices.

6. Super-octonionic geometry suggests that the twistorial construction of scattering amplitudes in $\mathcal{N}=4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in an appropriate extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

   If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of super-twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic: indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N}=4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the extension of rationals considered. The rest would be dictated by symmetries and integrations over various moduli spaces, which should be number theoretically universal so that residue calculus strongly suggests itself.

7. What is the connection with super conformal variant of Yangian symmetry, whose generalization in TGD framework is highly suggestive? Twistorial construction of scattering amplitudes at the level of M^8 looks highly promising idea and could also realize Yangian supersymmetry. The conjecture is that the twistorial amplitudes decompose to M^4 and CP_2 parts with similar structure with E^4 spin (electroweak isospin) replacing ordinary spin and that the integrands in Grassmannians emerging from the conservation of M^4 and E^4 4-momenta are identical in the two cases and thus guarantee Yangian supersymmetry in both sectors. The only difference would be due to the product of delta functions associated with the "negative helicities" (weak isospins with negative sign) expressible as a delta function in the complement of $SU(3)$ Cartan algebra $U(1)\times U(1)$ by using exponential map.

It is appropriate to close with a question about fundamentals.

1. The basic structure at M^8 side consists of complexified octonions. The metric tensor for the complexified inner product for complexified octonions (no complex conjugation with respect to i for the vectors in the inner product) can be taken to have any signature $(\epsilon_1,...,\epsilon_8)$, $\epsilon_i=\pm 1$. By allowing some coordinates to be real and some coordinates imaginary one obtains effectively any signature from say purely Euclidian signature. What matters is that the restriction of complexified metric to the allowed sub-space is real. These sub-spaces are linear Lagrangian manifolds for Kähler form representing the commuting imaginary unit i. There is analogy with wave mechanics. Why M^8 -actually M^4 - should be so special real section? Why not some other signature?

2. The first observation is that the CP_2 point labelling tangent space is independent of the signature so that the problem reduces to the question why M^4 rather than some other signature $(\epsilon_1,..,\epsilon_4)$. The intersection of real subspaces with different signatures and same origin $(t,r)=0$ is the common sub-space with the same signature. For instance, for $(1,-1,-1,-1)$ and $(-1,-1,-1,-1)$ this subspace is 3-D $t=0$ plane sharing with CD the lower tips of CD. For $(-1,1,1,1)$ and $(1,1,1,1)$ the situation is same. For $(1,-1,-1,-1)$ and $(1,1,-1,-1)$ $z=0$ holds in the intersection having as common with the lower boundary of CD the boundary of 3-D light-cone. One obtains in a similar manner boundaries of 2-D and 1-D light-cones as intersections.

3. What about CDs in various signatures? For a fully Euclidian signature the counterparts for the interiors of CDs reduce to 4-D intervals $t \in [0, T]$ and their exteriors and thus the space-time varieties representing incoming particles reduce to pairs of points $(t, r) = (0, 0)$ and $(t, r) = (T, 0)$: it does not make sense to speak about external particles. For other signatures the external particles correspond to 4-D surfaces and dynamics makes sense. The CDs associated with the real sectors intersect at boundaries of lower dimensional CDs: these lower-dimensional boundaries are analogous to subspaces of Big Bang (BB) and Big Crunch (BC).

4. I have not found any good argument for selecting $M^4 = M^{1,3}$ as a unique signature. Should one allow also other real sections? Could the quantum numbers be transferred between sectors of different signature at BB and BC? The counterpart of Lorentz group acting as a symmetry group depends on signature and would change in the transfer. Conservation laws should be satisfied in this kind of process if it is possible. For instance, in the leakage from $M^4 = M^{1,3}$ to Mi, j, say $M^{2,2}$, the intersection would be $M^{1,2}$. Momentum components for which signature changes, should vanish if this is true. Angular momentum quantization axis normal to the plane is defined by two axis with the same signature. If the signatures of these axes are preserved, angular momentum projection in this direction should be conserved. The amplitude for the transfer would involve integral over either boundary component of the lower-dimensional CD.

   Could the leakage between signatures be detected as disappearance of matter for CDs in elementary particle scales or lab scales?

5. One can also raise a question about the role of WCW geometry as a continuous infinite-D geometry: could the discretization by cognitive representations making WCW effectively discrete mean its loss? It seems that this cannot be the case. At least in the real sector continuum must be present and the discretization reflects only the discreteness of cognitive representations. In principle continuous WCW could make sense also in p-adic sectors of the adele.

   The identification of space-time surfaces as zero loci of polynomials generalizes to rational functions and even transcendental functions although the existence of the p-adic counterparts of these functions requires additional conditions. Could one interpret the representation in terms of polynomials and possibly rational functions as an approximation? Could the hierarchy of approximations obtained in this manner give rise to a hierarchy of hyper-finite factors of type II_1 defining a hierarchy of measurement resolutions [K99]?

Chapter 4

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part III

4.1 Introduction

In the third chapter about $M^8 - H$ duality the question whether the space-time surfaces in M^8 allow a global slicing by string world sheets X^2 defined by an integrable distribution of local tangent spaces $M^2(x) \subset M^4$ and their orthogonal duals or whether there is only a discrete set of surfaces X^2 is discussed. Discrete set is obtained by requiring that space-time surface or its normal space contains string world sheet as a complex (commutative) sub-manifold. By the strong form of holography (SH) this is enough to deduce the image of $X^4 \subset M^8$ in H from the boundary data consisting of the H-images of X^2 and metrically 2-D light-like partonic orbits X^3_L of topological dimension $D = 3$.

Also the relation of $M^8 - H$ duality to p-adic length scale hypothesis and dark matter hierarchy are discussed and it is shown that the notion of p-adic length scale emerging from p-adic mass calculations emerges also geometrically.

The fermionic aspects of $M^8 - H$ duality are discussed: the basic purely number theoretic elements are the octonionic realization of M^8 spinors and the replacement of Dirac equation as a partial differential equation with an algebraic equation for octonionic spinors. Dirac equation for octonionic spinors is analogous to the algebraic momentum space variant of the ordinary Dirac equation. This provides also considerable understanding about the bosonic aspects of $M^8 - H$ duality. In particular, the pre-images of $X^3_L \subset X^4 \subset H$ in M^8 correspond to mass shells for massless octonionic spinor modes realized as light-like 3-surfaces in M^8. One can say that M^8 picture realizes the momentum space dual of the modified Dirac equation in $X^4 \subset H$. Twistor Grassmannian picture supports the view that spinor modes also in H are localized to $X^3_L \subset X^4$, and obey the modified Dirac equation associated with Chern-Simons term.

Cognitive representations is the third basic topic of the chapter. Cognitive representations are identified as sets of points in an extension of rationals for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [L42, L27, L33].

The notion is applied in various cases and the connection with $M^8 - H$ duality is rather loose.

1. Extensions of rationals are essentially coders of information. There the possible analogy of extensions of rationals with genes deserves discussion. Extensions, which are not extensions of extensions would be analogous to genes. The notion of conserved gene as number theoretical analogy for Galois extensions as the Galois group of extension which is normal subgroup of

Galois extension.

2. The work of Peter Scholze [A174] based on the notion of perfectoid has raised a lot of interest in the community of algebraic geometers. One application of the notion relates to the attempt to generalize algebraic geometry by replacing polynomials with analytic functions satisfying suitable restrictions. Also in TGD this kind of generalization might be needed at the level of $M^4 \times CP_2$ whereas at the level of M^8 algebraic geometry might be enough. The notion of perfectoid as an extension of p-adic numbers Q_p allowing all p:th roots of p-adic prime p is central and provides a powerful technical tool when combined with its dual, which is function field with characteristic p.

   Could perfectoids have a role in TGD? The infinite-dimensionality of perfectoid is in conflict with the vision about finiteness of cognition. For other p-adic number fields Q_q, $q \neq p$ the extension containing p:th roots of p would be however finite-dimensional even in the case of perfectoid. Furthermore, one has an entire hierarchy of almost-perfectoids allowing powers of p^m:th roots of p-adic numbers. The larger the value of m, the larger the number of points in the extension of rationals used, and the larger the number of points in cognitive representations consisting of points with coordinates in the extension of rationals. The emergence of almost-perfectoids could be seen in the adelic physics framework as an outcome of evolution forcing the emergence of increasingly complex extensions of rationals [L34].

3. The construction of cognitive representation represents a well-known mathematical problem of finding the points of space-time surface with embedding space coordinates in given extension of rationals. Number theorist Minhyong Kim [A156, A167] has speculated about very interesting general connection between number theory and physics. The reading of a popular article about Kim's work revealed that number theoretic vision about physics provided by TGD has led to a very similar ideas and suggests a concrete realization of Kim's ideas [L70]. In the following I briefly summarize what I call identification problem. The identification of points of algebraic surface with coordinates, which are rational or in extension of rationals, is in question. In TGD framework the embedding space coordinates for points of space-time surface belonging to the extension of rationals defining the adelic physics in question are common to reals and all extensions of p-adics induced by the extension. These points define what I call cognitive representation, whose construction means solving of the identification problem.

   Cognitive representation defines discretized coordinates for a point of "world of classical worlds" (WCW) taking the role of the space of spaces in Kim's approach. The symmetries of this space are proposed by Kim to help to solve the identification problem. The maximal isometries of WCW necessary for the existence of its Kähler geometry provide symmetries identifiable as symplectic symmetries. The discrete subgroup respecting extension of rationals acts as symmetries of cognitive representations of space-time surfaces in WCW, and one can identify symplectic invariants characterizing the space-time surfaces at the orbits of the symplectic group.

4. One expects that large cognitive representations are winners in the number theoretical fight for survival. Strong form of holography suggests that it is enough to consider cognitive representations restricted to string world sheets and partonic 2-surfaces. If the 2-surface possesses large group of symmetries acting in extension of rationals, one can have large cognitive representations as orbit of point in extension. Examples of highly symmetric 2-D surfaces are geodesic spheres assignable to partonic 2-surfaces and cosmic strings and elliptic curves assignable with string world sheets and cosmic strings [L81].

5. Rationals and their extensions give rise to a unique discretizations of space-time surface (for instance) - cognitive representation - having interpretation in terms of finite measurement resolution. There are howevever many open questions. Should one allow only octonionic polynomials defined as algebraic continuations of real polynomials or should one allow also analytic functions and regard polynomials as approximations. Zeta functions are especially interesting analytic functions and Dekekind zetas characterize extensions of rationals and one can pose physically motivated questions about them [L58].

4.2 About $M^8 - H$-duality, p-adic length scale hypothesis and dark matter hierarchy

$M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy as phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ are basic assumptions of TGD, which all reduce to number theoretic vision. In the sequel $M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy are discussed from number theoretic perspective.

Several new results emerge. Strong form of holography (SH) allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to H: SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

M^8 duality allows to relate p-adic length scales L_p to differences for the roots of the polynomial defining the extension defining "special moments in the life of self" assignable causal diamond (CD) central in zero energy ontology (ZEO). Hence p-adic length scale hypothesis emerges both from p-adic mass calculations and $M^8 - H$ duality. It is proposed that the size scale of CD correspond to the largest dark scale nL_p for the extension and that the sub-extensions of extensions could define hierarchy of sub-CDs. Skyrmions are an important notion if nuclear and hadron physics, $M^8 - H$ dyality suggests an interpretation of skyrmion number as winding number as that for a map defined by complex polynomial.

4.2.1 Some background

A summary of the basic notions and ideas involved is in order.

p-Adic length scale hypothesis

In p-adic mass calculations [K50] real mass squared is obtained by so called canonical identification from p-adic valued mass squared identified as analog of thermodynamical mass squared using p-adic generelization of thermodynamics assuming super-conformal invariance and Kac-Moody algebras assignable to isometries ad holonomies of $H = M^4 \times CP_2$. This implies that the mass squared is essentially the expectation value of sum of scaling generators associated with various tensor factors of the representations for the direct sum of super-conformal algebras and if the number of factors is 5 one obtains rather predictive scenario since the p-adic temperature T_p must be inverse integer in order that the analogs of Boltzmann factors identified essentially as p^{L_0/T_p}.

The p-adic mass squared is of form $Xp + O(p^2)$ and mapped to $X/p + O(1/p^2)$. For the p-adic primes assignable to elementary particles ($M_{127} = 2^{127} - 1$ for electron) the higher order corrections are in general extremely small unless the coefficient of second order contribution is larger integer of order p so that calculations are practically exact.

Elementary particles seem to correspond to p-adic primes near powers 2^k. Corresponding p-adic length - and time scales would come as half-octaves of basic scale if all integers k are allowed. For odd values of k one would have octaves as analog for period doubling. In chaotic systems also the generalization of period doubling in which prime $p = 2$ is replaced by some other small prime appear and there is indeed evidence for powers of $p = 3$ (period tripling as approach to chaos). Many elementary particles and also hadron physics and electroweak physics seem to correspond to Mersenne primes and Gaussian Mersennes which are maximally near to powers of 2.

For given prime p also higher powers of p define p-adic length scales: for instance, for electron the secondary p-adic time scale is .1 seconds characterizing fundamental bio-rhythm. Quite generally, elementary particles would be accompanied by macroscopic length and time scales perhaps assignable to their magnetic bodies or causal diamonds (CDs) accompanying them.

This inspired p-adic length scale hypothesis stating the size scales of space-time surface correspond to primes near half-octaves of 2. The predictions of p-adic are exponentially sensitive to the value of k and their success gives strong support for p-adic length scale hypothesis. This hypothesis applied not only to elementary particle physics but also to biology and even astrophysics and cosmology. TGD Universe could be p-adic fractal.

Dark matter as phases of ordinary matter with $h_{eff}=nh_0$

The identification of dark matter as phases of ordinary matter with effective Planck constant $h_{eff}=nh_0$ is second key hypothesis of TGD. To be precise, these phases behave like dark matter and galactic dark matter could correspond to dark energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes.

There are good arguments in favor of the identification $h=6h_0$ [L22, L50]. "Effective" means that the actual value of Planck constant is h_0 but in many-sheeted space-time n counts the number of symmetry related space-time sheets defining space-time surface as a covering. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is nh_0.

M^8-H duality

M^8-H duality ($H=M^4\times CP_2$) [L67] has taken a central role in TGD framework. M^8-H duality allows to identify space-time regions as "roots" of octonionic polynomials P in complexified M^8 - M^8_c - or as minimal surfaces in $H=M^4\times CP_2$ having 2-D singularities.

Remark:O_c,H_c,C_c,R_c will be used in the sequel for complexifications of octonions, quaternions, etc.. number fields using commuting imaginary unit i appearing naturally via the roots of real polynomials.

The precise form of M^8-H duality has however remained unclear. Two assumptions are involved.

1. Associativity stating that the tangent or normal space of at the point of the space-time space-time surface M^8 is associative - that is quaternionic. There are good reasons to believe that this is true for the polynomial ansatz everywhere but there is no rigorous proof.
2. The tangent space of the point of space-time surface at points mappable from M^8 to H must contain fixed $M^2\subset M^4\subset M^8$ or an integrable distribution of $M^2(x)$ so that the 2-surface of M^4 determined by it belongs to space-time surface.

The strongest, global form of M^8-H duality states that $M^2(x)$ is contained to tangent spaces of X^4 at all points x. Strong form of holography (SH) states allows also the option for which this holds true only for 2-D surfaces - string world sheets and partonic 2-surfaces - therefore mappable to H and that SH allows to determined $X^4\subset H$ from this data. In the following a realization of this weaker form of M^8-H duality is found. Note however that one cannot exclude the possibility that also associativity is true only at these surfaces for the polynomial ansatz.

Number theoretic origin of p-adic primes and dark matter

There are several questions to be answered. How to fuse real number based physics with various p-adic physics? How p-adic length scale hypothesis and dark matter hypothesis emerge from TGD?

The properties of p-adic number fields and the strange failure of complete non-determinism for p-adic differential equations led to the proposal that p-adic physics might serve as a correlate for cognition, imagination, and intention. This led to a development of number theoretic vision which I call adelic physics. A given adele corresponds to a fusion of reals and extensions of various p-adic number fields induced by a given extension of rationals.

The notion of space-time generalizes to a book like structure having real space-time surfaces and their p-adic counterparts as pages. The common points of pages defining is back correspond to points with coordinates in the extension of rationals considered. This discretization of space-time surface is in general finite and unique and is identified as what I call cognitive representation. The Galois group of extension becomes symmetry group in cognitive degrees of freedom. The ramified primes of extension are exceptionally interesting and are identified as preferred p-adic primes for the extension considered.

The basic challenge is to identify dark scale. There are some reasons to expect correlation between p-adic and dark scales which would mean that the dark scale would depend on ramified primes, which characterize roots of the polynomial defining the extensions and are thus not defined completely by extension alone. Same extension can be defined by many polynomials. The naïve guess is that the scale is proportional to the dimension n of extension serving as a measure for algebraic complexity (there are also other measures). p-Adic length scales L_p would be proportional

nL_p, p ramified prime of extension? The motivation would be that quantum scales are typically proportional to Planck constant. It turns out that the identification of CD scale as dark scale is rather natural.

4.2.2 New results about $M^8 - H$ duality

In the sequel some new results about $M^8 - H$ duality are deduced. Strong form of holography (SH) allows to weaken the assumptions making possible $M^8 - H$ duality. It would be enough to map only certain complex 2-D sub-manifolds of quaternionic space-time surface in M^8 to H: SH would allow to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds would be determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and they form a discrete set.

Strong form of holography (SH)

Ordinary 3-D holography is forced by general coordinate invariance (GCI) and loosely states that the data at 3-D surfaces allows to determined space-time surface $X^4 \subset H$. In ZEO 3-surfaces correspond to pairs of 3-surfaces with members at the opposite light-like boundaries of causal diamond (CD) and are analogous to initial and final states of deterministic time evolution as Bohr orbit.

This poses additional strong conditions on the space-time surface.

1. The conjecture is that these conditions state the vanishing of super-symplectic Noether charges for a sub-algebra of super-symplectic algebra SC_n with radial conformal weights coming as n-multiples of those for the entire algebra SC and its commutator $[SC_n, SC]$ with the entire algebra: these conditions generalize super conformal conditions and one obtains a hierarchy of realizations.

   This hierarchy of minimal surfaces would naturally corresponds to the hierarchy of extensions of rationals with n identifiable as dimension of the extension giving rise to effective Planck constant. At the level of Hilbert spaces the inclusion hierarchies for extensions could also correspond to the inclusion hierarchies of hyper-finite factors of type I_1 [K99] so that $M^8 - H$ duality would imply beautiful connections between key ideas of TGD.

2. Second conjecture is that the preferred extremals (PEs) are extremals of both the volume term and Kähler action term of the action resulting by dimensional reduction making possible the induction of twistor structure from the product of twistor spaces of M^4 and CP_2 to 6-D S^2 bundle over X^4 defining the analog of twistor space. These twistor spaces must have Kähler structure since action for 6-D surfaces is Kähler action - it exists only in these two cases [A150] so that TGD is unique.

Strong form of holography (SH) is a strengthening of 3-D holography. Strong form of GCI requires that one can use either the data associated either with

- light-like 3-surfaces defining partonic orbits as surfaces at which signature of the induced metric changes from Euclidian to Minkowskian or
- the space-like 3-surfaces at the ends of CD to determine space-time surface as PE (in case that it exists).

This suggests that the data at the intersections of these 2-surfaces defined by partonic 2-surfaces might be enough for holography. A slightly weaker form of SH is that also string world sheets intersecting partonic orbits along their 1-D boundaries is needed and this form seems more realistic.

SH allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to H: SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

Space-time as algebraic surface in M_c^8 regarded complexified octonions

The octonionic polynomial giving rise to space-time surface as its "root" is obtained from ordinary real polynomial P with rational coefficients by algebraic continuation. The conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees associativity and one can formulate this as rather convincing argument [?] Space-time surface X_c^4 is identified as a 4-D root for a H_c-valued "imaginary" or "real" part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from M_c^8 to M^8. One could drop the subscripts "$_c$" but in the sequel they will be kept.

M_c^4 appears as a special solution for any polynomial P. M_c^4 seems to be like a universal reference solution with which to compare other solutions.

One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have M^4 projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of P. For monic polynomials these time values are algebraic integers and Galois group permutes them.

One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [?, ?] suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexifiation of octonions. i appears also in the roots defining complex extensions of rationals.

How do the solutions assignable to the opposite boundaries of CD relate to each other?

CD has two boundaries. The polynomials associated with them could be different in the general formulation discussed in [L86, L95] but they could be also same. How are the solutions associated with opposite boundaries of CD glued together in a continuous manner?

1. The polynomials assignable to the opposite boundaries of CD are allowed to be polynomials of o *resp.* $(o - T)$: here T is the distance between the tips of CD.

2. CD brings in mind the realization of conformal invariance at sphere: the two hemispheres correspond to powers of z and $1/z$: the condition $z = \overline{1/z}$ at unit circle is essential and there is no real conjugation. How the sphere is replaced with 8-D CD which is also complexified. The absence of conjugation looks natural also now: could CD contain a 3-surface analogous to the unit circle of sphere at which the analog of $z = \overline{1/z}$ holds true? If so, one has $P(o, z) = P(1/o, z)$ and the solutions representing roots fo $P(o, z)$ and $P(1/o, z)$ can be glued together.

   Note that $1/o$ can be expressed as $\overline{o}/o\overline{o}$ when the Minkowskian norm squared $\overline{o}o$ is non-vanishing and one has polynomial equation also now. This condition is true outside the boundary of 8-D light-cone, in particular near the upper boundary of CD.

   The counter part for the length squared of octonion in Minkowskian signature is light-one proper time coordinate $a^2 = t^2 - r^2$ for M_+^8. Replacing o which scaled dimensionless variable $o_1 = o/(T/2)$ the gluing take place along $a = T/2$ hyperboloid.

One has algebraic holomorphy with respect to o but also anti-holomorphy is possible. What could these two options correspond to? Could the space-time surfaces assignable to self and its time-reversal relate by octonionic conjugation $o \to \overline{o}$ relating two Fock vacuums annihilated by fermionic annihilation *resp.* creation operators?

In [L86, L95] the possibility that the sequence of SSFRs or BSFRs could involve iteration of the polynomial defining space-time surface - actually different polynomials were allowed for two boundaries. There are 3 options: each SSFR would involve the replacement $Q = P \circ .. \circ P \to P \circ Q$, the replacement occurs only when new "special moments in the life of self" defined by the roots of P as $t = r_n$ balls of cd, or the replacement can occur in BSFR when the metabolic resources do not allow to continue the iteration (the increase of h_{eff} during iteration increases the needed metabolic feed).

The iteration is compatible with the proposed picture. The assumption $P(0) = 0$ implies that iterates of P contain also the roots of P as roots - they are like conserved genes. Also the 8-D light-cone boundary remains invariant under iteration. Even more general function decompositions $P \to Q \to P$ are consistent with the proposed picture.

Brane-like solutions

One obtains also 6-D brane-like solutions to the equations.

1. In general the zero loci for imaginary or real part are 4-D but the 7-D light-cone δM^8_+ of M^8 with tip at the origin of coordinates is an exception [L37, L38, L39]. At δM^8_+ the octonionic coordinate o is light-like and one can write $o = re$, where 8-D time coordinate and radial coordinate are related by $t = r$ and one has $e = (1 + e_r)/\sqrt{2}$ such that one as $e^2 = e$.

   Polynomial $P(o)$ can be written at δM^8_+ as $P(o) = P(r)e$ and its roots correspond to 6-spheres S^6 represented as surfaces $t_M = t = r_N$, $r_M = \sqrt{r_N^2 - r_E^2} \leq r_N$, $r_E \leq r_N$, where the value of Minkowski time $t = r = r_N$ is a root of $P(r)$ and r_M denotes radial Minkowski coordinate. The points with distance r_M from origin of $t = r_N$ ball of M^4 has as fiber 3-sphere with radius $r = \sqrt{r_N^2 - r_E^2}$. At the boundary of S^3 contracts to a point.

2. These 6-spheres are analogous to 6-D branes in that the 4-D solutions would intersect them in the generic case along 2-D surfaces X^2. The boundaries $r_M = r_N$ of balls belong to the boundary of M^4 light-cone. In this case the intersection would be that of 4-D and 3-D surface, and empty in the generic case (it is however quite not clear whether topological notion of "genericity" applies to octonionic polynomials with very special symmetry properties).

3. The 6-spheres $t_M = r_N$ would be very special. At these 6-spheres the 4-D space-time surfaces X^4 as usual roots of $P(o)$ could meet. Brane picture suggests that the 4-D solutions connect the 6-D branes with different values of r_n.

   The basic assumption has been that particle vertices are 2-D partonic 2-surfaces and light-like 3-D surfaces - partonic orbits identified as boundaries between Minkowskian and Euclidian regions of space-time surface in the induced metric (at least at H level) - meet along their 2-D ends X^2 at these partonic 2-surfaces. This would generalize the vertices of ordinary Feynman diagrams. Obviously this would make the definition of the generalized vertices mathematically elegant and simple.

   Note that this does not require that space-time surfaces X^4 meet along 3-D surfaces at S^6. The interpretation of the times t_n as moments of phase transition like phenomena is suggestive. ZEO based theory of consciousness suggests interpretation as moments for state function reductions analogous to weak measurements ad giving rise to the flow of experienced time.

4. One could perhaps interpret the free selection of 2-D partonic surfaces at the 6-D roots as initial data fixing the 4-D roots of polynomials. This would give precise content to strong form of holography (SH), which is one of the central ideas of TGD and strengthens the 3-D holography coded by ZEO alone in the sense that pairs of 3-surfaces at boundaries of CD define unique preferred extremals. The reduction to 2-D holography would be due to preferred extremal property realizing the huge symplectic symmetries and making $M^8 - H$ duality possible as also classical twistor lift.

   I have also considered the possibility that 2-D string world sheets in M^8 could correspond to intersections $X^4 \cap S^6$? This is not possible since time coordinate t_M constant at the roots and varies at string world sheets.

   Note that the compexification of M^8 (or equivalently octonionic E^8) allows to consider also different variants for the signature of the 6-D roots and hyperbolic spaces would appear for $(\epsilon_1, \epsilon_i, .., \epsilon_8)$, $epsilon_i = \pm 1$ signatures. Their physical interpretation - if any - remains open at this moment.

5. The universal 6-D brane-like solutions S^6_c have also lower-D counterparts. The condition determining X^2 states that the C_c-valued "real" or "imaginary" for the non-vanishing Q_c-valued

"real" or "imaginary" for P vanishes. This condition allows universal brane-like solution as a restriction of O_c to M^4_c (that is CD_c) and corresponds to the complexified time=constant hyperplanes defined by the roots $t = r_n$ of P defining "special moments in the life of self" assignable to CD. The condition for reality in R_c sense in turn gives roots of $t = r_n$ a hyper-surfaces in M^2_c.

Explicit realization of $M^8 - H$ duality

$M^8 - H$ duality allows to map space-time surfaces in M^8 to H so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in M^8 and as minimal surfaces with 2-D singularities in H satisfying an infinite number of additional conditions stating vanishing of Noether charges for super-symplectic algebra actings as isometries for the "world of classical worlds" (WCW). Twistor lift allows variants of this duality. M^8_H duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of extensions of rationals defining an evolutionary hierarchy. This forms the basis for the number theoretical vision about TGD.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions.

1. Associativity condition for tangent-/normal space is the first essential condition for the existence of $M^8 - H$ duality and means that tangent - or normal space is quaternionic.
2. The tangent space of space-time surface and thus space-time surface itself must contain a preferred $M^2_c \subset M^4_c$ or more generally, an integrable distribution of tangent spaces $M^2_c(x)$ and similar distribution of their complements $E^2c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in R_c sense. $E^2_c(x)$ would correspond to partonic 2-surface.

One can imagine two realizations for this condition.

Option I: Global option states that the distributions $M^2_c(x)$ and $E^2_c(x)$ define slicing of X^4_c.

Option II: Only discrete set of 2-surfaces satisfying the conditions exist, they are mapped to H, and strong form of holography (SH) applied in H allows to deduce space-time surfaces in H. This would be the minimal option.

That the selection between these options is not trivial is suggested by following.

1. For massless extremals (MEs, topological light rays) parameterized by light-like vector vector k defining $M^2 \subset M^2 \times E^2 \subset M^4$ at each point and by space-like polarization vector ϵ depending on single transversal coordinate of E^2 [K8].
2. CP_2 coordinates have an arbitrary dependence on both $u = k \cdot m$ and $w = \epsilon \cdot m$ and can be also multivalued functions of u and w. Single light-like vector k is enough to identify M^2. CP_2 type extremals having metric and Kähler form of CP_2 have light-like geodesic as M^4 projection defining M^2 and its complement E^2 in the normal space.
3. String like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ are minimal surfaces and X^2 defines the distribution of $M^2(x) \subset M^4$. Y^2 ddefines the complement of this distribution.

   Option I is realized in all 3 cases. It is not clear whether M^2 can depend on position in the first 2 cases and also CP_2 point in the third case. It could be that only a discrete set of these string world sheets assignable to wormhole contacts representing massless particles is possible (**Option II**).

How these conditions would be realized?

1. The basic observation is that X^2c can be fixed by posing to the non-vanishing H_c-valued part of octonionic polynomial P condition that the C_c valued "real" or "imaginary" part in C_c sense for P vanishes. M^2_c would be the simplest solution but also more general complex sub-manifolds $X^2_c \subset M^4_c$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

   These surfaces would be like the families of curves in complex plane defined by $u = 0$ an $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. As found, there are also classes special global solutions for which the choice of M_c^2 is global and does not depend on space-time point. The interpretation would be in terms of modes of classical massless fields characterized by polarization and momentum. If the identification of M_c^2 is correct, these surfaces are however unstable against perturbations generating discrete string world sheets and orbits of partonic 2-surfaces having interpretation space-time counterparts of quanta. That fields are detected via their quanta was the revolutionary observation that led to quantum theory. Could quantum measurement induce the instability decomposing the field to quanta at the level of space-time topology?

3. One can generalize this condition so that it selects 1-D surface in X_c^2. By assuming that R_c-valued "real" or "imaginary" part of quaternionic part of P at this 2-surface vanishes. one obtains preferred M_c^1 or E_c^1 containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \rightarrow C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

   This option could be made possible by SH. SH states that preferred extremals are determined by data at 2-D surfaces of X^4. Even if the conditions defining X_c^2 have only a discrete set of solutions, SH at the level of H could allow to deduce the preferred extremals from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. Associativity and existence of $M^2(x)$ would be required only at the 2-D surfaces.

4. I have proposed that physical string world sheets and partonic 2-surfaces appear as singularities and correspond to 2-D folds of space-time surfaces at which the dimension of the quaternionic tangent space degenerates from 4 to 2 [L65] [K8]. This interpretation is consistent with a book like structure with 2-pages. Also 1-D real and imaginary manifolds could be interpreted as folds or equivalently books with 2 pages.

   For the singular surfaces the dimension quaternionic tangent or normal space would reduce from 4 to 2 and it is not possible to assign CP_2 point to the tangent space. This does not of course preclude the singular surfaces and they could be analogous to poles of analytic function. Light-like orbits of partonic 2-surfaces would in turn correspond to cuts.

5. What could the normal space singularity mean at the level of H? Second fundamental form defining vector basis in normal space is expected to vanish. This would be the case for minimal surfaces.

   (a) String world sheets with Minkowskian signature (in M^4 actually) are expected to be minimal surfaces. In this case T matters and string world sheets could be mapped to H by $M^8 - H$ duality and SH would work for them.

   (b) The light-like orbits of partonic 2-surfaces with Euclidian signature in H would serve as analogs of cuts. N is expected to matter and partonic 2-surfaces should be minimal surfaces. Their branching of partonic 2-surfaces is thus possible and would make possible (note the analogy with the branching of soap films) for them to appear as 2-D vertices in H.
   The problem is to identify the pre-images of partonic 2-surfaces in M^8. The light-likeness of the orbits of partonic 2-surfaces (induced 4-metric changes its signature and degenerates to 3-D) should be important. Could light-likeness in this sense define the pre-images partonic orbits in M^8?

Remark: It must be emphasized that SH makes possible $M^8 - H$ correspondence assuming that also associativity conditions hold true only at partonic 2-surfaces and string world sheets. Thus one could give up the conjecture that the polynomial ansatz implies that tangent or normal spaces are associative. Proving that this is the case for the tangent/normal spaces of these 2-surfaces should be easier.

Does $M^8 - H$ duality relate hadron physics at high and low energies?

During the writing of this article I realized that $M^8 - H$ duality has very nice interpretation in terms of symmetries. For $H = M^4 \times CP_2$ the isometries correspond to Poincare symmetries and color $SU(3)$ plus electroweak symmetries as holonomies of CP_2. For octonionic M^8 the subgroup $SU(3) \subset G_2$ is the sub-group of octonionic automorphisms leaving fixed octonionic imaginary unit invariant - this is essential for $M^8 - H$ duality. $SU(3)$ is also subgroup of $SO(6) \equiv SU(4)$ acting as rotation on $M^8 = M^2 \times E^6$. The subgroup of the holonomy group of $SO(4)$ for E^4 factor of $M^8 = M^4 \times E^4$ is $SU(2) \times U(1)$ and corresponds to electroweak symmetries. One can say that at the level of M^8 one has symmetry breaking from $SO(6)$ to $SU(3)$ and from $SO(4) = SU(2) \times SO(3)$ to $U(2)$.

This interpretation gives a justification for the earlier proposal that the descriptions provided by the old-fashioned low energy hadron physics assuming $SU(2)_L \times SU(2)_R$ and acting acting as covering group for isometries $SO(4)$ of E^4 and by high energy hadron physics relying on color group $SU(3)$ are dual to each other.

Skyrmions and $M^8 - H$ duality

I received a link (`https://tinyurl.com/ycathr3u`) to an article telling about research (`https://tinyurl.com/yddwhr2o`) carried out for skyrmions, which are very general condensed matter quasiparticles. They were found to replicate like DNA and cells. I realized that I have not clarified myself the possibility of skyrmions on TGD world and decided to clarify my thoughts.

1. What skyrmions are?

Consider first what skyrmions are.

1. Skyrmions are topological entities. One has some order parameter having values in some compact space S. This parameter is defined in say 3-ball such that the parameter is constant at the boundary meaning that one has effectively 3-sphere. If the 3rd homotopy group of S characterizing topology equivalence classes of maps from 3-sphere to S is non-trivial, you get soliton-llike entities, stable field configurations not deformable to trivial ones (constant value). Skyrmions can be assigned to space S which is coset space $SU(2)_L \times SU(2)_R/SU(2)_V$, essentially S^3 and are labelled by conserved integer-valued topological quantum number.
2. One can imagine variants of this. For instance, one can replace 3-ball with disk. $SO(3) = S^3$ with 2-sphere S^2. The example considered in the article corresponds to discretized situation in which one has magnetic dipoles/spins at points of say discretized disk such that spins have same direction about boundary circle. The distribution of directions of spin can give rise to skyrmion-like entity. Second option is distribution of molecules which do not have symmetry axis so that as rigid bodies the space of their orientations is discretized version of $SO(3)$. The field would be the orientation of a molecule of lattice and one has also now discrete analogs of skyrmions.
3. More generally, skyrmions emerge naturally in old-fashioned hadron physics, where $SU(2)_L \times SU(2)_R/SU(2)_V$ involves left-handed, right-handed and vectorial subgroups of $SO(4) = SU(2)_L \times SU(2)_R$. The realization would be in terms of 4-component field (π, σ), where π is charged pion with 3 components - axial vector - and σ which is scalar. The additional constraint $\pi \cdot \pi + \sigma^2 = constant$ defines 3-sphere so that one has field with values in S^3. There are models assigning this kind of skyrmion with nucleon, atomic nuclei, and also in the bag model of hadrons bag can be thought of as a hole inside skyrmion. These models seem to have something to do with reality so that a natural question is whether skyrmions might appear in TGD.

2. Skyrmion number as winding number

In TGD framework one can regard space-time as 4-surface in either octonionic M^8_c, c refers here to complexification by an imaginary unit i commuting with octonions, or in $M^4 \times CP_2$. For the solution surfaces M^8 has natural decomposition $M^8 = M^2 \times E^6$ and E^6 has $SO(6)$ as isometry group containing subgroup $SU(3)$ having automorphisms of octonions as subgroup leaving M^2 invariant. $SO(6) = SU(4)$ contains $SU(3)$ as subgroup, which has interpretation as isometries of CP_2 and counterpart of color gauge group. This supports $M^8 - H$ duality, whose most recent form is discussed in [L84].

The map $S^3 \to S^3$ defining skyrmion could be taken as a phenomenological consequence of $M^8 - H$ duality implying the old-fashioned description of hadrons involving broken $SO(4)$ symmetry (PCAC) and unbroken symmetry for diagonal group $SO(3)_V$ (CCV). The analog of $(\pi, sigma)$ field could correspond to a B-E condensate of pions $(\pi, sigma)$.

The obvious question is whether the map $S^3 \to S^3$ defining skyrmion could have a deeper interpretation in TGD framework. I failed to find any elegant formulation. One could however generalize and ask whether skyrmion like entities characterize by winding number are predicted by basic TGD.

1. In the models of nucleon and nuclei the interpretation of conserved topological skyrmion number is as baryon number. This number should correspond to the homotopy class of the map in question, essentially winding number. For polynomials of complex number degree corresponds to winding number. Could the degree $n = h_{eff}/h_0$ of polynomial P having interpretation as effective Planck constant and measure of complexity - kind of number theoretic IQ - be identifiable as skyrmion number? Could it be interpreted as baryon number too?

2. For leptons regarded as local 3 anti-quark composites in TGD based view about SUSY [L73] the same interpretation would make sense. It seems however that the winding number must have both signs. Degree is n is however non-negative.

   Here complexification of M^8 to M^8_c is essential. One an allow both holomorphic and anti-holomorphic continuations of real polynomials P (with rational coefficients) using complexification defined by commutative imaginary unit i in M^8_c so that one has polynomials $P(z)$ *resp.* $P(\overline{z})$ in turn algebraically continued to complexified octonionic polynomials $P(z, o)$ *resp.* $P(\overline{z}, o)$.

   Particles *resp.* antiparticles would correspond to the roots of octonionic polynomial $P(z, o)$ resp. $P(\overline{z}, o)$ meaning space-time geometrization of the particle-antiparticle dichotomy and would be conjugates of each other. This could give a nice physical interpretation to the somewhat mysterious complex roots of P.

3. More detailed formulation

To make this formulation more detailed on must ask how 4-D space-time surfaces correspond to 8-D "roots" for the "imaginary" ("real") part of complexified octonionic polynomial as surfaces in M^8_c.

1. Equations state the simultaneous vanishing of the 4 components of complexified quaternion valued polynomial having degree n and with coefficients depending on the components of O_c, which are regarded as complex numbers $x + iy$, where i commutes with octonionic units. The coefficients of polynomials depend on complex coordinates associated with non-vanishing "real" ("imaginary") part of the O_c valued polynomial.

2. To get perspective, one can compare the situation with that in catastrophe theory in which one considers roots for the gradient of potential function of behavior variables x^i. Potential function is polynomial having control variables as parameters. Now behavior variable correspond "imaginary" ("real") part and control variables to "real" ("imaginary") of octonionic polynomial.

   For a polynomial with real coefficients the solution divides to regions in which some roots are real and some roots are complex. In the case of cusp catastrophe one has cusp region with 3-D region of the parameter defined by behavior variable x and 2 control parameters with 3 real roots, the region in which one has one real root. The boundaries for the projection of 3-sheeted cusp to the plane defined by control variables correspond to degeneration of two complex roots to one real root.

   In the recent case it is not clear whether one cannot require the M^8_c coordinates for space-time surface to be real but to be in $M^8 = M^1 + iE^7$.

3. Allowing complex roots gives 8-D space-time surfaces. How to obtain real 4-D space-time surfaces?

   (a) One could project space-time surfaces to real M^8 to obtain 4-D real space-time surfaces. For M^8 this would mean projection to $M^1 + iE^7$ and in time direction the real part of root is accepted and is same for the root and its conjugate. For E^7 this would mean that imaginary part is accepted and means that conjugate roots correspond to different space-time surfaces and the notion of baryon number is realized at space-time level.

   (b) If one allows only real roots, the complex conjugation proposed to relate fermions and anti-fermions would be lost.

4. One can select for 4 complex M_c^8 coordinates X^k of the surface and the remaining 4 coordinates Y^k can be formally solved as roots of n:th degree polynomial with dynamical coefficients depending on X^k and the remaining Y^k. This is expected to give rise to preferred extremals with varying dimension of M^4 and CP_2 projections.

5. It seems that all roots must be complex.

   (a) The holomorphy of the polynomials with respect to the complex M_c^8 coordinates implies that the coefficients are complex in the generic point M_c^8. If so, all 4 roots are in general complex but do not appear as conjugate pairs. The naïve guess is that the maximal number of solutions would be n^4 for a given choice of M^8 coordinates solved as roots. An open question is whether one can select subset of roots and what happens at $t = r_n$ surfaces: could different solutions be glued together at them.

   (b) Just for completeness one can consider also the case that the dynamical coefficients are real - this is true in the E^8 sector and whether it has physical meaning is not clear. In this case the roots come as real roots and pairs formed by complex root and its conjugate. The solution surface can be divided into regions depending on the character of 4 roots. The n roots consist of complex root pairs and real roots. The members or complex root pairs are mapped to same point in E^8.

4. Could skyrmions in TGD sense replicate?

What about the observation that condensed matter skyrmions replicate? Could this have analog at fundamental level?

1. The assignment of conserved topological quantum number to the skyrmion is not consistent with replication unless the skyrmion numbers of outgoing states sum up to that of the initial state. If the system is open one can circumvent this objection. The replication would be like replication of DNA in which nucleotides of new DNA strands are brought to the system to form new strands.

2. It would be fascinating if all skyrmions would correspond to space-time surfaces at fundamental M^8 level. If so, skyrmion property also in magnetic sense could be induced by from a deeper geometric skyrmion property of the MB of the system. The openness of the system would be essential to guarantee conservation of baryon number. Here the fact that leptons and baryons have opposite baryon numbers helps in TGD framework. Note also ordinary DNA replication could correspond to replication of MB and thus of skyrmion sequences.

4.2.3 About p-adic length scale hypothesis and dark matter hierarchy

It is good to introduce first some background related to p-adic length scale hypothesis discussed in chapters of [K59] and dark matter hierarchy discussed in chapters [K40, K41], in particular in chatper [?].

General form of p-adic length scale hypothesis

The most general form of p-adic length scale hypothesis does not pose conditions on allowed p-adic primes and emerges from p-adic mass calculations [K21, K50, K61]. It has two forms corresponding to massive particles and massless particles.

1. For massive particles the preferred p-adic mass calculations based on p-adic thermodynamics predicts the p-adic mass squared m^2 to be proportional to p or its power- the real counterpart of m^2 is proportional to $1/p$ or its power. In the simplest case one has

$$m^2 = \frac{X}{p}\frac{\hbar}{L_0} \ ,$$

   where L_0 is apart from numerical constant the length R of CP_2 geodesic circle. X is a numerical constant not far from unity. $X \geq 1$ is small integer in good approximation. For instance for electron one has $x = 5$.

By Uncertainty Principle the Compton length of particle is characterizing the size of 3-surfaces assignable to particle are proportional to $\sqrt{p}$:

$$L_c(m) = \frac{\hbar}{m} = \sqrt{\frac{1}{X}}L_p \ , \quad L_p = \sqrt{p}L_0 = \ .$$

Here L_p is p-adic length scale and corresponds to minimal mass for given p-adic prime. p-Adic length scale would be would characterize the size of the 3-surface assignable to the particle and would correspond to Compton length.

2. For massless particles mass vanishes and the above picture is not possible unless there is very small mass coming from p-adic thermodynamics and determined by the size scale of CD - this is quite possible. The preferred time/spatial scales p-adic energy- equivalently 3-momentum are proportional to p-adic prime p or its power. The real energy is proportional to $1/p$. At the embedding space level the size of scale causal diamond (CD) [L72] would be proportional to p: $L = T = pL_0$, $L_0 = T_0$ for $c = 1$. The interpretation in terms of Uncertainty Principle is possible.

   There would be therefore two levels: space-time level and embedding space level . At the space-time level the primary p-adic length scale would be proportional to $\sqrt{p}$ whereas the p-adic length scale at embedding space-time would correspond to secondary p-adic length scale proportional to p. The secondary p-adic length scales would assign to elementary new physics in macroscopic scales. For electron the size scale of CD would be about .1 seconds, the time scale associated with the fundamental bio-rhythm of about 10 Hz.

3. A third piece in the picture is adelic physics [L42, L43] inspiring the hypothesis that effective Planck constant h_{eff} given by $h_{eff}/h_0 = n$, $h = 6h_0$, labels the phases of ordinary matter identified as dark matter. n would correspond to the dimension of extension of rationals.

   The connection between preferred primes and the value of $n = h_{eff}/h_0$ is interesting. One proposal is that preferred primes p in p-adic length scale hypothesis determining the mass scale of particle correspond to so called ramified primes, which characterize the extensions. The p-adic variant of the polynomial defining space-time surfaces in M^8 picture would have vanishing discriminant in order $O(p)$. Since discriminant is proportional to the product of differences of different roots of the polynomial, two roots would be very near to each other p-adically. This would be mathematical correlate for criticality in p-adic sense.

   $M^8 - H$ duality [L67, L63] leads to the prediction that the roots r_n of polynomial defining the space-time region in M^8 correspond to preferred time values $t = t_n =\propto r_n$- I have called $t = t_n$ "special moments in the life of self". Since the squares for the differences for the roots are proportional to ramified primes, these time differences would code for ramified primes assignable to the space-time surface. There would be several p-adic time scales involved and they would be coded by $t_{ij} = r_i - r_j$, whose moduli squared are divided by so called ramified primes defining excellent candidates for preferred p-adic primes. p-Adic physics would make itself visible at the level of space-time surface in terms of "special moments in the life of self".

4. p-Adic length scales emerge naturally from $M^8 - H$ duality [L67, L63]. Ramified primes would in M^8 picture appear as factors of time differences associated with "special moments in the life of self" associated with CD [L63]. One has $|t_i - t_j| \propto \sqrt{p_{ij}}$, p_{ij} ramified prime. It is essential that square root of ramified prime appears here.

   This suggests strongly that p-adic length scale hypothesis is realized at the level of space-time surface and there are several p-adic length scales present coded to the time differences. Knowing of the polynomial would give information about p-adic physics involved. If dark scales correlate with p-adic length scales as proposed, the definition of dark scale should assume the dependence of ramified primes quite generally rather than as a result of number theoretic survival of fittest as one might also think.

   The factors $t_i - t_j$ are proportional - not only to the typically very large p-adic prime p_{max} charactering the system - but also smaller primes or their powers. Could the scales in question be of form $l_p = \sqrt{X}\sqrt{p_{max}}L_0$ rather than p-adic length scales $L_{p_{ram}}$ defined by various ramified primes. Here X would be integer consisting of small ramified primes.

   p-Adic mass calculations predict in an excellent approximation the mass of the particle is given by $m = (\sqrt{X}/\sqrt{p})m_0$, X small integer and $m_0 = 1/L_0$. Compton length would be given by $L_c(p) = \sqrt{p}/\sqrt{X})L_0$. The identification $l_p = L_c(p)$ would be attractive but is not possible unless one has $X = 1$. In this case one would be considering p-adic length scale L_p. the interpretation in terms of multi-p-adicity seems to be the realistic option.

About more detailed form of p-adic length scale hypothesis

More specific form of p-adic length scale hypothesis poses conditions on physically preferred p-adic primes. There are several guesses for preferred primes. They could be primes near to integer powers 2^k, where k could be positive integer, which could satisfy additional conditions such as being odd, prime or be associated with Mersenne prime or Gaussian Mersenne. One can consider also powers of other small primes such as $p = 2, 3, 5$. p-Adic length scale hypothesis in is basic form would generalize the notion of period doubling. For odd values of k one would indeed obtain period doubling, tripling, etc... suggesting strongly chaos theoretic origin.

1. p-Adic length scale hypothesis in its basic form

Consider first p-adic length scale hypothesis in its basic form.

1. In its basic form states that primes $p \simeq 2^k$ are preferred p-adic primes and correspond by p-adic mass calculations p-adic length scales $L_p \equiv L(k) \propto \sqrt{p} = 2^{k/2}$. Mersenne primes and primes associated with Gaussian Mersennes as especially favored primes and charged leptons ($k \in \{127, 113, 107\}$) and Higgs boson ($k = 89$) correspond to them. Also hadron physics ($k = 107$) and nuclear physics ($k = 113$) correspond to these scales. One can assign also to hadron physics Mersenne prime and the conjecture is that Mersennes and Gaussian Mersennes define scaled variants of hadron physics and electroweak physics. In the length scale between cell membrane thickness fo 10 nm and nuclear size about 2.5 μm there are as many as 4 Gaussian Mersennes corresponding to $k \in \{151, 157, 163, 167\}$.

   Mersenne primes correspond to prime values of k and I have proposed that k is prime for fundamental p-adic length scales quite generally. There are also however also other p-adic length scales - for instance, for quarks k need not be prime - and it has remained unclear what criterion could select the preferred exponents k. One can consider also the option that odd values of k defined fundamental p-adic length scales.

2. What makes p-adic length scale hypothesis powerful is that masses of say scaled up variant of hadron physics can be estimated by simple scaling arguments. It is convenient to use electron's p-adic length scale and calculate other p-adic length scales by scaling $L(k) = 2^{(k-127)/2}L(127)$.

Here one must make clear that there has been a confusion in the definitions, which was originally due to a calculational error.

1. I identified the p-adic length scale $L(151)$ mistakenly as $L(151) = 2^{(k-127)/2}L_e(127)$ by using instead of $L(127)$ electron Compton length $L_e \simeq L(127/\sqrt{5}$. The notation for these scales would be therefore $L_e(k)$ identified as $L_e(k) = 2^{(k-127)/2}L_e(127)$ and I have tried to use it systematically but failed to use the wrong notation in informal discussions.
2. This mistake might reflect highly non-trivial physics. It is scaled up variants of L_e which seem to appear in physics. For instance, $L_e(151) \simeq 10$ nm corresponds to basic scale in living matter. Why the biological important scales should correspond to scaled up Compton lengths for electron? Could dark electrons with scaled up Compton scales equal to $L_e(k)$ be important in these scales? And what about the real p-adic length scales relate to these scales by a scaling factor $\sqrt{5} \simeq 2.23$?

2. Possible modifications of the p-adic length scale hypothesis

One can consider also possible modifications of the p-adic length scale hypothesis. In an attempt to understand the scales associated with INW structures in terms of p-adic length scale hypothesis it occurred to me that the scales which do not correspond to Mersenne primes or Gaussian Mersennes might be generated somehow from the these scales.

1. Geometric mean $L = \sqrt{L(k_1)L(k_2)}$ would length scale which would correspond to L_p with $p \simeq 2^{(k_1+k_2)/2}$. This is of the required form only if $k = k_1 + k_2$ is even so that k_1 and k_2

are both even or odd. If one starts from Mersennes and Gaussian Mersennes the condition is satisfied. The value of $k = (k_1 + k_2)/2$ can be also even.

Remark: The geometric mean $(127 + 107)/2 = 117$ of electronic and hadronic Mersennes corresponding to mass 16 MeV rather near to the mass of so called X boson [L24] (`https://tinyurl.com/ya3yuzeb`).

2. One can also consider the formula $L = (L(k_1)L(k_2)..L(k_n))^{1/n}$ but in this case the scale would correspond to prime $p \simeq 2^{k_1+...k_n)/n}$. Since $(k_1 + ..k_n)/n$ is integer only if $k_1 + ...k_n$ is proportional to n.

What about the allowed values of fundamental integers k? It seems that one must allow all odd integers.

1. If only prime values of k are allowed, one can obtain obtain for twin prime pair $(k-1, k+1)$ even integer k as geometric mean $\sqrt{k}$ if k is square. If prime k is not a member of this kind of pair, it is not possible to get integers $k-1$ and $k+1$. If only prime values of k are fundamental, one could assign to $k = 89$ characterizing Higgs boson weak bosons $k = 90$ possibly characterizing weak bosons. Therefore it seems that one must allow all odd integers with the additional condition already explained.

2. Just for fun one can check whether $k = 161$ forced by the argument related to electroweak scale and h_{eff} corresponds to a geometric mean of two Gaussian Mersennes. One has $k(k_1, k_2) = (k_2 + k_2)/2$ giving the list $k(151, 157) = 154)$, $k(151, 163) = 157$ Gaussian Mersenne itself, $k(151, 167) = 159$, $k(157, 163) = 160$, $k(157, 167) = 162$, $k(163, 167) = 165$. Unfortunately, $k = 161$ does not belong to this set. If one allows all odd values of k as fundamental, the problem disappears.

One can also consider refinements of p-adic length scale hypothesis in its basic form.

1. One can consider also a generalization of p-adic length scale hypothesis to allow length scales coming as powers of small primes. The small primes $p = 2, 3, 5$ assignable to Platonic solids would be especially interesting. $p = 2, 3, 5$ and also Fermat primes and Mersenne primes are maximally near to powers of two and their powers would define secondary and higher p-adic length scales. In this sense the extension would not actually bring anything new.

   There is evidence for the occurrence of long p-adic time scales coming as powers of 3 [I1, I2] (`http://tinyurl.com/ycesc5mq`) and [K62] (`https://tinyurl.com/y8camqlt`. Furthermore, prime 5 and Golden Mean are related closely to DNA helical structure. Portion of DNA with L(151) contains 10 DNA codons and is the minimal length containing an integer number of codons.

2. The presence of length scales associated with 1 nm and 2 nm thick structures encourage to consider the possibility of p-adic primes near integers $2^k3^l5^m$ defining generators of multiplicative ideals of integers. They do not satisfy the maximal nearness criterion anymore but would be near to integers representable as products of powers of primes maximally near to powers of two.

What could be the interpretation of the integer k appearing in $p \simeq 2^k$? Elementary particle quantum numbers would be associated with wormhole contacts with size scale of CP_2 whereas elementary particles correspond to p-adic size scale about Compton length. What could determine the size scale of wormhole contact? I have proposed that to p-adic length scale there is associated a scale characterizing wormhole contact and depending logarithmically on it and corresponds to $L_k = (1/2)log(p)L_0 = (k/2)log(2)L_0$. The generalization of this hypothesis to the case of $p \simeq 2^k3^l5^m...$ be straightforward and be $L_{k,l,m} = (1/2)(klog(2) + llog(3) + mlog(5) + ..)$.

Dark scales and scales of CDs and their relation to p-adic length scale hierarchy

There are two length scale hierarchies. p-Adic length scale hierarchy assignable to space-time surfaces and the dark hierarchy assignable to CDs. One should find an identification of dark scales and understand their relationship to p-adic length scales.

1. *Identification of dark scales*

The dimension n of the extension provides the roughest measure for its complexity via the formula $h_{eff}/h_0 = n$. The basic - rather ad hoc - assumption has been that n as dimension of extension defines not only h_{eff} but also the size scale of CD via $L = nL_0$.

This assumption need not be true generally and already the attempt to understand gravitational constant [L85] as a prediction of TGD led to the proposal that gravitational Planck constant $h_{gr} = n_{gr}h_0 = GMm/v_0$ [?] could be coded by the data relating to a normal subgroup of Galois group appearing as a factor of n.

The most general option is that dark scale is coded by a data related to extension of its sub-extension and this data involves ramified primes. Ramified primes depend on the polynomial defining the extension and there is large number polynomials defining the same extension. Therefore ramified ramifies code information also about polynomial and dynamics of space-time surface.

First some observations.

1. For Galois extension the order n has a natural decomposition to a product of orders n_i of its normal subgroups serving also as dimensions of corresponding extensions: $n = \prod_i n_i$. This implies a decomposition of the group algebra of Galois group to a tensor product of state spaces with dimensions n_i [L95].
2. Could one actually identify several dark scales as the proposed identifications of gravitational, electromagnetic, etc variants of h_{eff} suggest? The hierarchy of normal subgroups of Galois group of rationals corresponds to sub-groups with orders given by $N(i,1) = n_i n_{i-1}...n_{i-1}$ of n define orders for the normal subgroups of Galois group. For extensions of $k-1$:th extension of rationals one has $N(i,k) = n_i n_{i-1}...n_{i-k}$. The most general option is that these normal subgroups provide only the data allowing to associate dark scales to each of them. The spectrum of h_{eff} could correspond to the $\{N_{i,k}\}$ or at least the set $\{N_{i,1}\}$.
3. The extensions with prime dimension $n = p$ have no non-trivial normal subgroups and $n = p$ would hold for them. For these extensions the state space of group algebra is prime as Hilbert space and does not decompose to tensor product so that it would represent fundamental system. Could these extensions be of special interest physically? SSFRs would naturally involve state function reduction cascades proceeding downwards along hierarchy of normal subgroups and would represent cognitive measurements [L95].

The original guess was that dark scale $L_D = nL_p$, where n is the order n for the extensions and p is a ramified prime for the extension. A generalized form would allow $L_D = N(i,1)L_{p_k}$ for the sub-extension such that p_k is ramified prime for the sub-extension.

2. Can one identify the size scale of CD as dark scale?

It would be natural if the scale of CD would be determined by the extension of rationals. Or more generally, the scales of CD and hierarchy of sub-CDs associated with the extension would be determined by the inclusion hierarchy of extensions and thus correspond to the hierarchy of normal sub-groups of Galois group.

The simplest option would be $L_{CD} = L_D$ so that the size scales of sub-CD would correspond dark scales for sub-extension given by $L_{CD,i} = N(i,1)L_{p_k}$, p_k ramified prime of sub-extension.

1. The differences $|r_i - r_j|$ would correspond to differences for Minkowski time of CD. CD need not contain all values of hyperplanes $t = r_i$ and the evolution by SSFR would gradually bring in day-light all roots r_n of the polynomial P defining space-time surface as "very special moments in the life of self". If the size scale of CD is so large that also the largest value of $|r_i|$ is inside the upper or lower half of CD, the size scale of CD would correspond roughly to the largest p-adic length scale.

   CD contains sub-CDs and these could correspond to normal subgroups of Galois extension as extension of extension of ....
2. One can ask what happens when all special moments $t = r_n$ have been experienced? Does BSFR meaning death of conscious entity take place or is there some other option? In [L86] I considered a proposal for how chaos could emerge via iterations of P during the sequence of SSFRs.

   One could argue that when CD has reached by SSFRs following unitary evolutions a size for which all roots r_n have become visible, the evolution could continues by the replacement of

P with $P \circ P$, and so on. This would give rise to iteration and space-time analog for the approach to chaos.

3. Eventually the evolution by SSFRs must stop. Biological arguments suggests that metabolic limitations cause the death of self since the metabolic energy feed is not enough to preserve the distribution of values of h_{eff} (energies increase with $h_{eff} \propto Nn$, for N:th iteration and h_{eff} is reduced spontaneously) [L96].

4.3 Fermionic variant of $M^8 - H$ duality

The topics of this section is $M^8 - H$ duality for fermions. Consider first the bosonic counterpart of $M^8 - H$ duality.

1. The octonionic polynomial giving rise to space-time surface X^4 as its "root" is obtained from ordinary real polynomial P with rational coefficients by algebraic continuation. The conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees associativity and one can formulate this as rather convincing argument [L37, L38, L39]. Space-time surface X^4_c is identified as a 4-D root for a H_c-valued "imaginary" or "real" part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from M^8_c to M^8. One could drop the subscripts "$_c$" but in the sequel they will be kept.

   M^4_c appears as a special solution for any polynomial P. M^4_c seems to be like a universal reference solution with which to compare other solutions.

   One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have M^4 projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of P. For monic polynomials these time values are algebraic integers and Galois group permutes them.

2. One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [L42], suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexifiation of octonions. i appears also in the roots defining complex extensions of rationals.

   The generalization of the relationship between reals, extensions of p-adic number fields, and algebraic numbers in their intersection is suggestive. The "world of classical worlds" (WCW) would contain the space-time surfaces defined by polynomials with general real coefficients. Real WCW would be continuous space in real topology. The surfaces defined by rational or perhaps even algebraic coefficients for given extension would represent the intersection of real WCW with the p-adic variants of WCW labelled by the extension.

3. $M^8 - H$ duality requires additional condition realized as condition that also space-time surface itself contains 2-surfaces having commutative (complex) tangent or normal space. These surfaces can be 2-D also in metric sense that is light-like 3-D surfaces. The number of these surfaces is finite in generic case and they do not define a slicing of X^4 as was the first expectation. Strong form of holography (SH) makes it possible to map these surfaces and their tangent/normal spaces to 2-D surfaces $M4 \times CP_2$ and to serve as boundary values for the partial differential equations for variational principle defined by twistor lift. Space-time surfaces in H would be minimal surface apart from singularities.

Concerning $M^8 - H$ duality for fermions, there are strong guidelines: also fermionic dynamics should be algebraic and number theoretical.

1. Spinors should be octonionic. I have already earlier considered their possible physical interpretation. [L15].

2. Dirac equation as linear partial differential equation should be replaced with a linear algebraic equation for octonionic spinors which are complexified octonions. The momentum space

variant of the ordinary Dirac equation is an algebraic equation and the proposal is obvious: $P\Psi = 0$, where P is the octonionic continuation of the polynomial defining the space-time surface and multiplication is in octonionic sense. The conjugation in O_c is induced by the conjugation of the commuting imaginary unit i. The square of the Dirac operator is real if the space-time surface corresponds to the projection $O_c \to M^8 \to M^4$ with real time coordinate and imaginary spatial coordinates so that the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

The masslessness condition restricts the solutions to light-like 3-surfaces $m_{kl}P^kP^l = 0$ in Minkowskian sector analogous to mass shells in momentum space - just as in the case of ordinary massless Dirac equation. $P(o)$ rather than octonionic coordinate o would define momentum. These mass shells should be mapped to light-like partonic orbits in H.

3. This picture leads to the earlier phenomenological picture about induced spinors in H. Twistor Grassmann approach suggests the localization of the induced spinor fields at light-like partonic orbits in H. If the induced spinor field allows a continuation from 3-D partonic orbits to the interior of X^4, it would serve as a counterpart of virtual particle in accordance with quantum field theoretical picture.

4.3.1 $M^8 - H$ duality for space-time surfaces

It is good to explain $M^8 - H$ duality for space-time surfaces before discussing it in fermionic sector.

Space-time as 4-surface in $M^8_c = O_c$

One can regard real space-time surface $X^4 \subset M^8$ as a M^8--projection of $X^4_c \subset M^8_c = O_c$. M^4_c is identified as complexified quaternions H_c [L67, L84]. The dynamics is purely algebraic and therefore local an associativity is the basic dynamical principle.

1. The basic condition is associativity of $X^4 \subset M^8$ in the sense that either the tangent space or normal space is associative - that is quaternionic. This would be realized if X^4_c as a root for the quaternion-valued "real" or "imaginary part" for the O_c algebraic continuation of real analytic function $P(x)$ in octonionic sense. Number theoretical universality requires that the Taylor coefficients are rational numbers and that only polynomials are considered.

   The 4-surfaces with associative normal space could correspond to elementary particle like entities with Euclidian signature (CP_2 type extremals) and those with associative tangent space to their interaction regions with Minkowskian signature. These two kinds space-time surfaces could meet along these 6-branes suggesting that interaction vertices are located at these branes.

2. The conditions allow also exceptional solutions for any polynomial for which both "real" and "imaginary" parts of the octonionic polynomial vanish. Brane-like solutions correspond to 6-spheres S^6 having $t = r_n$ 3-ball B^3 of light-cone as M^4 projection: here r_n is a root of the real polynomial with rational coefficients and can be also complex - one reason for complexification by commuting imaginary unit i. For scattering amplitudes the topological vertices as 2-surfaces would be located at the intersections of X^4_c with 6-brane. Also Minkowski space M^4 is a universal solution appearing for any polynomial and would provide a universal reference space-time surface.

3. Polynomials with rational coefficients define EQs and these extensions form a hierarchy realized at the level of physics as evolutionary hierarchy. Given extension induces extensions of p-adic number fields and adeles and one obtains a hierarchy of adelic physics. The dimension n of extension allows interpretation in terms of effective Planck constant $h_{eff} = n \times h_0$. The phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ behave like dark matter and galactic dark matter could correspond to classical energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes. It is not completely clear whether number galactic dark matter must have $h_{eff} > h$. Dark energy in would correspond to the volume part of the energy of the flux tubes.

   There are good arguments in favor of the identification $h = 6h_0$ [L47]. "Effective" means that the actual value of Planck constant is h_0 but in many-sheeted space-time n counts the number of symmetry related space-time sheets defining X^4 as a covering space locally. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is nh_0.

The ramified primes of extension in turn are identified as preferrred p-adic primes. The moduli for the time differences $|t_r - t_s|$ have identification as p-adic time scales assignable to ramified primes [L84]. For ramified primes the p-adic variants of polynomials have degenerate zeros in $O(p) = 0$ approximation having interpretation in terms of quantum criticality central in TGD inspired biology.

4. During the preparation of this article I made a trivial but overall important observation. Standard Minkowski signature emerges as a prediction if conjugation in O_c corresponds to the conjugation with respect to commuting imaginary unit i rather than octonionic imaginary units as though earlier. If the space-time surface corresponds to the projection $O_c \to M^8 \to M^4$ with real time coordinate and imaginary spatial coordinates the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

Realization of $M^8 - H$ duality

$M^8 - H$ duality allows to $X^4 \subset M^8$ to $X^4 \subset H$ so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in M^8 and as minimal surfaces with 2-D preferred 2-surfaces defining holography making possible $M^8 - H$ duality and possibly appearing as singularities in H. The dynamics of minimal surfaces, which are also extremals of Kähler action, reduces for known extremals to purely algebraic conditions analogous to holomorphy conditions in string models and thus involving only gradients of coordinates. This condition should hold generally and should induce the required huge reduction of degrees of freedom proposed to be realized also in terms of the vanishing of super-symplectic Noether charges already mentioned [K76].

Twistor lift allows several variants of this basic duality [L79]. M^8_H duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of EQs defining an evolutionary hierarchy. This forms the basics for the number theoretical vision about TGD.

As already noticed, $X^4 \subset M^8$ would satisfy an infinite number of additional conditions stating vanishing of Noether charges for a sub-algebra $SSA_n \subset SSA$ of super-symplectic algebra SSA actings as isometries of WCW.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions [L67].

1. Associativity condition for tangent-/normal spaces is the first essential condition for the existence of $M^8 - H$ duality and means that tangent - or normal space is associative/quaternionic.
2. Each tangent space of X^4 at x must contain a preferred $M^2_c(x) \subset M^4_c$ such that $M^2_c(x)$ define an integrable distribution and therefore complexified string world sheet in M^4_c. This gives similar distribution for their orthogonal complements $E^2c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in R_c sense. $E^2_c(x)$ would correspond to partonic 2-surface. This condition generalizes for X^4 with quaternionic normal space. A possible interpretation is as a space-time correlate for the selection of quantization axes for energy (rest system) and spin.

One can imagine two realizations for the additional condition.

Option I: Global option states that the distributions $M^2_c(x)$ and $E^2_c(x)$ define a slicing of X^4_c.

Option II: Only a discrete set of 2-surfaces satisfying the conditions exist, they are mapped to H, and strong form of holography (SH) applied in H allows to deduce $X^4 \subset H$. This would be the minimal option.

It seems that only **Option II** can be realized.

1. The basic observation is that X^2_c can be fixed by posing to the non-vanishing H_c-valued part of octonionic polynomial P condition that the C_c-valued "real" or "imaginary" part in C_c

sense for P vanishes. M_c^2 would be the simplest solution but also more general complex sub-manifolds $X_c^2 \subset M_c^4$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

These surfaces would be like the families of curves in complex plane defined by $u = 0$ an $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. SH makes possible $M^8 - H$ duality assuming that associativity conditions hold true only at 2-surfaces including partonic 2-surfaces or string world sheets or perhaps both. Thus one can give up the conjecture that the polynomial ansatz implies the additional condition globally.

   SH indeed states that PEs are determined by data at 2-D surfaces of X^4. Even if the conditions defining X_c^2 have only a discrete set of solutions, SH at the level of H could allow to deduce the PEs from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. The existence of $M^2(x)$ would be required only at the 2-D surfaces.

3. There is however a delicacy involved: X^2 might be 2-D only metrically but not topologically! The 3-D light-like surfaces X_L^3 indeed have metric dimension $D = 2$ since the induced 4-metric degenerates to 2-D metric at them. Therefore their pre-images in M^8 would be natural candidates for the singularities at which the dimension of the quaternionic tangent or normal space reduces to $D = 2$ [L65] [K8]. If this happens, SH would not be quite so strong as expected. The study of fermionic variant of $M^8 - H$-duality supports this conclusion.

One can generalize the condition selecting X_c^2 so that it selects 1-D surface inside X_c^2. By assuming that R_c-valued "real" or "imaginary" part of complex part of P sense at this 2-surface vanishes. One obtains preferred M_c^1 or E_c^1 containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as a complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \rightarrow C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

4.3.2 What about $M^8 - H$ duality in the fermionic sector?

During the preparation of this article I become aware of the fact that the realization $M^8 - H$ duality in the fermionic sector has remained poorly understood. This led to a considerable integration of the ideas about $M^8 - H$ duality also in the bosonic sector and the existing phenomenological picture follows now from $M^8 - H$ duality. There are powerful mathematical guidelines available.

Octonionic spinors

By supersymmetry, octonionicity should have also fermionic counterpart.

1. The interpretation of M_c^8 as complexified octonions suggests that one should use complexified octonionic spinors in M_c^8. This is also suggested by SO(1,7) triality unique for dimension $d = 8$ and stating that the dimensions of vector representation, spinor representation and its conjugate are same and equal to $D = 8$. I have already earlier considered the possibility to interpret M^8 spinors as octonionic [L15]. Both octonionic gamma matrices and spinors have interpretation as octonions and gamma matrices satisfy the usual anti-commutation rules. The product for gamma matrices and gamma matrices and spinors is replaced with non-associative octonionic product.

2. Octonionic spinors allow only one M^8-chirality, which conforms with the assumption of TGD inspired SUSY that only quarks are fundamental fermions and leptons are their local composites [L73].

3. The decomposition of $X^2 \subset X^4 \subset M^8$ corresponding to $R \subset C \subset Q \subset O$ should have analog for the O_c spinors as a tensor product decomposition. The special feature of dimension $D = 8$ is that the dimensions of spinor spaces associated with these factors are indeed 1, 2, 4, and 8 and correspond to dimensions for the surfaces!

   One can define for octonionic spinors associative/co-associative sub-spaces as quaternionic/co-quaternionic spinors by posing chirality conditions. For $X^4 \subset M_c^8$ one could define the analogs of projection operators $P_\pm = (1 \pm \gamma_5)/2$ as projection operators to either factor of the spinor space as tensor product of spinor space associated with the tangent and normal spaces of

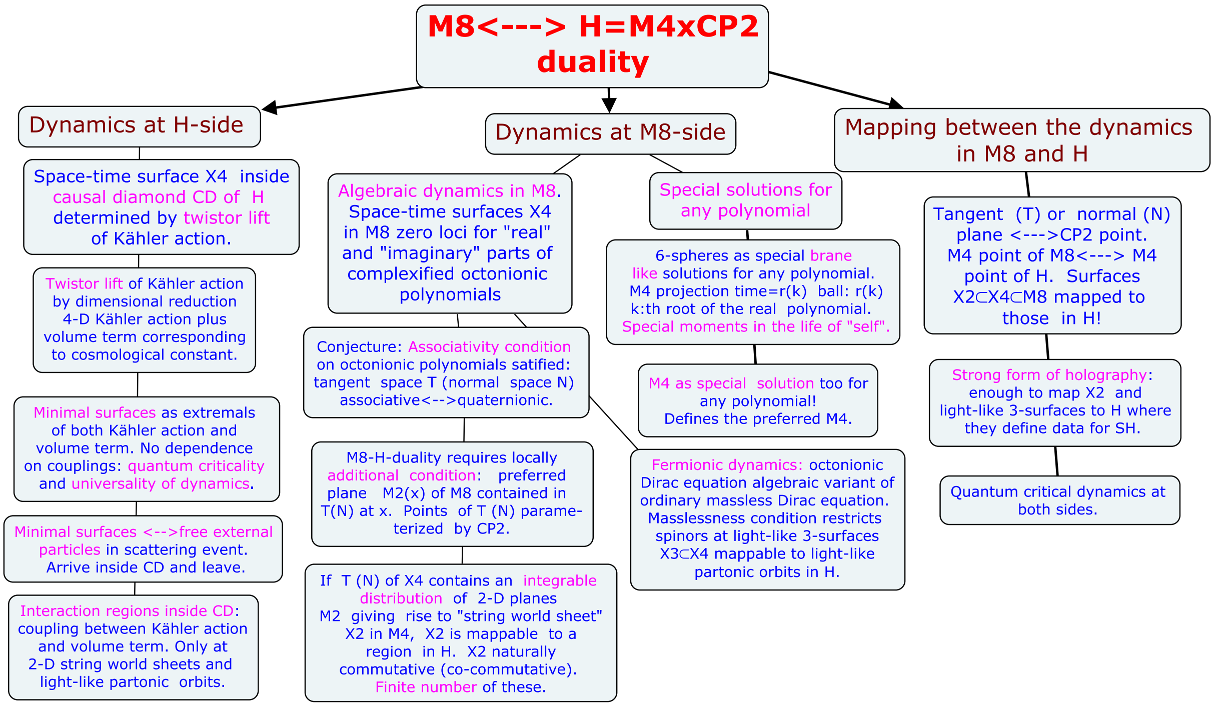

Figure 4.1: $M^8 - H$ duality.

X^4: the analog of γ_5 would correspond to tangent or normal space depending on whether tangent or normal space is associative. For the spinors with definite chirality there would be no entanglement between the tensor factors. The condition would generalize the chirality condition for massless M^4 spinors to a condition holding for the local M^4 appearing as tangent/normal space of X^4.

4. The chirality condition makes sense also for $X^2 \subset X^4$ identified as complex/co-complex surface of X^4. Now γ_5 is replaced with γ_3 and states that the spinor has well-defined spin in the direction of axis defined by the decomposition of X^2 tangent space to $M^1 \times E^1$ with M^1 defining real octonion axis and selecting rest frame. Interpretation in terms of quantum measurement theory is suggestive.

What about tangent space quantum numbers in M^8 picture. In H-picture they correspond to spin and electroweak quantum numbers. In M^8 picture the geometric tangent space group for a rest system is product $SU(2) \times SU(2)$ with possible modifications due to octonionicity reducing tangent space group to those respecting octonionic automorphisms.

What about the sigma matrices for the octonionic gamma matrices? The surprise is that the commutators of M^4 sigma matries and those of E^4 sigma matrices close to the sama $SO(3)$ algebra allowing interpretation as representation for quaternionic automorphisms. Lorentz boosts are represented trivially, which conforms with the fact that octonion structure fixes unique rest system. Analogous result holds in E^4 degrees of freedom. Besides this one has unit matrix assignable to the generalize spinor structure of CP_2 so that also electroweak $U(1)$ factor is obtained.

One can understand this result by noticing that octonionic spinors correspond to 2 copies of a tensor products of the spinor doublets associated with spin and weak isospin. One has $2 \otimes 2 = 3 \oplus 1$

so that one must have $1 \oplus 3 \oplus 1 \oplus 3$. The octonionic spinors indeed decompose like $1+1+3+\overline{3}$ under $SU(3)$ representing automophisms of the octonions. $SO(3)$ could be interpreted as $SO(3) \subset SU(3)$. $SU(3)$ would be represented as tangent space rotations.

Dirac equation as partial differential equation must be replaced by an algebraic equation

Algebraization of dynamics should be also supersymmetric. The modified Dirac equation in H is linear partial differential equation and should correspond to a linear algebraic equation in M^8.

1. The key observation is that for the ordinary Dirac equation the momentum space variant of Dirac equation for momentum eigenstates is algebraic! Could the interpretation for $M^8 - H$ duality as an analog of momentum-position duality of wave mechanics considered already earlier make sense! This could also have something to do with the dual descriptions of twistorial scattering amplitudes in terms of either twistor and momentum twistors. Already the earlier work excludes the interpretation of the octonionic coordinate o as 8-momentum. Rather, $P(o)$ has this interpretation and o corrresponds to embedding space coordinate.
2. The first guess for the counterpart of the modified Dirac equation at the level of $X^4 \subset M^8$ is $P\Psi = 0$, where Ψ is octonionic spinor and the octonionic polynomial P defining the space-time surface can be seen as a generalization of momentum space Dirac operator with octonion units representing gamma matrices. If associativity/co-associativity holds true, the equation becomes quaternionic/co-quaternionic and reduces to the 4-D analog of massless Dirac equation and of modified Dirac equation in H. Associativity hols true if also Ψ satisfies associativity/co-associativity condition as proposed above.
3. What about the square of the Dirac operator? There are 3 conjugations involved: quaternionic conjugation assumed in the earlier work, conjugation with respect to i, and their combination. The analog of octonionic norm squared defined as the product $o_c o_c^*$ with conjugation with respect to i only, gives Minkowskian metric $m_{kl} o^k \overline{o}^l$ as its real part. The imaginary part of the norm squared is vanishing for the projection $O_c \to M^8 \to M^4$ so that time coordinate is real and spatial coordinates imaginary. Therefore Dirac equation allows solutions only for the M^4 projection X^4 and M^4 ($M8$) signature of the metric can be said to be an outcome of quaternionicity (octonionicity) alone in accordance with the duality between metric and algebraic pictures.

   Both $P^\dagger P$ and PP should annihilate Ψ. $P^\dagger P\Psi = 0$ gives $m_{kl} P^k \overline{P}^l = 0$ as the analog of vanishing mass squared in M^4 signature in both associative and co-associative cases. $PP\Psi = 0$ reduces to $P\Psi = 0$ by masslessness condition. One could perhaps interpret the projection $X_c^4 \to M^8 \to M^4$ in terms of Uncertainty Principle.

   There is a $U(1)$ symmetry involved: instead of the plane M^8 one can choose any plane obtained by a rotation $exp(i\phi)$ from it. Could it realize quark number conservation in M^8 picture?

   For $P = o$ having only $o = 0$ as root $Po = 0$ reduces to $o^\dagger o = 0$ and o takes the role of momentum, which is however vanishing. 6-D brane like solutions S^6 having $t = r_n$ balls $B^3 \subset CD_4$ as M^4 projections one has $P = 0$ so that the Dirac equation trivializes and does not pose conditions on Ψ. o would have interpretation as space-time coordinates and $P(o)$ as position dependent momentum components P^k.

   The variation of P at mass shell of M_c^8 (to be precise) could be interpreted in terms of the width of the wave packet representing particle. Since the light-like curve at partonic 2-surface for fermion at X_L^3 is not a geodesic, mass squared in M^4 sense is not vanishing. Could one understand mass squared and the decay width of the particle geometrically? Note that mass squared is predicted also by p-adic thermodynamics [K50].
4. The masslessness condition restricts the spinors at 3-D light-cone boundary in $P(M^8)$. $M^8 - H$ duality [L67] suggests that this boundary is mapped to $X_L^3 \subset H$ defining the light-like orbit of the partonic 2-surface in H. The identification of the images of $P_k P^k = 0$ surfaces as X_L^3 gives a very powerful constraint on SH and $M^8 - H$ duality.
5. Also at 2-surfaces $X^2 \subset X^4$ an the variant Dirac equation would hold true and should commute with the corresponding chirality condition. Now $D^\dagger D\Psi = 0$ gives 2-D variant

of masslessness condition with 2-momentum components represented by those of P. 2-D masslessness locates the spinor to a 1-D curve X^1_L. Its H-image would naturally contain the boundary of the string word sheet at X^3_L assumed to carry fermion quantum numbers and also the boundary of string world sheet at the light-like boundary of CD_4. The interior of string world sheet in H would not carry induced spinor field.

6. The general solution for both 4-D and 2-D cases can be written as $\Psi = P\Psi_0$, Ψ_0 a constant spinor - this in a complete analogy with the solution of modified Dirac equation in H. P depends on position: the WKB approximation using plane waves with position dependent momentum seems to be nearer to reality than one might expect.

The phenomenological picture at H-level follows from the M^8-picture

Remarkably, the partly phenomenological picture developed at the level of H is reproduced at the level of M^8. Whether the induced spinor fields in the interior of X^4 are present or not, has been long standing question since they do not seem to have any role in the physical picture. The proposed picture answers this question.

Consider now the explicit realization of $M^8 - H$-duality for fermions.

1. SH and the expected analogy with the bosonic variant of $M^8 - H$ duality lead to the first guess. The spinor modes in $X^4 \subset M^8$ restricted to X^2 can be mapped by $M^8 - H$-duality to those at their images $X^2 \subset H$, and define boundary conditions allowing to deduce the solution of the modified Dirac equation at $X^4 \subset H$. X^2 would correspond to string world sheets having boundaries X^1_L at X^3_L.

   The guess is not quite correct. Algebraic Dirac equation requires that the solutions are restricted to the 3-D and 1-D mass shells $P_kP^k = 0$ in M^8. This should remain true also in H and X^3_L and their 1-D intersections X^1_L with string world sheets remain. Fermions would live at boundaries. This is just the picture proposed for the TGD counterparts of the twistor amplitudes and corresponds to that used in twistor Grassmann approach!

   For 2-D case constant octonionic spinors Ψ_0 and gamma matrix algebra are equivalent with the ordinary Weyl spinors and gamma matrix algebra and can be mapped as such to H. This gives one additional reason for why SH must be involved.

2. At the level of H the first guess is that the modified Dirac equation $D\Psi = 0$ is true for D based on the modified gamma matrices associated with both volume action and Kähler action. This would select preferred solutions of modified Dirac equation and conform with the vanishing of super-symplectic Noether charges for SSA_n for the spinor modes. The guess is not quite correct. The restriction of the induced spinors to X^3_L requires that Chern-Simons action at X^3_L defines the modified Drac action.

3. The question has been whether the 2-D modified Dirac action emerges as a singular part of 4-D modified Dirac action assignable to singular 2-surface or can one assign an independent 2-D Dirac action assignable to 2-surfaces selected by some other criterion. For singular surfaces $M^8 - H$ duality fails since tangent space would reduce to 2-D space so that only their images can appear in SH at the level of H.

   This supports the view that singular surfaces are actually 3-D mass shells M^8 mapped to X^3_L for which 4-D tangent space is 2-D by the vanishing of $\sqrt{g_4}$ and light-likeness. String world sheets would correspond to non-singular $X^2 \subset M^8$ mapped to H and defining data for SH and their boundaries $X^1_L \subset X^3_L$ and $X^1_L \subset CD_4$ would define fermionic variant of SH.

What about the modified Dirac operator D in H?

1. For X^3_L modified Dirac equation $D\Psi = 0$ based on 4-D action S containing volume and Kähler term is problematic since the induced metric fails to have inverse at X^3_L. The only possible action is Chern-Simons action S_{CS} used in topological quantum field theories and now defined as sum of C-S terms for Kähler actions in M^4 and CP_2 degrees of freedom. The presence of M^4 part of Kähler form of M^8 is forced by the twistor lift, and would give rise to small CP breaking effects explaining matter antimatter asymmetry [L73]. S_{C-S} could emerge as a limit of 4-D action.

The modified Dirac operator D_{C-S} uses modified gamma matrices identified as contractions $\Gamma^{\alpha}_{CS} = T^{\alpha k}\gamma_k$, where $T^{\alpha k} = \partial L_{CS}/\partial(\partial_\alpha h^k)$ are canonical momentum currents for S_{C-S} defined by a standard formula.

2. CP_2 part would give conserved Noether currents for color in and M^4 part Poincare quantum numbers: the apparently small CP breaking term would give masses for quarks and leptons! The bosonic Noether current $J_{B,A}$ for Killing vector j^k_A would be proportional to $J^{\alpha}_{B,A} = T^{\alpha}_k j_A k$ and given by $J_{B,A} = \epsilon^{\alpha\beta\gamma}[J_{\beta\gamma}A_k + A_\beta J_{\gamma k}]j^k_A$.
   Fermionic Noether current would be $J_{F,A} = \overline{\Psi}J^\alpha\Psi$ 3-D Riemann spaces allow coordinates in which the metric tensor is a direct sum of 1-D and 2-D contributions and are analogous to expectation values of bosonic Noether currents. One can also identify also finite number of Noether super currents by replacing $\overline{\Psi}$ or Ψ by its modes.

3. In the case of X^3_L the 1-D part light-like part would vanish. If also induced Kähler form is non-vanishing only in 2-D degrees of freedom, the Noether charge densities J^t reduce to $J^t = JA_k j^k_A$, $J = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}$ defining magnetic flux. Modified Dirac operator would reduce to $D = JA_k\gamma^k D_t$ and 3-D solutions would be covariantly constant spinors along the light-like geodesics parameterized by the points 2-D cross section. One could say that the number of solutions is finite and corresponds to covariantly constant modes continued from X^1_L to X^3_L. This picture is just what twistor Grassmannian approach led to [L56].

A comment inspired by the ZEO based quantum measurement theory

I cannot resist the temptation to make a comment relating to quantum measurement theory inspired by zero energy ontology (ZEO) extending to a theory of consciousness [L72, L95, L96].

I have proposed [L84, L86] that the time evolution by "big" state function reductions (BSFRs) could be induced by iteration of real polynomial P - at least in some special cases. The foots of the real polynomial P would define a fractal at the limit of larger number of iterations. The roots of n-fold iterate $\circ^n P$ would contain the inverse images under $\circ^{-n+1}P$ of roots of P and for $P(0) = 0$ the inverse image $\circ^n P$ would consist of inverse images under $\circ^{-k}P$, $k = 0, ..., n-1$, of roots of P.

Also the mass shells for $\circ^n P$ would be unions of inverses images under $\circ^{-k}P$, $k = 0, , ..., n-1$, of roots of P. This gives rather concrete view about evolution of M^4 projections of the partonic orbits. A rough approximate expression for the largest root of real P approximated as $P(x) \simeq a_n x^n + an - 1ix^{n-1}$ for large x is $x_{max} \sim a_n/a_{n-1}$. For $\circ^n P$ one obtains the same estimate. This suggests that the size scales of the partonic orbits are same for the iterates. The mass shells would not differ dramatically: could they have an interpretation in terms of mass splitting?

The evolution by iteration would add new partonic orbits and preserve the existing ones: this brings in mind conservation of genes in biological evolution. This is true also for a more general evolution allowing general functional decomposition $Q \to Q \circ P$ to occur in BSFR.

What next in TGD?

The construction of scattering amplitudes has been the dream impossible that has driven me for decades. Maybe the understanding of fermionic $M^8 - H$ duality provides the needed additional conceptual tools. The key observation is utterly trivial but far reaching: there are 3 possible conjugations for octonions corresponding to the conjugation of commutative imaginary unit or of octonionic imaginary units or both of them. 1st norm gives a real valued norm squared in Minkowski signature natural at M^8 level! Second one gives a complex valued norm squared in Euclidian signature. 1st and 2nd norms are equivalent for octonions light-like with respect to the first norm. The 3rd conjugation gives a real-valued Euclidian norm natural at the level of Hilbert space.

1. M^8 picture looks simple. Space-time surfaces in M^8 can be constructed from real polynomials with real (rational) coefficients, actually knowledge of their roots is enough. Discrete data - roots of the polynomial!- determine space-time surface as associative or co-associative region! Besides this one must pose additional condition selecting 2-D string world sheets and 3-D light-like surfaces as orbits of partonic 2-surfaces. These would define strong form of holography (SH) allowing to map space-time surfaces in M^8 to $M^4 \times CP_2$.

2. Could SH generalize to the level of scattering amplitudes expressible in terms of n-point functions of CFT?! Could the n points correspond to the roots of the polynomial defining space-time region!

   Algebraic continuation to quaternion valued scattering amplitudes analogous to that giving space-time sheets from the data coded SH should be the key idea. Their moduli squared are real - this led to the emergence of Minkowski metric for complexified octonions/quaternions) would give the real scattering rates: this is enough! This would mean a number theoretic generalization of quantum theory.

3. One can start from complex numbers and string world sheets/partonic 2-surfaces. Conformal field theories (CFTs) in 2-D play fundamental role in the construction of scattering string theories and in modelling 2-D statistical systems. In TGD 2-D surfaces (2-D at least metrically) code for information about space-time surface by strong holography (SH) .

   Are CFTs at partonic 2-surfaces and string world sheets the basic building bricks? Could 2-D conformal invariance dictate the data needed to construct the scattering amplitudes for given space-time region defined by causal diamond (CD) taking the role of sphere S^2 in CFTs. Could the generalization for metrically 2-D light-like 3-surfaces be needed at the level of "world of classical worlds" (WCW) when states are superpositions of space-time surfaces, preferred extremals?

The challenge is to develop a concrete number theoretic hierarchy for scattering amplitudes: $R \rightarrow C \rightarrow H \rightarrow O$ - actually their complexifications.

1. In the case of fermions one can start from 1-D data at light-like boundaries LB of string world sheets at light-like orbits of partonic 2-surfaces. Fermionic propagators assignable to LB would be coded by 2-D Minkowskian QFT in manner analogous to that in twistor Grassmann approach. n-point vertices would be expressible in terms of Euclidian n-point functions for partonic 2-surfaces: the latter element would be new as compared to QFTs since point-like vertex is replaced with partonic 2-surface.

2. The fusion (product?) of these Minkowskian and Euclidian CFT entities corresponding to different realization of complex numbers as sub-field of quaternions would give rise to 4-D quaternionic valued scattering amplitudes for given space-time sheet. Most importantly: there moduli squared are real for both norms.

   It is not quite clear whether one must use the 1st Minkowskian norm requiring "time-like" scattering amplitudes to achieve non-negative probabilities or use the 3rd norm to get the ordinary positive-definite Hilbert space norm. A generalization of quantum theory (CFT) from complex numbers to quaternions (quaternionic "CFT") would be in question.

3. What about several space-time sheets? Could one allow fusion of different quaternionic scattering amplitudes corresponding to different quaternionic sub-spaces of complexified octonions to get octonion-valued non-associative scattering amplitudes. Again scattering rates would be real. This would be a further generalization of quantum theory.

There is also the challenge to relate M^8- and H-pictures at the level of WCW. The formulation of physics in terms of WCW geometry [K76, L78] leads to the hypothesis that WCW Kähler geometry is determined by Kähler function identified as the 4-D action resulting by dimensional reduction of 6-D surfaces in the product of twistor spaces of M^4 and CP_2 to twistor bundles having S^2 as fiber and space-time surface $X^4 \subset H$ as base. The 6-D Kähler action reduces to the sum of 4-D Kähler action and volume term having interpretation in terms of cosmological constant.

The question is whether the Kähler function - an essentially geometric notion - can have a counterpart at the level of M^8.

1. SH suggests that the Kähler function identified in the proposed manner can be expressed by using 2-D data or at least metrically 2-D data (light-like partonic orbits and light-like boundaries of CD). Note that each WCW would correspond to a particular CD.

2. Since 2-D conformal symmetry is involved, one expects also modular invariance meaning that WCW Kähler function is modular invariant, so that they have the same value for $X^4 \subset H$ for which partonic 2-surfaces have induced metric in the same conformal equivalence class.

3. Also the analogs of Kac-Moody type symmetries would be realized as symmetries of Kähler function. The algebra of super-symplectic symmetries of the light-cone boundary can be regarded as an analog of Kac-Moody algebra. Light-cone boundary has topology $S^2 \times R_+$ where R_+ corresponds to radial light-like ray parameterized by radial light-like coordinate r. Super symplectic transformations of $S^2 \times CP_2$ depend on the light-like radial coordinate r, which is analogous to the complex coordinate z for he Kac-Moody algebras.

   The infinitesimal super-symplectic transformations form algebra SSA with generators proportional to powers r^n . The Kac-Moody invariance for physical states generalizes to a hierarchy of similar invariances. There is infinite fractal hierarchy of sub-algebras $SSA_n \subset SSA$ with conformal weights coming as n-multiples of those for SSA. For physical states SSA_n and $[SSA_n, SSA]$ would act as gauge symmetries. They would leave invariant also Kähler function in the sector WCW_n defined by n. This would define a hierarchy of sub- WCWs of the WCW assignable to given CD.

   The sector WCW$_n$ could correspond to extensions of rationals with dimension n, and one would have inclusion hierarchies consisting of sequences of n_i with n_i dividing n_{i+1}. These inclusion hierarchies would naturally correspond to those for hyper-finite factors of type II$_1$ [K99].

4.4 Cognitive representations and algebraic geometry

The general vision about cognition is realized in terms of adelic physics as physics of sensory experience and cognition [L42, L41]. Rational points and their generalization as ratios of algebraic integers for geometric objects would define cognitive representations as points common for real and various p-adic variants of the space-time surface. The finite-dimensionality for induced p-adic extensions allows also extensions of rationals involving root of e and its powers. This picture applies both at space-time level, embedding space level, and at the level of space-time surfaces but basically reduces to embedding space level. Hence counting of the (generalized) rational points for geometric objects would be determination of the cognitive representability.

4.4.1 Cognitive representations as sets of generalized rational points

The set of rational points depends on the coordinates chosen and one can argue that one must allow different cognitive representations and classify them according to their effectiveness.

How uniquely the M^8_c coordinates can be chosen?

1. Polynomial property allows only linear transformations of the complex octonionic coordinates with coefficients which belong to the extension of rationals used. This poses extremely strong restrictions on the allowed representations once the quaternionic moduli defining a foliation of M^4_0 is chosen. One has therefore moduli space of quaternionic structures. One must also fix the time axis in M^4 assignable to real octonions.
2. One can also define several inequivalent octonionic structures and associate a moduli space to these. The moduli space for octonionic structures would correspond to the space of $M^4_0 \subset M^8$s as quaternionic planes containing fixed M^2_0. One can allow even allow Lorentz transforms mixing real and imaginary octonionic coordinates. It seems that these moduli are not relevant at the level of H.

What could the precise definition of rationality?

1. The coordinates of point are rational in the sense defined by the extension of rationals used. Suppose that one considers parametric representations of surfaces as maps from space-time surface to embedding space. Suppose that one uses as space-time coordinates subset of preferred coordinates for embedding space. These coordinate changes cannot be global and one space-time surface decomposes to regions in which different coordinates apply.
2. The coordinate transformations between over-lapping regions are birational in the sense that both the map and its inverse are in terms of rational functions. This makes the notion of rationality global.
3. When cognitively easy rational parametric representations are possible? For algebraic curves with $g \geq 2$ in CP_2 represented as zeros of polynomials this cannot the case since the number

of rational points is finite for instance for $g \geq 2$ surfaces. There is simple explanation for this. Solving second complex coordinate in terms of the other one gives it as an algebraic function for $g \geq 2$: this must be the reason for the loss of dense set of rational points. For elliptic surfaces $y^2 - x^3 - ax - b = 0$ y^2 is however polynomial of x and one can find rational parametric representation by taking y^2 as coordinate [L33]. For $g = 0$ one has linear equations and one obtains dense set of rational points. For conic sections one can also have dense set of rational points but not always. Generalizing from this it would seem that the failure to have rational parametric representation is the basic reason for the loss of dense set of rational points.

This picture does not work for general surfaces but generalizes for algebraic varieties defined by several polynomial equations. The co-dimension $d_c = 1$ case is however unique and the most studied one since for several polynomial equations one encounters technical difficulties when the intersection of the surfaces defined by the d_c polynomials need not be complete for $d_c > 1$. In the recent situation one has $d_c = 4$ but octonion analyticity could be powerful enough symmetry to solve the problem of non-complete intersections by eliminating them or providing a physical interpretation for them.

4.4.2 Cognitive representations assuming $M^8 - H$ duality

Many questions should be answered.

1. Can one generalize the results applying to algebraic varieties? Could the general vision about rational and potentially dense set of rational points generalize?. At M^8 side the description of space-time surfaces as algebraic varieties indeed conforms with this picture. Could one understand SH from the fact that real analyticity octonionic polynomials are determined by ordinary polynomial real coordinate completely? In information theoretic sense sense SH reduces to 1-D holography and the polynomial property makes the situation effectively discrete since finite number of points of real axis allows to determine the octonionic polynomial completely! It is a pity that one cannot measure octonionic polynomial directly!
2. Also the notion of Zariski dimension should make sense in TGD at M^8 side. Preferred extremals define the notion of closed set for given CD at M^8 side? It would indeed seem that one define Zariski topology at the level of M^8_c. Zariski topology would require 4-surfaces, string world sheets, or partonic 2-surfaces and even 1-D curves. This picture conforms with the recent view about TGD and resembles the M-theory picture, where one has branes. SH suggests that the analog of Zariski dimension of space-time surface reduces to that for strings world sheets and partonic 2-surfaces and that even these are analogous to 1-D curves by complex analyticity. Integrability of TGD and preferred extremal property would indeed suggest simplicity.
   $M^8 - H$ hypothesis suggests that these conjectures make sense also at H side. String world sheets, partonic 2-surface, space-like 3-surfaces at the ends of space-time surface at boundaries of CD, and light-like 3-surfaces correspond to closed sets also at the level of WCW in the topology most natural for WCW.
3. Also the problems related to Minkowskian signature could be solved. String world sheets are problematic because of the Minkowskian signature. They however have the topology of disk plus handles suggesting immediately a vision about cognitive representations in terms of rational points. One can can complexify string world sheets and it seems possible to apply the results of algebraic geometry holding true in Euclidian signature. This would be analogous to the Wick rotation used in QFTs and also in twistor Grassmann approach.
4. What about algebraic geometrization of the twistor lift? How complex are twistor spaces of M^4, CP_2 and space-time surface? How can one generalize twistor lift to the level of M^8. S^2 bundle structure and the fact that S^2 allows a dense set of rational suggests that the complexity of twistor space is that of space-time surface itself so that the situation actually reduces to the level of space-time surfaces.

Suppose one accepts $M^8 - H$ duality requiring that the tangent space of space-time surface at given point x contains $M^2(x)$ such that $M^2(x)$ define an integrable distribution giving rise to string world sheets and their orthogonal complements give rise to partonic 2-surfaces. This would give rise to a foliation of the space-time surface by string world sheets and partonic 2-surface conjecture on basis of the properties of extremals of Kähler action. As found these foliations could correspond to quaternion structures that is allowed choices of quaterionic coordinates.

Should one define cognitive representations at the level of M^8 or at the level of $M^4 \times CP_2$? Or both? For M^8 option the condition that space-time point belongs to an extension of rationals applies at the level of M^8 coordinates. For $M^4 \times CP_2$ option cognitive representations are at the level of M^4 and CP_2 parameterizing the points of M^4 and their tangent spaces. The formal study of partial differential equations alone does not help much in counting the number of rational points. One can define cognitive representation in very many ways, and some cognitive representation could be preferred only because they are more efficient than others. Hence both cognitive representations seems to be acceptable.

Some cognitive representations are more efficient than others. General coordinate invariance (GCI) at the level of cognition is broken. The precise determination of cognitive efficiency is a challenge in itself. For instance, the use of coordinates for which coordinate lines are orbits of subgroups of the symmetry group should be highly efficient. Only coordinate transformations mediated by bi-rational maps can take polynomial representations to polynomial representations. It might well be that only a rational (in generalized sense) sub-group G_2 of octonionic automorphisms is allowed. For rational surfaces allowing parametric representation in terms of polynomial functions the rational points form a dense set.

The cognitive resolution for a dense set of rational points is unrealistically high since cognitive representation would contain infinite number of points. Hence one must tighten the notion of cognitive representation. The rational points must contain a fermion. Fermions are indeed are identified as correlates for Boolean cognition [K20]. This would suggests a view in which cognitive representations are realized at the light-like orbits of partonic 2-surfaces at which Minkowskian associative and Euclidian co-associative space-time surfaces meet. The general wisdom is that rational points are localized to lower-dimensional sub-varieties (Bombieri-Lang conjecture): this conforms with the view that fermion lines reside at the orbits of partonic 2-surfaces.

4.4.3 Are the known extremals in H easily cognitively representable?

Suppose that one takes TGD inspired adelic view about cognition seriously. If cognitive representations correspond to rational points for an extension of rationals, then the surfaces allowing large number of this kind of points are easily representable cognitively by adding fermions to these points. One could even speculate that mathematical cognition invents those geometric objects, which are easily cognitively representable and thus have a large number of rational points.

Could the known extremals of twistor lift be cognitively easy?

Also TGD is outcome of mathematical cognition. Could the known extremals of the twistor lift of Kähler action be cognitively easy? This is suggested by the fact that even such a pariah class theoretician as I am have managed to discover then! Positive answer could be seen as support for the proposed description of cognition!

1. If one believes in $M^8 - H$ duality and the proposed identification of associative and co-associative space-time surfaces in terms of algebraic surfaces in octonionic space M^8_c, the generalization of the results of algebraic geometry should give overall view about the cognitive representations at the level of M^8. In particular, surfaces allowing rational parametric representation (polynomials would have rational coefficients) would allow dense set or rational points since the images of rational points are rational. Rationals are understood here as ratios of algebraic integers in extension of rationals.

2. Also for H the existence of parameter representation using preferred H-coordinates and rational functions with rational coefficients implies that rational points are dense. If $M^8 - H$ correspondence maps the parametric representations in terms of rational functions to similar representations, dense set of rational points is preserved in the correspondence. There is however no obvious reason why $M^8 - H$ duality should have this nice property.
One can even play with the idea that the surfaces, which are cognitively difficult at the M^8 side, might be cognitively easy at H-side or vice versa. Of course, if the explicit representation as algebraic functions makes sense at M^8 side, this side looks cognitively ridiculously easy

as compared to H side. The preferred extremal property and SH can however change the situation.

3. At M^8 side and for a given point of M^4 there are several points of E^4 (or vice versa) if the degree of the polynomial is larger than $n = 1$ so that for the image of the surface H there are several CP_2 points for a given point of M^4 (or vice versa) depending on the choice of coordinates. This is what the notion of the many-sheeted space-time predicts.
4. The equations for the surface at H side are obtained by a composite map assigning first to the coordinates of $X^4 \subset M^8$ point of $M^4 \times E^4$, and then assigning to the points of $X^4 \subset M^8$ CP_2 coordinates of the tangent space of the point. At this step the slightly non-local tangent space information is fed in and the surfaces in $M^4 \times CP_2$ cannot be given by zeros of polynomials. The indeed satisfy instead of algebraic equations partial differential equations given by the Kähler action for the twistor lift TGD. Algebraic equations instead of partial differential equations suggests that the M^8 representation is much simpler than H-representation. On the other hand, reduction to algebraic equations at M^8 side could have interpretation in terms of the conjectured complete integrability of TGD [K8, K91].

Testing the idea about self-reference

In any case, it is possible to test the idea about self-reference by looking whether the known extremals in H are cognitively easy and even have a dense set of rational points in natural coordinates. Here I will consider the situation at the level of $M^4 \times CP_2$. It was already found that the known extremals can have inverse images in M^8.

1. Canonically imbedded M^4 with linear coordinates and constant CP_2 coordinates rational is the simple example about preferred extremal and it seems that TGD based cosmology at microscopic relies on these extremals. In this case it is obvious that one has a dense set of rational points at both sides. Could this somehow relate to the fact that physics as physics M^4 was discovered before general relativity?
   Canonically imbedded M^4 corresponds to a first order octonionic polynomial for which imaginary part is put to constant so that tangent space is same everywhere and corresponds to a constant CP_2 coordinate.
2. CP_2 type extremals have 4-D CP_2 projection and light-like geodesic line of M^4 as M^4 projection. One can choose the time parameter as a function of CP_2 coordinates in infinitely many ways. Clearly the rational points are dense in any CP_2 coordinates.
3. Massless extremals (MEs) are given as zeros of arbitrary functions of CP_2 coordinates and 2 M^4 coordinates representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant. In the general case light-like direction would define tangent space of string world sheet giving rise also to a distribution of ortogonal polarization planes. This is consistent with the general properties of the M^8 representation and corresponds to the decomposition of quaternionic tangent plane to complex plane and its complement. One can ask whether one should allow only polynomials with rational coefficients as octonionic polynomials.
4. String like objects $X^2 \times Y^2$ with $X^2 \subset M^4$ a minimal surface and Y^2 complex or Lagrangian surface of CP_2 are also basic extremals and their deformations in M^4 directions are expected to give rise to magnetic flux tubes.
   If Y^2 is complex surface with genus $g = 0$ rational points are dense. Also for $g = 1$ one obtains a dense set of rational points in some extension of rationals. For elliptic curves one has lattice of rational points. What happens for Lagrangian surfaces Y^2? In this case one does not have complex curves but real co-dimension 2 surfaces. There is no obvious objection why these surfaces would not be possible.
5. What about string world sheets? If the string world is static $M^2 \subset M^4$ one has a dense set of rational points. One however expects something more complex. If the string world sheet is rational map M^2 to its orthogonal complement E^2 one has rational surface. For rotating strings this does not make sense except for certain period of time. If the choice of the quaternion structure corresponds to a choice of minimal surface in M^4 as integrable distribution for $M^2(x)$, the coordinates associated with the Hamilton-Jacobi structure could make the situation simple.

If one restricts the consideration the intersections of partonic 2-surfaces and string world sheets at two boundaries of CD the situation simplifies and the question is only about the rationality of the M^4 coordinates at rational points of $Y^2 \subset CP_2$. This would simplify the situation enormously and might even allow to use existing knowledge.

6. The slicing of of space-time surfaces by string world sheets and partonic 2-surfaces required by Hamilton-Jacobi structure could be seen as a fibering analogous to that possessed by elliptic surfaces. This suggest that M^8 counterparts of spacetime surfaces are not of general type in Kodaira classification and that the number of rational points can be large. If the existence of Hamilton-Jacobi structure does not allow handles, this factor would be cognitively simple. This would however suggests that fermion number is not localized at the ends of strings only - as assumed in the construction of scattering amplitudes inspired by twistor Grassmann approach [K35] - but also to the interior of the light-like curves inside string world sheets.

4.4.4 Twistor lift and cognitive representations

What about twistor lift of TGD replacing space-time surfaces with their twistor spaces. Consider first M^8 side.

1. At M^8 side S^2 seems to introduce nothing new. One might expect that the situation does not change at H-side since space-time surfaces are obtained essentially by dimensional reduction and the possible problem relates to the choice of base space as section of is twistor bundle and the embedding of space-time as base space could have singularities at the boundary of Euclidian and Minkowskian space-time regions as discussed in [L33].
   At the side of M^8 the proposed induction of twistor structure is just a projection of the twistor sphere S^6 to its geodesic sphere and one has 4-D moduli space for geodesic spheres $S^2 \subset S^6$. If one interprets the choice of $S^2 \subset S^6$ as as a section in the moduli space, the moduli of S^2 can depend on the point of space-time surface. Note that there are is also a position dependent choice of preferred point of S^2 representing Kähler form, and this choice is good candidate for giving rise to Hamilton-Jacobi structures with position dependent M^2.
2. The notion of Kodaira dimension is defined also for co-dimension 4 algebraic varieties in M^8_c. The cognitively easiest spacetime surfaces would allow rational parametric representation with complex coordinates serving as parameters. If this is not possible, one has algebraic functions, which makes the situation much more complex so that the number of rational points would be small.
3. For some complex enough extensions of rationals the set of rational points can be dense. $g \geq 2$ genera are basic example and one expects also in more general case that polynomials involving powers larger than $n = 4$ make the situation problematic. The condition that real or imaginary part of real analytic octonionic polynomial is in question is a strong symmetry expected to faciliate cognitive representability.
4. The general intuitive wisdom from algebraic geometry is that the rational points are dense only in lower-dimensional sub-varieties (Bombieri-Lang and Vojta conjectures mentioned in the first section). The general vision inspired by SH and the proposal for the construction of twistor amplitudes indeed is that the algebraic points (rational in generalized sense) defining cognitive representations are associated with the intersections of string world sheets and partonic 2-surfaces to which fermions are assigned. This would suggest that partonic 2-surfaces and string world sheets contain the cognitive representation, which under additional conditions can contain very many points.
5. An interesting question concerns the M^8 counterparts of partonic 2-surfaces as space-time regions with Minkowskian and Euclidian signature. The partonic orbits representing the boundaries between these regions should be mapped to each other by M^8-H duality. This conforms with the fact that induced metric must have degenerate signature $(0,-1,-1,-1)$ at partonic orbits. Can one assume that the topologies of partonic 2-surfaces at two sides are identical? Consider partonic 2-surface of genus g in $M^4 \times CP_2$ - say at the boundary of CD. It should be inverse image of a 2-surface in $M^4 \times E^4$ such that the tangent space of this surface labelled by CP_2 coordinates is mapped to a 2-surface in $M^4 \times CP_2$. If the inverse of $M^8 - H$ correspondence is continuous one expects that g is preserved.

Consider next the H-side. There is a conjecture that for Cartesian product the Kodaira dimension is sum $d_K = \sum_i d_{K,i}$ of the Kodaira dimensions for factors. Suppose that CP_1 fiber as surface in the 12-D twistor bundle $T(M^4) \times T(CP_2)$ has Kodaira dimension $d_K(CP_1) = -\infty$ (it is expected to be rational surface) then the fact that the bundle decomposes to Cartesian product locally and rational points are pairs of rational points in the factors, is indeed consistent with the proposal. S^2 would give dense set of rational points in S^2 and the bundle would have infinite number of rational points.

In TGD context, it is however space-time surface which matters. Space-time surface as section of the bundle would not however have a dense set of points in the general case and the relevant Kodaira dimension be $d_K = d_K(X^4)$. One can of course ask whether the space-time surface as an algebraic section (not many of them) of the twistor bundle could chosen to be cognitively simple.

4.4.5 What does cognitive representability really mean?

The following considerations reflect the ideas inspired by Face Book debate with Santeri Satama (SS) relating to the notion of number and the notion of cognitive representation.

SS wants to accept only those numbers that are constructible, and SS mentioned the notion of demonstrability due to Gödel. According to my impression demonstrability means that number can be constructed by a finite algorithm or at least that the information needed to construct the number can be constructed by a finite algorithm although the construction itself would not be possible as digit sequence in finite time. If the constructibility condition is taken to extreme, one is left only with rationals.

As a physicists, I cannot consider starting to do physics armed only with rationals: for instance, continuous symmetries and the notion of Riemann manifold would be lost. My basic view is that we should identify the limitations of cognitive representability as limitations for what can exist. I talked about cognitive representability of numbers central in the adelic physics approach to TGD. Not all real numbers are cognitively representable and need not be so.

Numbers in the extensions of rationals would be cognitively representable as points with coordinates in an extension of rationals. The coordinates themselves are highly unique in the octonionic approach to TGD and different coordinates choices for complexified octonionic M^8 are related by transformations changing the moduli of the octonion structure. Hence one avoids problems with general coordinate invariance). Not only algebraic extensions of rationals are allowed. Neper number e is an exceptional transcendental in that e^p is p-adic number and finite-D extensions of p-adic numbers by powers for root of e are possible.

My own basic interest is to find a deeper intuitive justification for why algebraic numbers shoud be cognitively representable. The naïve view about cognitive representability is that the number can be produced in a finite number of steps using an algorithm. This would leave only rationals under consideration and would mean intellectual time travel to ancient Greece.

Situation changes if one requires that only the information about the construction of number can be produced in a finite number of steps using an algorithm. This would replace construction with the recipe for construction and lead to a higher abstraction level. The concrete construction itself need not be possible in a finite time as bit sequence but could be possible physically ($\sqrt{2}$ as a diagonal of unit square, one can of course wonder where to buy ideal unit squares). Both number theory and geometry would be needed.

Stern-Brocot tree associated with partial fractions indeed allows to identify rationals as finite paths connecting the root of S-B tree to the rational in question. Algebraic numbers can be identified as infinite periodic paths so that finite amount of information specifies the path. Transcendental numbers would correspond to infinite non-periodic paths. A very close analogy with chaos theory suggests itself.

Demonstrability viz. cognitive representability

SS talked about demonstrable numbers. According to Gödel demonstrable number would be representable by a formula G, which is provable in some axiom system. I understand this that G would give a recipe for constructing that number. In computer programs this can even mean infinite loop, which is easy to write but impossible to realize in practice. Here comes the possibility that demonstrability does not mean constructibility in finite number of steps but only a finite recipe for this.

The requirement that all numbers are demonstrable looks strange to me. I would talk about cognitive representability and reals and p-adic number fields emerge unavoidably as prerequisites for this notion: cognitive representation must be about something in order to be a representation.

About precise construction of reals or something bigger - such as surreals - containing them, there are many views and I am not mathematician enough to take strong stance here. Note however that if one accepts surreals as being demonstrable (I do not really understand what this could mean) one also accept reals as such. These delicacies are not very interesting for the formulation of physics as it is now.

The algorithm defining G defines a proof. But what does proof mean? Proof in mathematical sense would reduce in TGD framework be a purely cognitive act and assignable to the p-adic sectors of adele. Mathematicians however tend to forget that for physicist the demonstration is also experimental. Physicist does not believe unless he sees: sensory perception is needed. Experimental proofs are what physicists want. The existence of $\sqrt{2}$ as a diagonal of unit square is experimentally demonstrable in the sense of being cognitively representable but not deducible from the axioms for rational numbers. As a physicist I cannot but accept both sensory and cognitive aspects of existence.

Instead of demonstrable numbers I prefer to talk about cognitively representable numbers.

1. All numbers are cognizable (p-adic) or sensorily perceivable (real). These must form continua if one wants to avoid problems in the construction of physical theories, where continuous symmetries are in a key role.
   Some numbers but not all are also *cognitively representable* that is being in the intersection reals and p-adics - that is in extension of rationals if one allows extensions of p-adics induced by extensions of rationals. This generalizes to intersection of space-time surfaces with real/p-adic coordinates, which are highly unique linear coordinates at octonionic level so that objections relating to a loss of general coordinate invariance are circumvented. General coordinate transformations reduce to automorphisms of octonions.
   The relationship to the axiom of choice is interesting. Should axiom of choice be restricted to the points of complexified octonions with coordinates in extensions of rationals? Only points in the extensions could be selected and this selection process would be physical in the sense that fermions providing realization of quantum Boolean algebra would reside at these points [K20]. In preferred octonionic coordinates the M^8 coordinates of these points would be in given extension of rationals. At the limit of algebraic numbers these points would form a dense set of reals.
   Remark: The spinor structure of "world of classical worlds" (WCW) gives rise to WCW spinors as fermionic Fock states at given 3-surface. In ZEO many-fermion states have interpretation in terms of superpositions of pairs of Boolean statements $A \rightarrow B$ with A and B represented as many-fermion states at the ends of space-time surface located at the opposite light-like boundaries of causal diamond (CD). One could say that quantum Boolean logic emerges as square root of Kähler geometry of WCW.
   At partonic 2-surfaces these special points correspond to points at which fermions can be localized so that the representation is physical. Universe itself would come in rescue to make representability possible. One would not anymore try to construct mathematics and physics as distinct independent disciplines.
   Even observer as conscious entity is necessarily brought into both mathematics and physics. TGD Universe as a spinor field in WCW is re-created state function reduction by reduction and evolves: evolution for given CD corresponds to the increase of the size of extension of rationals in statistical sense. Hence also mathematics with fixed axioms is replaced with a q dynamical structure adding to itself new axioms discovery by discovery [L43, L42].

2. Rationals as cognitively representable numbers conforms with naïve intuition. One can however criticize the assumption that also algebraic numbers are such. Consider $\sqrt{2}$: one can simply define it as length of diagonal of unit square and this gives a meter stick of length $\sqrt{2}$: one can represent any algebraic number of form $m + n\sqrt{2}$ by using meter stricks with length of 1 and $\sqrt{2}$. Cognitive representation is also sensory representation and would bring in additional manner to represent numbers.

Note that algebraic numbers in n-dimensional extension are points of n-dimensional space and their cognitive representations as points on real axis obtained by using the meter sticks assignable to the algebraic numbers defining base vectors. This should generalize to the roots arbitrary polynomials with rational or even algebraic coefficients. Essentially projection form n-D extension to 1-D real line is in question. This kind of projection might be important in number theoretical dynamics. For instance, quasi-periodic quasi-crystals are obtained from higher-D periodic crystals as projections.

n-D algebraic extensions of p-adics induced by those of rationals might also related to our ability to imagine higher-dimensional spaces.

3. In TGD Universe cognitive representability would emerge from fundamental physics. Extensions of rationals define a hierarchy of adeles and octonionic surfaces are defined as zero loci for real or imaginary parts (in quaternionic sense) of polynomials of real argument with coefficients in extension continued to octonionic polynomials [L36]. The zeros of real polynomial have a direct physical interpretation and would represent algebraic numbers physically. They would give the temporal positions of partonic 2-surfaces representing particles at light-like boundary of CD.
4. Note that all calculations with algebraic numbers can be done without using approximations for the genuinely algebraic numbers defining the basis for the extension. This actually simplifies enormously the calculation and one avoids accumulating errors. Only at the end one represents the algebraic units concretely and is forced to use rational approximation unless one uses above kind of cognitive representation.

For these reasons I do not feel any need to get rid of algebraics or even transcendentals. Sensory aspects of experience require reals and cognitive aspects of experience require p-adic numbers fields and one ends up with adelic physics. Cognitive representations are in the intersection of reality and various p-adicities, something expressible as formulas and concrete physical realizations or at least finite recipes for them.

What the cognitive representability of algebraic numbers could mean?

Algebraic numbers should be in some sense simple in order to be cognitively representable.

1. For rationals representation as partial fractions produces the rational number by using a finite number of steps. One starts from the top of Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq`) and moves to right or left at each step and ends up to the rational number appearing only once in S-B tree.
2. Algebraic numbers cannot be produced in a finite number of steps. During the discussion I however realized that one can produce the information needed to construct the algebraic number in a finite number of steps. One steps to a new level of abstraction by replacing the object with the information allowing to construct the object using infinite number of steps but repeating the same sub-algorithm with finite number of steps: infinite loop would be in question.

   Similar abstraction takes place as one makes a step from the level of space-time surface to the level of WCW. Space-time surface with a continuum of points is represented by a finite number of WCW coordinates, in the octonionic representation of space-time surface by the coefficients of polynomial of finite degree belonging to an extension of rationals [L36]. Criticality conditions pose additional conditions on the coefficients. Finite number of algebraic points at space-time surface determines the entire space-time surface under these conditions! Simple names for complex things replacing the complex things is the essence of cognition!
3. The interpretation for expansions of numbers in given base suggests an analog with complexity theory and symbolic dynamics associated with division. For cognitively representable numbers the information about this dynamics should be coded by an algorithm with finite steps. Periodic orbit or fixed point orbit would be the dynamical analog for simplicity. Non-periodic orbit would correspond to complexity and possibly also chaos.

These ideas led to two approaches in attempt to understand the cognitive representability of algebraic numbers.

1. Generalized rationals in extensions of rationals as periodic orbits for the dynamics of division

The first approach allows to represent ratios of algebraic integers for given extension using periodic expansion in the base so that a finite amount of information is needed to code the number if one accepts the numbers defining the basis of the algebraic extension as given.

1. Rationals allow periodic expansion with respect to any base. For p-adic numbers the base is naturally prime. Therefore the information about rational is finite. One can see the expansion as a periodic orbit in dynamics determining the expansion by division m/n in given base. Periodicity follows from the fact that the output of the division algorithm for a given digit has only a finite number of outcomes so that the process begins to repeat itself sooner or later.
2. This generalizes to generalized rationals in given extension of rationals defined as ratios of algebraic integers. One can reduce the division to the construction of the expansion of ordinary rational identified as number theoretic norm $|N|$ of the denominator in the extension of rationals considered.
   The norm $|N|$ of N is the determinant $|N| = det(N)$ for the linear map of extension induced by multiplication with N. $det(N)$ is ordinary (possibly p-adic) integer. This is achieved by multiplying $1/N$ by $n-1$ conjugates of N both in numerator and denominator so that one obtains product of $n-1$ conjugates in the numerator and $det(N)$ in the denominator. The computation of $1/N$ as series in the base used reduces to that in the case of rationals.
3. One has now periodic orbits in n-dimensional space defined by algebraic extensions which for ordinary rationals reduced to periodic orbits in 1-D space. This supports the interpretation of numbers as orbits of number theoretic dynamics determining the next digit of the generalized rational for given base. This picture also suggests that transcendentals correspond to non-periodic orbits. Some transcendentals could still allow a finite algorithm: in this case the dynamics would be still deterministic. Some transcendentals would be chaotic.
4. Given expansion of algebraic number is same for all extensions of rationals containing the extension in question and the ultimate extension corresponds to algebraic numbers.

The problem of this approach is that the algebraic numbers defining the extension do not have representation and must be accepted as irreducibles.

2. Algebraic numbers as infinite periodic orbits in the dynamics of partial fractions

Second approach is based on partial fractions and Stern-Brocot tree (see `http://tinyurl.com/yb6ldekq`, see also `http://tinyurl.com/yc6hhboo`) and indeed allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. I had managed to not become aware of this possibility and am grateful for SS for mentioning the representation of algebraics in terms of S-B tree.

1. The definition S-B tree is simple: if m/n and m'/n' are any neighboring rationals at given level in the tree, one adds $(m+m')/(n+n')$ between them and obtains in this manner the next level in the tree. By starting from $(0/1)$ and $(1/0)$ as representations of zero and ∞ one obtains $(0/1)(1/1)(1/0)$ as the next level. One can continue in this manner ad infinitum. The nodes of S-B tree represent rational points and it can be shown that given rational appears only once in the tree.
   Given rational can be represented as a finite path beginning from 1/1 at the top of tree consisting of left moves L and right moves R and ending to the rational which appears only once in S-B tree. Rational can be thus constructured by a sequences $R^{a_0}L^{a_1}L^{a_2}....$ characterized by the sequence $a_0; a_1, a_2....$ For instance, $4/11 = 0 + 1/(2+x)$, $x = 1/(1+1/3)$ corresponds to $R^0L^2R^1L^{3-1}$ labelled by $0; 2, 1, 3$.
2. Algebraic numbers correspond to infinite but periodic paths in S-B tree in the sense that some sequence of L:s and R:s characterized by sequences of non-negative integers starts to repeat itself. Periodicity means that the information needed to construct the number is finite.
   The actual construction as a digit sequence representing algebraic number requires infinite amount of time. In TGD framework octonionic physics would come in rescue and construct algebraic numbers as roots of polynomials having concrete interpretations as coordinate values assignable to fermions at partonic 2-surfaces.
3. Transcendentals would correspond to non-periodic infinite sequences of L:s and R:s. This does not exclude the possibility that these sequences are expressible in terms of some rule involving

finite number of steps so that the amount of information would be also now finite. Information about number would be replaced by information about rule.
This picture conforms with the ideas about transition to chaos. Rationals have finite paths. A possible dynamical analog is particle coming at rest due to the dissipation. Algebraic numbers would correspond to periodic orbits possible in presence of dissipation if there is external feed of energy. They would correspond to dynamical self-organization patterns.
Remark: If one interprets the situation in terms of conservative dynamics, rationals would correspond to potential minima and algebraic numbers closed orbits around them.
The assignment of period doubling and p-pling to this dynamics as the dimension of extension increases is an attractive idea. One would expect that the complexity of periodic orbits increases as the degree of the defining irreducible polynomial increases. Algebraic numbers as maximal extension of rationals possibly also containing extension containing all rational roots of e and transcendentals would correspond to chaos.
Transcendentals would correspond to non-periodic orbits. These orbits need not be always chaotic in the sense of being non-predictable. For instance, Neper number e can be said to be p-adically algebraic number (e^p is p-adic integer albeit infinite as real integer). Does the sequence of L:s and R:s allow a formula for the powers of L and R in this case?

4. TGD should be an integrable theory. This suggests that scattering amplitudes involve only cognitive representations as number theoretic vision indeed strongly suggests [L36]. Cognitively representable numbers would correspond to the integrable sub-dynamics [L46]. Also in chaotic systems both periodic and chaotic orbits are present. Complexity theory for characterization of real numbers exists. The basic idea is that complexity is measured by the length of the shortest program needed to code the bit sequences coding for the number.

Surreals and ZEO

The following comment is not directly related to cognitive representability but since it emerged during discussion, I will include it. SS favors surreals (see `http://tinyurl.com/86jatas`) as ultimate number field containing reals as sub-field. I must admit that my knowledge and understanding of surreals is rather fragmentary.

I am agnostic in these issues and see no conflict between TGD view about numbers and surreals. Personally I however like very much infinite primes, integers, and rationals over surreals since they allow infinite numbers to have number theoretical anatomy [K84]. A further reason is that the construction of infinite primes resembles structurally repeated second quantization of the arithmetic number field theory and could have direct space-time correlate at the level of many-sheeted space-time. One ends up also to a generalization of real number. Infinity can be seen as something related to real norm: everything is finite with respect to various p-adic norms.

Infinite rationals with unit real norm and various p-adic norms bring in infinitely complex number theoretic anatomy, which could be even able to represent even the huge WCW and the space of WCW spinor fields. One could speak of number theoretical holography or algebraic Brahman=Atman principle. One would have just complexified octonions with infinitely richly structure points.

Surreals are represented in terms of pairs of sets. One starts the recursive construction from empty set identified as 0. The definition says that the pairs (.|.) of sets defining surreals x and y satisfy $x \leq y$ if the left hand part of x as set is to left from the pair defining y and the right hand part of y is to the right from the pair defining x. This does not imply that one has always $x<y$, $y<x$ or $x=y$ as for reals.

What is interesting that the pair of sets defining surreal x is analogous to a pair of states at boundaries of CD defining zero energy state. Is there a connection with zero energy ontology (ZEO)? One could perhaps say at the level of CD - forgetting everything related to zero energy states - following. The number represented by CD_1 - say represented as the distance between its tip - is smaller than than the number represented by CD_2, if CD_1 is inside CD_2. This conforms with the left and righ rule if left and right correspond to the opposite boundaries of CD. A more detailed definition would presumably say that CD_1 can be moved so that it is inside CD_2.

What makes this also interesting is that CD is the geometric correlate for self, conscious entity, also mathematical mental image about number.

4.5 Galois groups and genes

In an article discussing a TGD inspired model for possible variations of G_{eff} [L51], I ended up with an old idea that subgroups of Galois group could be analogous to conserved genes in that they could be conserved in number theoretic evolution. In small variations such as above variation Galois subgroups as genes would change only a little bit. For instance, the dimension of Galois subgroup would change.

The analogy between subgoups of Galois groups and genes goes also in other direction. I have proposed long time ago that genes (or maybe even DNA codons) could be labelled by $h_{eff}/h = n$. This would mean that genes (or even codons) are labelled by a Galois group of Galois extension (see `http://tinyurl.com/zu5ey96`) of rationals with dimension n defining the number of sheets of space-time surface as covering space. This could give a concrete dynamical and geometric meaning for the notion of gene and it might be possible some day to understand why given gene correlates with particular function. This is of course one of the big problems of biology.

4.5.1 Could DNA sequence define an inclusion hierarchy of Galois extensions?

One should have some kind of procedure giving rise to hierarchies of Galois groups assignable to genes. One would also like to assign to letter, codon and gene and extension of rationals and its Galois group. The natural starting point would be a sequence of so called intermediate Galois extensions E^H leading from rationals or some extension K of rationals to the final extension E. Galois extension has the property that if a polynomial with coefficients in K has single root in E, also other roots are in E meaning that the polynomial with coefficients K factorizes into a product of linear polynomials. For Galois extensions the defining polynomials are irreducible so that they do not reduce to a product of polynomials.

Any sub-group $H \subset Gal(E/K))$ leaves the intermediate extension E^H invariant in element-wise manner as a sub-field of E (see `http://tinyurl.com/y958drcy`). Any subgroup $H \subset Gal(E/K))$ defines an intermediate extension E^H and subgroup $H_1 \subset H_2 \subset ...$ define a hierarchy of extensions $E^{H_1} > E^{H_2} > E^{H_3}...$ with decreasing dimension. The subgroups H are normal - in other words $Gal(E)$ leaves them invariant and $Gal(E)/H$ is group. The order $|H|$ is the dimension of E as an extension of E^H. This is a highly non-trivial piece of information. The dimension of E factorizes to a product $\prod_i |H_i|$ of dimensions for a sequence of groups H_i.

Could a sequence of DNA letters/codons somehow define a sequence of extensions? Could one assign to a given letter/codon a definite group H_i so that a sequence of letters/codons would correspond a product of some kind for these groups or should one be satisfied only with the assignment of a standard kind of extension to a letter/codon?

Irreducible polynomials define Galois extensions and one should understand what happens to an irreducible polynomial of an extension E^H in a further extension to E. The degree of E^H increases by a factor, which is dimension of E/E^H and also the dimension of H. Is there a standard manner to construct irreducible extensions of this kind?

1. What comes into mathematically uneducated mind of physicist is the functional decomposition $P^{m+n}(x) = P^m(P^n(x))$ of polynomials assignable to sub-units (letters/codons/genes) with coefficients in K for a algebraic counterpart for the product of sub-units. $P^m(P^n(x))$ would be a polynomial of degree $n+m$ in K and polynomial of degree m in E^H and one could assign to a given gene a fixed polynomial obtained as an iterated function composition. Intuitively it seems clear that in the generic case $P^m(P^n(x))$ does not decompose to a product of lower order polynomials. One could use also polynomials assignable to codons or letters as basic units. Also polynomials of genes could be fused in the same manner.
2. If this indeed gives a Galois extension, the dimension m of the intermediate extension should be same as the order of its Galois group. Composition would be non-commutative but associative as the physical picture demands. The longer the gene, the higher the algebraic complexity would be. Could functional decomposition define the rule for who extensions and Galois groups correspond to genes? Very naïvely, functional decomposition in mathematical sense would correspond to composition of functions in biological sense.
3. This picture would conform with $M^8 - M^4 \times CP_2$ correspondence [L36] in which the construction of space-time surface at level of M^8 reduces to the construction of zero loci of polynomials

of octonions, with rational coefficients. DNA letters, codons, and genes would correspond to polynomials of this kind.

4.5.2 Could one say anything about the Galois groups of DNA letters?

A fascinating possibility is that this picture could allow to say something non-trivial about the Galois groups of DNA letters.

1. Since $n = h_{eff}/h$ serves as a kind of quantum IQ, and since molecular structures consisting of large number of particles are very complex, one could argue that n for DNA or its dark variant realized as dark proton sequences can be rather large and depend on the evolutionary level of organism and even the type of cell (neuron viz. soma cell). On the other, hand one could argue that in some sense DNA, which is often thought as information processor, could be analogous to an integrable quantum field theory and be solvable in some sense. Notice also that one can start from a background defined by given extension K of rationals and consider polynomials with coefficients in K. Under some conditions situation could be like that for rationals.
2. The simplest guess would be that the 4 DNA letters correspond to 4 non-trivial finite groups with smaller possible orders: the cyclic groups Z_2, Z_3 with orders 2 and 3 plus 2 finite groups of order 4 (see the table of finite groups in `http://tinyurl.com/j8d5uyh`). The groups of order 4 are cyclic group $Z_4 = Z_2 \times Z_2$ and Klein group $Z_2 \oplus Z_2$ acting as a symmetry group of rectangle that is not square - its elements have square equal to unit element. All these 4 groups are Abelian. Polynomial equations of degree not larger than 4 can be solved exactly in the sense that one can write their roots in terms of radicals.
3. Could there exist some kind of connection between the number 4 of DNA letters and 4 polynomials of degree less than 5 for whose roots one an write closed expressions in terms of radicals as Galois found? Could it be that the polynomials obtained by a a repeated functional composition of the polynomials of DNA letters have also this solvability property?
   This could be the case! Galois theory states that the roots of polynomial are solvable by radicals if and only if the Galois group is solvable meaning that it can be constructed from abelian groups using Abelian extensions (see `https://cutt.ly/4RuXmGo`).
   Solvability translates to a statement that the group allows so called sub-normal series $1 < G_0 < G_1 ... < G_k$ such that G_{j-1} is normal subgroup of G_j and G_j/G_{j-1} is an abelian group. An equivalent condition is that the derived series $G \triangleright G^{(1)} \triangleright G^{(2)} \triangleright ...$ in which $j+1$:th group is commutator group of G_j ends to trivial group. If one constructs the iterated polynomials by using only the 4 polynomials with Abelian Galois groups, the intuition of physicist suggests that the solvability condition is guaranteed! Wikipedia article also informs that for finite groups solvable group is a group whose composition series has only factors which are cyclic groups of prime order.
   Abelian groups are trivially solvable, nilpotent groups are solvable, p-groups (having order, which is power prime) are solvable and all finite p-groups are nilpotent. Every group with order less than 60 elements is solvable. Fourth order polynomials can have at most S_4 with 24 elements as Galois groups and are thus solvable. Fifth order polynomials can have the smallest non-solvable group, which is alternating group A_5 with 60 elements as Galois group and in this case are not solvable. S_n is not solvable for $n > 4$ and by the finding that S_n as Galois group is favored by its special properties (see `https://arxiv.org/pdf/1511.06446.pdf`).
   A_5 acts as the group icosahedral orientation preserving isometries (rotations). Icosahedron and tetrahedron glued to it along one triangular face play a key role in TGD inspired model of bio-harmony and of genetic code [L16, L53]. The gluing of tetrahedron increases the number of codons from 60 to 64. The gluing of tetrahedron to icosahedron also reduces the order of isometry group to the rotations leaving the common face fixed and makes it solvable: could this explain why the ugly looking gluing of tetrahedron to icosahedron is needed? Could the smallest solvable groups and smallest non-solvable group be crucial for understanding the number theory of the genetic code.

An interesting question inspired by $M^8 - H$-duality [L36] is whether the solvability could be posed on octonionic polynomials as a condition guaranteeing that TGD is integrable theory in number theoretical sense or perhaps following from the conditions posed on the octonionic polynomials. Space-time surfaces in M^8 would correspond to zero loci of real/imaginary parts (in quaternionic sense) for octonionic polynomials obtained from rational polynomials by analytic continuation. Could solvability relate to the condition guaranteeing M^8 duality boiling down to the condition that the tangent spaces of space-time surface are labelled by points of CP_2. This requires that tangent or normal space is associative (quaternionic) and that it contains fixed complex subspace of octonions or perhaps more generally, there exists an integrable distribution of complex subspaces of octonions defining an analog of string world sheet.

What could the interpretation for the events in which the dimension of the extension of rationals increases? Galois extension is extensions of an extension with relative Galois group $Gal(rel) = Gal(new)/Gal(old)$. Here $Gal(old)$ is a normal subgroup of $Gal(new)$. A highly attractive possibility is that evolutionary sequences quite generally (not only in biology) correspond to this kind of sequences of Galois extensions. The relative Galois groups in the sequence would be analogous to conserved genes, and genes could indeed correspond to Galois groups [K26] [L36]. To my best understanding this corresponds to a situation in which the new polynomial P_{m+n} defining the new extension is a polynomial P_m having as argument the old polynomial $P_n(x)$: $P_{m+n}(x) = P_m(P_n(x))$.

What about the interpretation at the level of conscious experience? A possible interpretation is that the quantum jump leading to an extension of an extension corresponds to an emergence of a reflective level of consciousness giving rise to a conscious experience about experience. The abstraction level of the system becomes higher as is natural since number theoretic evolution as an increase of algebraic complexity is in question.

This picture could have a counterpart also in terms of the hierarchy of inclusions of hyperfinite factors of type II_1 (HFFs). The included factor M and including factor N would correspond to extensions of rationals labelled by Galois groups $Gal(M)$ and $Gal(N)$ having $Gal(M) \subset Gal(M)$ as normal subgroup so that the factor group $Gal(N)/Gal(M)$ would be the relative Galois group for the larger extension as extension of the smaller extension. I have indeed proposed [L54] that the inclusions for which included and including factor consist of operators which are invariant under discrete subgroup of $SU(2)$ generalizes so that all Galois groups are possible. One would have Galois confinement analogous to color confinement: the operators generating physical states could have Galois quantum numbers but the physical states would be Galois singlets.

4.6 Could the precursors of perfectoids emerge in TGD?

In algebraic-geometry community the work of Peter Scholze [A174] (see http://tinyurl.com/y7h2sms7) introducing the notion of perfectoid related to p-adic geometry has raised a lot of interest. There are two excellent popular articles about perfectoids: the first article in AMS (see http://tinyurl.com/ydx38vk4) and second one in Quanta Magazine (see http://tinyurl.com/yc2mxxqh). I had heard already earlier about the work of Scholze but was too lazy to even attempt to understand what is buried under the horrible technicalities of modern mathematical prose. Rachel Francon re-directed my attention to the work of Scholze (see http://tinyurl.com/yb46oza6). The work of Scholze is interesting also from TGD point of view since the construction of p-adic geometry is a highly non-trivial challenge in TGD.

1. One should define first the notion of continuous manifold but compact-open characteristic of p-adic topology makes the definition of open set essential for the definition of topology problematic. Even single point is open so that hopes about p-adic manifold seem to decay to dust. One should pose restrictions on the allowed open sets and p-adic balls with radii coming as powers of p are the natural candidates. p-Adic balls are either disjoint or nested: note that also this is in conflict with intuitive picture about covering of manifold with open sets. All this strangeness originates in the special features of p-adic distance function known as ultra-metricity. Note however that for extensions of p-adic numbers one can say that the Cartesian products of p-adic 1-balls at different genuinely algebraic points of extension along particular axis of extension are disjoint.
2. At level of M^8 the p-adic variants of algebraic varieties defined as zero loci of polynomials do not seem to be a problem. Equations are algebraic conditions and do not involve derivatives like partial differential equations naturally encountered if Taylor series instead of polynomials

are allowed. Analytic functions might be encountered at level of $H = M^4 \times CP_2$ and here p-adic geometry might well be needed.
The idea is to define the generalization of p-adic algebraic geometry in terms of p-adic function fields using definitions very similar to those used in algebraic geometry. For instance, generalization of variety corresponds to zero locus for an ideal of p-adic valued function field. p-Adic ball of say unit radius is taken as the basic structure taking the role of open ball in the topology of ordinary manifolds. This kind of analytic geometry allowing all power series with suitable restrictions to function field rather than allowing only polynomials is something different from algebraic geometry making sense for p-adic numbers and even for finite fields.

3. One would like to generalize the notion of analytic geometry even to the case of number fields with characteristic p (p-multiple of element vanishes), in particular for finite fields F_p and for function fields $F_p[t]$. Here one encounters difficulties. For instance, the factorial $1/n!$ appearing as normalization factor of forms diverges if p divides it. Also the failure of Frobenius homomorphism to be automorphism for $F_p[t]$ causes difficulties in the understanding of Galois groups.

The work of Scholze has led to a breakthrough in unifying the existing ideas in the new framework provided by the notion of perfectoid. The work is highly technical and involves infinite-D extension of ordinary p-adic numbers adding all powers of all roots p^{1/p^m}, $m = 1, 2....$ Formally, an extension by powers of p^{1/p^∞} is in question.

This looks strange at first but it guarantees that all p-adic numbers in the extension have p:th roots, one might say that one forms a p-fold covering/wrapping of extension somewhat analogous to complex numbers. This number field is called perfectoid since it is perfect meaning that Frobenius homomorphism $a \to a^p$ is automorphism by construction. $Frob$ is injection always and by requiring that p:th roots exist always, it becomes also a surjection.

This number field has same Galois groups for all of its extensions as the function field $G[t]$ associated with the union of function fields $G = F_p[t^{1/p^m}]$. Automorphism property of $Frob$ saves from the difficulties with the factorization of polynomials and p-adic arithmetics involving remainders is replaced with purely local modulo p arithmetics.

4.6.1 About motivations of Scholze

Scholze has several motivations for this work. Since I am not a mathematician, I am unable to really understand all of this at deep level but feel that my duty as user of this mathematics is at least to try!

1. Diophantine equations is a study of polynomial equations in several variables, say $x^2+2xy+y = 0$. The solutions are required to be integer valued: in the example considered $x = y = 0$ and $x = -y = -1$ is such a solution. For integers the study of the solution is very difficult and one approach is to study these equations modulo p that is reduced the equations to finite field G_p for any p. The equations simplify enormously since ane has $a^p = a$ in F_p. This identity in fact defines so called Frobenius homomorphism acting as automorphism for finite fields. This holds true also for more complex fields with characteristic p say the ring $F_p[t]$ of power series of t with coefficients in F_p.
The powers of variables, say x, appearing in the equation is reduced to at most x^{p-1}. One can study the solutions also in p-adic number fields. The idea is to find first whether finite field solution, that is solution modulo p, does exist. If this is the case, one can calculate higher powers in p. If the series contains finite number of terms, one has solution also in the sense of ordinary integers.

2. One of the related challenges is the generalization of the notion of variety to a geometry defined in arbitrary number field. One would like to have the notion of geometry also for finite fields, and for their generalizations such as $F_p[t]$ characterized by characteristic p ($px = 0$ holds true for any element of the field). For fields of characteristic 1 - extensions of rationals, real, and p-adic number fields) $xp = 0$ not hold true for any $x \neq 0$. Any field containing rationals as sub-field, being thus local field, is said to have characteristic equal to 1. For local fields the challenge is relatively easy.

3. The situation becomes more difficult if one wants a generalization of differential geometry. In differential geometry differential forms are in a key role. One wants to define the notion

of differential form in fields of characteristic p and construct a generalization of cohomology theory. This would generalize the notion of topology to p-adic context and even for finite fields of finite character. A lot of work has been indeed done and Grothendieck has been the leading pioneer.

The analogs of cohomology groups have values in the field of p-adic numbers instead of ordinary integers and provide representations for Galois groups for the extensions of rationals inducing extensions of p-adic numbers and finite fields.

In ordinary homology theory non-contractible sub-manifolds of various dimensions correspond to direct summands Z (group of integers) for homology groups and by Poincare duality those for cohomology groups. For Galois groups Z is replaced with Z_N. N depends on extension to which Galois group is associated and if N is divisible by p one encounters technical problems.

There are many characteristic p- and p-adic cohomologies such as etale cohomology, chrystalline cohomology, algebraic de-Rham cohomology. Also Hodge theory for complex differential forms generalizes. These cohomologies should be related by homomorphism and category theoretic thinking the proof of the homomorphism requires the construction of appropriate functor between them.

The integrals of forms over sub-varieties define the elements of cohomology groups in ordinary cohomology and should have p-adic counterparts. Since p-adic numbers are not well-ordered, definite integral has no straightforward generalization to p-adic context. One might however be able to define integrals analogous to those associated with differential forms and depending only on the topology of sub-manifold over which they are taken. These integrals would be analogous to multiple residue integrals, which are the crux of the twistor approach to scattering amplitudes in super-symmetric gauge theories. One technical difficulty is that for a field of finite characteristic the derivative of X^p is pX^{p-1} and vanishes. This does not allow to define what integral $\int X^{p-1}dX$ could mean. Also $1/n!$ appears as natural normalization factor of forms but if p divides it, it becomes infinite.

4.6.2 Attempt to understand the notion of perfectoid

Consider now the basic ideas behind the notion of perfectoid.

1. For finite finite fields F_p Frobenius homomorphism $a \to a^p$ is automorphism since one has $a^p = a$ in modulo p arithmetics. A field with this property is called perfect and all local fields are perfect. Perfectness means that an algebraic number in any extension L of perfect field K is a root of a separable minimal polynomial. Separability means that the number of roots in the algebraic closure of K of the polynomial is maximal and the roots are distinct.

2. All fields containing rationals as sub-fields are perfect. For fields of characteristic p $Frob$ need not be a surjection so that perfectness is lost. For instance, for $F_p[t]$ $Frob$ is trivially injection but surjective property is lost: $t^{1/p}$ is not integer power of t.

   One can however extend the field to make it perfect. The trick is simple: add to $F_p[t]$ all fractional powers t^{1/p^n} so that all p:th roots exist and $Frob$ becomes and automorphism. The automorphism property of $Frob$ allows to get rid of technical problems related to a factorization of polynomials. The resulting extension is infinite-dimensional but satisfies the perfectness property allowing to understand Galois groups, which play key role in various cohomology theories in characteristic p.

3. Let $K = Q_p[p^{1/p^\infty}]$ denote the infinite-dimensional extension of p-adic number field Q_p by adding all powers of p^m:th roots for all all $m = 1, 2, ...$ This is not the most general option: K could be also only a ring. The outcome is perfect field although it does not of course have Frobenius automorphism since characteristic equals to 1.

   One can divide K by p to get K/p as the analog of finite field F_p as its infinite-dimensional extension. K/p allows all p:th roots by construction and $Frob$ is automorphism so that K/p is perfect by construction.

   The structure obtained in this manner is closely related to a perfect field with characteristic p having same Galois groups for all its extensions. This object is computationally much more attractive and allows to prove theorems in p-adic geometry. This motivates the term perfectoid.

4. One can assign to K another object, which is also perfectoid but has characteristic p. The correspondence is as follows.

(a) Let F_p be finite field. F_p is perfect since it allows trivially all p:th roots by $a^p = a$. The ring $F_p[t]$ is however not prefect since t^{1/p^m} is not integer power of t. One must modify $F_p[t]$ to obtain a perfect field. Let $G_m = F_p[t^{1/p^m}]$ be the ring of formal series in powers of t^{1/p^m} defining also function field. These serious are called t-adic and one can define t-adic norm.

(b) Define t-adic function field K_b called the **tilt** of K as

$$K_b = \cup_{m=1,...}(K/p)[t^{1/pm}][t] \ .$$

One has all possible power series with coefficients in K/p involving all roots t^{1/p^m}, $m = 1, 2, ...$, besides powers of positive integer powers of t. This function field has characteristic p and all roots exist by construction and $Frob$ is automorphism. K_b/t is perfect meaning that the minimal polynomials for the for given analog of algebraic number in any of its extensions allows separable polynomial with maximal number of roots in its closure.

This sounds rather complicated! In any case, K_b/t has same number theoretical structure as $Q_p[p^{1/p^\infty}]/p$ meaning that Galois groups for all of its extensions are canonically isomorphic to those for extensions of K. Arithmetics modulo p is much simpler than p-adic arithmetic since products are purely local and there is no need to take care about remainders in arithmetic operations, this object is much easier to handle.
Note that also p-adic number fields fields Q_p as also $F_p = Q_p/p$ are perfect but the analog of $K_b = F_b[t]$ fails to be perfect.

4.6.3 Second attempt to understand the notions of perfectoid and its tilt

This subsection is written roughly year after the first version of the text. I hope that it reflects a genuine increase in my understanding.

1. Scholze introduces first the notion of perfectoid. This requires some background notions. The characteristic p for field is defined as the integer p (prime) for which $px = 0$ for all elements x. Frobenius homomorphism (Frob familiarly) is defined as $Frob : x \to x^p$. For a field of characteristic p $Frob$ is an algebra homomorphism mapping product to product and sum to sum: this is very nice and relatively easy to show even by a layman like me.
2. Perfectoid is a field having either characteristic $p = 0$ (reals, p-adics for instance) or for which $Frob$ is a surjection meaning that $Frob$ maps at least one number to a given number x.
3. For finite fields $Frob$ is identity: $x^p = x$ as proved already by Fermat. For reals and p-adic number fields with characteristic p=0 it maps all elements to unit element and is not a surjection. Field is perfect if it has either $p = 0$ (reals, p-adics) or if Frobenius is surjection. Finite fields are obviously perfectoids too.

Scholze introduces besides perfectoids K also what he calls tilt K_b of the perfectoid. K_b is infinite-D extension of p-adic numbers by iterated p:th roots p-adic numbers: the units of the extension correspond to the roots p^{1/p^k}. They are something between p-adic number fields and reals and leads to theorems giving totally new insights to arithmetic geometry. Unfortunately, my technical skills in mathematics are hopelessly limited to say anything about these theorems.

1. As we learned during the first student year of mathematics, real numbers can be defined as Cauchy sequences of rationals converging to a real number, which can be also algebraic number or transcendental. The elements in the tilt K_b would be this kind of sequences.
2. Scholze starts from (say) p-adic numbers and considers infinite sequence of iterates of $1/p$:th roots. At given step $x \to x^{1/p}$. This gives the sequence $(x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, ...)$ identified as an element of the tilt K_b. At the limit one obtains $1/p^\infty$ root of x.
Remark: For finite fields each step is trivial ($x^p = x$) so that nothing interesting results: one has $(x, x, x, x, ...)$

(a) For p-adic number fields the situation is non-trivial. $x^{1/p}$ exists as p-adic number for all p-adic numbers with unit norm having $x = x_0 + x_1p + ...$ In the lowest order $x \simeq x_0$ the root is just x since x is effectively an element of finite field in this approximation. One can develop the $x^{1/p}$ to a power series in p and continue the iteration. The sequence obtained defines an element of tilt K_b of field K, now p-adic numbers.

(b) If the p-adic number x has norm p^n, $n \neq 0$ and is therefore not p-adic unit, the root operation makes sense only if one performs an extension of p-adic numbers containing all the roots p^{1/p^k} . These roots define one particular kind of extension of p-adic numbers and the extension is infinite-dimensional since all roots are needed. One can approximate K_b by taking only finite number iterated roots.

3. The tilt is said to be fractal: this is easy to understand from the presence of the iterated p:th root. Each step in the sequence is like zooming. One might say that p-adic scale becomes p:th root of itself. In TGD the p-adic length scale L_p is proportional to $p^{1/2}$: does the scaling mean that the p-adic length scale would defined hierarchy of scales proportional to $p^{1/2kp}$: root of itself and approach the CP_2 scale since the root of p approaches unity. Tilts as extensions by iterated roots would improve the length scale resolution.

One day later after writing this I got the feeling that I might have vaguely understood one more important thing about the tilt of p-adic number field: changing of the characteristic 0 of p-adic number field to characteristics $p > 0$ of the corresponding finite field for its tilt. What could this mean?

1. Characteristic p (p is the prime labelling p-adic number field) means $px = 0$. This property makes the mathematics of finite fields extremely simple: in the summation one need not take care of the residue as in the case of reals and p-adics. The tilt of the p-adic number field would have the same property! In the infinite sequence of the p-adic numbers coming as iterated p:th roots of the starting point p-adic number one can sum each p-adic number separately. This is really cute if true!
2. It seems that one can formulate the arithmetics problem in the tilt where it becomes in principle as simple as in finite field with only p elements! Does the existence of solution in this case imply its existence in the case of p-adic numbers? But doesn't the situation remain the same concerning the existence of the solution in the case of rational numbers? The infinite series defining p-adic number must correspond a sequence in which binary digits repeat with some period to give a rational number: rational solution is like a periodic solution of a dynamical system whereas non-rational solution is like chaotic orbit having no periodicity? In the tilt one can also have solutions in which some iterated root of p appears: these cannot belong to rationals but to their extension by an iterated root of p.

The results of Scholze could be highly relevant for the number theoretic view about TGD in which octonionic generalization of arithmetic geometry plays a key role since the points of space-time surface with coordinates in extension of rationals defining adele and also what I call cognitive representations determining the entire space-time surface if $M^8 - H$ duality holds true (space-time surfaces would be analogous to roots of polynomials). Unfortunately, my technical skills in mathematics needed are hopelessly limited.

TGD inspires the question is whether this kind of extensions could be interesting physically. At the limit of infinite dimension one would get an ideal situation not realizable physically if one believes that finite-dimensionality is basic property of extensions of p-adic numbers appearing in number theoretical quantum physics (they would related to cognitive representations in TGD). Adelic physics [L42] involves all finite-D extensions of rationals and the extensions of p-adic number fields induced by them and thus also cutoffs of extensions of type K_b- which I have called precursors of K_b.

How this relates to Witt vectors?

Witt vectors provide an alternative representation of p-adic arithmetics of p-adic integers in which the sum and product are reduced to purely local digit-wise operations for each power of p for the components of Witt vector so that one need not worry about carry pinary digit.

1. The idea is to consider the sequence consisting pinary cutoffs to p-adic number $x mod p^n$ and identify p-adic integer as this kind of sequence as n approaches infinity. This is natural approach when one identifies finite measurement resolution or cognitive resolution as a cutoff in some power of p^n. One simply forms the numbers $X_n = x\ mod\ p^{n+1}$: for numbers $1, ..., p-1$ they are called Teichmueller representatives and only they are needed to construct the sequences for general x. One codes this sequence of pinary cutoffs to Witt vector.

2. The non-trivial observation made by studying sums of p-adic numbers is that the sequence $X_0, X_1, X_2, ...$ of approximations define a sequence of components of Witt vector as $W_0 = X_0$, $W_1 = X_0^p + pX_1$, $W_2 = X_0^(p^2) + pX_1^p + p^2X_2$, ... or more formally $W_n = Sum_{i<n} p^i ZX_i^[p^(n-i)]$.
3. The non-trivial point is that Witt vectors form a commutative ring with local digit-wise multiplication and sum modulo p: there no carry digits. Effectively one obtains infinite Cartesian power of finite field F_p. This means a great simplification in arithmetics. One can do the arithmetics using Witt vectors and deduce the sum and product from their product.
4. Witt vectors are universal. In particular, they generalize to any extension of p-adic numbers. Could Witt vectors bring in something new from physics point of view? Could they allow a formulation for the hierarchy of pinary cutoffs giving some new insights? For instance, neuro-computationalist might ask whether brain could perform p-adic arithmetics using a linear array of modules (neurons or neuron groups) labelled by $n = 1, 2, ...$ calculates sum or product for component W_n of Witt vector? No transfer of carry bits between modules would be needed. There is of course the problem of transforming p-adic integers to Witt vectors and back - it is not easy to imagine a natural realization for a module performing this transformation. Is there any practical formulation for say p-adic differential calculus in terms of Witt vectors?

I would seem that Witt vectors might relate in an interesting manner to the notion of perfectoid. The basic result proved by Petter Scholtze is that the completion $\cup_n Q_p(p^{1/p^n})$ of p-adic numbers by adding p^n:th roots and the completion of Laurent series $F_p((t))$ to $\cup_n F_p((t^{1/p^n}))$ have isomorphic absolute Galois groups and in this sense are one and same thing. On the other hand, p-adic integers can be mapped to a subring of $F_p(t)$ consisting of Taylor series with elements allowing interpretation as Witt vectors.

4.6.4 TGD view about p-adic geometries

As already mentioned, it is possible to define p-adic counterparts of n-forms and also various p-adic cohomologies with coefficient field taken as p-adic numbers and these constructions presumably make sense in TGD framework too. The so called rigid analytic geometry is the standard proposal for what p-adic geometry might be.

The very close correspondence between real space-time surfaces and their p-adic variants plays realized in terms of cognitive representations [L44, L43, L36] plays a key role in TGD framework and distinguishes it from approaches trying to formulate p-adic geometry as a notion independent of real geometry.

Ordinary approaches to p-adic geometry concentrate the attention to single p-adic prime. In the adelic approach of TGD one considers both reals and all p-adic number fields simultaneously.

Also in TGD framework Galois groups take key role in this framework and effectively replace homotopy groups and act on points of cognitive representations consisting of points with coordinates in extension of rationals shared by real and p-adic space-time surfaces. One could say that homotopy groups at level of sensory experience are replaced by Galois at the level of cognition. It also seems that there is very close connection between Galois groups and various symmetry groups. Galois groups would provide representations for discrete subgroups of symmetry groups.

In TGD framework there is strong motivation for formulating the analog of Riemannian geometry of $H = M^4 \times CP_2$ for p-adic variants of H. This would mean p-adic variant of Kähler geometry. The same challenge is encountered even at the level of "World of Classical Worlds" (WCW) having Kähler geometry with maximal isometries. p-Adic Riemann geometry and n-forms make sense locally as tensors but integrals defining distances do not make sense p-adically and it seems that the dream about global geometry in p-adic context is not realizable. This makes sense: p-adic physics is a correlate for cognition and one cannot put thoughts in weigh or measure their length.

Formulation of adelic geometry in terms of cognitive representations

Consider now the key ideas of adelic geometry and of cognitive representations.

1. The king idea is that p-adic geometries in TGD framework consists of p-adic balls of possibly varying radii p^n assignable to points of space-time surface for which the preferred embedding space coordinates are in the extension of rationals. At level of M^8 octonion property

fixes preferred coordinates highly uniquely. At level of H preferred coordinates come from symmetries.

These points define a cognitive representation and inside p-adic points the solution of field equations is p-adic variant of real solution in some sense. At M^8 level the field equations would be algebraic equations and real-p-adic correspondence would be very straightforward. Cognitive representations would make sense at both M^8 level and H level.

Remark: In ordinary homology theory the decomposition of real manifold to simplexes reduces topology to homology theory. One forgets completely the interiors of simplices. Could the cognitive representations with points labelling the p-adic balls could be seen as analogous to decompositions to simplices. If so, homology would emerge as something number theoretically universal. The larger the extension of rationals, the more precise the resolution of homology would be. Therefore p-adic homology and cohomology as its Poincare dual would reduce to their real counterparts in the cognitive resolution used.

2. M^8-H correspondence would play a key role in mapping the associative regions of space-time varieties in M^8 to those in H. There are two kinds of regions. Associative regions in which polynomials defining the surfaces satisfy criticality conditions and non-associative regions. Associative regions represent external particles arriving in CDs and non-associative regions interaction regions within CDs.

3. In associative regions one has minimal surface dynamics (geodesic motion) at level of H and coupling parameters disappear from the field equations in accordance with quantum criticality. The challenge is to prove that M^8-H correspondence is consistent with the minimal surface dynamics n H. The dynamics in these regions is determined in M^8 as zero loci of polynomials satisfying quantum criticality conditions guaranteeing associativity and is deterministic also in p-adic sectors since derivatives are not involved and pseudo constants depending on finite number of pinary digits and having vanishing derivative do not appear. M^8-H correspondence guarantees determinism in p-adic sectors also in H.

4. In non-associative regions M^8-H correspondence does not make sense since the tangent space of space-time variety cannot be labelled by CP_2 point and the real and p-adic H counterparts of these regions would be constructed from boundary data and using field equations of a variational principle (sum of the volume term and Kähler action term), which in non-associative regions gives a dynamics completely analogous to that of charged particle in induced Kähler field. Now however the field characterizes extended particle itself.

Boundary data would correspond to partonic 2-surfaces and string world sheets and possibly also the 3-surfaces at the ends of space-time surface at boundaries of CD and the light-like orbits of partonic 2-surfaces. At these surfaces the 4-D (!) tangent/normal space of space-time surface would be associative and could be mapped by M^8-H correspondence from M^8 to H and give rise to boundary conditions.

Due to the existence of p-adic pseudo-constants the p-adic dynamics determined by the action principle in non-associative regions inside CD would not be deterministic in p-adic sectors. The interpretation would be in terms of freedom of imagination. It could even happen that boundary values are consistent with the existence of space-time surface in p-adic sense but not with the existence of real space-time surfaces. Not all that can be imagined is realizable.

At the level of M^8 this vision seems to have no obvious problems. Inside each ball the same algebraic equations stating vanishing of $IM(P)$ (imaginary part of P in quaternionic sense) hold true. At the level of H one has second order partial differential equations, which also make sense also p-adically. Besides this one has infinite number of boundary conditions stating the vanishing of Noether charges assignable to sub-algebra super-symplectic algebra and its commutator with the entire algebra at the 3-surfaces at the boundaries of CD. Are these two descriptions really equivalent?

During writing I discovered an argument, which skeptic might see as an objection against M^8-H correspondence.

1. M^8 correspondence maps the space-time varieties in M^8 in non-local manner to those in $H=M^4\times CP_2$. CP_2 coordinates characterize the tangent space of space-time variety in M^8 and this might produce technical problems. One can map the real variety to H and find the points of the image variety satisfying the condition and demand that they define the "spine" of the p-adic surface in p-adic H.

2. The points in extensions of rationals in H need not be images of those in M^8 but should this be the case? Is this really possible? M^4 point in $M^4 \times E^4$ would be mapped to $M^4 \subset M^4 \times CP_2$: this is trivial. 4-D associative tangent/normal space at m containing preferred M^2 would be characterized by CP_2 coordinates: this is the essence of $M^8 - H$ correspondence. How could one guarantee that the CP_2 coordinates characterizing the tangent space are really in the extension of rationals considered? If not, then the points of cognitive representation in H are not images of points of cognitive representation in M^8. Does this matter?

Are almost-perfectoids evolutionary winners in TGD Universe?

One could take perfectoids and perfectoid spaces as a mere technical tool of highly refiner mathematical cognition. Since cognition is basic aspect of TGD Universe, one could also ask perfectoids or more realistically, almost-perfectoids, could be an outcome of cognitive evolution in TGD Universe?

1. p-Adic algebraic varieties are defined as zero loci of polynomials. In the octonionic M^8 approach identifying space-time varieties as zero loci for RE or IM of octonionic polynomial (RE and IM in quaternionic sense) this allows to define p-adic variants of space-time surfaces as varieties obeying same polynomial equations as their real counterparts provided the coefficients of octonion polynomials obtainable from real polynomials by analytic continuation are in an extension of rationals inducing also extension of p-adic numbers.
   The points with coordinates in the extension of rationals common to real and p-adic variants of M^8 identified as cognitive representations are in key role. One can see p-adic space-time surfaces as collections of "monads" labelled by these points at which Cartesian product of 1-D p-adic balls in each coordinate degree. The radius of the p-adic ball can vary. Inside each ball the same polynomial equations are satisfied so that the monads indeed reflect other monads. Kind of algebraic hologram would be in question consisting of the monads. The points in extension allow to define ordinary real distance between monads. Only finite number of monads would be involved since the number of points in extension tends to be finite. As the extension increases, this number increases. Cognitive representations become more complex: evolution as increase of algebraic complexity takes place.
2. Finite-dimensionality for the allowed extensions of p-adic number fields is motivated by the idea about finiteness of cognition. Perfectoids are however infinite-dimensional. Number theoretical universality demands that on only extensions of p-adics induced by those of rationals are allowed and defined extension of the entire adele. Extensions should be therefore be induced by the same extension of rationals for all p-adic number fields.
   Perfectoids correspond to an extension of Q_p apparently depending on p. This dependence is in conflict with number theoretical universality if real. This extension could be induced by corresponding extension of rationals for all p-adic number fields. For p-adic numbers Q_q $q \neq p$ all equation $a^{p^n} = x$ reduces to $a^n = x \ mod \ p$ and this in term to $a^m = x \ mod \ p$, $m = n \ mod \ p$. Finite-dimensional extension is needed to have all roots of required kind! This extension is therefore finite-D for all $q \neq p$ and infinite-D for p.
3. What about infinite-dimensionality of the extension. The real world is rarely perfect and our thoughts about it even less so, and one could argue that we should be happy with almost-perfectoids! "Almost" would mean extension induced by powers of p^{1/p^m} for large enough m, which is however not infinite. A finite-dimensional extension approaching perfectoid asymptotically is quite possible!
4. One could see the almost perfectoid as an outcome of evolution and perfectoid as the asymptotic states. High dimension of extension means that p-adic numbers and extension of rationals have large number of common numbers so that also cognitive representations contain a large number of common points. Maybe the p-adic number fields, which are evolutionary winners, have managed to evolve to especially high-dimensional almost-perfectoids! Note however that also the roots of e can be considered as extensions of rationals since corresponding p-adic extensions are finite-dimensional. Similar evolution can be considered also now.
   To get some perspective mote that for large primes such as $M_{127} = 2^{127} - 1$ characterizing electron the lowest almost perfectoid would give powers of $M_{127}^{1/M_{127}} = (2^{127} - 1)^{1/(2^{127}-1)} \sim 1 + log(2)2^{-120}$! The lattice of points in extension is extremely dense near real unit. The

density of of points in cognitive representations near this point would be huge. Note that the length scales comes as negative powers of two, which brings in mind p-adic length scale hypothesis [K59].

Although the octonionic formulation in terms of polynomials (or rational functions identifying space-time varieties as zeros or poles of $RE(P)$ or $IM(P)$ is attractive in its simplicity, one can also consider the possibility of allowing analytic functions of octonion coordinate obtained from real analytic functions. These define complex analytic functions with commutative imaginary unit used to complexify octonions. Could meromorphic functions real analytic at real axis having only zeros and poles be allowed? The condition that all p-adic variants of these functions exist simultaneously is non-trivial. Coefficients must be in the extension of rationals considered and convergence poses restrictions. For instance, e^x converges only for $|x|_p < 1$. These functions might appear at the level of H.

4.7 Secret Link Uncovered Between Pure Math and Physics

I learned about a possible existence of a very interesting link between pure mathematics and physics (see `http://tinyurl.com/y86bckmo`). The article told about ideas of number theorist Minhyong Kim working at the University of Oxford. As I read the popular article, I realized it is something very familiar to me but from totally different view point.

Number theoretician encounters the problem of finding rational points of an algebraic curve defined as real or complex variant in which case the curve is 2-D surface and 1-D in complex sense. The curve is defined as root of polynomials polynomials or several of them. The polynomial have typically rational coefficients but also coefficients in extension of rationals are possible.

For instance, Fermat's theorem is about whether $x^n + y^n = 1$, $n = 1, 2, 3, ...$ has rational solutions for $n \geq 1$. For $n = 1$, and $n = 2$ it has, and these solutions can be found. It is now known that for $n > 2$ no solutions do exist. Quite generally, it is known that the number is finite rather than infinite in the generic case.

A more general problem is that of finding points in some algebraic extension of rationals. Also the coefficients of polynomials can be numbers in the extension of rationals. A less demanding problem is mere counting of rational points or points in the extension of rationals and a lot of progress has been achieved in this problem. One can also dream of classifying the surfaces by the character of the set of the points in extension.

I have consider the identification problem earlier in [L36] and I glue here a piece of text summarizing some basic results. The generic properties of sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see `http://tinyurl.com/y9oq37ce`) states that a curve over Q with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere CP_1 in CP_2 has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A158, A169]).

   $g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in CP_2 with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in CP_2 (see `http://tinyurl.com/lovksny`) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

   $g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and CP_2 as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in CP_2 (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially dense set of rational points.

In my article [L36] providing TGD perspective about the role of algebraic geometry in physics, one can find basic results related to the identification problem including web links and references to literature.

4.7.1 Connection with TGD and physics of cognition

The identification problem is extremely difficult even for mathematicians - to say nothing about humble physicist like me with hopelessly limited mathematical skills. It is however just this problem which I encounter in TGD inspired vision about adelic physics [L43, L42, L36]. Recall that in TGD space-times are 4-surfaces in $H = M^4 \times CP_2$, preferred extremals of the variational principle defining the theory [K76, L56].

1. In this approach p-adic physics for various primes p provide the correlates for cognition: there are several motivations for this vision. Ordinary physics describing sensory experience and the new p-adic physics describing cognition for various primes p are fused to what I called adelic physics. The adelic physics is characterized by extension of rationals inducing extensions of various p-adic number fields. The dimension n of extension characterizes kind of intelligence quotient and evolutionary level since algebraic complexity is the larger, the larger the value of n is. The connection with quantum physics comes from the conjecture that n is essentially effective Planck constant $h_{eff}/h_0 = n$ characterizing a hierarchy of dark matters. The larger the value of n the longer the scale of quantum coherence and the higher the evolutionary level, the more refined the cognition.
2. An essential notion is that of cognitive representation [K62] [L42, L36]. It has several realizations. One of them is the representation as a set of points common to reals and extensions of various p-adic number fields induced by the extension of rationals. These space-time points have points in the extension of rationals considered defining the adele. The coordinates are the embedding space coordinates of a point of the space-time surface. The symmetries of embedding space provide highly unique embedding space coordinates.
3. The gigantic challenge is to find these points common to real number field and extensions of various p-adic number fields appearing in the adele.
4. If this were not enough, one must solve an even tougher problem. In TGD the notion of "world of classical worlds" (WCW) is also a central notion [K76]. It consists of space-time surfaces in embedding space $H = M^4 \times CP_2$, which are so called preferred extremals of the action principle of theory. Quantum physics would reduce to geometrization of WCW and construction of classical spinor fields in WCW and representing basically many-fermion states: only the quantum jump would be genuinely quantal in quantum theory.

   There are good reasons to expect that space-time surfaces are minimal surfaces with 2-D singularities, which are string world sheets - also minimal surfaces [L56, L65]. This gives nice geometrization of gauge theories since minimal surfaces equations are geometric counterparts for massless field equations.

   One must find the algebraic points, the cognitive representation, for all these preferred extremals representing points of WCW (one must have preferred coordinates for H - the symmetries of embedding space crucial for TGD and making it unique, provide the preferred coordinates)!
5. What is so beautiful is that in given cognitive resolution defined by the extension of rationals inducing the discretization of space-time surface, the cognitive representation defines the coordinates of the space-time surfaces as a point of WCW. In finite cognitive and measurement resolution this huge infinite-dimensional space WCW discretizes and the situation can be handled using finite mathematics.

4.7.2 Connection with Kim's work

So: what is then the connection with the work and ideas of Kim. There has been a lot of progress in understanding the problem: here I an only refer to the popular article.

1. One step of progress has been the realization that if one uses the fact that the solutions are common to both reals and various p-adic number fields helps a lot. The reason is that for rational points the rationality implies that the solution of equation representable as infinite

power series of p contains only finite number powers of p. If one manages to prove the this happens for even single prime, a rational solution has been found.
The use of reals and all p-adic numbers fields is nothing but adelic physics. Real surfaces and all its p-adic variants form pages of a book like structure with infinite number of pages. The rational points or points in extension of rationals are the cognitive representation and are points common to all pages in the back of the book.
This generalizes also to algebraic extensions of rationals. Solving the number theoretic problem is in TGD framework nothing but finding the points of the cognitive representation. The surprise for me was that this viewpoint helps in the problem rather than making it more complex.
There are however problematic situations in some cases the hypothesis about finite set of algebraic points need not make sense. A good example is Fermat for $x + y = 1$. All rational points and also algebraic points are solutions. For $x^2 + y^2 = 1$ the set of Pythagorean triangles characterizing the solutions is infinite. How to cope with these situations in which one has accidental symmetries as one might say?

2. Kim argues that one can make even further progress by considering the situation from even wider perspective by making the problem even bigger. Introduce what the popular article (see `http://tinyurl.com/y86bckmo`) calls the space of spaces. The space of string world sheets is what string models suggests. WCW is what TGD suggests. One can get a wider perspective of the problem of finding algebraic points of a surface by considering the problem in the space of surfaces and at this level it might be possible to gain much more understanding. The notion of WCW would not mean horrible complication of a horribly complex problem but possible manner to understand the problem!
The popular article mentioned in the beginning mentions so called Selmer varieties as a possible candidate for the space of spaces. From the Wikipedia article (see `http://tinyurl.com/y27so3f2`) telling about Kim one can find a link to an article [A156] related to Selmer varieties. This article goes over my physicist's head but might give for a more mathematically oriented reader some grasp about what is involved. One can find also a list of publications of Kim (see `http://people.maths.ox.ac.uk/kimm/`.
Kim also suggests that the spaces of gauge field configurations could provide the spaces of spaces. The list contains an article [A167] with title *Arithmetic Gauge Theory: A Brief Introduction* (see `http://tinyurl.com/y66mphkh`) , which might help physicist to understand the ideas. An arithmetic variant of gauge theory could provide the needed space of spaces.

4.7.3 Can one make Kim's idea about the role of symmetries more concrete in TGD framework?

The crux of the Kim's idea is that somehow symmetries of space of spaces could come in rescue in the attempts to understand the rational points of surface. The notion of WCW suggest in TGD framework rather concrete realization of this idea that I have discussed from the point of view of construction of quantum states.

1. A little bit more of zero energy ontology (ZEO) is needed to follow the argument. In ZEO causal diamonds (CDs) are central. CDs are defined as intersections of future and past directed light-cones with points replaced with CP_2 and forming a scale hierarchy are central. Space-time surfaces are preferred extremals with ends at the opposite boundaries of CD indeed looking like diamond. Symplectic group for the boundaries of causal diamond (CD) is the group of isometries of WCW [K76] [L56]. Maximal isometry group is required to guarantee that the WCW Kähler geometry has Riemann connection - this was discovered for loop spaces by Dan Freed [A121]. Its Lie algebra has structure of Kac- Moody algebra with respect to the light-like radial coordinate of the light-like boundary of CD, which is piece of light-cone boundary. This infinite-D group plays central role in quantum TGD: it acts as maximal group of WCW isometries and zero energy states are invariant under its action at opposite boundaries.
2. As one replaces space-time surface with a cognitive representation associated with an extension of rationals, WCW isometries are replaced with their infinite discrete subgroup acting in the number field define by the extension of rationals defining the adele. These discrete isometries do not leave the cognitive representation invariant but replace with it new one having the same

number of points and one obtains entire orbit of cognitive representations. This is what the emergence of symmetries in wider conceptual framework would mean.

3. One can in fact construct invariants of the symplectic group. Symplectic transformations leave invariant the Kähler magnetic fluxes associated with geodesic polygons with edges identified as geodesic lines of H. There are also higher-D symplectic invariants. The simplest polygons are geodesic triangles. The symplectic fluxes associated with the geodesic triangles define symplectic invariants characterizing the cognitive representation. For the twistor lift one must allow also M^4 to have analog of Kähler form and it would be responsible for CP violation and matter antimatter asymmetry [L30]. Also this defines symplectic invariants so that one obtains them for both M^4 and CP_2 projections and can characterize the cognitive representations in terms of these invariants. Note that the existence of twistor lift fixes the choice of H uniquely since M^4 and CP_2 are the only 4-D spaces allowing twistor space with Kähler structure [A150] necessary for defining the twistor lift of Kähler action.
   More complex cognitive representations in an extension containing the given extension are obtained by adding points with coordinates in the larger extension and this gives rise to new geodesic triangles and new invariants. A natural restriction could be that the polynomial defining the extension characterizing the preferred extremal via $M^8 - H$ duality defines the maximal extension involved.

4. Also in this framework one can have accidental symmetries. For instance, M^4 with CP_2 coordinates taken to be constant is a minimal surface, and all rational and algebraic points for given extension belong to the cognitive representation so that they are infinite. Could this has something to do with the fact that we understand M^4 so well and have even identified space-time with Minkowski space! Linear structure would be cognitively easy for the same reason and this could explain why we must linearize.
   CP_2 type extremals with light-like M^4 geodesic as M^4 projection is second example of accidental symmetries. The number of rational or algebraic points with rational M^4 coordinates at light-like curve is infinite - the situation is very similar to $x + y = 1$ for Fermat. Simplest cosmic strings are geodesic sub-manifolds, that is products of plane $M^2 \subset M^4$ and CP_2 geodesic sphere. Also they have exceptional symmetries.
   What is interesting from the point of view of proposed model of cognition is that these cognitively easy objects play a central role in TGD: their deformations represent more complex dynamical situations. For instance, replacing planar string with string world sheet replaces cognitive representation with a discrete or perhaps even finite one in M^4 degrees of freedom.

5. A further TGD based simplification would be $M^8 - H$ ($H = M^4 \times CP_2$) duality in which space-time surfaces at the level of M^8 are algebraic surfaces, which are mapped to surfaces in H identified as preferred extremals of action principle by the $M^8 - H$ duality [L36]. Algebraic surfaces satisfying algebraic equations are very simple as compared to preferred extremals satisfying partial differential equations but "preferred" is what makes possible the duality. This huge simplification of the solution space of field equations guarantees holography necessitated by general coordinate invariance implying that space-time surfaces are analogous to Bohr orbits. It would also guarantee the huge symmetries of WCW making it possible to have Kähler geometry.
   This suggests in TGD framework that one finds the cognitive representation at the level of M^8 using methods of algebraic geometry and maps the points to H by using the $M^8 - H$ duality. TGD and octonionic variant of algebraic geometry would meet each other.
   It must be made clear that now solutions are not points but 4-D surfaces and this probably means also that points in extension of rationals are replaced with surfaces with embedding space coordinates defining function in extensions of rational functions rather than rationals. This would bring in algebraic functions. This might provide also a simplification by providing a more general perspective. Also octonionic analyticity is extremely powerful constraint that might help.

4.8 Cognitive representations for partonic 2-surfaces, string world sheets, and string like objects

Cognitive representations are identified as points of space-time surface $X^4 \subset M^4 \times CP_2$ having embedding space coordinates in the extension of of rationals defined by the polynomial defined by the M^8 pre-image of X^4 under $M^8 - H$ correspondence [L37, L38, L75, L67, L64, L58]. Cognitive representations have become key piece in the formulation of scattering amplitudes [L69] . One might argue that number theoretic evolution as increase of the dimension of the extension of rationals favors space-time surfaces with especially large cognitive representations since the larger the number of points in the representation is, the more faithful the representation is.

One can pose several questions if one accepts the idea that space-time surfaces with large cognitive representations are survivors.

1. Preferred p-adic primes are proposed to correspond to the ramified primes of the extension [L77]. The proposal is that the p-adic counterparts of space-time surfaces are identifiable as imaginations whereas real space-time surfaces correspond to realities. p-Adic space-time surfaces would have the embedding space points in extension of rationals as common with real surfaces and large number of these points would make the representation realistic. Note that the number of points in extension does not depend on p-adic prime.
Could some extensions have an especially high number of points in the cognitive representation so that the corresponding ramified primes could be seen as survivors in number theoretical fight for survival, so to say? Galois group of the extension acts on cognitive representation. Galois extension of an extension has the Galois group of the original extension as normal subgroup so that ormal Galois group is analogous to a conserved gene.
2. Also the type of extremal matters. For instance, for instance canonically imbedded M^4 and CP_2 contain all points of extension. These surfaces correspond to the vanishing of real or imaginary part (in quaternionic sense) for a linear octonionic polynomial $P(o) = o$! As a matter of fact, this is true for all known preferred extremals under rather mild additional conditions. Boundary conditions posed at both ends of CD in ZEO exclude these surfaces and the actual space-time surfaces are expected to be their deformations.
3. Could the surfaces for which the number of points in cognitive representation is high, be the ones most easily discovered by mathematical mind? The experience with TGD supports positive answer: in TGD the known extremals [K8] are examples of such mathematical objects! If so, one should try to identify mathematical objects with high symmetries and look whether they allow TGD realization.
4. One must also specify more precisely what cognitive representation means. Strong form of holography (SH) states that the information gives at 2-D surfaces - string world sheets and partonic 2-surfaces - is enough to determine the space-time surfaces. This suggests that it is enough to consider cognitive representation restricted to these 2-surfaces. What kind of 2-surfaces are the cognitively fittest one? It would not be surprising if surfaces with large symmetries acting in extension were favored and elliptic curves with discrete 2-D translation group indeed turn out to be assigable string world sheets as singularities and string like objects. In the case of partonic 2-surfaces geodesic sphere of CP_2 is similar object.

All known extremals, in particular preferred extremals, are good candidates in this respect because of their high symmetries. By strong form of holography (SH) partonic 2-surfaces and string world sheets are expected to give rise to cognitive representations. Also cosmic strings are expected to carry them. Under what conditions these representations are large?

4.8.1 Partonic 2-surfaces as seats of cognitive representations

One can start from SH and look the situation more concretely. The situation for partonic 2-surfaces has been considered already earlier [L76, L63] but deserves a separate discussion.

1. Octonionic polynomials allow special solutions for which the entire polynomial vanishes. This happens at 6-sphere S^6 at the boundary of 8-D light-cone. S^6 is analogous to brane and has radius $R = r_n$, which is a root of the real polynomial with rational coefficients algebraically continued to the octonionic polynomial.

S^6 has the ball B^3 of radius r_n of the light-cone M^4_+ with time coordinate $t = r_n$ as analog of base space and sphere S^3 of E^4 with radius $R = \sqrt{r_n^2 - r^2}$, r the radial coordinate of B^3 as an analog of fiber. The analog of the fiber contracts to a point at the boundary of the light-cone. The points with B^3 projection and E^4 coordinates in extension of rationals belong to the cognitive representation. The condition that $R^2 = x_i x^i = r_n^2 - r^2$ is square of a number of extension is rather mild and allows infinite number of solutions.

2. The 4-D space-time surfaces X^4 are obtained as generic solutions of $Im(P(o)) = 0$ or $Re(P(o)) = 0$. Their intersection with S^6 - partonic 2-surface X^2 - is 2-D. The assumption is that the incoming and outgoing 4-D space-time surfaces representing orbits of particles in topological sense are glued together at X^2 and possibly also in their interiors. X^2 serves as an analog of vertex for 3-D particles. This gives rise to topological analogs of Feynman diagrams.
   In the generic case the number of points in cognitive representation restricted to X^2 is finite unless the partonic 2-surface X^2 is special - say correspond to a geodesic spere of S^6.
3. The discrete isometries and conformal symmetries of the cognitive representation restricted to X^2 possibly represented as elements of Galois group might play a role. For $X^2 = S^2$ the finite discrete subgroups of $SO(3)$ giving rise to finite tessellations and appearing in ADE correspondence might be relevant. For genera $g = 01, 2$ conformal symmetry Z_2 is always possible but for higher genera only in the case of hyper-elliptic surfaces- this used to explain why only $g = 0, 1, 2$ correspond to observed particles [K21] whereas higher genera could be regarded as many-particle states of handles having continuous mass spectrum. Torus is an exceptional case and one can ask whether discrete subgroup of its isometries could be realized.
4. In TGD inspired theory of consciousness [L44, L63] the moments $t = r_n$ corresponds to "very special moments in the life of self". They would be also cognitively very special - kind of eureka moments with a very large number of points in cognitive representation. The question is whether these surfaces might be relevant for understanding the nature of mathematical consciousness and how the mathematical notions emerge at space-time level.

4.8.2 Ellipticity

Surfaces with discrete translational symmetries is a natural candidate for a surface with very large cognitive representation. Are their analogs possible? The notions of elliptic function, curve, and surface suggest themselves as a starting point.

1. Elliptic functions (`http://tinyurl.com/gpugcnh`) have 2-D discrete group of translations as symmetries and are therefore doubly periodic and thus identifiable as functions on torus.
   Weierstrass elliptic functions $\mathcal{P}(z; \omega_1, \omega_2)$ (`http://tinyurl.com/ycu8oa4r`) are defined on torus and labelled by the conformal equivalence class $\lambda = \omega_1/\omega_2$ of torus identified as the ratio $\lambda = \omega_1/\omega_2$ of the complex numbers ω_i defining the periodicities of the lattice involved. Functions $\mathcal{P}(z; \omega_1, \omega_2)$ are of special interest as far as elliptic curves are considered and defines an embedding of elliptic curve to CP_2 as will be found.
   If the periods are in extension of rationals then values in the extension appear infinitely many times. Elliptic functions are not polynomials. Although the polynomials giving rise to octonionic polynomials could be replaced by analytic functions it seems that elliptic functions are not the case of primary interest. Note however that the roots r_n could be also complex and could correspond to values of elliptic function forming a lattice.
2. Elliptic curves (`http://tinyurl.com/lovksny`) are defined by the polynomial equation

$$y^2 = P(x) = x^3 + ax + b \ . \tag{4.8.1}$$

   An algebraic curve of genus 1 allowing 2-D discrete translations as symmetries is in question. If a point of elliptic curve has coordinates in extension of rationals then 2-D discrete translation acting in extension give rise to infinite number of points in the cognitive representation. Clearly, the 2-D vectors spanning the lattice defining the group must be in extension of rationals.

One can indeed define commutative sum $P + Q$ for the points of the elliptic curve. The detailed definition of the group law and its geometric illustration can be found in Wikipedia article (`http://tinyurl.com/lovksny`).

1. Consider real case for simplicity so that elliptic curve is planar curve. $y^2 = P(x) = x^3 + ax + b$ must be non-negative to guarantee that y is real. $P(x) \geq 0$ defines a curve in upper (x, y)

plane extending from some negative value x_{min} corresponding to $y^2 = P(x_{min}) = 0$ to the right. Given value of y can correspond to 3 real roots or 1 real root of $P_y(x) = y^2 - P(x)$. At the two extrema of $P_y(x)$ 2 real roots co-incide. The graph of $y = \pm\sqrt{P(x)}$ is reflection symmetric having two branches beginning from $(x_{min}, y = 0)$.

2. The negative $-P$ is obtained by reflection with respect to x-axis taking y_P to $-y_P$. Neutral element O is identified as point a infinity (assuming compactification of the plane to a sphere) which goes to itself under reflection $y \to -y$.
3. One assigns to the points P and Q of the elliptic curve a line $y = sx+d$ containing them so that one has $s = (y_p - y_Q)/(x_P - x_Q)$. In the generic case the line intersects the elliptic curve also at third point R since $P_{y=sx+d}(x)$ is third order polynomial having three roots (x_P, x_Q, x_R). It can happen that 2 roots are complex and one has 1 real root. At criticalityfor the transiton from 3 to 1 real roots one has $x_Q = x_R$.
   Geometrically one can distinguish between 4 cases.
   - The roots P, Q, R of $P_{y=sx+d}(x)$ are different and finite: one defines the sum as $P + Q = -R$.
   - $P \neq Q$ and $Q = R$ (roots Q and R are degenerate): $P + Q + Q = O$ giving $R = -P/2$.
   - P and Q are at a line parallel to y-axis and one has $R = O$: $P + Q + O = O$ and $P = -Q$.
   - P is double root of $P_{y=sx+d}(x)$ with tangent parallel to y-axis at the point $(x_{min}, y = 0)$ at which the elliptic curve begins so that one has $R = O$: $P + P + O = O$ gives $P = -P$. This corresponds to torsion.
4. Elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) define a generalization of elliptic curves and are defined for 4-D complex manifolds. Fiber is required to be smooth and has genus 1.

4.8.3 String world sheets and elliptic curves

In twistor lift of TGD space-time surfaces identifiable as minimal surfaces with singularities, which are string world sheets and partonic 2-surfaces. Preferred extremal property means that space-time surfaces are extremals of both Kähler action and volume action except at singularities.

Are string world sheets with very large number of points in cognitive representation possible? One has right to expect that string world sheets allow special kind of symmetries allowing large, even infinite number of points at the limit of large sheet and related by symmetries acting in the extension of rationals. If one of the points is in the extension, also other symmetry related points are in the extension. For a non-compact group, say translation one would have infinite number of points in the representation but the finite size of CD would pose a limitation to the number of points.

String world sheets are good candidates for the realization of elliptic curves.

1. The general conjecture is that preferred extremals allow what I call Hamilton-Jacobi structure for M^4 [K76]. The distribution of tangent spaces having decomposition $M^4(x) = M^2(x) \times E^2(x)$ would be integrable giving rise to a family of string world sheets Y^2 and partonic 2-surfaces X^2 more general than those defined above. X^2 and Y^2 are orthogonal to each other at each point of X^4. One can introduce local light-cone coordinates (u, v) for Y^2 and local E^2 complex coordinate w for X^2.
2. Space-time surface itself would be a deformation of M^4 with Hamilton-Jacobi structure in CP_2 direction. w coordinate as function $w(z)$ of CP_2 complex coordinate z or vice versa would define the string world sheet. This would be a transversal deformation of the basic string world sheet Y^2: stringy dynamics is indeed transversal.
3. The idea about maximal cognitive representation suggests that $w \leftrightarrow z$ correspondence defines elliptic curve. One would have $y^2 = P(x) = x^3 + ax + b$ with either $(y = w, x = z)$ or $(y = z, x = w)$. A natural conjecture is that for the space-time surface corresponding to a given extension K of rationals the coefficients a an b belong to K so that the algebraic complexity of string world sheet would increase in number theoretic evolution [L74]. The orbit of a algebraic point at string world sheet would be lattice made finite by the size of CD. Elliptic curves would define very special deformed string world sheets in space-time.
4. It is interesting to consider the pre-image of given point y ($y = w$ or $y = z$) covering point x. One has $y = \pm\sqrt{u}$, $u = P(x)$ corresponding to group element and its negative: there are

two points of covering given value of u. $u = P(x)$ covers 3 values of x. The values of x would belong to 6-fold covering of rationals. The number theoretic interpretation for the effective Planck constant $h_{eff} = nh_0$ states that n is the number of sheets for space-time surface as covering.
There is evidence that $h_{eff} = h$ corresponds to $n = 6$ [L22]. Could 6-fold covering of rationals be fundamental since it gives very large cognitive representation at the level of string world sheets?
For extensions K of rationals the x coordinates for the points of cognitive representation would belong to 6-D extension of K.

5. Ellipticity condition would apply on the string world sheets themselves. In the number theoretic vision string world sheets would correspond at M^8 level to singularities at which the quaternionic tangent space degenerates to 2-D complex space. Are these conditions consistent with each other? It would seem that the two conditions would select cognitively very special string world sheets and partonic 2-surfaces defining by strong form of holography (SH) space-time surface as a hologram in SH. Consciousness theorist interested in mathematical cognition might ask whether the notion of elliptic surfaces have been discovered just because it is cognitively very special. In the case of partonic 2-surfaces geodesic sphere of CP_2 is similar object.

4.8.4 String like objects and elliptic curves

String like objects - cosmic strings - and their deformations, are fundamental entities in TGD based cosmology and astrophysics and also in TGD inspired quantum biology. One can assign elliptic curves also to string like objects.

1. Quite generally, the products $X^2 \times Y^2 \subset M^4$ of string world sheets X^2 and complex surfaces Y^2 of CP_2 define extremals that I have called cosmic strings [K8].
2. Elliptic curves allow a standard embedding to CP_2 as complex surfaces constructible in terms of Weierstrass elliptic function $\mathcal{P}(z)$ (`http://tinyurl.com/ycu8oa4r`) satisfying the identity

$$[\mathcal{P}'(z)]^2 = [\mathcal{P}(z)]^3 - g_2\mathcal{P}(z) - g_3 \quad . \tag{4.8.2}$$

Here g_2 and g_3 are modular invariants. This identity is of the same form as the condition $y^2 = x^3 + ax + b$ with identifications $y = \mathcal{P}'(z), x = \mathcal{P}(z)$ and $(a = -g_2, b = -g_3)$. From the expression

$$y^2 = x(x-1)(x-\lambda) \tag{4.8.3}$$

in terms of the modular invariant $\lambda = \omega_1/\omega_2$ of torus one obtains

$$g_2 = \frac{4^{1/3}}{3}(\lambda^2 - \lambda + 1 \quad , \quad g_3 = \frac{1}{27}(\lambda+1)(2\lambda^2 - 5\lambda + 2) \quad . \tag{4.8.4}$$

Note that third root of a appears in the formula. The so called modular discriminant

$$\Delta = g_2^3 - 27g_3^2 = \lambda^2(\lambda-1)^2 \quad . \tag{4.8.5}$$

vanishes for $\lambda = 0$ and $\lambda = 1$ for which the lattice degenerates.
3. The embedding of the elliptic curve to CP_2 can be expressed in projective coordinates of CP_2 as

$$(z^1, z^2, z^3) = (\xi^1, \xi^2, 1) = (\frac{\mathcal{P}'(w)}{2}, \mathcal{P}(w), 1) \quad . \tag{4.8.6}$$

4.9 Are fundamental entities discrete or continuous and what discretization at fundamental level could mean?

There was an interesting FB discussion about discrete and continuum. I decided to write down my thoughts and emphasize those points that I see as important.

4.9.1 Is discretization fundamental or not?

The conversation inspired the question whether discreteness is something fundamental or not. If it is assumed to be fundamental, one encounters problems. The discrete structures are not unique. One has deep problem with the known space-time symmetries. Symmetries are reduced to discrete subgroup or totally lost. A further problem is the fact that in order to do physics, one must bring in topology and length measurements.

In discrete situation topology, in particular space-time dimension, must be put in via homology effectively already meaning use of embedding to Euclidian space. Length measurement remains completely ad hoc. The construction of discrete metric is highly non-unique procedure and the discrete analog of of say Einstein's theory (Regge calculus) is rather clumsy. One feeds in information, which was not there by using hand weaving arguments like infrared limit. It is possible to approximate continuum by discretization but discrete to continuum won't go.

In hype physics these hand weaving arguments are general. For instance, the emergence of 3-space from discrete Hilbert space is one attempt to get continuum. One puts in what is factually a discretization of 3-space and then gets 3-space back at IR limit and shouts "Eureka!".

4.9.2 Can one make discretizations unique?

Then discussion went to numerics. Numerics is for mathematicians same as eating for poets. One cannot avoid it but luckily you can find people doing the necessary programming if you are a professor. Finite discretization is necessary in numerics and is highly unique.

I do not have anything personal against discretization as a numerical tool. Just the opposite, I see finite discretization as absolutely essential element of adelic physics as an attempt to describe also the correlates of cognition in terms of p-adic physics with p-adic space-time sheets as correlates of "thought bubbles" [L42, L43]. Cognition is discrete and finite and uses rational numbers: this is the basic clue.

1. Cognitive representations are discretizations of (for instance) space-time surface. One can say that physics itself builds its cognitive representation in all scales using p-adic space-time sheets. They should be unique once measurement resolution is characterized if one is really talking about fundamental physics.
   The idea abou tp-adic physics as physics of cognition indeed led to powerful calculational recipes. In p-adic thermodynamics the predictions come in power series of p-adic prime p and for the values of p assignable to elementary particles the two lowest terms give practically exact result [K50]. Corrections are of order 10^{-76} for electron characterized by Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38}$.
2. Adelic physics [L42] provides the formulation of p-adic physics: it is assumed that cognition is universal. Adele is a book like structure having as pages reals and extensions of various p-adic number fields induced by given extension of rationals. Each extension of rationals defines its own extension of the rational adele by inducing extensions of p-adic number fields. Common points between pages consist of points in extension of rationals. The books associated with the adeles give rise to an infinite library.
   At space-time level the points with coordinates in extension define what I call cognitive representation. In the generic case it is discrete and has finite number of points. The loss of general coordinate invariance is the obvious objection. In TGD however the symmetries of the embedding space fix the coordinates used highly uniquely. $M^8 - H$ duality ($H = M^4 \times CP_2$) and octonionic interpretation implies that M^8 octonionic linear coordinates are highly unique [L36, L67]. Note that M^8 must be complexified. Different coordinatizations correspond to different octonionic structures- to different moduli - related by Poincare transformations of M^8. Only rational time translations as transformations of octonionic real coordinate are allowed as coordinate changes respecting octonionic structure.
3. Discretization by cognitive representation is unique for given extension of rationals defining the measurement resolution. At the limit of algebraic numbers algebraic points form a dense set of real space-time surface and p-adic space-time surfaces so that the measurement resolution is ideal. One avoids the usual infinities of quantum field theories induced by continuous delta functions, which for cognitive representations are replaced with Kronecker deltas. This

seems to be the best that one can achieve with algebraic extensions of rationals. Also for transcendental extensions the situation is discrete.
This leads to a number theoretic vision about second quantization of induced spinor fields central for the construction of gamma matrices defining the spinor structure of "world of classical worlds" (WCW) providing the arena of quantum dynamics in TGD analogous to the super-space of Wheeler [K76]. One ends up to a construction allowing to understand TGD view about SUSY as necessary aspect of second quantization of fermions and leads to the conclusions that in the simplest scenario only quarks are elementary fermions and leptons can be seen as their local composites analogous to super partners.

4. Given polynomial defining space-time surfaces in M^8 defines via its roots extension of rationals. The hierarchy of extensions defines an evolutionary hierarchy. The dimension n of extension defines kind of IQ measuring algebraic complexity and n corresponds also to effective Planck constant labelling phases of dark matter in TGD sense so that a direct connection with physics emerges.
Embedding space assigns to a discretization a natural metric. Distances between points of metric are geodesic distances computed at the level of embedding space.

5. An unexpected finding was that the equations defining space-time surfaces as roots of real or imaginary parts of octonionic polynomials have also 6-D brane like entities with topology of S^6 as solutions [L63, L75]. These entities intersect space-time surfaces at 3-D sections for which linear M^4 time is constant. 4-D roots can be glued together along these branes. These solutions turn out to have an interpretation in TGD based theory of quantum measurement extending to a theory of consciousness. The interpretation as moments of "small" state function reductions as counterparts of so called weak measurements. They could correspond to special moments in the life of conscious entity.

4.9.3 Can discretization be performed without lattices?

For a systems obeying dynamics defined by partial differential equations, the introduction of lattices seems to be necessary aspect of discretization. The problem is that the replacement of derivatives with discrete approximations however means that there is no hope about exact results. In the general case the discretization for partial differential equations involving derivatives forces to introduce lattice like structures. This is not needed in TGD.

1. At the level of M^8 ordinary polynomials give rise to octonionic polynomials and space-time surfaces are algebraic surfaces for which imaginary or real part of octonionic polynomial in quaternionic sense vanishes. The equations are purely algebraic involving no partial derivatives and there is no need for lattice discretization.
For surfaces defined by polynomials the roots of polynomial are enough to fix the polynomials and therefore also the space-time surface uniquely: discretization is not an approximation but gives an exact result! This could be called number theoretical holography and generalizes the ordinary holography. Space-time surfaces are coded by the roots of polynomials with rational coefficients.

2. What about the field equations at the level of $H = M^4 \times CP_2$? $M^8 - H$ duality maps these surfaces to preferred extremals as 4-surfaces in H analogous to Bohr orbits. Twistor lift of TGD predicts that they should be minimal surfaces with 2-D singularities being also extremals of 4-D Kähler action. The field equations would reduce locally to purely algebraic conditions. In properly chosen coordinates for H they are expected to be determined in terms of polynomials coding for the same extension of rationals as their M^8 counterparts so that the degree should be same [L67]. This would allow to deduce the partial derivatives of embedding space for the image surfaces without lattice approximation.

3. The simplest assumption is that the polynomials have rational coefficients. Number theoretic universality allows to consider also algebraic coefficients. In both cases also WCW is discretized and given point -space-time surface in QCD has coordinates given by the points of the number theoretically universal cognitive representation of the space-time surface. Even real coefficients are possible. This would allow to obtain WCW as a continuum central for the construction of WCW metric but is not consistent with number theoretical universality.

Can one have polynomial/functions with rational coefficients and discretization of WCW without lattice but without losing WCW metric? Maybe the same trick that works at space-time level works also in WCW!

(a) The group WCW isometries is identified as symplectic transformations of $\delta M^4_\pm \times CP_2$ ($\delta M^4_\pm$ denotes light-cone boundary) containing the boundary of causal diamond CD. The Lie algebra *Sympl* of this group is analogous half-Kac Moody algebra having symplectic transformations of $S^2 \times CP_2$ as counterpart of finite-D Lie group has fractal structure containing infinite number of sub-algebras $Sympl_n$ isomorphic to algebra itself: the conformal weights assignable to radial light-like coordinate are n-multiples of those for the entire algebra. Note that conformal weights of *Sympl* are non-negative.

(b) One formulation for the preferred extremal property is in terms of infinite number of analogs of gauge conditions stating the vanishing of classical and also Noether charges for $Sympl_n$ and $[Sympl_n, Sympl]$. The conditions generalize to the super-counterpart of *Sympl* and apply also to quantum states rather than only space-time surfaces. In fact, while writing this I realized that - contrary to the original claim - also the vanishing of the Noether charges of higher commutators is required so that effectively $Sympl_n$ would define normal subgroup of *Sympl*. These conditions does not follow automatically.
The Hamiltonians of $Sympl(S^2 \times CP_2)$ are also labelled by the representations of the product of the rotation group $SO(3) \subset SO(3,1)$ of S^2 and color group $SU(3)$ together forming the analog of the Lie group defining Kac-Moody group. This group does not have have the fractal hierarchy of subgroups. The strongest condition is that the algebra corresponding to Hamiltonian isometries does not annihilate the physical states.
The space of states satisfying the gauge conditions is finite-D and that WCW becomes effectively finite-dimensional. A coset space associated with *Sympl* would be in question and it would have maximal symmetries as also WCW. The geometry of the reduced WCW, WCW_{red} could be deduced from symmetry considerations alone.

(c) Number theoretic discretization would correspond to a selection of points of this subspace with the coordinates in the extension of rationals.The metric of $\text{WCW}_{red,n}$ at the points of discretization would be known and no lattice discretization would be needed. The gauge conditions are analogous to massless Dirac equation in WCW and could be solved in the points of discretization without introducing the lattice to approximate derivatives. As a matter fact, Dirac equation can be formulated solely in terms of the generators of *Sympl*.

(d) This effectively restricts WCW to $WCW_{red,n}$ in turn reduced to its discrete subset - since infinite number of WCW coordinates are fixed. If this sub-space can be regarded as realization of infinite number of algebraic conditions by polynomials with rational coefficients one can assign to it extension of rationals defining naturally the discretization of $\text{WCW}_{red,n}$. This extension is naturally the same as for space-time surfaces involved so that the degree of polynomials defining $\text{WCW}_{red,n}$ would be naturally n and same as that for the polynomial defining the space-time surface. $\text{WCW}_{red,n}$ would decompose to union of spaces WCW_{red,E_n} labelled by extensions E_n of rationals with same dimension n.
There is analogy with gauge fixing. WCW_{red,E_n} is a coset space of WCW defined by the gauge conditions. One can represent this coset space as a sub-manifold of WCW by taking one representative point from each coset. This choice is not unique but one can hope finding a gauge choice realized by an infinite number of polynomials of degree n defining same extension of rationals as the polynomial defining the space-time surfaces in question.

(e) WCW spinor fields would be always restricted to finite-D algebraic surface of WCW_{red,E_n} expressible in terms of algebraic equations. Finite measurement resolution indeed strongly suggests that WCW spinor field mode is non-vanishing only in a region parameterized in WCW by finite number of parameters. There is also a second manner to see this. WCW_{red,E_n} could be also seen as $n+4$-dimensional surface in WCW.

(f) One can make this more concrete. Cognitive representation by points of space-time surface with coordinates in the extension - possibly satisfying additional conditions such as belonging to the 2-D vertices at which space-time surfaces representing different roots meet - provides WCW coordinates of given space-time surface. Minimum number of points corresponds to the dimension of extension so that the selection of coordinate can be redundant. As the values of these coordinates vary, one obtains coordinatization for the

sector of WCW_{red,E_n}. An interesting question is whether one could represent the distances of space-time surfaces in this space in terms of the data provided by the points of discretization.
An interesting question is whether one can represent the distances of space-time surfaces in this space in terms of the data provided by the points of cognitive representation. One can define distance between two disjoint surfaces as the minimum of distance between the points of 2-surfaces. Could something like this work now? The points would be restricted to the cognitive representations. Could one define the distance between two cognitive representations with same number N of points in the following manner.
Consider all bipartitions formed by the cognitive representations obtained by connecting their points together in 1-1 manner. There are N! bipartitions of this kind if the number of points is N. Calculate the sum of the squares of the embedding space distances between paired points. Find the bipartition for which this distance squared is minimum and define the distance between cognitive representations as this distance. This definition works also when the numbers of points are different.

(g) If there quantum states are the basic objects and there is nothing "physical" behind them one can ask how we can imagine mathematical structures which different from basic structure of TGD. Could quantum states of TGD Universe in some sense represent all mathematical objects which are internally consistent. One could indeed say that at the level of WCW all $n+4$-D manifolds can be represented concretely in terms of WCW spinor fields localized to n-D subspaces of WCW. WCW spinor fields can represent concept of 4-surface of $\text{WCW}_{red,n}$ as a quantum superposition of its instance and define at the same time $n+4$-D surfaces [L78] [L65, L70, L69, L78].

4.9.4 Simple extensions of rationals as codons of space-time genetic code

A fascinating idea is that extensions of rationals define the analog of genetic code for space-time surfaces, which would therefore represent number theory and also finite groups.

(a) The extensions of rationals define an infinite hierarchy: the proposal is that the dimension of extensions corresponds to the integer n characterizing subalgebra $Sympl_n$. This would give direct correspondence between the inclusions of HFFs assigned to the hierarchy of algebras $Sympl_n$ and hierarchy of extensions of rationals with dimension n.
Galois group for a extension of extension contains Galois group of extension as normal subgroup and is therefore *not simple*. Extension hierarchies correspond to inclusion hierarchies for normal subgroups. Simple Galois groups are in very special position and associated with what one might call simple extensions serving as fundamental building bricks of inclusion hierarchies. They would be like elementary particles and define fundamental space-time regions. Their Galois groups would act as groups of physical symmetries.

(b) One can therefore talk about elementary space-time surfaces in M^8 and their compositions by function composition of octonionic polynomials. Simple groups would label elementary space-time regions. They have been classified: (see `http://tinyurl.com/y3xh4hrh`). The famous Monster groups are well-known examples about simple finite groups and would have also space-time counterparts. Also the finite subgroups of Lie groups are special and those of $SU(2)$ are associated with Platonic solids and seem to play key role in TGD inspired quantum biology. In particular, vertebrate genetic code can be assigned to icosahedral group.

(c) There is also an analogy with genes. Extensions with simple Galois groups could be seen as codons and sequences of extension obtained by functional composition as analogs of genes. I have even conjectured that the space-time surfaces associated with genes could quite concretely correspond to extensions of extensions of ...

4.9.5 Are octonionic polynomials enough or are also analytic functions needed?

I already touched the question whether also analytic functions with rational coefficients (number theoretical universality) might be needed.

(a) The roots of analytic functions generate extension of rationals. If the roots involve transcendental numbers they define infinite extensions of rationals. Neper number e is very special in this sense since e^p is ordinary p-adic number for all primes p so that the induced extension is finite-dimensional. One could thus allow it without losing number theoretical universality. The addition of π gives infinite-D extension but one could do by adding only roots of unity to achieve finite-D extensions with finite accuracy of phase measurement. Phases would be number theoretically universal but not angles.

(b) One could of course consider only transcendental functions with rational roots. Trigonometric function $sin(x/2\pi)$ serves as a simple example. One can also argue that since physics involves in an essential manner trigonometric functions via Fourier analysis, the inclusion of analytic functions with algebraic roots must be allowed.

(c) What about analytic functions as limits of polynomials with rational coefficients such that the number of roots becomes infinite at the limit? Also their imaginary and real part can vanish in quaternionic sense and could define space-time surfaces - analogs of transcendentals as space-time surfaces. It is not clear whether these could be allowed or not.

Could one have a universal polynomial like function giving algebraic numbers as the extension of rationals defined by its algebraic roots? Could Riemann zeta (see `http://tinyurl.com/nfbkrsx`) code algebraic numbers as an extension via its roots. I have conjectured that roots of Riemann zeta are algebraic numbers: could they span all algebraic numbers?

It is known that the real or imaginary part of Riemann zeta along $s = 1/2$ critical line can approximate any function to arbitrary accuracy: also this would fit with universality. Could one think that the space-time surface defined as root of octonionic continuation of zeta could be universal entity analogous to a fixed point of iteration in the construction of fractals? This does not look plausible.

4. One can construct iterates of Riemann zeta having at least the same roots as zeta by the rule

$$\begin{aligned} &f_0(s) = \zeta(s) \ , \\ &f_n(s) = \zeta(f_{n-1}(s)) - \zeta(0), \ \ \zeta(0) = -1/2 \ . \end{aligned} \quad (4.9.1)$$

ζ is not a fixed point of this iteration as the fractal universality would suggest. The set of roots however is. Should one be happy with this.

5. Riemann zeta has also counterpart in all extensions of rationals known as Dedekind zeta (see `http://tinyurl.com/y5grktv`) [L40, L77, L68]. Riemann zeta is therefore not unique. One can ask whether Dedekind zetas associated with simple Galois groups are special and whether Dedekind zetas associated with extensions of extensions of .... can be constructed by using the Dedekind zetas of simple extensions. How do the roots of Dedekind zeta depend on the associated extension of rationals? How the roots of Dedekind zeta for extension of extension defined by composite of two polynomials depend on extensions involved? Are the roots union for the roots associated with the composites?

6. What about forming composites of Dedekind zetas? Categorical according to my primitive understanding raises the question whether a composition of extensions could correspond to a composition of functions. Could Dedekind zeta for a composite of extensions be obtained from a composite of Dedekind zetas for extensions? Requiring that roots of extension E_1 are preserved would give formula

$$\zeta_{D,E_1E_2} = \zeta_{D,E_1} \circ \zeta_{D,E_2} - \zeta_{D,E_1}(0) \ . \quad (4.9.2)$$

The zeta function would be obtained by an iteration of simple zeta functions labelled by simple extensions. The inverse image for the set of roots of ζ_{D,E_1} under ζ_{D,E_2} that is the set $\zeta^{-1}_{D,E_2}(roots(\zeta_{D,E_1})$ would define also roots of ζ_{D,E_1E_2}. This looks rather sensible.

But what about iteration of Riemann zeta, which corresponds to trivial extension? Riemann ζ is not invariant under iteration although its roots are. Should one accept this and say that it is the set of roots which defines the invariant. Could one say that the iterates of various Dedekind zetas define entities which are somehow universal.

Chapter 5

Could quantum randomness have something to do with classical chaos?

5.1 Introduction

There was an interesting guest post by Tim Palmer in the blog of Sabine Hosssenfelder (`http://tinyurl.com/yx7htn3u`).

5.1.1 Palmer's idea

Consider first what was said in the post "Undecidability, Uncomputability and the Unity of Physics. Part 1" by Tim Palmer.

1. I understood (perhaps mis-) that the idea is to reduce quantum randomness to classical chaos. If this is taken to mean that quantum theory reduces to chaos theory, I will not follow. The precise rules of quantum measurement having interpretation as measurements performed for the observables - typically generators of symmetries - are very restrictive and it is extremely difficult to image that classical physics could explain them. Quantum theory is much more than probability theory. Probabilities are essentially moduli squared for probability amplitudes and this gives rise to interference and entanglement. Therefore the idea of reducing state function reduction (SFR) and quantum randomness to classical chaos does not look promising. One could however consider the possibility classical chaos is in some sense as a correlate for quantum randomness or associated with state function reductions.
2. The difficulty to combine general relativity (GRT) to quantum gravity was mentioned. The difficulty is basically due to the loss of Poincare symmetries in curved space-time. Already string models solve the problem by assuming that strings live in M^{10} or its spontaneous compactification. Strings are however 2-D, not 4-D, and this leads to a catastrophe. In TGD $H = M^4 \times CP_2$ allows to have Poincare invariance and conservation laws are not lost. In QFT picture this means that the existence of energy guarantees existence of Hamiltonian defining time evolution operator and S-matrix.
3. It was noticed that chaos in quantum theory cannot be assigned to Schrödinger equation. This is true and applies quite generally to unitary time evolution generated by unitary S-matrix acting linearly. It as also noticed that in statistical mechanism Liouville operator defines a linear equation for phase space probability distribution analogous to Schrödinger equation. Liouville equation allows the classical system to be non-linear and chaotic. Could Schrödinger equation in some sense replace Liouville equation in in quantum theory since phase space ceases to make sense by Uncertainty Principle.
   Could Schrödinger equation allow in some sense non-linear chaotic classical systems? In Copenhagen interpretation no classical system exists except at macroscopic limit as an approximation. One has only wave function coding for the knowledge about physical system changing in

quantum measurement. There is no classical reality and there are no classical orbits of particle since one gives up the notion of Bohr orbit. Could Bohr orbit be more than approximation?

The author considers also the question about definition of chaos.

1. Chaos is difficult to define in GRT. The replacement time coordinate with its logarithm exponentially growing difference becomes linearly growing and one does not have chaos. By general coordinate invariance this definition of chaos does not therefore make sense.
2. Strange attractors are typical asymptotic situations in chaotic systems and can make sense also in general relativity (GRT). They represent lower dimensional manifolds to which the dynamics of the system is restricted asymptotically. It is not possible to predict to which strange attractor the chaotic dynamical system ends up. This definition of chaos makes sense also in GRT.

Remark: One must remember that the notion of chaos is often used in misleading sense. The increase of complexity looks like chaos for external observer but need not have anything to do with genuine chaos.

5.1.2 Could TGD allow realization of Palmer's idea in some form?

It came as a surprise to me that these to notions could a have deep relationship in TGD framework.

1. Strong form of Palmer's idea stating that quantum randomness reduces to classical chaos certainly fails but one can consider weaker forms of the idea. Even these variants fail in Copenhagen interpretation since strictly speaking there is no classical reality, only wave function coding for the knowledge about the system. Bohr orbits should be more than approximation and in TGD framework space-time surface as preferred extremal of action is analogous to Bohr orbit and classical physics defined by Bohr orbits is an exact part of quantum theory.
2. In the zero energy ontology (ZEO) of TGD the idea works in weaker form and has very strong implications for the more detailed understanding of ZEO and $M^8-M^4\times CP_2$ duality. Ordinary ("big") state functions (BSFRs) meaning the death of the system in a universal sense and re-incarnation with opposite arrow of time would involve quantum criticality accompanied by classical chaos assignable to the correspondence between geometric time and subjective time identified as sequence of "small" state function reductions (SSFRs) as analogs of weak measurements. The findings of Minev *et al* [L59] give strong support for this view [L59] and Libet's findings about active aspects of consciousness [J1] can be understood if the act of free will corresponds to BSFR.

M^8 picture identifies 4-D space-time surfaces X^4 as roots for "imaginary" or "real" part of octonionic polynomial P_2P_1 obtained as a continuation of real polynomial $P_2(L-r)P_1(r)$, whose arguments have origin at the the tips of B and A and roots a the light-cone boundaries associated with tips. Causal diamond (CD) is identified intersection of future and past directed light-cones light-cones A and B. In the sequences of SSFRs $P_2(L-r)$ assigned to B varies and $P_1(r)$ assigned to A is unaffected. L defines the size of CD as distance $\tau = 2L$ between its tips.

Besides 4-S space-time surfaces there are also brane-like 6-surfaces corresponding to roots $r_{i,k}$ of $P_i(r)$ and defining "special moments in the life of self" having $t_i = r_{i,k}$ ball as M^4_+ projection. The number of roots and their density increases rapidly in the sequence of SSFRs. The condition that the largest root belongs to CD gives a lower bound to it size L as largest root. Note that L increases.

Concerning the approach to chaos, one can consider three options.

Option I: The sequence of steps consisting of unitary evolutions followed by SSFR corresponds to a functional factorization at the level of polynomials as sequence $P_2 = Q_1 \circ Q_2 \circ ...Q_n$. The size L of CD increases if it corresponds to the largest root, also the tip of active boundary of CD must shift so that the argument of P_2 $L-r$ is replaced in each iteration step to with updated argument with larger value of L identifiable as the largest root of P_2.

Option II: A completely unexpected connection with the iteration of analytic functions and Julia sets, which are fractals assigned also with chaos interpreted as complexity emerges. In a reasonable approximation quantum time evolution by SSFRs could be induced by an iteration of a polynomial or even an analytic function: $P_2 = P_2 \rightarrow P_2^{\circ 2} \rightarrow ....$ For $P_2(0) = 0$ the roots of the iterate consists of inverse images of roots of P_2 by $P_2^{\circ -k}$ for $k = 0, ..., N-1$.

Suppose that M^8 and X^4 are complexified and thus also $t = r$ and "real" X^4 is the projection of X_c^4 to real M^8. Complexify also the coefficients of polynomials P. If so, the Mandelbrot and Julia sets (http://tinyurl.com/cplj9pe and http://tinyurl.com/cvmr83g) characterizing fractals would have a physical interpretation in ZEO.

Chaos is approached in the sense that the inverse images of the roots of P_2 assumed to belong to filled Julia set approaching to points of Julia set of P_2 as the number N of iterations increases in statistical sense. The size L as largest root of $P_2^{\circ N}$ would increase with N if CD is assumed to contain all roots. The density of the roots in Julia set increases near L since the size of CD is bounded by the size Julia set. One could perhaps say that near the $t = L$ in the middle of CD the life of self when the size of CD has become almost stationary, is the most intensive.

Option III: A conservative option is to consider only real polynomials $P_2(r)$ with real argument r. Only non-negative real roots r_n are of interest whereas in the general case one considers all values of r. For a large N the inverse iterates of the roots of P_2 would approach to the real Julia set obtained as a real projection of Julia set for complex iteration.

How the size L of CD is determined and when can BSFR occur?

Option I: If L is minimal and thus given by the largest root of $P_2^{\circ N}$ in Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). Should L be smaller than the sizes of Julia sets of both$A$ and B if the iteration gives no roots outside Julia set.

Could BSFR become probable when L as the largest allowed root for $P_2^{\circ N}$ is larger than the size of Julia set of A? There would be no more new "special moments in the life of self" and this would make death and re-incarnation with opposite arrow of time probable. The size of CD could decrease dramatically in the first iteration for P_1 if it is determined as the largest allowed root of P_1: the re-incarnated self would have childhood.

Option II: The size of CD could be determined in SSFR statistically as an allowed root of P_2. Since the density of roots increases, one would have a lot of choices and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time. BSFR would occur for the same reason as for the first option.

The fact that fractals quite generally assignable to iteration (http://tinyurl.com/ctmcdx5) appear everywhere gives direct support for the ZEO based view about consciousness and self-organization and would give a completely new meaning for "self" in "self-organization" [L68]. Fractals, quantum measurement theory, theory of self-organization, and theory of consciousness would be closely related.

5.2 Could classical chaos and state function reduction relate to each other in TGD Universe?

In the sequel the idea about connection between chaos in some sense and state function reductions as they are understood in ZEO is discussed.

5.2.1 Classical physics is an exact part of quantum physics in TGD

Concerning the relation between classical and quantum the situation changes in TGD framework. Classical physics becomes an exact part of quantum theory. In zero energy ontology (ZEO) quantum states are superpositions of space-time surfaces preferred extremals of basic variational principle connecting 3-surfaces at opposite boundaries of causal diamond (CD). This solves the well-known basic problem of quantum measurement theory. Unitary time evolution operator or its generalization are totally different things from classical time evolution defined by highly non-linear field equations. There is nothing preventing quantum counterpart of chaos - it need not be classical chaos at space-time level but could correspond to some other form of chaos. Ordinary state function reduction in ZEO involves naturally quantum criticality involving long range quantum fluctuations assignable to chaotic systems so that the correlation between classical chaos defined in proper manner and state function reduction might make sense.

5.2.2 TGD space-time and $M^8 - H$ duality

$M^8 - H$ duality combined with zero energy ontology (ZEO) is central for the TGD inspired proposal for the connection between chaos and quantum.

Basic vision

Consider first what TGD space-time is.

1. In TGD framework space-times can be regarded 4-surfaces in $H = M^4 \times CP_2$ or in complexifiation of octonionic M^8. Linear Minkowski coordinates or Robertson-Walker coordinates for light-cone (used in TGD based cosmology) provide highly unique coordinate choice and this problem disappears. Exponential divergence in M^4 coordinates could be used as a symptom for a chaotic behavior.
2. The solutions of field equations are preferred extremals satisfying extremely powerful additional conditions giving rise to a huge generalization of the ordinary 2-D conformal symmetry to 4-D context. In fact, twistor twist of TGD predicts that one has minimal surfaces, which are also extremals of 4-D Kähler action apart from 2-D singularities identifiable as string world sheets and partonic 2-surfaces having a number theoretical interpretation. The huge symmetries act as maximal isometry group of "world of classical worlds" (WCW) consisting of preferred extremals connecting pair of 3-surfaces, whose members are located at boundaries of causal diamond (CD). These symmetries strongly suggest that TGD represents completely integrable system and thus non-chaotic and diametrical opposite of a chaotic system. Therefore the chaos - if present - would be something different.

$M^8 - H$ duality suggests an analogous picture at the level of M^8. $M^8 - H$ duality in itse most restrictive form states that space-time surfaces are characterized by "roots" of rational polynomials extended to complexified octonionic ones by replacing the real coordinate by octonionic coordinate o [L37, L38, L39].

1. One can define the imaginary and real parts $IM(P)$ and $RE(P)$ of $P(o)$ in octonionic sense by using the decomposition of octonions $o = q_1 + I_4 q_2$ to two quaternions so that $IM(P)$ and $RE(P)$ are quaternion valued. For 4-D space-time surfaces one has either $IM(P) = 0$ or $RE(P) = 0$ in the generic case. The curve defined by the vanishing of imaginary or real part of complex function serves as the analog.
2. If the condition $P(0) = 0$ is satisfied, the boundary of δM^8_+ of M^8 light-cone is special. By the light-likeness of δM^8_+ points the polynomial $P(o)$ at δM^8_+ reduces to ordinary real polynomial $P(r)$ of the radial M^4 coordinate r identifiable as linear M^4 time coordinate t: $r = t$.
   Octonionic roots $P(o) = 0$ at M^8 light-cone reduce to roots $t = r_n$ of the real polynomial $P(r)$ and give rise to 6-D exceptional solutions with $IM(P) = RE(P) = 0$ vanish. The solutions are located to δM^8_+ and have topology of 6-sphere S^6 having 3-balls B^3 with $t = r_n$ as of M^4_+ projections. The "fiber" at point of B^3 with radial M^4 coordinate $r_M \leq r_n$ is 3-sphere $S^3 \subset E^4 \subset M^8 = M^4 \times E^4$ contracting to point at the δM^4_+.
   These 6-D objects are analogous to 5-branes in string theory and define "special moments in the life of self". At these surfaces the 4-D "roots" for $IM(P)$ or $RE(P)$ intersect and intersection is 2-D partonic surface having interpretation as a generalization of vertex for particles generalized to 3-D surfaces (instead of strings). In string theory string world sheets have boundaries at branes. Strings are replaced with space-time surfaces and branes with "special moments in the life of self".
   Quite generally, one can consider gluing 4-D "roots" for different polynomials P_1 and P_2 at surface $t = r_n$ when r_n is common root. For instance, P and its iterates $P^{\circ N}$ having r_n and the lower inverse iterates as common roots can be glued in this manner.
3. It is possible complexify M^8 and thus also r. Complexification is natural since the roots of P are in general complex. Also 4- space-time surface is complexified to 8-D surface and real space-time surface can be identified as its real projection.

To sum up, space-time surfaces would be coded a polynomial with rational or at most algebraic coefficients. Essentially the discrete data provided by the roots r_n of P would dictate the space-time surface so that one would have extremely powerful form of holography.

One can consider generalizations of the simplest picture.

1. One can also consider a generalization of polynomials to general analytic functions F of octonions obtained as octonionic continuation of a real function with rational Taylor coefficients: the identification of space-time surfaces as "roots" of $IM(F)$ or $RE(F)$ makes sense.
2. What is intriguing that for space-time surfaces for which $IM(F_1) = 0$ and $IM(F_2) = 0$, one has $IM(F_1F_2) = RE(F_1)IM(F_2)+IM(F_1)RE(F_2) = 0$. One can multiply space-time surfaces by multiplying the polynomials. Multiplication is possible also when one has $RE(F_1) = 0$ and $IM(F_2) = 0$ or $RE(F_2) = 0$ or $IM(F_1) = 0$ since one has $RE(F_1F_2) = RE(F_1)RE(F_2) - IM(F_1)IM(F_2) = 0$.
For $IM(F) = 0$ type space-time surfaces one can even define polynomials analytic functions of the space-time surface with rational Taylor coefficients. One could speak of functions having space-time surface as argument, space-time surface itself would behave like number.
3. One can also form functional composites $P \circ Q$ (also for analytic functions with complex coefficients). Since $P \circ Q$ at $IM(Q) = 0$ surface is quaternionic, its image by P is quaterionic and satisfies $IM(P \circ Q) = 0$ so that one obtains a new solution. One can iterate space-time surfaces defined by $Im(P) = 0$ condition by iterating these polynomials to give $P, P^{circ2}, ..., P^{\circ N}$... From $IM(P) = 0$ solutions one obtains a solutions with $RE(Q) = 0$ by multiplying the M^8 coordinates with I_4 appearing in $o = q_1 + I_4q_2$.
The $Im(P) = 0$ solutions can be iterated to give $P \to P \circ P \to$ .., which suggests that the sequence of SSFRs could at least approximately correspond to the dynamics of iterations and generalizations of Mandelbrot and Julia sets and other complex fractals and also their space-time counterparts. Chaos (or rather, complexity theory) including also these fractals could be naturally part of TGD!

Building many-particle states at the level of M^8

The polynomials defining surfaces in M^8 are defined in preferred M^8 coordinates with preferred selection of M^8 time axis M^1 as real octonionic axis and one octonionic imaginary axes characterizing subspace $M^2 \subset M^8$. $M^4 \subset M^8$ is quaternionic subspace containing M^2. Different choices of $M^4 \sup M^2$ are labelled by points of CP_2 and $M^8 - H$ duality maps these choices to points of CP_2.

The origin of $M8$ coordinates coordinates must be at M^1 so that the 8-D Poincare symmetry reduces to time translations and rotations of around spatial coordinate axis M^2 respecting the rationality of polynomial coefficients or in more general case the extension of rationals associated with the coefficients. This corresponds to a selection of quantization axis for energy and angular momentum and could have a deeper meaning in quantum measurement theory.

The Lorentz transformations of M^4 change the direction of time axis and also M^2 in the general case and generate new octonionic structure and quaternonic structure. One should understand how space-time regions as roots of octonionic polynomials with different rest frames relate to each other.

The intuitive picture is that each particle as a region determined by octonionic polynomial corresponds to its own CD and rest frame determined by its 4-momentum in fixed coordinate frame for M^4. Also quantization axis of spin fixed. One can assign CD for to interacting many particle system with common rest frame. One can speak of external (incoming and outgoing) free particles with their own CDs characterizing their rest systems. The challenges is to related the polynomials P_n associated with the external particles to the polynomial characterizing the interacting system.

1. Assume that the polynomial defining the CD is product P_1P_2 of polynomials P_1 and P_2 assignable to its active and passive boundaries with origins of octonionic coordinates at the tips $t = 0$ and $t = \tau$ of CD. If the space-time surface reduces to the root of P_1 at passive boundary and root of $IM(P_2)$ at active boundary, one could say that the 3-surfaces at these boundaries correspond to P_1 and P_2 asymptotically. If these conditions are true everywhere, one has two un-correlated space-time surfaces, which does not make sense. $IM(P_1)RE(P_2) + RE(P_1)IM(P_2) = 0$ indeed allows more general solutions than $IM(P_1) = 0$ and $IM(P_2) = 0$ everywhere. The fact that the boundaries correspond to special 6-D brane like solutions in M^8 suggests that it is possible to pose the boundary condition $IM(P_1) = 0$ *resp.* $IM(P_2) = 0$ at the boundaries.
2. The formation of products is possible also at the boundaries so that one can assume that P_i at the boundary of many-particle CD is with product $P_i = \prod_k P_{ik}$. The boundary conditions

would read read $P_{ik} = 0$ at active *resp.* passive boundary of many-particle CD respectively. The interpretation would be that P_{ik} corresponds to an external particle which is in interacting state at active boundary. In the interior of many-particle CD only the condition $Im(P_1P_2) = 0$ would hold true so that interactions of particles would have algebraic description.

3. One should also understand how the external particles characterized by CDs with different rest frame are glued to the boundary of many-particle CD. Assume that M^4 is same for all these particles so that CP_2 coordinates are same. The boundaries of 4-D CDs are 3-D light-cones with different origins so that their M^4 intersection is 2-D defining a 2-D surface at the boundary of CD. The interpretation in terms of partonic 2-surface suggests itself. The partonic 2-surfaces of free particle and its interacting variant would be same at the intersection.
   The gluing should correspond to a root $t = r_n$ of polynomial defining a "special moment in the life of self". The roots of P_1 and its Lorentz boots as values of coordinates at light-radial geodesic are related by Lorentz boost and are not same in general. One could require that the root r_n and its Lorentz boost belong to the 2-D interaction of two light-cones and thus define two points of partonic 2-surface. These points would not be identical and the interaction would be non-local in the scale of partonic 2-surface. It seems that the condition that root r_n and Lorentz boost $L(rm)$ co-incide would pose too strong constraints on external momenta.

5.2.3 In what sense chaos/complexity could emerge in TGD Universe?

Consider now in what sense chaos (or complexity, one must be precise here) could emerge in TGD framework?

1. Chaos (or complexity) could be an approximate property emerging in number theoretical discretization for cognitive representations labelled by extensions of rationals as the dimension of extension and therefore algebraic complexity increases ad the number of points in cognitive representation as points of M^8 with coordinates in the extension of rationals increases. The minimal number of points corresponds to the degree of the polynomial determining the extension. At the limit of maximal complexity the extension would consists of algebraic numbers and the cognitive representation would be dense subset of space-time surface. It is not clear whether the roots r_n are also dense along time axis.

2. Also transcendental extensions of rationals can be considered. Typically they are infinite-D in both real and p-adic sectors. Exponential function is however number-theoretically completely unique. Neper number e and its roots define infinite-D extensions of rationals but - rather remarkably - finite-dimensional extensions of p-adic numbers since e^p is ordinary p-adic number. Extension of rationals would become infinite-D but the induced extensions of rationals would remain finite-D in accordance with the idea that cognition is always finite-D.
   Could one allow e and its roots and thus exponential functions besides polynomials? Could exponential divergence be the hallmark of chaos or perhaps the first step in the transition to transcendental chaos (or rather, complexity)? Could chaos (complexity) in real sense be possible for extensions of rationals generated by a root of e? One can however argue that the finite dimension of induced p-adic extensions means that cognitive chaos is not yet present.
   For general transcendentals the dimensions of p-adic extensions are infinite and one would have also cognitive chaos (infinite complexity). Could the transition to chaos means the emergence of analytic functions with rational coefficients having also roots, which are transcendentals. Chaos would mean that one can only approximated f analytic function as a polynomial giving approximation for the roots. By $M^8 - H$ duality these roots would correspond to values of M^4 time inside light-cone, preferred moments of time [L63]. These would become transcendental and in general p-adic extension would become infinite-D.

3. An interesting analogy with real numbers emerges. Real numbers have expansion in powers of any integer, in particular any prime p. The sequence defined by the coefficients of the expansion are analogous to an orbit of a discrete dynamical system. For transcendentals the expansion is unpredictable and analogous to a chaotic orbit.
   For rationals this expansion is periodic so that one has analog of a periodic orbit. This applies also to expansion of rationals formed from the integers in finite-D extensions of rationals. One must of course accept that the algebraic numbers defining the roots do not allow periodic expansion but one can do all calculations in extension and perform approximation only at

end of computation. Therefore the extensions of rationals represent also islands of order in the ocean of trancendental chaos. Could one see he gradual increase of the dimension of extension of rationals as a transition to chaos: of course, chaos would be wrong term since increase in algebraic complexity, which corresponds to evolution in TGD Universe is in question. Cognition becomes more and more refined.

4. As found, space-time surfaces behave like numbers and one can have functions having space-time surface as argument. Could the picture about emergence of chaos for reals be translated to the level of space-time surfaces identified as "roots" of octonion analytic function in M^8? The polynomial space-time surfaces would represent islands of order in chaos defined by general analytic functions with rational Taylor coefficients.

Can one imagine a connection between quantum randomness and chaos?

To my view, the reduction of quantum randomness to classical chaos is definitely excluded. Quantum classical correspondence allows however to consider a looser connection between quantum randomness and chaos.

1. The following considerations lead to a formulation of a more precise view about the sequence of steps consisting of a unitary evolution followed by SSFR as a a model of self. M^8-H duality involving representation of space-time surface in terms of a polymial with rational coefficients leads to an approximate model of the quantal time evolution by SSFRs as quantum counterpart for an iteration of a polynomial map, and gives a direct connection with chaos as algebraic complexity in the sense of generalization of Mandelbrot and Julia sets (`http://tinyurl.com/cplj9pe` and `http://tinyurl.com/cvmr83g`).
   The identification of time evolution as iteration $P \rightarrow P^{\circ 2} \rightarrow ...$ is very probably only an approximation. More general picture would assume that the corresponds to a functional factorization of P as $P = P_1 \circ P_2 \circ ... \circ P_n$. Even this assumption can be only approximate.
2. The fixed points of iteration would correspond to asymptotics for the evolution of space-time surface defined by iteration and approach of CD to a fixed point CD. This conforms with the idea that fixed points of iteration as representations of fractals, criticality and chaos. Chaos understood as genuine chaos could correspond to a fluctuation of the arrow of time in the sequence of SSFRs as a fixed point of iteration is reached.

It must be of course made clear that the view about M^8-H duality already considered and the view about the emergence of fractals to be discussed are only one of the many options that one can imagine and involve many poorly understood aspecs. Only time will tell whether the proposals work and how they must be improved.

Chaos and time

TGD Universe has gigantic symmetries [K24, K76] and looks like a completely integrable system and the idea about genuine chaos at space-time level does not look attractive. M^8-H duality suggests that chaos - actually complexity - in the sense of Mandelbrot fractals looks more promising idea. ZEO int turn suggests that chaos could be associated with the relationship between geometric and subjective time in the sense that the orderings of the two times would not be strictly identical.

1. Often the chaos is taken to mean increase of complexity (Mandelbrot and Julia sets), which actually means a diametric opposite of chaos. In TGD framework a more promising connection is between finite measurement resolution and complexity as that for extension of rationals. For trivial extensions of rationals the points of cognitive representation have rational M^8 (and becase also H-) coordinates. All other points fail to have a cognitive representation. For extensions of rationals the number of points in cognitive representations increases: the increase of cognitive complexity has actually nothing to do with emergence of a genuine chaos. Here one must be however very cautious and one must consider ZEO view about state function reduction in detail to see what happens.
2. M^8-H duality allows to consider a concrete example. The roots r_n of real rational polynomials P or even analytic functions correspond "special moments in the life of self". Could the increase of complexity be understood in terms of what happens for the roots. The number of these moments equals to the degree n of P and cognitive representation more and more complex

since the dimension of extension equals to n: this could occur in BSFRs at least. The clock defined by the moments roots $t = r_k$ could become more precise. It will be found that in presence of quantum criticality the emerging complexity could also correspond to a genuine chaos.

3. One can define clock time as a temporal distance τ between tips of CD after "small" state function reduction (SSFR), which corresponds to weak measurement in standard picture. Passive boundary and the states at the passive boundary of CD remain unchanged (generalized Zeno effect) and the states at active boundary is change. Also the distance between tips of CD changes but increases in statistical sense.
   The statistical nature of the change implies that the ordering for subjective time as sequence of SSFRs is not quite the same as that for τ (one could of course assume that only increase of the CD size is possible in BSFR but this would be an ad hoc assumption). This corresponds to a kind of quantum randomness due to the state function reductions. If the number of roots is large and the average time chronon is small, the changes of time order could occur often. Could this have interpretation as a genuine chaos in short time scales due to SSFRs? This need not correspond to a genuine chaos at the level of space-surfaces as preferred extremals. Chaos as algebraic complexity could however increases and would be consistent with complete integrability: this happends in n increases in BSFRs.

Chaos in death according to ZEO

The assignment of a genuine chaos to death looks natural from what we know about biological death. Could this assignment make sense in ZEO where BSFR corresponds in a well-defined sense to death?

1. Recall that BSFR corresponds to ordinary state function reduction in which the arrow of geometric time identifiable as distance between the tips of CD changes: self dies and re-incarnates with an opposite arrow of time. The active boundary of CD becomes passive. The passive boundary becomes active and the size of CD starts to statistically increase in opposite time direction in SSFRs. The former passive boundary CD can remain at the critical moment but could also shift towards the former active boundary - the re-incarnated self would have small CD and could have "childhood."
   The continual increase of CD looks strange. Also our mental images would increase in size and unless one makes special assumptions (say that the average change of the size of CD is proportional to its size (scaling)) one ends up with difficulties. Time evolution as stepwise scaling would be indeed natural.
2. Under what conditions does BSFR - death and reincarnation - occur? A quantum criticality implying instability against BSFR should be involved. The size scales of CD as temporal distances τ between its tips would have critical values τ_{cr} at which death of self in this universal sense could take place. τ_{cr} could be integer multiple of CP_2 length scale with allowed integers being primes of preferred primes allowed by p-adic length scale hypothesis. Criticality indeed involves long range fluctuations assigned with chaotic behavior: the simplest example is the transition to chaos in convection as energy feed to the system increases.
3. A concrete model for SSFRs [L72] suggests that one can assign to CD temperature T satisfying $T \propto 1/\tau$ so that the evolution of self would correspond to T as analog of cosmic temperature. Death could correspond to a critical temperature T_{cr} (τ_{cr}) and would be unavoidable. The quantum criticality assignable to death could correspond to the emergence of a genuine temporal chaos. The time order would become more and more ill-defined, and time τ would go forth and back so that eventually one would $\tau = \tau_{crit}$ as size of CD and death would occur. This however requires that the number of roots r_n increases so that also their density increases. This requires that the degree of the polynomial P defining the extension increases. Can this be consistent with the assumption that passive boundary does not change?
   Remark: Why I take this seriously is that I have had near death experience being in clinically unconscious but actually conscious state and I experienced quite literally the flow of time forth and back and was fighting to preserve the usual arrow of time.
4. This picture applies to all BSFRs and SSFRs and therefore to ordinary state functions reductions in all scales: the findings of Minev *et al* [L59] can be understood if the arrow of time

indeed changes [L59]. There would be a connection between state function reductions and chaos understood as genuine chaos. The idea that this chaos corresponds to a strange attractor at space-time level is not plausible. Rather it could be analogous to chaos in the sense of an attractor of iteration of complex function by functional decomposition. Fixed point is also a fractal and corresponds to criticality.

What gives rise to the lethal quantum criticality, BSFR, and death?

What could give rise to quantum criticality leading to death and reincarnation of self as BSFR?

1. If P remains the same during SSFRs, one could think that once the CD size is so large that all "special moments in the life of self" have been experienced as time values $\tau = r_n$, the system is ready to die. But how could this give rise to quantum criticality?
2. Assume that CD is defined as the intersection of future and past light-cones and the polynomial P corresponds to a product $P_1(r)P_2(L-r)$ of polynomials associated with these two light-cones such that P_i vanishes at the tip of its light-cone corresponding to $r = 0$ *resp.* $L - r = 0$. P_1 associated with the passive boundary of CD would not change in SSFRs but P_2 associated with the active boundary would change. Most importantly its degree would increase and the number of roots and their density would increase too. Eventually the density of active roots would become so high that death as BSFR is bound to occur as event $\tau = \tau_{cr}$.
   Remark: One can consider two options: real M^8 and real r or complexified M^8 and complex r.
3. As already noticed, if the space-time surface reduces to the root of P_1 at passive boundary and root of P_2 at active boundary, one could say that the 3-surfaces at these boundaries correspond to P_1 and P_2 asymptotically. The fact that the boundaries correspond to special 6-D brane like solutions in M^8 sugests that the boundary conditions are possible.
4. The statistically increasing extension of rationals would correspond to "personal" evolution for the changing part of self during life cycle. Note that $n = h_{eff}/h_0$ corresponds to the scale of quantum coherence thus increasing. This extension would define the evolutionary level of the unchanging part ("soul") during the next re-incarnation.

Could polynomial iteration approximate quantum time evolution by SSFRs in statistical sense?

I have considered rather concrete models for the counterpart of S-matrix for given space-time surface [L54, L56, L73] but the deeper understanding of the sequence of SSFRs is still lacking although quite concrete proposals already exists.

Number theoretical vision suggests that also the time evolution by SSFRs should reduce to number theory being induced by some natural number theoretical dynamics.

1. The most general option is that in each SSFR a superposition over extensions defined by various polynomials with varying rational coefficients is generated. The idea about the correspondence of the sequence of SSFRs with a functional decomposition of polynomials is however attractive.
2. The sequence of unitary evolutions brings strongly in mind the iteration $U \to U^2 \to U^3$.... One can however consider also the possibly $U \to U_1U \to U_2U_1U$.... The obvious guess for the iteration of U is that it is induced by a functional iteration of polynomial P_2 assigned to the active boundary of CD $P_2 \to P_2 \circ P_2 \to$ .... The more general option would not be iteration anymore but a composition of form $P_2 \to P_3 \circ P_2 \to$ .....
   The boundary conditions at the boundary of CD and at gluing points - possibly $t = r_n$ surfaces to which 6-branes are assignable as special solutions and identified as "special moments in the life of self" could make the superpositions of functional composites more probable contributions in the superposition. The polynomial $P \circ Q$ has same roots as Q (for $P(0) = Q(0) = 0$) and this favors conservative state function reductions preserving the state already achieved.
   Iteration would be even more conservative option. If the solutions assignable to P and Q are to be glued together along brane with $t = r_n$ they must share r_n as root. This would favor iterations if one has superposition over different rational coefficient values for P and Q with fixed degree.
   Remark: Also critical points of Q as zeros of derivative are preserved in $Q \to P \to Q$ as one finds by applying chain rule. For iteration both the new critical points/roots of $P \circ P^{\circ k}$

are inverse images of critical points/roots of $P^{\circ k}$. Only roots are of significance in the picture considered.

3. Superpositions of different iterates generated in the unitary time evolution preceding SSFR are required by the model of temporal chaos. SSFR selects extension of rationals and thus fixed iteration. In statistical sense the degree of iteration is bound to increase so that in statistical sense quantum iteration reduces to classical one. At the limit of fixed point of iteration the number of critical points $t = p_n$ and roots $t = r_n$ of the iterate increases as also their density along time axis and temporal chaos emerges leading to fluctuation of CD size τ.
4. Iteration of the real polynomial P satisfying $P(0) = 0$ would mean that one would have a series extensions obtained as powers of generating extension: E, $E \circ E$, $E \circ E \circ E$,... conserving the roots of E provided the polynomials involved vanish at origin: $P(0) = 0$. The proposal has been that biological evolution corresponds to a more general series of extensions $E_1, E_2 \circ E_1, E_3 \circ E_2 \circ E_1, ...$ Also now Galois groups in the series of them would be conserved. I have proposed that Galois groups are analogs of conserved genes [L36, L39].

The proposed picture is only one possibility to interpret evolution of self as iteration leading to chaos in the proposed sense.

1. One could argue that the polynomial $P_{nk} = P_n \circ .... \circ P_n$ associated with the active boundary remains the same during SSFRs as long as possible. This because the increase of degree from nk to $n(k+1)$ in $P_{nk} \rightarrow P_{nk} \circ P_n$ increases h_{eff} by factor $(k+1)/k$ so that the metabolic feed needed to preserve the value of h_{eff} increases.
   Rather, when all roots of the polynomials P assignable to the active boundary of CD are revealed in the gradual increase of CD preserving P_{nk}, the transition $P_{nk} \rightarrow P_{nk} \circ P_n$ could occur provided the metabolic resources allow this. Otherwise BSFR occurs and self dies and re-incarnates. The idea that BSFR occurs when metabolic resources are not available is discussed in [L96].
2. Could $P_{nk} \rightarrow P_{nk} \circ P_n$ occur only in BSFRs so that the degree n of P would be preserved during single life cycle of self - that n can increase only in BSFRs was indeed the original guess.

5.2.4 Basic facts about iteration of real polynomials

The iteration of real polynomials and also more general functions can be understood graphically. Assign to a x point $y = f(x)$ of the graph and reflect through the line $y = x$ and project to the graph to obtain the image point $x_1 = f(x)$. Fixed points $x = f(x)$ correspond to the intersections of the line $y = x$ and graph $y = f(x)$. The magnitude $|df/dx|$ at the intersection point determines whether it is attractor ($|df/dx| < 1$ or repellor ($|df/dx| \geq 1$) in which case large jumps in the value of x can occur, as one can easily check. Quite generally iteration in the part of the graph below (above) $y = x$ decreases (increases) x. Real polynomial $c - x^2$ provides a simple example.

Feigenbaum discovered by iterating logistic map numerically (`http://tinyurl.com/u3zwmar`) that the approach to chaos - not only for logistic map but - for real functions $f(x)$ with one quadratic maximum and depending on a varied parameter a is universal. Period-doubling bifurcations occur at parameter values satisfying at the limit $n \rightarrow \infty$

$$\frac{a_{N-1} - a_{N-2}}{a_N - a_{N-1}} \rightarrow 4.669201609... \ .$$

Second universality relates to the widths of tines - distances between the branches of bifurcation - appearing in the sequence of bifurcations. The ratio between width of the tine to widths of its sub-tines approaches at the limit $N \rightarrow \infty$ to constant given by

$$\alpha = 2.502907875095892822283902873218... \ .$$

.

In TGD framework conservative option would correspond to real M^4 so that the coordinates t and r would be real and the polynomials P_1 an P_2 would have real coefficients. The time evolution by iterations of P_2 would reduce to an iteration of a real polynomial P_2.

The number of real roots is in general smaller than the degree n of the polynomial. Only non-negative roots can be considered since one as $r \geq 0$ and $r = 0$ is a root. This condition could generalize to complex polynomials of complexified r as a condition $Re(r_c) \geq 0$ guaranteeing that roots are in the upper half plane for the variable $z = ir_c$.

The real polynomial $P(x)$ of degree n one has either positive or negative values between neighboring roots and at least one extremum between them. The n roots of $P_n(x)$ gives rise to Nn roots in N:th iteration and only non-negative ones are allowed. Since the roots are below the axis $y = s$, the root is obtained from the inverse of the roots by reflecting with respect to $y = x$ and projecting to the graph. The inverse of this operation increases the root. One has special case of complex iteration.

5.2.5 What about TGD analogs of Mandelbrot -, Julia-, and Fatou sets?

What about the interpretation of Mandelbrot -, Julia-, and Fatou sets (http://tinyurl.com/cplj9pe and http://tinyurl.com/cvmr83g) in the proposed picture? Could the iteration of P_2 define analogs of Mandelbrot and Julia fractals? This would give the long-sought-for connection between quantum physics and Mandelbrot and Julias sets, which are simply too beautiful objects to lack a physical application. Period-doubling bifurcations (http://tinyurl.com/t2swmdg) are involved with the iteration of real functions and relate closely to the complex fractals when the polynomials considered have real coefficients.

1. In the simplest situation both Mandelbrot and Julia sets are fractals associated with the iteration of complex polynomial $P_c(z) = z^2 + c$ where z and c are complex numbers (note that in TGD would have $c = 0$ in this case). One can consider also more general polynomials and even rational functions, in particular polynomial $f = P_2$ defined earlier, and replace $z = 0$ with any critical point satisfying $df/dz = 0$. Even meromorphic transcendental functions can be considered: what is required that the image contains the domain.
2. Mandelbrot set M is defined as the region of the plane spanned by the values of c for which the iteration starting from the critical point z_{cr} does not lead to infinity. Physically the restriction to Mandelbrot set looks natural.
3. For rational functions Julia set J_c (http://tinyurl.com/cplj9pe corresponds to a fixed value of c, and is defined as points z for which are unstable in the sense that for an arbitrary small perturbation of z iteration can lead to infinity. Inside J_c the iteration is repelling: $|f(w) - f(z)| > |w - z|$ for all w in neighbourhood of z within J_c. One can say that the behavior is chaotic within J_c and regular in its complement - Fatou set. Julia set can contain also cycles and iteration in J_c leads to these cycles. These cycles are analogs of the limit cycles appearing in the iteration of real-valued function discovered by Feigenbaum (http://tinyurl.com/u3zwmar).
   For polynomials Julia set can be identified as the boundary of the filled Julia set consisting of points for which iterates remain bounded. Also the inverse iterates in this set remain bounded. The filled J_c - denote it by $J_{c,in}$ - can be regarded as a set of points, which are inverse images of fixed points of the polynomial. All points except at most two points of J_c can be regarded as points in the limiting set for the union $\cup_n f^{-n}(z)$ of the inverse images for the points z in filled Julia set. Julia set and its complement Fatou set are invariant under both P and P^{-1} and therefore also under their functional powers. Julia set is the set of pre-images for practically any point of J_c: this can be used for computational purposes. If I have understood correctly there can be single exceptional point for which this is not the case. J_c can be regarded as a fractal curve. For parameter values inside M J_c is connected, which seems counter intuitive. For c outside the M, Julia set is a discrete Cantor space, Fatou dust.
   What is remarkable from TGD point of view is that the new roots obtained in N:th step of iteration are $N - 1$:th inverse images of the roots of P. Since polynomial iteration takes sufficiently distant points to ∞, its inverse does the opposite so that the roots of $P^{\circ N}$ are bounded: this strongly suggests that the roots of $P^{\circ N}$ are in J_c if those of P_2 are. One can say that the situation becomes chaotic at the large N limit since the number of roots increases without bound.
4. Fatou set F_c can be identified as the complement of Julia set. Fatou set fills the complex plane densely and has disjoint components, which contain at least one point with $df/dz = 0$ unless Fatou set contains $z = \infty$. Note however that critical point is ot fixed point as in gradient dynamics. This allows to deduce the number or at least upper bound for the number

of components of Fatou set, which equals to the degree n of polynomial in the generic case. All components have entire J_c accumulation points. Since the points of J_c are infinitely near to more than 2 disjoint sets for Fatou set with more than 2 components, J_c cannot be a smooth curve in this case being thus fractal. However, the Julia set of $P = z^2 + c$ is also fractal although Fatou set has only two components corresponding to the critical point $z = 0$ and $z = \infty$.

A couple of examples are in order: for $P(z) = z^2$ Julia set is unit circle S^1 and Fatou set has interior and exterior of S^1 as its components. The cycles in Julia set correspond to roots of unity and the orbits of other points form dense sets of unit circle. For $P(z) = z^2 - 2$ Julia set is the interval $(-2, 2)$ having fixed points as its ends. Fatou set has only one component as the complement of Julia set. For $P(z) = z^2 + c$, c complex Julia set is in general fractal. Hence the roots of the polynomial need not belong to Julia set.

Emergence of Mandelbrot and Julia sets from ZEO assuming $M^8 - H$ duality

Consider now the application to TGD assuming $M^8 - H$ duality [L37, L38, L39, L67] .

1. In TGD framework complex numbers $x + iy$ emerge in the complexification of M^8 and i commutes with octonionic units. If space-time surfaces are identified as real projection of their complexified variants obtained as roots of polynomials one can consider also polynomials with complexified coefficients c. Note that c would be complex rational but one can also consider complex algebraic numbers. The most general situation corresponds to analytic functions with complex rational Taylor coefficients. Complex argument with complex coefficients is possible space-time surface is identified by projection the complex space-time surface to real part of complexified M^8 [L37, L38, L38].
2. The complexified light-like coordinate r at the active boundary CD defines the analog of z plane in which iterates of P_2 act. r corresponds directly to the complexified linear time coordinate t of M^8 (time-axis connects tips of CD) and the roots r_n of P_2 correspond to the "special moments in the life of self" as time values $t = r_n$. Assume that $P_2(0)$ vanishes so that r_n are also roots of iterates.
3. Julia set J_c bounds filled Julia set $J_{c,in}$ of the complexified r-plane, whose interior points remain inside $J_{c,in}$ in the iterations by fixed P_2. Julia set J_c is connected but the Fatou set as its complement has several components labelled by the $n-1$ points p_k satisfying $dP_2(z)/dz = 0$ and by $z = \infty$ so that Fatou set has n components. The inverse iterates of roots need not belong to Fatou sets not containing ∞ or to the filled Julia set.
4. There are several Mandelbrot sets and the extrema of P_2 satisfying $dP_2/dr = 0$ label them. The extrema of P_2 are also extrema of its iterates. There are $n-1$ extrema p_k. In the real case they can be classified as either attractors or repellors but in complex situation they correspond to saddle points. Denote by $M(p_n)$ the region of parameter space of polynomial coefficients c for which the iteration starting starting at $p(n)$ does not lead outside it.
   In the real case the iteration of P_2 leads to the attractors $t = p_k$. In complex case the situation is not so simple and the basic of attraction is replaced with the Fatou set $F_c(p_k)$.
   Since c parameterizes points in the space of polynomials characterizing space-time surfaces in TGD, Mandelbrot set can be defined as a sub-space of "world of classical worlds" (WCW). Inside $M(p_n)$ the iteration maps r_n to a point $M_{in}(r_n)$. Note that also new roots emerge in each iteration and the Mandelbrot set for the iterates contains more components.

Remark: In TGD only the roots of P_2 are interesting. The roots of iterates are inverse iterates of roots of P_2.

Could one understand the size of CD and its evolution during the iteration of P_2?

1. Consider first the situation for real time $t = r$ and real polynomials. Since the boundary of CD contains only the roots $t = r_n$, the simplest guess is that the size of CD corresponds to the largest root of $P_2^{\circ N}$. The size of CD would increase in the iterations. The inverse images of the roots approach to Julia set so that the real counterpart of Julia set is important for understanding the asymptotic situation. Mandelbrot set defines the coefficient values for which iteration does not lead to infinity.

2. The situation is essentially the same for complexified time. The size of CD would correspond to the modulus for the largest of the iterate root and increases during iteration. The size of CD approaches to that for a point in Julia set.

Could the iteration lead to a stationary size of CD?

One can represent an objection to the idea that quantum iteration of P_2 could be more than an approximation.

1. Suppose that the size of CD is determined by the maximum for the iterates of the roots of P_2. Suppose that the parameters c are fixed and belong to Mandelbrot set $M(p_k)$. For given c there is therefore an upper for $\tau = 2r$ given by $r = r_{max}(c, p_k)$ for the Fatou set $F_c(p_k)$. One gets stuck to fixed τ since maximal root cannot become larger than $r_{max}(c)$ in the iteration. Note that in this situation the number of roots of $P_2^{\circ k}$ increases and if they corresponds to "special moments in the life of self", this could lead to quantum criticality and occurrence of BSFR.
2. Fluctuations of τ in the sequences of SSFRs is possible if superpositions of iterates are allowed. This could cause BSFR would occur and eventually second BSFR would eventually lead to the original situation. If P_2 is not modified, the iteration continues and one is still at criticality. BSFR soon occurs and same repeats itself.

Is this situation acceptable? Maybe - I have considered the possibility that the size of CD remains below some upper bound [L72, L60]. The selves such as our mental images could continue to live in the geometric past and memories would be communications with them. Or should one get rid of this situation? How?

1. Assume that SSFR creates a superposition of iterates with varying values of parameters c belonging to the Mandelbrot set $M(P_2)$. The value of $r_{max}(c, p_k)$ depends on c and it is possible to increase the value of τ in statistical sense if SSFR selects the values of c suitably. The value of L would be however given by maximal root and would remain below the maximum r_{max} of $r_{max}(c, p_k)$ in $M(P_2)$ if c belongs to $M(P_2)$. $\tau = 2L$ would remain below the maximum for the size of $J_c(P_2)$ in $M(P_2)$. One would get stuck if this size is finite, which is the case if $r_{max}(c, p_k)$ is bounded as function of c and p_k?
   Is $r_{max}(c, p_k)$ bounded? The polynomials with given degree of can have arbitrarily large roots and critical points in the same extension of rationals. Therefore it might be possible to avoid getting stuck if there is no restriction on the size of the roots of P_2 in the superposition over different values of c.

When death occurs and can self have a childhood?

I hope that talking about death and reincarnation does not irritate the reader too much. I use these terms as precisely defined technical terms applying universally. There are two extreme options for what happens to the former passive boundary in BSFR. The real situation could be between these two.

1. The first shift after reincarnation is to geometric past so that CD size increases.
2. The first shift is towards the former active boundary so that the size of CD decreases at least to the size of CD when the iteration of P_2 began. The reincarnated self would have "childhood" and would start from scratch so to say.

Consider P_1P_2 option. Suppose that time evolution is induced by iteration of either polynomial and maximal root defines the size of the size of CD. What happens to P_1?

1. Could the new functional iteration start from where it stopped in previous re-incarnation: if P_1 is n:th functional power of Q ($P_1 = Q^{\circ n}$), the first step would corresponds to $P_1 \to Q \circ P_1$. This conservative option does not quite correspond to the idea that one starts from scratch.
2. If P_1 can change, could one require that P_1 is replaced with a polynomial, which is minimal in the sense that it is not functional power of form $P_{1,new} = Q_{new}^{\circ n}$. Or could one even require that it is functional prime having prime valued degree: $n = p$. This would mean starting from scratch except that the algebraic extension of P_2 is fixed.

Probably these options represent only extreme situations. The most general option is that BSFR generates a state, which corresponds to a superposition of extensions of rationals characterized by polynomials P_2P_1, P_2 fixed, and from these one is selected.

Suppose that L as the size of CD is minimal and thus given by the largest root of $P_2^{\circ N}$ in the filled Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). Under what conditions can BSFR occur? Can the re-incarnated self have childhood?

1. One can argue that L should be smaller than the sizes of Julia sets of both A and B since the iteration gives no roots outside Julia set. This would require iteration to stop when the largest root of P_2^{circN} exceeds the size of the Julia set of A. When applied to B this condition would prevent BSFRs in the opposite time direction would prevent the growth of CD and it would become stationary. This condition looks too deterministic.
2. This picture suggests that the unitary evolution preceding SSFR creates a superposition of iterates $P_2^{\circ N}$ and that the size of CD as outcome of SSFR is determined statistically as a maximal root for $P_2^{\circ N}$ selected in the iteration. N could also decrease. Since the density of roots increases, one would have a lot of choices for the maximal root and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time.
3. Could BSFR become only probable as L as the largest root for the iterate $P_2^{\circ N}$ has exceeded the size of Julia set of A? A quantum analogy with super-cooling comes in mind. The size of CD boundary at side A would contain more volume than needed to store the information provided by the roots r_n and bring no new "special moments in the life of self" at A side. At B side the density of these moments would eventually become large enough so that the reduction of the size of CD destroying part of these moments would mean only a loss of precision. Could this make death and re-incarnation with an opposite arrow of time probable?
If $P_2^{\circ N}$ is achieved during the life cycle, the reduction in the size of CD in BSFR would reduce N to $N_1 < N$. For $P_1 = Q_1^M$ similar reduction of M to $M_1 < M$ would take place. If one returns to the situation when the iterated started, all new "special moments" are lost. Nothing would have been learned but one could start from scratch and live a childhood, as one might say.
In the proposed picture - one of many - the opposite boundaries of CD would correspond to both short and long range quantum fluctuations. Could this observation be raised to a guiding principle: could one even say that the opposite boundaries of CD give holistic and reductionistic representations.
4. Do the roots of $P_2^{\circ N}$ belonging to filled Julia set approach the Julia set as N increases? Or are they located randomly inside Julia set? Indeed, the inverse iterate of a root of P_2 is larger than the root as one finds graphically. The $P_2^{\circ N}$ does the same for the roots $P_2^{\circ N}$. If this argument is correct, the density of the roots is largest near Julia set and near the maximum $L - t = L - r$ near the corner of CD.
5. The proposed picture is interesting from the point of view of consciousness theory. Action would be near the corner of CD in the sense that conscious experience would gain most of its content in Minkowskian sense here whereas larger smaller values of $L - r$.
This does not mean paradox since the size of CD inreases and special moments already experienced are shifted to the future direction and would define the unchanging part - "soul" - of the next re-incarnation. This could be seen as wisdom gained in the previous life [L72].
6. Suppose that the approach to chaos in the iteration of P_2 indeed leads to death and re-incarnation. Can one avoid this or at least increase the span of life cycle? Could one start a new life by replacing P_2 with some polynomial Q_2 in the iteration so that the new iterates would be of form $Q_2^{N_2} \circ P_2^{\circ N_1}$. If the replacement is done sufficiently early, the development of chaos might be delayed since reaching the boundary of Julia set of Q would require quite a many iterations if its largest root is larger than that for P_2. This is also true if the degree of Q_2 is small enough.

Unexpected observations about memories

Some comments about memories in the model of self based on iteration.

1. The conscious activity is at the corner of CD in middle of CD if the new roots define “special moments in the life of self” as conscious experiences. The roots r_n of $P_2^{\circ N}$ defining already experienced special moments shift to Minkowskian geometric future as CD increases in size. Subjective memories are in Minkowskian future and become in re-incarnation stable memories about previous life!
   Subjective memories from recent and previous life could be obtained by communications with geometric future and past involving time reflection of the signal so that the constraints due to the finite light velocity can be overcome.
   One can ask whether self can have “remember” or “anticipate” also external world. If this is possible then the “memories” are indeed from geometric past and “anticipations” from geometric future.
2. The view about subjective memories raises interesting speculations (to be made with tongue in cheek). Consider an unlucky theoretician who believes that he has discovered wonderful theory and has used his lifetime to develop it. Unfortunately, colleagues have not shown a slightest to his theory. Although personal fame might not matter for him, he might be interested in knowing during his lifetime whether his life work will ever gain recognition. Is this possible in TGD Universe?
   Suppose that dreams involve sub-selves representing signals to Minkowskian future and their time reflection inside CD (re-incarnation). If sub-selves near the boundary of CD are able to send time signals to geometric future they might get information about the external world, maybe even about what colleagues think about the theory of unlucky theoretician. Dreams might allow to receive this information indirectly. Dreams might even involve meetings with colleagues of geometricfuture and if their behavior is very respectful, unlucky theoretician might wonder whether his work might have been recognized or is this only wishful thinking!
3. Usually it is thought the recollection of past is not good idea. One can however argue that it communication not only with subjective past but also with objective future (the world external to personal CD). This would give information about the external world of geometric future and also increase the span the time scale of conscious experience and of temporal quantum coherence. This might helpful or a theoretician not interested in fashionable thinking only.

5.3 Can one define the analogs of Mandelbrot and Julia sets in TGD framework?

The stimulus to this contribution came from the question related to possible higher-dimensional analogs of Mandelbrot and Julia sets (see this). The notion complex analyticity plays a key role in the definition of these notions and it is not all clear whether one can define these analogs.

I have already earlier considered the iteration of polynomials in the TGD framework [**?**] suggesting the TGD counterparts of these notions. These considerations however rely on a view of M^8-H duality which is replaced with dramatically simpler variant and utilizing the holography=holomorphy principle [L139] so that it is time to update these ideas.

This principle states that space-time surfaces are analogous to Bohr orbits for particles which are 3-D surfaces rather than point-like particles. Holography is realized in terms of space-time surfaces which can be regarded as complex surfaces in $H = M^4 \times CP_2$ in the generalized sense. This means that one can give H 4 generalized complex coordinates and 3 such generalized complex coordinates can be used for the 4-surface. These surfaces are always minimal surfaces irrespective of the action defining them as its externals and the action makes itself visible only at the singularities of the space-time surface.

5.3.1 Ordinary Mandelbrot and Julia sets

Consider first the ordinary Mandelbrot and Julia sets.

1. The simplest example of the situation is the map $f: z \to z^2+c$. One can consider the iteration of f by starting from a selected point z and look for various values of complex parameter c whether the iteration converges or diverges to infinity. The interface between the sets of the complex c-plane is 1-D Mandelbrot set and is a fractal. One can generalize the iteration to an arbitrary rational function f, in particular polynomials.

2. For polynomials of degree n also consider $n-1$ parameters c_i, $i = 1, ..., n$, to obtain $n-1$ complex-dimensional analog of Mandelbrot set as boundaries of between regions where the iteration lead or does not lead to infinity. For $n = 2$ one obtains a 4-D set.
3. One can also fix the parameter c and consider the iteration of f. Now the complex z-plane decomposes to two a finite region with a finite number of components and its complement, Fatou set. The iteration does not lead out from the finite region but diverges in the complement. The 1-D fractal boundary between these regions is the Julia set.

5.3.2 Holography= holomorphy principle

The generalization to the TGD framework relies heavily on holography=holomorphy principle.

1. In the recent formulation of TGD, holography required by the realization of General Coordinate Invariance is realized in terms of two functions f_1, f_2 of 4 analogs of generalized complex coordinates, one of them corresponds to the light-like (hypercomplex) M^4 coordinate for a surface $X^2 \subset M^4$ and the 3 complex coordinates to those of Y^2 orthogonal to X^2 and the two complex coordinates of CP_2.
   Space-time surfaces are defined by requiring the vanishing of these two functions: $(f_1, f_2) = (0, 0)$. They are minimal surfaces irrespective of the action as long it is general coordinate invariant and constructible in terms of the induced geometry.
2. In the number theoretic vision of TGD, $M^8 - H$-duality [L139] maps the space-time as a holomorphic surface $X^4 \subset H$ is mapped to an associative 4-surface $Y^4 \subset M^8$. The condition for holography in M^8 is that the normal space of Y^4 is quaternionic.
   In the number theoretic vision, the functions f_i are naturally rational functions or polynomials of the 4 generalized complex coordinates. I have proposed that the coefficients of polynomials are rationals or even integers, which in the most stringent approach are smaller than the degree of the polynomial. In the most general situation one could have analytic functions with rational Taylor coefficients.
   The polynomials $f_i = P_i$ form a hierarchy with respect to the degree of P_i, and the iteration defined is analogous to that appearing in the 2-D situation. The iteration of P_i gives a hierarchy of algebraic extensions, which are central in the TGD view of evolution as an increase of algebraic complexity. The iteratikon would also give a hierarchy of increasingly complex space-time surface and the approach to chaos at the level of space-time would correspond to approach of Mandelbrot or Julia set.
3. In the TGD context, there are 4-complex coordinates instead of 1 complex coordinate z. The iteration occurs in H and the vanishing conditions for the iterates define a sequence of 4-surfaces. The initial surface is defined by the conditions $(f_1, f_2) = 0$. This set is analogous to the set $f(z) = 0$ for ordinary Julia sets.
   One could consider the iteration as $(f_1, f_2) \rightarrow (f_1 \circ f_1, f_2 \circ f_2)$ continued indefinitely. One could also iterate only f_1 or f_2. Each step defines by the vanishing conditions a 4-D surface, which would be analogous to the image of the $z = 0$ in the 2-D iteration. The iterates form a sequence of 4-surfaces of H analogous to a sequence of iterates of z in the complex plane.
   The sequence of 4-surfaces also defines a sequence of points in the "world of classical worlds" (WCW) analogous to the sequence of points $z, f(z), ...$ This conforms with the idea that 3-surface is a generalization of point-like particles, which by holography can be replaced by a Bohr orbit-like 4-surface.
4. Also in this case, one can see whether the iteration converges to a finite result or not. In the zero energy ontology (ZEO), convergence could mean that the iterates of X^4 stay within a causal diamond CD having a finite volume.

5.3.3 The counterparts of Mandelbrot and Julia sets at the level of WCW

What the WCW analogy of the Mandelbrot and Julia sets could look like?

1. Consider first the Mandelbrot set. One could start from a set of roots of $(f_1, f_2) = (c_1, c_2)$ equivalent with the roots of $(f_1 - c_1, f_2 - c_2) = (0, 0)$. Here c_1 and c_2 define complex parameters analogous to the parameter c of the Mandelbrot sent. One can iterate the two functions for all

pairs (c_1, c_2). One can look whether the iteration converges or not and identify the Mandelbrot set as the critical set of parameters (c_1, c_2). The naive expectation is that this set is 3-D dimensional fractal.

2. The definition of Julia set requires a complex plane as possible initial points of the iteration. Now the iteration of $(f_1, f_2) = 0$ fixes the starting point (not necessarily uniquely since 3-D surface does not fix the Bohr orbit uniquely: this is the basic motivation for ZEO). The analogy with the initial point of iteration suggests that we can assume $(f_1, f_2) = (c_1, c_2)$ but this leads to the analog of the Mandelbrot set. The notions coincide at the level of WCW.
3. Mandelbrot and Julia sets and their generalizations are critical in a well-defined sense. Whether iteration could be relevant for quantum dynamics is of course an open question. Certainly it could correspond to number theoretic evolution in which the dimension of the algebraic extension rapidly increases. For instance, one could one consider a WCW spinor field as a wave function in the set of converging iterates. Quantum criticality would correspond to WCW spinor fields restricted to the Mandelbrot or Julia sets.

Could the 3-D analogs of Mandelbrot and Julia sets correspond to the light-like partonic orbits defining boundaries between Euclidean and Minkowskian regions of the space-time surface and space-time boundaries? Can the extremely complex fractal structure as sub-manifold be consistent with the differentiability essential for the induced geometry? Could light-likeness help here.

5.3.4 Do the analogs of Mandelbrot and Julia sets exist at the level of space-time?

Could one identify the 3-D analogs of Mandelbrot and Julia sets for a given space-time surface? There are two approaches.

1. The parameter space (c_1, c_2) for a given initial point h of H for iterations of $f_1 - c_1, f_2 - c_2)$ defines a 4-D complex subspace of WCW. Could one identify this subset as a space-time surface and interpret the coordinates of H as parameters? If so, there would be a duality, which would represent the complement of the Fatou set (the thick Julia set) defined as a subset of WCW as a space-time surface!
2. One could also consider fixed points of iteration for which iteration defines a holomorphic map of space-time surface to itself. One can consider generalized holomorphic transformations of H leaving X^4 invariant locally. If they are 1-1 maps they have interpretation as general coordinate transformations. Otherwise they have a non-trivial physical effect so that the analog of the Julia set has a physical meaning. For these transformations one can indeed find the 3-D analog of Julia set as a subset of the space-time surface. This set could define singular surface or boundary of the space-time surface.

5.3.5 Could Mandelbrot and Julia sets have 2-D analogs in TGD?

What about the 2-D analogs of the ordinary Julia sets? Could one identify the counterparts of the 2-D complex plane (coordinate z) and parameter space (coordinate c).

1. Hamilton-Jacobi structure defines what the generalized complex structure is [L131] and defines a slicing of M^4 in terms of integrable distributions of string world sheets and partonic 2-surfaces transversal or even orthogonal to each other. Partonic 2-surface could play the role of complex plane and string world sheet the role of the parameter space or vice versa.

   Partonic 2-surfaces *resp.* and string world sheet having complex *resp.* hyper-complex structures would therefore be in a key role. $M^8 - H$ duality maps these surfaces to complex *resp.* co-complex surfaces of octonions having Minkowskian norm defined as number theoretically as $Re(o^2)$.
2. In the case of Julia sets, one could consider generalized holomorphic transformations of H mapping X^4 to itself as a 4-surface but not reducing to 1-1 maps. If f_2 (f_1) acts trivially at the partonic 2-surface Y^2 (string world sheet X^2), the iteration reduces to that for f_1 (f_2). Within string world sheets and partonic 2-surfaces the iteration defines Julia set and its hyperbolic analog in the standard way. One can argue that string world sheets and partonic

2-surfaces should correspond to singularities in some sense. Singularity could mean this fixed point property.

The natural proposal is that the light-like 3-surfaces defining boundaries between Euclidean and Minkowskian regions of the space-time surface define light-like orbits of the partonic 2-surface. And string world sheets are minimal surfaces having light-like 1-D boundaries at the partonic 2-surface having physical interpretation as world-lines of fermions.

One could also iterate only f_1 or f_2 allow the parameter c of the initial value of f_1 to vary. This would give the analog of Mandelbrot set as a set of 2-D surfaces of H and it might have dual representation as a 2-surface.

3. The 2-D analog of the Mandelbrot set could correspond to a set of 2-surfaces obtained by fixing a point of the string world sheet X^2. Also now one could consider holomorphic maps leaving the space-time surface locally but not acting 1-1 way. The points of Y^2 would define the values of the complex parameter c remaining invariant under these maps. The convergence of the iteration of f_1 in the same sense as for the Mandelbrot fractal would define the Mandelbrot set as a critical set. For the dual of the Mandelbrot set X^2 and Y^2 would change their roles.

Chapter 6

Breakthrough in understanding of $M^8 - H$ duality

6.1 Introduction

$M^8 - H$ duality [L67, L63, L64, L89] has become a cornerstone of quantum TGD but several aspects of this duality are still poorly understood.

6.1.1 Development of the idea about $M^8 - H$ duality

A brief summary about the development of the idea is in order.

1. The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent/normal space and contains a complex co-complex surface, it can be mapped to a 4-surface in $M^4 \times CP_2$.
2. Later emerged the idea that octonionic analyticity realized in terms of a real polynomials P algebraically continued to polynomials of complexified octonion might realize the dream [L37, L38, L39]. The original idea was that the vanishing condition for the real/imaginary part of P in quaternion sense could give rise to co-assocative/associative sense.
   $M^8 - H$ duality concretizes number theoretic vision [L43, L42] summarized as adelic physics fusing ordinary real number based physics for the correlates of sensory experience and various p-adic physics ($p = 2, 3, ...$) as physics for the correlates of cognition. The polynomials of real variable restricted to be rational valued defines an extension or rationals via the roots of the polynomials and one obtains an evolutionary hierachy associated with these extensions increasing in algebraic complexity. These extensions induce extensions of p-adic numbers and the points of space-time surface in M^8 with coordinates in the extension of rationals define cognitive representations as unique discretizations of the space-time surface.
3. The realization of the general coordinate invariance in TGD framework [K42, K24, K76, L98] [L94] motivated the idea that strong form of holography (SH) in H could allow realizing $M^8 - H$ duality by assuming associativity/co-associativity conditions only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits at which the signature of the induced metric changes from Minkowskian to Euclidian.

6.1.2 Critical re-examination of the notion

In this article $M^8 - H$ duality is reconsidered critically.

1. The healthy cold shower was the learning that quaternion (associative) sub-spaces of quaternionic spaces are geodesic manifolds [A106]. The distributions of quaternionic normal spaces are however always integrable. Hence, co-associativity remains the only interesting option. Also the existence of co-commutative sub-manifolds of space-time surface demanding the existence of a 2-D integrable sub-distribution of subspaces is possible. This learning experience motivated a critical examination of the $M^8 - H$ duality hypothesis.

2. The basic objection is that for the conjectured associative option, one must assign to each state of motion of a particle its own octonionic structure since the time axis would correspond to the octonionic real axis. It was however clear from the beginning that there is an infinite number of different 4-D planes O_c in which the number theoretical complex valued octonion inner product reduces to real - the number theoretic counterpart for Riemann metric. In the co-associative case this is the only option. Also the Minkowski signature for the real projection turns out to be the only physically acceptable option. The mistake was to assume that Euclidian regions are co-associative and Minkowskian regions associative: both must be co-associative.
3. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [L37, L38, L39] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [L63, L72].
4. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X = 0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.
   The expectation was wrong! The equations $X = 0$ and $Y = 0$ involve the same(!) complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X = 0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.
   There are solutions $X = 0$ and $Y = 0$ only if the two polynomials considered have a common a_c^2 as a root! Also now the solutions are complex mass shells H^7_c.
5. How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .
   A co-associative X^4_c is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines $M^8 - H$ duality even if the co-associativity in geometric sense were not satisfied.
   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

It this, still speculative, picture is correct, it would fulfil the original dream about solving classical TGD exactly in terms of roots for real/imaginary parts of octonionic polynomials in M^8 and by mapping the resulting space-time surfaces to H by $M^8 - H$ duality. In particular, strong form of holography (SH) would not be needed at the level of H, and would be replaced with a dramatically stronger number theoretic holography.

Octonionic Dirac equation, which is purely algebraic equation and the counterpart for ordinary Dirac equation in momentum space, serves as a second source of information.

1. The first implication is that O_c has interpretation as an analog of momentum space for quarks: this has profound implications concerning the interpretation. The space-time surface in M^8 would be analog of Fermi ball. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.
2. Cognitive representations are defined by points of M^8 with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta of quarks. In the generic case the cognitive representations are finite. If the points of M^8 correspond to quark momenta, momentum conservation is therefore expected to make the scattering trivial.

However, a dramatic implication of the reduction of the co-associativity conditions to the vanishing of ordinary polynomials Y is that by the Lorentz invariance of roots of P, the 3-D mass shells ofd M^4 have an infinite number of points in a cognitive representation defined by points with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta. This is what makes interesting scattering amplitudes for massive quarks possible.

3. What is the situation for the images of M^4 points under the effective local CP_2 element defined by local $SU(3)$ element g preserving the mass squared and mapping H^3 to $g(H^3)$? If g is expressible in terms of rational functions with rational coefficients, algebraic points are mapped to algebraic points. This is true also in the interior of M^4.
   This would mean a kind of cognitive explosion for massive quark momenta. Without the symmetry one might have only forward scattering in the interior of X^4_r. Note that massless quarks can however arrive at the boundary of CD which also allows cognitive representation with an infinite number of points.
4. In the number theoretic approach, kinematics becomes a highly non-trivial part of the scattering. The physically allowed momenta would naturally correspond to algebraic integers in the extension E of rationals defined by P. Momentum conservation and on-mass-shell conditions together with the condition that momenta are algebraic integers in E are rather strong. The construction of Pythagorean squared generalize to the case of quaternions provides a general solutions to the conditions: the solutions to the conditions are combinations of momenta which correspond to squares of quaternions having algebraic integers as components.
5. The original proposal was that local $G_{2,c}$ element $g(x)$ defines a vanishing holomorphic gauge field and its restriction to string world sheet or partonic 2-surface defines conserved current. M^8-H duality however requires that local $SU(3)$ element with the property that image point is invariant under $U(2)$ is required by M^8-H duality defines $X^4 \subset M^8$.
   In any case, these properties suggest a Yangian symmetry assignable to string world sheets and partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The generators of the Yangian algebra have a representation as Hamiltonians which are in involution. They define conserved charges at the orbits for a Hamiltonian evolution defined by any combination of these the Hamiltonians. ZEO suggests a concrete representation of this algebra in terms of quark and antiquark oscillator operators. This algebra extends also to super-algebra. The co-product of the associated Yangian would give rise to zero energy states defining as such the scattering amplitudes.

6.1.3 Octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of the ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = qq = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of p p are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

Before continuing, I must apologize for the still fuzzy organization of the material related to $M^8 - H$ duality. The understanding of its details has been a long and tedious process, which still continues, and there are unavoidably inaccuracies and even logical inconsistencies caused by the presence of archeological layers present.

6.2 The situation before the cold shower

The view about $M^8 - H$ duality before the cold shower - leading to what I dare to call a breakthrough - helps to gain idea about the phenomenological side of $M^8 - H$ duality. Most of the phenomenology survives the transition to a more precise picture. This section is however not absolutely necessary for what follows it.

6.2.1 Can one deduce the partonic picture from $M^8 - H$ duality?

The M^8 counterparts for partons and their light like orbits in H can be understood in terms of octonionic Dirac equation in M^8 as an analog for the algebraic variant of ordinary Dirac equation at the level of momentum space [L89, L88] but what about the identification of partonic 2-surfaces as interaction vertices at which several partonic orbits meet? Can one deduce the phenomenological view about elementary particles as pairs of wormhole contacts connected by magnetic flux tubes from $M^8 - H$ duality? There is also the question whether partonic orbits correspond to their own sub-CDs as the fact that their rest systems correspond to different octonionic real axes suggests.

There are also some questions which have become obsolote. For instance: qhy should the partonic vertices reside at $t = r_n$ branes? This became obsolste with the realization that M^8 is analogous to momentum space so that the identification as real octonionic coordinate corresponds now to a component of 8-momentum identifiable as energy. Furthermore, the assumption the associativity of the 4-surface in M^8 had to be replaced with-co-associtivity and octonionic real coordinate does not have interpretation as time coordinate is associative surface

$M^8 - H$ duality indeed conforms with the phenomenological picture about scattering diagrams in terms of partonic orbits [L98, L97] [L97, L98] [L98], and leads to the view about elementary particles as pairs of Euclidian wormhole contacts associated with flux tubes carrying monopole flux.

6.2.2 What happens to the "very special moments in the life of self"?

The original title was "What happens at the "very special moments in the life of self?" but it turned out that "at" must be replaced with "to". The answer to the new question would be "They disappear from the glossary".

The notion of "very special moments in the life of self" (VPM) [L63, L72] makes sense if M^8 has interpretation as an 8-D space-time. M^4 projections of VPMs were originally identified as hyperplanes $t = r_n$, where t is time coordinate and r_n is a root of the real polynomial defining octonionic polynomial as its algebraic continuation.

The interpretation of M^8 as cotangent space of H was considered from the beginning but would suggest the interpretation of M^8 as the analog of momentum space. It is now clear that this interpretation is probably correct and that $M^8 - H$ duality generalizes the momentum-position duality of wave mechanics. Therefore one should speak of $E = r_n$ plane and simply forget the misleading term VMP. VPMs would correspond to constant values of the M^8 energy assignable to M^4 time coordinate.

The identification of space-time surface as co-associative surface with quaternionic normal space containing integrable distribution of 2-D commutative planes essential for $M^8 - H$ duality is also in conflict with the original interpretation. Also the modification of $M^8 - H$ duality in M^4 degrees of freedom forced by Uncertainty Principle [L110] has led to the conclusion that VMPs need not have a well-defined images in H.

6.2.3 What does SH mean and its it really needed?

SH has been assumed hitherto but what is its precise meaning?

1. Hitherto, SH at the level of H is believed to be needed: it assumes that partonic 2-surfaces and/or string world sheets serve as causal determinants determining X^4 via boundary conditions.
   (a) The normal or tangent space of X^4 at partonic 2-surfaces and possibly also at string world sheets has been assumed to be associative that is quaternionic. This condition should be true at the entire X^4.

(b) Tangent or normal space has been assumed to contain preferred M^2 which could be replaced by an integrable distribution of $M^2(x) \subset M^4$. At string world sheets only the tangent space can be associative. At partonic 2-surfaces also normal space could be associative. This condition would be true only at string world sheets and partonic 2-surfaces so that only these can be mapped to H by $M^8 - H$ duality and continued to space-time surfaces as preferred extremals satisfying SH.

The current work demonstrates that although SH could be used at the level of SH, this is not necessary. Co-associativity together with co-commutativity for string world sheets allows the mapping of the real space-time surfaces in M^8 to H implying exact solvability of the classical TGD.

6.2.4 Questions related to partonic 2-surfaces

There are several questions related to partonic 2-surfaces.

Q1: What are the M^8 pre-images of partons and their light-like partonic orbits in H?
It will be found that the octonionic Dirac equation in M^8 implies that octo-spinors are located to 3-D light-like surfaces Y^3_r - actually light-cone boundary and its 3-D analogs at which number theoretic norm squared is real and vanishes - or to the intersections of X^3_r with the 6-D roots of P in which case Dirac equation trivializes and massive states are allowed. They are mapped to H by $M^8 - H$ duality.
Remark: One can ask whether the same is true in H in the sense that modified Dirac action would be localized to 3-D light-like orbits and 3-D ends of the space-time surfaces at the light-like boundaries of CD having space-like induced metric. Modified Dirac action would be defined by Cherm-Simons term and would force the classical field equations for the bosonic Chern-Simons term. If the interior part of the modified Dirac action is absent, the bosonic action is needed to define the space-time surfaces as extremals. They would be minimal surfaces and universal by their holomorphy and would not depend on coupling parameters so that very general actions can have them as preferred extremals. This issue remains still open.
The naïve - and as it turned out, wrong - guess was that the images of the light-like surfaces should be light-like surfaces in H at the boundaries of Minkowskian and Euclidian regions (wormhole contacts). In the light-like case Y^3_r corresponds to the light-cone boundary so that this would be the case. X^3_r however turns out to correspond to a hyperboloid in M^4 as an analog of a mass shell and is not identifiable as a partonic orbit.
It turned out that the complex surface X^4_c allows real sections in the sense that the number theoretic complex valued metric defined as a complex continuation of Minkowski norm is real at 4-D surfaces: call them Z^4_r. They are bounded by a 3-D region at Z^3_r at which the value of norm squared vanishes. This surface is an excellent candidate for the pre-image of the light-like orbit of partonic 2-surface serving as a topological vertex. One has therefore strings worlds sheets, partonic 2-surfaces and their light-like orbits and they would connect the "mass shells" at X^4_r. All ingredients for SH would be present.
The intersections of Z^3_r with X^3_r identifiable as the section of X^4_r $a = constant$ hyperboloid would give rise to partonic 2-surfaces appearing as topological reaction vertices.
The assumption that the 4-D tangent space at these light-like 3-surfaces is co-associative, would give an additional condition determining the image of this surface in H, so that the boundary conditions for SH would become stronger. One would have boundary conditions at light-like partonic orbits. Note that string world sheets are assumed to have light-like boundaries at partonic orbits.

Q2: Why should partonic 2-surfaces appear as throats of wormhole contact in H? Wormhole contacts do not appear in M^8.

1. In M^8 light-like orbits are places where the Minkowskian signature changes to Euclidian. Does $M^8 - H$ duality map the images of these coinciding roots for Euclidian and Minkowskian branches to different throats of the wormhole contact in H so that the intersection would disappear?
2. This is indeed the case. The intersection of Euclidian and Minkowskian branches defines a single 3-surface but the tangent and normal spaces of branches are different. Therefore

their H images under $M^8 - H$ duality for the partonic 2-surface are different since normal spaces correspond to different CP_2 coordinates. These images would correspond to the two throats of wormhole contact so that the H-image by SH is 2-sheeted. One would have wormhole contacts in H whereas in M^8 the wormhole contact would reduce to a single partonic 2-surface.

3. The wormhole contact in H can have only Euclidian signature of the induced metric. The reason is that the M^4 projections of the partonic surfaces in H are identical so that the points with same M^4 coordinates have different CP_2 coordinates and their distance is space-like.

Q3: In H picture the interpretation of space-time surfaces as analogs of Feynman graphs assumes that several partonic orbits intersect at partonic 2-surfaces. This assumption could be of course wrong.This raises questions.

What the pre-images of partonic 2-surfaces are in M^8? Why should several partonic orbits meet at a given partonic 2-surface? Is this needed at all?

The space-time surface X^4_r associated intersects the surface X^6_r associated with different particle - say with different value of mass along 2-D surface. Could this surface be identified as partonic 2-surface X^2_r? This occurs symmetrically so that one has a pair of 2-surfaces X^2_r. What does this mean? Could these surface map to the throats of wormhole contact in H?

Why several partonic surfaces would co-incide in topological reaction vertex at the level of H? At this moment is is not clear whether this is forced by M^8 picture.

Octonionic Dirac equation implies that M^8 has interpretation as analog of momentum space so that interaction vertices are replaced by multilocal vertices representing momenta and propagators become local being in this sense analogous to vertices of QFT. One could of course argue that without the gluing along ends there would be no interactions since the interactions in X^6_r for two 3-surfaces consist in the generic case of a discrete set of points. One could also ask whether the surfaces Y^3_r associated with the space-time surfaces X^4_r associated with incoming particles must intersect along partonic 2-surface rather than at discrete set of points.

The meeting along ends need not be true at the level of M^8 since the momentum space interpretation would imply that momenta do not differ much so that particles should have identical masses: for this to make sense one should assume that the exchanged virtual particles are massless. One other hand, if momenta are light-like for Y^3_r, this might be the case.

Q4: Why two wormhole contacts and monopole flux tubes connecting them at the level of H? Why monopole flux?

1. The tangent spaces of the light-like orbits have different light-like direction. Intuitively, this corresponds to different directions of light-like momenta. Momentum conservation requires more than one partonic orbit changing its direction meeting at partonic 2-surface. By light-likeness, the minimum is 2 incoming and two outgoing lines giving a 4-vertex. This allows the basic vertices involving Ψ and $\overline{Psi}$ at opposite throats of wormhole contacts. Also a higher number of partonic orbits is possible.
2. A two-sheeted closed monopole flux tube having wormhole contacts as its "ends" is suggested by elementary particle phenomenology. Since M^8 homology is trivial, there is no monopole field in M^8. If M^8-H duality is continuous it maps homologically trivial partonic 2-surfaces to homologically trivial 2-surfaces in H. This allows the wormhole throats in H to have opposite homology charges. Since the throats cannot correspond to boundaries there must be second wormhole contact and closed flux tube.
3. What does the monopole flux for a partonic 2-surface mean at the level of M^8? The distribution of quaternionic 4-D tangent/normal planes containing preferred M^2 and associated with partonic 2-surface in M^8 would define a homologically on-trivial 2-surface in CP_2. The situation is analogous to a distribution of tangent planes or equivalently normal vectors in S^2.

Q4: What is the precise form of $M^8 - H$ duality: does it apply only to partonic 2-surfaces and string world sheets or to the entire space-time surfaces?

$M^8 - H$ duality is possible if the X^4 in M^8 contains also integrable distribution of complex tangent or normal 2-planes at which 4-D tangent space is quaternionic/associative. String world sheets and partonic 2-surfaces define these distributions.

The minimum condition allowed by SH in H is that string world sheets and there is a finite number of partonic 2-surfaces and string world sheets. In this case only these 2-surfaces can be mapped to H and SH assigns to them a 4-D space-time surface. The original hypothesis was that these surfaces define global orthogonal slicings of the X^4 so that $M^8 - H$ duality could be applied to the entire X^4. This condition is probably too strong.

6.3 Challenging $M^8 - H$ duality

$M^8 - H$ duality involves several alternative options and in the following arguments possibly leading to a unique choice are discuses.

1. Are both associativity and co-associativity possible or is only either of these options allowed? Is it also possible to pose the condition guaranteeing the existence of 2-D complex sub-manifolds identifiable as string world sheets necessary to map the entire space-time surface from M^8 to H? In other words, is the strong form of holography (SH) needed in M^8 and/or H or is it needed at all?
2. The assignment of the space-time surface at the level of M^8 to the roots of real or imaginary part (in quaternionic sense) of octonionic polynomial P defined as an algebraic continuation of real polynomial is an extremely powerful hypothesis in adelic physics [L42, L43] and would mean a revolution in biology and consciousness theory.
   Does P fix the space-time surface with the properties needed to realize $M^8 - H$ duality or is something more needed? Does the polynomial fix the space-time surface uniquely - one would have extremely strong number theoretic holography - so that one would have number theoretic holography with coefficients of a real polynomial determining the space-time surface?
3. $M^8 - H$ duality involves mapping of $M^4 \subset M^8$ to $M^4 \subset H$. Hitherto it has been assumed that this map is direct identification. The form of map should however depend on the interpretation of M^8. In octonionic Dirac equation M^8 coordinates are in the role of momenta [L89]. This suggests the interpretation of M^8 as an analog of 8-D momentum space. If this interpretation is correct, Uncertainty Principles demands that the map $M^4 \subset M^8 \rightarrow M^4 \subset H$ is analogous to inversion mapping large momenta to small distances.
4. Twistor lift of TGD [K91] is an essential part of the TGD picture. Ttwistors and momentum twistors provide dual approaches to twistor Grassmann amplitudes. Octonionic Dirac equation suggests that M^8 and H are in a similar dual relation. Could $M^8 - H$ duality allow a generalization of twistorial duality to TGD framework?

6.3.1 Explicit form of the octonionic polynomial

What does the identification of the octonionic polynomial P as an octonionic continuation of a polynomial with real or complexified coefficients imply? In the following I regard M^8_c as O^8_c and consider products for complexified octonions.

Remark: In adelic vision the coefficients of P must be rationals (or at most algebraic numbers in some extension of rationals).

One interesting situation corresponds to the real subspace of O_c spanned by $\{I_0, iI_k\}, = 1, ..7$, with a number theoretic metric signature $(1, -1, -1..., -1)$ of M^8 which is complex valued except at in various reals subspaces.This subspace is associative. The original proposal was that Minkowskian space-time regions as projections to this signature are associative whereas Euclidian regions are co-associative. It however turned out that associative space-time surfaces are physically uninteresting.

The canonical choice $(iI_0, I_1, I_2, iI_3, I_4, iI_5, I_6, iI_7)$ defining the complexification of the tangent space represents a co-associative sub-space realizing Minkowski signature. It turns out that both Minkowskian and Euclidian space-time regions must be co-associative .

Surprises

The explicit calculation of the octonionic polynomial yielded a chilling result. If one poses (co-)associativity conditions as vanishing of the imaginary or real part in quaterionic sense: $Im_Q(P) = 0$ or $Re_Q(P) = 0$, the outcome is that the space-time surface is just M^4 or E^4. Second chilling result is that quaternionic sub-manifolds are geodesic sub-manifolds. This led to the question how to modify the (co-)associativity hypothesis.

The vision has been that space-time surfaces can be identified as roots for the imaginary (co-associative) part $Im_Q(O)$ or real part $Im_Q(O)$ of octonionic polynomial using the standard decomposition $(1, e_1, e_2, e_3)$.

1. The naïve counting of dimensions suggests that one obtains 4-D surfaces. The surprise was that also 6-D brane like entities located at the boundary of M^8 light-cone and with topology of 6-sphere S^6 are possible. They correspond to the roots of a real polynomial $P(o)$ for the choice $(1, iI_1, ..., iI_7)$. The roots correspond to the values of the real octonion coordinate interpreted as values of linear M^4 time in the proposal considered. Also for the canonical proposal one obtains a similar result. In O_c they correspond to 12-D complex surfaces X_c^6 satisfying the same condition conditions $x_0^2+r^2=0$ and $P(x_0)=0$.
2. There was also another surprise. As already described, the general form for the octonionic polynomial $P(o)$ induced from a real polynomial is extremely simple and reduces to $X(t^2,r^2)I_0 + iY(t^2,r^2)Im(o)$. There are only two complex variables t and r^2 involved and the solutions of $P=0$ are 12-D complex surfaces X_c^6 in O_c. Also the special solutions have the same dimension.
3. In the case of co-associativity 8 conditions are needed for $Re_Q(P)=0$: note that $X=0$ is required. The naive expectation is that this gives a complex manifold X_c^4 with 4-D real projection X_r^4 as an excellent candidate for a co-associative surface.
   The expectation turned out to be wrong: in absence of any additional conditions the solutions are complex 7-dimensional mass shells! This is due to the symmetries of the octonionic polynomials as algebraic continuation of a real polynomial.
4. The solution of the problem is to change the interpretation completely. One must assign to the 7-D complex mass shell H_c^7 a 3-D complex mass shell H_c^3.
   One can do this by assuming space-time surface is surface intersecting the7-D mass shell obtained as a deformation of $M_c^4 \subset M_c^8$ by acting with local $SU(3)$ gauge transformation and requiring that the image point is invariant under $U(2)$. If the 4-D complex mass squared remains invariant in this transformation, X_c^4 intersects H_c^7.
   With these assumptions, a local CP_2 element defines X_c^4 and X_r^4 is obtained as its real projection in M^4. This definition assigns to each point of M^4 a point of CP_2 so that M^8-H duality is well-defined.
   One obtains holography in which the fixing of 3-D mass shells fixes the 4-surface and also assigns causal diamond with the pair of mass shells with opposite energies. If the space-time surface is analog of Bohr orbit, also its preimage under M^8-H duality should be such and P would determine 4-surface highly uniquely [L114] and one would have number theoretic holography.

General form of P and of the solutions to $P=0$, $Re_Q(P)=0$, and $Im_Q(P)=0$

It is convenient to introduce complex coordinates for O_c since the formulas obtained allow projections to various real sections of O_c.

1. To see what happens, one can calculate o_c^2. Denote o_c by $o_c = tI_0 + \overline{o_c}$ and the norm squared of $\overline{o}$ by r^2, where $r^2=\sum o_k^2$ where o_k are the complex coordinates of octonion. Number theoretic norm squared for o_c is t^2+r^2 and reduces to a real number in the real sections of O_c. For instance, in the section $(I_1, iI3, iI_5, iI_7)$ the norm squared is $-x_1^2+x_3^2+x_5^2+x_7^2$ and defines Minkowskian norm squared.
   For o^2 one has:
   $$o^2 = t^2 - r^2 + 2t\overline{o} \equiv X_2 + \overline{Y}_2 \ .$$
   For o^3 one obtains
   $$o^3 = tX_2 - \overline{o}\cdot\overline{Y}_2 + t\overline{Y}_2 + X_2\overline{o} \ .$$
   Clearly, $Im_Q(o^n)$ has always the same direction as $Im_Q(o)$. Hence one can write in the general case
   $$o^n = X + Y\overline{o} \ . \quad (6.3.1)$$

This trivial result was obtained years ago but its full implications became evident only while preparing the current article. The point is that the solutions to associativity/co-associativity conditions by putting $Re(Q(P) = 0$ or $Im_Q(P) = 0$ are trivial: just M^4 or E^4. What goes wrong with basic assumptions, will be discussed later.
Remark: In M^8 sub-space one has imaginary $\overline{o}$ is proportional to the commuting imaginary unit.

2. It is easy to deduce a recursion formula for the coefficients for X and Y for n:th power of o_c. Denote by t the coordinate associated with the real octonion unit (not time coordinate). One obtains

$$\begin{array}{l} o_c^n = X_n I_0 + Y_n \overline{o} \ , \\ X_n = tX_{n-1} - rY_{n-1} \ , \\ Y_n = tY_{n-1} + rX_{n-1} \ . \end{array} \tag{6.3.2}$$

In the co-associative case one has $t = 0$ or possibly constant $t = T$ (note that in the recent interpretation t does not have interpretation as time coordinate). The reason is that the choice of octonionic coordinates is unique apart from translation along the real axis from the condition that the coefficients of P remain complex numbers in powers of the new variable.

3. The simplest option correspond to $t = 0$. One can criticize this option since the quaternionicity of normal space should not be affected if t is constant different from zero. In any case, for $t = 0$ the recursion formula gives for the polynomial $P(o_c)$ the expression

$$P(o_c) = \sum (-1)^n r^{2n} (p_{2n-1} I_0 + p_{2n} \overline{o}) \ . \tag{6.3.3}$$

Denoting the even and of odd parts of P by P_{even} and P_{odd}, the roots $r_{k,odd}$ of $X = Re(P(o_c))$ are roots P_{odd} and roots $r_{k,even}$ of $Y = Im(P(o_c))$ are roots of P_{even}. Co-associativity gives roots of X and the roots of P as simultaneous roots of P_{odd} and P_{even}. The interpretation of roots is as in general complex mass squared values.
In the general case, the recursion relation would give the solution

$$\begin{array}{l} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = A^n \begin{pmatrix} t \\ r \end{pmatrix} \\ A = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \end{array} \tag{6.3.4}$$

One can diagonalize the matrix appearing in the iteration by solving the eigenvalues $\lambda_{\pm} = t \pm ir$ and eigenvectors $X_{\pm} = (\pm i, 1)$ and by expressing $(X_1, Y_1) = (t, r)$ in terms of the eigenvectors as $(t, r) = ((it + r)X_+ + (r - it)X_-)/2$. This gives

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (t+ir)^n i - (t-ir)^n i \\ (t+ir)^n + (t-ir)^n \end{pmatrix} \tag{6.3.5}$$

This gives

$$\begin{array}{l} P(o_c) = P_1 I_0 + P_2 \overline{o} \ , \\ P_1(r) = \sum X_n p_n r^{2n} \ , \\ P_2(r) = \sum Y_n p_n r^{2n} \ . \end{array} \tag{6.3.6}$$

For the restriction to M_c^4, r^2 reduces to complex 4-D mass squared given by the root r_n. In general case r^2 corresponds to complex 8-D mass squared. All possible signatures are obtained by assuming M_c^8 coordinates to be either real or imaginary (the number theoretical norm squared is real with this restriction).

How does one obtain 4-D space-time surfaces?

Contrary to the naive expections, the solutions of the vanishing conditions for the $Re_Q(P)$ ($Im_Q(P)$) (real (imaginary) part in quaternionic sense) are 7-D complex mass shells $r^2 = r_{n,1}$ as roots of $P_1(r) = 0$ or $r^2 = r_{n,2}$ of $P_2(r) = 0$ rather than 4-D complex surfaces (for a detailed discussion see [K17]) A solution of both conditions requires that P_1 and P_2 have a common root but the solution remains a 7-D complex mass shell! This was one of the many cold showers during the development of the ideas about $M^8 - H$ duality! It seems that the adopted interpretation is somehow badly wrong. Here zero energy ontology (ZEO) and holography come to the rescue.

1. Could the roots of P_1 or P_2 define only complex mass shells of the 4-D complex momentum space identifiable as M_c^4? ZEO inspires the question whether a proper interpretation of mass shells could be as pre-images of boundaries of cd:s (intersections of future and past directed light-cones) as pairs of mass shells with opposite energies. If this is the case, the challenge would be to understand how X_c^4 is determined if P does not determine it.
   Here holography, considered already earlier, suggests itself: the complex 3-D mass shells belonging to X_c^4 would only define the 3-D boundary conditions for holography and the real mass shells would be mapped to the boundaries of cds. This holography can be restricted to X_R^4. Bohr orbit property at the level of H suggests that the polynomial P defines the 4-surface more or less uniquely.
2. Let us take the holographic interpretation as a starting point. In order to obtain an X_c^4 mass shell from a complex 7-D light-cone, 4 complex degrees of freedom must be eliminated. $M^8 - H$ duality requires that X_c^4 allows M_c^4 coordinates.
   Note that if one has $X_c^4 = M_c^4$, the solution is trivial since the normal space is the same for all points and the H image under $M^8 - H$ duality has constant $CP_2 = SU(3)/U(2)$ coordinates. X_c^4 should have interpretation as a non-trivial deformation of M_c^4 in M^8.
3. By $M^8 - H$ duality, the normal spaces should be labelled by $CP_2 = SU(3)/U(2)$ coordinates. $M^8 - H$ duality suggests that the image $g(p)$ of a momentum $p \in M_c^4$ is determined essentially by a point $s(p)$ of the coset space $SU(3)/U(2)$. This is achieved if M_c^4 is deformed by a local $SU(3)$ transformation $p \rightarrow g(p)$ in such a way that each image point is invariant under $U(2)$ and the mass value remains the same: $g(p)^2 = p^2$ so that the point represents a root of P_1 or P_2.
   Remark: I have earlier considered the possibility of G_2 and even $G_{2,c}$ local gauge transformation. It however seems that that local $SU(3)$ transformation is the only possibility since G_2 and $G_{2,c}$ would not respect $M^8 - H$ duality. One can also argue that only real $SU(3)$ maps the real and imaginary parts of the normal space in the same manner: this is indeed an essential element of $M^8 - H$ duality.
4. This option defines automatically $M^8 - H$ duality and also defines causal diamonds as images of mass shells $m^2 = r_n$. The real mass shells in H correspond to the real parts of r_n. The local $SU(3)$ transformation g would have interpretation as an analog of a color gauge field. Since the H image depends on g, it does not correspond physically to a local gauge transformation but is more akin to an element of Kac-Moody algebra or Yangian algebra which is in well-defined half-algebra of Kac-Moody with non-negative conformal weights.

The following summarizes the still somewhat puzzling situation as it is now.

1. The most elegant interpretation achieved hitherto is that the polynomial P defines only the mass shells so that mass quantization would reduce to number theory. Amusingly, I started to think about particle physics with a short lived idea that the d'Alembert equation for a scalar field could somehow give the mass spectrum of elementary particles so that the issue comes full circle!
2. Holography assigns to the complex mass shells complex 4-surfaces for which $M^8 - H$ duality is well-defined even if these surfaces would fail to be 4-D co-associative. These surfaces are expected to be highly non-unique unless holography makes them unique. The Bohr orbit property of their images in H indeed suggests this apart from a finite non-determinism [L114]. Bohr orbit property could therefore mean extremely powerful number theoretical duality for which the roots of the polynomial determine the space-time surface almost uniquely. $SU(3)$ as color symmetry emerges at the level of M^8. By $M^8 - H$ duality, the mass shells are mapped to the boundaries of CDs in H.
3. Do we really know that X_r^4 co-associative and has distribution of 2-D commuting subspaces of normal space making possible $M^8 - H$ duality? The intuitive expectation is that the answer is affirmative [A106]. In any case, $M^8 - H$ duality is well-defined even without this condition.

4. The special solutions to $P = 0$, discovered already earlier, are restricted to the boundary of CD_8 and correspond to the values of energy (rather than mass or mass squared) coming as roots of the real polynomial P. These mass values are mapped by inversion to "very special moments in the life of self" (a misleading term) at the level of H as special values of light-cone proper time rather than linear Minkowski time as in the earlier interpretation [L63]. The new picture is Lorenz invariant.

Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \to M^4 \subset H$ from identification to inversion. The equations for $Re_Q(P) = 0$ reduce to simultaneous roots of the real polynomials defined by the odd and even parts of P having interpretation as complex values of mass squared mapped to light-cone proper time constant surfaces in H. This leads to the idea that the formulation of scattering amplitudes at M^8 levels provides the counterpart of momentum space description of scattering whereas the formulation at the level of H provides the counterpart of space-time description.

This picture combined with zero energy ontology (ZEO) leads also to a view about quantum TGD at the level of M^8. Local $SU(3)$ element has properties suggesting a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

6.3.2 The input from octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = q\overline{q} = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of $p\,p$ are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

One of the big surprises was that the cognitive representations for both light-like boundary and X_r^4 are not generic meaning that they would consist of a finite set of points but infinite due to the Lorentz symmetry: a kind of cognitive explosion would happen by the Lorentz symmetry. The natural assumption is that for a suitable momentum unit, physical momenta satisfying mass shell conditions are algebraic integers in the extension of rationals defined by P. Periodic boundary conditions in turn suggest that for the physical states the total momenta are ordinary integers and this leads to Galois confinement as a universal mechanism for the formation of bound states.

Hamilton-Jacobi structure and Kähler structure of $M^4 \subset H$ and their counterparts in $M^4 \subset M^8$

The Kähler structure of $M^4 \subset H$, forced by the twistor lift of TGD, has deep physical implications and seems to be necessary. It implies that for Dirac equation in H, modes are eigenstates of only the longitudinal momentum and in the 2 transversal degrees of freedom one has essentially harmonic oscillator states [L110, L106], that is Gaussians determined by the 2 longitudinal momentum components. For real longitudinal momentum the exponents of Gaussians are purely imaginary or purely real.

The longitudinal momentum space $M^2 \subset M^4$ and its orthogonal complement E^2 is in a preferred role in gauge theories, string models, and TGD. The localization of this decomposition leads to the notion of Hamilton-Jacobi (HJ) structure of M^4 and the natural question is how this relates to Kähler structures of M^4. At the level of H spinors fields only the Kähler structure corresponding to constant decomposition $M^2 \oplus E^2$ seems to make sense and this raises the question how the H-J structure and Kähler structure relate. TGD suggests the existence of two geometric structure in M^4: HJ structure and Kähler structure. It has remained unclear whether HJ structure and Kähler structure with covariantly constant self-dual Kähler form are equivalent notions or whether there several H-J structures accompaning the Kähler structure.

In the following I argue that H-J structures correspond to different choices of symplectic coordinates for M^4 and that the properties of $X^4 \subset H$ determined bt $M^- H$ duality make it natural to to choose particular symplectic coordinates for M^4.

Consider first what H-J structure and Kähler structure could mean in H.

1. The H-J structure of $M^4 \subset H$ would correspond to an integrable distribution of 2-D Minkowskian sub-spaces of M^4 defining a distribution of string world sheets $X^2(x)$ and orthogonal distribution of partonic 2-surfaces $Y^2(x)$. Could this decomposition correspond to self-dual covariantly Kähler form in M^4?
   What do we mean with covariant constancy now? Does it mean a separate covariant constancy for the choices of $M^2(x)$ and $Y^2(x)$ or only of their sum, which in Minkowski coordinates could correspond to a constant electric and magnetic fields orthogonal to each other?
2. The non-constant choice of $M^2(x)$ ($E^2(x)$) cannot be covariantly constant. One can write $J(M^4) = J(M^2(x)) \oplus J(E^2(x)$ corresponding to decomposition to electric and magnetic parts. Constancy of $J(M^2(x)$ would require that the gradient of $J(M^2(x)$ is compensated by the gradient of an antisymmetric tensor with square equal to the projector to $M^2(x)$. Same condition holds true for $J(E^2(x))$. The gradient of the antisymmetric tensor would be parallel to itself implying that the tensor is constant.
3. H-J structure can only correspond to a transformation acting on J but leaving $J_{kl}dm^k dm^l$ invariant. One should find analogs of local gauge transformations leaving J invariant. In the case of CP_2, these correspond to symplectic transformations and now one has a generalization of the notion. The M^4 analog of the symplectic group would parameterize various decompositions of $J(M^4)$.
   Physically the symplectic transformations define local choices of 2-D space $E^2(x)$ of transversal polarization directions and longitudinal momentum space M^2 emerging in the construction of extremals of Kähler action.
4. For the simplest Kähler form for $M^4 \subset H$, this decomposition in Minkowski coordinates would be constant: orthogonal constant electric and magnetic fields. This Kähler form extends to its number theoretical analog in M^8. The local $SU(3)$ element g would deform M^4 to $g(M^4)$ and define an element of local CP_2 defining $M^8 - H$ duality. g should correspond to a symplectic transformation of M^4.

Consider next the number theoretic counterparts of H-J- and Kähler structures of $M^4 \subset H$ in $M^4 \subset M^8$.

1. In M^4 coordinates H-J structure would correspond to a constant $M^2 \times E^2$ decomposition. In M^4 coordinates Kähler structure would correspond to constant E and B orthogonal to each other. Symplectic transformations give various representations of this structure as H-J structures.
2. The number theoretic analog of H-J structure makes sense also for $X^4 \subset M^8$ as obtained from the distribution of quaternionic normal spaces containing 2-D commutative sub-space at each point by multiplying then by local unit $I_4(x)$ orthogonal to the quaternionic units $\{1, I_1 = I_2 = I_3\}$ with respect to octonionic inner product. There is a hierarchy of CDs and the choices of these structures would be naturally parameterized by G_2.
   This would give rise to a number theoretically defined slicing of $X^4_c \subset M^8_c$ by complexified string world sheets X^2_c and partonic 2-surfaces Y^2_c orthogonal with respect to the octonionic inner product for complexified octonions.
3. In $M^8 - H$ duality defined by $g(p) \subset SU(3)$ assigns a point of CP_2 to a given point of M^4. $g(p)$ maps the number theoretic H-J to H-J in $M^4 \subset M^8$. The space-time surface itself - that

is $g(p)$ - defines these symplectic coordinates and the local $SU(3)$ element g would naturally define this symplectic transformation.

4. For $X^4 \subset M^8$ g reduces to a constant color rotation satisfying the condition that the image point is $U(2)$ invariant. Unit element is the most natural option. This would mean that g is constant at the mass and energy shells corresponding to the roots of P and the mass shell is a mass shell of M^4 rather than some deformed mass shell associated with images under $g(p)$. This alone does not yet guarantee that the 4-D tangent space corresponds to M^4. The additional physically very natural condition on g is that the 4-D momentum space at these mass shells is the same. $M^8 - H$ duality maps these mass shells to the boundaries of these cd:s in M^4 (CD= $cd \times CP_2$). This conforms with the identification of zero energy states as pairs of 3-D states at the boundaries of CD.

This generalizes the original intuitive but wrong interpretation of the roots r_n of P as "very special moments in the life of self" [L63].

1. Since the roots correspond to mass squared values, they are mapped to the boundaries of cd with size $L = \hbar_{eff}/m$ by $M^8 - H$ duality in M^4 degrees of freedom. During the sequence of SSFRs the passive boundary of CD remains does not shift only changes in size, and states at it remain unaffected. Active boundary is shifted due to scaling of cd.
   The hyperplane at which upper and lower half-cones of CD meet, is shifted to the direction of geometric future. This defines a geometric correlate for the flow of experienced time.
2. A natural proposal is that the moments for SSFRs have as geometric correlates the roots of P defined as intersections of geodesic lines with the direction of 4-momentum p from the tip of CD to its opposite boundary (here one can also consider the possibility that the geodesic lines start from the center of cd). Also energy shells as roots $E = r_n$ of P are predicted. They decompose to a set of mass shells $m_{n.,k}$ with the same $E = r_n$: similar interpretation applies to them.
3. What makes these moments very special is that the mass and energy shells correspond to surfaces in M^4 defining the Lorentz quantum numbers. SSFRs correspond to quantum measurements in this basis and are not possible without this condition. At $X^4 \subset M^8$ the mass squared would remain constant but the local momentum frame would vary. This is analogous to the conservation of momentum squared in general relativistic kinematics of point particle involving however the loss of momentum conservation.
4. These conditions, together with the assumption that g is a rational function with real coefficients, strongly suggest what I have referred to as preferred extremal property, Bohr orbitology, strong form of holography, and number theoretical holography.

In principle, by a suitable choice of M^4 one can make the momentum of the system light-like: the light-like 8-momentum would be parallel to M^4. I have asked whether this could be behind the fact that elementary particles are in a good approximation massless and whether the small mass of elementary particles is due to the presence of states with different mass squares in the zero state allowed by Lorentz invariance.

The recent understanding of the nature of right-handed neutrinos based on M^4 Kähler structure [L106] makes this mechanism un-necessary but poses the question about the mechanism choosing some particular M^4. The conditions that $g(p)$ leaves mass shells and their 4-D tangent spaces invariant provides this kind of mechanism. Holography would be forced by the condition that the 4-D tangent space is same for all mass shels representing inverse images for very special moments of time.

What about string world sheets and partonic 2-surfaces?

One can apply the above arguments also to the identification of 2-D string world sheets and partonic 2-surfaces.

1. One has two kinds of solutions: M^2 and 3-D surfaces of X^4 as analogs of 6-brane. The interpretation for 3-D *resp.* 2-D branes as a light-like 3-surface associated with the octonionic Dirac equation representing mass shell condition *resp.* string world sheet is attractive.
2. M^2 would be replaced with an integrable distribution of $M^2(x)$ in local tangent space $M^4(x)$. The space for the choices of $M^2(x)$ would be S^3 corresponding to the selection of a preferred quaternion imaginary unit equal to the choices of preferred octonion imaginary unit.

The choices of the preferred complex subspace $M^2(x)$ at a given point would be characterized by its normal vector and parameterized by sphere S^2: the interpretation as a quantization axis of angular momentum is suggestive. One would have space $S^3 \times S^2$. Also now the integrability conditions $de_A = 0$ would hold true.

3. String world sheets could be regarded as analogs of superstrings connecting 3-D brane like entities defined by the light-like partonic orbits. The partonic 2-surfaces at the ends of light-like orbits defining also vertices could correspond to the 3-surfaces at which quaternionic 4-surfaces intersect 6-branes.

6.3.3 Is (co-)associativity possible?

The number theoretic vision relying on the assumption that space-time surfaces are 8-D complex 4-surfaces in o_c^8 determined as algebraic surfaces for octonionic continuations of real polynomials, which for adelic physics would have coefficients which are rational or belong to an extension of rationals. The projections to subspaces Re^8 of o_c^8 defined as space for which given coordinate is purely real or imaginary so that complexified octonionic norm is real would give rise to real 4-D space-time surfaces. $M^8 - H$ duality would map these surfaces to geometric objects in $M^4 \times CP_2$. This vision involves several poorly understood aspects and it is good to start by analyzing them.

Challenging the notions of associativity and co-associativity

Consider first the notions of associativity *resp.* co-associativity equivalent with quaternionicity *resp.* co-quaternionicity. The original hope was that both options are possible for surfaces of real sub-spaces of O_c ("real" means here that complexified octonionic metric is real).

1. The original idea was that the associativity of the tangent space or normal space of a real space-time surface X^4 reduces the classical physics at the level of M^8 to associativity. Associativity/co-associativity of the space-time surface states that at each point of the tangent-/normal space of the real space-time surface in O is quaternionic. The notion generalizes also to $X_c^4 \subset O_c^8$. (Co-)associativity makes sense also for the real subspaces space of O with Minkowskian signature.
2. It has been however unclear whether (co-)associativity is possible. The cold shower came as I learned that associativity allows only for geodesic sub-manifolds of quaternionic spaces about which octonions provide an example [A106]. The good news was that the distribution of co-associative tangent spaces always defines an integrable distribution in the sense that one can find sub-manifold for which the associative normal space at a given point has tangent space as an orthogonal complement. Should the number theoretic dynamics rely on co-associativity rather than associativity?
3. Minkowskian space-time regions have been assumed to be associative and to correspond to the projection to the standard choice for basis as $\{1, iI_1, iI_2, iI_3\}$. The octonionic units $\{1, I_1, I_2, I_3\}$ define quaternionic units and associative subspace and their products with unit I_4 define the orthogonal co-associative subspace as $\{I_4, I_5 = I4I_1, I_6 = I_4I_2, I_7 = I_4I_3\}$. This result forces either to weaken the notion of associativity or to consider alternative identifications of Minkowskian regions, which can be co-associative: fortunately, there exists a large number of candidates.

The article [A106] indeed kills the idea about the associativity of the space-time surface. The article starts from a rather disappointing observation that associative sub-manifolds are geodesic sub-manifolds and therefore trivial. Co-associative quaternion sub-manifolds are however possible. With a motivation coming from this observation, the article discusses what the author calls RC quaternionic sub-manifolds of quaternion manifolds. For a quaternion manifold the tangent space allows a realization of quaternionic units as antisymmetric tensors. These manifolds are constant curvature spaces and typically homogeneous spaces.

1. Quaternion sub-manifold allows a 4-D integrable distribution of quaternion units. The normal complement of this distribution is expressible in terms of the second fundamental form and the condition that it is trivial implies that the second fundamental form is vanishing so that one has a geodesic submanifold. Quaternionic sub-manifolds are thus too trivial to be interesting. As a diametric opposite, one can also define totally real submanifolds for which the normal space

contains a distribution of quaternion units. In this case the distribution is always integrable. This case is much more interesting from the TGD point of view.

2. Author introduces the notion of CR quaternion sub-manifold $N \subset M$, where M is quaternion manifold with constant sectional curvatures. N has quaternion distribution D in its tangent spaces if the action of quaternion units takes D to itself. $D^{\perp}$ is the co-quaternionic orthogonal complement D in the normal space N. D would take also $D^{\perp}$ to itself. $D^{\perp}$ can be expressed in terms of the components of the second fundamental form and vanishes for quaternion sub-manifolds.
3. Author deduces results about CR quaternion sub-manifolds, which are very interesting from the TGD point of view.
   (a) Sub-manifold is CR quaternion sub-manifold only if the curvature tensor of R_M of the embedding space satisfies $R_M(D, D, D^{\perp}, D\ \ ) = 0$. The condition is trivial if the quaternion manifold is flat. In the case of octonions this would be the case.
   (b) D is integrable only if the second fundamental form restricted to it vanishes meaning that one has a geodesic manifold. Totally real distribution $D^{\perp}$ is always integrable to a co-associative surface.
   (c) If $D^{\perp}$ integrates to a minimal surface then N itself is a minimal surface.

Could one consider RC quaternion sub-manifolds in TGD framework? Both octonions and their complexifixation can be regarded as quaternionic spaces. Consider the real case.

1. If the entire D is quaternionic then N is a geodesic sub-manifold. This would leave only E^4 and its Minkowskian variants with various signatures. One could have however 4-D totally real (co-associative) space-time surfaces. Simple arguments will show that the intersections of the conjectured quaternionic and co-quaternionic 4-surfaces have 2- and 3-D intersections with 6-branes.
   Should one replace associative space-time surfaces with CR sub-manifolds with $d \leq 3$ integrable distribution D whereas the co-quaternionic surfaces would be completely real having 4-D integrable $D^{\perp}$? Could one have 4-D co-associative surfaces for which $D^{\perp}$ integrates to $n \geq 1$-dimensional minimal surface (geodesic line) and the X^4 itself is a minimal surface?
   Partially associative CR manifold do not allow M^8H duality. Only co-associative surfaces allow it and also their signature must be Minkowskian: the original idea [L67, L37, L38, L39] about Euclidian (Minkowskian) signature for co-associative (associative) surfaces was wrong.
2. The integrable 2-D sub-distributions D defining a distribution of normal planes could define foliations of the X^4 by 2-D surfaces. What springs to mind is foliations by string world sheets and partonic 2 surfaces orthogonal to them and light-like 3-surfaces and strings transversal to them. This expectation is realized.

How to identify the Minkowskian sub-space of O_c?

There are several identifications of subspaces of O_c with Minkowskian signature. What is the correct choice has been far from obvious. Here symmetries come in rescue.

1. Any subspace of O^c with 3 (1) imaginary coordinates and 1 (3) real coordinates has Minkowskian signature in octonionic norm algebraically continued to O_c (complex valued continuation of real octonion norm instead of real valued Hilbert space norm for O_c). Minkowskian regions should have local tangent space basis consisting of octonion units which in the canonical case would be $\{I_1, iI_3, iI_5, iI_7\}$, where i is commutative imaginary unit. This particular basis is co-associative having whereas its complement $\{iI_0, I_2, I_4, I_6\}$ is associative and has also Minkowskian signature.
2. The size of the isometry group of the subspace of M^8_c depends on whether the tangent basis contains real octonion unit 1 or not. The isometry group for the basis containing I_0 is $SO(3)$ acting as automorphisms of quaternions and $SO(k, 3-k)$ when $3-k$ units are proportional to i. The reason is that G_2 (and its complexification $G_{2,c}$) and its subgroups do not affect I_0. For the tangent spaces built from 4 imaginary units I_k and iI_l the isometry group is $SO(k, 4-k) \subset G_{2,c}$.
   The choice therefore allows larger isometry groups and also co-associativity is possible for a suitable choice of the basis. The choice $\{I_1, iI_3, iI_5, iI_7\}$ is a representative example, which

will be called canonical basis. For these options the isometry group is the desired $SO(1,3)$ as an algebraic continuation of $SO(4) \subset G_2$ acting in $\{I_1, I_3, I_5, I_7\}$, to $SO(1,3) \subset G_{2,c}$.
Also Minkowskian signature - for instance for the original canonical choice $\{I_0, iI_1, iI_2, iI_3\}$ - can have only $SO(k, 3-k)$ as isometries. This is the basic objection against the original choice $\{I_0, iI_1, iI_2, iI_3\}$. This identification would force the realization of $SO(1,3)$ as a subgroup of $SO(1,7)$. Different states of motion for a particle require different octonion structure with different direction of the octonion real axis in M^8. The introduction of the notion of moduli space for octonion structures does not look elegant. For the option $\{I_1, iI_3, iI_5, iI_7\}$ only a single octonion structure is needed and $G_{2,c}$ contains $SO(1,3)$.
Note that also the signatures $(4,0)$, $(0,4)$ and $(2,2)$ are possible and the challenge is to understand why only the signature (1,3) is realized physically.

Co-associative option is definitely the only physical alternative. The original proposal for the interpretation of the Minkowski space in terms of an associative real sub-space of M^4 had a serious problem. Since time axis was identified as octonionic real axis, one had to assign different octonion structure to particles with non-parallel moment: $SO(1,7)$ would relate these structures: how to glue the space-time surfaces with different octonion structures together was the problem. This problem disappears now. One can simply assign to particles with different state of motion real space-time surface defined related to each other by a transformation in $SO(1,3) \subset G_{2,c}$.

The condition that $M^8 - H$ duality makes sense

The condition that $M^8 - H$ duality makes sense poses strong conditions on the choice of the real sub-space of M^8_c.

1. The condition that tangent space of O_c has a complexified basis allowing a decomposition to representations of $SU(3) \subset G_2$ is essential for the map to $M^8 \to H$ although it is not enough. The standard representation of this kind has basis $\{\pm iI_0 + I_1\}$ behaving like $SU(3)$ singlets $\{I_2 + \epsilon iI_3, I_4 + \epsilon iI_5, \epsilon I_6 \pm iI_7\}$ behaves like SU(3) triplet 3 for $\epsilon = 1$ and its conjugate $\overline{3}$ for $\epsilon = -1$. $G_{2,c}$ provides new choices of the tangent space basis consistent with this choice. $SU(3)$ leaves the direction I_1 unaffected but more general transformations act as Lorentz transformation changing its direction but not leaving the M^4 plane. Even more general $G_{2,c}$ transformations changing M^4 itself are in principle possible.
Interestingly, for the canonical choice the co-associative choice has $SO(1,3)$ as isometry group whereas the complementary choice failing to be associative correspond to a smaller isometry group $SO(3)$. The choice with M^4 signature and co-associativity would provide the highest symmetries. For the real projections with signature $(2,2)$ neither consistent with color structure, neither full associativity nor co-associativity is possible.
2. The second essential prerequisite of $M^8 - H$ duality is that the tangent space is not only (co-)associative but contains also (co-)complex - and thus (co-)commutative - plane. A more general assumption would be that a co-associative space-time surface contains an integrable distribution of planes $M^2(x)$, which could as a special case reduce to M^2.
The proposal has been that this integrable distribution of $M^2(x)$ could correspond to string sheets and possibly also integrable orthogonal distribution of their co-complex orthogonal complements as tangent spaces of partonic 2-surfaces defining a slicings of the space-time surface. It is now clear that this dream cannot be realized since the space-time surface cannot be even associative unless it is just E^4 or its Minkowskian variants.
3. As already noticed, any distribution of the associative normal spaces integrates to a co-associative space-time surface. Could the normal spaces also contain an integrable distribution of co-complex planes defined by octonionic real unit 1 and real unit $I_k(x)$, most naturally I_1 in the canonical example? This would give co-commutative string world sheet. Commutativity would be realized at the 2-D level and associativity at space-time level. The signature of this plane could be Minkowskian or Euclidian. For the canonical example $\{I_1, iI_3, iI_5, iI_7\}$ the 2-D complex plane in quaternionic sense would correspond to $(a \times 1, +n_2I_2 + n_4I_6 + n_6I_6$, where the unit vector n_i has real components and one has $a = 1$ or $a = i$ is forced by the complexification as in the canonical example.
Since the distribution of normal planes integrates to a 4-surface, one expects that its sub-distribution consting of commutative planes integrates to 2-D surface inside space-time surface

and defines the counterpart of string worlds sheet. Also its normal complement could integrate to a counterpart of partonic 2-surface and a slicing of space-time surface by these surfaces would be obtained.

4. The simplest option is that the commutative space does not depend on position at X^4. This means a choice of a fixed octonionic imaginary unit, most naturally I_1 for the canonical option. This would make $SU(3)$ and its sub-group $U(2)$ independent of position. In this case the identification of the point of $CP_2 = SU(3)/U(2)$ labelling the normal space at a given point is unique.
   For a position dependent choice $SU(3)(x)$ it is not clear how to make the specification of $U(2)(x)$ unique: it would seem that one must specify a unique element of $G_2(x)$ relating $SU(3)(x)$ to a choice at special point x_0 and defining the conjugation of both $SU(3)(x)$ and $U(2)(x)$. Otherwise one can have problems. This would also mean a unique choice for the direction of time axis in O and fixing of $SO(1,3)$ as a subgroup of $G_{2,c}$. Also this distribution of associative normal spaces is integrable. Physically this option is attractive but an open question is whether it is consistent with the identification of space-time surfaces as roots $Re_Q(P) = 0$ of P.

Co-associativity from octonion analyticity or/and from G_2 holography?

Candidates for co-associative space-time surfaces X^4_r are defined as restrictions X^4_r for the roots X^4_c of the octonionic polynomials such that the O_c coordinates in the complement of a real co-associative sub-space of O_c vanish or are constant. Could the surfaces X^4_r or even X^4_c be co-associative?

1. X^4_r is analogous to the image of real or imaginary axis under a holomorphic map and defines a curve in complex plane preserving angles. The tangent vectors of X^4_r and X^4_c involve gradients of all coordinates of O_c and are expressible in terms of all octonionic unit vectors. It is not obvious that their products would belong to the normal space of X^4_r a strong condition would be that this is the case for X^4_c.
2. Could octonion analyticity in the proposed sense guarantee this? The products of octonion units also in the tangent space of the image would be orthogonal to the tangent space. Ordinary complex functions preserve angles, in particular, the angle between x- and y-axis is preserved since the images of coordinate curves are orthogonal. Octonion analyticity would preserve the orthogonality between tangent space vectors and their products.
3. This idea could be killed if one could apply the same approach to associative case but this is not possible! The point is that when the real tangent space of O_c contains the real octonion unit, the candidate for the 4-D space-time surface is a complex surface X^2_c. The number theoretic metric is real only for 2-D X^2_r so that one obtains string theory with co-associativity replaced with co-commutativity and $M^4 \times CP_2$ with $M^2 \times S^2$. One could of course ask whether this option could be regarded as a "sub-theory" of the full theory.

My luck was that I did not realize the meaning of the difference between the two cases first and realized that one can imagine an alternative approach.

1. G_2 as an automorphism group of octonions preserves co-associativity. Could the image of a co-associative sub-space of O_c defined by an octonion analytic map be regarded as an image under a local G_2 gauge transformation. $SU(3) \subset G_2$ is an especially interesting subgroup since it could have a physical interpretation as a color gauge group. This would also give a direct connection with $M^8 - H$ duality since $SU(3)$ corresponds to the gauge group of the color gauge field in H.
2. One can counter-argue that an analog of pure gauge field configuration is in question at the level of M^8. But is a pure gauge configuration for $G_{2,c}$ a pure gauge configuration for G_2? The point is that the $G_{2,c}$ connection $g^{-1}\partial_\mu g$ trivial for $G_{2,c}$ contains by non-linearity cross terms from $g_2 g, c = g_{2,1} + ig_{2,2}$, which are of type $Re = X[g_{2,1}, g_{2,1}] - X[g_{2,2}, g_{2,2}] = 0$ and $Im = iZ[g_{2,1}, g_{2,2}] = 0$. If one puts $g_{2,2}$ contributions to zero, one obtains $Re = X[g_{2,1}, g_{2,1}]$, which does not vanish so that $SU(3)$ gauge field is non-trivial.
3. X^4_r could be also obtained as a map of the co-associative M^4 plane by a local $G_{2,c}$ element. It will turn out that $G_{2,c}$ could give rise to the speculated Yangian symmetry [L32] at string

world sheets analogous to Kac-Moody symmetry and gauge symmetry and crucial for the construction of scattering amplitudes in $M8$.

4. The decomposition of the co-associative real plane of O_c should contain a preferred complex plane for $M^8 - H$ duality to make sense. $G_{2,c}$ transformation should trivially preserve this property so that SH would not be necessary at H side anymore.

There is a strong motivation to guess that the two options are equivalent so that $G_{2,c}$ holography would be equivalent with octonion analyticity. The original dream was that octonion analyticity would realize both associative and co-associative dynamics but was exaggeration!

Does one obtain partonic 2-surfaces and strings at boundaries of ΔCD_8?

It is interesting to look for the dimensions of the intersections of the light-like branes at the boundary of CD_8 giving rise to the boundary of CD_4 in M^4 to see whether it gives justification for the existing phenomenological picture involving light-like orbits of partonic 2-surfaces connected by string world sheets.

1. Complex light-cone boundary has dimension $D = 14$. $P = 0$ as an additional condition at δCD_8 gives 2 complex conditions and defines a 10-D surface having 5-D real projections.
2. The condition $Im_Q(P) = 0$ gives 8 conditions and gives a 2-D complex surface with 1-D real projection. The condition $Re_Q(P) = 0$ gives 3 complex conditions since $X = 0$ is already satisfied and the solution is a 4-D surface having 2-D real projection. Could the interpretation be in terms of the intersection of the orbit of a light-like partonic surface with the boundary of CD_8?
3. Associativity is however not a working option. If only co-associative Minkowskian surfaces allowing mapping to H without SH are present then only 4-D space-time surfaces with Minkowskian signature, only partonic 2-surfaces and their light-like orbits would emerge from co-associativity. This option would not allow string world sheets for which there is a strong intuitive support. What could a co-complex 2-surface of a co-associative manifold mean? In the co-associative case the products of octonion imaginary units are in the normal space of space-time surface. Could co-complex surface $X_c^2 \subset X_c^4$ be defined by an integrable co-complex sub-distribution of co-associative distribution. The 4-D distribution of normal planes is always integrable.
   Could the 2-D sub-distributions of co-associative distribution integrate trivially and define slicings by string world sheets or partonic 2-surfaces. Could the distribution of string distributions and its orthogonal complement be both integrable and provide orthogonal slicings by string world sheets and partonic 2-surfaces? String world sheets with Minkowskian signature should intersect the partonic orbits with Euclidian signature along light-like lines. This brings in mind the orthogonal grid of flow lines defined by the $Re(f) = 0$ and $Im(f) = 0$ lines of an analytic function in plane.
4. In this picture the partonic 2-surfaces associated with light-like 3-surface would be physically unique and could serve as boundary values for the distributions of partonic 2-surfaces. But what about string world sheets connecting them? Why would some string world sheets be exceptional? String world sheets would have a light-like curve as an intersection with the partonic orbit but this is not enough.
   Could the physically special string world sheets connect two partonic surfaces? Could the string associated with a generic string world sheet be like a flow line in a hydrodynamic flow past an obstacle - the partonic 2-surface? The string as a flowline would go around the obstacle along either side but there would be one line which ends up to the object.

Interactions would correspond geometrically to the intersections of co-associative space-time surfaces X_r^4 associated with particles and corresponding to different real sub-spaces of O_c related by Lorentz boost in $SO(1,3) \subset G_{2,c}$. In the generic case the intersection would be discrete. In the case that X and Y have a common root the real surfaces $X_r^4 \subset X_r^6$ associated with quarks and depending on their state of motion would reside inside the same 6-D surface X_r^6 and have a 2-D surface X_r^2 as intersection. Could this surface be interpreted as a partonic 2-surface? One must however bear in mind that partonic 2-surfaces as topological vertices are assumed to be non-generic in the sense that the light-like partonic orbits meet at them. At the level of H, the intersections would be partonic 2-surfaces X^2 at which the four 3-D partonic orbits would meet along their ends. Does this hold true at the level of M^8? Or can it hold true even at the level H?

The simplest situation corresponds to 4 external quarks. There are 6 different intersections. Not all of them are realized since a given quark can belong only to a single intersection. One must have two disjoint pairs -say 12 and 34. Most naturally positive *resp.* negative energy quarks form a pair. These pairs are located in different half-cones. The intersections would give two partonic 2-surfaces and this situation would be generic. This suggests a modification of the description of particle reaction in M^8 . $M^8 - H$ duality suggests a similar description in H.

What could be the counterparts of wormhole contacts at the level of M^8?

The experience with H, in particular the presence of extremals with Euclidian signature of the induced metric and identified as building bricks of elementary particles, suggest that also the light-like 3-surfaces in M^8_c could have a continuation with an Euclidian signature of the number theoretic metric with norm having real values only for the projections to planes allowing real coordinates.

The earlier picture has been that the wormhole contacts as CP_2 type extremals correspond to co-associative regions and their exteriors to associative regions. If one wants $M^8 - H$ duality in strong form and thus without need for SH, one should assume that both these regions are co-associative.

1. The simplest option is that the real Minkowskian time coordinate becomes imaginary. Instead of the canonical (I_1, iI_3, iI_5, iI_7) the basis would be (iI_1, iI_3, iI_5, iI_7) having Euclidian signature and SO(4) as isometry group. The signature would naturally change at light-like 3-surface the time coordinate along light-like curves becomes zero - proper time for photon vanishes - and can ransforms continuously from real to imaginary.
2. Wormhole contacts in H behave like pairs of magnetic monopoles with monopole charges at throats. If one does not allow point-like singularity, the monopole flux must go to a parallel Minkowskian space-time sheet through the opposite wormhole throat. Wormhole contact with effective magnetic charge would correspond in M^8_c to a distribution of normal 4-planes at the partonic 2-surfaces analogous to the radial magnetic field of monopole at a sphere surrounding it. To avoid singularity of the distribution, there must be another light-like 3-surface M^8 such that its partonic throat has a topologically similar distribution of normal planes.

In the case of X^3_c dimension does not allow co-quaternion structure: could they allow 4-D co-associative sub-manifolds? It will be found that this option is not included since co-associative tangent space distributions in a quaternion manifold (now O) are always integrable.

6.3.4 Octonionic Dirac equation and co-associativity

Also the role of associativity concerning octonionic Dirac equation in M^8 must be understood. It is found that co-associativity allows very elegant formulation and suggests the identification of the points appearing as the ends of quark propagator lines in H as points of boundary of CD representing light-like momenta of quarks. Partonic vertices would involve sub-CDs and momentum conservation would have purely geometric meaning bringing strongly in mind twistor Grassmannian approach [B26, B22, B27]. I have discussed the twistor lift of TGD replacing twistors as fields with surfaces in twistor space having induced twistor structure in [K91, K79, L56] [L78, L79].

Octonionic Dirac equation

The following arguments lead to the understanding of co-associativity in the case of octonion spinors. The constant spinor basis includes all spinors but the gamma matrices appearing in the octonionic Dirac equation correspond to co-associative octonion units.

1. At the level of O_c the idea about massless Dirac equation as partial differential equation does not make sense. Dirac equation must be algebraic and the obvious idea is that it corresponds to the on mass shell condition for a mode of ordinary Dirac equation with well-define momentum: $p^k\gamma_k\Psi = 0$ satisfying $p^kp_k = 0$. This suggests that octonionic polynomial P defines the counterpart of $p^k\gamma_k$ so that gamma matrices γ_k would be represented as octonion components. Does this make sense?

2. Can one construct octonionic counterparts of gamma matrices? The imaginary octonion units I_k indeed define the analogs of gamma matrices as $\gamma_k \equiv iI_k$ satisfying the conditions $\{\gamma_k, \gamma_l\} = 2\delta_{kl}$ defining Euclidian gamma matrices. The problem is that one has $I_0 I_l k + I_k I_0 = 2I_k$. One manner to solve the problem would be to consider tensor products $I_0\sigma_3$ and $I_k\sigma_2$ where σ_3 and $sigma_2$ are Pauli's sigma matrices with anti-commutation relations $\{\sigma_i, \sigma_j\} = \delta_{i,j}$. Note that I_k do not allow a matrix representation.
   Co-associativity condition suggests an alternative solution. The restriction of momenta to be co-associative and therefore vanishing component p^0 as octonion, would selects a sub-space spanned by say the canonical choice $\{I_2, iI3, iI_5, iI_7\}$ satisfying the anticommutation relations of Minkowskian gamma matrices. Octonion units do not allow a matrix representation because they are not associative. The products for a co-associative subset of octonion units are however associative $(a(bc) = (ab)c$ so that they can be mapped to standard gamma matrices in Minkowski space. Co-associativity would allow the representation of 4-D gamma matrices as a maximal associative subset of octonion units.
3. What about octonionic spinors. The modes of the ordinary Dirac equation with a well-defined momentum are obtained by applying the Dirac operator to an orthogonal basis of constant spinors u_i to give $\Psi = p^k\gamma_k u_i$. Now the counterparts of constant spinors u_i would naturally be octonion units $\{I_0, I_k\}$: this would give the needed number 8 of real spinor components as one has for quark spinors.
   Dirac equation reduces to light-likeness conditions $p^k p_k = 0$ and p_k must be chosen to be real - if pk are complex, the real and imaginary parts of momentum are parallel. One would obtain an entire 3-D mass shell of solution and a single mode of Dirac equation would correspond to a point of this mass shell.
   Remark: Octonionic Dirac equation is associative since one has a product of form $(p_k\gamma_k)^2 u_i$ and octonion products of type $x^2 y$ are associative.
4. p^k would correspond to the restriction of $P(o_c)$ to M^4 as sub-space of octonions. Since co-associativity implies $P(o_c) = Y(o_c)o_c$ restricted to counterpart of M^4 (say subspace spanned by $\{I_2, iI3, iI_5, iI_7\}$), Dirac equation reduces to the condition $o^k o_k = 0$ in M^4 defining a light-cone of M^4. This light-cone is mapped to a curved light-like 3-surface X^3 in o_c as $o_c \to P(o_c) = Y o_c$. $M^8 - H$ duality maps points of space-time surface on M^8 H and therefore the light-cone of M^4 corresponds to either light-like boundary of CD. It seems that the image of X^3 in H has M^4 projection to the light-like boundary of CD.
   Co-associative space-time surfaces have 3-D intersections X^3 with the surface $P = 0$: the conjecture is that X^3 corresponds to a light-like orbit of partonic 2-surfaces in H at which the induced metric signature changes. At X^3 one has besides $X = 0$ also $Y = 0$ so that octonionic Dirac equation $P(o_c)\Psi = P^k I_k\Psi = Yp^k I_k\Psi = 0$ is trivially satisfied for all momenta $p^k = o^k$ defined by the M^4 projections of points of X^3 and one would have $P^k = Yp^k = 0$ so that the identification of P^k as 4-momentum would not allow to assign non-vanishing momenta to X^3. The direction of p^k is constrained only by the condition of belonging to X^3 and the momentum would be in general time-like since X^3 is inside future light-cone.
   $Y = 0$ condition conforms with the proposal that X^3 defines a boundary of Minkowskian and Euclidian region: Euclidian mass shell condition for real P^k requires $P^k = 0$. The general complex solution to $P^2 = 0$ condition is $P = P_1 + iP_2$ with $P_1^2 = P_2^2$.

A single mode of Dirac equation with a well-defined value of p^k as the analog of 4-momentum would correspond to a selection of single time-like point at X^3 or light-like point at the light-like boundary of CD. X^3 intersects light-cone boundary as part of boundary of 7-D light-cone. The picture about scattering amplitudes - consistent with the view about cognitive representations as a unique discretization of space-time surface - is that quarks are located at discrete points of partonic 2-surfaces representing the ends of fermionic propagator lines in H and that one can assign to them light-like momenta.

Challenging the form of $M^8 - H$ duality for the map $M^4 \subset M^8$ to $M^4 \subset H$

The assumption that the map $M^4 \subset M^8$ to $M^4 \subset H$ in $M^8 - H$ duality is a simple identification map has not been challenged hitherto.

1. Octonionic Dirac equation forces the identification of M^8 as analog of 8-D momentum space and the earlier simple identification is in conflict with Uncertainty Principle. Inversion al-

lowed by conformal invariance is highly suggestive: what comes first in mind is a map $m^k \rightarrow \hbar_{eff} m^k/m^k m_k$.
At the light-cone boundary the map is ill-defined. Here on must take as coordinate the linear time coordinate m^0 or equivalently radial coordinate $r_M = m^0$. In this case the map would be of fporm $t \rightarrow \hbar_{eff}/m^0$: m^0 has interpretation as energy of massless particle.
The map would give a surprisingly precise mathematical realization for the intuitive arguments assigning to mass a length scale by Uncertainty Principle.

2. Additional constraints on $M^8 - H$ duality in M^4 degrees of freedom comes from the following argument. The two half-cones of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T - o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T - o)$: the space-time surfaces at half-cones would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - \overline{o}^2$ only, the condition is identically satisfied for both options.
There are two options for the identification of the coordinate t.
Option a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time.
At the level of M^8, The M^4 projection of CD_8 is a union of future and past directed light-cones with a common tip rather than CD_4. Both incoming and outgoing momenta have the same origin automatically. This identification is the natural one at the level of M^8.
Option b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The half-cone at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.
If **Option a)** is realized at the level of M^8 and **Option b)** at the level of H, as seems natural, a time translation $m^0 \rightarrow m^0 + 2T$ of the past directed light-cone in $M^4 \subset H$ is required in order to to give upper half-cone of CD_4.

3. The map of the momenta to embedding space points does not prevent the interpretation of the points of M^8 as momenta also at the level of H since this information is not lost. One cannot identify p^k as such as four-momentum neither at the level of M^8 nor H as suggested by the naïve identification of the Cartesian factors M^4 for M^8 and H. This problem is circumvented by a conjugation in M_c^8 changing the sign of 3-momentum. The light-like momenta along the light-cone boundary are non-physical but transform to light-like momenta arriving into light-cone as the physical intuition requires.
Therefore the map would have in the interior of light-cone roughly the above form but there is still a question about the precise form of the map. Does one perform inversion for the M^4 projection or does one take M^4 projection for the inversion of complex octonion. The inversion of M^4 projection seems to be the more plausible option. Denoting by $P(o_c)$ the real M^4 projection of X^4 point one therefore has:

$$P(o_c) \rightarrow \hbar_{eff} \frac{\overline{P(o_c)}}{P(o_c) \cdot P(o_c)} \quad . \tag{6.3.7}$$

Note that the conjugation changes the direction of 3-momentum.
At the light-cone boundary the inversion is ill-defined but Uncertainty Principle comes in rescue, and one can invert the M^4 time coordinate:

$$Re(m^0) = t \rightarrow \hbar_{eff} \frac{1}{t} \quad . \tag{6.3.8}$$

A couple of remarks are in order.

1. The presence of $\hbar_{eff}$ instead of $\hbar$ is required by the vision about dark matter. The value of $\hbar_{eff}/h_0$ is given by the dimension of extension of rationals identifiable as the degree of P.
2. The image points $\overline{p}^k$ in H would naturally correspond to the ends of the propagator lines in the space-time representation of scattering amplitudes.

The information about momenta is not lost in the map. What could be the interpretation of the momenta $\overline{p}^k$ at the level of H?

1. Super-symplectic generators at the partonic vertices in H do not involve momenta as labels. The modes of the embedding space spinor field assignable to the ground states of super-symplectic representations at the boundaries of CD have 4-momentum and color as labels. The identification of $\overline{p}^k$ as this momentum label would provide a connection with the classical picture about scattering events.
   At the partonic 2-surfaces appearing as vertices, one would have a sum over the ground states (spinor harmonics). This would give integral over momenta but $M^8 - H$ duality and number theoretic discretization would select a finite subset and the momentum integral would reduce to a discrete sum. The number of M^8 points with coordinates in a given extension of rationals is indeed finite.
2. $M^4 \subset M^8$ could be interpreted as the space of 4-momenta labeling the spinor harmonics of M^8. Same would apply at the level of H: spinor harmonics would correspond to the ground states of super-symplectic representations.
3. The interpretation of the points of M^4_c as complex 4-momenta inspires the question whether the interpretation of the imaginary part of the momentum squared in terms of decay decay width so that M^8 picture would code even information about the dynamics of the particles.

6.4 How to achieve periodic dynamics at the level of $M^4 \times CP_2$?

Assuming $M^8 - H$ duality, how could one achieve typical periodic dynamics at the level of H - at least effectively?

It seems that one cannot have an "easy" solution to the problem?

1. Irreducible polynomials which are products of monomials corresponding to roots r_n which are in good approximation evenly spaced $r_n = r_0 + nr_1\Delta r_n$ would give "very special moments in the life of self" as values of M^4 time which are evenly spaced [L67, L63]. This could give rise to an effective periodicity but it would be at the level of M^8, not H, where it is required.
2. Is it enough that the periodic functions are *only* associated with the spinor harmonics of H involved with the construction of scattering amplitudes in H [L97]? For the modified Dirac equation [K100] the periodic behavior is possible. Note also that the induced spinors defining ground states of super-symplectic representations are restrictions of second quantized spinors of H proportional to plane waves in M^4. These solutions do not guarantee quantum classical correspondence.

6.4.1 The unique aspects of Neper number and number theoretical universality of Fourier analysis

Could one assume more general functions than polynomials at the level of H? Discrete Fourier basis is certainly an excellent candidate in this respect but does it allow number theoretical universality?

1. Discrete Fourier analysis involves in the Euclidian geometry periodic functions $exp(2\pi x)$, n integer and in hyperbolic geometry exponential functions $exp(kx)$.
   Roots of unity $exp(i2\pi/n)$ allow to generalize Fourier analysis. The p-adic variants of $exp(ix)$ exist for rational values of $x = k2\pi/n$ for $n = K$ if $exp(i2\pi/K)$ belongs to the extension of rationals. $x = k = 2pi/n$ does not exist as a p-adic number but $exp(x) = exp(i2\pi/n)$ can exist as phase replacing x as coordinate in extension of p-adics. One can therefore define Fourier basis $\{exp(inx)|n \in Z\}$ which exist at discrete set of rational points $x = k/n$
   Neper number e is also p-adically exceptional in that e^p exists as a p-adic number for all primes p. One has a hierarchy of finite-D extensions of p-adic numbers spanned by the roots $e^{1/n}$. Finiteness of cognition might allow them. Hyperbolic functions $exp(nx)$, $n = 1, 2...$ would have values in extension of p-adic number field containing $exp(1/N)$ in a discrete set of points $\{x = k/N|k \in Z\}$.
2. (Complex) rationality guarantees number theoretical universality and is natural since CP_2 geometry is complex. This would correspond to the replacement $x \to exp(ix)$ or $x \to exp(x)$ for powers x^n. The change of the signature by replacing real coordinate x with ix would automatically induce this change.

3. Exponential functions are in a preferred position also group theoretically. Exponential map maps $g \rightarrow exp(itg)$ the points of Lie algebra to the points of the Lie group so that the tangent space of the Lie algebra defines local coordinates for the Lie group. One can say that tangent space is mapped to space itself. M^4 defines an Abelian group and the exponential map would mean replacing of the M^4 coordinates with their exponential, which are p-adically more natural. Ordinary Minkowski coordinates have both signs so that they would correspond to the Lie algebra level.
4. CP_2 is a coset space and its points are obtained as selected points of $SU(3)$ using exponentiation of a commutative subalgebra t in the decomposition $g = h + t + \bar{t}$ in the Lie-algebra of $SU(3)$. One could interpret the CP_2 points as exponentials and the emergence of exponential basis as a basis satisfying number theoretical universality.

6.4.2 Are CP_2 coordinates as functions of M^4 coordinates expressible as Fourier expansion

Exponential basis is not natural at the level of M^8. Exponential functions belong to dynamics, not algebraic geometry, and the level H represents dynamics.

It is the dependence of CP_2 coordinates on M^4 coordinates, where the periodicity is needed. The map of the tangent spaces of $X^4 \subset M^8$ to points of CP_2 is slightly local since it depends on the first derivatives crucial for dynamics. Could this bring in dynamics and exponential functions at the level of H?

These observations inspire the working hypothesis that CP_2 points as functions of M^4 coordinates are expressible as polynomials of hyperbolic and trigonometric exponentials of M^4 coordinates.

Consider now the situation in more detail.

1. The basis for roots of e would be characterized by integer K in $e^{1/K}$. This brings in a new parameter characterizing the extension of rationals inducing finite extensions of p-adic numbers. K is analogous to the dimension of extension of rationals: the p-adic extension has dimension $d = Kp$ depending on the p-adic prime explicitly.
2. If CD size T is given, $e^{-T/K}$ defines temporal and spatial resolution in H. K or possibly Kp could naturally correspond to the gravitational Planck constant [L48] [K10] [?] $K = n_{gr} = \hbar_{gr}/h_0$.
3. In [L99] many-sheetedness with respect to CP_2 was proposed to correspond to flux tubebundles in M^4 forming quantum coherent structures. A given CP_2 point corresponds to several M^4 points with the same tangent space and their number would correspond to the number of the flux tubes in the bundle.
   Does the number of these points relate to K or Kp? p-Adic extension would have finite dimension $d = Kp$. Could $d = Kp$ be analogous to a degree of polynomial defining the dimension of extension of rationals? Could this be true in p-adic length scale resolution $O(p^2) = 0$ The number of points would be Kp and very large. For electron one has $p = M_{127} = 2^{127} - 1$.
4. The dimension n_A Abelian extension associated with EQ would naturally satisfy $n_A = K$ since the trigonometric and hyperbolic exponentials are obtained from each other by replacing a real coordinate with an imaginary one.
5. There would be two effective Planck constants. $h_{eff} = nh_0$ would be defined by the degree n of the polynomial P defining $X^4 \subset M^8$. $\hbar_{gr} = n_{gr}h_0$ would define infra-red cutoff in M^4 as the size scale of CD in $H = M^4 \times CP_2$. n *resp.* $n_{gr} = Kp$ would characterize many-sheetedness in M^4 *resp.* CP_2 degrees of freedom.

6.4.3 Connection with cognitive measurements as analogs of particle reactions

There is an interesting connection to the notion of cognitive measurement [L99, L100, L105].

1. The dimension n of the extension of rationals as the degree of the polynomial $P = P_{n_1} \circ P_{n_2} \circ ...$ is the product of degrees of degrees n_i: $n = \prod_i n_i$ and one has a hierarchy of Galois groups G_i

associated with $P_{n_i} \circ ....$ G_{i+1} is a normal subgroup of G_i so that the coset space $H_i = G_i/G_{i+1}$ is a group of order n_i. The groups H_i are simple and do not have this kind of decomposition: simple finite groups appearing as building bricks of finite groups are classified. Simple groups are primes for finite groups.

2. The wave function in group algebra $L(G)$ of Galois group G of P has a representation as an entangled state in the product of simple group algebras $L(H_i)$. Since the Galois groups act on the space-time surfaces in M^8 they do so also in H. One obtains wave functions in the space of space-time surfaces. G has decomposition to a product (not Cartesian in general) of simple groups. In the same manner, $L(G)$ has a representation of entangled states assignable to $L(H_i)$ [L99, L105].

This picture leads to a model of analysis as a cognitive process identified as a cascade of "small state function reductions" (SSFRs) analogous to "weak" measurements.

1. Cognitive measurement would reduce the entanglement between $L(H_1)$ and $L(H_2)$, the between $L(H_2)$ and $L(H_3)$ and so on. The outcome would be an unentangled product of wave functions in $L(H_i)$ in the product $L(H_1) \times L(H_2) \times ....$ This cascade of cognitive measurements has an interpretation as a quantum correlate for analysis as factorization of a Galois group to its prime factors. Similar interpretation applies in M^4 degrees of freedom.
2. This decomposition could correspond to a replacement of P with a product $\prod_i P_i$ of polynomials with degrees $n = n_1 n_2 ...$, which is irreducible and defines a union of separate surfaces without any correlations. This process is indeed analogous to analysis.
3. The analysis cannot occur for simple Galois groups associated with extensions having no decomposition to simpler extensions. They could be regarded as correlates for irreducible primal ideas. In Eastern philosophies the notion of state empty of thoughts could correspondto these cognitive states in which SSFRs cannot occur.
4. An analogous process should make sense also in the gravitational sector and would mean the splitting of $K = n_A$ appearing as a factor $n_{gr} = Kp$ to prime factors so that the sizes of CDs involved with the resulting structure would be reduced. This process would reduce to a simultaneous measurement cascade in hyperbolic and trigonometric Abelian extensions. The IR cutoffs having interpretation as coherence lengths would decrease in the process as expected. Nature would be performing ordinary prime factorization in the gravitational degrees of freedom.

Cognitive process would also have a geometric description.

1. For the algebraic EQs, the geometric description would be as a decay of n-sheeted 4-surface with respect to M^4 to a union of n_i-sheeted 4-surfaces by SSFRs. This would take place for flux tubes mediating all kinds of interactions.
   In gravitational degrees of freedom, that is for trascendental EQs, the states with $n_{gr} = Kp$ having bundles of Kp flux tubes would deca to flux tubes bundles of $n_{gr,i} = K_i p$, where K_i is a prime dividing K. The quantity $log(K)$ would be conserved in the process and is analogous to the corresponding conserved quantity in arithmetic quantum field theories (QFTs) and relates to the notion of infinite prime inspired by TGD [K84].
2. This picture leads to ask whether one could speak of cognitive analogs of particle reactions representing interactions of "thought bubbles" i.e. space-time surfaces as correlates of cognition. The incoming and outgoing states would correspond to a Cartesian product of simple subgroups: $G = \prod_i^{\times} H_i$. In this composition the order of factors does not matter and the situation is analogous to a many particle system without interactions. The non-commutativity in general case leads to ask whether quantum groups might provide a natural description of the situation.
3. Interestingly, Equivalence Principle is consistent with the splitting of gravitational flux tube structures to smaller ones since gravitational binding energies given by Bohr model in $1/r$ gravitational potential do not depend on the value of $\hbar_{gr}$ if given by Nottale formula $\hbar_{gr} = GMm/v_0$ [L113]. The interpretation would be in terms of spontaneous quantum decoherence taking place as a decay of gravitational flux tube bundles as the distance from the source increases.

6.4.4 Still some questions about $M^8 - H$ duality

There are still on questions to be answered.

1. The map $p^k \rightarrow \ m^k = \hbar_{eff} p^k/p \cdot p$ defining $M^8 - H$ duality is consistent with Uncertainty Principle but this is not quite enough. Momenta in M^8 should correspond to plane waves in H.
   Should one demand that the momentum eigenstate as a point of cognitive representation associated with $X^4 \subset M^8$ carrying quark number should correspond to a plane wave with momentum at the level of $H = M^4 \times CP_2$? This does not make sense since $X^4 \subset CD$ contains a large number of momenta assignable to fundamental fermions and one does not know which of them to select.
2. One can however weaken the condition by assigning to CD a 4-momentum, call it P. Could one identify P as
   (a) the total momentum assignable to either half-cone of CD
   (b) or the sum of the total momenta assignable to the half-cones?

The first option does not seem to be realistic. The problem with the latter option is that the sum of total momenta is assumed to vanish in ZEO. One would have automatically zero momentum planewave. What goes wrong?

1. Momentum conservation for a single CD is an ad hoc assumption in conflict with Uncertainty Principle, and does not follow from Poincare invariance. However, the sum of momenta vanishes for non-vanishing planewave when defined in the entire M^4 as in QFT, not for planewaves inside finite CDs. Number theoretic discretization allows vanishing in finite volumes but this involves finite measurement resolution.
2. Zero energy states represent scattering amplitudes and at the limit of infinite size for the large CD zero energy state is proportional to momentum conserving delta function just as S-matrix elements are in QFT. If the planewave is restricted within a large CD defining the measurement volume of observer, four-momentum is conserved in resolution defined by the large CD in accordance with Uncertainty Principle.
3. Note that the momenta of fundamental fermions inside half-cones of CD in H should be determined at the level of H by the state of a super-symplectic representation as a sum of the momenta of fundamental fermions assignable to discrete images of momenta in $X^4 \subset H$.

$M^8 - H$-duality as a generalized Fourier transform

This picture provides an interpretation for $M^8 - H$ duality as a generalization of Fourier transform.

1. The map would be essentially Fourier transform mapping momenta of zero energy as points of $X^4 \subset CD \subset M^8$ to plane waves in H with position interpreted as position of CD in H. CD and the superposition of space-time surfaces inside it would generalize the ordinary Fourier transform . A wave function localized to a point would be replaced with a superposition of space-time surfaces inside the CD having interpretation as a perceptive field of a conscious entity.
2. $M^8 - H$ duality would realize momentum-position duality of wave mechanics. In QFT this duality is lost since space-time coordinates become parameters and quantum fields replace position and momentum as fundamental observables. Momentum-position duality would have much deeper content than believed since its realization in TGD would bring number theory to physics.

How to describe interactions of CDs?

Any quantum coherent system corresponds to a CD. How can one describe the interactions of CDs? The overlap of CDs is a natural candidate for the interaction region.

1. CD represents the perceptive field of a conscious entity and CDs form a kind of conscious atlas for M^8 and H. CDs can have CDs within CDs and CDs can also intersect. CDs can have shared sub-CDs identifiable as shared mental images.
2. The intuitive guess is that the interactions occur only when the CDs intersect. A milder assumption is that interactions are observed only when CDs intersect.

3. How to describe the interactions between overlapping CDs? The fact the quark fields are induced from second quantized spinor fields in in H *resp.* M^8 solves this problem. At the level of H, the propagators between the points of space-time surfaces belonging to different CDs are well defined and the systems associated with overlapping CDs have well-defined quark interactions in the intersection region. At the level of M^8 the momenta as discrete quark carrying points in the intersection of CDs can interact.

Zero energy states as scattering amplitudes and subjective time evolution as sequence of SSFRs

This is not yet the whole story. Zero energy states code for the ordinary time evolution in the QFT sense described by the S-matrix. What about subjective time evolution defined by a sequence of "small" state function reductions (SSFRs) as analogs of "weak" measurements followed now and then by BSFRs? How does the subjective time evolution fit with the QFT picture in which single particle zero energy states are planewaves associated with a fixed CD.

1. The size of CD increases at least in statistical sense during the sequence of SSFRs. This increase cannot correspond to M^4 time translation in the sense of QFTs. Single unitary step followed by SSFR can be identified as a scaling of CD leaving the passive boundary of the CD invariant. One can assume a formation of an intermediate state which is quantum superposition over different size scales of CD: SSFR means localization selecting single size for CD. The subjective time evolution would correspond to a sequence of scalings of CD.
2. The view about subjective time evolution conforms with the picture of string models in which the Lorentz invariant scaling generator L_0 takes the role of Hamiltonian identifiable in terms of mass squared operator allowing to overcome the problems with Poincare invariance. This view about subjective time evolution also conforms with super-symplectic and Kac-Moody symmetries of TGD.

   One could perhaps say that the Minkowski time T as distance between the tips of CDs corresponds to exponentiated scaling: $T = exp(L_0 t)$. If t has constant ticks, the ticks of T increase exponentially.

The precise dynamics of the unitary time evolutions preceding SSFRs has remained open.

1. The intuitive picture that the scalings of CDs gradually reveal the entire 4-surface determined by polynomial P in M^8: the roots of P as "very special moments in the life of self" would correspond to the values of time coordinate for which SSFRs occur as one new root emerges. These moments as roots of the polynomial defining the space-time surface would correspond to scalings of the size of both half-cones for which the space-time surfaces are mirror images. Only the upper half-cone would be dynamical in the sense that mental images as sub-CDs appear at "geometric now" and drift to the geometric future.
2. The scaling for the size of CD does *not* affect the momenta associated with fermions at the points of cognitive representation in $X^4 \subset M^8$ so that the scaling is not a genuine scaling of M^4 coordinates which does not commute with momenta. Also the fact that L_0 for super symplectic representations corresponds to mass squared operator means that it commutes with Poincare algebra so that M^4 scaling cannot be in question.
3. The Hamiltonian defining the time evolution preceding SSFR could correspond to an exponentiation of the sum of the generators L_0 for super-symplectic and super-Kac Moody representations and the parameter t in exponential corresponds to the scaling of CD assignable to the replaced of root r_n with root r_{n+1} as value of M^4 linear time (or energy in M^8). L_0 has a natural representation at light cone boundaries of CD as scalings of light-like radial coordinate.
4. Does the unitary evolution create a superposition over all over all scalings of CD and does SSFR measure the scale parameter and select just a single CD?

   Or does the time evolution correspond to scaling? Is it perhaps determined by the increase of CD from the size determined by the root r_n as "geometric now" to the root r_{n+1} so that one would have a complete analogy with Hamiltonian evolution? The scaling would be the ratio r_{n+1}/r_n which is an algebraic number.

   Hamiltonian time evolution is certainly the simplest option and predicts a fixed arrow of time during SSFR sequence. L_0 identifiable essentially as a mass squared operator acts like conjugate for the logarithm of the logarithm of light-cone proper time for a given half-cone. One can assume that L_0 as the sum of generators associated with upper and lower half-cones if the fixed state at the lower half-cone is eigenstate of L_0.

How does this picture relate to p-adic thermodynamics in which thermodynamics isdetermined by partition function which would in real sector be regarded as a vacuum expectation value of an exponential $exp(iL_0t)$ of a Hamiltonian for imaginary time $t = i\beta$ $\beta = 1/T$ defined by temperature. L_0 is proportional to mass squared operator.

1. In p-adic thermodynamics temperature T is dimensionless parameter and $\beta = 1/T$ is integer valued. The partition function as exponential $exp(-H/T)$ is replaced with $p^{\beta L_0)}$, $\beta = n$, which has the desired behavior if L_0 has integer spectrum. The exponential form $e^{L_0/T_R)}$, $\beta_R = nlog(p)$ equivalent in the real sector does not make sense p-adically since the p-adic exponential function has p-adic norm 1 if it exists p-adically.
2. The time evolution operator $exp(-iL_0t)$ for SSFRs (t would be the scaling parameter) makes sense for the extensions of p-adic numbers if the phase factors for eigenstates are roots of unity belonging to the extension. $t = 2\pi k/n$ since L_0 has integer spectrum. SSFRs would define a clock. The scaling $exp(t) = exp(2\pi k/n)$ is however not consistent with the scaling by r_{n-1}/r_n.
   Both the temperature and scaling parameter for time evolution by SSFRs would be quantized by number theoretical universality. p-Adic thermodynamics could have its origins in the subjective time evolution by SSFRs.
3. In the standard thermodynamics it is possible to unify temperature and time by introducing a complex time variable $\tau = t + i\beta$, where $\beta = 1/T$ is inverse temperature. For the space-time surface in complexified M^8, M^4 time is complex and the real projection defines the 4-surface mapped to H. Could thermodynamics correspond to the imaginary part of the time coordinate?
   Could one unify thermodynamics and quantum theory as I have indeed proposed: this proposal states that quantum TGD can be seen as a "complex square root" of thermodynamics. The exponentials $U = exp(\tau L_0/2)$ would define this complex square root and thermodynamical partition function would be given by $UU^\dagger = exp(-\beta L_0)$.

6.5 Can one construct scattering amplitudes also at the level of M^8?

$M^8 - H$ duality suggests that the construction is possible both at the level of H and M^8. These pictures would be based on differential geometry on one hand and algebraic geometry and number theory on the other hand. The challenge is to understand their relationship.

6.5.1 Intuitive picture

H picture is phenomenological but rather detailed and M^8 picture should be its pre-image under $M^8 - H$ duality. The following general questions can be raised.

1. Can one construct the counterparts of the scattering amplitudes also at the level of M^8?
2. Can one use $M^8 - H$ duality to map scattering diagrams in M^8 to the level of H?

Consider first the notions of CD and sub-CD.

1. The intuitive picture is that at the level of H that one must surround partonic vertices with sub-CDs, and assign the external light-like momenta with the ends of propagator lines from the boundaries of CD and other sub-CDs. The incoming momenta $\overline{p}^k$ would be assigned to the boundary of sub-CD.
2. What about the situation in M^8? Sub-CDs must have different origin in the general case since the momentum spectrum would be shifted. Therefore the sub-CDs have the same tip - either upper or lower tip, and have as their boundary part of either boundary of CD. A hierarchy of CDs associated with the same upper or lower tip is suggestive and the finite maximal size of CD in H gives IR cutoff and the finite maximal size of CD in M^8 gives UV cutoff.

3. Momentum conservation at the vertices in M^8 could decompose the diagram to sub-diagrams for which the momentum conservation is satisfied. On the basis of QFT experience, one expects that there are some minimal diagrams from which one can construct the diagram: in the TGD framework this diagram would describe 4-quark scattering. The condition that the momenta belong to the extension of rationals gives extremely strong constraints and it is not clear that one obtains any solutions to the conditions unless one poses some conditions on the polynomials assigned with the two boundaries of CD.
   The two half-cones (HCs) of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T-o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T-o)$: the space-time surfaces at HCs would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - r^2$ only, the condition is identically satisfied for both options.
   There are two options for the identification of the coordinate t.
   Option (a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time. The M^4 projection of CD_8 gives a union of future and past directed light-cones with a common tip rather than CD_4 in M^4 at the level of M^8 . Both incoming and outgoing momenta have the same origin automatically. This identification seems to be the natural one at the level of M^8.
   Option (b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The HC at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.
   If Option (a) is realized at the level of M^8 and Option b) at the level of H, as seems natural, a time translation of the past directed light-cone by T in $M^4 \subset H$ is required to give CD_4. The momentum spectra of the two HCs differ only by sign and at least a scattering diagram in which all points are involved is possible. In fact all the pairs of subsets with opposite momenta are allowed. These however correspond to a trivial scattering. The decomposition to say 4-vertices with common points involving momentum space propagator suggests a decomposition into sub-CDs. The smaller the sub-CDs at the tips of the CD, the smaller the momenta are and the better is the IR resolution.
4. The proposal has been that one has a hierarchy of discrete size scales for the CDs. Momentum conservation gives a constraint on the positions of quarks at the ends of propagator lines in M^8 mapped to a constraint for their images in H: the sum of image points in H is however not vanishing since inversion is not a linear map.
5. QFT intuition would suggest that at the level of M^8 the scattering diagrams decompose to sub-diagrams for which momentum conservation is separately satisfied. If two such sub-diagrams A and B have common momenta, they correspond to internal lines of the diagram involving local propagator D_p, whose non-local counterpart at the level of H connects the image point to corresponding point of all copies of B.
   The usual integral over the endpoint of the propagator line $D(x, y)$ at space-time level should correspond to a sum in which the H image of B is shifted in M^4. Introduction of a large number of copies of H image of the sub-diagram looks however extremely ugly and challenges the idea of starting from the QFT picture.
   What comes in mind is that all momenta allowed by cognitive representation and summing up to zero define the scattering amplitude as a kind of super-vertex and that Yanigian approach allows this construction.

6.5.2 How do the algebraic geometry in M^8 and the sub-manifold geometry in H relate?

Space-time surfaces in H have also Euclidian regions - in particular wormhole contacts - with induced metric having Euclidian signature due to the large CP_2 contribution to the induced metric. They are separated from Minkowskian regions by a light-like 3-surfaces identifiable as partonic orbits at which the induced metric becomes degenerate.

1. The possible M^8 counterparts of these regions are expected to have Euclidian signature of the number theoretic metric defined by complexified octonion inner product, which must be real in these regions so that the coordinates for the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ are either imaginary or real. This allows several signatures.
2. The first guess is that the energy p^0 assignable to I_1 becomes imaginary. This gives tachyonic p^2. The second guess is that all components of 3-momentum $\{iI_3, iI_5, iI_7\}$ become imaginary meaning that the length of 3-momentum becomes imaginary.
3. One cannot exclude the other signatures, for instance the situation in which 1 or 2 components of the 3-momentum become imaginary. Hence the transition could occur in 3 steps as $(1,-1,-1-,1) \rightarrow (1,1,-1,-1) \rightarrow (1,1,1,-1) \rightarrow (1,1,1,1)$. The values of $p^2 \equiv Re(p_c^2)$ would be non-negative and also their images in $M^4 \subset H$ would be inside future light-cone. This could relate to the fact that all these signatures are possible inthe twistor Grassmannian approach.
4. These regions belong to the complex mass shell $p_c^2 = r_n = m_0^2 = r_n$ appearing as a root to the co-associativity condition $X = 0$. This gives the conditions

$$\begin{aligned} Re(p_c)\cdot Im(p_c^2) &= Im(r_n) \ , \\ Re(p_c^2) \equiv p^2 &= Im(p_c^2) + m_n^2 \ , \\ m_n^2 &\equiv Re(r_n) \geq 0 \ . \end{aligned} \tag{6.5.1}$$

Consider first the case $(1,1,1,1)$.

1. The components of p_c are either real or imaginary. Using the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ the components of p_c are real in the Minkowskian region and imaginary in the totally time-like Euclidian region. One has for the totally time-like momentum $p = (p_0, iIm(p_3))$ in the canonical basis.
This would give
$$Re(p_c^2) \equiv p^2 = p_0^2 = -Im(p_3)^2 + m_n^2 \ . \tag{6.5.2}$$
The number theoretic metric is Euclidian and totally time-like but one has $p^2 \geq 0$ in the range $[m_0^2, 0]$. This region is a natural counterpart for an Euclidian space-time region in H. The region $p^2 \geq m_0^2$ has Minkowskian signature and counterpart for Minkowskian regions in H. The region $0 \leq p^2 < m_0^2$ is a natural candidate for an Euclidian region in M^4.
Remark: A possible objection is that Euclidian regions in O_c are totally time-like and totally space-like in H.
2. The image of these regions under the map $Re(p^k) \rightarrow M^k$ under inversion plus octonionic conjugation defined as $p^k \rightarrow \hbar_{eff}\overline{p^k}/p^2$ (to be discussed in more detail in the sequel) consists of points M^k in the future light-cone of $M^4 \subset H$. The image of the real Euclidian region of O_c with $p^2 \in [0, m_0^2)$ is mapped to the region $M^kM_k < \hbar_{eff}^2/m_0^2$ of $M^4 \subset H$.
3. The contribution of CP_2 metric to the induced metric is space-like so that it can become Euclidian. This would naturally occur in the image of a totally time-like Euclidian region and this region would correspond to small scales $M^kM_k < \hbar_{eff}^2/m_0^2$. The change of the signature should take place at the orbits of partonic 2-surfaces and the argument does not say anything about this. The boundary of between the two regions corresponds to momenta $p = (p_0, 0)$ which is is a time-like line perhaps identifiable as the analog of the light-like geodesic defining the M^4 projection of CP_2 type extremal, which is an idealized solution to actual field equations.

This transition does not explain the M^8 counterpart of the 3-D light-like partonic orbit to which the light-light geodesic thickens in the real situation?

The above argument works also for the other signatures of co-associative real sub-spaces and provides additional insights. Besides the Minkowskian signature, 3 different situations with signatures $(1,1,1,1)$, $(1,-1,1,1)$, and $(1,-1-1,1)$ with non-space-like momentum squared are possible.

The following formulas list the signatures, the expressions of real momentum squared, and dimension D of the transition transition $Im(p_c^2) = 0$ as generalization of partonic orbit and the possible identification of the transition region.

$$
\begin{array}{lll}
\textbf{Signature} & p^2, & D \\
(+,-,-,+): & (p^0)^2-(p^1)^2-(p^2)^2=-Im(p^3)^2+m_n^2 & 3\ , \\
\textbf{Identification} & \text{partonic orbit} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,-,+,+): & (p^0)^2-(p^1)^2=-Im(p^2)^2-Im(p^3)^2+m_n^2\ , & 2\ , \\
\textbf{Identification} & \text{string world sheet} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,+,+,+): & (p^0)^2=-Im(p^1)^2-Im(p^2)^2-Im(p^3)^2+m_n^2\ , & 1\ . \\
\textbf{Identification} & \text{string boundary} & .
\end{array}
\qquad (6.5.3)
$$

Since the map of the co-associative normal space to CP_2 does not depend on the signature, M^8-H duality is well defined for all these signatures. One can ask whether a single transition creates partonic orbit, two transitions a string world sheet and 3 transitions ends of string world sheet inside partonic orbit or even outside it.

6.5.3 Quantization of octonionic spinors

There are questions related to the quantization of octonionic spinors.

1. Co-associative gamma matrices identified as octonion units are associative with respect to their octonionic product so that matrix representation is possible. Do second quantized octonionic spinors in M^8 make sense? Is it enough to second quantize them in M^4 as induced octonionic spinors? Are the anti-commutators of oscillator operators Kronecker deltas or delta functions in which case divergence difficulties might be encountered? This is not needed since the momentum space propagators can be identified as those for E^8_c restricted to X^4_r as a subspace with real octonion norm.

   The propagators are just massless Dirac propagators for the choice of M^4 for which light-like M^8 momentum reduces to M^4 momentum. Could one formulate the scattering amplitudes using only massless inverse propagators as in the twistor Grassmannian approach?This does not seem to be the case.

2. Could the counterpart of quark propagator as inverse propagator in M^8 as the idea about defining momentum space integrals as residue integrals would suggest? This would allow on-mass-shell propagation like in twistor diagrams and would conform with the idea that inversion relates M^8 and H descriptions. This is suggested by the fact that no integration over intermediate virtual momenta appears in the graphs defined by the algebraic points of the pre-images of the partonic 2-surfaces X^2_r .

How to identify external quarks? Note that bosons would consist of correlated quark-antiquark pairs with the propagator obtained as a convolution of quark propagators. The correlation would be present for the external states and possibly also for the states in the diagram and produced by topologically.

1. The polynomial P and the $P=0$ surface with 6-D real projection X^6_r is not affected by octonion automorphisms. Quarks with different states of motion would correspond to the same P but to different choices of M^4 as co-associative subspace for M^8_c. P could be seen as defining a class of scattering diagrams. P determines the vertices.

2. The space-time surface associated with a quark carrying given 4-momentum should be obtainable by a Lorenz transformation in $SO(3,1) \subset G_{2,c}$ to give it light-like M^4 so that complexified octonionic automorphisms would generate 3-surfaces representing particles. If $M^4 \subset M^8$ and thus the CD associated with the quark is chosen suitably, the quark is massless. Any incoming particle would be massless in this frame.

   Lorentz invariance however requires a common Lorentz frame provided by the CD. The momentum of a quark in CD would be obtained by $G_{2,c}$ transformation. In the frame of CD the external quark momenta arriving to the interior of CD at vertices associated with $X^3_r \cap Y^3_r$ are time-like. Momentum conservation would hold in this frame. The difference between massive constituent quarks and massless current quarks could be understood as reflecting M^8 picture.

To sum up, the resulting picture is similar to that at the level of H these diagrammatic structures would be mapped to H by momentum inversion. Quantum classical correspondence would be very detailed providing both configuration space and momentum space pictures.

6.5.4 Does $M^8 - H$ duality relate momentum space and space-time representations of scattering amplitudes?

It would seem that the construction of the scattering amplitudes is possible also at the level of M^8 [L97]. M^8 picture would provide momentum representation of scattering diagrams whereas H picture would provide the space-time representation.

Consider first a possible generalization of QFT picture involving propagators and vertices.

1. At first it seems that it is not possible to talk about propagation at the level of momentum space: in positive energy ontology nothing propagates in momentum space if the propagator is a free propagator D_p! In ZEO this is not quite so. One can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD): one has momentum space propagation from p to $-p$! The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. The momentum space propagator D_p would actually result from the pairing of creation creation operators with the opposite values of p and the notation $D(p,-p)$ would be more appropriate.
2. In QFT interaction vertices are local in space-time but non-local in momentum space. The n-vertex conserves the total momentum. Therefore one should just select points of M^8 and they are indeed selected by cognitive representation and assign scattering amplitude to this set of points. To each point one could assign momentum space propagator of quark in M^8_c but it would not represent propagation! The vertex would be a multilocal entity defined by the vertices defining the masses involved at light cone boundary and mass shells.
   The challenge would be to identify these vertices as poly-local entities. In the QFT picture there would be a set of n-vertices with some momenta common. What could this mean now? One would have subset sets of momenta summing up to zero as vertices. If two subsets have a common momentum this would correspond to a propagator line connecting them. Should one decompose the points of cognitive representation so that it represents momentum space variant of Feynman graph? How unique this decomposition is and do this kind of decompositions exist unless one poses the condition that the total momenta associated with opposite boundaries sum up to zero as done in ZEO. A given n-vertex in the decomposition means the presence of sub-CDs for which the external momenta sum up to zero. This poses very tight constraints on the cognitive representation, and one can wonder they can be satisfied if the cognitive representation is finite as it is in the generic case.
3. Note that for given a polynomial P allowing only points in cognitive representation, one would *not* have momentum space integrations as in QFT: they could however come from integrations over the polynomial coefficients and would correspond to integration of WCW. In adelic picture one allows only rational coefficients for the polynomials. This strongly suggests that the twistor Grasmmannian picture [B22, B27, B48, B13] in which residue integral in the momentum space gives as residues inverse quark propagators at the poles. M^8 picture would represent the end result of this integration and only on mass shell quarks would be involved. One could even challenge the picture based on propagators and vertices and start from Yangian algebra based on the generalization of local symmetries to multilocal symmetries [A97, A173] [B18] [L32].
4. In the case of H restriction of the second quantized free quark field of H to space-time surface defines the propagators. In the recent case one would have a second quantized octonionic spinor field in M^8. The allowed modes of H spinor field are just the co-associative modes for fixed selection of M^4 analogous to momentum space spinors and restricted to Y^3_r. One could speak of wave functions at Y^3_r, which is very natural since they correspond to mass shells.
   The induced spinor field would have massless part corresponding to wave functions at the M^4 light-cone boundary and part corresponding to X^3 at which the modes would have definite mass. $P=0$ would select a discrete set of masses. Could second quantization have the standard meaning in terms of anti-commutation relations posed on a free M^8 spinor field. In the case of M^8_c one avoids normal ordering problems since there is no Dirac action. The anti-

commutators however have singularities of type 7-D delta function. The anti-commutators of oscillator operators at the same point are the problem. If only a single quark oscillator operator at a given point of M^8 is allowed since there is no local action in coordinate space with the interaction part producing the usual troubles.

5. Could one perform a second quantization for E^8 spinor field using free Dirac action? Could one restrict the expansion of the spinor field to co-associative space-time surfaces giving oscillator operators at the points of cognitive representation with the additional restriction to the pre-image of given partonic 2-surface, whose identification was already considered. Scattering amplitudes would involve n-vertices consisting of momenta summing up to zero and connected to opposite incoming momenta at the opposite sides of the HCs with the same tip in M^8. Scattering amplitude would decompose to sub-diagrams defining a cluster decomposition, and would correspond to sub-CDs. The simplest option is that there are no internal propagator lines. The vanishing of the total momenta poses stringent conditions on the points of cognitive representation.

   Normal ordering divergences can however produce problems for this option in the case of bosonic charges bilear in oscillator operators. At the level of H the solution came from a bilocal modified Dirac action leading to bilocal expressions for conserved charges. Now Yangian symmetry suggests a different approach: local vertices in momentum space can involve only commuting oscillator operators.

   Indeed, in ZEO one can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD). The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. As already noticed, also the momentum space propagator $D_p = D(p, -p)$ would be also a bi-local object.

6. This is not enough yet. If there is only a single quark at given momentum, genuine particle creation is not possible and the particle reactions are only re-arrangements of quarks but already allowing formation of bosons as bound states of quarks and antiquarks. Genuine particle creation demands local composites of several quarks at the same point p having interpretation as a state with collinear momenta summing up to p and able to decay to states with the total momentum p. This suggests the analog of SUSY proposed in [L73]. Also Yangian approach is highly suggestive.

   To sum up, momentum conservation together with the assumption of finite cognitive representations is the basic obstacle requiring new thinking.

6.5.5 Is the decomposition to propagators and vertices needed?

One can challenge the QFT inspired picture.

1. As already noticed, the relationship $P_1(t) = P(2T-t)$ makes it possible to satisfy this condition at least for the entire set of momenta. This does not yet allow non-trivial interactions without posing additional conditions on the momentum spectrum. This does not look nice. One can ask whether there is a kind of natural selection leading to polynomials defining space-time surfaces allowing cognitive representations with vertex decompositions and polynomials $P(t)$ and $P_r(t)$ without this symmetry? This idea looks ugly. Or could evolution start from simplest surfaces allowing 4 vertices and lead to an engineering of more complex scattering diagrams from these?
2. The map of momentum space propagators regarded as completely local objects in M^8 to H propagators is second ugly feature. The beauty and simplicity of the original picture would be lost by introducing copies of sub-diagrams mapped to the various translations in H.
3. The Noether charges of the Dirac action in H fail to give rise to 4-fermion vertex operator. The theory would be naturally just free field theory if one assumes cognitive representations.

The first heretic question is whether the propagators are really needed at the level of momentum space. This seems to be the case.

1. In ZEO the propagators pair creation and operators with opposite 4-momenta assignable to the opposite HCs of CD having conjugate fermionic vacua (Dirac sea of negative energy fermions and Dirac sea of positive energy fermions) so that momentum space propagators $D(p, -p)$ are non-local objects. The propagators would connect positive and negative energy fermions

at the opposite HCs and this should be essential in the formulation of scattering amplitudes. They cannot be avoided.

2. The propagators would result from the contractions of fermion oscillator operators giving a 7-D delta function at origin in continuum theory. This catastrophe is avoided in the number theoretic picture. Since one allows only points with M^8 coordinates in an extension of rationals, one can assume Kronecker delta type anti-commutators. Besides cognitive representations, this would reflect the profound difference between momentum space and space-time.
   This would also mean that the earlier picture about the TGD analog of SUSY based on local composites of oscillator operators [L73] makes sense at the level of M^8. The composites could be however local only for oscillator operators associated with the HC of CD. With the same restriction they could be local also in the$H$ picture.

What about vertices? Could Yangian algebra give directly the scattering amplitudes? This would simplify dramatically the $M^8 - H$ duality for transition amplitudes. For this option the $P_1(t) = P(2T - t)$ option required by continuity would be ideal.

1. Without vertices the theory would be a free field theory. The propagators would connect opposite momenta in opposite HCs of CD. Vertices are necessary and they should be associated with sub-CDs. Unless sub-CDs can have different numbers of positive and negative energy quarks at the opposite HCs, the total quark number is the same in the initial and final states if quarks and antiquarks associated with bosons as bound states of fermion and antiquark are counted. This option would require minimally 4-quark vertex having 2 fermions of opposite energies at the two hemi-spheres of the CD. A more general option looks more plausible. One obtains non-trivial scattering amplitudes by contracting fermions assigned to the boundary P (F) past (future) HC of CD to the past (future) boundary P_{sub} (F_{sub}) of a sub-CD. Sub-CD and CD must have an opposite arrow of time to get the signs of energies correctly.
   Sub-CDs would thus make particle creation and non-trivial scattering possible. There could be an arbitrary number of sub-CDs and they should be assignable to the pre-images of the partonic 2-surfaces X_r^2 if the earlier picture is correct. The precise identification of the partonic 2-surfaces is still unclear as also the question whether light-like orbits of partonic 2-surfaces meet along their ends in the vertices.
2. As in the case of H, one could assign the analogs of n-vertices at pre-images of partonic 2-surfaces at X_r^2 representing the momenta of massive modes of the octonionic Dirac equation and belonging to the cognitive representations. The idea is to use generators of super-Yangian algebra to be discussed later which are both bosonic and fermionic. The simplest construction would assign these generators to the vertices as points in cognitive representation.
   An important point is that Yangian symmetry would be a local symmetry at the level of momentum space and correspond to non-local symmetry at the level of space-time rather than vice versa as usually. The conserved currents would be local composites of quark oscillator operators with same momentum just as they are in QFTs at space-time level representing parallelly propagating quarks and antiquarks.
   The simplest but not necessary assumption is that they are linear and bilinear in oscillator operators associated with the same point of M^8 and thus carrying 8-momenta assignable to the modes of E^8 spinor field and restricted to the co-associative 4-surface. Their number of local composites is finite and corresponds to the number 8 of different states of 8-spinors of given chirality.
   Also a higher number of quarks is possible, and this was indeed suggested in [L73]. The proposal was that instance leptons would correspond to local composites of 3 quarks. The TGD based view about color allows this. These states would be analogous to the monomials of theta parameters in the expansion of super-field. The H picture allows milder assumptions: leptonic quarks reside at partonic 2-surface at different points but this is not necessary.
3. Instead of super-symplectic generators one has $G_{2,c}$ as the complexified automorphism group. Also the Galois group of the extension acts as an automorphism group and is proposed to have a central role in quantum TGD with applications to quantum biology [L26, L95]. As found, $G_{2,c}$ acts as an analog of gauge or Kac-Moody group. Yangian has analogous structure but the analogs of conformal weights are non-negative.
4. The identification of the analogs of the poly-local vertex operators as produces of charges generators associated with FHC anbd PHC is the basic challenge. They should consist of

quark creation operators (annihilation operators being associated as creation operators at the opposite HC) and be generators of infinitesimal symmetries which in number theoretic physics would correspond instead of isometries of WCW to the octonionic automorphism group G_2 complexified to $G_{2,c}$ containing also the generators of $SO(4) \subset G_2$ and thus also those of Lorentz group $SO(1,3) \subset G_{2,c}$.
The construction Noether charges of E^8 second quantized spinor field at momentum space representation gives bilinear expressions in creation and annihilation operators associated with opposite 3-momenta and would have a single fermion in a given HC. This is not enough: there should be at least 4 fermions.

What strongly suggests itself are Yangian algebras [A97] [L32] having poly-local generators and considered already earlier and appearing in the twistor Grassmannian approach [B22, B27]. The sums of various quantum numbers would vanish for the vertex operators. These algebras are quantum algebras and the construction of n-vertices could involve co-algebra operation. What is new as compared to Lie algebras is that Yangian algebras are quantum algebras having co-algebra structure allowing to construct n-local generators representing scattering amplitudes. It might be possible replace oscillator operators with the quantum group counterparts.

6.5.6 Does the condition that momenta belong to cognitive representations make scattering amplitudes trivial?

Yangian symmetry is associated with 2-D integrable QFTs which tend to be physically rather uninteresting. The scattering is in the forward direction and only phase shifts are induced. There is no particle creation. If the relationship $P_1(t) = P(2T - t)$ is applied the momentum spectra for FHC and PHC differ only by the sign. If all momenta are involved and the cognitive representations are finite, the situation would be the same! Also the existence of cluster compositions involving summations of subsets of momenta to zero is implausible. Something seems to go wrong!

The basic reason for the problem is the assumption that the momenta belong to cognitive representations assumed to be finite as they indeed are in the generic case. But are they finite in the recent situation involving symmetries?

1. The assumption that all possible momenta allowed by cognitive representation are involved, allows only forward scattering unless there are several subsets of momenta associated with either HC such that the momenta sum-up to the same total momentum. This would allow the change of the particle number. The subsets S_i with same total momentum p_{tot} in the final state could save as final states of subsets S_j with the same total momentum p in the initial state. What could be the number theoretical origin of this degeneracy?
2. In the generic case the cognitive representation contains only a finite set of points (Fermat theorem, in which one considers rational roots of $x^n + y^n = z^n$, $n > 2$ is a basic example of this) . There are however special cases in which this is not true. In particular, M^4 and its geodesic sub-manifolds provide a good example: all points in the extension of rationals are allowed in M^4 coordinates (note that there are preferred coordinates in the number theoretic context).
The recent situation is indeed highly symmetric due to the Lorentz invariance of space-time surfaces as roots reducing the equations to ordinary algebraic equations for a single complex variable. $X = 0$ condition gives as a result $a_c^2 = constant$ complex hyperboloid with a real mass hyperboloid as a real projection. $a_c^2 = r_n$ is in the extension of rationals as a root of n:th order polynomial. One has the condition $Re(m^2)^2 - Im(m^2) = Re(r_n)$ giving X_r^4 a slicing by real mass hyperboloids. If $Im(m)$ and the spatial part of $Re(m)$ belongs to the extension, one has for real time coordinate $t = \sqrt{r_M^2 + Im(m^2) + r_n}$. If $r_M^2 + Im(m)^2 + r_n$ is a square in the extension also t belongs to the extension. Cognitive representation would contain an infinite number of points and the it would be possible to have non-trivial cluster decompositions. Scattering amplitude would be a sum over different choices of the momenta of the external particles satisfying momentum conservation condition.
As found, the intersection of X_r^4 and X_r^6 is either empty or X_r^4 belongs to X_r^6, Cognitive representations would have an infinite number of points also now by the previous argument. Partonic 2-surfaces at X_r^3 would be replaced with 3-D surfaces in X_r^4 in this situation and would contain a large number of roots. The partonic 2-surfaces would be still present and

correspond to the intersections of incoming space-time surfaces of quarks inside X^6_r. These surfaces would also contain the vertices.

3. Could number theoretic evolution gradually select space-time surfaces for which the number theoretic dynamics involving massive quarks is possible? First would be generic polynomials for which X^3_r would be empty and only massless quarks arriving at the light-cone boundary would be possible. After that surfaces allowing non-empty X^3_r and massive quarks would appear. There is a strong resemblance with the view about cosmological evolution starting from massless phases and proceeding as a sequence of symmetry breakings causing particle massivation. Now the massivation would not be caused by Higgs like fields but have purely number theoretic interpretation and conform with the p-adic mass calculations [K50].
   Also a cognitive explosion would occur since these space-time surfaces would be cognitively superior after the emergence of massive quarks. If this picture has something to do with reality, the space-time surfaces contributing to the scattering amplitudes would be very special and interactions could be seen as a kind of number theoretical resonance phenomenon.
4. Even is not enough to obtain genuine particle reaction instead of re-arrangements: one must have also local composites of collinear quarks at the same momentum p identifiable as the sum of parallel momenta discussed in [L73]. This kind of situation is also encountered for on-mass-shell vertices in twistor Grassmannian approach. The local composites could decay to local composites with a smaller number of quarks but respecting momentum conservation. Here the representations of Yangian algebra would come in rescue.

6.5.7 Momentum conservation and on-mass-shell conditions for cognitive representations

Momentum conservation and on-mass shell-conditions are very powerful for cognitive representations, which in the generic case are finite. At mass shells the cognitive representations consist of momenta in the extension of rationals satisfying the condition $p^2 = Re(r_n)$, r_n a complex root of X, which is polynomial of degree n in p^2 defined by the odd part of P. If $\sqrt{Re(r_n)}$ does not belong to the extension defined by P, it can be extended to contain also $\sqrt{Re(r_n)}$.

For Pythagorean triangles in the field of rationals, mass shell condition gives for the momentum components in extension an equation analogous to the equation $k^2 + l^2 = m^2$, which can be most easily solved by noticing that the equation has rotation group $SO(2)$ consisting of rational rotation matrices as symmetries. The solutions are of form $(k = r^2 - s^2, l = 2rs, m = r^2 + s^2)$. By $SO(2)$ invariance, one can choose the coordinate frame so that one has $(k, l) = (r^2 + s^2, 0)$. By applying to this root a rational rotation with $cos(\phi) = (r^2 - s^2)/(r^2 + s^2), sin(\phi) = 2rs/(r^2 + s^2)$ to obtain the general solution $(k = r^2 - s^2, l = 2rs, n = r^2 + s^2)$. The expressions for k and l can be permuted, which means replacing ϕ with $\phi - pi/2$. For a more general case $k^2 + l^2 = n$ one can replace n with $\sqrt{n}$ so that one has an extension of rationals.

For the hyperbolic variants of Pythagorean triangles, one has $k^2 - l^2 = m^2$ or equivalently $l^2 + m^2 = k^2$ giving a Pythagorean triangle. The solution is $k = r^2 + s^2, l = r^2 - s^2, m^2 = 2rs$. The expressions for l and ma can be permuted. Rotation is replaced with 2-D Lorentz boost $cosh(\eta) = (r^2 + s^2)/(r^2 - s^2)$ and $sinh(\eta) = 2rs/(r^2 - s^2)$ with rational matrix elements.

Consider now the 4-D case.

1. The algebra behind the solution depends in no manner on the number field considered and makes sense even for the non-commutative case if m and n commute. Hence one can apply the Pythagorean recipe also in 4-D case to the extension of rationals defined by P by adding to it $\sqrt{r_n}$.
2. Assume that a Lorentz frame can be chosen to be the rest frame in which one has $p = (E = \sqrt{Rer_n}, 0)$ (this might not be possible always). As in the Pythagorean case, there must be a consistency condition. Now it would be of form $E = \sqrt{r_n} = p_0^2 - p_1^2 - p_2^2 - p_3^2$ in the extension defined by $\sqrt{r_n}$. It is not clear whether this condition can be solved for all choices of momentum components in the extension or assuming that algebraic integers of extension are in question. One can also consider an option in which one has algebraic integer divided by some integer N. p-Adic considerations would suggest that prime powers $N = p^k$ might be interesting.

The solutions $\sqrt{r_n} = p_1^2 - p_2^2$ represent a special case. The general solution is obtained by making Lorenz transformation with a matrix with elements in the discrete subgroup of Lorentz group with matrix elements in the extension of rationals.

3. The solutions would define a discretization of the mass shell (3-D hyperbolic space) defined as the orbit of the infinite discrete subgroup of $SO(1,3)$ considered - perhaps the subgroup of $SL(2,C)$ with matrix elements identified as algebraic integers.

If the entire subgroup of SL(2,C) with matrix elements in the extension of rationals is realized, the situation would correspond effectively to a continuous momentum spectrum for infinite cognitive representations. The quantization of momenta is however physically a more realistic option.

1. An interesting situation corresponds to momenta with the same time component, in which case the group would be a discrete subgroup of $SO(3)$. The finite discrete symmetry subgroups act as symmetries of Platonic solids and polygons forming the ADE hierarchy associated to the inclusions of hyperfinite factors of type II_1 and proposed to provide description of finite measurement resolution in TGD framework.
2. The scattering would be analogous to diffraction and only to the directions specified by the vertices of the Platonic solid. Platonic solids, in particular, icosahedron appear also in TGD inspired quantum biology [L16, L91], and also in Nature. Could their origin be traced to $M^8 - H$ duality mapping the Platonic momentum solids to H by inversion?

A more general situation would correspond to the restriction to a discrete non-compact sub-group $\Gamma \subset SL(2,C)$ with matrix elements in the extension of rationals. $Sl(2,C)$ has a representation as Möbius transformations of upper half-plane H^2 of complex plane acting as conformal transformations whereas the action in H^3 is as isometries. The Möbius transformation acting as isometries of H^2 corresponds to $SL(2,Z)$ having also various interesting subgroups, in particular congruence subgroups.

1. Subgroups Γ of the modular group $SL(2,Z)$ define tessellations (analogs of orindary lattices in a curved space) of both H^2 and H^4. The fundamental domain [A30] (`https://cutt.ly/ahBrtT5`) of the tessellation defined by $\Gamma \subset SL(2,C)$ contains exactly one point at from each orbit of Γ. The fundamental domain is analogous to lattice cell for an Euclidian 3-D lattice.
   Γ must be small enough since the orbits would be otherwise dense just like rationals are a dense sub-set of reals. In the case of rationals this leaves into consideration tje modular subgroup $SL(2,Z)$ or its subgroups. In the recent situation an extension of the modular group allowing matrix elements to be algebraic integers of the extension is natural. Physically this would correspond to the quantization of momentum components as algebraic integers. The tessellation in M^8 and its image in H would correspond to reciprocal lattice and lattice in condensed matter physics.
2. So called uniform honeycombs [A53, A39, A88] (see `https://cutt.ly/xhBwTph`, `https://cutt.ly/lhBwPRc`, and `https://cutt.ly/OhBwUOO`) in H^3 assignable to $SL(2,Z)$ can be regarded as polygons in 4-D space and H^3 takes the roles of sphere S^2 for platonic solids for which the tessellation defined by faces is finite.
   The four regular compact honeycombs in H^3 for which the faces and vertex figures (the faces meeting the vertex) are finite are of special interest physically. In the Schönflies notation characterizing polytopes (tessellations are infinite variants of them) they are labelled by (p,q,r), where p is the number of vertices of face, q is the number of faces meeting at vertex, and s is the number of cells meeting at edge.
   The regular compact honeycombs are listed by (5,3,4), (4,3,5), (3,5,3), (5,3,5). For Platonic solids (5,3) characterizes dodecahedron, (4,3) cube, and (3,5) for icosahedron so that these Platonic solids serve as basic building bricks of these tessellations. Rather remarkably, icosahedral symmetries central in the TGD based model of genetic code [L16, L91], characterize cells for 3 uniform honeycombs.

Consider now the momentum conservation conditions explicitly assuming momenta to be algebraic integers. It is natural to restrict the momenta to algebraic integers in the extension of rationals defined by the polynomial P. This allows linearization of the constraints from momentum conservation quite generally.

Pythagorean case allows to guess what happens in 4-D case.

1. One can start from momentum conservation in the Pythagorean case having interpretation in terms of complex integers $p = (r+is)^2 = r^2 - s^2 + 2irs$. The momenta in the complex plane are squares of complex integers $z = r + is$ obtained by map $z \to w = z^2$ and complex integers. One picks up in the w-plane integer momenta for the incoming and outgoing states satisfying the conservation conditions $\sum_i P_{out,i} = \sum_k P_{in,k}$: what is nice is that the conditions are linear in w-plane. After this one checks whether the inverse images $\sqrt{P_{out,i}}$ and $\sqrt{P_{in,i}}$ are also complex integers.
2. To get some idea about constraints, one can check what CM system for a 2-particle system means (it is not obvious whether it is always possible to find a CM system: one could have massive particles which cannot form a rest system). One must have opposite spatial momenta for $P_1 = (r_1 + is_1)^2$ and $P_2 = (r_2 + is_2)^2$. This gives $r_{s1} = r_2 s_2$. The products $r_i s_i$ correspond to different compositions of the same integer N to factors. The values of $r_i^2 + si^2$ are different.
3. In hyperbolic case one obtains the same conditions since the roles of $r^2 - s^2$ and $r^2 + s^2$ in the conditions are changed so that $r^2 - s^2$ corresponds now to mass mass mass and differs for different decomposition of N to factors. The linearization of the conservation conditions generalizes also to the algebraic extensions of rationals with integers replaced by algebraic integers.

The generalization to the 4-D case is possible in terms of octonions.

1. Replace complex numbers by quaternions $q = q_0 + \overline{q}$. The square of quaternion is $q^2 = q_0^2 - \overline{q} \cdot \overline{q} + 2iq_0\overline{q}$. Allowed momenta for given mass correspond to points in q^2-plane. Conservation conditions in the q^2 plane are linear and satisfied by quaternionic integers, which are squares. So that in the q^2 plane the allowed momenta form an integer lattice and the identification as a square selects a subset of this lattice. This generalizes also to the algebraic integers in the extension of rationals.

2. What about the co-associative case corresponding to the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$? Momenta would be as co-associative octonion o but o^2 is a quaternion in the plane defined by $\{I_0, iI_2, iI_4, iI_6\}$. o representable in terms of a complexified quaternion $q = q_0 + i\overline{q}$ as $o = I_4 q$ and the in general complex values norm squared is give by $o\overline{o}$ with conjugation of octonionic imaginary units but not i: this gives Minkowskian norm squared. This reduces the situation to the quaternionic case.
3. In this case the CM system for two-particle case corresponds to the conditions $q_{1,0}\overline{q}_1 = q_{2,0}\overline{q}_2$ implying that q_1 and q_2 have opposite directions and $q_{1,0}|\overline{q}_1| = q_{2,0}|\overline{q}_2|$. The ratio of the lengths of the momenta is integer. Now the squares $q_{i,0}|\overline{q}_i|^2$, $i = 1, 2$ are factorizations of the same integer N. Masses are in general different.
4. The situation generalizes also to complexified quaternions - the interpretation of the imaginary part of momentum might be in terms of a decay width - and even to general octonions since associativity is not involved with the conditions.

6.5.8 Further objections

The view about scattering amplitudes has developed rather painfully by objections creating little shocks. The representation of scattering amplitudes is based on quark oscillator operator algebra. This raises two further objections.

The non-vanishing contractions of the oscillator operators are necessary for obtaining non-trivial scattering amplitudes but is this condition possible to satisfy.

1. One of the basic deviations of TGD from quantum field theories (QFTs) is the hypothesis that all elementary particles, in particular bosons, can be described as bound states of fermions, perhaps only quarks. In TGD framework the exchange of boson in QFT would mean an emission of a virtual quark pair and its subsequent absorption. In ZEO in its basic form this seems to be impossible.
2. If scattering corresponds to algebra morphism mapping products to products of co-products - the number of quarks in say future HC is higher than in the past HC as required. But how to obtain non-vanishing scattering amplitudes? There should be non-vanishing counterparts

of propagators between points of FHC but this is not possible if only creation operators are present in a given HC as ZEO requires. All particle reactions would be re-arrangements of quarks and antiquarks to elementary fermions and bosons (OZI rule of the hadronic string model: `https://en.wikipedia.org/wiki/OZI_rule`). The emission of virtual or real bosons requires the creation of quark antiquark pairs and seems to be in conflict with the OZI rule.

3. It would be natural to assign to quarks and bosons constructed as their bound states non-trivial inner product in a given HC of CD. Is this possible if the counterparts of annihilation operators act as creation operators in the opposite HC? Can one assign inner product to a given boundary of CD by assuming that hermitian conjugates of quark oscillator operators act in the dual Hilbert space of the quark Fock space? Could this dual Hilbert space relate to the Drinfeld's double?

How could one avoid the OZI rule?

1. Is it enough to also allow annihilation operators in given HC? Bosonic $G_{2,c}$ generators could involve them. The decay of boson to quark pair would still correspond to re-arrangement but one would have inner product for states at given HC. The creation of bosons would still be a problem. Needless to say, this option is not attractive.

2. A more plausible solution for this problem is suggested by the phenomenological picture in which quarks at the level of H are assigned with partonic 2-surfaces and their orbits, string world sheets, and their boundaries at the orbits of partonic 2-surfaces. By the discussion in the beginning of this section, these surfaces could correspond at the level of M^8 to space-time regions of complexified space-time surface with real number theoretic metric having signature (+,+,-,-), (+,+,+,-), (+,+,+,+) having 2,3, or 4 time-like dimensions. They would allow non-negative values of mass squared and would be separated from the region of Minkowskian signature by a transition region space-time region with dimension $D \in \{3, 2, 1\}$ mapped to CP_2.

   In these regions one would have 1, 2, or 3 additional energy like momentum components $p_i = E_i$. E_i. Could the change of sign for E_i transform creation operator to annihilation operator as would look natural. This would give bosonic states with a non-vanishing norm and also genuine boson creation. What forces to take this rather radical proposal seriously that it conforms with the phenomenological picture.

   In this region one could have a non-trivial causal diamond CD with signature (+,+,-,-), (+,+,+,-). For the signature (+,+,+,+) CD reduces to a point with a vanishing four-momentum and would correspond to CP_2 type extremals (wormhole contacts). Elementary fermions and bosons would consist of quarks in regions with signature (+,+,-,-) and (+,+,+,-). It would seem that the freedom to select signature in twistorial amplitude is not mere luxury but has very deep physical content.

One can invent a further objection. Suppose that the above proposal makes sense and allows to assign propagators to a given HC. Does Yangian co-product allow a construction of zero energy states giving rise to scattering amplitudes, which typically have a larger number of particles in the future HC (FHC) than in past HC (PHC) and represent a genuine creation of quark pairs?

1. One can add to the PHC quarks and bosons one-by-one by forming the product super $G(2, c)$ generators assignable to the added particles. To the FHC one would add the product of co-products of these super $G(2, c)$ generators (co-product of product is product of co-products as an algebra morphism).

2. By the basic formula of co-product each addition would correspond to a superposition of two states in FHC. The first state would be the particle itself having suffered a forward scattering. Second state would involve 2 generators of super $G_{2,c}$ at different momenta summing up to that for the initial state, and represent a scattering $q \to q + b$ for a quark in the initial state and scattering $b \to 2b$, $b \to 2b$, or $b \to 2q$ for a boson in the initial state.

   Number theoretic momentum conservation assuming momenta to be algebraic integers should allow processes in which quark oscillator operators are contracted between the states in FHC and PHC or between quarks in the FHC.

3. Now comes the objection. Suppose that the state in PC consists of fundamental quarks. Also the FC containing the product of co-products of quarks must contain these quarks

with the same momenta. But momentum conservation does not allow anything else in FC! The stability of quarks is a desirable property in QFTs but something goes wrong! How to solve the problem?

Also now phenomenological picture comes to the rescue and tells that elementary particles - as opposed to fundamental fermions - are composites of fundamental fermions assignable to flux tubes like structures involving 2 wormhole contacts. In particular, quarks as elementary particles would involve quark at either throat of the first wormhole contact and quark-antiquark pair associated with the second wormhole contact. The state would correspond to a quantum superposition of different multilocal momentum configurations defining multi-local states at M^8 level. The momentum conservation constraint could be satisfied without trivializing the scattering amplitudes since the contractions could occur between different components of the superposition - this would be essential.

Note also that at H level there can be several quarks at a given wormhole throat defining a multilocal state in M^8: one could have a superposition of these states with different momenta and again different components of the wave function could contract. By Uncertainty Principle the almost locality in H would correspond to strong non-locality in M^8. This could be seen as an approximate variant of the TGD variant of H variant of SUSY considered in [L73].

Could the TGD variant of SUSY proposed in [L73] but realized at the level of momentum space help to circumvent the objection? Suppose that the SUSY multiplet in M^8 can be created by a local algebraic product possessing a co-product delocalizing the local product of oscillator operators at point p in PC and therefore represents the decay of the local composite to factors with momenta at p_1 and $p - p_1$ in FC. This would not help to circumvent the objection. Non-locality and wave functions in momentum space is needed.

6.6 Symmetries in M^8 picture

6.6.1 Standard model symmetries

Can one understand standard model symmetries in M^8 picture?

1. $SU(3) \subset G_2$ would respect a given choice of time axis as preferred co-associative set of imaginary units ($I_2 \subset \{I_2, iI_3, iIb, iI_7\}$ for the canonical choice). The labels would therefore correspond to the group $SU(3)$. $SU(3)_c$ would be analogous to the local color gauge group in the sense that the element of local $SU(3)_c$ would generate a complecofied space-time surface from the flat and real M^4. The real part of pure $SU(3)_c$ gauge potential would not however reduce to pure $SU(3)$ gauge potential. Could the vertex factors be simply generators of $SU(3)$ or $SU(3)_c$?

2. What about electroweak quantum numbers in M^8 picture? Octonionic spinors have spin and isospin as quantum numbers and can be mapped to H spinors. Bosons would be bound states of quarks and antiquarks at both sides.

   How could electroweak interactions emerge at the level of M^8? At the level of H an analogous problem is met: spinor connection gives only electroweak spinor connection but color symmetries are isometries and become manifest via color partial waves. Classical color gauge potentials can be identified as projections of color isometry generators to the space-time surface.

   Could electroweak gauge symmetries at the level of M^8 be assigned with the subgroup $U(2) \subset SU(3)$ of $CP_2 = SU(3)/U(2)$ indeed playing the role of gauge group? There is a large number of space-time surfaces mapped to the same surface in H and related by a local $U(2)$ transformation. If this transformation acted on the octonionic spinor basis, it would be a gauge transformation but this is not the case: constant octonion basis serves as a gauge fixing. Also the space-time surface in M^8 changes but preserves its "algebraic shape".

6.6.2 How the Yangian symmetry could emerge in TGD?

Yangian symmetry [A97, A173] appears in completely 2-D systems. The article [B33] (`https://arxiv.org/pdf/1606.02947.pdf`) gives a representation which is easy to understand by a physicist like me whereas the Wikipedia article remains completely incomprehensible to me.

Yangian symmetry is associated with 2-D QFTs which tend to be physically rather uninteresting. The scattering is in forward direction and only phase shifts are induced. There is no particle creation. Yangian symmetry appears in 4-D super gauge theories [B18] and in the twistor approach to scattering amplitudes [B19, B28, B22, B27]. I have tried to understand the role of Yangian symmetry in TGD [L32].

Yangian symmetry from octonionic automorphisms

An attractive idea is that the Yangian algebra having co-algebra structure could allow to construct poly-local conserved charges and that these could define vertex operators in M^8.

1. Yangian symmetry appears in 2-D systems only. In TGD framework strings world sheets could be these systems as co-commutative 2-surfaces of co-associative space-time surface.
2. What is required is that there exists a conserved current which can be also regarded as a flat connection. In TGD the flat connection would a connection for $G_{2,c}$ or its subgroup associated with the map taking standard co-associative sub-space of O_c for which the number theoretic norm squared is real and has Minkowski signature (M^4 defined by the canonical choice $\{I_2, iI_3, iI_5, iI_7\}$.
   The recent picture about the solution of co-associativity conditions fixes the subgroup of G_2 to $SU(3)$. X^4 corresponds to element g of the local $SU(3)$ acting on preferred $M^4 \subset M_c^8$ with the additional condition that the 4-surface $X^4 \subset M^8$ is invariant under $U(2) \subset SU(3)$ so that each point of X^4 corresponds to a CP_2 point. At the mas shells as roots of a polynomial P, g reduces to unity and the 4-D tangent space is parallel to the preferred M^4 on which g acts. One can induce this flat connection to string world sheet and holomorphy of g at this surface would guarantee the conservation of the current given by $j_{0)} = g^{-1}dg$.
3. Under these conditions the integral of the time component of current along a space-like curve at string world sheets with varying end point is well-defined and the current

$$j_{1)}(x) = \epsilon\mu\nu j_{0),\nu}(x) - \frac{1}{2}[j^{\mu}_{0)}(x,t), \int^x j^0_{0)}(t,y)dy]$$

   is conserved. This is called the current at first level. Note that the currents have values in the Lie algebra considered. It is essential that the integration volume is 1-D and its boundary is characterized by a value of single coordinate x.
4. One can continue the construction by replacing j_0 with j_1 in the above formula and one obtains an infinite hierarchy of conserved currents $j_{n)}$ defined by the formula

$$j_{n+1)}(x) \quad = \quad \epsilon\mu\nu j_{n),\nu}(x) - \frac{1}{2}[j^{\mu}_{n)}(x,t), \int^x j^0_{n)}(t,y)dy] \tag{6.6.1}$$

   The corresponding conserved charges Q_n define the generators of Yangian algebra.
5. 2-D metric appears in the formulas. In the TGD framework one does not have Riemann metric - only the number theoretic metric which is real only at real space-time surfaces already discussed. Is the (effective) 2-dimensionality and holomorphy enough to avoid the possible problems? Holomorphy makes sense also number theoretically and implies that the metric disappears from the formulas for currents. Also current conservation reduces to the statement of that current is equivalent to complex differential form.
6. Conserved charges would however require a 1-D integral and number theory does not favor this. The solution of the problem comes from the observation that one can construct a slicing of string world sheet to time-like curves as Hamiltonian orbits with Hamiltonian belonging to the Yangian algebra and defined by the conserved current by standard formula $j^\alpha = J^{\alpha\beta}\partial_\beta H$ in terms of Kähler form defined by the 2-D Kähler metric of string world sheet. This generalizes to Minkowskian signature and also makes sense for partonic 2-surfaces. Hamiltonians become the classical conserved charges constant along the Hamiltonian orbit. This gives an infinite hierarchy of conserved Hamiltonian charges in involution. Hamiltonian can be any combination of the Hamiltonians in the hierarchy and labelled by a non-negative integer and the label of $G_{2,c}$ generator. This is just what integrability implied by Yangian algebra means. Co-associativity and co-commutativity would be the deeper number theoretic principles implying the Yangian symmetry.

7. Could one formulate this argument in dimension $D = 4$? Could one consider instead of local current the integral of conserved currents over 2-D surfaces labelled by single coordinate x for a given value of t? If the space-time surface in M^8 (analog of Fermi sphere) allows a slicing by orthogonal strings sheets and partonic 2-surfaces, one might consider the fluxes of the currents $g^{-1}dg$ over the 2-D partonic 2-surfaces labelled by string coordinates (t, x) as effectively 2-D currents, whose integrals over x would give the conserved charge. Induced metric should disappear from the expressions so that fluxes of holomorphic differential forms over partonic 2-surface at (t, x) should be in question. Whether this works is not clear.

One should interpret the above picture at the level of momentum space instead of ordinary space-time. The roles of momentum space and space-time are changed. At this point, one can proceed by making questions.

1. One should find a representation for the algebra of the Hamiltonians associated with $g(x)$ defining the space-time surface. The charges are associated with the slicings of string world sheets or partonic 2-surfaces by the orbits of Hamiltonian dynamics defined by a combination of conserved currents so that current conservation becomes charge conservation. These charges are labelled by the coordinate x characterizing the slices defined by the Hamiltonian orbits and from these one can construct a non-local basis discrete basis using Fourier transform.
2. What the quantization of these classical charges - perhaps using fermionic oscillator operators in ZEO picture for which the local commutators vanish - could mean (only the anti-commutators of creation operators associated with the opposite half-cones of CD with opposite momenta are non-vanishing)? Do the Yangian charges involve only creation operators of either type with the same 8-momentum as locality at M^8 level suggests? Locality is natural l since these Yangian charges are analogous to charges constructed from local currents at space-time level.
3. Could the Yangian currents give rise to poly-local charges assignable to the set of vertices in a cognitive representation and labelled by momenta? Could the level n somehow correspond to the number n of the vertices and could the co-product Δ generate the charges? What does the tensor product appearing in the co-product really mean: do the sector correspond to different total quark numbers for the generators? Is it a purely local operation in M^8 producing higher monomials of creation operators with the same momentum label or is superposition over Hamiltonian slices by Fourier transform possibly involved ?

How to construct quantum charges

One should construct quantum charges. In the TGD framework the quantization of $g(x)$ is not an attractive idea. Could one represent the charges associated with g it in terms of quark oscillator operators induced from the second quantized E^8 spinors so that propagators would emerge in the second quantization? Analogs of Kac Moody representations but with a non-negative spectrum of conformal weights would be in question. Also super-symplectic algebra would have this property making the formulation of the analogs of gauge conditions possible, and realizing finite measurement resolution in terms of hierarchy of inclusions of hyper-finite factors of type II_1 [K99, K33]. The Yangian algebra for $G_{2,c}$ or its subgroup could be the counterpart for these symmetries at the level of H.

The following proposal for the construction for the charges and super-charges of Yangian algebra in terms of quark oscillator operators is the first attempt.

1. One knows the Lie-algebra part of Yangian from the Poisson brackets of Hamiltonians associated with string world sheet slicing and possibly also for a similar slicing for partonic 2-surfaces. One should construct a representation in terms of quark ocillator operators in ZEO framework for both Lie-algebra generators and their super-counterparts. Also co-product should be needed.
2. The oscillator operators of E^8 spinor field located at the points of X^4 are available. The charges must be local and describe states with non-linear quarks and antiquarks.
   One must construct conserved charges as currents associated with the Hamiltonian orbits. Bosonic currents are bilinear in quark and antiquark oscillator operators and their super counterparts linear in quark or antiquark oscillator operators.

3. Since the system is 2-D one can formally assume in Euclidian signature (partonic 2-surface) Kähler metric $g^{z\overline{z}}$ and Kähler form $J^{z\overline{z}} = igz\overline{z}$, which is antisymmetric and real in real coordinates ($J^{kl} = -J^{lk}$) knowing that they actually disappear from the formulas. One can also define gamma matrices $\Gamma_\alpha = \gamma_k \partial_\alpha p^k$ as projections of embedding space gamma matrices to the string world sheet. In the case of string world sheet one can introduce light-like coordinates (u, v) as analogous of complex coordinates and the only non-vanishing component of the metric is g^{uv}.
4. The claim is that the time components J^u_n the bosonic currents

$$J^\alpha_n = b^\dagger_p \overline{v}(p) \Gamma^\alpha H_n u(p) a^\dagger \tag{6.6.2}$$

at the Hamiltonian curves with time coordinate t define conserved charges ($\alpha \in \{u, v\}$ at the string world sheet).
Remark: v_p corresponds to momentum $-p$ for the corresponding plane wave in the Fourier expansion of quark field but the physical momentum is p and the point of M^8 that this state corresponds.
Therefore one should have

$$\frac{J^u_n}{du} = 0 \tag{6.6.3}$$

One can check by a direct calculation what additional conditions are possibly required by this condition.
5. The first point is that H_n is constant if $v = constant$ coordinate line is a Hamiltonian orbit. Also oscillator operators creating fermions and antifermions are constant. The derivative of $u(p)$ is

$$\frac{du(p)}{du} = \frac{\partial u(p)}{\partial_{p^k}} \frac{dp^k}{du} \quad .$$

. u_p is expressible as $u_p = Du_a$, where D is a massless Dirac operator in M^8 and u_a is a constant 8-D quark spinor with fixed chirality. D is sum of M^4- and E^4 parts and M^4 part is given by $D(M^4) = \gamma^k p_k$ so that one has $dp^k/dt = \gamma_r dp^r/dt$.
This gives

$$\frac{d(\Gamma^u H_n u(p))}{du} = g^{uv} \gamma_k \partial_v p^k \frac{du(p)}{du} = g^{uv} \partial_u p \cdot \partial_v p \quad .$$

If the tangent curves of u and v are orthogonal in the induced metric and $v = 0$ constant lines are Hamiltonian orbits the bosonic charges are conserved.
One can perform a similar calculation for $d\frac{d\overline{v}(p)}{du}$ and the result is vanishing.
One must also have $dg^{uv}/du = 0$. This should reduce to the covariant constancy of g^{uv}. If the square root of the metric determinant for string world sheet is included it cancels g^{uv}.
6. From the bosonic charges one construct corresponding fermionic super charges by replacing the fermionic or anti-quark oscillator operator part with a constant spinor.

The simplest option is that partonic 2-surfaces contain these operators at points of cognitive representation. One can ask whether co-product could forces local operators having a higher quark number. What is clear that this number is limited to the number $n = 0$ of spin degrees of $n = 8$.

1. The commutators of bosonic and fermionic charges are fermionic charges and co-product would in this case be a superposition of tensor products of bosonic and fermionic charges, whose commutator gives bosonic charge. Now however the bosonic and fermionic charges commute in the same half-cone of CD. Does this mean that the tensor product in question must be tensor product for the upper and lower half-cones of CD?
For instance, in the fermionic case one would obtain superposition over pairs of fermions at say lower half-cone and bosons at the upper half-cone. The momenta would be opposite meaning that a local bosonic generator would have total momentum $2p$ at point p and fermionic generator at opposite cone would have momentum $-p$. The commutator would have momentum p as required. In this manner one could create bosons in either half-cone.

2. One can also assign to the bosonic generators a co-product as a pair of bosonic generators in opposite half-cones commuting to the bosonic generator. Assume that bosonic generator is at lower half-cone. Co-product must have a local composite of 4 oscillator operators in the lower half-cone and composite of 2 oscillator operators in the upper half-cone. Their anti-commutator contracts two pairs and leaves an operator of desired form. It therefore seems.
   Statistics allows only generators with a finite number of oscillator operators corresponding to 8 spin indices, which suggests an interpretation in terms of the proposed SUSY [L73]. The roots of P are many-sheeted coverings of M^4 and this means that there are several 8-momenta with the same M^4 projection. This degree of freedom corresponds to Galois degrees of freedom.
3. Only momenta in cognitive representation are allowed and momentum is conserved. The products of generators appearing in the sum defining the co-product of a given generator T, which is a local composite of quarks, would commute or anti-commute to T, and their momenta would sum-up to the momentum associated with T. The co-product would be poly-local and receive contributions from the points of the cognitive representation. Also other quantum numbers are conserved.

About the physical picture behind Yangian and definition of co-product

The physical picture behind the definition of Yangian in the TGD framework differs from that adopted by Drinfeld, who has proposed - besides a general definition of the notion of quantum algebra - also a definition of Yangian. In the Appendix Drinfeld's definition is discussed in detail: this discussion appears almost as such in [L32].

1. Drinfeld proposes a definition in terms of a representation in terms of generators of a free algebra to which one poses relations [B41]. Yangian can be seen as an analog of Kac-Moody algebra but with generators labelled by integer $n \geq 0$ as an analog of non-negative conformal weight. Also super-symplectic algebra has this property and its Yangianization is highly suggestive. The generators of Yangian as algebra are elements J^A_n, $n \geq 0$, with $n = 0$ and $n = 1$. Elements J^A_0 define the Lie algebra and elements J^A_1 transform like Lie-algebra elements so that commutators at this level are fixed.
   Remark: I have normally used generator as synonym for the element of Lie algebra: I hope that this does not cause confusion
   The challenge is to construct higher level generators J^A_n. Their commutators with JA^0 with J^An are fixed and also the higher level commutators can be guessed from the additivity of n and the transformation properties of generators J^A_n. The commutators are very similar to those for Kac-Moody algebra. In the TGD picture the representation as Hamiltonians fixes these commutation relations as being induced by a Poisson bracket. The Lie-algebra part of Yangian can be therefore expressed explicitly.
2. The challenge is to understand the co-product Δ. The first thing to notice is that Δ is a Lie algebra homomorphism so that one has $\Delta(XY) = \Delta(X)\Delta(Y)$ plus formulas expressing linearity. The intuitive picture is that Δ adds a tensor factor and is a kind of time reversal of the product conserving total charges and the total value of the weight n. Already this gives a good overall view about the general structure of the co-commutation relations.
   The multiplication of generators by the unit element Id of algebra gives the generator itself so that $\Delta(J_A)$ should involve part $Id \otimes J^A \oplus J^A \otimes Id$. Generators are indeed additive in the ordinary tensor product for Lie-algebra generators - for instance, rotation generators are sums of those for the two systems. However, one speaks of interaction energy: could the notion of "interaction quantum numbers" make sense quite generally. Could this notion provide some insights to proton spin puzzle [C1] meaning that quark spins do not seem to contribute considerably to proton spin? A possible TGD based explanation is in terms of angular momentum associated with the color magnetic flux tubes [K54], and the formulation of this notion at M^8 level could rely on the notion of "interaction angular momentum".
   The time reversal rule applied to $[J^m_A, J^n_B] \propto f_{ABC}J^{m+n}_C$ suggests that $\Delta(T^n_A)$ contains a term proportional to $f_{CBA}J^m_C \otimes J^{n-m}_B$. This would suggest that co-product as a time reversal involves also in the case of J^0_A the term $k_1 f_{CBA}J^0_C \otimes J^0_B$, where k_1 as an analog of interaction energy.
   Drinfeld's proposal does not involve this term in accordance with Drinfeld's intuition that co-product represents a deformation of Lie-algebra proportional to a parameter denoted by

$\hbar$, which need not (and cannot!) actually correspond to $\hbar$. This view could be also defended by the fact that J_0^A do not create physical states but only measures the quantum numbers generated by J_A^n, $n > 0$. TGD suggests interpretation as the analog of the interaction energy.

3. In Drinfeld's proposal, the Lie-algebra commutator is taken to be $[J_A^0, J_B^0] = kf_{ABC}J_C^0$, $k = 1$. Usually one thinks that generators have the dimension of $\hbar$ so that dimensional consistency requires $k = \hbar$. It seems that Drinfeld puts $\hbar = 1$ and the $\hbar$ appearing in the co-product has nothing to do with the actual $\hbar$.
   The conservation of dimension applied to the co-product would give $k_1 = 1/\hbar$! What could be the interpretation? The scattering amplitudes in QFTs are expanded in powers of gauge coupling strengths $\alpha = g^2/4\pi\hbar$. In ZEO co-product would be essential for obtaining non-trivial scattering amplitudes and the expansion in terms of $1/\hbar$ would emerge automatically from the corrections involving co-products - in path integral formalism this expansion emerges from propagors
   This view would also conform with the vision that Mother Nature loves her theoreticians. The increase of $h_{eff}/h_0 = n$ as dimension of extension of rationals would be Mother Nature's way to make perturbation theory convergent [K32]. The increase of the degree of P defining the space-time surface increases the algebraic complexity of the space-time surface but reduces the value of α as a compensation.

4. Drinfeld gives the definition of Yangian in terms of relations for the generating elements with weight $n = 0$ and $n = 1$. From these one can construct the generators by applying Δ repeatedly. Explicit commutation relations are easier to understand by a physicist like me, and I do not know whether the really nasty looking representation relations - Drinfeld himself calls "horrible" [B33] - are the only manner to define the algebra. In the TGD framework the definition based on the idea about co-product as a strict time reversal of product would mean deviation in the $n = 0$ sector giving rise to an interaction term having natural interpretation as analog of interaction energy.

5. Drinfeld proposes also what is known as Drinfeld's double [A175] (see http://tinyurl.com/y7tpshkp) as a fusion of two Hopf algebras and allowing to see product and co-product as duals of each other. The algebra involves slight breaking of associativity characterized by Drinfeld's associator. ZEO suggests [K44] that the members of Drinfeld's double correspond to algebra and co-algebra located at the opposite half-cones and there are two different options. Time reversal occurring in "big" state functions reductions (BSFRs) would transform the members to each other and change the roles of algebra and co-algebra (fusion would become decay).

In the TGD framework there is also an additional degree of freedom related to the momenta in cognitive representation, which could be regarded also as a label of generators. The idea that commutators and co-commutators respect conservation of momentum allows the fixing of the general form of Δ. Co-product of a generator at momentum p ina given half-cone would be in the opposite half-cone and involve sum over all momentum pairs of generators at p_1 and p_2 with the constraint $p_1 + p_2 + p = 0$.

Summation does not make sense for momenta in the entire extension of rationals. The situation changes if the momenta are algebraic integers for the extension of rationals considered: quarks would be particles in a number theoretic box. In the generic case, very few terms - if any - would appear in the sum but for space-time surfaces as roots of octonionic polynomials this is not the case. The co-products would as such define the basic building bricks of the scattering amplitudes obtained as vacuum expectation reducing the pairs of fermions in opposite half-cones to propagators.

6.7 Appendix: Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

6.7.1 Yang-Baxter equation (YBE)

YBE has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A173]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [B18] (see http://tinyurl.com/mcvvcqp) and in conformal field theories and models for anyons. Yangian symmetry appears also in the twistor Grassmann approach to scattering amplitudes [B19, B28] and thus involves YBE. At the same time new invariants for links were discovered and a new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [B41] (see http://tinyurl.com/l4z6zyr, where one can also find a list of references). YBE was first discovered by McGuire (1964) and 3 years later by Yang in a quantum mechanical many-body problem involving a delta function potential $\sum_{i<j}\delta(x_i-x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as a building brick 2-particle scattering matrix - R-matrix. YBE emerged for the R-matrix as a consistency condition for factorization. Baxter discovered in 1972 a solution of the eight vertex model in terms of YBE. Zamolodchikov pointed out that the algebraic mechanism behind factorization of 2-D QFTs is the same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed a quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. The interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of http://tinyurl.com/kyw6xu6). In 2-D quantum field theories and in models for basic gate in topological quantum computation the R-matrix is unitary. R-matrix can be regarded as an endomorphism mapping $V_1\otimes V_2$ to $V_2\otimes V_1$ representing permutation of the particles.

YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v) = R_{12}(v)R_{13}(u+v)R_{23}(u) \tag{6.7.1}$$

having interpretation as associativity condition for quantum algebras.

At the limit $u,v\rightarrow\infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braiding operation replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on the basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V\otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V\otimes V$ the condition states that $R(0)$ is proportional to the permutation matrix P for the factors.

General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued to meromorphic functions in the complex plane with poles forming an Abelian group. R-matrices can be

classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have a pole at origin and elliptic solutions have a lattice of poles. In [B41] (see `http://tinyurl.com/l4z6zyr`) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [B41] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.
3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anythingto do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of a Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be as interesting as Yangian: in this case co-product Δ does not seem to have a natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$. For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.
5. The article of Jimbo discusses also a fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs the R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

6.7.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \rightarrow A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \rightarrow C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Also ϵ satisfies associativity condition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [B41] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n + 1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n+1 \geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretations improve the understanding of classical correspondence (QCC).

Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian for $\mathcal{N}=4$ SUSY [B18], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

However, it must be emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC} J_0^B \times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \times J_1^A + f^A_{BC} J_0^B \times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n=0$ and $n=1$. The first half of these relations discussed in very clear manner in [B18] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f^{AB}_C J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f^{AB}_C J^{(1)C} \quad . \tag{6.7.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right]+\left[J^{(1)B},\left[J^{(1)C},\ J^A\right]\right]+\left[J^{(1)C},\left[J^{(1)A},\ J^B\right]\right] \\
&\qquad = \frac{1}{24}\, f^{ADK} f^{BEL} f^{CFM} f_{KLM}\{J_D,J_E,J_F\} \quad , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\ \left[J^C,J^{(1)D}\right]\right]+\left[\left[J^{(1)C},J^{(1)D}\right],\ \left[J^A,J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24}(f^{AGL} f^{BEM} f^{CD}_K \\
&\qquad + f^{CGL} f^{DEM} f^{AB}_K) f^{KFN} f_{LMN}\{J_G,J_E,J_F\} \quad .
\end{aligned} \tag{6.7.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

The right hand side often has coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant and as noticed in the main text has dimension of $1/\hbar$. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A} = J^A$ the first Serre relation reduces to Jacobi identity and second to the antisymmetry of the Lie bracket. The right hand side involved completely symmetrized trilinears $\{J_D, J_E, J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra, whose elements are labeled by a non-negative integer n. The generators obtained in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purpose of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exist also for continuum one-dimensional index).

Under certain consistency conditions, a discrete one-dimensional lattice provides a representation for the Yangian algebra. One assumes that each lattice point allows a representation

R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} = f^A_{BC} \sum_{i<j} J_i^B J_j^C \ . \quad (6.7.4)$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C \end{aligned} \quad (6.7.5)$$

Δ allows to imbed Lie algebra into the tensor product in a non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N} = 4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B18].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can involve besides the unit operator also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrized tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the following form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes the identity matrix and $PU(n|m)$ and $SU(n|m)$ are the same. This does not happen for $n = m$: this is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as a new physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f_{AB}^{C'} \sum_{i<j} J_i^A J_j^B \\ &= g_{CC'} f_{AB}^{C'} g^{AA'} g^{BB'} \sum_{i<j} J_{A'}^i J_{B'}^j \quad . \end{aligned} \tag{6.7.6}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

6.8 Conclusions

$M^8 - H$ duality plays a crucial role in quantum TGD and this motivated a critical study of the basic assumptions involved.

6.8.1 Co-associativity is the only viable option

The notion of associativity of the tangent or normal space as a number theoretical counterpart of a variational principle. This is not enough in order to have $M^8 - H$ duality. The first guess was that the tangent space is associative and contains a commutative 2-D sub-manifold to guarantee $M^8 - H$ duality.

1. The cold shower came as I learned that 4-D associative sub-manifolds of quaternion spaces are geodesic manifolds and thus trivial. Co-associativity is however possible since any distribution of associative normal spaces integrates to a sub-manifold. Typically these sub-manifolds are minimal surfaces, which conforms with the physical intuitions. Therefore the surface X_r^4 given by holography should be co-associative. By the same argument space-time surface contains string world sheets and partonic 2-surfaces as co-complex surfaces.
2. $X = Re_Q(o) = 0$ and $Y = Im_Q(P) = 0$ allow M^4 and its complement as associative/co-associative subspaces of O_c. The roots $P = 0$ for the complexified octonionic polynomials satisfy two conditions $X = 0$ and $Y = 0$.
   Surprisingly, universal solutions are obtained as brane-like entities X_c^6 with real dimension 12, having real projection X_r^6 ("real" means that the number theoretic complex valued octonion norm squared is real valued).
   Equally surprisingly, the non-universal solutions to the conditions to $X = 0$ correspond complex mass shells with real dimension 6 rather than 8. The solutions to $X = Y = 0$ correspond to common roots of the two polynomials involved and are also 6-D complex mass shells.
   The reason for the completely unexpected behavior is that the equations $X = 0$ and $Y = 0$ are reduced by Lorentz invariance to equations for the ordinary roots of polynomials for the complexified mass squared type variable. The intersection is empty unless X and Y have a common root and X_r^4 belongs to X_r^6 for a common root.

How to associate to the polynomial P a real 4-surface satisfying the conditions making $M8 - H$-duality?

1. P would fix complex mass shells in terms of its roots but not the 4-surfaces, contrary to the original expectations. The fact that the 3-D mass shells belong to the same M^4 and also their tangent spaces are parallel to M^4 together with rationality conditions for local $SU(3)$ element suggests number theoretical holography.
2. The key observation is that G_2 as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space $M^c \subset O_c$ gives a co-associative four-surface as a real projection. Also octonion analyticity allows G_2 gauge transformation. If X^4 is the image M^4

by a local $SU(3)$ element such that it also remains invariant under $SU(2)$ at each point, one obtains automatically $M^8 - H$ duality.
The image of X^4 under M^8-H duality depends on g so that gauge invariance is not in question. The plausible interpretation in case of $SU(3)$ is in terms of Kac-Moody - or even Yangian symmetry. Note that at QFT limit the gauge potentials defined at H level as projections of Killing vector fields of $SU(3)$ are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

The study of octonionic Dirac equation shows that the solutions correspond to momenta at mass shells $m^2 = r_n$ obtained as roots of the polynomial P and that co-associativity is an essential for the octonionic Dirac equation. This conforms with the reduction of everything to algebraic conditions at the level of M^8.

6.8.2 Construction of the momentum space counter parts of scattering amplitudes in M^8

The construction of scattering amplitudes in M^8 was the main topic of this article. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. 0

1. The fact that $SU(3)$ gauge transformation with boundary conditions defined by the mass shells as roots of polynomial P defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differing from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anti-commutators since the cognitive representation is discrete.

2. The main objection is that the scattering amplitudes are trivial if quark momenta belong to cognitive representations, which are finite in the generic case. This would be the case also in 2-D integrable theories. The objection can be circumvented. First, the huge symmetries imply that cognitive representations can contain a very large - even an infinite - number of points. At partonic 2-surface this number could reduce to finite. Equally importantly, local composites of quark oscillation operators with collinear quark momenta are possible and would be realized in terms of representations of Yangian algebra for $G_{2,c}$ serving as the counterpart for super-symplectic and Kac-Moody algebras at the level of H.
3. ZEO leads to a concrete proposal for the construction of zero energy states - equivalently scattering amplitudes - by using a representation of Yangian algebra realized in terms of positive and negative energy quarks in opposite half-cones. Co-product plays a key role in the construction. Also the proposed local composites of quarks proposed in [L73] make sense.
4. Momentum conservation conditions and mass shell conditions combined with the requirement that the momenta are algebraic integers in the extension of rationals determined by the polynomial P look rather difficult to solve. These conditions however linearize in the sense that one can express the allowed momenta as squares of integer quaternions.

Also the construction of scattering amplitudes in M^8 is considered. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. The fact that $G_{2,c}$ gauge transformation defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differering from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anticommutators since the cognitive representation is discrete.

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Chapter 7

New findings related to the number theoretical view of TGD

7.1 Introduction

TGD could be seen as a holy trinity of three visions about quantum physics based on physics as geometry, physics as number theory, and physics as topology.

Quite recently I gave a talk on TGD and TGD inspired theory of consciousness and was asked about the motivations for the number theoretic vision. My response went roughly as follows.

1. The attempt to find a mathematical description for the physical correlates of cognition could have led to the vision of quantum TGD as a number theory. What are the possibly geometric/number theoretic/topological correlates of thought bubbles?

   A bold guess could have been p-adic numbers, $p = 2, 3, 5, 7, ...$ provide natural mathematical correlates for cognition. Rationals, algebraic extensions of rationals, and the extensions of p-adic number fields induced by them are natural candidates as also complex numbers, quaternions, and octonions. Also finite number fields emerged quite recently as natural ingredients of the number theoretic vision [K85, K86, K84] [L127].

   As a matter of fact, I ended up to a proposal that p-adic physics provides the correlates of cognition via a different route, by p-adic mass calculations based on p-adic thermodynamics, which turned out to have surprisingly high predictive power due to the number theoretic existence conditions [K50].
2. Sensory experience corresponds to real number based physics. There is a strong correlation between cognition and sensory experience, but it is not perfect. Sensations arouse thoughts, but cognition is also able to dream and imagine.
3. Cognition includes mathematical thought. The concretization of mathematical thinking as computation requires discretization. This suggests that discretization should correspond to what one might call a cognitive representation transforming thoughts to sensory percepts and it should have a number theoretic representation.
4. Mathematical thinking is able to imagine spaces with an arbitrary dimension, while the dimension of the perceptual world is fixed and is the dimension of three-space. How does cognition achieve this?
5. Cognition has evolved. Why and how can this be the case?
6. If the correlates of cognition are part of reality, then cognition must be optimally efficient. How?

This leads to the following questions and answers.

1. Could p-adic spacetime surfaces represent thought bubbles, equivalent to real 4-surfaces? They are a number-theoretic concept, they also involve a different topology than the sense-world, and p-adic space-time surfaces would be examples of algebraic geometry.
2. How is cognition able to imagine? p-Adic differential equations are non-deterministic: integration constants, which by definition have vanishing derivatives are only piecewise constants. Could this make imagination possible [K62]??

3. How the strong correlation between cognition and sensory experience could be realized? All p-adic number fields and their extensions must be allowed. Consider first the simplet book involving only reals and p-adic number fields. p-Adic number fields Q_p, $p = 2, 3, 5, ...$ can be combined into a book, an adele [L43, L42]. Different number fields as extensions of rationals represent the pages of this book. Real numbers correspond to sensory experience and various p-adic number fields to cognition. The back of the book corresponds to rational numbers that are common to all chapters.
   Every algebraic extension of rationals defines extensions of p-adic number fields. The p-adic pages of the algebraically extended book are algebraic extensions of various p-adic number fields. One obtains an infinite library with books labeled by algebraic extensions of rationals. Now the back of the book consists of algebraic numbers for the extension generated by the roots of a polynomial with integer coefficients. The back of the book gives a cognitive representation, a number theoretic discretization of the 4-surface that is unique for a given extension. The bigger the extension, the more accurate the discretization. Cognitive evolution would correspond to a refinement of cognitive representation induced by the increase of the order of the polynomial defining the extension.
4. How is cognition able to imagine higher-dimensional mathematical objects that do not exist at the level of sensory experience? algebraic extensions for p-adic numbers can have an arbitrarily high dimension if the corresponding polynomial has high enough degree. One can have p-adic 4-surfaces for which the associated algebraic dimension is arbitrarily high! p-Adic cognition is liberated from the chains of matter!
5. Why is evolution related to cognition? One gets an infinite number of books labeled algebraic extensions, a whole library. Does the evolution of cognition present a hierarchy? The bigger the algebraic extension, the better the approximation to real numbers and thus to sensory experience.
6. Can p-adic cognition be maximally effective? Here p-adic thermodynamics suggests the answer. p-Adic mass calculations assign to each elementary particle a p-adic prime. For instance, for electrons it is Mersenne prime $p = M_{127} = 2^{127} - 1 \simeq 10^{38}$. p-Adic mass squared value is expansion powers of p and its real counterpart is power series in negative powers of p. This series converges extremely rapidly for large primes such as $p \simeq 10^{38}$ and two lowest orders give a practically exact answer so all errors would be due to the assumptions of the model rather than due to computations.

How to realize number theoretic physics?

1. Number theoretic discretization does not resonate with the idea of general coordinate invariance. For $H = M^4 \times CP_2$ allows linear Minkowski coordinates but CP_2 coordinates are not linear although also now complex coordinates consistent with the isometries of $SU(3)$ are natural.
   What about M^8 or possibly its complexification suggested by twistorial considerations and also by the fact that classical TGD predicts that Euclidian space-time regions give an imaginary contribution to the conserved four momenta. M^8 allows highly unique linear Minkowski coordinates and the idea that M^8_c corresponds to complexified octonions is very natural. The automorphism group G_2 of octonions poses additional conditions.
2. This leads to the idea that number theoretic physics is realized at the level of M^8_c and that it is dual to the geometric physics realized at the level of H and that these physics are related by $M^8 - H$ duality mapping 4-D surfaces in M^8 to H. TGD can be regarded as a wave mechanics for point-like particles replaced with 3-D surfaces in H, which, by the failure of complete determinism for holography, must be replaced by analogs of Bohr orbits. Wave mechanics is characterized by momentum-position duality, which naturally generalizes to $M^8 - H$ duality [L82, L83, L125, L127].
3. The physics in M^8_c should be purely algebraic as is also the ordinary physics at the level of momentum space for free fields. This physics should make sense also in all p-adic number fields. This suggests that polynomials with integer coefficients, in particular their roots, together with number theoretic holography based on associativity, partially characterize the 4-surfaces in M^8, which would make sense also as their p-adic variants.
   It is not clear whether the p-adicization is needed at the level of H: it might be enough to have it only at the level of M^8 so that only the p-adic variants of M^8 would be needed.

The geometric vision of TGD is rather well-understood (see for instance [L110]), but one need not think long to realize that there is still a lot of fog in the number theoretic vision (see for instance [K85, K86, K84] and [L82, L83, L101, L125, L127]).

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?
2. The detailed realization of $M^8 - H$ duality [L82, L83] involves uncertainties: in particular, how the complexification of M^8 to M^8_c can be consisted with the reality of $M^4 \subset H$.
3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?
4. How unique is the choice of 3-D surfaces at the mass shells $H^3_m \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.
5. The understanding of 3-geometries is essential for the understanding of the holography in both M^8 and H. The mathematical understanding of 3-geometries is at a surprisingly high level: the prime 3-manifolds can be constructed using 8 building bricks. Do these building bricks, model geometries, have counterparts as prefered extremals of action in the TGD framework. The known extremals $X^4 \subset H$ satisfying holography should be analogues of Bohr orbits [L114]. They are proposed to satisfy a 4-D generalization of 2-D holomorphy and apart from lower-D singularities would be the same for any general coordinate invariant action based on induced geometry and spinor structure. They would be minimal surfaces both in H and M^8 except at singularities at which the details of the action principle would matter [L136]. This suggests that the preferred extremals could have maximal isometries and provide topological invariants as also do the model geometries in the classification of 3-geometries.

7.2 What does one mean with M^8 physics?

In TGD, the point-like particle is replaced by a 3-surface $X^3 \subset H = M^4 \times CP_2$, and the holography required by the general coordinate invariance requires the replacment of the 3-surfaces with the analogues of Bohr trajectories passing through them. The Bohr trajectories are not completely deterministic as already the case of hydrogen atoms demonstrates. The "World of Classical Worlds" (WCW) is thus the space of generalized Bohr orbits as the counterpart of the superspace of Wheeler which originally inspired the notion of WCW [L135, L136, L132, L133]).

In wave mechanics, the duality between the descriptions using momentum and position space applies in wave mechanics but does not generalize to field theory. The $M^8 - H$ duality [L82, L83] can be seen as a generalization of this duality. M^8 is the momentum space counterpart and $H = M^4 \times CP_2$ is the position space counterpart in this duality.

7.2.1 Physical interpretation of the 4-surfaces of the space M^8 and their singularities

The physical interpretation of 4-surfaces in the complexifixation of the momentum space M^8 is far from straightforward. There are many reasons for the complexification.

1. Complexified octonionicity requires that M^8, or equivalently E^8, is complexified: one has $M^8_c = E^8_c$ giving as its subspaces various real subspaces with various signatures of the number theoretical. The metric obtained from the Minkowski norm $\delta_{kl} z^k z^l$, where z^k are 8 complex coordinates. M^4 with signature $(1, -1, -1, -1)$ is in a special physical role and one can of course ask, whether also other signatures might be important.
2. If complex roots are allowed for polynomials P determining together with associativity the holography, complexification must be allowed. Virtual momenta could therefore be complex, but Galois confinement says that the total momenta of physical states are real and have integer components in the momentum scale determined by the size of the causal diamond

(CD). Physical intuition suggests that the imaginary parts of the momenta code for the decay width of the particle. This is natural if the imaginary part is associated with the energy in the rest system.

3. The conserved momenta given by Noether's theorem at the level of H have real parts assignable to Minkowskian space-time regions. The fact that $\sqrt{g_4}$ appears in the integral defining a conserved quantity differs in Minkowski and Euclidean regions by an imaginary unit suggests that the contributions to momenta from the Euclidian regions are imaginary. The momenta from the Minkowskian space-time regions can be transferred to the light-like boundaries between Minkowskian and Euclidian regions identified as light-like partonic orbits. Quantum-classical correspondence requires that the classical total momenta, like all conserved quantities, correspond to the total momenta of the fermion state.
   Euclidian regions most naturally correspond to CP_2 type extremals as preferred extremals. They can be regarded as singularities resulting in the blow up of tip-like cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in M^8 . This would suggest that the real parts of momenta are associated with the Minkowskian regions of space-time surfaces and imaginary parts to the Euclidian regions. This applies also to other conserved quantities.

7.2.2 Number theoretic holography

Number theoretic holography has two forms.

1. The weak $3 \rightarrow 4$ form corresponds to the ordinary holography $Y^3 \subset M^8 \rightarrow Y^4 \subset M^8$, which is by $M^8 - H$ duality equivalent of the holography for $X^3 \subset H \rightarrow X^4 \subset H$ for space-time surfaces. The proposed interpretation of Y^3 is as a fundamental region of H^3/Γ.
2. For the strong $2 \rightarrow 4$ form of the holography Y^3 is determined by a 2-D data defined by the boundary of the fundamental region of H^3/Γ. The proposal to be considered is that the boundary of the fundamental region of H^3/Γ can be identified as 2-D hyperbolic space H^2/Γ.

Consider next the weak form of the holography.

1. The 4-surface $Y^4 \subset M^8_c$ is determined from number-theoretic dynamics and is an associative surface, i.e. its normal space is associative and therefore quaternionic.
2. There are also commutative 2-D surfaces that most naturally correspond to string world sheets, and for them commutativity of tangent space (as analog of associativity) as subspace of normal space of Y^2 defines holography. Holographic data corresponds now to strings connecting wormhole contacts assignable to Euclidian singularities inside $Y^3 \subset H^3_m$. One can also consider the possibility that partonic 2-surfaces correspond to co-commutative 2-surfaces. The situation is not completely clear here.
3. One must also identify the 3-surfaces $Y^3 \subset H^3_m$ defining the holography. Holography is subject to very strong conditions and I have proposed that these surfaces are hyperbolic 3-manifolds X^3 obtained as coset spaces H^3/Γ, where G is suitably chosen discrete but infinite subgroup of $SL(2,C)$ acting as Lorentz transformations in H^3. The spaces H^3/Γ are fundamental domains of H^3 tessellations.
   $Y^3 = H^3/\Gamma$ is counterpart for the unit cells of a lattice in E^3, which effectively has this topology and geometry due to boundary conditions stating that G leaves various "field configurations" invariant. The situation is the same as in the case of ordinary condensed matter, where periodic boundary conditions for a cube as a unit cell make it effectively a 3-torus.
   Also the crystal-like structures consisting of a finite number of copies of the fundamental domain of H^3/Γ glued together are possible choices for Y^3. They would be analogous to the unit cells of the lattices of Euclidian space E^3 or finite crystals formed from them. Therefore the analog of solid state physics would be realized at the fundamental level.
   One can also consider closed 3-manifolds $Y^3 = H^3/\Gamma$ obtained by gluing two copies of the fundamental region with different S^3 coordinates connected together along their 2-D boundaries. The gluing could be performed by a cylinder of $S^3 \subset E^4 \subset M^4 \times E^4$ connecting the boundaries.
4. The quantum state at H^3_m consists of several Galois singlets assignable to 3-surfaces Y^3_i. The total momenta for X^3_i would be real and have integer valued components for the momentum unit defined by the size scale of CD involved. This condition is analogous to the periodic boundary conditions.

5. Quantum classical correspondence requires that the many-fermion state on H^3_m, characterized partially by momenta, which are in the algebraic extension of rationals associated with the polynomial P, determines $Y^3_i \subset H^3_m$. For a given Y^3_i, the accompanying fermions correspond to the points of H^3_m. The classical momentum of the state given by Noether theorem in H would the sum of the fermionic momenta.
An attractive idea is that at least a subset of the fermionic momenta corresponds to cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`), which can be visualized geometrically as vertices of an algebraic surface at which the direction of normal space is ill-defined.
The cusps correspond to parabolic subgroups $P \subset G \subset SL(2,C)$ (`https://rb.gy/b5t55`), where the $\Gamma \subset SL(2,C)$ defines the fundamental domain H^3/Γ of the tessellation. Parabolic subgroups P are automorphic to the subgroup of translations of upper half-plane H generated by $SL(2,C)$ matrix $(1a;0,1)$, a a real algebraic number. This particular P acts as Möbius transformations in H representing hyperbolic space H^2. The cusp singularities would encode at least a subset of fermionic momenta of the state into the hyperbolic geometry of Y^3_i. Each fermion would correspond to its own parabolic generator in the subgroup G.
In the TGD view of hadron physics [L130], the fermions associated with the cusps could be identified as analogs of valence quarks. They would be associated with singularities identifiable as light-like 3-D partonic orbits serving as boundaries of 4-D CP_2 type extremals with Euclidian signature of the induced metric.
Also fermionic momenta, which have algebraic integers as components but do not correspond to cusps, can be considered. These would be naturally associated with strings predicted to connect cusps at the throats of different wormhole contacts. The blow-up would be now 2-sphere relating to cusp singularity like line charge to point charge. It is not clear whether the sea partons could correspond to these string-like singularities. In any case, hyperbolic 3-manifolds have string-like singularities connecting cusps.
6. If Y^3_i corresponds to a Galois singlet, then its total 4-momentum is real and integer-valued and should be mapped to a discrete plane wave in the finite lattice defined by the crystal like structure formed by the copies $X^3(Y^3)$ in H^3_a given by inversion. Each Galois singlet Y^3_i would define such a plane wave and one can imagine a hierarchy of such structures just as in the case of condensed matter with crystals of different sizes. The analogy with condensed matter physics suggests that Γ is a lattice. This follows also from the condition that H^3/Γ has a finite volume.
7. This picture would suggest that also $X^3(Y^3_i)$ is hyperbolic manifold of its fundamental region and perhaps isometric with Y^3. This would mean a geometric realization of Lorentz invariance analogous to the dual of conformal invariance encountered in the twistorialization.

7.2.3 Quantum classical correspondence for momenta

Quantum classical correspondence for momenta and also other conserved charges poses very strong conditions.

1. Noether charges for the classical action define momenta and other conserved charges. The classical contribution is a c-number. In addition, quantum contributions from fermions are included. They correspond to the momenta related to the second quantized spinor modes of H and from the orbital degrees of freedom associated with the "world of classical worlds" (WCW). The fermionic contributions are related to the ground states of the super symplectic representation characterized in terms of spinor modes for H spinor fields [K24, K76] [L127].
2. Are the classical contributions separate and do they add up to the total momentum? The fact that classical contributions are separately conserved, does not support this view.
3. Quantum classical correspondence would mean that the classical total momentum is the sum of the fermionic momenta determined by the multi-fermion state. This would hold quite generally for Cartan algebra of observables. For example, in the case of hadrons, the dominant classical contribution could correspond to the gluon sea, that is to multi-gluon state with gluons expressible in terms of quark-antiquark pairs. This picture is consistent with QCD and is therefore perhaps a more realistic guess.

4. Wormhole contact has Euclidean induce metric and the related classical conserved momentum is naturally imaginary. Could the sum of the imaginary parts of complex fermionic momenta of fermions for a wormhole throat correspond to the classical imaginary momentum assignable with the wormhole contact? Could the imaginary part of the fermionic momentum be assigned with the end of the euclidean string inside CP_2 type extremals, while the real momentum would be assigned with an end of a Minkowskian string?

5. Quantum-classical correspondence would be realized if the fermionic conserved four-momenta on the H side corresponded to M^8 points at hyperbolic 3-surfaces H^3_m. Their inversions in the M^8-H duality would be points of M^4_c of the spacetime surface $H^3_m \subset M^4_c \times CP_2$. It would seem that one must map only the real parts of the momenta at H^3_m to H^3_a.
   It would also seem that H^3_m must be associated with the M^4 projection of M^4_c. Whether the variant of H^3_m for complex valued m^2 makes even sense is not obvious.

7.2.4 The analog of time evolution in M^8 as a coupling constant evolution conserving dual quantum numbers

The proposal that 4-D surfaces appear at the level of M^8 suggests that it makes sense to talk about dynamics also in M^8 and the 4-D analogies of space-time surfaces make sense. This does not fit the usual classical intuition.

The twistor picture for conformal invariant field theories predicts that conformal invariance has a dual counterpart. This would mean that 4-momenta and other Poincare charges in H have dual counterparts in M^8. In TGD, the dual counterparts would be obtained by inversion from the defining the M^8-H duality and mapped to points of the space-time surface at the mass shells H^3_a in H. They would be analogs of the representation of lattice momenta as points of the heavenly sphere in crystal diffraction.

1. In zero energy ontology (ZEO), the time evolution at the level of H by "small" state function reductions (SSFRs), which are analogous to the so called weak measurements introduced by quantum opticians, would correspond to time evolution in terms of scalings rather than time translations. They would scale the size of the causal diamond (CD) and leave the passive boundary of CD invariant. These analogs of time evolutions would be generated by the scaling generator L_0. This would naturally also apply to M^8. This time evolution would be induced by scalings of the mass-scale, which need not be identical.

2. Could the "energy evolution", by identifying the square of the mass as the counterpart of time, correspond to the development related to the renormalization group? M^8-H duality would map the renormalization group evolution from M^8 to time evolution in H. This is quite a strong prediction.

3. Mass squared values for the fundamental fermions would not define particle masses but mass scales. 4-momenta for physical particles would correspond to total momenta for many fermion states, which obey Galois confinement, which requires that the momentum components are integers, when the mass unit is defined by the size scale of CD.

4. What would be the interpretation of the mass shells M^4_c determined by the roots of the polynomial P in the coupling constant evolution? Could the related hyperbolic 3-manifolds correspond to fixed points for the coupling constant evolution? With these mass values, something special would happen. Could H^3_a correspond to critical moments of light-cone proper time a when the SSFRs occur and a new unitary time evolution begins and ends with the next SSFR, as I have suggested?

5. What about the M^8 side? Could one talk about conserved quantities with respect to the evolution determined by scalings? Could these dual charges, dual momenta, and. also the charges related to E^4 isometries, be invariants of the renormalization group evolution?
   I have proposed that the $SO(4)$ symmetry of of hadrons in old-fashioned hadron physics involving notions like conserved vector current (CVC) and partially conserved axial current (PCAC) could correspond to the color symmetry of higher energy hadron physics by M^8-H duality in which the natural conserved charges on M^8 side are associated with the product of isometry groups of M^4 and E^4 and perhaps even with $SO(1,7) \times T^8$ or G_2 as automorphism group of octonions. At H side one would have a product of Poincare group and color group.

Also holonomy groups are involved. At least $SO(4)$ symmetry could define invariants of the coupling constant evolution in M^8.

Consider now a more detailed interpretation of 4-surfaces $Y^4 \subset M^8$ in terms of a generalized coupling constant evolution.

1. The changes $m_i^2 \to m_{i+1}^2$ for the roots of P would define a discrete evolution in both M^8 and H. Discrete coupling constant evolution affects the mass resolution and brings in or deletes details and therefore would induce changes for the representations of the states. The 4-surfaces in M^8 would represent renormalization group flows. The failure of a complete determinism is expected and could be interpreted in terms of phase transitions occurring at critical masses.
2. A given mass shell m_i^2 determined by a root of P would define a discrete mass scale for the evolution having perhaps an interpretation as a fixed point or a critical point of the coupling constant evolution. It would be natural to assume that the evolution induced by the change of resolution conserves other total quantum numbers than 4-momentum.
3. What about the conservation of 4-momentum? At $m^2 = m_{i+1}^2$ the value of mass squared for fundamental fermions defining the mass scale changes. The structure of the state must change in $m_i^2 \to m_{i+1}^2$ if 4-momentum conservation is assumed.
The normalization group evolution for the mass m^2 of the physical state, is typically logarithmic in QFTs, and must be distinguished from the discrete evolution for the mass scale m_i^2. Hence the change of m^2 in $m_i^2 \to m_{i+1}^2$ is expected to be small. This could be realized if n corresponds to a (possibly normal) subgroup of the Galois group of P perhaps spanned by the roots $m_k^2 \leq m_i^2$.
Could the phase transition $m_i^2 \to m_{i+1}^2$ change the rest energy of the state? Does the change require an energy feed between the CD and its environment as ordinary phase transitions require? This is not the case if CD is interpreted as a perceptive field rather than a physical system.
4. Does it make sense to talk about the conservation of dual momentum $X^k = \sum_i X_i^k$, $X_i^k = Re(\hbar_{eff} p^k/p_{k,i} p_i^k) = Re(\hbar_{eff} p^k/m_i^2)$? The conservation of momentum p^k does not imply the conservation of dual momentum since it is proportional to $1/m_i^2$: X^k would scale as $1/m_i^2$. The size of the CD is assumed to increase in statistical sense during the sequence of small state function reductions (SSFRs). The increase of the size of the CD would gradually make the mass shells inside it visible.
$M^4 \subset H$ center of mass position X^k therefore changes $m_i^2 \to mi + 1^2$ unless h_{eff} is not scaled to compensate the change $m_i^2 \to mi + 1^2$ in the formula for X_i^k. The integer n in $h_{eff} = nh_0$ is assumed to correspond to the order of the Galois group of P. It could also correspond to the order of a subgroup of the Galois group of P defined by the roots m_k^2, $k \leq i$. If so, h_{eff} would increase in evolution and one can even imagine a situation in which $Re(h_{eff}/m_i^2)$ remains constant.

7.3 $M^8 - H$ duality

The proposed $M^8 - H$ duality maps 4-surfaces $Y^4 \subset M_+^4 \subset M^8 = M^4 \times E^4$ to space-time surfaces $X^4 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$.

7.3.1 $M^8 - H$ duality as inversion

$M^8 - H$ duality relates also the hyperbolic spaces $H_m^3 \subset M_+^4 \subset M^8 = M^4 \times E^4$ and $H_a^3 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$. The hyperbolic space $H_m^3 \subset M_+^4 \subset M^8$ corresponds to the mass shell for which mass squared value m^2 is a root of a polynomial P. The hyperbolic space $H_a^3 \subset M^4 \subset= M^4 \subset M^4 \times CP_2$ corresponds to light-cone proper time constant surface $t^2 - r^2 = a^2$.

1. The root of P is in general a complex algebraic number. The first guess is that $M^8 - H$ duality is defined by inversion $p^k \to m^k = \hbar_{eff} p^k/p^l p_l$. Or briefly, $p \to \hbar_{eff} p/m^2$. The light-cone proper time $a = h_{eff}/m$ characterized the hyperboloid $H_a^3 \subset M^4$. $H_m^3 \to H_a^3$ is consistent with the Uncertainty Principle. In this case the image would be complex. This creates interpretational problems. There is no need for the complexification of CP_2, which also suggests that the image un H_a^3 must be real.

2. One can consider the possibility that only the real projection of the complex variant of H^3_m to $H^3_{Re(m)}$ is involved. The image of the real part $Re(p^k)$ in H^3_a obtained by inversion would be real but would not code information about the imaginary part $Im(p^k)$.
3. One could however take the real part of a complex inversion to get a point in H^3_a. $Re(\hbar_{eff}p^k/p^2$ would code information about the imaginary value of m^2.

Inversion fails at the light-cone boundary with $m = 0$. In this case, the inversion must be defined as the inversion of the energy of the massless state: $p^k \rightarrow \hbar_{eff}Re(p^k/E^2)$.

7.3.2 The technical problems posed $M^8 - H$ duality the complexification of M^8

Complexification of M^8 is highly desirable in the number-theoretic vision. But how to deal with the fact that fermion momenta, for which with components are algebraic integers in the algebraic extension determined by a polynomial P, are in general complex?

1. Without additional conditions, the mass shells in $M^4_c \subset M^8_c$ for complex m^2 as a root of P are 6-D. There are 2 conditions coming from the conditions fixing the value of $Re(m^2) = Re(p^2) - Im(p^2)$ and $Im(m^2) = 2Re(p) \cdot Im(p)$. If one only energy is complex, the dimension of the mass shell is 3. This looks natural. The preferred time axis would be determined by the rest system for massive states. A possible interpretation for the imaginary part is as decay width in the rest system.
2. The complexified mass shells of complexified $M^4 \subset H$ must be considered. Does this make sense? Since the CP_2 point labelling tangent space of Y^4 does not depend on complexification there is no need to consider complexification of CP_2. Therefore the natural conclusion is that also the $M^4 \subset H$ images should be real.

The inversion $p^k \rightarrow p^k/m_rp_rp_s$ is the simplest realization for $M^8 - H$ duality and would naturally fit into a generalization of 2-D conformal invariance to 4-D context. $h_{eff} = nh_0$ hierarchy comes along in a natural way. The polynomial P determines the algebraic extension and the value of h_{eff}. The size of the CD would scale like h_{eff} on the H side. There would be no scaling on the M^8 side.

1. The first thing to notice is that one could understand classically complex momenta. On the H side, Euclidean regions could give an imaginary contribution to the classical momentum.
2. The complex inversion $p^k \rightarrow p^k/m_rp_rp_s$ maps complex H^3_m to complex H^3_a. What would be the interpretation of the complex M^4_c coordinates? The same problem is also encountered in twistorization. One can ask whether a complex time coordinate corresponds to, for example, the inverse temperature?
   However, since no complexification is needed for CP_2, it seems that the only natural option is that the $M^4 \subset H$ image is real.
3. One can consider 3 options guaranteeing the reality.
   (a) Only the real parts of the complex $M^4 \subset M^8$ momenta are mapped to H. The information about the imaginary parts would be lost.
   (b) The complex algebraic integer valued momenta are allowed and the real part $Re(p^k \rightarrow p^k/m_rp_rp_s)$ of the complex inversion defines the image points in H. The $M^4 \subset H$ complexification would not be needed for this option but the information about the imaginary part of the momenta would not be lost.
   (c) By Galois confinement, the physical multiparticle states consist of momenta with integer value components using the momentum unit assignable to CD at the M^8 level of space with mass shells. These would define the 3-D data for the holography, which determines the 4-surface $Y^4 \subset M^8$ through the associativity of the normals space of Y^4. Only the real, integer-valued momenta of Galois confined states would be mapped from the mass shells of $M8$ to their images in H.
   The information about the fermion composition of the many-particle states would be lost completely. Therefore this option does not look realistic.
   The realization of this view might be possible however. 4-momenta determine the 3-surfaces Y^3 with real mass shells H^3 as data for associative holography. Momenta could

correspond to point-like singularities on $Y^3 \subset H^3$ and these should be assigned as CP_2 type extremals at H side as blow-ups of the singularities.

To conclude, the option $p^k \rightarrow Re(p^k \rightarrow p^k/m_r p_r p_s)$ seems to be the physically realistic option.

7.3.3 Singularities and $M^8 - H$ duality

Consider next the description of the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in $M^8 - H$ duality. The condition that information is not lost, requires that the map is given by $p^k \rightarrow Re(p^k/m_{rs}p^r p^s)$.

1. The cusps in $M^8 - H$ duality would be mapped to a 3-D surface of CP_2. It would correspond to the 3-D section CP_2 of the extremum, which corresponds to a wormhole contact associated with fermions at its throats.
   At H^3_m there is a temptation to assign to the cusp singularity, identified as a blow-up, the 3-D sphere $S^3 \subset E^4$ of the normal space E^4 defined by the mass shell condition. The simplest option is that this sphere is mapped to $U(2)$ invariant sphere $S^3 \subset CP_2$ for which the radius would be fixed by the mass squared value.
   The metric of H^3/Γ is singular at the cusp. The elimination of the singularity requires that one must allow a hole Z^3 around the cusp. The boundary of X^3 can have any genus. The size scale of the hole should be determined by the mass squared value.
   This view conforms with the physical picture of the CP_2 type extremal as an orbit of an Euclidian wormhole contact connecting two Minkowskian space-time sheets. S^3 would be replaced with $S^3 \setminus Z^3$ mapped to CP_2, where it corresponds to a wormhole throat having arbitrary genus.
   This picture would suggest that a given cusp singularity can correspond only to a single wormhole throat. This is not in conflict with the recent view of what elementary particles having wormhole contacts as composites should be. Composite, involving 2 wormhole contacts (required by the conservation of the monopole flux forming a loop involving two space-time sheets) and therefore 2 wormhole throats, can have spin varying from 0 to 2 which conforms with the popular wisdom that elementary particle spins vary in this range.
2. In the case of string-like objects $Y^2 \times R \subset H^3_m$, that is $S^2 \times R$ and their higher genus counterparts $H^2/\Gamma \times R$, the counterpart of the blow-up would be $Y^2 \subset S^3 \subset E^4$. Y^2 would be mapped to $X^2 \subset CP_2$ such that the radius assignable to S^2 or the size scale assigned to H^2/Γ would correspond to the mass squared.
3. Fermion trajectories at the partonic orbits would be light-like curves defining boundaries of string world sheets. CP_2 extremal would be associated with a fermionic cusp by holography and $M^8 - H$ duality.
   Fundamental fermion as an analog of valence quark [L130] could correspond to a cusp at the boundary of the string world sheet. Cusps would be related to the boundary of X^3_a composed of partonic 2-surfaces.
4. In principle, fermion momenta in the interior of $Y^3 \subset H^3_m$ are also possible. The picture given by hadron physics would indicate that the interior contribution corresponds to the sea partons. They can also be associated with string world sheets and correspond to virtual bosons appearing as fermion-antifermion pairs. These singularities would be string-like objects of the form $X^2 \times R$ and $X^2 \subset CP_2$ would replace the sphere of CP_2. One could say that fermions are delocalized at string.

7.3.4 Realization of the Uncertainty Principle

Inversion alone is not enough to realize Uncertainty Principle (UP), which requires that $M^8 - H$ duality is analogous to the Fourier transform. However, with the help of H^3 tessellations, it is possible to understand how the UP is realized in a finite measurement resolution.

The invariance of points of H^3/Γ under the subgroup $G \subset SL(2, C)$ is analogous to the periodic boundary conditions that replace the cubic unit cell of a crystal lattice with a torus. Now the tessellation of H^3, which quantizes the momenta, would be replaced by H^3/Γ. The momentum lattice having the fundamental region for H^3_m/G as unit cell would be mapped by inversion to a position lattice having the fundamental region H^3_a/G as a unit cell. A point in H^3_m would correspond to an analog of plane wave as a superposition of all positions of $X^3(Y^3)$ in a part of the tessellation in H^3_a: a wave function in finite crystal. One would have a superposition of 3 surfaces X^3_a corresponding to different lattice points multiplied by the phase factor. For a multi-fermion state the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) assigned to the momenta of fermions would characterize H^3/G so that the information about ("valence") fermion state would be code geometrically. Similar coding would be realized also for the string-like entities $H^2/\Gamma \times R$ at H^3/Γ. What is new and surprising, and also challenges the interpretation, is that the genus of H^2/Γ would code for the momenta of many-fermion states. Does the number of fermion-antifermion pairs correlate with the genus which in turn is proposed to label fermion families? There would be one fermion-antifermion pair per single handle. This would conform with the quantum classical correspondence. The proposed explanation [K21] for the number of observed fermion families would be in terms of hyper-ellipticity meaning that Z_2 acts as a conformal symmetry for all genera smaller than 3. Genus two would correspond to a formation of a bound state of two handles. Could this mean a formation of a graviton-like bound state of 2 fermion pairs and that higher spin states are not possible as bound states of handle. If fermions correspond to cusp singularities surrounded by holes, this picture might make sense: fermion antifermion pair would correspond to two holes connected by a handle.

The M^8-H duality maps the surface $Y^4 \subset M^8$ to the space-time surface $X^4 \subset H$. The point of $M^4 \subset H$ is obtained as the real part of the inversion of the point of the M^4 projection of the surface Y^4.

There would be a direct analogy to the physics of condensed matter.

A hyperbolic 3-manifold would correspond to a fundamental domain of a tessellation. It would be the equivalent of a unit cell both in position space and momentum space. These unit cells would correspond to each other at the H^3 level by M^8-H duality. Both would involve discretization. By finite momentum and position resolution UP would be reduced to the interior of the finite tessellation analogous to finite crystal. Quantum-classical correspondence and inversion are consistent with the realization of the UP related to Bohr's orbitology. Momenta in H^3_m would be mapped to equivalents of plane waves, i.e. superpositions of positions of the fundamental region in the tessellation. This picture generalizes to the multi-fermion states. Each fermion momentum defines a cusp and fermionic statistics makes it possible to avoid several cusps at the same points. Fermions for which other quantum numbers, such as spin differ, can however have the same momentum. They should correspond to the same cusp. How can this make sense? Could S^3 be involved somehow. Could they correspond to different holes in S^3 whose sizes and locations correlate with the other quantum numbers somehow? I have considered this problem earlier in the twistor picture where spin corresponds to a geometric degree of freedom in twistor space, which has identification at the level of M^8. The space of causal diamonds (CDs) as a kind of spine of WCW is discussed in [L135]. Lorentz transformations also occur at the level of CDs. The moduli of CD correspond to cm degrees of freedom in H. The finite volume of CD allows states for which Poincare quantum numbers are not exactly opposite for the boundaries of CD. Therefore the values of the total Poincare quantum numbers can be assigned to the CD. Only at the limit of infinitely large CDs does the zero energy property become exact. Therefore the CD wave function carries genuine information. At the p-adic level, translations and Lorentz transformations have the same effect as transformations of a compact group. Translations or Lorentz transformations of order $O(p^n)$ do not increase the p-adic norm of a point.

7.4 Holography

4-D general coordinate invariance forces holography at the level of $H = M^4 \times CP_2$ and one can regard space-time surfaces as analogues of Bohr orbits determined almost uniquely by 3-D surfaces. Quantum TGD is therefore very much like wave mechanics with point-like particles replaced with 3-surfaces in turn replaced with 4-D Bohr orbits. In fact, a wave-mechanical toy model for TGD would replace electron wave functions in atoms with wave functions in the space of its Bohr orbits.

M^8 is analogous to the momentum space in wave mechanics and the 4-surfaces in M^8 obey number theoretical holography based on associativity.

7.4.1 What does one mean with holography?

Consider now a more precise definition of holography.

1. The standard form of holography as $3 \rightarrow 4$ assigning to a 3-surface at the boundary of causal diamond (CD) an almost unique 4-D surface is the weakest form of holography. The non-uniqueness of the holography forces zero energy ontology (ZEO) in which analogues of Bohr orbits are basic geometrical objects.
2. $2 \rightarrow 4$ holography is the strongest form of holography. I have called it strong holography (SH). The 2-D partonic surfaces and possibly also the string world sheets would encode the data about the 4-surface and also the data about quantum holography. The strong form of holography could be realized as super symplectic and super-Kac Moody invariance and super-conformal invariance being minimally broken. Only the scaling generator L_0 would not annihilate the states. This condition is however too strong.
3. For the weaker form of SH super-symplectic and conformal symmetries are broken such that the algebras A_n (there are several of them), whose conformal weights are n-multiples of the conformal weights of the entire algebra A, and $[A_n, A]$ annihilate the physical states [L127, L69]. This requires half-algebra with non-negative conformal weights.
   The breaking hierarchy labelled by the values of n makes sense also for the ordinary conformal invariance but to my best albeit non-professional knowledge is not considered as a physical option. Hierarchies corresponding to the inclusion hierarchies of rational extensions and HFFs are obtained.

Both holographies set very strong conditions for the 3-surfaces appearing as holographic data.

Role of polynomials

At the level of M^8 physics is algebraic as it is also for the momentum space in the case of free field theory and reduces to algebraic conditions like mass shell condition and orthogonality of polarization vector and momentum. Polynomials P having integer coefficients determined the physics.

1. P as such does not fix the 4-surface nor even the 3-surface defining the data for number theoretic holography.
2. The polynomial P must have integer coefficients to guarantee number theoretical universality in the sense that they make sense also in p-adic number fields. If the coefficients are smaller than the degree of P, also finite fields become natural mathematical structures in TGD so that all number fields are involved. The roots of P give rise to the mass shells in M^8_c with mass squared values defined by the roots of P. The roots define an extension of rationals.
3. Polynomials are also characterized by ramified primes as the divisors of the discriminant of the polynomial determined by the product of the differences of its roots [L136]. They are not a property of the algebraic extensions. They depend on P and the exponent of the classical action is proposed to correspond to the discriminant D. Ramified primes are identified as p-adic primes playing a central role in p-adic mass calculations [K50].

The role of fermions

Quantum classical correspondence requires that the 3-surfaces Y^3 at the mass shells are determined by the quantum numbers of fermions associated with the quantum states. What assumptions could provide this additional data and how could this data be coded to the geometry of Y^3?

The data in question correspond to fermion momenta, spins and electroweak quantum numbers. Color does not appear as spin-like quantum numbers but corresponds to color partial waves in CP_2. Consider next how momenta are coded to the properties of 3-surfaces.

1. I have proposed that the 3-surfaces Y^3 in H^3_m could correspond to the fundamental domains of tessellations of H^3. The unit cell of ordinary crystal in E^3 serves as an analog for the fundamental domain of a tessellation in H^3. The disjoint components of Y^3 would naturally

correspond to surfaces Y^3_i at H^3_m and would correspond to fundamental domains of analogs of finite crystals formed by gluing them together.
The points of $E^4 \subset M^4 \times E^4$ correspond to 3-sphere with radius determined by mass m and for given Y^3_i the values of E^4 coordinates would be constant. A stronger condition would be that the values are the same for all Y^3_i. At the cusp points the point would be replaced by S^3, which could touch two disjoint sheets with different values of S^3 coordinates. Since the metric becomes singular at cusp, a natural proposal is that a small hole is drilled around the point and to S^3 and they are glued along their boundaries. The scale of the hole would be determined by the mass.

2. TGD predicts as basic objects also string-like objects $X^2 \times R \subset H$ and their deformations to magnetic flux tubes. By $M^8 - H$ duality they are expected to be present also in M^8 in particular at the hyperboloids H^3_m.
There are two kinds of string-like objects depending on whether their CP_2 projection is homologically trivial or not. In the latter case the string carries monopole flux.

Quantum classical correspondence suggests that the momenta of fermions as points of $H^3_m \subset M^8$ are coded into the geometry of Y^3 as singularities. $M^8 - H$ duality based on inversion in turn maps the momenta to singular points of H^3_a.

1. Singularities would be naturally cusps as analogs of tips of algebraic surfaces allowing all normal spaces of Y^4 at the singularity: M^8 duality would assign a 3-D subset of CP_2 to the tip.
2. Is it possible to have singularities, where the throats of the opposite wormhole throats touch? Or could the wormhole throats of the incoming partons fuse to single throat? This could occur in the topological counterpart of 3-vertex describing pair annihilation to a single particle. The singularities emerging in this way could relate to the description of the creation of fermion-antifermion pairs and would also define defects essential for exotic differential structures occurring only in dimension $D = 4$ [L121].

Can one code also spin to geometry or should it be regarded as a fermionic quantum number?

1. At the level of H one would have a product of twistor spaces $T(M^4)$ and $T(CP_2)$: these twistor spaces are unique in the sense that they have Kähler structure. This makes H a unique choice for the embedding space.
Twistorialization replaces space-time surface with 6-D surface $X^6 \subset T(M^4) \times T(CP_2)$ having S^2-bundle structure as possessed also by $T(M^4)$ and $T(CP_2)$. Spinor description of spin and electroweak isospin is replaced by a wave function in twistor spheres S^2.
The embedding corresponds to dimensional reduction producing X^6 as S^2 bundle. The twistor spheres associated with M^4 and CP_2 must be identified by the embedding of $X^6 \subset T(M^4) \times T(CP_2)$.
The identification of the twistor spheres forces spin doublets to correlate almost completely with electroweak spin doublets apart from the directions of the two spins. This picture allows only spin- and electroweak spin doublest. Does this force a complete correlation between the values of spin and electroweak to be identical or do the details of the identification for the the embedding of $X^6 \subset T(M^4) \times T(CP_2)$ allow to regard spin and electroweak spin as independent?
The identification of the two twistor spheres is not unique. The spin rotations and possibly also electroweak spin rotations (, which are not isometries) changes the identification of the two twistor spheres. This would make spin and electroweak spin independent quantum numbers. One can argue that only the relative rotation of the two spheres matters. Could this mean that electroweak spin axes can be thought of being completely fixed. Electroweak quantization axes are indeed completely fixed physically.
2. Something similar could happen at the level of M^8. Now one must consider twistor spaces of M^4 and E^4 and similar embedding of a 6-D surface $X^6 \subset M^8$ as twistor space with S^2 bundle structure.
In M^8 one would have an algebraic description of spin and electroweak spin instead of a wave function at S^2. A direction of S^2 would define a quantization axis and the diametrically opposite points of S^2 associated with it would provide a geometric correlate for the spin and electroweak spin values of fermion. The relative rotations of the twistor spheres of M^4

and E^4 associated with their identification are also now possible so that the two quantum numbers can be regarded as independent but with the electroweak quantization axes fixed.

In the twistorial picture one would have $5 \rightarrow 6$ weak holography or even $4 \rightarrow 6$ strong almost unique holography.

7.4.2 What kind of 3-geometries are expected in the TGD framework?

To get a wider perspective, it is good to have an overall view of the Geometrization conjecture of Thurston `https://rb.gy/9x3pm` proven by Perelman by studying Ricci flows. Geometrization theorem implies Poincare conjecture and so called spherical space conjecture.

The inspiration comes from the classification of 2-D manifolds expressed by uniformization theorem (`https://rb.gy/ts8va`). There are only 3 closed simply connected Riemann manifolds: sphere, disk, and hyperbolic plane. These are constant curvature spaces with corresponding Lie groups of isometries. One can obtain connected closed 2-manifolds with a nontrivial fundamental group by identifying the points related by a discrete subgroup of isometries.

In the case of the hyperbolic plane the isometry group is infinite and gives rise to a non-trivial fundamental group. For the hyperbolic plane, one obtains 2-manifolds with nonvanishing genus allowing a negative constant curvature. Constant curvature can be normalized to be -1, 1 or 0 in various cases. For non-vanishing curvature, the area serves as a topological invariant. For torus this is not the case.

The following provides the summary of my non-professional understanding of the 3-D case. The TGD inspired comments rely on what I know from the universal preferred extremals of practically any variational principle which is general coordinate invariant and can be constructed from the induced geometric quantities. They are always minimal surfaces outside 3- or lower-dimensional singularities at which the field equations depend on the action. The known extremals are discussed in [K8, K14, L114].

1. Thurston's conjecture `https://rb.gy/9x3pm` states that every oriented and closed irreducible (prime) 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with a finite volume which becomes a topological invariant in geometric topology. For instance, knots give rise to a 3-manifold in this way.
   An important difference is that the closed 3-manifold decomposes to a union of different types of 3-manifolds rather than only of single type as in the 2-D case.
2. The notion of model geometry is essential. There exists a diffeomorphism to X/Γ for some model geometry such that Γ is a discrete subgroup of a Lie group of isometries acting in G. There are 8 types of model geometries.
3. Irreducible 3-manifolds appear as building bricks of 3-manifolds using connected sum. There are 8 types of model geometries for closed prime 3-manifolds, which by definition do not allow a connected sum decomposition. These geometries are E^3, S^3, H^3, $S^2\times R$, $H^2\times R$, $SL(2,"R")$, Nil, and $Solve$.
4. The model geometries allow a constant curvature metric. The finite volume of the manifold becomes a topological characteristic if one has constant curvature equal to ± 1.
5. All types except one, $S^2 \times R$, which corresponds to a string-like objects in TGD, allow a 3-D Lie group as subgroup of isometries (Bianchi group).
6. All model geometries except hyperbolic manifold (`https://rb.gy/snpft`) and Solv manifold are Seifert fiber spaces (`https://rb.gy/uxszk`), which are fibered by S^1 fiber. Hyperbolic manifolds are atoroidal but have an infinite fundamental group since Γ must be infinite from the finite volume property. Atoroidality means that there is no embedding of torus which would not be parallel to the boundary of the hyperbolic manifold.
   The finite volume property of H^3/Γ also requires that Γ is a lattice: this implies a deep analogy with condensed matter physics. The group elements in the TGD frame-work be $SL(2,C)$ matrices with elements which are algebraic integers in an extension of rationals defined by the polynomial P defining 4-surface in M^8. Note that also momentum components are predicted to be algebraic integers using a unit defined by the scale of the causal diamond (CD).

   TGD leads to the proposal [K81] that the H^3 lattices could appear in cosmological scales and explain "God's fingers" [?] discovered by Halton Arp. They are astrophysical objects appearong along a line and having quantized redshifts.

7. One can form the spaces of the orbits for a discrete subgroup $\Gamma \subset G$ to obtain 3-manifolds with non-trivial fundamental group or orbifolds as in the case of S^3 and $S^2 \times R$. For hyperbolic 3-manifolds, the fundamental group is infinite and generated by elements of parabolic subgroups of G (`https://rb.gy/b5t55`). Cusp point and cusp neighborhood (`https://rb.gy/fd4dz`) are related to the infinite part of the fundamental group. Since parabolic subgroups $P \subset \Gamma$ are infinite groups, the fundamental group of the hyperbolic manifold is infinite.
8. One can decompose an irreducible 3-manifold to pieces, which are either Seifert manifolds or atoroidal. All 8 model geometries except hyperbolic geometries and so called Solv manifolds are Seifert manifolds.
   Solv manifolds are fiber spaces over a circle with 2-D plane with Minkowski signature as a fiber. In TGD solv manifolds could correspond to the so-called massless extremals [K8, K14, K66] serving representing classical radiation fields having only Fourier components with wave vectors in a single direction: laser beam is a good analog for them. They are not embeddable to H^3.
   In Ricci flows the hyperbolic pieces expand whereas other pieces contract so that asymptotically the manifold becomes hyperbolic. In fact, the collapse occurs in some caes in a finite time as found already by Richard S. Hamilton. The flow "kills" the positive curvature geometries S^3 and $S^2 \times R$ in the connected sum. What is left at large times is "thick-thin" decomposition. The "thick" piece is a hyperbolic geometry whereas the "thin" piece is a so-called graph manifold.

Hyperbolic manifolds and Seifert fiber spaces

Hyperbolic space and Seifert fiber space (`https://rb.gy/uxszk`) are in a central role in the TGD framework and therefore deserve short discussion. The following just gives the basic definitions and brief TGD inspired comments.

1. Hyperbolic manifolds

A hyperbolic n-manifold (`https://rb.gy/2esup`) is a complete Riemannian n-manifold of constant sectional curvature. Every complete, connected, simply-connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space H^n. As a result, the universal cover of any closed manifold M of constant negative curvature -1.

Every hyperbolic manifold (`https://rb.gy/snpft`) can be written as H^n/Γ, where Γ is a torsion-free discrete group of isometries on H^n. That is, Γ is a discrete subgroup of $SO^+_{1,n}$. The manifold has a finite volume if and only if Γ is a lattice.

Its "thick–thin" decomposition has a "thin" part consisting of tubular neighborhoods of closed geodesics and ends which are the product of a Euclidean $(n-1)$-manifold and the closed half-ray. The manifold is of finite volume if and only if its "thick" part is compact.

> *In the TGD framework, the lattice structure is natural and would mean that the elements of the matrices of Γ are algebraic extensions in the extension of rational defined by the polynomial P determining Y^4. The tubular neighborhoods of the "thin" part would correspond to string-like objects (tubular neighborhoods) as geodesics whereas the ends would correspond to cusp singularities inducing blow-up as 3-surface $S^3 \subset E^3$.*
> *At the level of H the tubular neighborhoods correspond to a string-like object and their ends to CP_2 type extremals serving as building bricks of elementary particles. Hadronic strings would represent examples of string-like objects and all elementary particles would involve them as monopole flux tubes connecting wormhole contacts.*

For $n > 2$ the hyperbolic structure on a finite volume hyperbolic n-manifold is unique by Mostow rigidity theorem and so geometric invariants are in fact topological invariants. One of these geometric invariants used as a topological invariant is the hyperbolic volume of a knot or link complement, which can allow us to distinguish two knots from each other by studying the geometry of their respective manifolds.

> *The identification of geometric invariants as topological invariants conforms with the TGD vision about "holy trinity" geometry-number theory-topology. Number theory would leak in through the identification of Γ as a lattice determined by the polynomial P.*

2. *Seifert fiber spaces*

A Seifert manifold `https://rb.gy/uxszk` is a closed 3-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

A standard fibered torus corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$ (with the natural fibering by circles). If $a = 1$ the middle fiber is called ordinary, while if $a > 1$ the middle fiber is called exceptional. A compact Seifert fiber space has only a finite number of exceptional fibers.

The set of fibers forms a 2-dimensional orbifold, denoted by B and called the base — also called the orbit surface — of the fibration. It has an underlying 2-dimensional surface B_0, but may have some special orbifold points corresponding to the exceptional fibers.

The definition of Seifert fibration can be generalized in several ways. The Seifert manifold is often allowed to have a boundary (also fibered by circles, so it is a union of tori). When studying non-orientable manifolds, it is sometimes useful to allow fibers to have neighborhoods that look like the surface bundle of a reflection (rather than a rotation) of a disk, so that some fibers have neighborhoods looking like fibered Klein bottles, in which case there may be one-parameter families of exceptional curves. In both of these cases, the base B of the fibration usually has a non-empty boundary. 6 of the 8 basic geometries of Thurston are Seifert fiber spaces.

In the TGD framework, the Seifert fiber spaces would correspond to string-like objects, which appear as several variants.

The eight simply connected 3-geometries appearing in the Thurston's conjecture from the TGD point of view

This section contains as almost verbatim the description of the 8 Thurston geometries provided by the Wikipedia article `https://rb.gy/9x3pm`. There is a good reason for this: I am not a professional and do not understand the technical details. There is a good reason for not giving a mere Wikipedia link: I have added comments relating to the TGD based identification of these model geometries as 3-surfaces.

It turns out that the geometries could correspond to fundamental regions of H^3, to energy $E = constant$ (M^4 time $t = constant$ in H) surfaces $D^3 \subset M^4_+ \subset M^8$, to string-like objects $X^2 \times R$ allowing Seifert fiber space structure, or to masless extremals with structure $M^2 \times E^2$ with M^2 and E^2 corresponding to the orthogonal planes defined by light-like momentum and polarization vector.

First some definitions:

1. A model geometry is a simply connected smooth manifold X together with a transitive action of a Lie group G on X having compact stabilizers (the isotropy group of a point is compact).
2. A model geometry is called maximal if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.This condition can be aso included in the definition of a model geometry.
3. A geometric structure on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X; this is a special case of a complete (G, X)-structure. If a given manifold admits a geometric structure, then it admits a structure, whose model is maximal.
   One can say that the spaces X provide the raw material from which one obtains various 3-geometries by identifications using a discrete subgroup of G.

A 3-dimensional model geometry X is relevant for the geometrization conjecture if it is maximal and if there is at least one compact manifold with a geometric structure modelled on X. Thurston classified the 8 model geometries satisfying these conditions; they are listed below and are sometimes called Thurston geometries. (There are also uncountably many model geometries without compact quotients.)

There is a connection with the Bianchi groups, which are the 3-dimensional Lie groups. Most Thurston geometries can be realized as a left invariant metric on a Bianchi group. However, $S^2 \times R$ does not allow Bianchi geometry; Euclidean space corresponds to two different Bianchi groups; and there are an uncountable number of solvable non-unimodular Bianchi groups, most of which give model geometries having no compact representatives.

1. Spherical geometry S^3

The point stabilizer is $O(3,R)$, and the group G is the 6-dimensional Lie group $O(4,R)$, with 2 components. The corresponding manifolds are exactly the closed 3-manifolds with a finite fundamental group. Examples include the 3-sphere, the Poincaré homology sphere, and lens spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type IX. Manifolds with this geometry are all compact, orientable, and have the structure of a Seifert fiber space (often in several ways). The complete list of such manifolds is given in the article on spherical 3-manifolds. Under Ricci flow, manifolds with this geometry collapse to a point in finite time.

In the TGD framework S^3 geometry could be associated with cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) of hyperbolic 3-manifold and represent the blow-up of a the cusp to S^3 which can be regarded as sphere in $E^4 \subset M^8 = M^4 \times E^4$. This is mapped to a 3-sphere of CP_2 in $M^8 - H$-duality.

2. Euclidean geometry E^3

The point stabilizer is $O(3,R)$, and the group G is the 6-dimensional Lie group $R^3 \times O(3,R)$, with 2 components. Examples are the 3-torus, and more generally the mapping torus of a finite-order automorphism of the 2-torus; see torus bundle. There are exactly 10 finite closed 3-manifolds with this geometry, 6 orientable and 4 non-orientable. This geometry can be modeled as a left invariant metric on the Bianchi groups of type I or VII_0.

Finite volume manifolds with this geometry are all compact, and have the structure of a Seifert fiber space (sometimes in two ways). The complete list of such manifolds is given in the article on Seifert fiber spaces. Under Ricci flow, manifolds with Euclidean geometry remain invariant.

In M^8 one has two kinds of roots of polynomials. For the first option they correspond mass square values defining mass shells H^3 . For the second option applying to the light-cone boundary as mass shell, energy E replaces mass and roots correspond to discrete energies. $E = constant$ surface corresponds to E^3 as 3-balls inside the light-cone.

3. Hyperbolic geometry H^3

The point stabilizer is $O(3,R)$, and the group G is the 6-dimensional Lie group $O^+(1,3,R)$, with 2 components. There are enormous numbers of examples of these, and their classification is not completely understood. The example with the smallest volume is the Weeks manifold. Other examples are given by the Seifert–Weber space, or "sufficiently complicated" Dehn surgeries on links, or most Haken manifolds.

The geometrization conjecture implies that a closed 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, and has an infinite fundamental group. This geometry can be modeled as a left invariant metric on the Bianchi group of type V or $\mathrm{VII}_{h\neq 0}$. Under Ricci flow, manifolds with hyperbolic geometry expand.

In TGD H^3 has an interpretation as a mass shell in $M^4 \subset M^8$ determined by the roots of the polynomial P or as a light-cone proper time constant hyperboloid in M^4.

This geometry does not allow Seifert fiber space structure unlike most other geometries.

4. The geometry of $S^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O(3,R) \times R \times Z/2Z$, with 4 components. The four finite volume manifolds with this geometry are: $S^2 \times S^1$, the mapping torus of the antipode map of S^2, the connected sum of two copies of 3-dimensional projective space, and the product of S^1 with two-dimensional projective space.

The first two are mapping tori of the identity map and antipode map of the 2-sphere, and are the only examples of 3-manifolds that are prime but not irreducible. The third is the only example of a non-trivial connected sum with a geometric structure. This is the only model geometry that cannot be realized as a left invariant metric on a 3-dimensional Lie group.

Finite volume manifolds with this geometry are all compact and have the structure of a Seifert fiber space (often in several ways). Under normalized Ricci flow manifolds with this geometry converge to a 1-dimensional manifold.

In the TGD framework, these surfaces could correspond to the simplest string-like objects for which S^2 corresponds to a geodesic sphere (homologically trivial or non-trivial) with a finite length connecting fundamental regions of H^3 or finite tessellations formed by them. S^2, which would correspond to a 2-D surface in CP_2 would be the base and string the fiber. One might argue that S^2 is more natural as fiber.

5. The geometry of $H^2 \times R$

The point stabilizer is $O(2,R) \times Z/2Z$, and the group G is $O^+(1,2,R) \times R \times Z/2Z$, with 4 components. Examples include the product of a hyperbolic surface with a circle, or more generally the mapping torus of an isometry of a hyperbolic surface.

Finite volume manifolds with this geometry have the structure of a Seifert fiber space if they are orientable. (If they are not orientable the natural fibration by circles is not necessarily a Seifert fibration: the problem is that some fibers may "reverse orientation"; in other words their neighborhoods look like fibered solid Klein bottles rather than solid tori.) The classification of such (oriented) manifolds is given in the article on Seifert fiber spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type III. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

In the TGD context, these geometries would correspond to closed string-like objects for which the CP_2 projection is a 2-surface with genus $g > 0$. Seifert fiber space property corresponds to closed strings.

6. The geometry of the universal cover of $SL(2,"R")$

The universal cover of $SL(2,R)$ is denoted $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. It fibers over H^2, and the space is sometimes called "Twisted $H^2 \times R$". The group G has 2 components. Its identity component has the structure $(\mathbf{R} \times \widetilde{\mathrm{SL}}_2(\mathbf{R}))/\mathbf{Z}$. The point stabilizer is $O(2,R)$.

Examples of these manifolds include: the manifold of unit vectors of the tangent bundle of a hyperbolic surface, and more generally the Brieskorn homology spheres (excepting the 3-sphere and the Poincare dodecahedral space). This geometry can be modeled as a left invariant metric on the Bianchi group of type VIII or III. Finite volume manifolds with this geometry are orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

Also now the interpretation as a closed string-like entity is possible in TGD.

7. Nil geometry

This fibers over E^2, and so is sometimes known as "Twisted $E2 \times R$". It is the geometry of the Heisenberg group. The point stabilizer is $O(2,R)$. The group G has 2 components, and is a semidirect product of the 3-dimensional Heisenberg group by the group $O(2,R)$ of isometries of a circle. Compact manifolds with this geometry include the mapping torus of a Dehn twist of a 2-torus, or the quotient of the Heisenberg group by the "integral Heisenberg group". This geometry can be modeled as a left invariant metric on the Bianchi group of type II.

Finite volume manifolds with this geometry are compact and orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow, compact manifolds with this geometry converge to R^2 with the flat metric.

In TGD also this geometry might be assigned with a closed string-like object as all Seifert fiber spaces.

8. Sol geometry

This geometry (also called Solv geometry) fibers over the line with fiber the plane, and is the geometry of the identity component of the group G. The point stabilizer is the dihedral group of order 8. The group G has 8 components, and is the group of maps from 2-dimensional Minkowski space to itself that are either isometries or multiply the metric by - 1. The identity component has a normal subgroup R^2 with quotient R, where R acts on R^2 with 2 (real) eigenspaces, with distinct real eigenvalues of product 1.

This is the Bianchi group of type VI_0 and the geometry can be modeled as a left invariant metric on this group. All finite volume manifolds with solv geometry are compact. The compact manifolds with solv geometry are either the mapping torus of an Anosov map of the 2-torus (such a map is an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct, such as

$$\left(\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}\right)) ,$$

or quotients of these by groups of order at most 8. The eigenvalues of the automorphism of the torus generate an order of a real quadratic field, and the solv manifolds can be classified in terms of the units and ideal classes of this order. Under normalized Ricci flow compact manifolds with this geometry converge (rather slowly) to R^1.

> *Unlike in the case of Seifert fiber spaces, a plane or disk appears as a fiber. Could one consider the possibility whether boundary conditions guaranteeing conservation laws could allow string-like objects for which the cross section is disk rather than a closed 2-surface? The appearance of isometries of 2-D Minkowski space suggests that the disk X^2 must have Minkowski signature so that the embedding to H^3 would not be possible.*
> *Could one assign this structure to massless extremals [K8, K66], which in the TGD framework define the representations for classical radiation fields, which involve the decomposition $M^4 = M^2 \times D^2$. The circle $S^1 \subset D^2$ defining its boundary would define the base space. Boundaries would be light-like and might allow to solve the boundary conditions. It is not clear how to obtain the counterparts of massless extremals at the M^8 level.*

7.4.3 $3 \rightarrow 4$ form of holography

One can consider two forms of holography. The first, weak, form corresponds to the ordinary $3 \rightarrow$ holography in which 3-D boundaries provide the data defining the 4-surface. The second, strong form, corresponds to $2 \rightarrow 4$ holography in which conformal boundaries provide the data defining the 4-surface. In this section the $3 \rightarrow 4$ form of the holography is considered.

Fundamental domains of hyperbolic tessellations as data for $3 \rightarrow 4$ holography

Good candidates for the surfaces Y_i^3 are fundamental domains assignable to hyperbolic 3-manifolds H^3/Γ_i represented as surfaces in $H^3 \subset M^4 \subset M^8$ (or its complexification). In the case of string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. The treatment of this case is a rather straightforward modification of the first case so that the discussion is restricted to H^3/Γ_i.

The surfaces Y_i^3 would correspond topologically to many-particle states of free particles. Holography would induce topological interactions in the interior of Y^4 and $X^4(Y^4)$. The momenta (positions) of the fermions analogous to valence quarks correspond to the cusp singularities.

For fundamental fermions momenta would have components, which are algebraic integers. Galois confinement states that the momenta for many-fermion states are ordinary integers. This poses a condition for H^3/Γ and it would be interesting to understand what the condition means.

The degrees of freedom orthogonal to H^3 correspond to a complexified sphere S^3 of E^4, whose radius squared corresponds to the square of the complex mass squared.

1. Hyperbolicity is a generic property of 3-manifolds and probably preserved in small enough deformations. In other words, deformations of hyperbolic 3-manifold X_i^3 probably allow a hyperbolic metric although the induced metric for the deformation is not in general hyperbolic. Deformation of the hyperbolic manifold (`https://rb.gy/snpft`) could take place in its evolution defining Y_i^4 and $X^4(Y_i^4)$ and could lead to, for example, to singularities such as the touching of different surfaces and interaction vertices at which partonic 2-surfaces meet.

2. There is an interesting connection to the geometrization conjecture of Thurston (https://rb.gy/9x3pm), especially with the work of the Russian mathematician Grigori Perelman, who studied 3-D Ricci flows (https://rb.gy/n6qlv) for metrics and proved that, apart from scaling, they lead to hyperbolic geometries.
Interestingly, hyperbolic manifolds decompose into "thin" and "thick" pieces and the "thin" piece corresponds to cusp neighborhoods (https://rb.gy/fd4dz). This decomposition brings in mind the notions of valence partons and sea partons with sea partons, in particular gluons assignable to the interior of Y_i^3 and giving the dominant contribution to the hadron mass.

Consider now what one can assume about Y_i^3.

1. The simplest assumption is that the S^3 coordinates are constant for Y_i^3 identified as the fundamental domain of a tessellation defined by H^3/Γ. It would represent a piece of H^3.
Could one consider the allowance of S^3 deformations H_d^3 of H^3 in the direction of S^3, which are invariant under G so that the space H_d^3/G would exist. They would define what mathematicians would call a model of hyperbolic geometry.
2. Can one allow for a given Y_i^3 a multiple covering of H^3 by copies of Y_i^3 with different constant values of S^3 coordinates? Could this state correspond topologically to a many-sheeted covering naturally associated with the polynomial P?
An interesting possibility is that Galois symmetry implies the existence of several copies of Y_i^3 with different S^3 coordinates as the orbit of the Galois group or its sub-group. Z_2 would be the simplest Galois group and give two sheets.

3-D data for $3 \rightarrow 4$ holography with 3-surfaces as hyperbolic 3-manifolds

It is good to start with questions.

1. Could the 3-surfaces X^3 associated with the mass shells $H_m^3 \subset M^8$ appearing as holographic data be fundamental domains (analogs of unit cell for crystals) of the tessellation H^3/Γ? Could a fermionic many-particle state for an algebraic extension determined by a given polynomial P assign a singularity to the fundamental domain and fix it?
The TGD view of hadron physics provides some clues. Gluon sea consists of gluons identifiable as fermion-antifermion pairs and fermions and antifermions. Here is the data for the given hyperbolic 3-manifold of singularities. The valence fermions could reside at throats and virtual sea gluons could be associated with strings $Y^2 \times R$ inside flux tubes and would give to the classical string tension?
Hyperbolic 3-manifolds also have string-like singularities connecting the cusp singularities. In the physical picture of TGD, these would correspond to strings connecting wormhole throats of different wormhole contacts which in turn would correspond to blow-ups of cusps.
2. Is the situation the same in M^8 and H? Could 4-surfaces assignable to the X_i^3 be minimal surfaces in both H and M^8 having a generalization of holomorphic structure to dimension 4? It would be possible to map X_i^3 to each other by inversion. Note that M^8-H correspondence would map the $M^4 \subset M^8$ projections of the points of Y^4 by inversion to H also in the interior of 4-surface.
Could this realize the dual conformal invariance proposed by the twistors, which would therefore be behind the analogy of Langlands duality and M^8-H duality?

7.4.4 Strong form of the hyperbolic holography

Holography roughly means an assignment of, not necessarily a unique 4-surface, to a set of 3-surfaces at the mass shells defined by roots of the polynomial P. The 4-surface is analogous to Bohr orbit.

A stronger form of the holography would be approximate $2 \rightarrow 4$ holography suggesting that the 3-surfaces allow $2 \rightarrow 3$ holography, which need not be completely deterministic. To understand what is involved one must have an idea about what kind of 3-surfaces could be involved.

1. Irreducible closed 3-surfaces Y_i^3 at H_m^3 consist of regions of 8 different types. Could these regions correspond to model geometries or at least have the symmetries of model geometries? This conjecture is natural if the 3-surfaces $Y_i^3 \subset H_m^3 \subset M^4 \subset M^8$ belong to (possibly complex) mass shells of M^4. In this case, the composites of fundamental regions of hyperbolic manifolds (https://rb.gy/snpft) as analogs of finite crystals would be natural.

The interiors of these regions would correspond to the "thick" part of the 3-manifold whereas the cusp singularities and string singularities as boundaries of string world sheets would correspond to the "thin" part. The blow-ups of cusp singularities would give rise to 3-D regions of CP_2.

2. Also monopole flux tubes connecting hyperbolic regions to form a network should be involved. Here the natural model geometries would be of type $S^2 \times R$ or $H^2 \times R$ with the ends of R at the two hyperbolic regions. By replacing H^2 with H^2/Γ, one would obtain higher flux tubes with a cross section having a higher genus.

The natural idea is that hyperbolic holography gives rise to $2 \rightarrow 3$ holography. In the case of H^3/Γ, the holography would assign H^3/Γ its fundamental region Y^3 to H^2/Γ.

In the case of H^2/Γ, applying for string-like objects, holography would assign $Y^2 \times R$ to a union of circles H^1/Γ defining its boundary. The rule would be simple: Y^n/Γ is a union of fundamental regions Y_i^n having H^{n-1}/Γ as boundary.

Hyperbolic holography from H^2/G to the fundamental domain of H^3/Γ

The representation of M^4 momenta in terms of bispinors is possible only for massless particles. This raises the question whether one must assume a strong form of holography in which 2-D surfaces at the boundaries of H^3_m dictate the 4-D surface almost completely. The hyperbolic 2-manifold H^2/G should define the boundary for Y_i^3 identifiable as a fundamental domain Y^3 of a hyperbolic 3-manifold H^3/Γ.

1. This would conform with the proposed realization of super-symplectic invariance and Kac-Moody type symmetries for light-like partonic orbits meaning that the interior degrees of freedom associated with the 3-surfaces X_i^3 and light-like orbits of partonic 2-surfaces are eliminated with a suitable gauge choice formulated in terms of a generalization the Virasoro and Kac-Moody conditions [L125, L127].
2. Physically this would mean that the fermion momenta at cusp point are light-like. This would conform with the view that fermions move along light-like curves inside the light-like partonic orbit.
3. If hyperbolic holography makes sense, the above formulation for H^2 would generalize to the case of H^3. Cusp neighborhood U/P as a projection $U \rightarrow H^2/G$ has a counterpart for H^3/Γ and the fundamental domain for H^2/G extends to a fundamental domain for H^3/Γ. For H^2/G as boundary it would correspond to the condition $p_3 > 0$ for the momentum component in the chosen direction.
4. The cusp singularity is analogous to a cusp of an algebraic surface. This suggests that near the cusp point of H^3/Γ the metric behaves like the induced metric of 3-D cusp in 4-D space. Near the cusp one has $t = k\sqrt{\rho}$ where t and ρ are vertical coordinate and transversal coordinates of the cusp in 4-D space. The radial component of the induced metric orthogonal to tip direction should behave like $g_{\rho\rho} = 1 + k^2/4\rho$ and the radial distance from the tip would diverge logarithmically. One cold say that this point is missing so that the hyperbolic manifold is compact but not closed since it has boundaries. The singularity of the metric is a good motivation for cutting off a small ball around the singularity in M^4 and a small ball from S^3 and for gluing the two together along boundaries. At the level of H this would correspond to wormhole throat.

7.4.5 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units

I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M_c^8.

In the following M^8-H duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u,v), which are analogous to z and $\overline{z}$. Any analytic map $u \rightarrow f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M_c^8

$Y^4 \subset M_c^8$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. M^8-H duality assigns to the normal space $N(y)$ a point of CP_2. M_c^4 point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M_c^4 \subset M_c^8$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. Is this deformation of H^3 in imaginary time direction equivalent with a region of H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \ . \tag{7.4.1}$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \rightarrow Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \rightarrow 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \rightarrow SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

1. This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.
2. A physically motivated proposal is that the real parts of $SU(3)_c$ gauge potential and color gauge field can be lifted to H and the lifts are equal to the classical gauge potentials and color gauge field proposed in H. Color gauge potentials in H are proportional to the isometry generators of the color gauge field and the components of the color gauge field are proportional to the products of color Hamiltonians with the induced Kähler form.
3. The color gauge field $Re(G)$ obeys the formula $Re(G) = dRe(A)+[Re(A), Re(A)] = [Re(A), Re(A)]$ and does not vanish since the contribution of $[Im(A), Im(A)]$ cancelling the real part is absent. The lift of $A_R = g^{-1}dg$ to H is determined by g using M^4 coordinates for Y^4 . The M^4 coordinates $p^k(M^8)$ having interpretation as momenta are mapped to the coordinates m^k of H by the inversion

$$I : m^k = \hbar_{eff} Re(\frac{p^k}{p^2}) \ , \ p^2 \equiv p^k p_k \ ,$$

where p^k is complex momentum. $Re(A)_H$ is obtained by the action of the Jacobian

$$dI^k_l = \frac{\partial p^k}{\partial m^l} \ ,$$

as

$$A_H = dI \cdot Re(A_{M^8}) \ .$$

dI^k_l can be calculated as the inverse of the Jacobian $\partial m^k/\partial Re(p)^l$. Note that $Im(p^k)$ is expressible in terms of $Re(p^k)$.
For $Im(p^k) = 0$ the Jacobian for I reduces to that for $m^k = \hbar_{eff}\frac{p^k}{p^2}$ and one has

$$\frac{\partial m^k}{\partial p^l} = \frac{\hbar_{eff}}{p^2}(\delta^k_l - \frac{p^k p_l}{p^2}) \ .$$

This becomes singular for $m^2 = 0$. The nonvanishing of $Im(p^k)$ however saves from the singularity.
4. The $M^8 - H$ duality obeys a different formula at the light-cone boundaries associated with the causal diamond: now one has $p^0 = \hbar_{eff}/m^0$. This formula should be applied for $m^2 = 0$ if this case is encountered. Note that number theoretic evolution for masses and classical color gauge fields is directly coded by the mass squared values and holography.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.
2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is

analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.
These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.
4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.
5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M^8_c$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?
2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.
3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically. In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have nonvanishing G_2 gauge fields.
   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X^3_i.

7.5 Singularities, quantum classical correspondence, and hyperbolic holography

The point-like fermions and their 1-D trajectories appear as singularities of the minimal surfaces [L114]. Strings that connect fermions located at their ends, and string world sheets in the interior of X^4 appear also as singularities. Also partonic 2-surfaces separating Minkowskian and Euclidian regions should correspond to singularities of X^3_i and their light-like radii.

There would therefore be singularities in dimensions $D = 0, 1, 2, 3$. These singularities should relate to the fundamental domains $Y^3_i \subset H^3_m \subset M^8$ and holography would suggest that they correspond to the singularities of 3-D hyperbolic manifolds (`https://rb.gy/snpft`).

7.5.1 Cusp singularities and fermionic point singularities

The singularities should be associated with hyperbolic manifolds Y^3_i identified as fundamental domains of coset spaces H^3/Γ , that is, as effective geometries H^3/Γ defined by the boundary conditions for various "fields". In the same way as, for example, a torus geometry appears in condensed matter physics for a unit cell of lattice.

Cusp singularity (https://rb.gy/fd4dz) is a natural candidate for a point-like singularity and geometrically corresponds to a cusp. For abstract Riemann geometry, the cusp property would correspond to a singularity of the metric for a cusp (tip) and mean that the radial component of the metric diverges at the tip.

Consider first the basic concepts and ideas in the case of 2-D hyperbolic space H^2 and corresponding hyperbolic manifold H^2/G case.

1. Riemann surface can be regarded as a coset space H^2/G, which is represented as a fundamental region for a tessellation of H^2.
2. Cusp singularities of H^2/G correspond to parabolic subgroups P (https://rb.gy/b5t55) generated by a parabolic element for $G \subset SL(2,C)$. Parabolic subgroup P is isomorphic to a discrete group of translations along, say, the real axis as a boundary of the upper half-plane and is noncompact. It is represented as Möbius transformations induced by the matrices $(1, n : 0, n)$. P can be regarded as a subgroup generated by a Lorentz boost in a fixed direction. The cusp singularity results from the identification of points related by the elements of P, which form a non-compact group. Let U denote the set with $Im(z) > 1$ which corresponds to the set $p^3 > 0$ in momentum space. U and $P(U)$ are disjoint. The cusp neighborhood (https://rb.gy/fd4dz) can be identified as the set U/P which is the projection of U to H/G.
3. In the simplest situation, one has $G \subset SL(2,Z) \subset SL(2,R) \subset SL(2,C))$, where $S(2,R)$ leaves the real axis invariant. Z could be replaced by an algebraic extension for rationals of algebraic integers in this extension.

$SL(2,C)$ and therefore also $SL(2,R)$ acts in M^4 as Lorentz transformations.

1. A given M^4 momentum has the representation $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2)$. The representation is unique apart from a complex scaling of z_i so that $z = z_1/z_2$ can be taken as a complex coordinate for the plane and $SL(2,C)$ acts as Möbius transformation. $SL(2,R)$ leaves the real axis invariant.
   The automorphism of sigma matrices induced by $SL(2,C)$ transformation in turn induces Lorentz transformation in momentum space.
2. Under what conditions can bi-spinors correspond to M^4 coordinates? Bi-spinor can be assumed to be of the form $(z_1, z_2) = (z, 1)$. From the formula $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2) = (z, 1)$ one can deduce an expression of the condition $Im(z_1/z_2) = Im(z_1) > 1$ in terms of p^k. The condition implies that the z-component of momentum satisfies $p^z = z\overline{z} - 1 > 0$.
   The description of M^4 momenta in terms of bi-spinors and H^2 identified as upper half-plane, denoted by H, is possible only for massless particles.

What happens at cusp singularity

What happens at the cusp singularity?

1. The normal space of the singularity is completely ill-defined as the direction of the electric field of a point-like charge. If so, CP_2 would always be a companion to the cusp. CP_2 would be a blow-up of the cusp points of X^3_i as a hyperbolic manifold (https://rb.gy/snpft). One would have $X^3_i \subset H^3$ and the cusp points would correspond to a 3-D sub-manifold of CP_2 defined by the normal spaces at the cusp singularity.
2. In the interior of the space-time surface the 3-D submanifold of CP_2 would extend to CP_2 type extremal with a light-like M^4 projection or its deformation. Several cusp singularities (see https://rb.gy/0p30o and https://rb.gy/fd4dz) could be associated with a single CP_2 type extremal representing wormhole contact. This corresponds to the view that wormhole throats can carry more than one fermion although the recent model assumes only a single fermion.
3. CP_2 type extremal defines a wormhole contact connecting two Minkowskian space-time sheets in $H = M^4 \times CP_2$. This would mean that the 3-D submanifold of CP_2 as a blow up is deformed to CP_2 type extremal with 2 throats at opposite sheets: at them the Euclidian induced metric transforms to Minkowskian signature.
   The conservation of monopole flux indeed forces the presence of two Minkowskian space-time sheets in the picture based on H. If the throat as a boundary of the 3-D region of $X^4 \subset H$

involves an incoming radial monopole flux, there must be another throat CP_2, where this flux runs to the other space-time sheet.

4. How could the throats connecting the two Minkowskian space-time sheets emerge in the M^8 picture? Should one allow several copies of Y_i^3 with the same H^3 projection but different constant S^3 coordinates and with a common cusp point. The blow-up would give several copies of 3-D regions of CP_2, and in holography they would define wormhole contact with 2 or even more throats.
   The simplest view is that quarks are the only fundamental fermions and leptons correspond to wormhole contacts carrying three antiquarks. They could have three throats associated with the same CP_2 type extremal but this is not the only possibility.

The singularities associated with string-like objects

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma\times R$ and $H^2/\Gamma\times R$. For $S^2\times R$ the spaces S^2/Γ, Γ a finite non-trivial subgroup of $SO(3)$ are orbifolds: the faces of Platonic solids are basic examples. For P^2/Γ one obtains $g>0$ 2-manifolds with constant curvature metric with negative curvature.

The physical interpretation would be that $S^2\times R$ and $H^2/\Gamma\times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma\times R$ and $H^2/\Gamma\times R$. For $S^2\times R$ the spaces S^2/Γ are orbifolds if Γ is finite non-trivial subgroup of $SO(3)$: the triangular, quadrilateral, and pentagonal faces of Platonic solids are key examples. From these one can build finite lattices at S^2. For P^2/Γ one obtains $g>0$ 2-manifolds with constant curvature metric with a negative curvature.

The physical interpretation would be that $S^2\times R$ and $H^2/\Gamma\times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

What could be the quantal counterpart for the geometric holography? This has been a long standing open question. Suppose that the strong form of holography is realized.

1. In [L130], I considered quantal holography as a counterpart of geometric holography discussed in this article. This led to a suggestion that valence quarks at the wormhole throats could pair with pairs of dark quark and antiquark at strings associated with magnetic flux tubes in the interior of the hadronic 3-surfaces. Could these strings correspond to string-like singularities assignable to geodesic lines inside fundamental regions of $H^3/Gamma$?
2. The flux tubes were proposed to have an effective Planck constant $h_{eff}>h$. The correspondence between valence quarks and dark quarks was proposed to be holographic. The spin and electroweak quantum numbers of dark antiquark would be opposite to those of valence quark and dark quark would have quantum numbers valence quark. There would be entanglement in color degrees of freedom for valence quark and dark antiquark to form color single: this would screen the color of valence quark and transfer it to the magnetic body. The holography in this way would allow a convergent perturbation theory. Nature would be theoretician friendly: a phase transition increasing h_{eff}, transferring color to dark quarks, and reducing color coupling strength to $\alpha_s=2^2/4\pi\hbar_{eff}$ would occur.
   Whether the dark quark-antiquark pairs as analog for gluon pairs as explanation for hadron mass could explain most of hadron mass remained open: if the classical conserved quantities are identical with the quantum contribution from fermions for Cartan algebra, this could be the case. Whether they could correspond to sea gluons remains also an open question.
3. Quantal holography allowing to obtain a convergent perturbation theory might be realized quite generally, also for leptons which correspond to color partial waves in CP_2 neutralized by super symplectic generator [K50, L56] [L69].

It should be noticed that leptonic dark holography would be very natural if leptons consist of 3 antiquarks [L102]. This option would explain matter-antimatter asymmetry in a new way. Antimatter would be identifiable leptons. For the simplest option, the 3 antiquarks would be associated with a single single wormhole throat. The generalized Kähler structure assignable to M^4 in twistor lift [L115, L116] allows a CP violation, which could favor the condensation of quarks to baryons and antiquarks to leptons.

There are however objections against this idea. The considerations of this article inspire the question whether a single wormhole throat can carry only a single quark assignable to the cusp singularity, as suggested already earlier. Two wormhole contacts would be required. This is required also by the fact that stable wormhole contact must carry a monopole flux and monopole flux flux loops must be closed. Uncertainty Principle would suggest that the flux tube must have length of order lepton Compton length. Can this be consistent with the point-like nature of leptons? These arguments favor the option in which leptons and quarks as opposite H chiralities of H-spinors are the fundamental fermions.

Other kinds of point-like singularities and analogy with Fermi surface

Point-like singularities as cusps would naturally correspond to fundamental fermions at the light-like orbits of partonic 2-surfaces.

1. The 2-D boundaries of the fundamental region Y_i^3 associated with H^3/Γ would be analogues for 2-D pieces of the Fermi surface corresponding to atomic energy levels as energy bands.
   In condensed matter physics, the energy shells can deform and the components of the Fermi surface can touch. These singularities are central to topological physics.
2. At M^8 level the 2-D boundaries of Y_i^3 are analogues of energy bands. The evolution defined by the number theoretic holography, identifiable as a coupling constant evolution at the level of M^8, induces deformations of Y_i^3. One expects that this kind of touching singularities take place.
   At the level of H this would correspond to simple touching of the outer boundaries of the physical objects. In particular, these touchings could take place at the partonic 2-surfaces identified as vertices at which severa partonic orbits meet as the partonic surfaces as their ends are glued to single surface just like the ends of lines of a vertex of Feynman diagram are glued together along their ends.
   Could the meeting of fermion and antifermion cusp singularity in this way relate to an annihilation to a boson regarded as a fermion antifermion pair?
3. One can of course challenge the assumption that all fermions correspond to cusps, which correspond to parabolic subgroups of $G \subset SL(2,C)$ (`https://rb.gy/b5t55`). The proposal that all momenta, whose components are algebraic integers for the extension defined by P, are possible. What could be the interpretation of fermions which do not correspond to cusp.
   What the addition of a fermion to a particular allowed momentum could mean? Could it mean that its momentum defines a parabolic subgroup of G? Or is it true only for the "thin" part of Y_i^3 perhaps representing analogs of valence quarks.
   Or could the non-singular momenta correspond to the momenta for the analogues of sea partons, in particular analogs of sea gluons as fermion-antifermion pairs so that their total momentum would dominate in the total momentum of hadron. These would correspond to the "thick" part of Y_i^3. Could these interior momenta correspond to states delocalized at the string world sheets in the interior of monopole flux tubes and also states delocalized in the interiors of the flux tubes. Are these fermions present too?
   The presence of these states should be coded by the geometry of the hyperbolic manifold H^3/Γ (`https://rb.gy/snpft`) and Y_i^3 as its fundamental domain. Somehow the group $G \subset SL(2,C)$ should be responsible for this coding.

7.5.2 About the superconformal symmetries for the light-like orbits of partonic 2-surfaces

Are the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) giving rise to CP_2 type extremals and the fermion momenta inside string world sheets and flux tubes associated with Y_i^3 sufficient to fix the 3-surfaces Y_i^3 in turn fixing number-theoretic holography?

1. The total energy for the classical action associated with these two kinds of fermions should correspond to the "sea" (thick part) and "valence fermions" assigned to the cusps (thin part).
2. Supersymplectic invariance and generalized conformal and Kac-Moody invariance assignable to light-like partonic orbits allows a large number of alternatives for the light-like surfaces

[L110, L127, L125]. If supersymplectic and Kac-Moody symmetries act as gauge symmetries, the surfaces related by these symmetries are physically equivalent.
The proposal is that these symmetries are partially broken and there is a hierarchy of breakings labelled by subalgebras $A_n \subset A$ of these algebras. The vanishing conditions for classical and quantal charges for A_n and $[A_n, A]$ serve as gauge conditions and also select the partonic 3-surfaces.
Interpretation of the partially broken gauge symmetries giving rise to dynamical symmetries is in terms of number theoretical measurement resolution and inclusion of hyperfinite factors of type II_1. These hierarchies relate to the hierarchies of extensions of rationals defined by the polynomials P defining the space-time surfaces apart from the effect of fermions.
If the preferred extremal property means generalization of holomorphy from 2-D case to 4-D case, one can conclude that the preferred extremals differ only at the singularities of space-time surfaces such as partonic orbits where the entire action comes into play. The regions outside the singularities would be universal: the minimal surface property would realize the 4-D generalization of the holomorphy.

3. Could different choices of the classical action, which determine the expressions of the classical and fermionic (quantal) Noether charges in terms of the modified Dirac action, correspond to different gauge choices selecting singular surfaces, in particular the CP_2 type extremals differently?
The standard view would suggest that the change of the parameters of the action at the level of H corresponds to coupling constant evolution, which in the TGD framework is discrete and in terms of p-adic length scales. On the other hand, the existence of dual M^4 conformal invariance suggests that the coupling constant evolution at the level of M^8 is realized as "energy" evolution by using associativity as a dynamical principle. Can these two views be consistent?
Note that the discriminant of the polynomial P is proposed to correspond to the exponent of action [L115, L117, L127, L125, L120]. The discriminant should change if the action changes. Does this mean that the change of the (effective) action in the discrete coupling constant evolution changes the polynomial?

7.6 Birational maps as morphisms of cognitive structures

https://en.wikipedia.org/wiki/Birational$_g$*eometry* and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself.

1. In the TGD framework, the algebraic extensions E are defined by rational polynomials P at the level of M^8_c identifiable as complexified octonions. E defines a unique discretization for the number theoretically preferred coordinates of M^8_c by the condition that the M^8 coordinates have values in E: I call these discretizations cognitive representations. They make sense also in the extensions of p-adic number fields induced by E serving as correlates of cognition in TGD inspired theory of conscious experience. Birational maps respect the extension E associated with the cognitive representations and map cognitive representations to cognitive representation of same kind. They are clearly analogous to morphisms in category theory.
2. M^8-H duality [L82, L83, L134, L138] is a number theoretic analogue of momentum-position duality. M^8_c serves as the analog of momentum space and $H = M^4 \times CP_2$ as the analog of position space. $M^8 - H$ duality maps the 4-surface defined in M^8_c by number theoretic holography based on 3-D data to a 4-D space-time surface in H.
3. Should $M^8 - H$ duality respect the algebraic extension? If so, it would map the cognitive representation defined by points belonging to 4-D surface $Y^4 \subset M^8$ with the values of preferred coordinates in E to points of $M^4 \subset H$ with coordinate values in E. One could say that $M^8 - H$ duality respects the number theoretical character of cognitive representations. The precise meaning of this intuition is however far from clear.

There are also questions related to the choice of preferred coordinates in which the cognitive representation is defined.

1. Number theoretic constraints fix the preferred coordinates at M^8 side rather uniquely and this induces a preferred choice also on $M^4 \subset H$. For hyperbolic spaces (mass shells) a cognitive

explosion happens and a natural question whether cognitive explosion happens also for the light-like curves assignable to the partonic orbits. If the light-like curve is geodesic, the explosion indeed occurs. For more general light-like curves this is not the case always: could these more general light-like curves be related by a birational map to light-like geodesics?

2. At the H side one can also imagine besides standard Minkowski coordinates also other physically preferred choices of coordinates: are they also theoretically preferred? The notion of Hamilton-Jacobi structure [L131] suggests that in the case of M^4 Hamilton-Jacobi coordinates are very natural for the holomorphic realization of holography. If these are allowed, a natural condition would be that the Hamilton-Jacobi coordinates are related to each other by birational maps mapping the point of E to points of E so that cognitive representations are mapped to cognitive representations.

7.6.1 M^8-H duality, holography as holomorphy, Hamilton-Jacobi structures, and birational maps as cognitive morphisms

In the sequel the questions raised in the introduction are considered. The basic notions are M^8-H duality [L82, L83, L134, L138], holography as a generalized holomorphy [L129, L137], Hamilton-Jacobi structures [L131], and birational maps as cognitive morphisms.

About more precise definitions of the basic concepts

Consider first more precise definitions of various notions involved.

1. What are the preferred coordinates of M^8_c in which the cognitive representation is constructed? M^8_c has a number theoretic interpretation in terms of complexified octonions and physical interpretation as 8-D momentum space. Linear Minkowski coordinates are number-theoretically preferred since octonionic multiplication and other arithmetic operations have a very simple form in these coordinates. Also the number theoretic automorphisms respect the arithmetic operations. The allowed automorphisms correspond to the group G_2 which is a subgroup of $SO(1,7)$. Physically Minkowski space coordinates are preferred coordinates in the momentum space and also in $M^4 \subset H$.
2. How the algebraic extension of rationals, call it E, is determined? The proposal is that rational polynomials characterize partially the 3-D data for number theoretic holography [L134]. The roots of a rational polynomial P define an algebraic extension of rationals, call it E. A stronger, physically motivated, condition on P is that its coefficients are integers smaller than the degree of P.
   The roots of P define mass shells $H^3_c \subset M^4_c \subset M^8_c$, which in turn assign to the mass shells a 4-D surface Y^4 of M^8_c going through the mass shells by associative holography requiring that the normal space of Y^4 is associative, that is quaternionic. It has been be assumed that the roots are complex although also the condition that the roots are real can be considered. The imaginary unit i associated with the roots commutes with the octonionic imaginary units.
3. How the cognitive representation is defined? The points of $Y^4 \subset M^8_c$ with M^4 coordinates in E define a unique discretization of Y^4, called a cognitive representation, making sense also in the extensions of p-adic number fields induced by E. In general, the number of algebraic points in the interior of Y^4 is discrete and even finite but at the mass shells H^3 a cognitive explosion takes place. All points of H^3 with coordinates in E are algebraic.
   The algebraic points with coordinates, which are algebraic *integers* are physically and cognitively in very special role in number theoretic physics and make sense also as points of various p-adic number fields making possible number theoretical universality. The points of H^3 have interpretation as momenta and for physical states the total momentum as sum of momenta at mass shells defined by the roots of P has components which are integers, called Galois confinement [L115, L116], would define fundamental mechanism for the formation of bound states.
4. M^8-H duality maps the points of $H^3_c \subset M^4_c \subset M^8_c$ to points of $H^3 \subset M^4 \subset M^4 \times CP_2 = H$ by a map, which is essentially an inversion: this form is motivated by Uncertainty Principle: for the most recent formulation of the duality see [L138]. This map is a birational map and takes points of E points of E. Also the points of cognitive representation belonging to the

interior of $Y^4 \subset M_c^8$ are mapped to the interior of $X^4 \subset M_c^8$. One can ask whether the discrete set of points of cognitive representations in the interiors are of special physical interest, say having interpretation as interaction vertices.

Questions to be pondered

There are many questions to be considered.

1. Also partonic orbits in $X^4 \subset H$ define 3-D holographic data in H. What are these partonic orbits? The simplest partonic orbits have light-like M^4 projection but also more general light-like H projection can be considered (note the analogy with a 2-D rigid body rotating along a light-like geodesic of H). A general light-like geodesic of H is a combination of time-like geodesic of M^4 and space-like geodesic of CP_2.
   A point of the light-like partonic orbit correspond at the level of M^8 to the 3-D blowup of a point of M^8 at which the quaternionic normal space parametrized by CP_2 point is not unique so that the normal spaces for a 3-D section of CP_2, whose union along (probably light-like) geodesic is CP_2 with two holes corresponding to the ends of the partonic orbit. This singularity is highly analogous to the singularity of the electric field of a point charge. Partonic orbits define part of the 3-D holographic data.
2. Could one associate cognitive representations also to the partonic orbits? Could also partonic orbits allow a cognitive explosion? The simplest way to guarantee light-likeness for the H projection is as a light-like geodesic and this indeed allows an infinite number of algebraic points in Minkowski coordinates. Same applies to more general light-like orbits. One would have at least 1-D explosion of the cognitive representation.
3. What can one say about the CP_2 and M^4 projections of the partonic 2-surface? Could also these projections to X^2 and Y^2 allow an infinite number of points with coordinates in E or do these kinds of points have some special physical meaning, say as vertices for particle reactions? Concerning cognitive representation, the blow-up would mean that the point has an infinite but discrete set of quaternionic normal spaces at the level of M^8. Since the partonic surface can have an arbitrary complex sub-manifold as CP_2, there is indeed information to be cognitively represented.

The most general cognitively preferred coordinate choices for space-time surfaces and H?

In the case of M_c^8, number theoretical considerations fix the preferred coordinates highly uniquely. In the case of H the situation is not so obvious and one cannot exclude alternative coordinate choices related by a birational map.

A possible motivation comes from the following argument.

1. String world sheets are candidates for singularities analogous to partonic orbits. At a given point of the string world sheet a blow up to a 2-D complex sub-manifold of CP_2 would occur. This would mean that the normal spaces of the point in M_c^8 form this sub-manifold. Cosmic strings are the simplest objcts of this kind. Monopole flux tubes are deformations of the cosmic strings and allow also an interpretation in terms of maps from M^4 to CP_2.
   If string world sheets define part of the data needed to define holography, one could argue that it makes sense to assign cognitive explosion to the string world sheet.
2. Cognitive explosion takes place if the string world sheets are 2-D geodesic submanifolds of H. Planes $M^2 \subset M^4$ represent the simplest example. A more complex example is obtained by taking a space-like geodesic in H and rotating it along a time-like geodesic of H. One can also take a light-like geodesic in H and rotate it along a light-like geodesic in dual light-like direction (ruler surface would be in question). In which case the gluing of the string world sheet along the boundary to the partonic orbit could be possible.
   One might perhaps think of building string world sheets by gluing these kinds of ultrasimple regions along their boundaries so that one would have edges. An interpretation as a discretization would be appropriate. One might even go further and argue that the cognitive

explosion explains why we are able to think of these kinds of regions in terms of simple formulas. One might argue that number theoretic physics realizes exactly what is usually regarded as approximation. One can however wonder whether life is so simple.

This argument encourages to consider a more complex option allowing more general string world sheets.

1. In the case of M^4 projection, the notion of the Hamilton-Jacobi structure [L131], generalizing the notion of ordinary complex structure, is highly interesting in this respect. It involves a generalization of complex coordinates involving local decompositions $M^4(x) = M^2(x) \times E^2(x)$ of the 4-D tangent space of M^4. The integrable distribution of $E^2(x)$ corresponds to complex coordinates $(w, \overline{w}$ integrating to a partonic 2-surface whereas the integrable distribution of $M^2(x)$ to light-like coordinate pairs (u, v) integrating to a string world sheet in M^4.
   Cognitive representation mean that the discretized values of the Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ are in E. Hamilton-Jacobi structure generalizes also to the level of $X^4 \subset H$ and now Y^2 can also correspond to CP_2 projection as in the case of cosmic strings and magnetic flux tubes. Note that in TGD one can use a subset of H coordinates as coordinates of X^4.
2. The simplest assumption is that the 1-D parton orbit corresponds to a light-like geodesic but could one map light-like geodesics to more general light-like curves by a birational map? Hamilton-Jacobi structure gives rise to a pair of curved (u, v) of light-like coordinates: could it relate to the standard flat light-like coordinates of M^2 by a birational map? Could a birational map relate standard complex coordinates of E^2 to the pair $(w, \overline{w})$? Could one also consider more general birational maps of $M^4 \to M^4$? If so, the Hamilton-Jacobi structures would be related by maps respecting algebraic extensions and cognitive representations. This would give a very powerful constraint on the Hamilton-Jacobi structures.

In the case of CP_2, projective coordinates are group-theoretically highly unique and determined apart from color rotations. Could one require that the CP_2 projection Y^2 associated with the partonic 2-surface and cosmic string or magnetic flux tube involves cognitive explosion. Are the allowed M^4 and CP_2 projections related by birational maps? Note that color rotations are birational maps.

These considerations suggest the following speculative view.

1. $M^8 - H$ duality, when restricted to 3-D holographic data at both sides, is analogous to a birational map expressible in terms of rational functions and respects the number theoretical character of cognitive representations.
2. Cognitive explosion occurs for the holographic data (this is very natural from the information theoretic perspective): this includes also string world sheets. Hamilton-Jacobi structures in the same cognitive class, partially characterized by the extension E of rationals, are related by a birational map.
3. $M^8 - H$ duality maps the quaternionic normal spaces to points of CP_2 and is an example of a birational map in M^4 degrees of freedom. It is not however easy to guess how the number theoretic holography is realized explicitly and how the 4-surfaces in M^8 are mapped to holomorphic 4-surfaces in H.
4. An interesting additional aspect relates to the non-determinism of partonic orbits due to the non-determinism of the light-likeness condition deriving from the fact that the action is Chern-Simons-Kähler action. The deformation of the partonic orbit induces the deformation of time derivatives of H coordinates at the boundary of $\delta M^4_+ \times CP_2$ to guarantee that boundary conditions at the orbit are realized. This suggests a strong form of holography [L137]. Already the 3-surfaces at $\delta M^4_+ \times CP_2$ or partonic orbits would be enough as holographic data. This in turn suggests that the analog of birational cognitive correspondence between the holographic data at $\delta M^4_+ \times CP_2$ and at partonic orbits.

7.6.2 Appendix: Some facts about birational geometry

Birational geometry has as its morphisms birational maps: both the map and its inverse are expressible in terms of rational functions. The coefficients of polynomials appearing in rational functions are in the TGD framework rational. They map rationals to rationals and also numbers of given extension E of rationals to themselves (one can assign to each space-time region an extension defined by a polynomial).

Therefore birational maps map cognitive representations, defined as discretizations of the space-time surface such that the points have physically/number theoretically preferred coordinates in E, to cognitive representations. They therefore respect cognitive representations and are morphisms of cognition. They are also number-theoretically universal, making sense for all p-adic number fields and their extensions induced by E. This makes birational maps extremely interesting from the TGD point of view.

The following lists basic facts about birational geometry as I have understood them on the basis of Wikipedia articles about birational geometry and Enriques-Kodaira classification. I have added physics inspired associations with TGD.

Birational geometries are one central approach to algebraic geometry.

1. They provide classification of complex varieties to equivalence classes related by birational maps. The classification complex curves (real dimension 2) reduces to the classification of projective curves of projective space CP_n determined as zeros of a homogeneous polynomial. Complex surfaces (real dimension 4) are of obvious interest in TGD: now however the notion of complex structure is generalized and one has Hamilton-Jacobi structure.
2. In TGD, a generalization of complex surfaces of complex dimension 2 in the embedding space $H = M^4 \times CP_2$ of complex dimension 4 is considered. What is new is the presence of the Minkowski signature requiring a combination of hypercomplex and complex structures to the Hamilton-Jacobi structure. Note however the space-time surfaces also have counterparts in the Euclidean signature $E^4 \times CP_2$: whether this has a physical interpretation, remains an open question. Second representation is provided as 4-surfaces in the space M^8_c of complexified octonions and an attractive idea is that $M^8 - H$ duality corresponds to a birational mapping of cognitive representations to cognitive representations.
3. Every algebraic variety is birationally equivalent with a sub-variety of CP_n so that their classification reduces to the classification of projective varieties of CP_n defined in terms of homogeneous polynomials. $n = 2$ (4 real dimensions) is of special relevance from the TGD point of view. A variety is said to be rational if it is birationally equivalent to some projective variety: for instance CP_2 is rational.
4. A concrete example of birational equivalence is provided by stereographic projections of quadric hypersurfaces in n+1-D linear space. Let p be a point of quadric. The stereographic projection sends a point q of the quadric to the line going through p and q, that is a point of CP_n in the complex case. One can select one point on the line as its representative. Another exammple is provided by Möbius transformations representing Lorentz group as transformations of complex plane.

The notion of a minimal model is important.

1. The basic observation is that it is possible to eliminate or add singularities by using birational maps of the space in which the surface is defined to some other spaces, which can have a higher dimension. The zeros of a birational map can be used to eliminate singularities of the algebraic surface of dimension n by blowups replacing the singularity with CP_n. Poles in turn create singularities. Peaks and self-intersections are examples of singularities.
   The idea is to apply birational maps to find a birationally equivalent surface representation, which has no singularities. There is a very counter-intuitive formal description for this. For instance, complex curves of CP_2 have intersections since their sum of their real dimensions is 4. The same applies to 4-surfaces in H. My understanding is as follows: the blowup for CP_2 makes it possible to get rid of an intersection with intersection number 1. One can formally say that the blow up by gluing a CP_1 defines a curve which has negative intersection number -1.
2. In the TGD framework, wormhole contacts have the same metric and Kähler structure as CP_2 and light-like M^4 projection (or even H projection). They appear as blowups of singularities of 4-surfaces along a light-like curve of M^8. The union of the quaternionic/associative normal spaces along the curve is not a line of CP_2 but CP_2 itself with two holes corresponding to the ends of the light-like curve. The 3-D normal spaces at the points of the light-like curve are not unique and form a local slicing of CP_2 by 3-D surfaces. This is a Minkowskian analog of a blow-up for a point and also an analog of cut of analytic function.

The Italian school of algebraic geometry has developed a rather detailed classification of these surfaces. The main result is that every surface X is birational either to a product $\mathbb{P}^1 \times$

CforsomecurveCortoaminimalsurfaceY.Preferredextremalsareindeedminimalsurfacessothatspace−
timesurfacesmightdefine minimalmodels.Theabsenceofsingularities(typicallypeaksorself−intersections)characteriz

There are several birational invariants listed in the Wikipedia article. Many of them are rather technical in nature. The canonical bundle K_X for a variety of complex dimension n corresponds to n:th exterior power of complex cotangent bundle that is holomorphic n-forms. For space-time surfaces one would have $n = 2$ and holomorphic 2-forms.

1. Plurigenera corresponds to the dimensions for the vector space of global sections $H_0(X, K_X^d)$ for smooth projective varieties and are birational invariants. The global sections define global coordinates, which define birational maps to a projective space of this dimension.
2. Kodaira dimension measures the complexity of the variety and characterizes how fast the plurigenera increase. It has values $-\infty, 0, 1, ..n$ and has 4 values for space-time surfaces. The value $-\infty$ corresponds to the simplest situation and for $n = 2$ characterizes CP_2 which is rational and has vanishing plurigenera.
3. The dimensions for the spaces of global sections of the tensor powers of complex cotangent bundle (holomorphic 1-forms) define birational invariants. In particular, holomorphic forms of type $(p, 0)$ are birational invariants unlike the more general forms having type (p, q). Betti numbers are not in general birational invariants.
4. Fundamental group is birational invariant as is obvious from the blowup construction. Other homotopy groups are not birational invariants.
5. Gromow-Witten invariants are birational invariants. They are defined for pseudo-holomorphic curves (real dimension 2) in a symplectic manifold X. These invariants give the number of curves with a fixed genus and 2-homology class going through n marked points. Gromow-Witten invariants have also an interpretation as symplectic invariants characterizing the symplectic manifold X.

   In TGD, the application would be to partonic 2-surfaces of given genus g and homology charge (Kähler magnetic charge) representatable as holomorphic surfaces in $X = CP_2$ containing n marked points of CP_2 identifiable as the loci of fermions at the partonic 2-surface. This number would be of genuine interest in the calculation of scattering amplitudes.

What birational classification could mean in the TGD framework?

1. Holomorphic ansatz gives the space-time surfaces as Bohr orbits. Birational maps give new solutions from a given solution. It would be natural to organize the Bohr orbits to birational equivalence classes, which might be called cognitive equivalence classes. This should induce similar organization at the level of M^8_c.
2. An interesting possibility is that for certain space-time surfaces CP_2 coordinates can be expressed in terms of preferred M^4 coordinates using birational functions and vice versa. Cognitive representation in M^4 coordinates would be mapped to a cognitive representation in CP_2 coordinates.
3. The interpretation of $M^8 - H$ duality as a generalization of momentum position duality suggests information theoretic interpretation and the possibility that it could be seen as a cognitive/birational correspondence. This is indeed the case M^4 when one considers linear M^4 coordinates at both sides.
4. An intriguing question is whether the pair of hypercomplex and complex coordinates associated with the Hamilton-Jacobi structure could be regarded as cognitively acceptable coordinates. If Hamilton-Jacobi coordinates are cognitively acceptable, they should relate to linear M^4 coordinates by a birational correspondence so that $M^8 - H$ duality in its basic form could be replaced with its composition with a coordinate transformation from the linear M^4 coordinates to particular Hamilton-Jacobi coordinates. The color rotations in CP_2 in turn define birational correspondences between different choices of Eguchi-Hanson coordinates.

   If this picture makes sense, one could say that the entire holomorphic space-time surfaces, rather than only their intersections with mass shells H^3 and partonic orbits, correspond to cognitive explosions. This interpretation might make sense since holomorphy has a huge potential for generating information: it would make TGD exactly solvable.

Part II

RIEMANN ZETA AND PHYSICS

Chapter 8

Riemann Hypothesis and Physics

8.1 Introduction

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

8.1.1 Super-Conformal Invariance And Generalization Of Hilbert-Polya Hypothesis

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

8.1.2 Zero Energy Ontology And RH

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

8.1.3 Miscellaneous Ideas

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

Logarithmic waves for zeros of zeta as complex algebraic numbers?

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B2] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s]+iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2+iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-symplectic weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

These approaches reflect the evolution of the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ would code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time would be coded.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L13]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

8.2 General Vision

Quantum TGD has inspired several strategies of proof of the Riemann hypothesis. The first strategy is based on the modification of Hilbert Polya hypothesis by requiring that the physical system in question has super-conformal transformations as its symmetries. Second strategy is based on considerations based on TGD inspired quantum theory of cognition and a generalization of the number concept inspired by it. Together with some physical inputs one ends up to a hypothesis that Riemann Zeta is well defined in all number fields near its zeros provided finite-dimensional extensions of p-adic numbers are allowed. This hypothesis generalizes the earlier hypothesis assuming that the extensions are trivial or at most algebraic. Third strategy is based on, what I call, Universality Principle.

There are also strong physical motivations to say something explicit about the spectrum of zeros and here p-adicization program inspires the hypothesis the numbers q^{iy}, q prime, belong to a finite algebraic extension of p-adic number field R_p for every prime p. The findings about the correlations of the spectrum of zeros inspire very concrete hypothesis about the spectrum of zeros as a union of translates of the same basic spectrum and this hypothesis is supported by the physical identification of the zeros of Zeta as super-symplectic conformal weights.

8.2.1 Generalization Of The Number Concept And Riemann Hypothesis

The hypothesis about p-adic physics as physics of cognition leads to a generalization of the notion of number obtained by gluing reals and various p-adic number fields together along rational numbers common to all of them. This structure is visualizable as a book like structure with pages represented by the number fields and the rim of the book represented by rationals. Even this structure can be generalized by allowing all finite-dimensional extensions of p-adic numbers including also those containing transcendental numbers and performing similar identification. Kind of fractal book might serve as a visualization of this structure.

In TGD inspired theory of consciousness intentions are assumed to correspond to quantum jumps involving the transformation of p-adic space-time sheets to real ones. An intuitive expectation is p-adic and real space-time sheets to each other must have a maximum number of common rational points. The building of idealized model for this transformation leads to the problem of defining functions having Taylor series with rational coefficients and continuable to both real and p-adic functions from a subset of rational numbers (or points of space-time sheet with rational coordinates). In this manner one ends up with the hypothesis that p-adic space-time sheets correspond to finite-dimensional extensions of p-adic numbers, which can involve also transcendental numbers such as e. This leads to a series of number theoretic conjectures.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B2] suggesting strongly that e and its $p-1$ powers at least should belong to extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s]+iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. A hierarchy of number theoretical conjectures stating that a finite number of iterated logarithms about transcendentals appearing in the extension forms a closed system under the operation of taking logarithms. Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the complex numbers p^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of p and any zero $1/2+iy$ of ζ.

8.2.2 Modified Form Of Hilbert-Polya Hypothesis

Super-conformal invariance inspires a strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of super-conformal algebra. The eigenfunctions of D^+ are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line and possibly those having $Re[s] > 1/2$.

A possible proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the states corresponding to zeros of ζ span a space with a hermitian metric. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal invariance in appropriate sense implies Riemann hypothesis. Indeed, a rather rigorous proof of Riemann hypothesis results from the observation that certain generator of conformal algebra permutes the two zeros located symmetrically with respect to the critical line. If the action of this generator exponentiates, Riemann hypothesis follows since exponentiation would imply the existence of infinite number of zeros along a line parallel to $Re[s]$-axis. One can formulate this argument rigorously using first order differential equation, and if one forgets all the preceding refined philosophical arguments, one can prove Riemann hypothesis using twenty line long analytic argument! Perhaps Ramajunan could have made this!

As already noticed, the state space metric can be made positive definite provided Riemann hypothesis holds true. Thus the system in question might quite well serve as a concrete physical model for quantum critical systems possessing super-conformal invariance as both dynamical and gauge symmetry.

8.2.3 Riemann Hypothesis In Zero Energy Ontology

Zeta reduces to a product $\zeta(s) = \prod_p Z_p(s)$ of partition functions $Z_p(s) = 1/[1-p^{-s}]$ over particles labelled by primes p. This relates very closely also to infinite primes and one can talk about Riemann gas with particle momenta/energies given by $log(p)$. s is in general complex number and for the zeros of the zeta one has $s = 1/2+iy$. The imaginary part y is non-rational number. At $s = 1$ zeta diverges and for $Re(s) \leq 1$ the definition of zeta as product fails. Physicist would interpret this as a phase transition taking place at the critical line $s = 1$ so that one cannot anymore talk about Riemann gas. Should one talk about Riemann liquid? Or - anticipating what follows- about quantum liquid? What the vanishing of zeta could mean physically? Certainly the thermodynamical interpretation as sum of something interpretable as thermodynamical probabilities apart from normalization fails.

The basic problem with this interpretation is that it is only formal since the temperature parameter is complex. How could one overcome this problem?

A possible answer emerged as I read the interview.

1. One could interpret zeta function in the framework of TGD - or rather in zero energy ontology (ZEO) - in terms of square root of thermodynamics! This would make possible the complex analog of temperature. Thermodynamical probabilities would be replaced with probability amplitudes.
2. Thermodynamical probabilities would be replaced with complex probability amplitudes, and Riemann zeta would be the analog of vacuum functional of TGD which is product of exponent of Kähler function - Kähler action for Euclidian regions of space-time surface - and exponent of imaginary Kähler action coming from Minkowskian regions of space-time surface and

defining Morse function. In QFT picture taking into account only the Minkowskian regions of space-time would have only the exponent of this Morse function: the problem is that path integral does not exist mathematically. In thermodynamics picture taking into account only the Euclidian regions of space-time one would only the exponent of Kähler function and would lose interference effects fundamental for QFT type systems. In quantum TGD both Kähler and Morse are present. With rather general assumptions the imaginary part and real part of exponent of vacuum functional are proportional to each other and to sum over the values of Chern-Simons action for 3-D wormhole throats and for space-like 3-surfaces at the ends of CD. This is non-trivial.

3. Zeros of zeta would in this case correspond to a situation in which the integral of the vacuum functional over the "world of classical worlds" (WCW) vanishes. The pole of ζ at $s = 1$ would correspond to divergence fo the integral for the modulus squared of Kähler function.

What the vanishing of the zeta could mean if one accepts the interpretation quantum theory as a square root of thermodynamics?

1. What could the infinite value of zeta at $s = 1$ mean? The The interpretation in terms of square root of thermodynamics implied following. In zero energy ontology zeta function function decomposition to $\prod_p Z_p$ corresponds to a product of single particle partition functions for which one can assigns probabilities $p^{-s}/Z_p(s)$ to single particle states. This does not make sense physically for complex values of s.
2. In ZEO one can however assume that the complex number p^{-sn} define the entanglement coefficients for positive and negative energy states with energies $nlog(p)$ and -$nlog(p)$: n bosons with energy log(p) just as for black body radiation. The sum over amplitudes over over all combinations of these states with some bosons labelled by primes p gives Riemann zeta which vanishes at critical line if RH holds.
3. One can also look for the values of thermodynamical probabilities given by $|p^{-ns}|^2 = p{-}n$ at critical line irrespective of zero. The sum over these gives for given p the factor $p/(p-1)$ and the product of all these factors gives $\zeta(1) = \infty$. Thermodynamical partition function diverges. The physical interpretation is in terms of Bose-Einstein condensation.
4. What the vanishing of the trace for the matrix coding for zeros of zeta defined by the amplitudes is physically analogous to the statement $\int \Psi dV = 0$ and is indeed true for many systems such as hydrogen atom. But what this means? Does it say that the zero energy state is orthogonal to vacuum state defined by unit matrix between positive and negative energy states? In any case, zeros and the pole of zeta would be aspects of one and same thing in this interpretation. This is an something genuinely new and an encouraging sign. Note that in TGD based proposal for a strategy for proving Riemann hypothesis, similar condition states that coherent state is orthogonal to "false" tachyonic vacuum.
5. RH would state in this framework that all zeros of ζ correspond to zero energy states for which thermodynamical partition function diverges. Another manner to say this is that the system is critical. (Maximal) Quantum Criticality is indeed the key postulate about TGD Universe and fixes the Kähler coupling strength characterizing the theory uniquely (plus possible other free parameters). Quantum Criticality guarantees that the Universe is maximally complex. Physics as generalized number theory would suggest that also number theory is quantum critical! When the sum over numbers proportional to probabilities diverges, the probabilities are considerably different from zero for infinite number of states. At criticality the presence of fluctuations in all scales implying fractality indeed implies this. A more precise interpretation is in terms of Bose-Eisntein condensation.
6. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does

not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

8.3 Riemann Hypothesis And Super-Conformal Invariance

Hilbert and Polya [A151] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator H such that the eigenvalues of this operator correspond to the imaginary parts of the nontrivial zeros $z = x + iy$ of ζ. One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-hermitian operator D^+ contains the non-trivial zeros of ζ. The eigen states in question are eigen states of an annihilation operator type operator D^+ and analogous to the so called coherent states encountered in quantum physics [A154]. In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of ζ is constructed.

1. The construction relies crucially on the interpretation of the vanishing of ζ as an orthogonality condition in a hermitian metric which is is a priori more general than Hilbert space inner product.
2. Second basic element is the scaling invariance motivated by the belief that ζ is associated with a physical system which has super-conformal transformations [A153] as its symmetries.

The core elements of the construction are following.

1. All complex numbers are candidates for the eigenvalues of D^+ (formal hermitian conjugate of D) and genuine eigenvalues are selected by the requirement that the condition $D^\dagger = D^+$ holds true in the set of the genuine eigenfunctions. This condition is equivalent with the hermiticity of the metric defined by a function proportional to ζ.
2. The eigenvalues turn out to consist of $z = 0$ and the non-trivial zeros of ζ and only the eigenfunctions corresponding to the zeros with $Re[s] = 1/2$ define a subspace possessing a hermitian metric. The vanishing of ζ tells that the "physical" positive norm eigenfunctions (in general *not* orthogonal to each other), are orthogonal to the "un-physical" negative norm eigenfunction associated with the eigenvalue $z = 0$.

The proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the space $\mathcal{V}$ spanned by the states corresponding to the zeros of ζ inside the critical strip has a hermitian induced metric. Riemann hypothesis follows also from the requirement that the induced metric in the spaces subspaces $\mathcal{V}_s$ of $\mathcal{V}$ spanned by the states Ψ_s and $\Psi_{1-\bar{s}}$ does not possess negative eigenvalues: this condition is equivalent with the positive definiteness of the metric in $\mathcal{V}$. Conformal invariance in the sense of gauge invariance allows only the states belonging to $\mathcal{V}$. Riemann hypothesis follows also from a restricted form of a dynamical conformal invariance in $\mathcal{V}$. This allows the reduction of the proof to a standard analytic argument used in Lie-group theory.

8.3.1 Modified Form Of The Hilbert-Polya Conjecture

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the hermiticity of the Hilbert-Polya operator. This means introduction of the non-hermitian operators D^+ and D which are hermitian conjugates of each other such that D^+ has the nontrivial zeros of ζ as its complex eigenvalues

$$D^+\Psi = z\Psi. \tag{8.3.1}$$

The counterparts of the so called coherent states [A154] are in question and the eigenfunctions of D^+ are not expected to be orthogonal in general. The following construction is based on the idea that D^+ also allows the eigenvalue $z = 0$ and that the vanishing of ζ at z expresses the orthogonality of the states with eigenvalue $z = x + iy \neq 0$ and the state with eigenvalue $z = 0$ which turns out to have a negative norm.

The trial

$$D = L_0 + V, \quad D^+ = -L_0 + V$$
$$L_0 = t\frac{d}{dt}, \qquad V = \frac{dlog(F)}{d(log(t))} = t\frac{dF}{dt}\frac{1}{F} \tag{8.3.2}$$

is motivated by the requirement of invariance with respect to scalings $t \to \lambda t$ and $F \to \lambda F$. The range of variation for the variable t consists of non-negative real numbers $t \geq 0$. The scaling invariance implying conformal invariance (Virasoro generator L_0 represents scaling which plays a fundamental role in the super-conformal theories [A153]) is motivated by the belief that ζ codes for the physics of a quantum critical system having, not only super-symmetries [A107], but also super-conformal transformations as its basic symmetries.

8.3.2 Formal Solution Of The Eigenvalue Equation For OperatorD^+

One can formally solve the eigenvalue equation

$$D^+\Psi_z = \left[-t\frac{d}{dt} + t\frac{dF}{dt}\frac{1}{F}\right]\Psi_z = z\Psi_z. \tag{8.3.3}$$

for D^+ by factoring the eigenfunction to a product:

$$\Psi_z = f_z F. \tag{8.3.4}$$

The substitution into the eigenvalue equation gives

$$L_0 f_z = t\frac{d}{dt}f_z = -zf_z \tag{8.3.5}$$

allowing as its solution the functions

$$f_z(t) = t^z. \tag{8.3.6}$$

These functions are nothing but eigenfunctions of the scaling operator L_0 of the super-conformal algebra analogous to the eigen states of a translation operator. A priori all complex numbers z are candidates for the eigenvalues of D^+ and one must select the genuine eigenvalues by applying the requirement $D^\dagger = D^+$ in the space spanned by the genuine eigenfunctions.

It must be emphasized that Ψ_z is *not* an eigenfunction of D. Indeed, one has

$$D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z. \tag{8.3.7}$$

This is in accordance with the analogy with the coherent states which are eigen states of annihilation operator but not those of creation operator.

8.4 Miscellaneous Ideas About Riemann Hypothesis

This section contains ideas about Riemann hypothesis which I regard as miscellaneous. I took them rather seriously for about more than decade ago but seeing them now makes me blush. I do not however have heart to throw away all these pieces of text away so that "miscellaneous" is a good attribute serving as a warning for the reader.

8.4.1 Universality Principle

The function, what I call $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-s})$ appearing in the product formula have value in the algebraic extension. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in the algebraic extension of the rationals, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle generalizes the original sharpened form of the Riemann hypothesis: the real parts of the phases p^{-iy} are rational. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of Riemann hypothesis becomes however extremely implausible and one could consider the possibility of regarding Riemann Hypothesis as an axiom.

8.4.2 How To Understand Riemann Hypothesis

The considerations of the preceding subsection lead to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

Some approaches to RH

It is appropriate to list various approaches to RH that I have considered during years.

1. *Coherent state approach to RH*

In this approach (see the preprint in [L1] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [H1]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. *The approach based on number theoretical universality*

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \ .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by the units defined by infinite primes (to be discussed later) would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

1. At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short $(k < -1/2)$ or long distances $(k > -1/2)$.
2. Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.
3. The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

4. The approach based on zero energy ontology

The approach based on zero energy ontology is the newest one and generalizes the thermodynamical approach by replacing thermodynamics with its square root. The amplitudes p^s define quantities proportional to time-like entanglement coefficients between positive and negative energy parts of a zero energy state having opposite energies given by $\pm log(p)$. The hypothesis that the sum over moduli squared for the coefficients diverges states that the zero energy state is not normalizable and has a physical interpretation as a critical state representing Bose-Einstein condensation. The additional condition that zero of zeta is in question is analogous to the condition $\int \Psi dV = 0$ and should be given a better physical justification. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$.

Connection with the conjecture of Berry and Keating

The idea that the imaginary parts y for the zeros of Riemann zeta function correspond to eigenvalues of some Hermitian operator H is not new. Berry and Keating [A107] however proposed quite recently that the Hamilton in question is super-symmetric and given by

$$H = xp - \frac{i}{2} \ . \qquad (8.4.1)$$

Here the momentum operator p is defined as $p = -id/dx$ and x has non-negative real values.

H can be indeed expressed as a square $H = Q^2$ of a Hermitian super symmetry generator Q:

$$\begin{aligned} Q &= \sqrt{i}\,[ix\sigma_1 + p\sigma_2] + \sqrt{\frac{i}{2}}\sigma_3 \ , \\ \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \\ \sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \end{aligned} \tag{8.4.2}$$

By a direct calculation one finds that the following relationship holds true:

$$Q^2 = \begin{pmatrix} xp + \frac{i}{2} & 0 \\ 0 & xp - \frac{i}{2} \end{pmatrix} \ .$$

The eigen spinors of Q can be written as

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = x^{-iy} \begin{pmatrix} x^{1/2} \\ \sqrt{\frac{y}{i}}x^{-1/2} \end{pmatrix} \ .$$

The eigenvalues of Q are $q = \sqrt{y}$. For $y \geq 0$ the eigenvalues are real so that Q is Hermitian when inner product is defined appropriately. Obviously y is eigenvalue of Hamiltonian.

Orthogonality requirement for the solutions of the Dirac equation requires that the inner product reduces to the inner product for plane waves $exp(iu)$, $u = log(x)$. This is achieved if inner product for spinors $\psi_i = (u_i, v_i)$ is defined as

$$\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{dx}{x} \left[\overline{u}_1 v_2 + \overline{v}_1 u_2 \right] \ . \tag{8.4.3}$$

In the basis formed by solutions of Dirac equation this inner product is indeed positive definite as one finds by a direct calculation.

The actual spectrum assumed to give the zeros of the Riemann Zeta function however remains open without additional hypothesis. An attractive hypothesis motivated by previous considerations is that the sharpened form of Riemann hypothesis stating that n^{iy} exists for any number field provided finite-dimensional extensions are allowed for the zeros of Riemann zeta function, holds true. This implies that x^{iy} satisfies the same condition for any rational value of x. $x^{\pm 1/2}$ in turn belongs to the infinite-dimensional algebraic extension Q_C^∞ of complex rationals, when x is rational. Therefore the solutions of Dirac equation, being of form $x^{iy}x^{\pm 1/2}$, exist for all number fields for rational values of argument x.

Connection with arithmetic quantum field theory and quantization of time

There is also a very interesting connection with arithmetic quantum field theory and sharpened form of Riemann hypothesis. The Hamiltonian for a bosonic/fermionic arithmetic quantum field theory is given by

$$H = \sum_p log(p) a_p^\dagger a_p \ . \tag{8.4.4}$$

where $a_p^\dagger$ and a_p satisfy standard bosonic/fermionic anti-commutation relations

$$\{a^\dagger_{p_1}, a_{p_2}\}_\pm = \delta(p_1, p_2) \ . \tag{8.4.5}$$

Here $\pm$ refers to anti-commutator/commutator. The sum of Hamiltonians defines super-symmetric arithmetic QFT. The states of the bosonic QFT are in one-one correspondence with non-negative integers and the decomposition of a non-negative integer to powers or prime corresponds to the decomposition of state to many boson states corresponding to various modes p. Analogous statement holds true for fermionic QFT.

The matrix element for the time development operator $U(t) \equiv exp(iHt)$ between states $|m\rangle$ and $|n\rangle$ can be written as

$$\langle m|U(t)|n\rangle \quad = \quad \delta(m,n)n^{it} \ . \tag{8.4.6}$$

Same form holds true both in bosonic and fermionic QFT: s. These matrix elements are defined for all number fields allowing finite-dimensional extensions if this holds true for n^{it} so that the allowed values of t corresponds to zeros of Riemann Zeta if one accepts Universality Principle. Similar statement holds in the case of fermionic QFT. One can say that the durations for the time evolutions are quantized in a well defined sense and allowed values of time coordinate correspond to the zeros of Riemann zeta function!

The result is very interesting from the point of view of quantum TGD since it would mean that $U(t)$ allows for the preferred values of the time parameter p-adicization ($p \ mod \ 4 = 3$) obtained by mapping the diagonal phases to their p-adic counterparts by phase preserving canonical identification. For phases this map means only the re-interpretation of the rational phase factor as a complexified p-adic number. For these quantized values of the time parameter time evolution operator of the arithmetic quantum field theory makes sense in all p-adic number fields besides complex numbers.

In the case of Berry's super-symmetric Hamiltonian the assumption that p^{iy} exists in all number fields with finite extensions allowed and the requirement that same holds true for the time evolution operator implies that allowed time durations for time evolution are given by $t = log(n)$. This means that there is nice duality between Berry's theory and arithmetic QFT. The allowed time durations (energies) in Berry's theory correspond to energies (allowed time durations) in arithmetic QFT.

8.4.3 Stronger Variants For The Sharpened Form Of The RiemannHypothesis

The previous form of the sharpened form of Riemann hypothesis was preceded by conjectures, which were much stronger. The strongest variant of the sharpening is that the phases p^{iy} are complex rational numbers for all primes and for all zeros ζ. A weaker form assumes that these phases belong to the square root allowing infinite-dimensional extension of rationals. Although these conjectures are probably unrealistic, they deserve a brief discussion.

Could the phases p^{iy} exist as complex rationals for the zeros of ζ?

The set $z = n/2 + iy$, $n > 0$ such that p^{-iy} is Pythagorean phase, is the set in which both real Riemann zeta function and the p-adic counterparts of Z_p exist for $p \ mod \ 4 = 3$. They exists also for $p \ mod \ 4 = 1$, if one *defines* $exp(ix) \equiv cos(x) + \sqrt{-1}sin(x)$: $\sqrt{-1}$ would be ordinary p-adic number for $p \ mod \ 4 = 1$. One could also allow phase factors in square root allowing algebraic extension of p-adics.

What is important that $x = 1/2$ is the smallest value of x for which the p-adic counterpart of $Z_B(p, x_p)$ exists. Already Riemann showed that the nontrivial zeros of Riemann Zeta function lie symmetrically around the line $x = 1/2$ in the interval $0 \leq x \leq 1$.

If one assumes that the zeros of Riemann zeta belong to the set at which the p-adic counterparts of Riemann zeta are defined, Riemann hypothesis follows in sharpened form.

1. Sharpened form of Riemann hypothesis does not necessarily exclude zeros with $x = 0$ or $x = 1$ as zeros of Riemann zeta unless they are explicitly excluded. It is however known that the lines $x = 0$ and $x = 1$ do not contains zeros of Riemann Zeta so that sharpened form implies also Riemann hypothesis.
2. The sharpening of the Riemann hypothesis following from p-adic considerations implies that the phases p^{iy} exist as rational complex phases for all values of $p \, mod \, 4 = 3$ when y corresponds to a zero of Riemann Zeta. Obviously the rational phases p^{iy} form a group with respect to multiplication isomorphic with the group of integers in case that y does not vanish. The same is also true for the phases corresponding to integers continuing only powers of primes $p \, mod \, 4 = 3$ phase factor.
3. A stronger form of sharpened hypothesis is that all primes p and all integers are allowed. This would mean that each zero of the Riemann Zeta would generate naturally group isomorphic with the group of integers. Pythagorean phases form a group and should contain this group as a subgroup. It might be that very simple number theoretic considerations exclude this possibility. If not, one would have infinite number of conditions on each zero of Riemann function and much sharper form of Riemann hypothesis which could fix the zeros of Riemann zeta completely:
   The zeros of Riemann Zeta function lie on axis $x = 1/2$ and correspond to values of y such that the phase factor p^{iy} is rational complex number for all values of prime $p \, mod \, 4 = 3$ or perhaps even for all primes p.
   Of course, the proposed condition might be quite too strong. A milder condition is that $U_p(x_p)$ is rational for single value of p only: this would mean that the zeros of Riemann Zeta would correspond to Pythagorean angles labeled by primes. One can consider also the possibility that p^{iy} is rational for all y but for some primes only and that these preferred primes correspond to the p-adic primes characterizing the effective p-adic topologies realized in the physical world.
4. If this hypothesis is correct then each zero defines a subgroup of Pythagorean phases and also zeros have a natural group structure. Pythagorean phases contain an infinite number of subgroups generated by integer powers of phase. Each such subgroup has some number N of generators such that the subgroup is generated as products of these phases. From the fact that Pythagorean phases are in a one-one correspondence with rationals, it is obvious that there exists large number of subgroups of this kind. Every zero defines infinite number of Pythagorean phases and there are infinite number of zeros. The entire group generated by the phases is in one-one correspondence with the pairs (p, y).
5. If n^{iy} are rational numbers, there must exist embedding map f: $(n, y) \to (r, s)$ from the set of phases n^{iy} to Pythagorean phases characterized by rationals $q = r/s$:

   $$(r, s) = (f_1(n, y), f_2(n, y)) \ .$$

   The multiplication of Pythagorean phases corresponds to certain map g

   $$\begin{aligned} &(r_1, s_1) \circ (r_2, s_2) = [g_1(r_1, s_1; r_2, s_2), \ g_2(r_1, s_1; r_2, s_2)] \\ &= (r_1 r_2 - s_1 s_2, r_1 s_2 + r_2 s_1) \equiv (r, s) \end{aligned}$$

   such that the values of r and s associated with the product can be calculated. Thus the product operation rise to functional equations giving constraints on the functional form of the map f.
   i) Multiplication of n^{iy_1} and n^{iy_2} gives rise to a condition

   $$f(n, y_1) \circ f(n, y_2) = f(n, y_1 + y_2) \ .$$

   ii) Multiplication of n_1^{iy} and n_2^{iy} gives rise to a condition

   $$f(n_1, y) \circ f(n_2, y) = f(n_1 n_2, y) \ .$$

This variant of the sharpened form of the Riemann hypothesis has turned out to be unnecessarily strong. Universality Principle requires only that the real parts of the factors $p^{-x}p^{-iy}$ are rational numbers: this means that allowed phases correspond to triangles whose two sides have integer-valued length squared whereas the third side has integer-valued length.

Sharpened form of Riemann hypothesis and infinite-dimensional algebraic extension of rationals

The proposed variant for the sharpened form of Riemann hypothesis states that the zeros of Riemann zeta are on the line $x = 1/2$ and that p^{iy}, where p is prime, are complex rational (Pythagorean) phases for zeros. Furthermore, Riemann hypothesis is equivalent with the corresponding statement for the fermionic partition function Z_F. If the sharpened form of Riemann hypothesis holds true, the value of $Z_F(z)$ in the set of zeros $z = 1/2 + iy$ of Z_F can be interpreted as a complex (vanishing) image of certain function $Z_F^{\infty}(1/2 + iy)$ having values in the infinite-dimensional algebraic extension of rationals defined by adding the square roots of all primes to the set of rational numbers.

1. The general element q of the infinite-dimensional extension Q_C^{∞} of complex rationals Q_C can be written as

$$\begin{aligned} q &= \sum_U q_U e_U \ , \\ e_U &= \prod_{i \in U} \sqrt{p}_i \ . \end{aligned} \tag{8.4.7}$$

Here q_U are complex rational numbers, U runs over the subsets of primes and e_U are the units of the algebraic extension analogous to the imaginary unit. One can map the elements of Q_C^{∞} to reals by interpreting the generating units $\sqrt{p}$ as real numbers. The real images $(e_U)_R$ of e_U are thus real numbers:

$$e_U \rightarrow [e_U]_R = \prod_i \sqrt{p_i} \ .$$

2. The value of $Z_F(z)$ at $z = 1/2 + iy$ can be written as

$$Z_F(z = 1/2 + iy) = \sum_U \left[\frac{1}{e_U}\right]_R \times (e_U^2)^{-iy} \ . \tag{8.4.8}$$

Here $(e_U)_R$ means that e_U are interpreted as real numbers.

3. If one restricts the set of values of $z = 1/2 + iy$ to such values of y that p^{iy} is complex rational for every value of p, then the value of $Z_F(1/2 + iy)$ can be also interpreted as the real image of the value of a function $Z_F(Q_\infty|z = 1/2 + iy)$ restricted to the set of zeros of Riemann zeta and having values at Q_C^{∞}:

$$\begin{aligned} Z_F(1/2 + iy) &= [Z_F(Q_\infty|1/2 + iy)]_R \ , \\ Z_F(Q_\infty|1/2 + iy) &\equiv \sum_U \frac{1}{e_U} \times (e_U^2)^{-iy} \ . \end{aligned} \tag{8.4.9}$$

Note that $Z_F(Q_\infty|z = 1/2 + iy)$ cannot vanish as element of Q_∞. One can also define the Q_C^{∞} valued counterparts of the partition functions $Z_F(p, 1/2 + iy)$

$$\begin{aligned} Z_F(Q_\infty|1/2 + iy) &= \prod_p Z_F(Q_\infty|p, z = 1/2 + iy) \ , \\ Z_F(Q_\infty|1/2 + iy) &\equiv 1 + p^{-1/2}p^{-iy} \ , \\ Z_F(p, 1/2 + iy) &= [Z_F(Q_\infty|p, 1/2 + iy)]_R \ . \end{aligned} \tag{8.4.10}$$

$Z_F(Q_\infty|1/2 + iy)$ and $Z_F(Q_\infty|p, 1/2 + iy)$ belong to Q_C^{∞} only provided p^{iy} is Pythagorean phase.

4. The requirement that p^{iy} is rational does not yet imply Riemann hypothesis. One can however strengthen this condition. The simplest condition is that the real image of $Z_F(Q_\infty|1/2 + iy)$ is complex rational number for any value of Z_F. A stronger condition is that the complex images of the functions

$$\frac{Z_F^{\infty}}{\prod_{p \in U} Z_p^{\infty}}$$

are complex rational and U is finite set of primes. The complex counterparts of these functions are given by

$$\left[\frac{Z_F^{\infty}}{\prod_{p\in U} Z_p^{\infty}}\right]_R = \frac{Z_F}{\prod_{p\in U} Z_F(p,..)} \quad . \tag{8.4.11}$$

Obviously these conditions can be true only provided that $Z_F(1/2+iy)$ vanishes identically for allowed values of y. This implies that sharpened form of Riemann hypothesis is true. "Physically" this means that the fermionic partition function restricted to any subset of integers not divisible by some finite set of primes, has real counterpart which is complex rational valued.

8.4.4 Are The Imaginary Parts Of The Zeros Of Zeta LinearlyIndependent Of Not?

Concerning the structure of the weight space of super-symplectic algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-symplectic algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the propagator does not exist mathematically for the entire super-symplectic algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned}
p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \quad , \\
e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \quad , \quad r = r(p,y) \quad , \quad s = s(p,y) \quad , \\
e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \quad , \quad m = m(p,y) \quad , \quad n = n(p,y) \quad .
\end{aligned} \tag{8.4.12}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1 \ mod\ 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \ , \ log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \ . \tag{8.4.13}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a, b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a, b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-symplectic algebra generated by generators labeled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi|q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of ways labeled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} = \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} \ . \tag{8.4.14}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \to p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \ . \tag{8.4.15}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the product of $U_q U_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \rightarrow U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1-q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{8.4.16}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)}\frac{\zeta'}{\zeta} = -\sum_q Q(q,p)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{8.4.17}$$

This function exists if the p-adic norms rational numbers $Q(q,p)$ approach to zero for $q \rightarrow \infty$: $|Q(q,p)|_p \rightarrow 0$ for $q \rightarrow \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q,p)$. The simplest guess would be $Q(q,p) \propto p^q$ for $q \rightarrow \infty$.

Correlation functions for the spectrum of zeros favors the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [A165]. They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

1. The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.
2. Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K + \Delta]$ and its mirror image $[-K - \Delta, -K]$ is the correlation function for the interval $[2K + \Delta]$ and depends only on the variable $2K + s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the smoothed out spectral density for the zeros given by

$$\langle\rho(k)\rangle = \frac{1}{2\pi}log(\frac{k}{2\pi}) \ . \tag{8.4.18}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi})-1) \ . \tag{8.4.19}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples q_iX_p, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{8.4.20}$$

the number $N_P(Y,p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi})-1) \ , \tag{8.4.21}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{8.4.22}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parameterization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

8.5 Could Local Zeta Functions Take The Role Of Riemann Zeta In TGD Framework?

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.
2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.
3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights or perhaps more naturally, to complex square roots of real conformal weights [K32]. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

It was also found that there are good reasons for expecting that the zetas in question should have only a finite number zeros. In the same section the self-referentiality hypothesis for ζ was proposed on basis of physical arguments. In this section (written before the emergence of self-referentiality hypothesis) the situation will be discussed from different view point.

8.5.1 Local Zeta Functions And Weil Conjectures

Riemann Zeta is not the only zeta [A1, A99]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A93] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A93].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

8.5.2 Local Zeta Functions And TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

Why local zeta functions could be natural in TGD framework?

In TGD framework Kähler-Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Ifthe Kähler-Dirac operator indeed assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the Kähler-Dirac operator and radial super-symplectic conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the mass squared assignable to the modes of the Kähler-Dirac operator, whose ground state part codes information about four-surface [K100] could in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the Kähler-Dirac operator and super-symplectic conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p,k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.
2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.
3. A rather fascinating self referentiality would result: the generalized eigen values of the Kähler-Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-symplectic conformal weight or a value of generalized eigenvalue of Kähler-Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

8.5.3 Galois Groups, Jones Inclusions, And Infinite Primes

Langlands program [K46, A126] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [A127]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW -spinor fields. One can also speak about WCW spinor s invariant under Galois group.
2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.
3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \rightarrow K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.
4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths $L_e(k) = \sqrt{5}L(k)$ associated with Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a way to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [A57]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.
2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L5] allows the embedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of embedding space.
3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of embedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L5].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L6] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the embedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the embedding of space-time surface to embedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

8.5.4 About Hurwitz Zetas

The action of modular group SL(2, Z) on Riemann zeta [A75] is induced by its action on theta function [A83]. The action of the generator $\tau \rightarrow -1/\tau$ on theta function is essential in providing the functional equation for Riemann Zeta. Usually the action of the generator $\tau \rightarrow \tau+1$ on Zeta is not considered explicitly. The surprise was that the action of the generator $\tau \rightarrow \tau+1$ on Riemann Zeta does not give back Riemann zeta but a more general function known as Hurwitz zeta $\zeta(s,z)$ for $z=1/2$. One finds that Hurwitz zetas for certain rational values of argument define in a well defined sense representations of fractional modular group to which quantum group can be assigned naturally. Could they allow to code the value of the quantum phase $q=exp(i2\pi/n)$ to the solution spectrum of the Kähler-Dirac operator D? As already shown the answer to this question is negative. Despite this Hurwitz zetas deserve a closer examination.

Definition

Hurwitz zeta is obtained by replacing integers m with $m+z$ in the defining sum formula for Riemann Zeta:

$$\zeta(s,z) = \sum_m (m+z)^{-s} \quad . \tag{8.5.1}$$

Riemann zeta results for $z=n$ apart from finite number of terms.
Hurwitz zeta obeys the following functional equation for rational $z=m/n$ of the second argument [A40]:

$$\zeta(1-s,\frac{m}{n}) = \frac{2\Gamma(s)}{2\pi n}^s \sum_{k=1}^{n} cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})\zeta(s,\frac{k}{n}) \quad . \tag{8.5.2}$$

The representation of Hurwitz zeta in terms of θ [A40] is given by the equation

$$\int_0^\infty \left[\theta(z,it)-1\right] t^{s/2}\frac{dt}{t} = \pi^{(1-s)/2}\Gamma(\frac{1-s}{2})\left[\zeta(1-s,z)+\zeta(1-s,1-z)\right] \quad . \tag{8.5.3}$$

By the periodicity of theta function this gives for $z=n$ Riemann zeta apart from finite number of terms.

The action of $\tau \rightarrow \tau + 1$ transforms $\zeta(s,0)$ to $\zeta(s,1/2)$

The action of the transformations $\tau \rightarrow \tau + 1$ on the integral representation of Riemann Zeta [A75] in terms of θ function [A83]

$$\theta(z;\tau) - 1 = 2\sum_{n=1}^{\infty}\left[exp(i\pi\tau)\right]^{n^2} cos(2\pi nz) \quad (8.5.4)$$

is given by

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^{\infty}\left[\theta(0;it) - 1\right] t^{s/2}\frac{dt}{t} \quad . \quad (8.5.5)$$

Using the first formula one finds that the shift $\tau = it \rightarrow \tau + 1$ in the argument θ induces the shift $\theta(0;\tau) \rightarrow \theta(1/2;\tau)$. Hence the result is Hurwitz zeta $\zeta(s,1/2)$. For $\tau \rightarrow \tau + 2$ one obtains Riemann Zeta.

Thus $\zeta(s,0)$ and $\zeta(s,1/2)$ behave like a doublet under modular transformations. Under the subgroup of modular group obtained by replacing $\tau \rightarrow \tau + 1$ with $\tau \rightarrow \tau + 2$ Riemann Zeta forms a singlet. The functional equation for Hurwitz zeta relates $\zeta(1-s,1/2)$ to $\zeta(s,1/2)$ and $\zeta(s,1) = \zeta(s,0)$ so that also now one obtains a doublet, which is not surprising since the functional equations directly reflects the modular transformation properties of theta functions. This doublet might be the proper object to study instead of singlet if one considers full modular invariance.

Hurwitz zetas form n-plets closed under the action of fractional modular group

The inspection of the functional equation for Hurwitz zeta given above demonstrates that $\zeta(s,m/n)$, $m = 0,1,...,n$, form in a well-defined sense an n-plet under fractional modular transformations obtained by using generators $\tau \rightarrow -1/\tau$ and $\tau \rightarrow \tau + 2/n$. The latter corresponds to the unimodular matrix $(a,b;c,d) = (1,2/n;0,1)$. These matrices obviously form a group. Note that Riemann zeta is always one member of the multiplet containing n Hurwitz zetas.

These observations bring in mind fractionization of quantum numbers, quantum groups corresponding to the quantum phase $q = exp(i2\pi/n)$, and the inclusions for hyper-finite factors of type II_1 partially characterized by these quantum phases. Fractional modular group obtained using generator $\tau \rightarrow \tau + 2/n$ and Hurwitz zetas $\zeta(s,k/n)$ could very naturally relate to these and related structures.

Acknowledgements

I am grateful for my son Timo for stimulating discussions relating to p-adic numbers. I would like also to thank Carlos Castro for interesting email conversations relating to Riemann Zeta and for bringing in my attention the possible connection with Golden Mean and for informing me about Hilbert-Polya hypothesis. I want to express my gratitude to Matthew Watkins for providing me information about connections between Golden Mean and Riemann Zeta and for generous help, in particular for pointing out several inaccuracies in the proof of Riemann hypothesis. Also Prof. Masud Chaichian and Doc. Claus Montonen are acknowledged for encouragement and help.

Chapter 9

Does Riemann Zeta Code for Generic Coupling Constant Evolution?

9.1 Introduction

During years I have made several attempts to understand coupling evolution in TGD framework.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K42]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.
   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K102] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking
2. p-Adic mass calculations for 2 decades ago [K50] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity [L54]. Needless to say these attempts were premature and a hoc.
3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [?] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

Quite recently the number theoretic interpretation of coupling constant evolution [K98] [L17] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K94]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.
Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.
The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s = 2$. The trivial poles for $s = 2n$, $n = 1, 2, ..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff} = n \times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s = 2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?
It turns out that at p-adic length scale $k = 131$ ($p \simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K98]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k = 127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w = w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `https://en.wikipedia.org/wiki/M\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{o\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\relax\let\ignorespaces\relax\accent127o\egroup\spacefactor\accent@spacefactorbius_transformation`) with real coefficients (element of $GL(2,R)$) so that one as $\zeta_F((as+b)/(cs+d))$. Rather general arguments force it to be and element of $GL(2,Q)$, $GL(2,Z)$ or maybe even $SL(2,Z)$ ($ad - bc = 1$) satisfying additional constraints. Since TGD predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2,Z)$ and by a scaling factor K.
Could one understand the general qualitative features of color and weak coupling contant

evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs + d$ and color confinement with the zero of $as + b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as + b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as + b/)(cs + d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p \simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a, b, c, d). In the sequel this vision is discussed in more detail.

9.2 Fermionic Zeta As Partition Function And Quantum Criticality

Riemann zeta has formal interpretation as a partition function $\zeta = Z_B = \prod 1/(1 - p^s)$ for a gas of bosons with energies coming as integer multiples of $log(p)$, for given mode labelled by prime p. I have proposed different interpretation based on the fermionic zeta ζ_F based on its representation as a product

$$\zeta_F = \prod_p (1 + p^s)$$

of single fermion partition functions associated with fermions with energy $log(p)$ (by Fermi statistics the fermion number is 0 or 1). In this framework the *poles* (not zeros!) of the fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ (the value of k turns out to be $k = 1/2$) (this identity is trivial to deduce) correspond to $s/2$, where s is either trivial or non-trivial zero of zeta (denominator), or the pole of zeta at $s = 1$ (numerator). Trivial poles are negative integers $s = -1 - 2, -3...$ suggesting an interpretation as conformal weights. This interpretation is proposed also for the nontrivial poles.

ζ_F emerges naturally in TGD, where the only fundamental (to be distinguished from elementary) particles are fermions. The assignment of physics to *poles* rather than zeros of ζ_F is also natural. The interpretation inspired by the structure of super-symplectic algebra is as conformal weights associated with the representations of extended super-conformal symmetry associated with super-symplectic algebra defining symmetries of TGD at the level of “World of Classical Worlds” (WCW).

“Conformal confinement” states that the sum of conformal weights of particles in given state is real. I discovered the idea for decade ago but gave it up to end up with it again. The fractal structure of superconformal algebra conforms with quantum criticality: infinite hierarchy of symmetry breakings to sub-symmetry isomorphic to original one! The conformal structure is infinitely richer than the ordinary one since the algebra in question has infinite number of generating elements labelled by all zeros of zeta rather than a handful of conformal weights ($n = -2, ... + 2$ for Virasoro algebra). Kind of Mandelbrot fractal is in question. There is however deviation from the ordinary conformal symmetry since real conformal weights can have only one sign (for generating elements all negative conformal weights $n = -1, -2, -.$ are realized as poles of $1/\zeta(2s)$ but $n = 1$ realized as pole of $\zeta(s)$ is the only positive conformal weight). Situation is therefore not quite identical with that in conformal field theories although also conformal field theories realizes only positive conformal weights (positivity is a convention) and have also some tachyonic conformal weights which are negative.

The problem of all attempts to interpret zeros of zeta relates to the fact that zeros are *not* purely imaginary but possess the troublesome real part $Re(s) = 1/2$. This led me to consider coherent states instead of eigenstates of Hamiltonian in my proposal, which I christened a strategy for proving Riemann hypothesis [K77], [L2]. Zeta has phase at the critical line so the interpretation as a partition function can be only formal. So called Z function defined at critical line and obtained by extracting the phase of zeta out, is real at critical line.

In TGD framework the solution of these problems is provided by zero energy ontology (ZEO). Quantum theory is "complex square root" of thermodynamics and means that partition function becomes a complex entity having also a phase. The well-known function

$$\xi(s) = \frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(s/2)(\zeta(s)$$

assignable to Riemann zeta having same zeros and basic symmetries has at critical line phase equal ± 1 except at zeros where the phase can be defined only as a limit depending the direction from which the zero is approached. Fermionic partion function $\zeta_F(s)$ has a complex phase and it is not clear whether it makes sense to assign with it the analog of $\xi(s)$. Ordinary partition function is modulus squared for the generalized partition function.

Why does the partition function interpretation does demand poles?

1. In ordinary thermodynamics the vanishing of partition function makes sense only at the limit of zero temperature when all Boltzmann weights approach to zero. By subtracting the energy of the lowest energy state from the energies the partition function becomes non-vanishing also in this case. Hence the idea that partition function vanishes does not look very attractive. The varying sign is even worse problem.
2. Since the temperature interpreted as $1/s$ in the partition function is not infinite could mean that one has analog of Hagedorn temperature (see `http://tinyurl.com/pvkbrum`): the degeneracy of states increases exponentially with temperature and at Hagedorn temperature compensates the s exponential decreases of Boltzmann weights so that partition function is sum of infinite number of terms approaching to unity. Hagedorn temperature relates by strong form of holography to magnetic flux tubes behaving as strings with infinite number of degrees of freedom. One would have quantum critical system possessing supersymplectic symmetry and other superconformal symmetries predicted by TGD [K24, K23, K91].
3. The temperature is complex for non-trivial zeros. This requires a generalization of thermodynamics by making partition function complex. Modulus squared of this function takes the role of an ordinary partition function. One can allow in the case of Kähler action the replacement of argument s with $ks + b$ without giving up the basic features of U(1) coupling constant evolution. Here one can allow rational numbers k and b. The inverse temperature for $\zeta_F(ks + b)$ is identified as $\beta = 1/T = k(s + b)$. It turns out that in the model for coupling constant evolution the scaling factor $k = 1/2$ is required. b is not completely fixed.
   Complex temperature is indeed the natural quantity to consider in ZEO. The real part of temperature at critical line equals to $Re(\beta) = (s+b)/4k$, with b rational or integer for $\zeta_F(w = k(s + b))$ at poles assignable with the zeros of $\zeta(2k(s + b))$ in denominator. Imaginary part
   $$Im\,[\beta] = \frac{1}{T} = \frac{1}{2k}(b + \ frac12 + iy) \tag{9.2.1}$$
   of the inverse temperature does not depend on b. Infinite number of critical temperatures is predicted and a discrete coupling constant evolution takes place already at the level of basic quantum TGD rather than emerging only at the QFT limit - I have also considered the possibility that coupling constant evolution emerges at the QFT limit only [L54]. One could even allow Möbius transformation with real coefficients in the argument of ζ_F and that this could allow the understanding of the evolutions of weak and colour coupling constants.
   $\zeta_F(w)$ at $s = -(n - b)/k$ are also present. For $s = 1/T$ they would correspond to negative temperatures $\beta = (-n + b)/k$? In the real context and for Hamiltonian with a fixed sign this looks weird. Preferred extremals can be however dominated by either electric or magnetic fields and the sign of the action density depends on this.
4. Interestingly, in p-adic thermodynamics p-adic temperatures has just the values $T = -1/n$ if one defines p-adic Boltzmann weight as $exp(-E/T) \rightarrow p^{-E/T}$, with $E = n \geq 0$ conformal weight. The condition that weight approaches zero requires that T identified in this is as real integer negative for p-adic thermodynamics! Trivial poles would correspond to p-adic

thermodynamics and non-trivial poles to ordinary real thermodynamics! Note that the earlier convention is that $T = 1/n$ is positive: the change of the sign is just a convention. Could the hierarchy of p-adic thermodynamics labelled by p-adic primes corresponds to the sequence of critical zeros of zeta? Number theoretic vision indeed leads to this proposal [L17], [K98]. The factor $1/(1-p^n)$ at the real poles $s = -2n$ would exist p-adically in p-adic number field Q_p so that the factors of zeta would correspond to adelic decomposition of the partition function. At critical line in turn $1/1+p^{1/2+iy}$ would exist for zeros y for which p^{iy} is root of unity (note that $p^{1/2}$ is somewhat problematic for Q_p: does it make sense to speak about an extension of Q_p containing *sqrtp* or is the extension just the same p-adic number field but with different definition of norm?). That p^{iy} is root of unity for some set $C(p)$ of zeros y associated with p was proposed in [L17], [K98]. Now $C(p)$ would consist of single zero $y = y(p)$.

9.2.1 Could The Spectrum Of Kähler Couplings Strength Correspond To Poles Of $\zeta_F(s/2)$?

The idea that the spectrum of conformal weights for supersymplectic algebra is given by the poles of ζ_F is not new [L17].

Poles of $\zeta_F(ks)$ ($k = /2$ turns out to be the correct choice) have also interpretation as complexified temperatures. Kähler action can be interpreted as a complexified partition function and the inverse $1/\alpha_K$ of Kähler coupling appears in the role of critical inverse temperature β. The original hypothesis was that Kähler coupling strength has only single value. The hierarchy of quantum criticalities and its assignment with number theoretical hierarchy of algebraic extensions of rationals led to consider the possibility that Kähler coupling strength has a spectrum corresponding to a hierarchy of critical temperatures. Quantum criticality and Hagedorn temperature for magnetic flux tubes as string like objects are indeed key elements of TGD.

The hypothesis to be studied is that the values $1/\alpha_K$ correspond to poles of

$$\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$$

with the identification $1/\alpha_K = ks$. The model for coupling constant evolution however favors $k = 1/2$ predicting that poles correspond to zeros of zeta in the denominator of ζ_F and $s = 2$ in its numerator. For $k = 1/2$ only even negative integers would appear in the spectrum and there would be pole at $s = 2$. Here one onr also allow the sift $ks \rightarrow ks + b$, b integer without shifting the imaginary parts of poles crucial for the coupling constant evolution. This induces a shift $Re[s] \rightarrow kRe[s] + b$ for the real parts of poles.

For nontrivial poles this requires the replacement of temperature with a complex temperature. Therefore also $1/\alpha_K$ becomes complex. This is just what the ZEO inspired idea about quantum theory as complex square root of thermodynamics suggests. Kähler action is also complex already for real values of $1/\alpha_K$ since Euclidian *resp.* Minkowskian regions give real/imaginary contribution to the Kähler action.

The poles of ζ_F would appear both as spectrum of complex critical temperatures $\beta = 1/T = 1/\alpha_K$ and as spectrum of supersymplectic conformal weights. ζ_F is complex along the critical line containing the complex poles. This makes sense only in ZEO. ξ function associated with ζ is real at critical line but the problems are vanishing at finite temperature, indefinite sign, and also the fact that partition function interpretation fails at positive real axis. This does not conform with the intuitive picture about partition function defined in terms of Boltzmann weights.

9.2.2 The Identification Of $1/\alpha_K$ As Inverse Temperature Identified As Pole Of ζ_F

Let us list the general assumptions of the model based on the identification of $1/\alpha_K$ as a complexified inverse temperature in turn identified as zero of ζ_F.

1. I have earlier considered the number theoretical vision based on the assumption that vacuum functional identified as exponent of Kähler action receiving real/imaginary contributions from Euclidian/Minkowskian space-time regions exists simultaneously in all number fields. This is in spirit with the idea of integrability meaning that functional integral reduces to a sum over exponents of Kähler action associated with stationary points. What is nice that by the Kähler

property of WCW metric Gaussian and metric determinants cancel [K42, K98] and one indeed obtains a discrete sum over exponentials making sense also in p-adic sectors, where ordinary integration does not make sense. Number theoretic universality is realized if one allows the extension of rationals containing also some roots of e if the exponent reduces to a product of root of unity and product of rational powers of e (e^p is ordinary p-adic number) and integer powers of primes p. It is perhaps needless to emphasize the importance of this result.
The criticism is obvious: how does one know, which preferred extremals have a number theoretically universal action exponent? For calculational purposes it might not be necessary to know this. The easy option would be that all preferred extremals are number theoretically universal: this cannot be however the case if the values of $1/\alpha_K$ correspond to zeros of ζ. Second option is that in the sum over preferred extremals those which do not have a number theoretically universal exponent give a vanishing net contribution and are effectively absent. The situation brings in mind the reduction of momentum spectrum of a particle in a box to momenta equal to $k = n2\pi/L$, L the length of the box. The contributions of other plane waves in integrals vanish since they are dropped away by boundary conditions.
Strong form of number theoretic universality requires that the exponent of Kähler action reduces to a product of rational power of some prime p or $e^{m/n}$ and a root of unity [K98], [L17]. This might be too strong a condition and weaker condition allows also powers of p mapped to real sector and vice versa by canonical identification. One could pose root of unity condition for the phase of $exp(S_K)$ as a boundary condition at the ends of causal diamond (CD) stating that some integer power of the exponent of Kähler action for the given value of α_K is real. If $exp(K)$ contains $e^{m/n}$ factor but no p^n factors, the reality of the n^{th} power of $exp(i\pi K)$ would reveal this. Single p^n factor in absence of $e^{m/n}$ factor could be detected by requiring that the exponent $exp(iyK)$ is real for some y (imaginary part of zero of zeta with p^{iy} a root of unity).

2. The assumption that $1/\alpha_K$ corresponds to a nontrivial zero of zeta has strong constraints on the values of the reduced Kähler action $S_{K,red} = \alpha_K S_K$ for which the classical field equations do not depend on α_K at all. The reason is that the S_K must be proposal to $1/\alpha_K$ to achieve number theoretical universality. Number theoretical universality thus implies that preferred extremals depend on $1/\alpha_K$ - this is something very quantal. The proportionality $1/\alpha_K$ to $h_{eff} = n \times h$ is highly suggestive. It does not destroy number theoretical universality for given preferred extremal.
3. $1/\alpha_K$ has form $1/\alpha_K = s = a+ib = (1/2k)(1/2+iy/2)$ for nontrivial poles, $1/\alpha_K = s = -n/k$ for trivial poles of $1/\zeta(2s)$, and $1/\alpha_K = s = 1/k$ for the pole of ζ. $k = 1/2$ is the physically preferred choice.
Kähler action can be written as a sum of Euclidian and Minkowskian contributions: $K = K_E + iK_M$. For non-trivial poles in the case of $1/\alpha_K = ks$ one has
$$K = s \times (K_E + iK_M) = \frac{1}{k} \times \left[\frac{K_E}{2} - yK_M + i(\frac{K_M}{2} + yK_E)\right] \quad . \tag{9.2.2}$$
Here $K_{red} = K_E + iK_M$ is *reduced* Kähler action. This option generalizes directly the original proposal.
4. For trivial poles $s = -n/k$ and $s = 1/k$ one has
$$K = \frac{s}{k} \times K_{red} = \frac{s}{k} \times (K_E + iK_M) \ . \tag{9.2.3}$$
5. For real poles universality holds true without additional conditions since the multiplication of $1/\alpha_K$ by the scaling factor $-n_2/n_1$ does not spoil number theoretical universality. One can of course consider this condition. It predicts that the K_{red} is scaled by n_1/n_2 in the transition $n_2 \to n_1$. For nontrivial poles K_{red} is scaled by the complex ratio s_2/s_1.
An attractive possibility is that the hierarchy of Planck constants corresponds to this RG evolution. n would correspond to the number of sheets in the n-sheeted covering for which sheets co-incide at the ends of space-time at the boundaries of CD. Therefore p-adic and $h_{eff} = n \times h$ hierarchies would find a natural interpretation in terms of zeros of ζ_F. To avoid confusion let us make clear that the values of $n = h_{eff}/h$ would not correspond to trivial poles.

Number theoretical universality could be realized in terms of RG invariance leaving the vacuum functional invariant but deforming the vacuum extremal. The hierarchy of Planck constants and p-adic length scale hierarchy could be interpreted as RG flows along real axis and critical line.

1. The grouping of poles to 4 RG orbits corresponding to non-trivial poles $y > 0$ and $y < 0$, to poles $s = -n/k < 0$, and $s = 1/k$ looks natural. The differential equations for RG evolution of Kähler action would be replaced with a difference equation relating the values of Kähler action for two subsequent critical poles of ζ_F.
2. Number theoretical universality allows to relate Minkowskian and Euclidian contributions K_M and K_E to each other. Earlier I have not even tried to deduce any correlation between them although the boundary conditions at light-like wormhole throats at which the signature of the induced metric changes, probably give strong constraints.
   The strongest form of the number theoretical universality condition assumes
   $$K_{red} = K_{red,E} + iK_{red,M} = \alpha_K K_1 = \frac{K_1}{s} = K(\alpha_K = 1) \ , \ s = \frac{1}{\alpha_K} \ . \qquad (9.2.4)$$
   K_1 satisfies the number theoretic universality meaning that $exp(K_1) = expK(\alpha_K = 1)$ reduces to a product of powers primes, root of e and root of unity.
   This ansatz has the very remarkable property that α_K disappears from the vacuum functional completely so that the RG action can be regarded as a symmetry leaving vacuum function invariant. This operation however changes the preferred extremal and reduced Kähler action so that the situation is non-classical. RG orbit would start from the pole $s = 1$ and contain complex poles.
3. The large CP breaking suggested by complexity of α_K would disappear at the level of vacuum functional and appears only at the level of preferred extremals. If this is to conform with the quantum classical correspondence, correlation functions, which must break CP symmetry receive this breaking from preferred extremals. $s = 1/2k$ and complex poles belong to the same orbit. This ansatz is not necessary for poles $s = 1/k$ and $s = -n/k$ for which number theoretic universality conditions are satisfied irrespective of the value of s.
4. A more realistic looking solution is obtained by assuming that complex poles correspond to separate orbit or even that positive and negative values of y correspond to separate orbits. RG flow would begin from the lowest zero of zeta at either side of real axis. This gives
   $$K_{red} = \frac{\alpha_K}{\alpha_{K,0}} \times K_{red}(\alpha_{K,0}) \ . \qquad (9.2.5)$$
   Also now the vacuum functional is invariant and preferred extremal changes in RG evolution. In accordance with quantum classical correspondence one has however a breaking of CP symmetry also at the level of vacuum functional due to the complexity of $\alpha_{K,0}$ unless $K_{red}(\alpha_{K,0})$ is proportional to $\alpha_{K,0}$.

Remark: The above arguments must be modified if one includes to the action cosmological volume term strongly suggested by twistor lift of TGD.

9.3 About Coupling Constant Evolution

p-Adic mass calculations inspired the hypothesis that the continuous coupling constant evolution in QFTs reduces in TGD framework to a discrete p-adic coupling constant evolution but assuming that α_K is absolute RG invariant. Therefore the hypothesis that the evolution of $1/\alpha_K$ defined by the non-trivial poles of ζ_F corresponds to the p-adic coupling constant evolution deserves a serious consideration.

1. p-Adic length scale hypothesis in the strong form states that primes $p \simeq 2^k$, k prime, correspond to physically preferred p-adic length scales. This would suggest that non-trivial zeros $s_1, s_2, s_3, ..$ taken in increasing order for magnitude correspond to primes $k = 2, 3, 5, 7...$ as suggested also in [L17], [K98]. This allows to assign to each zero s_n a unique prime: $p \leftrightarrow y(p)$ and this suggests more precise of the earlier hypothesis to state that $p^{iy(p)}$ is root of unity. The class of zeros associated with p would contain single zero.
   Discrete p-adic length scale evolution would thus correspond to the evolution of non-trivial zeros. The evolution associated with the hierarchy of Planck constants could only multiple Kähler action with integer. To make this more concrete one must consider detailed physical interpretation.

2. $1/\alpha_K$ corresponds to $U(1)$ coupling of standard model: $\alpha_K = \alpha(U(1)) \equiv 1/\alpha_1$. Kähler action could be seen as analogous to a Hamiltonian associated with electroweak U(1) symmetry. $U(1)$ gauge theory is not asymptotically free and this correspond to the fact that $Im(1/\alpha_K) = y$ approaches in UV to the lowest zero $y = 14.12...$ In IR y diverges, which conforms with $U(1)$ gauge theory symmetry.
Electromagnetic coupling corresponds to

$$\frac{1}{\alpha_{em}} = \frac{1}{\alpha_K cos^2(\theta_W)} \quad . \tag{9.3.1}$$

The challenge is to understand also the evolution of $cos^2(\theta_W)$ allowing in turn to understand the entire electroweak evolution.

3. The values of electroweak couplings at the length scale of electron ($k = 127$ or at 4 times longer length scale $k = 131$ ($L(131) = .1$ Angstrom) are well-known and this provides a killer test for the model. Depending on whether one assumes fine structure constant to correspond to $L(127)$ associated with electron or to 4 times long length scale $L(131)$ one has too options. $L(131)$ allows to reproduce fine structure constant with a value of $p = sin^2(\theta_W)$ deviating only .7 per cent from its measured value in this length scale! If this is not a mere nasty accident, Riemann zeta might code the entire electroweak physics and perhaps even strong interactions!
The first guess is that UV asymptotia for the Weinberg angle is same as for GUTS: $p \to 3/8$ for $p = 2$ giving $1/\alpha_{em} \to 22.61556016$. IR asymptotia corresponds to $p \to 0$ implying $1/\alpha_{em} = 1/\alpha_K$. Notice that the evolution is rather fast in extreme UV. In extreme IR it becomes slow. It turns out that the UV behavior of Weinberg angle does not conform with this naïve expectation.

4. Since p-adic length scale is proportional to $1/p^{1/2}$ it is enough to obtain RG evolution for coupling constnt as function of p. One obtains reasonably accurate understanding about the evolution by deducing an estimate for pdy/dp . This is obtained as $pdy/dp = (dy/dN)(dN/dk)p(dk/dp)$.
   - $p \simeq 2^k$ implies $k \simeq log(p)/log(2)$ and $pdk/dp \simeq 1/log(2)$.
   - The approximate formula for the number $N(y)$ of zeros smaller than y is given by

     $$N(y) \sim u \times log(u) \quad , \quad u = \frac{y}{2\pi}$$

     giving

     $$\frac{dN}{dy} \sim \frac{1}{2\pi} \times (log(u) - 1), \quad u = \frac{y}{2\pi} \quad .$$

   - The number $\pi(k)$ of primes smaller than k is given by

     $$N(k) \sim \frac{k}{log(k)}$$

     giving

     $$\frac{dN(y)}{dk} \sim \frac{1}{log(k)} - \frac{1}{log(k)^2} \quad .$$

   By combining the formulas, one obtains

   $$p\frac{dy}{dp} = \beta = \frac{2\pi}{log(2)} \times (\frac{1}{log(y/2\pi)} - 1) \times (\frac{1}{log(k)} - \frac{1}{log(k)^2}) \quad , \quad k = \frac{log(p)}{log(2)} \quad . \tag{9.3.2}$$

   The beta function for the evolution as function of p-adic length scale differs by factor 2 from this one. Note that also double logarithms appear in the formula. Note that beta function depends on y logarithmically making the equation rather nonlinear. This dependence can be shifted to the left hand side and by replacing y which appropriation chosen function of it one obtains

   $$p\frac{dN(y)}{dp} = \beta_1 = \frac{1}{log(k)} - \frac{1}{log(k)^2} \quad , \quad k = \frac{log(p)}{log(2)} \quad . \tag{9.3.3}$$

5. Coupling constant evolution would take place at the level of single space-time sheet. Observations involve averaging over space-time sheet sizes characterized by p-adic length scales so that a direct comparison with experimental facts is not quite easy and requires a concrete statistical model.

The entire electroweak $U(1)$ coupling constant evolution would be predicted exactly from number theory. Physics would represent mathematics rather than vice versa. Concerning experimental testing a couple of remarks are in order.

1. An open question is how much many-sheetedness of space-time affects situation: one expects kind of statistical average of say Weinberg angles over p-adic length scales coming from a superposition over space-time sheets of many-sheeted space-time. Space-time with single sheet is not easy to construct experimentally although mathematically it is extremely simple system as compared to the space-time of GRT.
2. The discreteness of the coupling constant evolution at fundamental level is one testable prediction. There is no continuous flow but sequence of phases with fixed point behavior with discrete phase transitions between them. At QFT limit one expects that continuous coupling constant evolution emerges is statistical average.
3. Later it will be found that the entire electroweak evolution can be predicted and this prediction is certainly testable.

9.3.1 General Description Of Coupling Strengths In Terms Of Complex Square Root Of Thermodynamics

The above picture is unsatisfactory in the sense that it says nothing about the evolution of other electroweak couplings and of color coupling strength. Does number theory fix also them rather than only $U(1)$ coupling? And what about color coupling strength α_s?

Here quantum TGD as a complex square root of thermodynamics vision helps.

1. Kähler action reduces for preferred extremals to Abelian Chern-Simons action localized at the ends of space-time surfaces at boundaries of causal diamond (CD) and possibly contains terms also at light-like orbits of partonic 2-surfaces. This corresponds to almost topological QFT property of TGD.
2. Kähler action contains additional boundary terms which serve as analogs for Lagrangian multipler terms fixing the numbers of various particles in thermodynamics. Now they fix the values of isometry charges for instance, or force the symplectic charges for a sub-algebra to vanish. Lagrangian multiplies can be written in the form μ_i/T in ordinary thermodynamics: μ_i denotes the chemical potentials assignable to particle of type i. Number theoretical universality strongly favors similar representation now. For instance, this would give

$$\frac{1}{\alpha_{em}} = \frac{\mu_{em}}{\alpha_K} \ , \ \mu_{em} = \frac{1}{cos^2(\theta_W)} \ . \tag{9.3.4}$$

In the same manner SU(2) coupling strength given by

$$\frac{1}{\alpha_W} = \frac{\mu_W}{\alpha_K} = \frac{cot^2(\theta_W)}{\alpha_K} \tag{9.3.5}$$

would define $cot^2(\theta_W)$ as analog of chemical potential.
3. In the case of weak interactions Chern-Simons term for induced SU(2) gauge potentials as a boundary term would be the analog of Kähler action having interpretation as Lagrangian multiplier term. In color degrees of freedom also an analog of Chern-Simons term would be in question and would be associated with the classical color gauge field defined by H_AJ, where H_A is Hamiltonian of color isometry in CP_2 and J is induced Kähler form.
4. The conditions for number theoretical universality would become more complex as also RG invariance interpreted in terms of number theoretical universality.

This picture assuming a linear relationship between generic coupling strength α and α_K in terms of chemical potential is not yet general enough.

9.3.2 Does ζ_F With $GL(2,Q)$ Transformed Argument Dictate The Evolution Of Other Couplings?

It seems that one cannot avoid dynamics totally. The dynamics at (quantum) criticality is however universal. This raises the hope that the evolution of coupling constant is universal and does not depend on the details of the dynamics at all. This could also explain the marvellous successes of QED and standard model

At criticality the dynamics reduces to conformal invariance by quantum criticality, and this inspires the idea about the values of coupling constant strength as poles of a meromorphic function obtained from ζ_F by a conformal transformation of the argument. After all, what one must understand is the relationship between $1/\alpha_W$ and $1/\alpha_K$, and the linear relationship between them can be seen as a simplifying assumption and an approximation.

The values of generic coupling strength - call it just α (to be not confused with α_{em}) without specifying the interaction - would still correspond to poles of $\zeta_F(s)$ but with a transformed argument s. Conformal transformation would relate various coupling constant evolutions to each other and allow to combine them together in a unique manner. Discreteness is of course absolutely essential. The analysis of the situation leads to a surprisingly simple picture about the coupling constant evolutions for weak and color coupling strengths.

1. By the symmetry of ζ_F under the reflection with respect to x-axis one can restrict the consideration to globally defined conformal transformations of the upper half plane identifiable as Möbius tranformations $w = (as + b)/(cs + d)$ with the real matrix coefficients (a, b, c, d). One can express the transformation as a product of an overall scaling by factor k and $GL(2,R)$ transformation with $ad - bc = 1$. Number theoretical universality demands that k and the coefficients a, b, c, d of $GL(2,R)$ matrix are real rationals. Number theoretically $GL(2,Q)$ is attractive and one can consider also the possibility that the transformation matrix $GL(2,Z)$ matrix with a, b, c, d integers. $SL(2,Z)$ is probably too restrictive option.
2. The Möbius transformation $w = (as + b)/(cs + d)$ acts on zeros of ζ mapping the discrete coupling constant evolution for $1/\alpha_K$ to that for $1/\alpha_W$ or $1/\alpha_s$. The transformed coupling constant depends logarithmically on p-adic length scale via $1/\alpha_K$ supporting the interpretation in terms of RG flow induced by that for $1/\alpha_K$ - something very natural since Kähler action is in special role in TGD framework since it determines the dynamics of preferred extremals.
3. Asymptotically (long length scales) one has $w \to a/c$ for $a \neq 0$ so that both at critical line and real axis one has accumulation of critical points to $w = a/c$! Thus for the option allowing only very large value of coupling strength in IR one has

$$w = K \times \frac{as + b}{cs + d} \quad , \quad ad - bc = 1 \quad \text{(Option 1)} \quad . \tag{9.3.6}$$

$a/c = 0$ ($a = 0$) corresponds to a diverging coupling strength (for color interactions and for weak interactions for vanishing Weinberg angle) and corresponds to $w = K \times b/cs + d$. $ad - bc = 1$ gives $b = -c = 1$ and if one accepts the IR divergence of coupling constant, one has

$$w = \frac{K}{-s + d} \quad \text{(Option 2)} \quad . \tag{9.3.7}$$

The only free parameters are the rational $K > 0$ and integer d. w has pole at $s = d$ mapped to 1 by ζ_F.

To gain physical insight consider the situation at real axes.

1. The real poles $s = -n/k$ and $s = 1/k$ are mapped to poles on real axes and the reflection symmetry with respect to x-axis is respected. Negative poles would be thus mapped to negative poles for $d \in 0, 1$ and $k < 0$. One could also require that the pole $s = 1$ is mapped to positive pole. For option 2 it is mapped to $w = +\infty$.
2. For option 1 this is true if one has $cs + d < 0$ and $as + b > 0$. The other manner to satisfy the conditions is $cs + d > 0$ and $as + b < 0$ for $s = -1, -2, ...$ By replacing the (a, b, c, d) with $(-a, -b, -c, -d)$ these conditions can be transformed to each other so that it is enough to consider the first conditions. The first form of the condition requires $c > 0$ and $a < 0$.
The condition that $s = 1/k$ goes to a positive pole gives $c/k+d > 0$ and $a/k+b > 0$. Altogether this gives for the two Options the conditions

$$\begin{aligned} w &= K\times \frac{as+b}{cs+d} < 0 \ , \\ k &> 0 \ , a<0 \ , \ c>0 \ , \ \frac{c}{k}+d>0 \ , \ \frac{a}{k}+b>0 \ . \ \text{(Option 1)} \ , \end{aligned} \tag{9.3.8}$$

and

$$w = \frac{K}{-s+\frac{1}{k}} < 0 \ , \ k>0 \ . \ \text{(Option 2)} \tag{9.3.9}$$

3. For option 2 $s = 1/k$ phase is mapped to $w = +\infty$. Coupling strength vanishes in this phase: this brings in mind the asymptotic freedom for QCD realized at extreme UV? In long scales α would behave like $1/\alpha_K$ and diverge suggesting that Option 2 provides at least an idealized description of QCD. The scaling parameter K would remain the only free parameter.
   For option 1 α can become arbitrary large in long scales but remains finite. The analog of asymptotically free phase is replaced with that having non-vanishing inverse coupling strength $w = (a+b)/(c+d)$. The interpretation could be in terms of weak coupling constant evolution. The non-vanishing of the parameter a would distinguish between weak and strong coupling constant evolution.
   By feeding in information about the evolution of weak and color coupling strengths, one can deduce information about the values of K and a.

Whether the analogs of weak and Chern-Simons actions can satisfy the number theoretical universality, when the transformation is non-linear is far from obvious since the induced gauge fields are not independent.

9.3.3 Questions About Coupling Constant Evolution

The simplest hypothesis conforming with the general form of Yang-Mills action is $1/\alpha_K = s$, where s is zero of zeta. With the identification $1/\alpha_K = 1/\alpha_{U(1)}$ this predicts the evolution of U(1) coupling and one obtains excellent prediction in p-adic length scale $k = 131$ ($L(131 \simeq 10^{-11}$ meters).

How general is the formula for $1/\alpha_K$?

Is the simplest linear form for $1/\alpha_K$ general enough? Consider first the most general form of $2\pi/\alpha_K$ taking as input the fact that its imaginary is equal to $1/\alpha_{U(1)}$ and corresponds to imaginary part y of zero of zeta at critical line.

Linear Möbius transformations $w = (as+b)/d$ with real coefficients do not affect $Im[s]$ and therfore the inverse of the imaginary part of the Kähler coupling strength which corresponds to th inverse of the measured U(1) coupling strength. The general formula for complex Kähler coupling strength would be

$$w = s + \frac{b}{d} \tag{9.3.10}$$

in the case of $SL(2,Q)$ giving $Re[1/\alpha_K] = 1/2 + b/d$. This would correspond to the analog of the inverse temperature appearing in the real exponent of Kähler function. For $SL(2,Z)$ on obtains

$$w = s + b \ , \ b \in Z \ . \tag{9.3.11}$$

This gives $Re[1/\alpha_K] = 1/2 + b$.

Does the reduction to Chern-Simons term give constraints

The coefficient of non-Abelian Chern-Simons action is quantized to integer and one can wonder whether this has any implications in TGD framework.

1. The Minkowskian term in Kähler action reduces to Abelian Chern-Simons term for Kähler action. In non-Abelian case the coefficient of Chern-Simons action (see `http://tinyurl.com/y7nfaj67`) is $k_1/4\pi$, where k_1 is integer.
   In Abelian case the triviality of gauge transformations does not give any condition on the phase factor so that in principle no conditions are obtained. One can however look what this condition gives. The coefficient of Chern-Simons term coming from in Kähler action is $1/(8\pi\alpha_K)$. For non-Abelian Chern-Simons theory with n fermions one obtains action $k \rightarrow k-n/2$. Depending on gauge group k_1 can vanish modulo 2 or 4. For the zeros at the real axes this would give the condition
   $$\frac{s}{2} = s + \frac{b}{d} = Re[\frac{1}{\alpha_K}] = 2k_1 \ , \quad s = -2n < 0 \ \text{or} \ s = 2 \ , \qquad (9.3.12)$$
   which is identically satisfied for integer valued b/d. Thus it seems that $SL(2,Z)$ is forced by the Chern-Simons argument in the case of Kähler action, which is however not too convincing for U(1).
   For non-trivial zeros it is not at all clear whether one certainly cannot apply the condition since there is also a contribution yS_E to the imaginary part. In any case, the condition would be
   $$\frac{Re[s]}{2} = 1/2 + \frac{b}{d} = Re[\frac{1}{\alpha_K}] = 2k_1 \ . \qquad (9.3.13)$$
   b/d must be half odd integer to satisfy the condition so that one would have $SL(2,Z)$ instead of $SL(2,Q)$. This is however in conflict with the Chern-Simons condition at real axis.
2. $w = s + b/d$ implies that the trivial poles $s = -2n$, $n > 0$, at the real axes are shifted to $s = -2n + b/d$ and become fractional. The poles at $s = 2$ is shifted to $2 + b/d$.

In the non-Abelian case one expects also Chern-Simons term but now emerging as an analog of Lagrange multiplier term rather than fundamental action reducing to Chern-Simons term. For $w = (as+b)(cs+d)$ the poles at real axis are mapped to rational numbers $w = (am+b)/(cm+d)$, $m = -2n$ or $m = 2$. Chern-Simons action would suggest integers. Gauge transformations would transform the action by a phase which is a root of unity. Vacuum functional is ZEO an analog of wave function as a square root of action exponential. Can one allow the wave function to be a finitely-many valued section in bundle?

Does the evolution along real axis corresponds to a confining or topological phase?

At real axis the imaginary part of s vanishes. Since it corresponds to the inverse of the gauge coupling strength, one can ask whether the proper interpretation is in terms of confining phase in which gauge coupling is literally infinite and it does not make sense to speak of perturbation theory. Instead one would have a phase in which Minkowski part of the Kähler action contributes only to the imaginary Chern-Simons term but not to the real part of the action. Topological QFT also based on Chern-Simons action also suggests itself.

The vanishing of gauge coupling strength is not a catastrophe now since the real part is non-vanishing. What looks strange that this phase is obtained also for Kähler coupling strength. Could this interpreted in terms of the fact that induced gauge potentials are not independent dynamical degrees of freedom but expressible in terms of CP_2 coordinates.

The spectrum of $1/\alpha_K$ at real axis has the $-2n + \frac{b}{d}$ and $2 + \frac{b}{d}$ and is integer or half-odd integer valued by the conditions on Chern-Simons action. One could make the entire spectrum integer value by a proper choice of b/d.

Integer valuedness forced by Chern-Simons condition leads to ask whether the situation could relate to hierarchy of Planck constants. This cannot be the case. One can assign to each value of y p-adic coupling constant labelled by prime k ($p \simeq 2^k$) a hierarchy of Planck constants $h_{eff} = n \times h$. If number theoretical universality is realized for $n = 1$, it is realized for all values of n and one can say that one has $1/\alpha = n/\alpha$ fora generic coupling strength α.

p-Adic temperature $T = 1/n$ using $log(p)$ as a unit correspond to the temperature parameter defined by α_K: the values of both are inverse integers. p-Adic thermodynamics might therefore provide a proper description for the confining phase as also the success of p-adic mass calculations encourages to think.

The sign of $1/\alpha_K$ is not fixed for the simplest option. The shift by $\frac{b}{d}$ could fix the sign to be negative. There is however no absolute need for a fixed sign since in Minkowskian regions the sign of Kähler action density depends on whether magnetic or electric fields dominate. In Euclidian regions the sign is always positive. Since the real part of Kähler action receives contributions from both Euclidian and Minkowskian regions it can can well have both signs so that for preferred extremals the signs of the real part of Kähler coupling strength and proper Kähler action compensate each other.

9.4 A Model For Electroweak Coupling Constant Evolution

In the following a model for electroweak coupling constant evolution using as inputs Weinberg angle at p-adic length scale $k = 127$ of electron or at four times longer scale $k = 131$ and in weak length scale $k = 89$ is developed.

9.4.1 Evolution Of Weinberg Angle

Concerning the electroweak theory, a key question is whether the notion of Weinberg angle still makes sense or whether one must somehow generalize the notion. Experimental data plus the prediction for $1/\alpha_{U(1)}$ as zero of zeta suggest that Weinberg angle varies. For instance, the value of$1/\alpha_{U(1)}$ for $k = 89$ corresponds to weak length scale and is 87.4 whereas fine structure constant is around 127. This gives $sin^2(\theta_W) \sim .312$, which is larger than standard model value.

1. Assume that the coupling constant evolutions for $1/\alpha_{em}$ and $1/\alpha_W$ correspond to different Möbius transformations acting in a nonlinear manner to s. Tangent of Weinberg angle is defined as the ratio of weak and U(1) coupling constants: $tan(\theta_W) = g_W/g_{U(1)}$ and it expresses the vectorial character of electromagnetic coupling. One can write

$$sin^2(\theta_W) = \frac{1}{1+X} \ , \ X = \frac{\alpha_{U(1)}}{\alpha_W} \ . \tag{9.4.1}$$

One can write the ansätze for for the coupling strengths as imaginary parts of complexified ones:

$$\begin{aligned} \frac{1}{\alpha_{U(1)}} &= Im[s+b] = y \ , \ s = \frac{1}{2} + iy \\ \frac{1}{\alpha_W} &= Im[\frac{a_W s + b_W}{c_W s + d_W}] = \frac{Dy}{c^2(\frac{1}{4}+y^2)+cd+d^2} \ , \\ D &= ad-bc \ . \end{aligned} \tag{9.4.2}$$

Here $GL(2,Q)$ matrices are assumed and determinant $D = ad - bc$ is allowed to differ from unity. From this one obtains for the Weinberg angle the expression

$$sin^2(\theta_W(y)) = \ = \frac{1}{1+[\frac{c^2}{D}(y^2+\frac{1}{4})+\frac{d}{c}+(\frac{d}{c})^2]} \ , \ D = ad-bc \ .$$

As the physical intuition suggests, Weinberg angle approaches zero at long length scales ($y \rightarrow \infty$). The value at short distance limit (the lowest zero $y_0 = 14.13$ at critical line) assignable to $p = 2$ is given by

$$sin^2(\theta_W(y_1)) \ = \ \frac{1}{1+\frac{c^2}{D}[(y_1^2+\frac{1}{4}+\frac{d}{c}+\frac{d}{c})^2]} \ .$$

Note that Weinberg angle decreases monotonically with y. The choices for which c^2/D are equivalent but the parameters (a, b, c, d) can be chosen nearer to integers for large enough D.

2. How to fix the parameters D, c, d?

   (a) The first guess $D = ad - bc = 1$ would reduces the unknown parameters to c, d. This does not however allow even approximately integer valued parameters a, b, cd.

   (b) The GUT value of Weinberg angle at this limit is $sin^2(\theta_W) = 3/8$. TGD suggests that the values of Weinberg angle correspond to Pythagorean triangles (see http://tinyurl.com/o7c4pkt). The lowest primitive Pythagorean triangle (side lengths are coprimes, (see http://tinyurl.com/j6ojlko) corresponds to the triplet (3,4,9) forming the trunk of the 3-tree formed by the primitive Pythagorean triangles with 3 triangles emanating at each node) and to $sin^2(\theta_W) = 9/25$ slightly smaller than the GUT value. The problem is that y_0 is not a rational number and for rational values of c, d the equation for Weinberg angule cannot be satisfied.

   (c) An alternative more reliable option is to use as input Weinberg angle at intermediate boson length scale $k = 89$ which corresponds to $y(24) = 87.4252746$. The value of fine structure constant at Z^0 boson length scale is about $1/\alpha_{em}(89) \simeq 127$. From this one would obtain

$$sin^2(\theta_W(k=89)) = 1 - \frac{y_{24}}{\alpha_{em}(89)} = 1 - \frac{\alpha_{U(1)}(24)}{\alpha_{em}(89)} \simeq 0.3116, \quad . \tag{9.4.3}$$

   One can obviously criticize the rather large value of the Weinberg angle forced by the value of $y(24)$ as being smaller than the experimental value. Experiments however suggests that Weinberg angle starts to increase after Z^0 pole. Gauge theory limit corresponds to a limit at which the sheets of many-sheeted are lumped together and one obtains a statistical average and the contributions of longer scale might increase the value of $1/\alpha_{U(1)}(24)$ and therefore reduce the value of the effective Weinberg angle.

   (d) Another input is the value of fine structure constant either at $k = 127$ corresponding to electron's p-adic length scale or at $k = 131$ ($L(131) = 10^{-11}$ meters and four times the p-adic length scale of electron) fixed by the condition that fine structure constant $\alpha_{em} = \alpha_{(U(1)}cos^2(\theta_W)$ corresponds its low energy value $1/\alpha_{em} = 137.035999139$ assigned often to electron length scale. From $y(32(= 1/\alpha_{U(1)} = 105.446623$ or $y(31) = 103.725538$ and $1/\alpha_{em}(131) = 137.035999139$ one can estimate the value of Weinberg angle as

$$\begin{aligned} sin^2(\theta_W(k=131)) &= 1 - \frac{y_{32}}{\alpha_{em}(131)} \simeq 0.23052 \ \text{ or} \\ sin^2(\theta_W(k=130)) &= 1 - \frac{y_{32}}{\alpha_{em}(127)} \quad . \end{aligned} \tag{9.4.4}$$

   It turns out that the first option does not work unless one assumes $1/alpha_{em}(k = 89) \leq 125.5263$ rather than $1/alpha_{em}(k = 89) \simeq 127$. The deviation is about 1-2 per cent. Second option works with a minimal modification for $1/alpha_{em}(k = 89) \simeq 127$.

   (e) The value of $y(1)$ is $y_1 = 14.13472$. The two latter conditions give rise to the following series of equations

$$\begin{aligned} X(k) &= cot^2(\theta_W)(k) = \frac{c^2}{D}(y^2(k) + A) \ , \ A = \frac{1}{4} + \frac{d}{c} + (\frac{d}{c})^2 \ , \\ \frac{X(24)}{X(K)} &\equiv Y = \frac{cot^2(\theta_W)(24)}{cot^2(\theta_W)(K)} = \frac{y^2(24) + A}{y^2(K) + A} \ , \\ A &= \frac{Y(y^2(K) - y^2(24))}{1 - Y} \quad . \end{aligned} \tag{9.4.5}$$

   Here K is either $K = 31$ or $K = 32$ corresponding to the p-adic length scale $k = 127$ or 131. It turns out that only $K = 31$ works fo $1/\alpha_{em}(89) = 127$.
   Also following parameters can be expressed in terms of the data.

$$\begin{aligned} \frac{c^2}{D} &= \frac{cot^2(\theta_W)(K)}{y^2(K)+A} \ , \\ \frac{d}{c} &= \frac{1}{2}\left(-1+\sqrt{A}\right) \ , \\ sin^2(\theta_W)(1) &= \frac{1}{1+X(1)} \ , \ X(1) = \frac{c^2}{D}\left(y^2(1)+A\right) \ . \end{aligned} \tag{9.4.6}$$

If the parameters a, b, c, d are integers, the equations cannot be satisfied exactly. For $K = 32$ it turns out that parameter A is negative for $1/alpha_{em}(k = 89) \leq 125.5263$. For $K = 31$ still negative and small so that $A = 0$ is the natural choice breaking slightly the conditions. **Table 9.1** represent both options.

(f) For $D = 1$ one has $c^2 \simeq 0.0002894$, which is very near to zero and not an integer. c must be non-vanishing to obtain a running Weinberg angle. For the general value of D the role c is taken by c^2D as an invariant fixed by the input data. $c \to c = 2$ requires $D = 1 \to int(4/c^2) = 138$. $D = 139$ almost equally good. One has $d/c = -0.5$ for $A = 0$ so that one would have $d = -1, c = 2$ for mimimum option. The condition $ad - bc = -a - 2b = D$ allows to estimate the values of the integer valued parameters a and b and get additional constraint on integer D. The values are not completely unique without additional conditions, say $b = 1$. This would give $a = -D+2 = -137$ for $D = 139$ (one cannot avoid association with the famous “137”!).

3. Consider now the physical predictions. The evolution of Weinberg angle is depicted in the tables **9.1** and **9.2** for $k = 127$ model whereas tables **9.3** and **9.4** give the predictions of $k = 131$ model. The value of Weinberg angle at electron scale $k = 127$ is predicted to be $sin^2(\theta_w) \simeq 0.2430$ deviating from its measured value by 5 per cent. For $k = 131$ the Weinberg angle deviates .7 per cent from the measured value but the value of $1/\alpha_{em}(k = 89)$ is about 1 per cent too small.
The expression for the predicted value of Weinberg angle at p-adic length scale $p = 2$ is $sin^2(\theta_W)_{p=2} \simeq 0.9453368487$, which is near to its maximal value and much larger than the $sin^2(\theta_W)_{p=2} \simeq 0.375$ of GUTs. This prediction was a total surprise but could be consistent with the new physics predicted by TGD predicting several scaled up copies of hadron physics above weak scale.
A related surprise at the high energy end was that $1/\alpha_{em}$ begins to increase again at $k = 13$ and is near to fine structure constant at $k = 11$! As if asymptotic freedom would apply to all couplings except $U(1)$ coupling. This behavior is due to the approach of $cos^2(\theta_W)$ to zero. One can of course ask whether $sin^2(\theta_W) = 1$ could be obtained for a suitable choice of the parameters. This can be achieved only for $y(1) = 0$ which is not possible since A the parameter A cannot be negative.

To sum up, experimental input allows to fix electroweak coupling constant evolution completely. The problematic feature of $k = 127$ model is the possibly too large value of Weinberg theta at low energies. The predicted scaled up copies of hadron physics could explain why Weinberg angle must increase at high energies. At electron length scale the 5 per cent too high value is somewhat disturbing. The many-sheeted space-time requiring lumping together of sheets to get space-time of General Relativity might help to understand why measured Weinberg angle is smaller than predicted. Average over sheets of different sizes could be in question.

9.4.2 Test For The Model Of Electroweak Coupling Constant Evolution

One can check whether the values of 100 lowest non-trivial zeros are consistent with their assignment with primes k in $p \simeq 2^k$ and whether the model is consistent with the value of fine structure constant $1/\alpha_{em} = 137.035999139$ and experimental value $P = .2312$ of Weinberg angle assigned either with electron's p-adic length scale $k = 127$ or $k = 131$ (0.1 Angstroms).

The tables below summarize the values of $1/\alpha_K$ identified as imaginary part of Riemann zero and $\alpha_{em} = \alpha_K(1 - P)$ for the model already discussed. P is .7 per cent smaller than the experimental value $P = .2312$ for $k = 131$. This agreement is excellent but it turns out that the model works only if fine structure constant corresponds to $\alpha_{em}(k)$ in electron length scale $k = 127$.

For $k = 127$ one obtains fine structure constant correctly for $P = 0.243078179077$ about 10 per cent larger than the experimental value. The predicted value of α_K at scale $k = 127$ changes from $\alpha_K = \alpha_{em}$ to $\alpha(U(1))$ due the presence of $cos^2(\theta_W) = .77$. One can wonder whether this is consistent with the p-adic mass calculations and the condition on CP_2 coming from the string tension of cosmic strings.

The predicted value of α_K changes at electron length scale by the introduction of $cos(\theta_W)$ factor. The formula for the p-adic mass squared involves second order contribution which cannot be predicted with certainty. This contribution is 20 per cent at maximum so that the change of α_K by 10 per cent can be tolerated.

Galactic rotation velocity spectrum gives also constraint on the string tension of cosmic strings and in this manner also to the value of the inverse $1/R$ of CP_2 radius to which p-adic mass scales are proportional. The size scale or large voids corresponds roughly to $k = 293$. From **Table 9.2** one has $1/\alpha_K = 167.2$. If the condition $\alpha_K \simeq \alpha_{em}$ holds true in long length scales, the scaling of $1/\alpha_K = 1/\alpha_{em}$ used earlier would be given by $r \simeq 167/137$ and would increase the string tension of cosmic strings by factor 1.2. This could be compensated by scaling $R^2_{CP_2}$ by the same factor. CP_2 mass scale would be scaled by factor $1/\sqrt{1.2} \simeq .9$. Also this can be tolerated. Note that maximal value cosmic string tension is assumed making sense only for the ideal cosmic strings with 2-D M^4 projection. Thickening of cosmic strings reduces their tension since magnetic energy per length is reduced.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 1	14.1347251	2	0.945336	258.5784
2	21.0220396	3	0.886600	185.3802
3	25.0108575	5	0.846706	163.1566
4	30.4248761	7	0.788698	143.9880
5	32.9350615	11	0.761068	137.8428
6	37.5861781	13	0.709786	129.5121
7	40.9187190	17	0.673584	125.3579
8	43.3270732	19	0.647955	123.0727
9	48.0051508	23	0.599889	119.9796
10	49.7738324	29	0.582401	119.1907
11	52.9703214	31	0.551851	118.1982
12	56.4462476	37	0.520249	117.6574
13	59.3470440	41	0.495203	117.5663
14	60.8317785	43	0.482855	117.6301
15	65.1125440	47	0.449024	118.1767
16	67.0798105	53	0.434344	118.5877
17	69.5464017	59	0.416691	119.2275
18	72.0671576	61	0.399493	120.0105
19	75.7046906	67	0.376117	121.3444
20	77.1448400	71	0.367315	121.9326
21	79.3373750	73	0.354389	122.8874
22	82.9103808	79	0.334500	124.5836
23	84.7354929	83	0.324876	125.5111
24	87.4252746	89	0.311321	126.9464
25	88.8091112	97	0.304627	127.7144
26	92.4918992	101	0.287691	129.8480
27	94.6513440	103	0.278326	131.1552
28	95.8706342	107	0.273213	131.9102
29	98.8311942	109	0.261303	133.7912
30	101.317851	113	0.251824	135.4198
31	103.725538	127	0.243078	137.0359
32	105.446623	131	0.237073	138.2133
33	107.168611	137	0.231264	139.4088
34	111.029535	139	0.218919	142.1486
35	111.874659	149	0.216337	142.7587

Table 9.1: Table represents the first 35 zeros of zeta identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the proposed model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 36	114.320220	151	0.209095	144.5436
37	116.226680	157	0.203677	145.9543
38	118.790782	163	0.196690	147.8767
39	121.370125	167	0.189990	149.8379
40	122.946829	173	0.186049	151.0495
41	124.256818	179	0.182861	152.0633
42	127.516683	181	0.175248	154.6123
43	129.578704	191	0.170659	156.2431
44	131.087688	193	0.167407	157.4452
45	133.497737	197	0.162390	159.3794
46	134.756509	199	0.159853	160.3964
47	138.116042	211	0.153349	163.1322
48	139.736208	223	0.150345	164.4624
49	141.123707	227	0.147838	165.6068
50	143.111845	229	0.144348	167.2548
51	146.000982	233	0.139481	169.6662
52	147.422765	239	0.137170	170.8597
53	150.053520	241	0.133037	173.0796
54	150.925257	251	0.131706	173.8183
55	153.024693	257	0.128579	175.6036
56	156.112909	263	0.124167	178.2452
57	157.597591	269	0.122123	179.5214
58	158.849988	271	0.120436	180.6009
59	161.188964	277	0.117374	182.6243
60	163.030709	281	0.115040	184.2239
61	165.537069	283	0.111970	186.4094
62	167.184439	293	0.110016	187.8511
63	169.094515	307	0.107811	189.5277
64	169.911976	311	0.106886	190.2468
65	173.411536	313	0.103056	193.3360
66	174.754191	317	0.101639	194.5256
67	176.441434	331	0.099898	196.0238
68	178.377407	337	0.097952	197.7472
69	179.916484	347	0.096444	199.1206
70	182.207078	349	0.094262	201.1698

Table 9.2: Table represents the zeros y_n of zeta in the range $n \in [35, 70]$ identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the proposed model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 1	14.1347251	2	0.943414	249.7949
2	21.0220396	3	0.882868	179.4744
3	25.0108575	5	0.841896	158.1927
4	30.4248761	7	0.782535	139.9074
5	32.9350615	11	0.754350	134.0732
6	37.5861781	13	0.702190	126.2089
7	40.9187190	17	0.665488	122.3238
8	43.3270732	19	0.639563	120.2072
9	48.0051508	23	0.591074	117.3933
10	49.7738324	29	0.573475	116.6964
11	52.9703214	31	0.542785	115.8544
12	56.4462476	37	0.511110	115.4580
13	59.3470440	41	0.486058	115.4744
14	60.8317785	43	0.473724	115.5892
15	65.1125440	47	0.439988	116.2700
16	67.0798105	53	0.425376	116.7369
17	69.5464017	59	0.407825	117.4423
18	72.0671576	61	0.390747	118.2878
19	75.7046906	67	0.367570	119.7045
20	77.1448400	71	0.358853	120.3232
21	79.3373750	73	0.346062	121.3225
22	82.9103808	79	0.326403	123.0862
23	84.7354929	83	0.316902	124.0459
24	87.4252746	89	0.303530	125.5263
25	88.8091112	97	0.296931	126.3164
26	92.4918992	101	0.280251	128.5057
27	94.6513440	103	0.271035	129.8435
28	95.8706342	107	0.266007	130.6152
29	98.8311942	109	0.254301	132.5350
30	101.317851	113	0.244992	134.1945
31	103.725538	127	0.236408	135.8390
32	105.446623	131	0.230518	137.0359
33	107.168611	137	0.224822	138.2504
34	111.029535	139	0.212726	141.0304
35	111.874659	149	0.210197	141.6489

Table 9.3: Table represents the first 35 zeros of zeta identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the $k = 131$ model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 36	114.320220	151	0.203108	143.4576
37	116.226680	157	0.197806	144.8861
38	118.790782	163	0.190972	146.8316
39	121.370125	167	0.184423	148.8150
40	122.946829	173	0.180571	150.0397
41	124.256818	179	0.177456	151.0641
42	127.516683	181	0.170022	153.6387
43	129.578704	191	0.165542	155.2850
44	131.087688	193	0.162368	156.4981
45	133.497737	197	0.157474	158.4494
46	134.756509	199	0.154999	159.4751
47	138.116042	211	0.148658	162.2333
48	139.736208	223	0.145730	163.5739
49	141.123707	227	0.143287	164.7270
50	143.111845	229	0.139887	166.3872
51	146.000982	233	0.135146	168.8158
52	147.422765	239	0.132897	170.0175
53	150.053520	241	0.128873	172.2522
54	150.925257	251	0.127578	172.9957
55	153.024693	257	0.124534	174.7923
56	156.112909	263	0.120242	177.4499
57	157.597591	269	0.118254	178.7336
58	158.849988	271	0.116613	179.8194
59	161.188964	277	0.113635	181.8541
60	163.030709	281	0.111367	183.4623
61	165.537069	283	0.108383	185.6594
62	167.184439	293	0.106483	187.1085
63	169.094515	307	0.104341	188.7935
64	169.911976	311	0.103443	189.5162
65	173.411536	313	0.099722	192.6201
66	174.754191	317	0.098346	193.8152
67	176.441434	331	0.096655	195.3201
68	178.377407	337	0.094766	197.0512
69	179.916484	347	0.093302	198.4305
70	182.207078	349	0.091184	200.4884

Table 9.4: Table represents the zeros y_n of zeta in the range $n \in [35, 70]$ identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the $k = 131$ model for $sin^2(\theta_W)$.

Chapter 10

TGD View about Coupling Constant Evolution?

10.1 Introduction

Atyiah has recently proposed besides a proof of Riemann Hypothesis also an argument claiming to derive the value of the structure constant (see http://tinyurl.com/y8xw8cey). The mathematically elegant arguments of Atyiah involve a lot of refined mathematics including notions of Todd exponential and hyper-finite factors of type II (HFFs) assignable naturally to quaternions. The idea that $1/\alpha$ could result by coupling constant evolution from π looks however rather weird for a physicist.

What makes this interesting from TGD point of view is that in TGD framework coupling constant evolution can be interpreted in terms of inclusions of HFFs with included factor defining measurement resolution [K99, K33]. An alternative interpretation is in terms of hierarchy of extensions of rationals with coupling parameters determined by quantum criticality as algebraic numbers in the extension [L42, L43].

In the following I will explain what I understood about Atyiah's approach. My critics includes the arguments represented also in the blogs of Lubos Motl (see http://tinyurl.com/ycq8fhsy) and Sean Carroll (see http://tinyurl.com/y87f8psg). I will also relate Atyiah's approach to TGD view about coupling evolution. The hasty reader can skip this part although for me it served as an inspiration forcing to think more precisely TGD vision.

There are two TGD based formulations of scattering amplitudes.

1. The first formulation is at the level of infinite-D "world of classical worlds" (WCW) [K76] uses tools like functional integral. The huge super-symplectic symmetries generalizing conformal symmetries raise hopes that this formulation exists mathematically and that it might even allow practical calculations some day. TGD would be an analog of integrable QFT.
2. Second - surprisingly simple - formulation [L52] is based on the analog of micro-canonical ensemble in thermodynamics (quantum TGD can be seen as complex square root of thermodynamics). It relates very closely to TGD analogs of twistorialization and twistor amplitudes [K91, K79].
   During writing I realized that this formulation can be regarded as a generalization of cognitive representations of space-time surfaces based on algebraic discretization making sense for all extensions of rationals to the level of scattering amplitudes. In the adelization the key question is whether it is necessary to define the p-adic counterparts of action exponentials. The number theoretical constraints seem hopelessly strong. One solution would be that the action exponentials for allow space-time surfaces equal to one. This option fails. The solution of the problem is however trivial. Kähler function can have only single minimum for given values of zero modes and the action exponentials cancel from scattering amplitudes completely in this case. This formulation allows a continuation to p-adic sectors and adelization [L42, L43]. Note that no conditions on α_K are obtained contrary to the first beliefs.

One can also understand the relationship of the two formulations in terms of M^8-H duality. This view allows also to answer to a longstanding question concerning the interpretation of the surprisingly successful p-adic mass calculations [K59]: as anticipated, p-adic mass calculations are carried out for a cognitive representation rather than for real world particles and the huge simplification explains their success for preferred p-adic prime characterizing particle as so called ramified prime for the extension of rationals defining the adeles.

The understanding of coupling constant evolution has been one of most longstanding problems of TGD and I have made several proposals during years. TGD view about cosmological constant turned out to be the solution of the problem.

1. The formulation of the twistor lift of Kähler action led to a rather detailed view about the interpretation of cosmological constant as an approximate parameterization of the dimensionally reduced 6-D Kähler action (or energy) allowing also to understand how it can decrease so fast as a function of p-adic length scale. In particular, a dynamical mechanism for the dimensional reduction of 6-D Kähler action giving rise to the induction of the twistor structure and predicting this evolution emerges.
   In standard QFT view about coupling constant evolution ultraviolet cutoff length serves as the evolution parameter. TGD is however free of infinities and there is no cutoff parameter. It turned out cosmological constant replaces this parameter and coupling constant evolution is induced by that for cosmological constant from the condition that the twistor lift of the action is not affected by small enough modifications of the moduli of the induced twistor structure. The moduli space for them corresponds to rotation group $SO(3)$. This leads to explicit evolution equations for α_K, which can be studied numerically.
2. I consider also the relationship to a second TGD based formulation of coupling constant evolution in terms of inclusion hierarchies of hyper-finite factors of type II_1 (HFFs) [K99, K33]. I suggest that this hierarchy is generalized so that the finite subgroups of $SU(2)$ are replaced with Galois groups associated with the extensions of rationals. An inclusion of HFFs in which Galois group would act trivially on the elements of the HFFs appearing in the inclusion: kind of Galois confinement would be in question.

Ramified primes are conjecture to correspond to the preferred p-adic primes characterizing particles. Ramified primes are special in the sense that their expression as a product of primes P_i of extension contains higher than first powers and the number P_i is smaller than the maximal number n defined by the dimension of the extension. It is not quite clear why ramified primes appear as preferred p-adic primes and in the following Dedekind zeta functions and what I call ramified zeta functions inspired by the interpretation of zeta function as analog of partition function are used in attempt to understand why ramified primes could be physically special.

The intuitive feeling is that quantum criticality is what makes ramified primes so special. In $O(p) = 0$ approximation the irreducible polynomial defining the extension of rationals indeed reduces to a polynomial in finite field F_p and has multiple roots for ramified prime, and one can deduce a concrete geometric interpretation for ramification as quantum criticality using $M^8 - H$ duality.

10.2 Criticism of Atyiah's approach

The basic idea of Atyiah is that π and the inverse of the fine structure constant $1/\alpha = 137.035999....$ are related by coupling constant evolution - that is renormalization - which is a basic operation in quantum field theory and has physical interpretation. For a physicist it is easy to invent objections.

1. In quantum field theory fine structure constant and all coupling strengths obey a continuous evolution as function of mass scale or length scale and one should predict the entire evolution rather than say its value at electron length scale. In TGD framework the coupling constant evolution becomes discrete and would basically labelled by the hierarchy of extensions of rationals.
2. π is purely geometric constant - kind of Platonic transcendental having very special role in the mathematical world order - whereas fine structure constant is a dynamical coupling parameter. Atyiah does not have any proposal for why these constants would be related in this manner. Also no explanation for what it would mean that the circumference of unit circle would grow from 2π to $2/\alpha$ is given.
   Remark: In TGD actually the coverings labelled by the value $h_{eff}/n_0 = n$ identified as the order of Galois group of extension of rationals defining given level of the hierarchy of

evolutionary levels (entanglement coefficients would belong to this extension as also S-matrix elements). The full angle using M^4 rotation angle as coordinate increases effectively to $n \times 2\pi$ for the covering spaces of extensions introducing n:th root of unity. In TGD would however have n instead of $1/(\alpha\pi)$.

3. That $1/\alpha \sim 137$ should have interpretation as renormalized value of angle π looks rather weird to me. The normalization would be very large and it is extremely difficult to see why $1/\pi$ have a role of fine structure constant say at high energy limit if one accepts coupling constant evolution and identifies $1/\alpha$ as the value of $1/\alpha$ at zero momentum transfer.

In fact, Atyiah proposes a discrete evolution of π to $1/\alpha$ defined by approximations of HFF as a finite-D algebra. Forgetting π as the starting point of the evolution, this idea looks beautiful. At first the idea that all numbers suffer a renormalization evolution, looks really cute. Coupling constant evolution is however not a sequence of approximations but represents a genuine dependence of coupling constants on length scale.

Remark: In TGD framework I propose something different.The length scale evolution of coupling constants would correspond to a hierarchy of inclusions of HFFs rather than a sequence of finite-D approximations approaching HFF. The included factor would represent measurement resolution. Roughly, the transformations of states by operations defined in included factor would leave state invariant in the measurement resolution defined by the included factor. Different values of coupling constant would correspond to different measurement resolutions.

1. Atyiah mentions as one of his inspirers the definition of 2π via a limiting procedure identifying it as the length of the boundary of n-polygon inside unit circle. Amusingly, I have proposed similar definition of 2π in p-adic context, where the introduction of π would give rise to infinite extension.

   Atyiah generalizes this definition to the area of quaternionic sphere so that the limiting procedure involves two integers. For sphere tessellations as analogs of lattices allow only Platonic solids. For torus one could have infinite hierarchy of tessellations [L49] allowing to define the area of torus in this manner. The value of n defined by the extension of rationals containing root of unity $exp(i2\pi/n)$ such that n is maximal. The largest n for the roots of unity appearing in the extension of p-adics would determine the approximation of 2π used.

2. Atyiah suggests a concrete realization for the coupling constant evolution of numbers, not only coupling constants. The evolution would correspond to a sequence of approximation to HFF converging to HFF. One can of course define this kind of evolution but to physicist it looks like a formal game only.

3. HFF is interpreted as an infinite tensor product of 2×2 complex Clifford algebras $M_2(C)$, which can be also interpreted as complexified quaternions. One defines the trace by requiring that the trace of infinite tensor product of unit matrices equals to 1. The usual definition of schoolbooks would given infinite power of 2, which diverges. The inner product is the product of the usual inner products for the factors of the tensor product labelled by n but divided by power $2^{-n_{max}}$ to guarantee that the trace of the identity matrix is unity as product of traces for factors otherwise equal to 2^n. In fact, fermionic Fock algebra familiar to physicist is HFF although in hidden manner.

   Remark: The appearance of quaternions is attractive from TGD point of view since in M^8-H duality the dynamics at the level of M^8 is determined by associativity of either tangent or normal space of 4-surface in M^8 and associativity is equivalent with quaternionicity [L36]. The hierarchy of HFFs is also basic piece of quantum TGD and realizable in terms of quaternions.

4. Atyiah tells there is an algebra isomorphism from complex numbers C to the subset of commuting matrices in HFF. One can define the map to C as either eigenvalue of the matrix and obtains to isomorphisms: t_+ and t_-. One can define the renormalization map $C \to C$ in terms of the inverse of $t_- \circ t_+^{-1}$ or its inverse. This would assign to a complex numbers z its normalized value.

   HFFs allow an excellent approximation by finite number of tensor factors and one can perform an approximation taking only finite number of tensor factors and at the limit of infinite number of factors get the desired normalization map. The approximation would be $t_-(n) \circ t_+(n)$. I must confess that I did not really understand the details of this argument.

   In any case, to me this does not quite correspond to what I understand with renormalization flow. Rather this is analogous to a sequence of approximations defining scattering amplitude as approximation containing only contributions up to power g^n. I would argue than one must consider the infinite sequence of inclusions of HFFs instead of a sequence of approximations defining HFF.

In this manner one would the renormalization map would be $t_-(n+1) \circ t_+^{-1}(n)$, where n now labels the hierarchy of HFFs in the inclusion hierarchy. $t_\pm n$ is now the exact map from commuting sub-algebra to complex numbers.

There is however a rather close formal resemblance since simple inclusions correspond to inclusions of the sub-algebra with one M_C^2 factor replaced with mere identity matrix.

5. The proposal of Atyiah is that this renormalization of numbers is mediated by so called Todd exponentiation used in the construction of the characteristic classes. This map would be defined in terms of generating function $G(x) = x/(1-exp-x)$ applied to $x = \pi$. If I understood anything about the explanation, this map is extended to infinite number of tensor factors defining the HFF and the outcome would be that $x = \pi$ for single tensor factor would be replaced with $1/\alpha$. Why Todd exponentiation? Atyiah also argues that one has $T(\pi)/\pi = T(\gamma)/\gamma$, where γ is Euler's constant. My mathematical education is so limited that I could not follow these arguments.
6. Atyiah also claims that the approximation $1/\alpha = 137$ assumed by Eddington to be exact has actually deeper meaning. There are several formulas in this approximation such as $1/\alpha = 2^0+2^3+2^7 = 1+8+128$. If I understood correctly, Atyiah tells that the numbers 1, 8, and 128 appear in the Bott periodicity theorem as dimensions of subsequent stable homotopy groups. My own favorite formula is in terms of Mersenne primes: $1/\alpha = M_2 + M_3 + M_7 = 3+7+127$. The next Mersenne prime would be M_{127} and corresponds to the p-adic length scale of electron.

   Remark: A fascinating numerological fact is that $p \simeq 2^k$, $k \simeq 137$, corresponds to the p-adic length scale near to Bohr radius: kind of cosmic joke one might say. Fine structure constant indeed emerged from atomic physics!

It would be of course marvellous if the renormalization would not depend on physics at all but here physicist protests.

1. The coupling constant evolutions for the coupling strengths of various interactions are different and depend also on masses of the particles involved. One might however hope that this kind of evolution might make sense for fundamental coupling constants of the theory. In TGD Kähler coupling strength $1/\alpha_K$ would be such parameter.
2. The quantum criticality of TGD Universe suggests that Atyiah's claim is true in a weaker sense. Quantum criticality is however a dynamical notion. I have actually proposed a model for the evolution of $1/\alpha_K$ based on the complex zeros of Riemann Zeta [L18] and also a generalization to other coupling strengths assuming that the argument of zeta is replaced with its Möbius transform.

   Very strong consistency conditions should be met. Preferred primes would be primes near prime power of 2 and ramified primes of extension, and also the zero of zeta in question should belong to the extension in question. I am of course the first to admit that this model is motivated more by mathematical aesthetics than concrete physical calculations.
3. The idea about renormalization evolution in this manner could - actually should - generalize. One can consider a maximal set of commuting set of observables in terms of tensor product of HFFs and define for them map to diagonal $n \times n$ matrices with complex eigenvalues. One would have infinite sum over the eigenvalues of diagonal matrices over factors: just as one has for many particle state in QFT containing contribution from all tensor factors which are now however ordered by the label n. The length scale evolution of these observables could be defined by the above formula for inclusion. Fine structure constant basically reduces to charge as eigenvalue of charge operator so that this could make sense.

   The beauty of this view would be that renormalization could be completely universal. In TGD framework quantum criticality (QC) indeed strongly suggests this universality in some sense. The hierarchy of extensions of rationals would define the discrete coupling constant evolution.

10.3 About coupling constant evolution in TGD framework

It is often forgotten that fine structure constant depends on length scale. When Eddington was working with the problem, it was not yet known that fine structure constant is running coupling constant. For continuous coupling constant evolution there is not much point to ponder why its value is what it is at say electron length scale. In TGD framework - adelic physics - coupling parameters however obey discrete length scale evolution deriving from the hierarchy of extensions of rationals. In this framework coupling constants are determined by quantum criticality implying that they do not run at all in the phase assignable to given extension of rational. They are analogous to critical temperature and determined in principle by number theory.

Two approaches to quantum TGD

There are two approaches to TGD: geometric and number theoretic. The "world of classical worlds" (WCW) is central notion of TGD as a geometrization of quantum physics rather than only classical physics.

1. WCW consists of 3-surfaces and by holography realized by assigning to these 3-surfaces unique 4-surfaces as preferred extremals. In zero energy ontology (ZEO) these 3-surfaces are pairs of 3-surfaces, whose members reside at opposite boundaries of causal diamond (CD) and are connected by preferred extremal analogous to Bohr orbit. The full quantum TGD would rely on real numbers and scattering amplitudes would correspond to zero energy states having as arguments these pairs of 3-surfaces. WCW integration would be involved with the definition of inner products.
2. The theory could be seen formally as a complex square root of thermodynamics with vacuum functional identified as exponent of Kähler function. Kähler geometry would allow to eliminate ill-defined Gaussian determinants and metric determinant of Kähler metric and they would simply disappear from scattering amplitudes. WCW is infinite-D space and one might argue that this kind of approach is hopeless. The point is however that the huge symmetries of WCW - super-symplectic invariance - give excellent hopes of really construction the scattering amplitudes: TGD would be integrable theory.
3. A natural interpretation would be that Kähler action as the analog of Hamiltonian defines the Kähler function of WCW and functional integral defined by it allows definition of full scattering amplitudes.

The number theoretic approach could be called adelic physics [L41, L43] providing also the physics of cognition.

1. At space-time level p-adicization as description of cognition requires discretization. Cognitive representations at space-time level consist of finite set of space-time points with preferred coordinates M^8 in extension of rationals inducing the extensions of p-adic number fields. These representations would realize the notion of finite measurement resolution. p-Adicization and adelization for given extension of rationals are possible only in this manner since these points can be interpreted as both real and p-adic numbers.
2. What about cognitive representations at the level of WCW? The discrete set of space-time points would replace the space-time surface with a finite discrete set of points serving also as its WCW coordinates and define the analog of discretization of WCW using polynomials in M^8 fixed by their values at these points [L36]. If the space-time surface is represented by a polynomial, this representation is all that is needed to code for the space-time surface since one can deduce the coefficients of a polynomial from its values at finite set of points. Now the coefficients belong to extension of rationals. If polynomials are replaced by analytic functions, polynomials provide approximation defining the cognitive representation.

While writing this I realized that what I have micro-canonical ensemble [L52] as kind of complex square root of its counterpart in thermodynamics can serve as a cognitive representation of scattering amplitudes. Cognitive representations of space-time surfaces would thus give also cognitive representations of WCW and micro-canonical ensemble would realize cognitive representations for the scattering amplitudes. Cognitive representations define only a hierarhcy of approximations. The exact description would involve the full WCW, its Kähler geometry, and vacuum functional as exponent of Kähler function.

The idea of micro-canonical ensemble as a subset of space-time surfaces with the same vanishing action would select a sub-set of surfaces with the same values of coupling parameters so that the fixing the coupling parameters together with preferred extremal property selects the subset with same value of action. There are two options to consider.

1. The real part of the action vanishes and imaginary part is multiple of 2π so that the action exponential is equal to unity. For the twistor lift this actually implies the vanishing of the entire action since volume term and Kähler term have the same phase (that of $1/\alpha_K$). The role of coupling parameters would be analogous to the role of temperature and applied pressure. In principle this condition is mathematically possible. The electric part of Kähler action in Minkowskian regions has sign opposite to magnetic part and volume term (actually magnetic S^2 part of 6-D Kähler action) so that these two contributions could cancel. The problem is that Kähler function would be constant and therefore also the Kähler metric.
2. I have also proposed [L52] that the analog of micro-canonical ensemble makes sense meaning that all space-time surfaces contributing to the scattering amplitude have the same action. As a consequence, the action exponential and the usual normalization factor would cancel each other and one would obtain just a sum over space-time surfaces with same action: otherwise action exponential would not appear in the scattering amplitudes - this is the case also in perturbative QFTs. This is crucial for the p-adicization and adelization since these exponential factors belong to the extension of rationals only under very strong additional conditions.
   This option has analog also at the level of WCW since Kähler function should have for give values of zero modes only single minimum so that localization in zero modes would mean that the action exponential cancels in the normalization of the amplitudes. It seems that this option is the only possible one.
   Note that the cancellation of the metric determinant and Gaussian determinant possible for Kähler metric with the exponent of Kähler function serving as vacuum functional reduces the perturbative integrations around the minima of Kähler action to a sum over exponents, and if only single minimum contributes for given values of the zero modes, the sum contains only single term.

10.3.1 Number theoretic vision about coupling constant evolution

Let us return to the question about the coupling constant evolution.

1. Each extension of rationals corresponds to particular values of coupling parameters determined by the extension so that it indeed makes sense to ponder what the spectrum of values for say fine structure constant is. In standard QFT this does not make sense.
2. Coupling constant evolution as a function of momentum or length scales reduces to p-adic coupling constant evolution in TGD as function of p-adic prime. Particles are characterized by preferred p-adic primes - for instance, electron corresponds to $M_{127} = 2^{127} - 1$ - the largest Mersenne prime which does not correspond to super-astronomical Compton length - and the natural identification is as so called ramified primes of extension.

Why the interpretation of p-adic primes as ramified primes?

1. As one increases length scale resolution particle decomposes to more elementary particles.
2. Particles correspond in TGD to preferred p-adic primes. This suggests that when a prime (ideal) of given extension is looked at improved precision determined by an extension of the orignal extension it decomposes into a product of primes. This indeed happens.
   The number of primes of the larger extension appearing in the decomposition to product equals to the dimension of extension as extension of the original extension. All these primes appear and only once in the generic case. Ramified primes of ordinary extension are however odd-balls. Some primes of extension are missing and some appear as higher powers than 1 in their decomposition.
3. Ramified primes are analogous to critical systems. Polynomial with a multiple root - now prime of extension appearing as higher power - corresponds to a critical system. TGD is quantum critical so that one expects that ramified primes are preferred physically and indeed correspond to quantum critical systems.
4. Only the momenta belonging to the extension of rationals are considered and one can identify them as real-valued or p-adic valued momenta. Coupling constants do not depend on the values of the momenta for given extension of rationals and are thus analogous to critical temperature. This involves interesting not totally resolved technical question inspired by p-adic mass calculations for which the p-adic mass squared value is mapped to its real value by canonical

identification $S\sum x_n p^n \to \sum x_n p^{-n}$. The correspondence is continuous and can be applied to Lorentz invariants appearing in scattering amplitudes [K60].
Could this correspondence be applied also to momenta rather than only mass squared values and Lorentz invariants? $M^8 - H$ correspondence [L36] selects fixed Poincare frame as moduli space for octonionic structures and at M^8 level this could make sense.

10.3.2 Cosmological constant and twistor lift of Kähler action

Cosmological constant Λ is one of the biggest problems of modern physics. Surprisingly, Λ turned out to provide the first convincing solution to the problem of understanding coupling constant evolution in TGD framework. In QFTs the independence of scattering amplitudes on UV cutoff length scale gives rise to renormalization group (RG) equations. In TGD there is however no natural cutoff length scale since the theory is finite. Cosmological constant should however evolve as a function of p-adic length scales and cosmological constant itself could give rise to the length scale serving in the role of cutoff length scale. Combined with the view about cosmological constant provided by twistor lift of TGD this leads to explicit RG equations for α_K and scattering amplitudes.

Cosmological constant has two meanings.

1. Einstein proposed non-vanishing value of Λ in Einstein action as a volume term at his time in order to get what could be regarded as a static Universe. It turned out that Universe expanded and Einstein concluded that this proposal was the greatest blunder of his life. For two decades ago it was observed that the expansion of the Universe acclerates and the cosmological constant emerged again. Λ must be extremely small and have correct sign in order to give accelerating rather decelerating expansion in Robertson-Walker cooordinate. Here one must however notice that the time slicing used by Einstein was different and fr this slicing the Universe looked static.
2. Λ can be however understood in an alternative sense as characterizing the dynamics in the matter sector. Λ could characterize the vacuum energy density of some scalar field, call it quintessense, proportional to 3- volume in quintessence scenario. This Λ would have sign opposite to that in the first scenario since it would appear at opposite side of Einstein's equations.

Cosmological constant in string models and in TGD

It has turned out that Λ could be the final nail to the coffin of superstring theory.

1. The most natural prediction of M-theory and superstring models is Λ in Einsteinian sense but with wrong sign and huge value: for instance, in AdS/CFT correspondence this would be the case. There has been however a complex argument suggesting that one could have a cosmological constant with a correct sign and even small enough size.
This option however predicts landscape and a loss of predictivity, which has led to a total turn of the philosophical coat: the original joy about discovering the unique theory of everything has changed to that for the discovery that there are no laws of physics. Cynic would say that this is a lottery win for theoreticians since theory building reduces to mere artistic activity.
2. Now however Cumrun Vafa - one of the leading superstring theorists - has proposed that the landscape actually does not exist at all [B43] (see `http://tinyurl.com/ycz7wvng`). Λ would have wrong sign in Einsteinian sense but the hope is that quintessence scenario might save the day. Λ should also decrease with time, which as such is not a catastrophe in quintessence scenario.
3. Theorist D. Wrase *et al* has in turn published an article [B16] (see `http://tinyurl.com/ychrhuxk`) claiming that also the Vafa's quintessential scenario fails. It would not be consistent with Higgs Higgs mechanism. The conclusion suggesting itself is that according to the no-laws-of-physics vision something catastrophic has happened: string theory has made a prediction! Even worse, it is wrong.
Remark: In TGD framework Higgs is present as a particle but p-adic thermodynamics rather than Higgs mechanism describes at least fermion massivation. The couplings of Higgs to fermions are naturally proportional their masses and fermionic part of Higgs mechanism is seen only as a way to reproduce the masses at QFT limit.

4. This has led to a new kind of string war: now inside superstring hegemony and dividing it into two camps. Optimistic outsider dares to hope that this leads to a kind of auto-biopsy and the gloomy period of superstring hegemony in theoretical physics lasted now for 34 years would be finally over.

String era need not be over even now! One could propose that both variants of Λ are present, are large, and compensate each other almost totally! First I took this as a mere nasty joke but I realized that I cannot exclude something analogous to this in TGD. It turned that this is not possible. I had made a delicate error. I thought that the energy of the dimensionally reduced 6-D Kähler action can be deduced from the resulting 4-D action containing volume term giving the negative contribution rather than dimensionally reducing the 6-D expression in which the volume term corresponds to 6-D magnetic energy and is positive! A lesson in non-commutativity!

The picture in which Λ in Einsteinian sense parametrizes the total action as dimensionally reduced 6-D twistor lift of Kähler action could be indeed interpreted formally as sum of genuine cosmological term identified as volume action. This picture has additional bonus: it leads to the understanding of coupling constant evolution giving rise to discrete coupling constant evolution as sub-evolution in adelic physics. This picture is summarized below.

The picture emerging from the twistor lift of TGD

Consider first the picture emerging from the twistor lift of TGD.

1. Twistor lift of TGD leads via the analog of dimensional reduction necessary for the induction of 8-D generalization of twistor structure in $M^4 \times CP_2$ to a 4-D action determining space-time surfaces as its preferred extremals. Space-time surface as a preferred extremal defines a unique section of the induced twistor bundle. The dimensionally reduced Kähler action is sum of two terms. Kähler action proportional to the inverse of Kähler coupling strength and volume term proportional to the cosmological constant Λ.
   Remark: The sign of the volume action is negative as the analog of the magnetic part of Maxwell action and *opposite* to the sign of the area action in string models.
   Kähler and volume actions should have opposite signs. At M^4 limit Kähler action is proportional to $E^2 - B^2$ In Minkowskian regions and to $-E^2 - B^2$ in Euclidian regions.
2. Twistor lift forces the introduction of also M^4 Kähler form so that the twistor lift of Kähler action contains M^4 contribution and gives in dimensional reduction rise to M^4 contributions to 4-D Kähler action and volume term.
   It is of crucial importance that the Cartesian decomposition $H = M^4 \times CP_2$ allows the scale of M^4 contribution to 6-D Kähler action to be different from CP_2 contribution. The size of M^4 contribution as compared to CP_2 contribution must be very small from the smallness of CP breaking [L45] [K79].
   For canonically imbedded M^4 the action density vanishes. For string like objects the electric part of this action dominates and corresponding contribution to 4-D Kähler action of flux tube extremals is positive unlike the standard contribution so that an almost cancellation of the action is in principle possible.
3. What about energy? One must consider both Minkowskian and Euclidian space-time regions and be very careful with the signs. Assume that Minkowskian and Eucidian regions have *same time orientation.*
   (a) Since a dimensionally reduced 6-D Kähler action is in question, the sign of energy density is positive Minkowskian space-time regions and of form $(E^2 + B^2)/2$. Volume energy density proportional to Λ is positive.
   (b) In Euclidian regions the sign of g^{00} is negative and energy density is of form $(E^2 - B^2)/2$ and is negative when magnetic field dominates. For string like objects the M^4 contribution to Kähler action however gives a contribution in which the electric part of Kähler action dominates so that M^4 and CP_2 contributions to energy have opposite signs.
   (c) 4-D volume energy corresponds to the magnetic energy for twistor sphere S^2 and is therefore positive. For some time I thought that the sign must be negative. My blunder was that I erratically deduced the volume contribution to the energy from 4-D dimensionally reduced action, which is sum of Kähler action and volume term rather than deducing it for

6-D Kähler action and then dimensionally reducing the outcome. A good example about consequences of non-commutativity!

The identification of the observed value of cosmological constant is not straightforward and I have considered several options without making explicit their differences even to myself. For Einsteinian option cosmological constant could correspond to the coefficient Λ of the volume term in analogy with Einstein's action. For what I call quintessence option cosmological constant Λ_{eff} would approximately parameterize the total action density or energy density.

1. Cosmological constant - irrespective of whether it is identified as Λ or Λ_{eff} - is extremely small in the recent cosmology. The natural looking assumption would be that as a coupling parameter Λ or Λ_{eff} depends on p-adic length scale like $1/L_p^2$ and therefore decreases in average sense as $1/a^2$, where a is cosmic time identified as light-cone proper time assignable to either tip of CD. This suggests the following rough vision.
   The increase of the thickness of magnetic flux tubes carrying monopole flux liberates energy and this energy can make possible increase of the volume so that one obtains cosmic expansion. The expansion of flux tubes stops as the string tension achieves minimum and the further increase of the volume would increase string tension. For the cosmological constant in cosmological scales the maximum radius of flux tube is about 1 mm, which is biological length scale. Further expansion becomes possible if a phase transition increasing the p-adic length scale and reducing the value of cosmological constant is reduced. This phase transition liberates volume energy and leads to an accelerated expansion. The space-time surface would expand by jerks in stepwise manner. This process would replace continuous cosmic expansion of GRT. One application is TGD variant of Expanding Earth model explaining Cambrian Explosion, which is really weird event [K34].
   One can however raise a serious objection: since the volume term is part of 6-D Kähler action, the length scale evolution of Λ should be dictated by that for $1/\alpha_K$ and be very slow: therefore cosmological constant identified as Einsteinian Λ seems to be excluded.
2. It however turns that it possible to have a large number of embedding of the twistor sphere into the product of twistor spheres of M^4 and CP_2 defining dimensional reductions. This set is parameterized by rotations sphere. The S^2 part of 6-D Kähler action determining Λ can be arbitrarily small. This mechanism is discussed in detail in [L55, L56] and leads also to the understanding of coupling constant evolution. The cutoff scale in QFT description of coupling constant evolution is replaced with the length scale defined by cosmological constant.

Second manner to increase 3-volume

Besides the increase of 3-volume of M^4 projection, there is also a second manner to increase volume energy: many-sheetedness. The phase transition reducing the value of Λ could in fact force many-sheetedness.

1. In TGD the volume energy associated with Λ is analogous to the surface energy in superconductors of type I. The thin 3-surfaces in superconductors could have similar 3-surface analogs in TGD since their volume is proportional to surface area - note that TGD Universe can be said to be quantum critical.
   This is not the only possibility. The sheets of many-sheeted space-time having overlapping M^4 projections provide second mechanism. The emergence of many-sheetedness could also be caused by the increase of $n = h_{eff}/h_0$ as a number of sheets of Galois covering.
2. Could the 3-volume increase during deterministic classical time evolution? If the minimal surface property assumed for the preferred extremals as a realization of quantum criticality is true everywhere, the conservation of volume energy prevents the increase of the volume. Minimal surface property is however assumed to fail at discrete set of points due to the transfer of conserved charged between Kähler and volume degrees of freedom. Could this make possible the increase of volume during classical time evolution so that volume and Kähler energy could increase?
   Remark: While writing this for the first time, I did not yet realize that if the action contains also parts associated with string world sheets and their light-like boundaries as M^8-H duality suggests, then the transfer of conserved quantities between space-time interior and string world sheets and string world sheets and their boundaries is possible, and implies the failure of the

minimal surface property at these surfaces. One can however formulated precisely the proposed option and it implies that also string world sheets are quantum critical and therefore minimal surfaces: the question whether this occurs everywhere or only for the portions of string world sheets near the boundaries of causal diamonds remains open [L66].

3. ZEO allows the increase of average 3-volume by quantum jumps. There is no reason why each "big" state function reduction changing the roles of the light-like boundaries of CD could not decrease the average volume energy of space-time surface for the time evolutions in the superposition. This can occur in all scales, and could be achieved also by the increase of $h_{eff}/h_0 = n$.

4. The geometry of CD suggests strongly an analogy with Big Bang followed by Big Crunch. The increase of the volume as increase of the volume of M^4 projection does not however seem to be consistent with Big Crunch. One must be very cautious here. The point is that the size of CD itself increases during the sequence of small state function reductions leaving the members of state pairs at passive boundary of CD unaffected. The size of 3-surface at the active boundary of CD therefore increases as also its 3-volume.

   The increase of the volume during the Big Crunch period could be also due to the emergence of the many-sheetedness, in particular due to the increase of the value of n for space-time sheets for sub-CDs. In this case, this period could be seen as a transition to quantum criticality accompanied by an emergence of complexity.

Is the cosmological constant really understood?

The interpretation of the coefficient of the volume term as cosmological constant has been a long-standing interpretational issue and caused many moments of despair during years. The intuitive picture has been that cosmological constant obeys p-adic length scale scale evolution meaning that Λ would behave like $1/L_p^2 = 1/p \simeq 1/2^k$ [K11].

This would solve the problems due to the huge value of Λ predicted in GRT approach: the smoothed out behavior of Λ would be $\Lambda \propto 1/a^2$, a light-cone proper time defining cosmic time, and the recent value of Λ - or rather, its value in length scale corresponding to the size scale of the observed Universe - would be extremely small. In the very early Universe - in very short length scales - Λ would be large.

A simple solution of the problem would be the p-adic length scale evolution of Λ as $\Lambda \propto 1/p$, $p \simeq 2^k$. The flux tubes would thicken until the string tension as energy density would reach stable minimum. After this a phase transition reducing the cosmological constant would allow further thickening of the flux tubes. Cosmological expansion would take place as this kind of phase transitions (for a mundane application of this picture see [K34]).

This would solve the basic problem of cosmology, which is understanding why cosmological constant manages to be so small at early times. Time evolution would be replaced with length scale evolution and cosmological constant would be indeed huge in very short scales but its recent value would be extremely small.

I have however not really understood how this evolution could be realized! Twistor lift seems to allow only a very slow (logarithmic) p-adic length scale evolution of Λ [L54]. Is there any cure to this problem?

1. The magnetic energy decreases with the area S of flux tube as $1/S \propto 1/p \simeq 1/2^k$, where $\sqrt{p}$ defines the transversal length scale of the flux tube. Volume energy (magnetic energy associated with the twistor sphere) is positive and increases like S. The sum of these has minimum for certain radius of flux tube determined by the value of Λ. Flux tubes with quantized flux would have thickness determined by the length scale defined by the density of dark energy: $L \sim \rho_{vac}^{-1/4}$, $\rho_{dark} = \Lambda/8\pi G$. $\rho_{vac} \sim 10^{-47}$ GeV^4 (see `http://tinyurl.com/k4bwlzu`) would give $L \sim 1$ mm, which would could be interpreted as a biological length scale (maybe even neuronal length scale).

2. But can Λ be very small? In the simplest picture based on dimensionally reduced 6-D Kähler action this term is not small in comparison with the Kähler action! If the twistor spheres of M^4 and CP_2 give the same contribution to the induced Kähler form at twistor sphere of X^4, this term has maximal possible value!

The original discussions in [K91, K11] treated the volume term and Kähler term in the dimensionally reduced action as independent terms and Λ was chosen freely. This is however not the case since the coefficients of both terms are proportional to $(1/\alpha_K^2)S(S^2)$, where $S(S^2)$ is the area of the twistor sphere of 6-D induced twistor bundle having space-time surface as base space. This are is same for the twistor spaces of M^4 and CP_2 if CP_2 size defines the only fundamental length scale. I did not even recognize this mistake.

The proposed fast p-adic length scale evolution of the cosmological constant would have extremely beautiful consequences. Could the original intuitive picture be wrong, or could the desired p-adic length scale evolution for Λ be possible after all? Could non-trivial dynamics for dimensional reduction somehow give it? To see what can happen one must look in more detail the induction of twistor structure.

1. The induction of the twistor structure by dimensional reduction involves the identification of the twistor spheres S^2 of the geometric twistor spaces $T(M^4) = M^4 \times S^2(M^4)$ and of T_{CP_2} having $S^2(CP_2)$ as fiber space. What this means that one can take the coordinates of say $S^2(M^4)$ as coordinates and embedding map maps $S^2(M^4)$ to $S^2(CP_2)$. The twistor spheres $S^2(M^4)$ and $S^2(CP_2)$ have in the minimal scenario same radius $R(CP_2)$ (radius of the geodesic sphere of CP_2. The identification map is unique apart from $SO(3)$ rotation R of either twistor sphere possibly combined with reflection P. Could one consider the possibility that R is not trivial and that the induced Kähler forms could almost cancel each other?

2. The induced Kähler form is sum of the Kähler forms induced from $S^2(M^4)$ and $S^2(CP_2)$ and since Kähler forms are same apart from a rotation in the common S^2 coordinates, one has $J_{ind} = J + RP(J)$, where R denotes a rotation and P denotes reflection. Without reflection one cannot get arbitrary small induced Kähler form as sum of the two contributions. For mere reflection one has $J_{ind} = 0$.
   Remark: It seems that one can do with reflection if the Kähler forms of the twistor spheres are of opposite sign in standard spherical coordinates. This would mean that they have have opposite orientation.
   One can choose the rotation to act on (y, z)-plane as $(y, z) \rightarrow (cy + sz, -sz + cy)$, where s and c denote the cosines of the rotation angle. A small value of cosmological constant is obtained for small value of s. Reflection P can be chosen to correspond to $z \rightarrow -z$. Using coordinates $(u = cos(\Theta), \Phi)$ for $S^2(M^4)$ and (v, Ψ) for $S^2(CP_2)$ and by writing the reflection followed by rotation explicitly in coordinates (x, y, z) one finds $v = -cu - s\sqrt{1-u^2}sin(\Phi)$, $\Psi = arctan[(su/\sqrt{1-u^2}cos(\Phi) + ctan(\Phi)]$. In the lowest order in s one has $v = -u - s\sqrt{1-u^2}sin(\Phi)$, $\Psi = \Phi + scos(\Phi)(u/\sqrt{1-u^2})$.

3. Kähler form J^{ind} is sum of unrotated part $J(M^4) = du \wedge d\Phi$ and $J(CP_2) = dv \wedge d\Psi$. $J(CP_2)$ equals to the determinant $\partial(v, \Psi)/\partial(u, \Phi)$. A suitable spectrum for s could reproduce the proposal $\Lambda \propto 2^{-k}$ for Λ. The S^2 part of 6-D Kähler action equals to $(J_{\theta\phi}^{ind})^2/\sqrt{g_2}$ and in the lowest order proportional to s^2. For small values of s the integral of Kähler action for S^2 over S^2 is proportional to s^2.
   One can write the S^2 part of the dimensionally reduced action as $S(S^2) = s^2F^2(s)$. Very near to the poles the integrand has $1/[sin(\Theta) + O(s)]$ singularity and this gives rise to a logarithmic dependence of F on s and one can write: $F = F(s, log(s))$. In the lowest order one has $s \simeq 2^{-k/2}$, and in improved approximation one obtains a recursion formula $s_n(S^2, k) = 2^{-k/2}/F(s_{n-1}, \log(s_{n-1})$ giving renormalization group evolution with k replaced by anomalous dimension $k_{n,a} = k + 2log[F(s_{n-1}, log(s_{n-1})]$ differing logarithmically from k.

4. The sum $J^{ind} = J + RP(J)$ defining the induced Kähler form in $S^2(X^4)$ is covariantly constant since both terms are covariantly constant by the rotational covariance of J.

5. The embeddings of $S^2(X^4)$ as twistor sphere of space-time surface to both spheres are holomorphic since rotations are represented as holomorphic transformations. Also reflection as $z \rightarrow 1/z$ is holomorphic. This in turn implies that the second fundamental form in complex coordinates is a tensor having only components of type $(1, 1)$ and $(-1, -1)$ whereas metric and energy momentum tensor have only components of type $(1, -1)$ and $(-1, 1)$. Therefore all contractions appearing in field equations vanish identically and $S^2(X^4)$ is minimal surface and Kähler current in $S^2(X^4)$ vanishes since it involves components of the trace of second fundamental form. Field equations are indeed satisfied.

6. The solution of field equations becomes a family of space-time surfaces parameterized by the values of the cosmological constant Λ as function of S^2 coordinates satisfying $\Lambda/8\pi G = \rho_{vac} = J \wedge (*J)(S^2)$. In long length scales the variation range of Λ would become arbitrary small.
7. If the minimal surface equations solve separately field equations for the volume term and Kähler action everywhere apart from a discrete set of singular points, the cosmological constant affects the space-time dynamics only at these points. The physical interpretation of these points is as seats of fundamental fermions at partonic 2-surface at the ends of light-like 3-surfaces defining their orbits (induced metric changes signature at these 3-surfaces). Fermion orbits would be boundaries of fermionic string world sheets.
   One would have family of solutions of field equations but particular value of Λ would make itself visible only at the level of elementary fermions by affecting the values of coupling constants. p-Adic coupling constant evolution would be induced by the p-adic coupling constant evolution for the relative rotations R combined with reflection for the two twistor spheres. Therefore twistor lift would not be mere manner to reproduce cosmological term but determine the dynamics at the level of coupling constant evolution.
8. What is nice that also $\Lambda = 0$ option is possible. This would correspond to the variant of TGD involving only Kähler action regarded as TGD before the emergence of twistor lift. Therefore the nice results about cosmology [K81] obtained at this limit would not be lost.

10.3.3 Does p-adic coupling constant evolution reduce to that for cosmological constant?

One of the chronic problems if TGD has been the understanding of what coupling constant evolution could be defined in TGD.

Basic notions and ideas

Consider first the basic notions and ideas.

1. The notion of quantum criticality is certainly central. The continuous coupling constant evolution having no counterpart in the p-adic sectors of adele would contain as a sub-evolution discrete p-adic coupling constant evolution such that the discrete values of coupling constants allowing interpretation also in p-adic number fields are fixed points of coupling constant evolution.
   Quantum criticality is realized also in terms of zero modes, which by definition do not contribute to WCW metric. Zero modes are like control parameters of a potential function in catastrophe theory. Potential function is extremum with respect to behavior variables replaced now by WCW degrees of freedom. The graph for preferred extremals as surface in the space of zero modes is like the surface describing the catastrophe. For given zero modes there are several preferred extremals and the catastrophe corresponds to the regions of zero mode space, where some branches of co-incide. The degeneration of roots of polynomials is a concrete realization for this.
   Quantum criticality would also mean that coupling parameters effectively disappear from field equations. For minimal surfaces (generalization of massless field equation allowing conformal invariance characterizing criticality) this happens since they are separately extremals of Kähler action and of volume term.
   Quantum criticality is accompanied by conformal invariance in the case of 2-D systems and in TGD this symmetry extends to its 4-D analogas isometries of WCW.
2. In the case of 4-D Kähler action the natural hypothesis was that coupling constant evolution should reduce to that of Kähler coupling strength $1/\alpha_K$ inducing the evolution of other coupling parameters. Also in the case of the twistor lift $1/\alpha_K$ could have similar role. One can however ask whether the value of the 6-D Kähler action for the twistor sphere $S^2(X^4)$ defining cosmological constant could define additional parameter replacing cutoff length scale as the evolution parameter of renormalization group.
3. The hierarchy of adeles should define a hierarchy of values of coupling strengths so that the discrete coupling constant evolution could reduce to the hierarchy of extensions of rationals and be expressible in terms of parameters characterizing them.

4. I have also considered number theoretical existence conditions as a possible manner to fix the values of coupling parameters. The condition that the exponent of Kähler function should exist also for the p-adic sectors of the adele is what comes in mind as a constraint but it seems that this condition is quite too strong.
   If the functional integral is given by perturbations around single maximum of Kähler function, the exponent vanishes from the expression for the scattering amplitudes due to the presence of normalization factor. There indeed should exist only single maximum by the Euclidian signature of the WCW Kähler metric for given values of zero modes (several extrema would mean extrema with non-trivial signature) and the parameters fixing the topology of 3-surfaces at the ends of preferred extremal inside CD. This formulation as counterpart also in terms of the analog of micro-canonical ensemble (allowing only states with the same energy) allowing only discrete sum over extremals with the same Kähler action [L52].
5. I have also considered more or less ad hoc guesses for the evolution of Kähler coupling strength such as reduction of the discrete values of $1/\alpha_K$ to the spectrum of zeros of Riemann zeta or actually of its fermionic counterpart [L18]. These proposals are however highly ad hoc.

Could the area of twistor sphere replace cutoff length?

As I started once again to consider coupling constant evolution I realized that the basic problem has been the lack of explicit formula defining what coupling constant evolution really is.

1. In quantum field theories (QFTs) the presence of infinities forces the introduction of momentum cutoff. The hypothesis that scattering amplitudes do not depend on momentum cutoff forces the evolution of coupling constants. TGD is not plagued by the divergence problems of QFTs. This is fine but implies that there has been no obvious manner to define what coupling constant evolution as a continuous process making sense in the real sector of adelic physics could mean!
2. Cosmological constant is usually experienced as a terrible head ache but it could provide the helping hand now. Could the cutoff length scale be replaced with the value of the length scale defined by the cosmological constant defined by the S^2 part of 6-D Kähler action? This parameter would depend on the details of the induced twistor structure. It was shown above that if the moduli space for induced twistor structures corresponds to rotations of S^2 possibly combined with the reflection, the parameter for coupling constant restricted to that to $SO(2)$ subgroup of $SO(3)$ could be taken to be taken $s = sin(\epsilon)$.
3. RG invariance would state that the 6-D Kähler action is stationary with respect to variations with respect to s. The variation with respect to s would involve several contributions. Besides the variation of $1/\alpha_K(s)$ and the variation of the $S(^2)$ part of 6-D Kähler action defining the cosmological constant, there would be variation coming from the variations of 4-D Kähler action plus 4-D volume term . This variation vanishes by field equations. As matter of fact, the variations of 4-D Kähler action and volume term vanish separately except at discrete set of singular points at which there is energy transfer between these terms. This condition is one manner to state quantum criticality stating that field equations involved no coupling parameters.
   One obtains explicit RG equation for α_K and Λ having the standard form involving logarithmic derivatives. The form of the equation would be
   $$\frac{dlog(\alpha_K)}{ds} = -\frac{S(S^2)}{(S_K(X^4)/Vol(X^4)) + S(S^2)} \frac{dlog(S(S^2))}{ds} \quad . \tag{10.3.1}$$
   It should be noticed that the choices of the parameter s in the evolution equation is arbitrary so that the identification $s = sin(\epsilon)$ is not necessary. Note that one must use Kähler action per volume.
   The equation contains the ratio $S(S^2)/(S_K(X^4) + S(S^2))$ of actions as a parameter. This does not conform with idea of micro-locality. One can however argue that this conforms with the generalization of point like particle to 3-D surface. For preferred extremal the action is indeed determined by the 3 surfaces at its ends at the boundaries of CD. This implies that the construction of quantum theory requires the solution of classical theory.
   In particular, the 4-D classical theory is necessary for the construction of scattering amplitudes, and one cannot reduce TGD to string theory although strong form of holography states that

the data about quantum states can be assigned with 2-D surfaces. Even more: M^8-H correspondence implies that the data determining quantum states can be assigned with discrete set of points defining cognitive representations for given adel This set of points depends on the preferred extremal!

4. How to identify quantum critical values of α_K? At these points one should have $dlog(\alpha_K)/ds = 0$. This implies $dlog(S(S^2)/ds = 0$, which in turn implies $dlog(\alpha_K)/ds = 0$ unless one has $S_K(X^4)+S(S^2) = 0$. This condition would make exponent of 6-D Kähler action trivial and the continuation to the p-adic sectors of adele would be trivial. I have considered also this possibility [L54].
   The critical values of coupling constant evolution would correspond to the critical values of S and therefore of cosmological constant. The basic nuisance of theoretical physics would determine the coupling constant evolution completely! Critical values are in principle possible. Both the numerator $J^2_{u\Phi}$ and the numerator $1/\sqrt{det(g)}$ increase with ϵ. If the rate for the variation of these quantities with s vary it is possible to have a situation in which the one has
   $$\frac{dlog(J^2_{u\Phi})}{ds} = -\frac{dlog(\sqrt{det(g)})}{ds} \quad . \tag{10.3.2}$$
5. One can make highly non-trivial conclusions about the evolution at general level. For the extremals with vanishing action and for which α_K is critical (vanishing derivate), also the second derivative of $d^2S(S^2)/ds^2 =$ holds true at the critical point. The QFT analogs of these points are points at which beta function develops higher order zero. The tip of cusp catastrophe is second analogy.
   The points at which that the action has minimum are also interesting. For magnetic flux tubes for which one has $S_K(X^4) \propto 1/S$ and $S_{vol} \propto S$ in good approximation, one has $S_K(X^4) = S_{vol}$ at minimum (say for the flux tubes with radius about 1 mm for the cosmological constant in cosmological scales). One can write
   $$\frac{dlog(\alpha_K)}{ds} = -\frac{1}{2}\frac{dlog(S(S^2))}{ds} \quad , \tag{10.3.3}$$
   and solve the equation explicitly:
   $$\frac{\alpha_{K,0}}{\alpha_K} = \frac{S(S^2)}{S(S^2)_0})^x \quad , \quad x = 1/2 \quad . \tag{10.3.4}$$
   A more general situation would correspond to a model with $x \neq 1/2$: the deviation from $x = 1/2$ could be interpreted as anomalous dimension. This allows to deduce numerically a formula for the value spectrum of $\alpha_{K,0}/\alpha_K$ apart from the initial values.
6. One can solve the equation also for fixed value of $S(X^4)/Vol(X^4)$ to get
   $$\frac{\alpha_{K,0}}{\alpha_K} = \frac{S(S^2)}{S(S^2)_0})^x \quad , \quad x = 1/2 \quad . \tag{10.3.5}$$
   $$\frac{\alpha_K}{\alpha_{K,0}} = \frac{S_K(X^4)/Vol(X^4)) + S(S^2)}{S_K(X^4)/Vol(X^4))} \quad . \tag{10.3.6}$$
   At the limit $S(S^2) =\rightarrow 0$ one obtains $\alpha_K \rightarrow \alpha_{K,0}$.
7. One should demonstrate that the critical values of s are such that the continuation to p-adic sectors of the adele makes sense. For preferred extremals cosmological constant appears as a parameter in field equations but does not affect the field equations expect at the singular points. Singular points play the same role as the poles of analytic function or point charges in electrodynamics inducing long range correlations. Therefore the extremals depend on parameter s and the dependence should be such that the continuation to the p-adic sectors is possible.
   A naïve guess is that the values of s are rational numbers. Above the proposal $s = 2^{-k/2}$ motivated by p-adic length scale hypothesis was considered but also $s = p^{-k/2}$ can be considered. These guesses might be however wrong, the most important point is that there is that one can indeed calculate $\alpha_K(s)$ and identify its critical values.

8. What about scattering amplitudes and evolution of various coupling parameters? If the exponent of action disappears from scattering amplitudes, the continuation of scattering amplitudes is simple. This seems to be the only reasonable option. In the adelic approach [L41] amplitudes are determined by data at a discrete set of points of space-time surface (defining what I call cognitive representation) for which the points have M^8 coordinates belong to the extension of rationals defining the adele.
Each point of $S^2(X^4)$ corresponds to a slightly different X^4 so that the singular points depend on the parameter s, which induces dependence of scattering amplitudes on s. Since coupling constants are identified in terms of scattering amplitudes, this induces coupling constant evolution having discrete coupling constant evolution as sub-evolution.

Could the critical values of α_K correspond to the zeros of Riemann Zeta?

Number theoretical intuitions strongly suggests that the critical values of $1/\alpha_K$ could somehow correspond to zeros of Riemann Zeta. Riemann zeta is indeed known to be involved with critical systems.

The naïvest ad hoc hypothesis is that the values of $1/\alpha_K$ are actually proportional to the non-trivial zeros $s = 1/2 + iy$ of zeta [L18]. A hypothesis more in line with QFT thinking is that they correspond to the imaginary parts of the roots of zeta. In TGD framework however complex values of α_K are possible and highly suggestive. In any case, one can test the hypothesis that the values of $1/\alpha_K$ are proportional to the zeros of ζ at critical line. Problems indeed emerge.

1. The complexity of the zeros and the non-constancy of their phase implies that the RG equation can hold only for the imaginary part of $s = 1/2 + it$ and therefore only for the imaginary part of the action. This suggests that $1/\alpha_K$ is proportional to y. If $1/\alpha_K$ is complex, RG equation implies that its phase RG invariant since the real and imaginary parts would obey the same RG equation.
2. The second - and much deeper - problem is that one has no reason for why $dlog(\alpha_K)/ds$ should vanish at zeros: one should have $dy/ds = 0$ at zeros but since one can choose instead of parameter s any coordinate as evolution parameter, one can choose $s = y$ so that one has $dy/ds = 1$ and criticality condition cannot hold true. Hence it seems that this proposal is unrealistic although it worked qualitatively at numerical level.

It seems that it is better to proceed in a playful spirit by asking whether one could realize quantum criticality in terms of of the property of being zero of zeta.

1. The very fact that zero of zeta is in question should somehow guarantee quantum criticality. Zeros of ζ define the critical points of the complex analytic function defined by the integral

$$X(s_0, s) = \int_{C_{s_0 \to s}} \zeta(s) ds \ , \tag{10.3.7}$$

where $C_{s_0 \to s}$ is any curve connecting zeros of ζ, a is complex valued constant. Here s does not refer to $s = sin(\epsilon)$ introduced above but to complex coordinate s of Riemann sphere.
By analyticity the integral does not depend on the curve C connecting the initial and final points and the derivative $dS_c/ds = \zeta(s)$ vanishes at the endpoints if they correspond to zeros of ζ so that would have criticality. The value of the integral for a closed contour containing the pole $s = 1$ of ζ is non-vanishing so that the integral has two values depending on which side of the pole C goes.
2. The first guess is that one can define S_c as complex analytic function $F(X)$ having interpretation as analytic continuation of the S^2 part of action identified as $Re(S_c)$:

$$\begin{aligned} S_c(S^2) &= F(X(s, s_0)) \ , \qquad X(s, s_0) = \int_{C_{s_0 \to s}} \zeta(s) ds \ , \\ S(S^2) &= Re(S_c) = Re(F(X)) \ , \\ \zeta(s) &= 0 \ , \qquad Re(s_0) = 1/2 \ . \end{aligned} \tag{10.3.8}$$

$S_c(S^2) = F(X)$ would be a complexified version of the Kähler action for S^2. s_0 must be at critical line but it is not quite clear whether one should require $\zeta(s_0) = 0$.

The real valued function $S(S^2)$ would be thus extended to an analytic function $S_c = F(X)$ such that the $S(S^2) = Re(S_c)$ would depend only on the end points of the integration path $C_{s_0 \rightarrow s}$. This is geometrically natural. Different integration paths at Riemann sphere would correspond to paths in the moduli space $SO(3)$, whose action defines paths in S^2 and are indeed allowed as most general deformations. Therefore the twistor sphere could be identified Riemann sphere at which Riemann zeta is defined. The critical line and real axis would correspond to particular one parameter sub-groups of $SO(3)$ or to more general one parameter subgroups.
One would have

$$\frac{\alpha_{K,0}}{\alpha_K} = (\frac{S_c}{S_0})^{1/2} \ . \tag{10.3.9}$$

The imaginary part of $1/\alpha_K$ (and in some sense also of the action $S_c(S^2)$) would determined by analyticity somewhat like the real parts of the scattering amplitudes are determined by the discontinuities of their imaginary parts.

3. What constraints can one pose on F? F must be such that the value range for $F(X)$ is in the value range of $S(S^2)$. The lower limit for $S(S^2)$ is $S(S^2) = 0$ corresponding to $J_{u\Phi} \rightarrow 0$.
The upper limit corresponds to the maximum of $S(S^2)$. If the one Kähler forms of M^4 and S^2 have same sign, the maximum is $2 \times A$, where $A = 4\pi$ is the area of unit sphere. This is however not the physical case.
If the Kähler forms of M^4 and S^2 have opposite signs or if one has RP option, the maximum, call it S_{max}, is smaller. Symmetry considerations strongly suggest that the upper limit corresponds to a rotation of 2π in say (y, z) plane ($s = sin(\epsilon) = 1$ using the previous notation). For $s \rightarrow s_0$ the value of S_c approaches zero: this limit must correspond to $S(S^2) = 0$ and $J_{u\Phi} \rightarrow 0$. For $Im(s) \rightarrow \pm\infty$ along the critical line, the behavior of $Re(\zeta)$ (see `http://tinyurl.com/y7b88gvg`) strongly suggests that $|X| \rightarrow \infty$. This requires that F is an analytic function, which approaches to a finite value at the limit $|X| \rightarrow \infty$. Perhaps the simplest elementary function satisfying the saturation constraints is

$$F(X) = S_{max} tanh(-iX) \ . \tag{10.3.10}$$

One has $tanh(x + iy) \rightarrow \pm 1$ for $y \rightarrow \pm\infty$ implying $F(X) \rightarrow \pm S_{max}$ at these limits. More explicitly , one has $tanh(-i/2 - y) = [-1 + exp(-4y) - 2exp(-2y)(cos(1) - 1)]/[1 + exp(-4y) - 2exp(-2y)(cos(1) - 1)]$. Since one has $tanh(-i/2 + 0) = 1 - 1/cos(1) < 0$ and $tanh(-i/2 + \infty) = 1$, one must have some finite value $y = y_0 > 0$ for which one has

$$tanh(-\frac{i}{2} + y_0) = 0 \ . \tag{10.3.11}$$

The smallest possible lower bound s_0 for the integral defining X would naturally to $s_0 = 1/2 - iy_0$ and would be below the real axis.

4. The interpretation of $S(S^2)$ as a positive definite action requires that the sign of $S(S^2) = Re(F)$ for a given choice of $s_0 = 1/2 + iy_0$ and for a propertly sign of $y - y_0$ at critical line should remain positive. One should show that the sign of $S = a \int Re(\zeta)(s = 1/2 + it)dt$ is same for all zeros of ζ. The graph representing the real and imaginary parts of Riemann zeta along critical line $s = 1/2 + it$ (see `http://tinyurl.com/y7b88gvg`) shows that both the real and imaginary part oscillate and increase in amplitude. For the first zeros real part stays in good approximation positive but the amplitude for the negative part increase be gradually. This suggests that S identified as integral of real part oscillates but preserves its sign and gradually increases as required.

A priori there is no reason to exclude the trivial zeros of ζ at $s = -2n$, $n = 1, 2, ...$

1. The natural guess is that the function $F(X)$ is same as for the critical line. The integral defining X would be along real axis and therefore real as also $1/\alpha_K$ provided the sign of S_c is positive: for negative sign for S_c not allowed by the geometric interpretation the square root would give imaginary unit. The graph of the Riemann Zeta at real axis (real) is given in MathWorld Wolfram (see `http://tinyurl.com/55qjmj`).
2. The functional equation

$$\zeta(1 - s) = \zeta(s) \frac{\Gamma(s/2)}{\Gamma((1 - s)/2)} \tag{10.3.12}$$

allows to deduce information about the behavior of ζ at negative real axis. $\Gamma((1-s)/2)$ is negative along negative real axis (for $Re(s) \leq 1$ actually) and poles at $n+1/2$. Its negative maxima approach to zero for large negative values of $Re(s)$ (see `http://tinyurl.com/clxv4pz`) whereas $\zeta(s)$ approaches value one for large positive values of s (see `http://tinyurl.com/y7b88gvg`). A cautious guess is that the sign of $\zeta(s)$ for $s \leq 1$ remains negative. If the integral defining the area is defined as integral contour directed from $s<0$ to a point s_0 near origin, S_c has positive sign and has a geometric interpretation.

3. The formula for $1/\alpha_K$ would read as $\alpha_{K,0}/\alpha_K(s=-2n) = (S_c/S_0)^{1/2}$ so that α_K would remain real. This integration path could be interpreted as a rotation around z-axis leaving invariant the Kähler form J of $S^2(X^4)$ and therefore also $S = Re(S_c)$. $Im(S_c) = 0$ indeed holds true. For the non-trivial zeros this is not the case and $S = Re(S_c)$ is not invariant.
4. One can wonder whether one could distinguish between Minkowskian and Euclidian and regions in the sense that in Minkowskian regions $1/\alpha_K$ correspond to the non-trivial zeros and in Euclidian regions to trivial zeros along negative real axis. The interpretation as different kind of phases might be appropriate.

What is nice that the hypothesis about equivalence of the geometry based and number theoretic approaches can be killed by just calculating the integral S as function of parameter s. The identification of the parameter s appearing in the RG equations is no unique. The identification of the Riemann sphere and twistor sphere could even allow identify the parameter t as imaginary coordinate in complex coordinates in $SO(3)$ rotations around z-axis act as phase multiplication and in which metric has the standard form.

Some guesses to be shown to be wrong

The following argument suggests a connection between p-adic length scale hypothesis and evolution of cosmological constant but must be taken as an ad hoc guess: the above formula is enough to predict the evolution.

1. p-Adicization is possible only under very special conditions [L41], and suggests that anomalous dimension involving logarithms should vanish for $s = 2^{-k/2}$ corresponding to preferred p-adic length scales associated with $p \simeq 2^k$. Quantum criticality in turn requires that discrete p-adic coupling constant evolution allows the values of coupling parameters, which are fixed points of RG group so that radiative corrections should vanish for them. Also anomalous dimensions Δk should vanish.
2. Could one have $\Delta k_{n,a} = 0$ for $s = 2^{-k/2}$, perhaps for even values $k = 2k_1$? If so, the ratio c/s would satisfy $c/s = 2^{k_1} - 1$ at these points and Mersenne primes as values of c/s would be obtained as a special case. Could the preferred p-adic primes correspond to a prime near to but not larger than $c/s = 2^{k_1} - 1$ as p-adic length scale hypothesis states? This suggest that we are on correct track but the hypothesis could be too strong.
3. The condition $\Delta d = 0$ should correspond to the vanishing of dS/ds. Geometrically this would mean that $S(s)$ curve is above (below) $S(s) = xs^2$ and touches it at points $s = x2^{-k}$, which would be minima (maxima). Intermediate extrema above or below $S = xs^2$ would be maxima (minima).

10.3.4 An alternative view about the coupling constant evolution in terms of cosmological constant

The above view about the evolution of cosmological constant relies crucially on the identification of $M^4 \times S^2$ as twistor space of M^4, and the assumption that the radii of twistor spheres $S^2(M^4$ and $S^2(CP_2)$ assignable to the twistor bundle of CP_2 are same.

One can however argue that the standard twistor space CP_3 of M^4 with Minkowskian signature (3,-3) is a more feasible candidate for the twistor space of M^4. Accepting this, one ends up to a modification of the above vision about coupling constant evolution [L76, L79]. The progress in understanding SUSY in TGD framework led also to a dramatic progress in the understanding of the coupling constant evolution [L73].

Getting critical about geometric twistor space of M^4

Let us first discuss the recent picture and how to modify it so that it is consistent with the hierarchy of CDs. The key idea is that the twistor space and its base space represents CD so that one obtains scale hierarchy of twistor spaces as a realization of broken scale invariance giving rise to the p-adic length scale hierarchy.

1. I have identified the twistor space of M^4 simply as $T(M^4) = M^4 \times S^2$. The interpretation would be at the level of octonions as a product of M^4 and choices of M^2 as preferred complex sub-space of octonions with S^2 parameterizing the directions of spin quantization axes. Real octonion axis would correspond to time coordinate. One could talk about the space of of light-like directions. Light-like vector indeed defines M^2. This view could be defended by the breaking of both translation and Lorentz invariance in the octonionic approach due to the choice of M^2 and by the fact that it seems to work.
   Remark: $M^8 = M^4 \times E^4$ is complexified to M^8_c by adding a commuting imaginary unit i appearing in the extensions of rationals and ordinary M^8 represents its particular sub-space. Also in twistor approach one uses often complexified M^4.
2. The objection is that it is ordinary twistor space identifiable as CP_3 with (3,-3) signature of metric is what works in the construction of twistorial amplitudes. CP_3 has metric as compact space and coset space. Could this choice of twistor space make sense after all as geometric twistor space?
   Here one must pause and recall that the original key idea was that Poincare invariance is symmetry of TGD for $X^4 \subset M^4 \times CP_2$. Now Poincare symmetry has been transformed to a symmetry acting at the level of M^8 in the moduli space of octonion structures defined by the choice of the direction of octonionic real axis reducing Poincare group to $T \times SO(3)$ consisting of time translations and rotations. Fixing of M^2 reducrs the group to $T \times SO(2)$ and twistor space can be seen as the space for selections of quantization axis of energy and spin.
3. But what about the space H? The first guess is $H = M^4_{conf} \times CP_2$. According to [B6] (see `http://tinyurl.com/y35k5wwo`) one has $M^4_{conf} = U(2)$ such that $U(1)$ factor is time- like and $SU(2)$ factor is space-like. One could understand $M^4_{conf} = U(2)$ as resulting by addition and identification of metrically 2-D light-cone boundaries at $t = \pm\infty$. This is topologically like compactifying E^3 to S^3 and gluing the ends of cylinder $S^3 \times D^1$ together to the $S^3 \times S^1$. The conformally compactified Minkowski space M^4_{conf} should be analogous to base space of CP_3 regarded as bundle with fiber S^2. The problem is that one cannot imagine an analog of fiber bundle structure in CP_3 having $U(2)$ as base. The identification $H = M^4_{conf} \times CP_2$ does not make sense.
4. In ZEO based breaking of scaling symmetry it is CD that should be mapped to the analog of M^4_{conf} - call it cd_{conf}. The only candidate is $cd_{conf} = CP_2$ with one hypercomplex coordinate. To understand why one can start from the following picture. The light-like boundaries of CD are metrically equivalent to spheres. The light-like boundaries at $t = \pm\infty$ are identified as in the case of M^4_{conf}. In the case of CP_2 one has 3 homologically trivial spheres defining coordinate patches. This suggests that cd_{conf} is simply CP_2 with second complex coordinate made hypercomplex. M^4 and E^4 differ only by the signature and so would do cd_{conf} and CP_2. The twistor spheres of CP_3 associated with points of M^4 intersect at point if the points differ by light-like vector so that one has and singular bundle structure. This structure should have analog for the compactification of CD. CP_3 has also bundle structure $CP_3 \to CP_2$. The S^2 fibers and base are homologically non-trivial and complex analogs of mutually orthogonal line and plane and intersect at single point. This defines the desired singular bundle structure via the assignment of S^2 to each point of CP_2.
   The M^4 points must belong to the interior of cd and this poses constraints on the distance of M^4 points from the tips of cd. One expects similar hierarchy of cds at the level of momentum space.
5. In this picture $M^4_{conf} = U(2)$ could be interpreted as a base space for the space of CDs with fixed direction of time axis identified as direction of octonionic real axis associated with various points of M^4 and therefore of M^4_{conf}. For Euclidian signature one would have base and fiber of the automorphism sub-group $SU(3)$ regarded as $U(2)$ bundle over CP_2: now one would have CP_2 bundle over $U(2)$. This is perhaps not an accident, and one can ask whether these spaces

could be interpreted as representing local trivialization of $SU(3)$ as $U(2) \times CP_2$. This would give to metric cross terms between $U(2)$ and CP_2.

6. The proposed identification can be tested by looking whether it generalizes. What the twistor space for entire M^8 would be? $cd = CD_4$ is replaced with CD_8 and the discussion of the preceding chapter demonstrated that the only possible identification of the twistor space is now is as the 12-D hyperbolic variant of HP_3 whereas $CD_{8,conf}$ would correspond to 8-D hyperbolic variant of HP_2 analogous to hyperbolic variant of CP_2.

The outcome of these considerations is surprising.

1. One would have $T(H) = CP_3 \times F$ and $H = CP_{2,H} \times CP_2$ where $CP_{2,H}$ has hyperbolic metric with metric signature $(1, -3)$ having M^4 as tangent space so that the earlier picture can be understood as an approximation. This would reduce the construction of preferred extremals of 6-D Kähler action in $T(H)$ to a construction of polynomial holomorphic surfaces and also the minimal surfaces with singularities at string world sheets should result as bundle projection. Since $M^8 - H$ duality must respect algebraic dynamics the maximal degree of the polynomials involved must be same as the degree of the octonionic polynomial in M^8.
2. The hyperbolic variant Kähler form and also spinor connection of hyperbolic CP_2 brings in new physics beyond standard model. This Kähler form would serve as the analog of Kähler form assigned to M^4 earlier, and suggested to explain the observed CP breaking effects and matter antimatter asymmetry for which there are two explanations [L73].

Some comments about the Minkowskian signature of the hyperbolic counterparts of CP_3 and CP_2 are in order.

1. Why the metric of CP_3 could not be Euclidian just as the metric of F? The basic objection is that propagation of fields is not possible in Euclidian signature and one completely loses the earlier picture provided by $M^4 \times CP_2$. The algebraic dynamics in M^8 picture can hardly replace it.
2. The map assigning to the point M^4 a point of CP_3 involves Minkowskian sigma matrices but it seems that the Minkowskian metric of CP_3 is not explicitly involved in the construction of scattering amplitudes. Note however that the antisymmetric bi-spinor metric for the spin 1/2 representation of Lorentz group and its conjugate bring in the signature. $U(2,2)$ as representation of conformal symmetries suggests $(2,2)$ signature for 8-D complex twistor space with 2+2 complex coordinates representing twistors.
   The signature of CP_3 metric is not explicitly visible in the construction of twistor amplitudes but analytic continuations are carried out routinely. One has also complexified M^4 and M^8 and one could argue that the problems disappear. In the geometric situation the signatures of the subspaces differ dramatically. As already found, analytic continuation could allow to define the variants of twistor spaces elegantly by replacing a complex coordinate with a hyperbolic one.
   Remark: For E^4 CP_3 is Euclidian and if one has $E^4_{conf} = U(2)$, one could think of replacing the Cartesian product of twistor spaces with $SU(3)$ group having $M^4_{conf} = U(2)$ as fiber and CP_2 as base. The metric of $SU(3)$ appearing as subgroup of quaternionic automorphisms leaving $M^4 \subset M^8$ invariant would decompose to a sum of M^4_{conf} metric and CP_2 metric plus cross terms representing correlations between the metrics of M^4_{conf} and CP_2. This is probably mere accident.

How the vision about coupling constant evolution would be modified?

The above described vision about coupling constant evolution in case of $T(M^4) = M^4 \times S^2$ would be modified since the interference of the Kähler form made possible by the same signature of $S^2(M^4)$ and $S^2(CP_2)$. Now the signatures are opposite and Kähler forms differ by factor i (imaginary unit commuting with octonion units) so that the induced Kähler forms do not interfere anymore. The evolution of cosmological constant must come from the evolution of the ratio of the radii of twistor spaces (twistor spheres).

1. $M^8 - H$ duality has two alternative forms with $H = CP_{2,h} \times CP_2$ or $H = M^4 \times CP_2$ depending on whether one projects the twistor spheres of $CP_{3,h}$ to $CP_{2,h}$ or M^4. Let us denote the twistor space $SU(3)/U(1) \times U(1)$ of CP_2 by F.

2. The key idea is that the p-adic length scale hierarchy for the size of 8-D CDs and their 4-D counterparts is mapped to a corresponding hierarchy for the sizes of twistor spaces $CP_{3,h}$ assignable to M^4 by M^8-H-duality. By scaling invariance broken only by discrete size scales of CDs one can take the size scale of CP_2 as a unit so that $r = R^2(S^2(CP_{3,h})/R(S^2(F))$ becomes an evolution parameter.
   Coupling constant evolution must correspond to a variation for the ratio of $r = R^2(S^2(CP_{3,h})/R(S^2(F))$ and a reduction to p-adic length scale evolution is expected. A simple argument shows that Λ is inversely proportional to constant magnetic energy assignable to $S^2(X^4)$ divided by $1/\sqrt{g_2(S^2)}$ in dimensional reduction needed to induce twistor structure. Thus one has $\Lambda \propto 1/r^2 \propto 1/L_p^2$. Preferred p-adic primes would be identified as ramified primes of extension of rationals defining the adele so that coupling constant evolution would reduce to number theory.
3. The induced metric would vanish for $R(S^2(CP_{3,h}) = R(S^2(F))$. Λ would be infinite at this limit so that one must have $R(S^2(CP_{3,h}) \neq R(S^2(F))$. The most natural assumption is that one $R(S^2(CP_{3,h}) > R(S^2(F))$ but one cannot exclude the alternative option. Λ behaves like $1/L_p^2$. Inversions of CDs with respect to the values of the cosmological time parameter $a = L_p$ would produce hierarchies of length scales, in particular p-adic length scales coming as powers of $\sqrt{p}$. CP_2 scale and the scale assignable to cosmological constant could be seen as inversions of each other with respect to a scale which is of order 10^{-4} meters defined by the density of dark energy in the recent Universe and thus biological length scale.
4. The above model for the length scale evolution of coupling parameters would reduce to that along paths at $S^2(CP_2)$ and would depend on the ends points of the path only, and also now the zeros of Riemann zeta could naturally correspond to the quantum critical points.

TGD vision about SUSY and coupling constant evolution

TGD view about SUSY leads to radical modification and re-interpretation of SUSY [L76, L73], and to a dramatic progress in the understanding of coupling constant evolution.

Quarks would be the only fundamental fermion fields, and leptons would be spartners of quarks identified as local composites of 3 quarks. Embedding space coordinates would have an expansion in terms of local super-monomials of quarks and antiquarks with vanishing baryon number and appearing as sums of monomial and its conjugate to guarantee hermiticity. Super-spinors would have similar expansion involving only odd quark numbers. This picture is forced by the requirement that propagators are consistent with the statistics of the spartner. Theta parameters would be replaced by creation and annihilation operators for quarks so that super-symmetrization would mean also second quantization. Number theoretic vision requires that only a finite number of Wick contractions of oscillator operators can vanish. These conditions have interpretation as conservation for the Noether currents of some symmetries.

This picture leads to a concrete view about S-matrix for the preferred extremals of a SUSY-variant of the basic action principle relying on the notion of super-variant of embedding space and super-variant of the modified Dirac action. Coupling constant evolution discretizes and would reduce to an increase of the finite number of non-vanishing Wick contractions interpreted as radiative corrections as the dimension of the extension of rationals defining the adele increases. This evolution reflects directly the corresponding evolution at the level of M^8 in terms of octonionic polynomials determining the extension of rationals involved. Whether this view is consistent with the above general vision remains to be seen.

10.3.5 Generalized conformal symmetry, quantum criticality, catastrophe theory, and analogies with thermodynamics and gauge theories

The notion of quantum criticality allows two realizations: as stationarity of S^2 part of the twistor lift of Kähler action and in terms of zeros of zeta are key elements in the explicit proposal for discrete coupling constant evolution reducing to that for cosmological constant.

Quantum criticality from different perspectives

Quantum criticality is however much more general notion, and one must ask how this view relates to the earlier picture.

1. At the real number side continuous coupling constant evolution makes sense. What does this mean? Can one say that quantum criticality makes possible only adelic physics together with large $h_{eff}/h_0 = n$ as dimension for extension of rationals. This hierarchy is essential for life and cognition.

   Can one conclude that living systems correspond to quantum critical values of $S(S^2)$ and therefore α_K and in-animate systems correspond to other values of α_K? But wouldn't his mean that one gives up the original vision that α_K is analogous to critical temperature. The whole point was that this would make physics unique?

   From mathematical view point also continuous α_K can make sense. α_K can be continuous if it corresponds to a higher-dimensional critical manifold at which two or more preferred extremals associated with the same parameter values co-incide - roots of polynomial $P(x,a,b)$ depending on parameters a,b serves as the canonical example. The degree of quantum criticality would vary and there would be a hierarchy of critical systems characterized by the dimension of the critical manifold. One would have a full analog of statistical physics. For mathematician this is the only convincing interpretation.

   2-D cusp catastrophe serves as a basic example helping to generalize [A129]. Cusp corresponds to the roots of $dP_4/dx = 0$ of third order polynomial $P_4(x,a,b)$, where (a,b) are control variables. The projection of region with 3 real roots to (a,b)-plane is bounded by critical lines forming a roughly V-shaped structure. d^2P_4/dx^2 vanishes at the edges of V, where two roots co-incide and d^3P_4/dx^3 vanishes at the tip of V, where 3 roots co-incide.

2. A hierarchy of quantum criticalities has been actually assumed. The hierarchy of representations for super-symplectic algebra realizing 4-D analog of super-conformal symmetries allows an infinite hierarchy of representations for which infinite-D sub-algebra isomorphic to a full algebra and its commutator with the full algebra annihilate physical states. Also classical Noether charges vanish. What is new is that conformal weights are non-negative integers. The effective dimensions of these systems are finite - at least in the sense that one one has finite-D Lie algebra (or its quantum counterpart) or corresponding Kac-Moody algebra as symmetries. This realization of quantum criticality generalize the idea that conformal symmetry accompanies 2-D criticality.

   This picture conforms also with the vision about hierarchy of hyper-finite-factors with included hyper-finite factor defining measurement resolution [K99]. Hyper-finiteness indeed means finite-dimensionality in excellent approximation.

TGD as catastrophe theory and quantum criticality as prerequisite for the Euclidian signature of WCW metric

It is good to look more precisely how the catastrophe theoretic setting generalizes to TGD.

1. The value of the twistor lift of Kähler action defining Kähler function very probably corresponds to a maximum of Kähler function since otherwise metric defined by the second derivatives could have non-Euclidian signature. One cannot however exclude the possibility that in complex WCW coordinates the (1,1) restriction of the matrix defined by the second derivatives of Kähler function could be positive definite also for other than minima.

   It would seem that one cannot accept several roots for given zero modes since one cannot have maximum of Kähler function for all of them. This would allow only the boundary of catastrophe region in which 2 or more roots co-incide. Positive definiteness of WCW metric would force quantum criticality.

   For given values of zero modes there would be single minimum and together with the cancellation of Gaussian and metric determinants this makes perturbation theory extremely simple since exponents of vacuum functional would cancel.

2. There is an infinite number of zero modes playing the role of control variables since the value of the induce Kähler form is symplectic invariant and there are also other symplectic invariants associated with the M^4 degrees of freedom (carrying also the analog of Kähler form for the twistor lift of TGD and giving rise to CP breaking). One would have catastrophe theory with

infinite number of control variables so that the number of catastrophes would be infinite so that standard catastrophe theory does not as such apply.

3. Therefore TGD would not be only a personal professional catastrophe but a catastrophe in much deeper sense. WCW would be a catastrophe surface for the functional gradient of the action defining Kähler function. WCW would consists of regions in which given zero modes would correspond to several minima. The region of zero mode space at which some roots identifiable as space-time surfaces co-incide would be analogous to the V-shaped cusp catastrophe and its higher-D generalizations. The question is whether one allows the entire catastrophe surface or whether one demands quantum criticality in the sense that only the union of singular sets at which roots co-incide is included.
4. For WCW as catastrophe surface the analog of V in the space of zero modes would correspond to a hierarchy of sub-WCWs consisting of preferred extremals satisfying the gauge conditions associated with a sub-algebra of supersymplectic algebra isomorphic to the full algebra. The sub-WCWs in the hierarchy of sub-WCWs within sub-WCWs would satisfy increasingly stronger gauge conditions and having decreasing dimension just like in the case of ordinary catastrophe. The lower the effective dimension, the higher the quantum criticality.
5. In ordinary catastrophe theory the effective number of behavior variables for given catastrophe can be reduced to some minimum number. In TGD framework this would correspond to the reduction of super-symplectic algebra to a finite-D Lie algebra or corresponding Kac-Moody (half-)algebra as modes of supersymplectic algebra with generators labelled by non-negative integer n modulo given integer m are eliminated as dynamical degrees of freedom by the gauge conditions: this would effectively leave only the modes smaller than m. The fractal hierarchy of these supersymplectic algebras would correspond to the decomposition of WCW as a catastrophe surface to pieces with varying dimension. The reduction of the effective dimension as two sheets of the catastrophe surface co-incide would mean transformation of some modes contributing to metric to zero modes.

RG invariance implies physical analogy with thermodynamics and gauge theories

One can consider coupling constant evolution and RG invariance from a new perspective based on the minimal surface property.

1. The critical values of Kähler coupling strength would correspond to quantum criticality of the S^2 part $S(S^2)$ of 6-D dimensionally reduced Kähler action for fixed values of zero modes. The relative S^2 rotation would serve as behavior variable. For its critical values the dimension of the critical manifold would be reduced, and keeping zero modes fixed a critical value of α_K would be selected from 1-D continuum.
   Quantum criticality condition might be fundamental since it implies the constancy of the value of the twistor lift of Kähler action for the space-time surfaces contributing to the scattering amplitudes. This would be crucial for number theoretical vision since the continuation of exponential to p-adic sectors is not possible in the generic case. One should however develop stronger arguments to exclude the continuous evolution of Kähler coupling strength in S^2 degrees of freedom for the real sector of the theory.
2. The extremals of twistor lift contain dependence on the rotation parameter for S^2 and this must be taken into account in coupling constant evolution along curve of S^2 connecting zeros of zeta. This gives additional non-local term to the evolution equations coming from the dependence of the embedding space coordinates of the preferred extremal on the evolution parameter. The derivative of Kähler action with respect to the evolution parameter is by chain rule proportional to the functional derivatives of action with respect to embedding space coordinates, and vanish if 4-D Kähler action and volume term are *separately* stationary with respect to variations. Therefore minimal surface property as implied by holomorphy guaranteeing quantum criticality as universality of the dynamics would be crucial in simplifying the equations! It does not matter whether there is coupling between Kähler action and volume term.

Could one find interpretation for the miminal surface property which implies that field equations are separately satisfied for Kähler action and volume term?

1. Quantum TGD can be seen as a "complex" square root of thermodynamics. In thermodynamics one can define several thermodynamical functions. In particular, one can add to energy E as term pV to get enthalpy $H = E + pV$, which remains constant when entropy and pressures are kept constant. Could one do the same now?
   In TGD V replaced with volume action and p would be a coupling parameter analogous to pressure. The simplest replacement would give Kähler action as outcome. The replacement would allow RG invariance of the modified action only at critical points so that replacement would be possible only there. Furthermore, field equations must hold true separately for Kähler action and volume term everywhere.

2. It seems however that one must allow singular sets in which there is interaction between these terms. The coupling between Kähler action and volume term could be non-trivial at singular sub-manifolds, where a transfer of conserved quantities between the two degrees of freedom would take place. The transfer would be proportional to the divergence of the canonical momentum current $D_\alpha(g^{\alpha\beta}\partial_\beta h^k)$ assignable to the minimal surface and is conserved outside the singular sub-manifolds.
   Minimal surfaces provide a non-linear generalization of massless wave-equation for which the gradient of the field equals to conserved current. Therefore the interpretation could be that these singular manifolds are sources of the analogs of fields defined by M^4 and CP_2 coordinates. In electrodynamics these singular manifolds would represented by charged particles. The simplest interpretation would be in terms of point like charges so that one would have line singularity. The natural identification of world line singularities would be as boundaries of string world sheets at the 3-D light-like partonic orbits between Minkowskian and Euclidian regions having induced 4-metric degenerating to 3-D metric would be a natural identification. These world lines indeed appear in twistor diagrams. Also string world sheets must be assumed and they are are natural candidates for the singular manifolds serving as sources of charges in 4-D context. Induced spinor fields would serve as a representation for these sources. These strings would generalize the notion of point like particle. Particles as 3-surfaces would be connected by flux tubes to a tensor network and string world sheets would connected fermion lines at the partonic 2-surfaces to an analogous network. This would be new from the standard model perspective.
   Singularities could also correspond to a discrete set of points having an interpretation as cognitive representation for the loci of initial and final states fermions at opposite boundaries of CD and at vertices represented topologically by partonic 2-surfaces at which partonic orbits meet. This interpretation makes sense if one interprets the embedding space coordinates as analogs of propagators having delta singularities at these points. It is quite possible that also these contributions are present: one would have a hierarchy of delta function singularities associated with string worlds sheets, their boundaries and the ends of the boundaries at boundaries of CD, where string world sheet has edges.

3. There is also an interpretation of singularities suggested by the generalization of conformal invariance. String world sheets would be co-dimension 2 singularites analogous to poles of analytic function of two complex coordinates in 4-D complex space. String world sheets would be co-dimension 2 singularities analogous to poles at light-like 3-surfaces. The ends of the world lines could be analogous of poles of analytic function at partonic 2-surfaces.
   These singularities could provide to evolution equations what might be called matter contribution. This brings in mind evolution equations for n-point functions in QFT. The resolution of the overall singularity would decompose to 2-D, 1-D and 0-D parts just like cusp catastrophe. One can ask whether the singularities might allow interpretation as catastrophes.

4. The proposal for analogs of thermodynamical functions suggests the following physical picture. More general thermodynamical functions are possible only at critical points and only if the extremals are miminal surfaces. The singularities should correspond to physical particles, fermions. Suppose that one considers entire scattering amplitude involving action exponential plus possible analog of pV term plus the terms associated with the fermions assigned with the singularities. Suppose that the coupling constant evolution from 6-D Kähler action is calculated *without* including the contribution of the singularities.
   The derivative of n-particle amplitude with respect to the evolution parameter contains a term coming from the action exponential receiving contributions only from the singularities and a

term coming from the operators at singularities. RG invariance of the scattering amplitude would require that the two terms sum up to zero. In the thermodynamical picture the presence of particles in many particle scattering amplitude would force to add the analog of pressure term to the Kähler function: it would be determined by the zero energy state.
One can of course ask how general terms can be added by requiring minimal surface property. Minimal surface property reduces to holomorphy, and can be true also for other kinds of actions such as the TGD analogs of electroweak and color actions that I considered originally as possible action candidates.
This would have interpretation as an analog for YM equations in gauge theories. Space-time singularities as local failure of minimal surface property would correspond to sources for H coordinates as analogs of Maxwell's fields and sources currents would correspond to fermions currents.

10.3.6 TGD view about inclusions of HFFs as a way to understand coupling constant evolution

The hierarchy of inclusions of HFFs is an alternative TGD inspired proposal for understanding coupling constant evolution and the intuitive expectation is that the inclusion hierarchies of extensions and their Galois groups contain the inclusion hierarchies of HFFs as special case. The included factor would define measurement resolution in well-defined sense. This notion can be formulated more precisely using Connes tensor product [A115, A190].

How Galois groups and finite subgroups of could $SU(2)$ relate

The hierarchy of finite groups associated with the inclusions of HFF corresponds to the finite subgroups of $SU(2)$. The set of these groups is very small as compared to the set of Galois groups - if I have understood correctly, any finite group can appear as Galois group. Should the hierarchy of inclusions of HFFs be replaced by much more general inclusion hierarchy? Is there a map projecting Galois groups to discrete subgroup of $SU(2)$?

By M^8-H duality quaternions appear at M^8 level and since $SO(3)$ is the automorphism group of quaternions, the discrete subgroups of $SU(2)$ could appear naturally in TGD. In fact, the appearance of quaternions as a basic building brick of HFFs in the simplest construction would fit with this picture.

It would seem that the elements of the discrete subgroups of $SU(2)$ must be in the extension of rationals considered. The elements of finite discrete subgroups G of $SU(2)$ are expressible in terms of rather small subset of extensions of rationals. Could the proper interpretation be that the hierarchy of extensions defines a hierarchy of discrete groups with elements in extension and the finite discrete subgroups in question are finite discrete subgroups of these groups. There would be correlation with the inclusion and extension. For instance, the fractal dimension of extension is an algebraic number defined in terms of root of unity so that the extension must contain this root of unity.

For icosahedron and dodecahedron the group action can be expressed using extension of rationals by $cos(\pi/n)$ and $sin(\pi/n)$ for $n=3,5$. For tetrahedron and cube one would have $n=2,3$. For tetrahedon, cube/octahedron and icosahedron basic geometric parameters are also expressible in terms of algebraic numbers in extension but in case of dodecahedron it is not clear for me whether the surface area proportional to $\sqrt{25+20\sqrt{5}}$ allows this (see http://tinyurl.com/p4rwc7).

It is very feasible that the finite sub-groups of also other Lie groups than $SU(2)$ are associated with the inclusions of HFFs or possibly more general algebras. In particular, finite discrete subgroups of color group $SU(3)$ should be important and extension of rationals should allow to represent these subgroups.

Once again about ADE correspondence

For a non-mathematician like me Mc-Kay correspondence is an inspiring and frustrating mystery (see http://tinyurl.com/y8jzvogn). What could be its physical interpretation?

Mac-Kay correspondence assigns to the extended Dynkin diagrams of ADE type characterizing Kac-Moody algebras finite subgroups of $SU(2)$, more precisely the McKay diagrams describing the tensor product decomposition rules for the fundamental representation of the finite subgroup of $SU(2)$. In the diagram irreps χ_i and χ_j are connected by n_{ij} arrows if χ_j appears n_{ij} times in the tensor product $V \otimes \chi_i$, where V is but need not be fundamental representation.

One can assign also to inclusions of HFFs of index $d \geq 4$ with ADE type Dynkin diagrams. To inclusions with index $d < 4$ one can assign subset of ADE type diagrams for Lie groups (rather than Kac-Moody groups) and they correspond to sub-groups of $SU(2)$. The correspondence generalizes to subgroups of other Lie groups.

1. As explained in [B39] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (dim(g) - r)/r$. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed. The Dynkin graphs of Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. E_6, E_7, and E_8 correspond to symmetry groups of tetrahedron, octahedron/cube, and icosahedron/dodecahedron. The group for octahedron/cube is missing: what could this mean?
   For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of $SU(2)$ and the interpretation proposed in [A190] is following.
   The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: A_∞ corresponding to $SU(2)$ itself, $A_{-\infty,\infty}$ corresponding to circle group $U(1)$, and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection.
   One can construct also inclusions for which the diagrams corresponding to finite subgroups $G \subset SU(2)$ are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with $n = 6, 7, 8$ for tetrahedron, cube, dodecahedron. These extensions correspond to ADE type Kac-Moody algebras.
   The extension is constructed by constructing first factor R as infinite tensor power of $M_2(C)$ (complexified quaternions). Sub-factor R_0 consists elements of of R of form $Id \otimes x$. $SU(2)$ preserves R_0 and for any subgroup G of $SU(2)$ one can identify the inclusion $N \subset M$ in terms of $N = R_0^G$ and $M = R^G$, where $N = R_0^G$ and $M = R^G$ consists of fixed points of R_0 and R under the action of G. The principal graph for $N \subset M$ is the extended Coxeter-Dynk graph for the subgroup G.
   Physicist might try to interpret this by saying that one considers only sub-algebras R_0^G and R^G of observables invariant under G and obtains extended Dynkin diagram of G defining an ADE type Kac-Moody algebra. Could the condition that Kac-Moody algebra elements with non-vanishing conformal weight annihilate the physical states state that the state is invariant under R_0 defining measurement resolution. Besides this the states are also invariant under finite group G? Could R_0^G and R^G correspond just to states which are also invariant under finite group G.
   Could this kind of inclusions generalize so that Galois groups would replace G. If this is possible it would assign to each Galois group an inclusion of HFFs and give a precise number theoretic formulation for the notion of measurement accuracy.
2. At M^8-side of $M^8 - H$ duality the construction of space-time surfaces reduces to data at finite set of points of space-time surface since they are defined by an octonionic extension of a polynomial of real variable with coefficients in extension of rationals. Space-time surfaces would have quaternionic tangent space or normal space. The coordinates of quaternions are restricted to extension of rationals and the subgroup of automorphisms reduce to a subgroup for which matrix elements belong to an extension of rationals.
   If the subgroup is finite, only the subgroups appearing in ADE correspondence are possible and the extension must be such that it allows the representation of this group. Does this mean that the extension can is obtained from an extension allowing this representation? For $\mathcal{M} : \mathcal{N} = 4$ case this sub-group would leave the states invariant.

10.3.7 Entanglement and adelic physics

In the discussion about fine structure constant I asked about the role entanglement in coupling constant evolution. Although entanglement does not have direct relationship to coupling constant evolution, I will discuss entanglement from number theoretic point of view since it enlightens the motivations of adelic physics.

1. For given extension of rationals determining the values of coupling parameters by quantum criticality, the entanglement coefficients between positive and negative energy parts of zero energy states are in the extension of rationals. All entanglement coefficients satisfy this condition.
2. Self the counterpart of observer in the generalization of quantum measurement theory - as conscious entity [L44] corresponds to sequence of unitary evolutions followed by weak measurements. The rule for weak measurements is that only state function for which the eigenvalues of the density matrix is in the extension of rationals can occur. In general they are in a higher-D extension as roots of N:th order polynomials, N the dimension of density matrix. Therefore state function reduction cannot occur in the generic case. State cannot decohere and entanglement is stable under weak measurements except in special situations when the eigenvalues of density matrix are in original extension.
3. The extension can change only in big state function reductions in which the arrow of clock time changes: this can be seen as an evolutionary step. From the point of view of consciousness theory big state function reduction means what might be called death and reincarnation of system in opposite time direction.
4. The number theoretical stabilization of entanglement at the passive boundary of CD makes possibility quantum computation in longer time scales than possible in standard quantum theory. $h_{eff}/h_0 = n$ equals to the dimension of extension of rationals and is therefore directly related to this.

   This could have profound technological implications.

1. Ordinary quantum computation as single unitary step is replaced by a sequence of them followed by the analog of weak measurement.
2. ZEO allows also quantum computations in opposite time direction. This might allow shorten dramatically the duration of quantum computations from the perspective of the observed since most of the computation could be done with opposite arrow of clock time.
   The philosophy of adelic physics is discussed in article in book published by Springer [L43, L42] (see `http://tinyurl.com/ybzkfevz` and `http://tinyurl.com/ybqpkwg9`).

10.4 Trying to understand why ramified primes are so special physically

Ramified primes (see `http://tinyurl.com/m32nvcz` and `http://tinyurl.com/y6yskkas`) are special in the sense that their expression as a product of primes of extension contains higher than first powers and the number of primes of extension is smaller than the maximal number n defined by the dimension of the extension. The proposed interpretation of ramified primes is as p-adic primes characterizing space-time sheets assignable to elementary particles and even more general systems.

In the following Dedekind zeta functions (see `http://tinyurl.com/y5grktvp`) as generalization of Riemann zeta [L54, L61] are studied to understand what makes them so special. Dedekind zeta function characterizes given extension of rationals and is defined by reducing the contribution from ramified reduced so that effectively powers of primes of extension are replaced with first powers.

If one uses the naïve definition of zeta as analog of partition function and includes full powers $P_i^{e_i}$, the zeta function becomes a product of Dedekind zeta and a term consisting of a finite number of factors having poles at imaginary axis. This happens for zeta function and its fermionic analog having zeros along imaginary axis. The poles would naturally relate to Ramond and N-S boundary conditions of radial partial waves at light-like boundary of causal diamond CD. The additional factor could code for the physics associated with the ramified primes.

The intuitive feeling is that quantum criticality is what makes ramified primes so special. In $O(p) = 0$ approximation the irreducible polynomial defining the extension of rationals indeed reduces to a polynomial in finite field F_p and has multiple roots for ramified prime, and one can deduce a concrete geometric interpretation for ramification as quantum criticality using $M^8 - H$ duality.

$M^8 - H$ duality central concept in following and discussed in [L36, L67, L63, L64] [L76]. Also the notion of cognitive representation as a set of points of space-time surface with preferred embedding space coordinates belonging to the extension of rationals defining the adele [L42] is important and discussed in [L70, L69, L75].

10.4.1 Dedekind zeta function and ramified primes

One can take mathematics and physical intuition guided by each other as a guideline in the attempts to understand ramified primes.

1. Riemann zeta can be generalized to Dedekind zeta function ζ_K for any extension K of rationals (see `http://tinyurl.com/y5grktvp`). ζ_K characterizes the extension - maybe also physically in TGD framework since zeta functions have formal interpretation as partition function. In the recent case the complexity is not a problem since complex square roots of partition functions would define the vacuum part of quantum state: one can say that quantum TGD is complex square root of thermodynamics.
   ζ_K satisfies the same formula as ordinary zeta expect that one considers algebraic integers in the extensions K and sums over non-zero ideals a - identifiable as integers in the case of rationals - with n^{-s} replaced with $N(a)^{-s}$, where $N(a)$ denotes the norm of the non-zero ideal. The construction of ζ_K in the extension of rationals obtained by adding i serves as an illustrative example (see `http://tinyurl.com/y563wcwv`). I am not a number theorists but the construction suggests a poor man's generalization strongly based on physical intuition.
2. The rules would be analogous to those used in the construction of partition function. $log(N(a))$ is analogous to energy and s is analogous to inverse temperature so that one has Boltzmann weight $exp(-log(N(a)s)$ for each ideal. Since the formation of ideals defined by integers of extension is analogous to that for forming many particle states labelled by ordinary primes and decomposing to primes of extension, the partition function decomposes to a product over partition functions assignable to ordinary primes just like in the case of Riemann zeta. Let K be an extension of rationals Q.
3. Each rational prime p decomposes in the extension as $p = \prod_{i=1,...g} P_i^{e_i}$, where n is the dimension of extension and e_i is the ramification degree. Let f_i be so called residue degree of P_i defined as the dimension of K mod P_i interpreted as extension of rational integers $Z \ mod \ p$. Then one has $\sum_1^g e_i f_i = n$.
   Remark: For Galois extensions for which the order of Galois group equals to the dimension n of the extension so that for given prime p one has $e_i = e$ and $f_i = f$ and $efg = n$.
4. Rational (and also more general) primes can be divided into 3 classes with respect to this decomposition.
   For ramified primes dividing the discriminant D associated with the polynomial ($D = b^2 - 4c$ for $P(x) = x^2 + bx + c$) one has $e_i > 1$ at least for one i so that $f_i = 0$ is true at least for one index. A simple example is provided by rational primes (determined by roots of $P(x) = x^2 + 1$ with discriminant -4): in this case $p = 2$ corresponds to ramified prime since on has $(1+i)(1-i) = 2$ and $1+i$ and $1-i$ differ only by multiplication by unit $-i$.
5. Split primes have n factors P_i and thus have $(e_i = 1, f_i = 1, g = n)$. They give a factor $(1 - p^{-s})^{-n}$. The physical analogy is n-fold degenerate state with original energy energy $nlog(p)$ split to states with energy $log(p)$.
   Inert primes are also primes of extension and there is no splitting and one has $(e_1 = e = 1, g = 1, f_1 = f = n)$. In this case one obtains factor $1/(1-p^{-ns})$. The physical analogy is n-particle bound state with energy $nlog(p)$.
6. For ramified primes the situation is more delicate. Generalizing from the case of Gaussian primes $Q[i]$ (see `http://tinyurl.com/y563wcwv`) ramified primes p_R would give rise to a factor

$$\prod_{i=1}^{g} \frac{1}{1 - p_R^{-f_i s}} \ .$$

g is the number of *distinct* ideals P_i in the decomposition of p to the primes of extension.
For Gaussian primes $p=2$ has $g=1$ since one can write $(2)=(1+i)(1-i)\equiv(1+i)^2$. This because $1+i$ and $1-i$ differ only by multiplication with unit $-i$ and thus define same ideal in $Q[i]$. Only the number g of distinct factors P_i in the decomposition of p matters.
One could understand this as follows. For the roots of polynomials ramification means that several roots co-incide so that the number of distinct roots is reduced. $e_i>1$ is analogous to the number co-inciding roots so that number if distinct roots would be 1 instead of e_i. This would suggests $k_i=1$ always. For ramified primes the number of factors Z_p the number $\sum_{i=1}^g f_ik_i=n$ for un-ramified case would reduce from to $\sum_{i=1}^g f_ik_i=n_d$, which is the number of distinct roots.

7. Could the physical interpretation be that there are g types of bound states with energies $f_ilog(p)$ appearing with degeneracy $e_i=1$ both in ramified and split case. This should relate to the fact that for ramified primes p L/p contains non-vanishing nilpotent element and is not counted. One can also say that the decomposition to primes of extension conserves energy: $\sum_{i=1,...,g} e_if_ilog(p)=n_dlog(p)$.
For instance, for Galois extensions $(e_i=e,f_i=f,g=n_d/ef)$ for given p the factor is $1/(1-p^{-es})^{fg}$: $efg=n_d$. If there is a ramification then all P_i are ramified. Note that e,f an g are factors of n_d.
8. One can can extract the factor $1/(1-p^{-s})$ from each of the 3 contributions and organize these factors to give the ordinary Riemann zeta. The number of ramified primes is finite whereas the numbers of split primes and inert primes are infinite. One can therefore extract from ramified primes the finite product

$$\zeta^1_{R,K}=\prod_{p_R}(1-p_R^{-s})\times\zeta^2_{R,K}\ \ ,\ \ \zeta^2_{R,K}=\prod_{p_R}[\prod_{i=1}^g\frac{1}{1-p^{-f_is}}]\ .$$

One can organize the remaining part involving infinite number of factors to a product of ζ and factors $(1-p^{-s})/(1-\prod p^{-s}))^n$ and $(1-p^{-s})/(1-p^{-ns})$ giving rise to zeta function -call it $\zeta_{si,K}$ - characterizing the extension. Note that $\zeta^2_{R,K}$ has interpretation as partition function and has pole of order n_d at origin.
One therefore can write the ζ_L as

$$\zeta_K=\zeta^1_{R,K}\times\zeta_{si,K}\times\zeta\ .$$

where $\zeta_{si,K}$ is the contribution of split and inert primes multiplied by $(1-p^{-s})$

ζ_L has pole only at $s=1$ and it carries in no obvious manner information about ramified primes. The naïve guess for ζ_L would be that also ramified primes p_R would give rise to a factor

$$\prod_{i=1}^{g}\frac{1}{(1-p_R^{-f_is})^{e_i}}\ .$$

One could indeed argue that at the limit when e_i prime ideals P_i of extension co-incide, one should obtain this expression. The resulting ζ function would be product

$$\zeta_{naive,K}=\zeta_{R,K}\zeta_K\ \ ,\qquad \zeta_{R,K}=\prod_{p_R}X(p_R)$$

$$X(p_R)=\prod_{i=1}^g\frac{1}{(1-p_R^{-f_is})^{e_i-1}}\ .$$

Note that the parameters e_i,f_i,g depend on p_R and that for Galois extensions one has $e_i=d,f_i=f$ for given p_R. $\zeta_{R,L}$ would have poles at along imaginary axis at points $s=-n2\pi/log(p)$. Ramified primes would give rise to poles along imaginary axis. As far as the proposed physical interpretation of ramified primes is considered, this form looks more natural.

Fermionic counterparts of Dedekind zeta and ramified ζ

One can look the situation also for the generalization of fermionic zeta as analog of fermionic partition function, which for rationals has the expression

$$\zeta^F(s)=\prod_p(1+p^{-s})=\frac{\zeta(s)}{\zeta(2s)}\ .$$

Supersymmetry of supersymmetric arithmetic QFT suggest the product of fermionic and bosonic zetas. Also the supersymmetry of infinite primes for which first level of hierarchy corresponds to irreducible polynomials suggests this. On the other hand, the appearance of only fermions as fundamental particles in TGD forces to ask whether the ramified part of fermionic zeta might be fundamental.

1. By an argument similar to used for ordinary zeta based on interpretation as partition function, one obtains the decomposition of the fermionic counterpart of ζ_K^F Dirichlet zeta to a product $\zeta_K^F = \zeta_{R,K}^F \zeta_{si,K}^F \zeta^F$ of ramified fermionic zeta $\zeta_{R,K}^F$, $\zeta_{si,K}^F$, and ordinary fermionic zeta ζ^F. The basic rule is simple: replace factors $1/(1-p^{-ks}$ appearing in ζ_K with $(1+p^{-ks})$ in ζ_K^F and extract ζ^F from the resulting expression. This gives

$$\zeta_{R,K}^{F,1} = \prod_{p_R}(1-p_R^{-s})\zeta_{R,K}^F \quad , \quad \zeta_{R,K}^F = \prod_{p_R}[\prod_{i=1}^g(1+p_R^{-f_is})] \quad .$$

where p_R is ramified prime dividing the discriminant. $\zeta_{R,K}^F$ is analogous to a fermionic partition function for a finite number of modes defined by ramified primes p_R of extension.

2. Also now one can wonder whether one should define ζ_K^F as a product in which ramified primes give factor

$$\prod_{p_R}[\prod_{i=1}^{g}(1+p_R^{-f_is})^{e_i}]$$

so that one would have

$$\zeta_{naive,K}^F = \zeta_{R,K}^F \zeta_K^F \quad , \qquad \zeta_R^F = \prod_{p_R} Y(p_R) \quad ,$$

$$Y(p_R) = \prod_{i=1}^g(1+p_R^{-f_is})^{e_i-1}$$

$\zeta_F(na\ddot{\imath}ve,K)$ would have zeros along imaginary axis serving as signature of the ramified primes.

About physical interpretation of $\zeta_{R,K}$ and $\zeta_{R,K}^F$

$\zeta_{R,K}$ and $\zeta_{R,K}^F$ are attractive from the view point of number theoretic vision and the idea that ramified primes are physically special. TGD Universe is quantum critical and in catastrophe theory the ramification for roots of polynomials is analogous to criticality. Maybe the ramification for p-adic primes makes them critical. $K/(p_R)$ has nilpotent elements, which brings in mind on mass shell massless particles.

1. $\zeta_{R,K}$ has poles at

$$s = i\frac{2n\pi}{log(p)f_i}$$

and $p_R^s = exp(in2\pi/f_i)$ is a root of unity, which conforms with the number theoretical vision. Only P_i with $e_i > 1$ contribute.

2. $Z_{R,K}^F$ has zeros

$$s = i\frac{(2n+1)\pi}{log(p)f_i}$$

and $p_R^s = exp(i(2n+1)\pi/f_i)$ is a root of unity. Zeros are distinct from the poles of $Z_{R,K}$.

3. The product $\zeta_{R,tot,K} = \zeta_{R,K}\zeta_{R,K}^F$ has the poles and zeros of $\zeta_{R,K}$ and $\zeta_{R,K}^F$. In particular, there is n:th order pole of $Z_{R,K}$ at $s=0$. The zeros of $z_{F,K}$ along imaginary axis at $p^{iy} = -1$ also survive in $\zeta_{R,tot,K}$.
$\zeta_{R,K}^F$ has only zeros and since fundamental fermions are primary fields in TGD framework, one could argue that only it carries physical information. On the other hand, supersymmetric arithmetic QFT [K84] and the fact that TGD allows SUSY [L73] suggests that the product $\zeta_{R,K} \times Z_{R,K}^F$ is more interesting.

From TGD point of view the ramified zeta functions $\zeta_{R,K}$, $\zeta_{R,K}^F$ and their product $\zeta_{R,K}\times\zeta_{R,K}^F$ look interesting.

1. $\zeta_{R,K}$ behaves like s^{-n_d}, $n_d = \sum_1^g (e_i - 1)$ near the origin. Could n_d-fold pole at $s = 0$ be interpreted in terms of a massless state propagating along light-cone boundary of CD in radial direction? This would conform with the proposal that zeros of zeta correspond to complex radial conformal weights for super-symplecti algebra. That ramified primes correspond to massless particles would conform with the identification of ramified prime as p-adic primes labelling elementary particles since in ZEO their mass would result from p-adic thermodynamics from a mixing with very massive states [L64].
   Besides this there would be stringy spectrum of real conformal weights along negative real axis and those coming as non-trivial zeros and these could correspond to ordinary conformal weights.
2. The zeros of $\zeta^F_{R,K}$ along imaginary axis might have interpretation as eigenvalues of Hamiltonian in analogy with Hilbert-Polya hypothesis. Maybe also the poles of $\zeta_{R,K}$ could have similar interpretation. The real part of zero/pole would not produce troubles (on the other hand, for waves along light-cone boundary it can be however absorbed to the integration measure.
3. A possible physical interpretation of the imaginary conformal weights could be as conformal weights associated with radial waves assignable to the radial light-like coordinate r of the light-cone boundary: r indeed plays the role of complex coordinate in conformal symmetry in the case of super-symplectic algebra suggested to define the isometries of WCW. Poles and zero could correspond to radial modes satisfying periodic/anti-periodic boundary conditions. The radial conformal weights s defined by the zeros of $\zeta^F_{R,K}$ would be number theoretically natural since one could pose boundary condition $p^{is(r/r_0)} = -1$ at $r = r_0$ requiring $p^{is} = -1$ at the corner of cd (maximum value of r in $CD = cd \times CP_2$.
   For the poles of $\zeta_{R,K}$ the periodic boundary condition $p^{is(r/r_0)} = 1$ is natural. The two boundary conditions could relate to Ramond an N-S representations of super-conformal algebras (see `http://tinyurl.com/y49y2ouj`). With this interpretation $s = 0$ would correspond to a radial plane-wave constant along light-like radial direction and therefore light-like momentum propagating along the boundary of CD. Other modes would correspond to other massless modes propagating to the interior of CD.
4. I have earlier considered an analogous interpretation for a subset zeros of zeta satisfying similar condition. The idea was that for given prime p as subset of $s = 1/2 + iy_i$ of non-trivial zeros ζ $p^s = p^{1/2+iy_i}$ is an algebraic number so that p^{iy_i} would be a root of unity. Zeros would decompose to subsets labelled by primes p. Also for trivial zeros of ζ (and also poles) the same holds true as for the zeros and poles ζ_R. This encourages the conjecture that the property is true also for L-functions.

The proposed picture suggests an assignment of "energy" $E = nlog(p)$ to each prime and separation of "ramified" energy $E_d = n_d log(p)$, $n_d = \sum_1^g f_i(e_i - 1)$, to each ramified prime. The interpretation as partition function suggests that that one has g types of states of f_i identical particles and energy $E_i = f_i log(p)$ and that this state is e_i-fold degenerate with energies $E_i = f_i log(p)$. For inert primes one would have $f_i = f = n$. For split primes one would have $e_i = 1, f_i = 1$. In case of ramified primes one can separate one of these states and include it to the Dedekind zeta.

Can one find a geometric correlate for the picture based on prime ideals?

If one could find a geometric space-time correlate for the decomposition of rational prime ideals to prime ideals of extensions, it might be also possible to understand why quantum criticality makes ramified primes so special physically and wha this means.

What could be correlate for f_i fundamental fermions behaving like single unit and what degeneracy for $e_i > 1$ does mean? One can look the situation first at the level of number fields Q and K and corresponding Galois group $Gal(K/Q)$, finite fields $F = Q/p$ and $F_i = K/P_i$, and corresponding Galois group $Gal(F_i/F)$. Appendix summarizes the basic terminology.

1. Inertia degree f_i is the number of elements of F_i/F_p ($F_i = K/P_i$ is extension of finite field $F_p = Q/p$). The Galois group $Gal(F_i/F_p)$ is identifiable as factor group D_i/I_i, where the *decomposition group* D_i is the subgroup of Galois group taking P_i to itself and the *inertia group* I_i leaving P_i point-wise invariant. The orbit under $Gal(F_i/F_p)$ in F_i/F_p would behave like single particle with energy $E_i = f_i log(p)$.

For inert primes with $f_i = n$ inertia group would be maximal. For split primes the orbits of ideals would consist of $f_i = 1$ points only and isotropy group would be trivial.

2. Ramification for primes corresponds intuitively to that for polynomials meaning multiple roots as is clear also from the expression $p = \prod P_i^{e_i}$. In accordance with the intuition about quantum criticality, ramification means that the irreducible polynomial reduced to a reducible polynomial in finite field Q/p has therefore a multiple roots with multiplicities e_i (see Appendix). For Galois extensions one has ($e_i = e, f_i = f$) Criticality would be seen at the level of finite field $F_p = Q/p$ associated with ramified prime p.

The interpretation of roots of corresponding octonionic polynomials as n-sheeted covering space like structures encourages to ask whether the independent tensor factors labelled by i suggested by the interpretation as a partition function could be assigned with the sheets of covering so that ramification with $e_i > 1$ would correspond to singular points of cognitive representation for which e_i sheets co-incide in some sense, maybe in finite field approximation ($O(p) = 0$). Galois groups indeed act on the coordinates of point of cognitive representation belonging to the extension K. In general the action does not take the point to a point belonging to a cognitive representation but one can consider quantum superpositions of cognitive representations.

This suggests an interpretation in terms of space-time surfaces accompanied by cognitive representation under Galois group. Quantum states would be superpositions of preferred extremals at orbits of Galois group and for cognitive representations the situation would be discrete.

1. To build a concrete connection between geometric space-time picture and number theoretic picture, one should find geometric counterparts of integers, ideals, and prime ideals. The analogs of prime ideals should be associated with the discretizations of space-time surfaces/cognitive representations in $O(p) = 0$ or $O(P_i) = 0$ approximation. Could one include only points of cognitive representations differing from zero in $O(p) = 0$ approximation and form quantum states as quantum superpositions of these points of cognitive representation?
   in $O(p) = 0$ approximation and for ramified primes irreducible polynomials would have multiple roots so that e_i sheets would co-incide at these points in $O(p) = 0$ approximation. Th conjecture that elementary particles correspond to this kind of singularities has been speculated already earlier with inspiration coming from quantum criticality.
2. In M^8 picture the octonionic polynomials obtained as continuation of polynomials with rational coefficients would be reduced to polynomials in finite field F_p. One can study corresponding discrete algebraic surfaces as discrete approximations of space-time surfaces.
3. One would like to have only single embedding space coordinate since the probability that all embedding space coordinates correspond to the same P_i is small. $M^8 - H$ duality reduces the number of embedding space coordinates characterizing partonic 2-surfaces containing vertices for fundamental fermions to single one identifiable as time coordinate.
   At the light-like boundary of 8-D CD in M^8 the vanishing condition for the real or imaginary part (quaternion) of octonionic polynomial $P(o)$ reduces to that for ordinary polynomial, and one obtains n roots r_n, which correspond to the values of M^4 time $t = r_n$ defining 6-spheres as analogs of branes. Partonic 2-surfaces corresponde to intersections of 4-D roots of $P(o)$ at partonic 2-surfaces. Galois group of the polynomial naturally acts on r_n labelling these partonic 2-surfaces by permuting them. One could form representations of Galois group using states identified as quantum superpositions of these partonic 2-surfaces corresponding to different values of $t = r_n$. Galois group leaves invariant the degenerate roots $t = r_n$.
4. The roots can be reduced to finite field F_p or K/P_i. Ramification would take place in this approximation and mean that e_i roots $t = r_n$ are identical in $O(p) = 0$ approximation. e_i time values $t = r_n$ would nearly co-incide. This gives more concrete contents to the statement of TGD inspired theory of consciousness that these time values correspond to very special moments in the life of self. Since this is the situation only approximately, one can argue that one must indeed count each root separately so that partition function must be defined as product of the contribution form ramified primes an Dedekind zeta.
   The assignment of fundamental fermions to the points of cognitive representations at partonic 2-surfaces assignable to the intersections of 4-D roots and universal 6-D roots of octonionic polynomials (brane like entities) conforms with this picture.
5. The analogs of 6-branes would give rise to additional degrees of freedom meaning effectively discrete non-determinism. I have speculated with this determinism with inspiration coming

from the original identification of bosonic action as Kähler action having huge 4-D spin glass degeneracy. Also the number theoretic vision suggest the possibility of interpreting preferred extremals as analogs of algebraic computations such that one can have several computations connecting given states [L32]. The degree n of polynomial would determine the number of steps and the degeneracy would correspond to n-fold degeneracy due to the discrete analogs of plane waves in this set.

What extensions of rationals could be winners in the fight for survival?

It would seem that the fight for survival is between extensions of rationals rather than individual primes p. Intuition suggests that survivors tend to have maximal number of ramified primes. These number theoretical speciei can live in the same extension - to "co-operate".

Before starting one must clarify some basic facts about extensions of rationals.

1. Extension of rationals are defined by an irreducible polynomial with rational coefficients. The roots give n algebraic numbers which can be used as a basis to generate the numbers of extension ast their rational linear combinations. Any number of extension can be expressed as a root of an irreducible polynomial. Physically it is is of interest, that in octonionic picture infinite number of octonionic polynomials gives rise to space-time surface corresponding to the same extension of rationals.
2. One can define the notion of integer for extension. A precise definition identifies the integers as ideals. Any integer of extension are defined as a root of a monic polynomials $P(x) = x^n + p_{n-1}x^{n-1}x^{n-1} + ... + p_0$ with integer coefficients. In octonionic monic polynomials are subset of octonionic polynomials and it is not clear whether these polynomials could be all that is needed.
3. By definition ramified primes divide the discriminant D of the extension defined as the product $D = \prod_{i\neq j}(r_i - r_j)$ of differences of the roots of (irreducible) monic polynomial with integer coefficients defining the basis for the integers of extension. Discriminant has a geometric interpretation as volume squared for the fundamental domain of the lattice of integers of the extension so that at criticality this volume interpreted as p-adic number would become small for ramified primes an vanish in $O(p)$ approximation. The extension is defined by a polynomial with rational coefficients and integers of extension are defined by monic polynomials with roots in the extension: this is not of course true for all monic polynomials polynomial (see `http://tinyurl.com/k3ujjz7`).
4. The scaling of the $n-1$-tuple of coefficients $(p_{n-1}, ..., p_1)$ to $(ap_{n-1}, a^2p_{n-1}....., a^np_0)$ scales the roots by a: $x_n \rightarrow ax_n$. If a is rational, the extension of rationals is not affected. In the case of monic polynomials this is true for integers k. This gives rational multiples of given root.
   One can decompose the parameter space for monic polynomials to subsets invariant under scalings by rational $k \neq 0$. Given subset can be labelled by a subset with vanishing coefficients $\{p_{i_k}\}$. One can get rid of this degeneracy by fixing the first non-vanishing p_{n-k} to a non-vanishing value, say 1. When the first non-vanishing p_k differs from p_0, integers label the polynomials giving rise to roots in the same extension. If only p_0 is non-vanishing, only the scaling by powers k^n give rise to new polynomials and the number of polynomials giving rise to same extension is smaller than in other cases.
   Remark: For octonionic polynomials the scaling symmetry changes the space-time surface so that for generic polynomials the number of space-time surfaces giving rise to fixed extension is larger than for the special kind polynomials.

Could one gain some understanding about ramified primes by starting from quantum criticality? The following argument is poor man's argument and I can only hope that my modest technical understanding of number theory does not spoil it.

1. The basic idea is that for ramified primes the minimal monic polynomial with integer coefficients defining the basis for the integers of extension has multiple roots in $O(p) = 0$ approximation, when p is ramified prime dividing the discriminant of the monic polynomial. Multiple roots in $O(p) = 0$ approximation occur also for the irreducible polynomial defining the extension of rationals. This would correspond approximate quantum criticality in some p-adic sectors of adelic physics.

2. When 2 roots for an irreducible rational polynomial co-incide, the criticality is exact: this is true for polynomials of rationals, reals, and all p-adic number fields. One could use this property to construct polynomials with given primes as ramified primes. Assume that the extension allows an irreducible olynomial having decomposition into a product of monomials = $x-r_i$ associated with roots and two roots r_1 and r_2 are identical: $r_1 = r_2$ so that irreducibility is lost.
   The deformation of the degenerate roots of an irreducible polynomial giving rise to the extension of rationals in an analogous manner gives rise to a degeneracy in $O(p) = 0$ approximation. The degenerate root $r_1 = r_2$ can be scaled in such a way that the deformation $r_2 = r_1(1+q))$, $q = m/n = O(p)$ is small also in real sense by selecting $n >> m$.
   If the polynomial with rational coefficients gives rise to degenerate roots, same must happen also for monic polynomials. Deform the monic polynomial by changing $(r_1, r_2 = r_1)$ to $(r_1, r_1(1+r))$, where integer r has decomposition $r = \prod_i p_i^{k_i}$ to powers of prime. In $O(p) = 0$ approximation the roots r_1 and r_2 of the monic polynomial are still degenerate so that p_i represent ramified primes.
   If the number of p_i is large, one has high degree of ramification perhaps favored by p-adic evolution as increase of number theoretic co-operation. On the other hand, large p-adic primes are expected to correspond to high evolutionary level. Is there a competition between large ramified primes and number of ramified primes? Large $h_{eff}/h_0 = n$ in turn favors large dimension n for extension.

3. The condition that two roots of a polynomial co-incide means that both polynomial $P(x)$ and its derivative dP/dx vanish at the roots. Polynomial $P(x) = x^n + p_{n-1}x^{n-1} + ..p_0$ is parameterized by the coefficients which are rationals (integers) for irreducible (monic) polynomials. $n-1$-tuple of coefficients $(p_{n-1}, ..., p_0)$ defines parameter space for the polynomials. The criticality condition holds true at integer points $n-1-D$ surface of this parameter space analogous to cognitive representation.
   The condition that critical points correspond to rational (integer) values of parameters gives an additional condition selecting from the boundary a discrete set of points allowing ramification. Therefore there are strong conditions on the occurrence of ramification and only very special monic polynomials are selected.
   This suggests octonionic polynomials with rational or even integer coefficients, define strongly critical surfaces, whose p-adic deformations define p-adically critical surfaces defining an extension with ramified primes p. The condition that the number of rational critical points is non-vanishing or even large could be one prerequisite for number theoretical fitness.

4. There is a connection to catastrophe theory, where criticality defines the boundary of the region of the parameter space in which discontinuous catastrophic change can take place as replacement of roots of $P(x)$ with different root. Catastrophe theory involves polynomials $P(x)$ and their roots as well as criticality. Cusp catastrophe is the simplest non-trivial example of catastrophe surface with $P(x) = x^4/4 - ax - bx^2/2$: in the interior of V-shaped curve in (a, b)-plane there are 3 roots to $dP(x) = 0$, at the curve 2 solutions, and outside it 1 solution. Note that now the parameterization is different from that proposed above. The reason is that in catastrophe theory diffeo-invariance is the basic motivation whereas in M^8 there are highly unique octonionic preferred coordinates.

If p-adic length scale hypothesis holds true, primes near powers of 2, prime powers, in particular Mersenne primes should be ramified primes. Unfortunately, this picture does not allow to say anything about why ramified primes near power of 2 could be interesting. Could the appearance of ramified primes somehow relate to a mechanism in which $p = 2$ as a ramified prime would precede other primes in the evolution. $p = 2$ is indeed exceptional prime and also defines the smallest p-adic length scale.

For instance, could one have two roots a and $a + 2^k$ near to each other 2-adically and could the deformation be small in the sense that it replaces 2^k with a product of primes near powers of 2: $2^k = \prod_i 2^{k_i} \rightarrow \prod_i p_i$, p_i near 2^{k_i}? For the irreducible polynomial defining the extension of rationals, the deforming could be defined by $a \rightarrow a + 2^k$ could be replaced by $a \rightarrow a + 2^k/N$ such that $2^k/N$ is small also in real sense.

10.4.2 Appendix: About the decomposition of primes of number field K to primes of its extension L/K

The followings brief summary lists some of the basic terminology related to the decomposition of primes of number field K in its extension.

1. A typical problem is the splitting of primes of K to primes of the extension L/K which has been already described. One would like to understand what happens for a given prime in terms of information about K. The splitting problem can be formulated also for the extensions of the local fields associated with K induced by L/K.
2. Consider what happens to a prime ideal p of K in L/K. In general p decomposes to product $p = \prod_{i=1}^{g} P_i^{e_i}$ of powers of prime ideals P_i of L. For $e_i > 1$ ramification is said to occur. The finite field K/p is naturally imbeddable to the finite field L/P_j defining its extension. The degree of the residue field extension $(L/P_i)/(K/p)$ is denoted by f_i and called inertia degree of P_i over p. The degree of L/K equals to $[L : K] = \sum e_i f_i$.
   If the extension is Galois extension (see http://tinyurl.com/zu5ey96), one has $e_i = e$ and $f_i = f$ giving $[L : K] = efg$. The subgroups of Galois group $Gal(L/K)$ known as decomposition group D_i and inertia group I_i are important. The Galois group of F_i/F equals to D_i/I_i.
   For Galois extension the Galois group $Gal(L/K)$ leaving p invariant acts transitively on the factors P_i permuting them with each other. Decomposition group D_i is defined as the subgroup of $Gal(L/K)$ taking P_i to itself.
   The subgroup of $Gal(L/K)$ inducing identity isomorphism of P_i is called inertia group I_i and is independent of i. I_i induces automorphism of $F_i = L/P_i$. $Gal(F_i/F)$ is isomorphic to D_i/I_i. The orders of I_i and D_i are e and ef respectively. The theory of Frobenius elements identifies the element of $Gal(F_i/F) = D_i/I_i$ as generator of cyclic group $Gal(F_i/F)$ for the finite field extension F_i/F. Frobenius element can be represented and defines a character.
3. Quadratic extensions $Q(\sqrt{n})$ are simplest Abelian extensions and serve as a good starting point (see http://tinyurl.com/zofhmb8) the discrimant $D = n$ for $p \ mod \ 4 = 1$ and $D = 4n$ otherwise characterizes splitting and ramification. Odd prime p of the extension not dividing D splits if and only if D quadratic residue modulo p. p ramifies if D is divisible by p. Also the theorem by Kronecker and Weber stating that every Abelian extension is contained in cyclotomic extension of Q is a helpful result (cyclotonic polynomials has as it roots all n roots of unity for given n)

Even in quadratic extensions L of K the decomposition of ideal of K to a product of those of extension need not be unique so that the notion of prime generalized to that of prime ideal becomes problematic. This requires a further generalization. One ends up with the notion of ideal class group (see http://tinyurl.com/hasyllh): two fractional ideals I_1 and I_2 of L are equivalent if the are elements a and b such that $aI_1 = bI_2$. For instance, if given prime of K has two non-equivalent decompositions $p = \pi_1\pi_2$ and $p = \pi_3\pi_4$ of prime ideal p associated with K to prime ideals associated with L, then π_2 and π_3 are equivalent in this sense with $a = \pi_1$ and $b = \pi_4$. The classes form a group J_K with principal ideals defining the unit element with product defined in terms of the union of product of ideals in classes (some products can be identical). Factorization is non-unique if the factor J_K/P_K - ideal class group - is non-trivial group. $Q(\sqrt{-5})$ gived a representative example about non-unique factorization: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ (the norms are 4×9 and 6×6 for the two factorizations so that they cannot be equivalent.

This leads to class field theory (see http://tinyurl.com/zdnw7j3 and http://tinyurl.com/z3s4kjn).

1. In class field theory one considers Abelian extensions with Abelian Galois group. The theory provides a one-to-one correspondence between finite abelian extensions of a fixed global field K and appropriate classes of ideals of K or open sub-groups of the idele class group of K. For example, the Hilbert class field, which is the maximal unramified abelian extension of K, corresponds to a very special class of ideals for K.
2. Class field theory introduces the adele formed by reals and p-adic number fields Q_p or their extensions induced by algebraic extension of rationals. The motivation is that the very tough problem for global field K (algebraic extension of rationals) defines much simpler problems for the local fields Q_p and the information given by them allows to deduce information about

K. This because the polynomials of order n in K reduce effectively to polynomials of order $n\ mod\ p^k$ in Q_p if the coefficients of the polynomial are smaller than p^k. One reduces monic irreducible polynomial f characterizing extension of Q to a polynomial in finite field F_p. This allows to find the extension Q_p induced by f.
An irreducible polynomial in global field need not be irreducible in finite field and therefore can have multiple roots: this corresponds to a ramification. One identifies the primes p for which complete splitting (splitting to first ordinary monomials) occurs as unramified primes.

3. Class field theory also includes a reciprocity homomorphism, which acts from the idele class group of a global field K, i.e. the quotient of the ideles by the multiplicative group of K, to the Galois group of the maximal abelian extension of K. Wikipedia article makes the statement "*Each open subgroup of the idele class group of K is the image with respect to the norm map from the corresponding class field extension down to K*". Unfortunately, the content of this statement is difficult to comprehend with physicist's background in number theory.

10.5 Appendix: Explicit formulas for the evolution of cosmological constant

What is needed is induced Kähler form $J(S^2(X^4)) \equiv J$ at the twistor sphere $S^2(X^4) \equiv S^2$ associated with space-time surface. $J(S^2(X^4))$ is sum of Kähler forms induced from the twistor spheres $S^2(M^4)$ and $S^2(CP_2)$.

$$J(S^2(X^4) \equiv J = P[J(S^2(M^4)) + J(S^2(CP_2))] \ , \tag{10.5.1}$$

where P is projection taking tensor quantity T_{kl} in $S^2(M^4)\times S^2(CP_2)$ to its projection in $S^2(X^4)$. Using coordinates y^k for $S^2(M^4)$ or $S^(CP_2)$ and x^μ forS^2, P is defined as

$$P: T_{kl} \rightarrow T_{\mu\nu} = T_{kl}\frac{\partial y^k}{\partial x^\mu}\frac{\partial y^l}{\partial x^\nu} \ . \tag{10.5.2}$$

For the induced metric $g(S^2(X^4)) \equiv g$ one has completely analogous formula

$$g = P[g(J(S^2(M^4)) + g(S^2(CP_2))] \ . \tag{10.5.3}$$

The expression for the coefficient K of the volume part of the dimensionally reduced 6-D Kähler action density is proportional to

$$L(S^2) = J^{\mu\nu}J_{\mu\nu}\sqrt{det(g)} \ . \tag{10.5.4}$$

(Note that $J_{\mu\nu}$ refers to S^2 part 6-D Kähler action). This quantity reduces to

$$L(S^2) = (\epsilon^{\mu\nu}J_{\mu\nu})^2\frac{1}{\sqrt{det(g)}} \ . \tag{10.5.5}$$

where $\epsilon^{\mu\nu}$ is antisymmetric tensor density with numerical values +,-1. The volume part of the action is obtained as an integral of K over S^2:

$$S(S^2) = \int_{S^2} L(S^2) = \int_{-1}^{1} du\int_0^{2\pi} d\Phi\frac{J^2_{u\Phi}}{\sqrt{det(g)}} \ . \tag{10.5.6}$$

$(u,\Phi) \equiv (cos(\Theta,\Phi)$ are standard spherical coordinates of S^2) varying in the ranges $[-1,1]$ and $[0,2\pi]$.

This the quantity that one must estimate.

10.5.1 General form for the embedding of twistor sphere

The embedding of $S^2(X^4) \equiv S^2$ to $S^2(M^4) \times S^2(CP_2)$ must be known. Dimensional reduction requires that the embeddings to $S^2(M^4)$ and $S^2(CP_2)$ are isometries. They can differ by a rotation possibly accompanied by reflection

One has

$$(u(S^2(M^4)), \Phi(S^2(M^4)) = (u(S^2(X^4), \Phi(S^2(X^4)) \equiv (u, \Phi) \ ,$$

$$\left[u(S^2(CP_2)), \Phi(S^2(CP_2))\right] \equiv (v, \Psi) = RP(u, \Phi)$$

where RP denotes reflection P following by rotation R acting linearly on linear coordinates (x,y,z) of unit sphere S^2). Note that one uses same coordinates for $S^2(M^4)$ and $S^2(X^4)$. From this action one can calculate the action on coordinates u and Φ by using the definite of spherical coordinates.

The Kähler forms of $S^2(M^4)$ resp. $S^2(CP_2)$ in the coordinates $(u = cos(\Theta), \Phi)$ resp.(v, Ψ) are given by $J_{u\Phi} = \epsilon = \pm 1$ resp. $J_{v\Psi} = \epsilon = \pm 1$. The signs for $S^2(M^4)$ and $S^2(CP_2)$ are same or opposite. In order to obtain small cosmological constant one must assume either

1. $\epsilon = -1$ in which case the reflection P is absent from the above formula (RP $\rightarrow$ R).
2. $\epsilon = 1$ in which case P is present. P can be represented as reflection $(x, y, z) \rightarrow (x, y, -z)$ or equivalently $(u, \Phi) \rightarrow (-u, \Phi)$.

Rotation R can represented as a rotation in (y,z)-plane by angle ϕ which must be small to get small value of cosmological constant. When the rotation R is trivial, the sum of induced Kähler forms vanishes and cosmological constant is vanishing.

10.5.2 Induced Kähler form

One must calculate the component $J_{u\Phi}(S^2(X^4)) \equiv J_{u\Phi}$ of the induced Kähler form and the metric determinant $det(g)$) using the induction formula expressing them as sums of projections of M^4 and CP_2 contributions and the expressions of the components of $S^2(CP_2)$ contributions in the coordinates for $S^2(M^4)$. This amounts to the calculation of partial derivatives of the transformation R (or RP) relating the coordinates (u, Φ) of $S^2(M^4)$ and to the coordinates (v, Ψ) of $S^2(CP_2)$.

In coordinates (u, Φ) one has $J_{u\Phi}(M^4) = \pm 1$ and similar expression holds for $J(v\Psi)S^2(CP_2)$. One has

$$J_{u\Phi} = 1 + \frac{\partial(v, \Psi)}{\partial(u, \Phi)} \ . \tag{10.5.7}$$

where right-hand side contains the Jacobian determinant defined by the partial derivatives given by

$$\frac{\partial(v,\Psi)}{\partial(u,\Phi)} = \frac{\partial v}{\partial u}\frac{\partial \Psi}{\partial \Phi} - \frac{\partial v}{\partial \Phi}\frac{\partial \Psi}{\partial u} \ . \tag{10.5.8}$$

10.5.3 Induced metric

The components of the induced metric can be deduced from the line element

$$ds^2(S^2(X^4) \equiv ds^2 = P[ds^2(S^2(M^4)) + ds^2(S^2(CP_2))] \ .$$

where P denotes projection. One has

$$P(ds^2(S^2(M^4))) = ds^2(S^2(M^4)) = \frac{du^2}{1-u^2} + (1-u^2)d\Phi^2 \ .$$

and

$$P[ds^2(S^2(CP_2))] = P[\frac{(dv)^2}{1-v^2} + (1-v^2)d\Psi^2] \ ,$$

One can express the differentials $(dv, d\Psi)$ in terms of $(du, d\Phi)$ once the relative rotation is known and one obtains

$$P[ds^2(S^2(CP_2))] = \frac{1}{1-v^2}[\frac{\partial v}{\partial u}du + \frac{\partial v}{\partial \Phi}d\Phi]^2 + (1-v^2)[\frac{\partial \Psi}{\partial u}du + \frac{\partial \Psi}{\partial \Phi}d\Phi]^2 \ .$$

This gives

$$\begin{aligned} & P[ds^2(S^2(CP_2))] \\ & = [(\tfrac{\partial v}{\partial u})^2 \tfrac{1}{1-v^2} + (1-v^2)(\tfrac{\partial \Psi}{\partial u})^2]du^2 \\ & +[(\tfrac{\partial v}{\partial \Phi})^2 \tfrac{1}{1-v^2} + (\tfrac{\partial \Psi}{\partial \Phi})^2 1-v^2]d\Phi^2 \\ & +2[\tfrac{\partial v}{\partial u}\tfrac{\partial v}{\partial \Phi}\tfrac{1}{(1-v^2)} + \tfrac{\partial \Psi}{\partial u}\tfrac{\partial \Psi}{\partial \Phi}(1-v^2)]dud\Phi \ . \end{aligned}$$

From these formulas one can pick up the components of the induced metric $g(S^2(X^4)) \equiv g$ as

$$\begin{aligned} g_{uu} &= \tfrac{1}{1-u^2} + (\tfrac{\partial v}{\partial u})^2 \tfrac{1}{1-v^2} + (1-v^2)(\tfrac{\partial \Psi}{\partial u})^2] \ , \\ g_{\Phi\Phi} &= 1-u^2 + (\tfrac{\partial v}{\partial \Phi})^2 \tfrac{1}{1-v^2} + (\tfrac{\partial \Psi}{\partial \Phi})^2(1-v^2) \\ g_{u\Phi} = g_{\Phi u} &= \tfrac{\partial v}{\partial u}\tfrac{\partial v}{\partial \Phi}\tfrac{1}{(1-v^2)} + \tfrac{\partial \Psi}{\partial u}\tfrac{\partial \Psi}{\partial \Phi}(1-v^2) \ . \end{aligned} \tag{10.5.9}$$

The metric determinant $det(g)$ appearing in the integral defining cosmological constant is given by

$$det(g) = g_{uu}g_{\Phi\Phi} - g_{u\Phi}^2 \ . \tag{10.5.10}$$

10.5.4 Coordinates (v, Ψ) in terms of (u, Φ)

To obtain the expression determining the value of cosmological constant one must calculate explicit formulas for (v, Ψ) as functions of (u, Φ) and for partial derivations of (v, Ψ) with respect to (u, Φ).

Let us restrict the consideration to the RP option.

1. P corresponds to $z \rightarrow -z$ and to

$$u \rightarrow -u \ . \tag{10.5.11}$$

2. The rotation R $(x, y, z) \rightarrow (x', y', z')$ corresponds to

$$x' = x, \quad y' = sz + cy = su + c\sqrt{1-u^2}sin(\Phi) \ , \quad z' = v = cu - s\sqrt{1-u^2}sin(\Phi) \tag{10.5.12}$$

Here one has $(s, c) \equiv (sin(\epsilon), cos(\epsilon)$, where ϵ is rotation angle, which is extremely small for the value of cosmological constant in cosmological scales.
From these formulas one can pick v and $\Psi = arctan(y'/x)$ as

$$v = cu - s\sqrt{1-u^2}sin(\Phi) \quad \Psi = arctan[\tfrac{su}{\sqrt{1-u^2}}cos(\Phi) + tan(\Phi)] \ . \tag{10.5.13}$$

3. RP corresponds to

$$v = -cu - s\sqrt{1-u^2}sin(\Phi) \quad \Psi = arctan[-\tfrac{su}{\sqrt{1-u^2}}cos(\Phi) + tan(\Phi)] \ . \tag{10.5.14}$$

10.5.5 Various partial derivatives

Various partial derivates are given by

$$\begin{aligned}
\frac{\partial v}{\partial u} &= -1 + s\frac{u}{\sqrt{1-u^2}}sin(\Phi) \ , \\
\frac{\partial v}{\partial \Phi} &= -s\frac{u}{\sqrt{1-u^2}}cos(\Phi) \ , \\
\frac{\partial \Psi}{\partial \Phi} &= (-s\frac{u}{\sqrt{1-u^2}}sin(\Phi) + c)\frac{1}{X} \ , \\
\frac{\partial \Psi}{\partial u} &= \frac{scos(\Phi)(1+u-u^2)}{(1-u^2)^{3/2}}\frac{1}{X} \ , \\
X &= cos^2(\Phi) + [-s\frac{u}{\sqrt{1-u^2}} + csin(\Phi)]^2 \ .
\end{aligned} \tag{10.5.15}$$

Using these expressions one can calculate the Kähler and metric and the expression for the integral giving average value of cosmological constant. Note that the field equations contain S^2 coordinates as external parameters so that each point of S^2 corresponds to a slightly different space-time surface.

10.5.6 Calculation of the evolution of cosmological constant

One must calculate numerically the dependence of the action integral S over S^2 as function of the parameter $s = sin(\epsilon)$). One should also find the extrema of S as function of s.

Especially interesting values are very small values of s since for the cosmological constant becomes small. For small values of s the integrand (see Eq. 10.5.6) becomes very large near poles having the behaviour $1/\sqrt{g} = 1/(sin(\Theta) + O(s))$ coming from $\sqrt{g}$ approaching that for the standard metric of S^2. The integrand remains finite for $s \neq 0$ but this behavior spoils the analytic dependence of integral on s so that one cannot do perturbation theory around $s = 0$. The expected outcome is a logarithmic dependence on s.

In the numerical calculation one must decompose the integral over S^2 to three parts.

1. There are parts coming from the small disks D^2 surrounding the poles: these give identical contributions by symmetry. One must have criterion for the radius of the disk and the natural assumption is that the disk radius is of order s.
2. Besides this one has a contribution from S^2 with disks removed and this is the regular part to which standard numerical procedures apply.

One must be careful with the expressions involving trigonometric functions which give rise to infinite if one applies the formulas in straightforward manner. These infinities are not real and cancel, when one casts the formulas in appropriate form inside the disks.

1. The limit $u \rightarrow \pm 1$ at poles involves this kind of dangerous quantities. The expression for the determinant appearing in $J_{u\Phi}$ remains however finite and $J^2_{u\phi}$ vanishes like s^2 at this limit. Also the metric determinant $1/\sqrt{g}$ remains finite expect at $s = 0$.
2. Also the expression for the quantity X in $\Psi = arctan(X)$ contains a term proportional to $1/cos(\Phi)$ approaching infinity for $\Phi \rightarrow \pi/2, 3\pi/2$. The value of $\Psi = arc(tan(X)$ remains however finite and equal to $\pm\Phi$ at this limit depending on on the sign of us.

Concerning practical calculation, the relevant formulas are given in Eqs. 10.5.5, 10.5.6, 10.5.7, 10.5.8, 10.5.9, 10.5.10, and 10.5.15.

The calculation would allow to test/kill the key conjectures already discussed.

1. There indeed exist extrema satisfying $dS(S^2)/ds = 0$.
2. These extrema are in one-one correspondence with the zeros of zeta.

There are also much more specific conjctures to be killed.

1. These extrema correspond to $s = sin(\epsilon) = 2{-}k$ or more generally $s = p^{-k}$. This conjecture is inspired by p-adic length scale hypothesis but since the choice of evolution parameter is to high extent free, the conjecture is perhaps too spesific.
2. For certain integer values of integer k the integral $S(S^2)$ of Eq. 10.5.6 is of form $S(S^2) = xs^2$ for $s = 2^{-k}$, where x is a universal numerical constant.
   This would realize the idea that p-adic length scales realized as scales associated with cosmological constant correspond to fixed points of renormalization group evolution implying that

radiative corrections otherwise present cancel. In particular, the deviation from $s = 2^{-d/2}$ would mean anomalous dimension replacing $s = 2^{-d/2}$ with $s^{-(d+\Delta d)/2}$ for $d = k$ the anomalies dimension Δd would vanish.
The condition $\Delta d = 0$ should be equivalent with the vanishing of the dS/ds. Geometrically this means that $S(s)$ curve is above (below) $S(s) = xs^2$ and touches it at points $s = x2^{-k}$, which would be minima (maxima). Intermediate extrema above or below $S = xs^2$ would be maxima (minima).

Chapter 11

About the role of Galois groups in TGD framework

11.1 Introduction

This chapter was inspired by the inverse problem of Galois theory [A180] (`https://cutt.ly/jmjpyDS`). Galois groups are realized as number theoretic symmetry groups realized physically in TGD [L43, L42]. Galois confinement is as analog of color confinement is proposed in TGD inspired quantum biology [L92, L103, L143, L90, L141, L142, L113].

Two instances of the inverse Galois problem, which are especially interesting in TGD, are following:

Q1: Can a given finite group appear as Galois group over Q? The answer is not known.

Q2: Can a given finite group G appear as a Galois group over some EQ? The answer to this question is positive as will be found and the extensions for a given G can be explicitly constructed.

The formulation adelic physics [L43, L42] is based on M^8-H duality in which space-time surface in complexified M^8 are coded by polynomials with rational coefficients. Adelic physics involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The idea allowing to answer this question is the requirement that TGD adelic physics is able to represent all finite groups as Galois groups of Q or some EQ acting physical symmetry group.

If the answer to **Q1** is positive, it is enough to have polynomials with coefficients in Q. It not, then also EQs are needed as coefficient fields for polynomials to get all Galois groups. Needless to say, the first option would be the more elegant one.

In the sequel the inverse problem is considered from the perspective of TGD.

1. M^8-H-duality, $H=M^4\times CP_2$ adelic physics [L43, L42] based on the identification of space-time surfaces X^4 in the complexified M^8 identifiable as complexified octonions. Normal space of X^4 is required to be associative/quaternionic and to contain an integrable distribution of commutative 2-D sub-spaces. At the level of H, twistor lift of TGD implies partial differential equations defined by a variational principle based on action, which is sum of volume term and Kähler action.
   The spacetime surfaces are preferred extremals [L108, L114] identifiable as minimal surfaces analogous to soap films except at the dynamically generated analogs of frames at which the minimal surface property fails and field equations hold true only for the full action. At the frame, the divergences of isometry currents for the volume term and Kähler action have delta function singularities which cancel each other.
   In M^8, space-time surfaces are determined as 4-D "roots" of polynomials of complex variable over rationals continued to an octonionic polynomial [L82, L83, L104, L114]. The Galois group over Q acts as a physical symmetry group permuting the sheets of the X^4 defined by the 4-D roots. This action induces symmetry also at the H-side of the duality and the Galois group defines a new kind of symmetry, which distinguishes between TGD and competing theories.

2. The notion of infinite primes is inspired by TGD [K84] and has a physical interpretation as a repeated second quantization of an arithmetic super-symmetric quantum field theory (QFT). This hierarchy corresponds to a hierarchy of multi-variable polynomials obtained by taking a polynomial of t_n and replacing its rational coefficients with rational functions of $t_1, ..t_{n-1}$ with rational coefficients. One can assign a Galois group to these polynomials and therefore also to infinite primes and integers.
3. The fraction of 2-groups with order not larger than n approaches unity at the limit $n \to \infty$. All 2-groups act as Galois groups of space-time surfaces. p-Adic length scale hypothesis states that primes near power of 2, define physically preferred p-adic length scales. The special role of 2-groups might explain why the p-adic length scale hypothesis [K50] is true.
4. Galois groups, in particular simple Galois groups acting on cognitive representations consisting of points, whose coordinates in a number theoretically preferred coordinate system of octonions belong to EQ, play a fundamental role in the TGD view of cognition [L95]. The TGD based model of genetic code [L16, L92] involves in an essential way the groups A_5 (icosahedron (I)), which is the smallest simple and non-commutative group, and A_4 (tetrahedron (T)). Genetic code has as building bricks Hamiltonian cycles of I and T. Genetic code relates to information and therefore to cognition so that the interpretation of these symmetry groups as Galois groups is suggestive.
   The most recent step of progress was the realization that genetic code can be represented in terms of icosa-tetrahedral tesselation of a hyperbolic 3-space H^3 [L103] and that the notion of genetic code generalizes dramatically. Also octahedron (O) is involved with the tesselation but plays a completely passive role. The question why the genetic code is a fusion of 3 icosahedral codes and of only a single tetrahedral code remained however poorly understood.
   The identification of the symmetry groups of the I, O, and T as Galois groups makes it possible to answer this question. Icosa-tetrahedral tesselation can be replaced with its 3-fold covering replacing $I/O/T$ with the corresponding symmetry group acting as a Galois group. T has only only a single Hamiltonian cycle and its 3-fold covering behaves as a single cycle.

11.2 Some background about Galois groups

11.2.1 Basic definitions

Galois extensions are by definition represented by the roots of polynomials with coefficients in K. By definition Galois group Gal leaves K invariant and permutes roots. For instance, complex conjugate roots are permuted.

There are two basic ways to construct Galois extensions L/K of a number field K.

1. The roots of irreducible polynomials over K (no rational roots in K) define a Galois extension. The order $ord(Gal)$ of Gal is equal to the dimension n of extension L: $ord(Gal) = n$.
2. If a number field L and its automorphism group Aut is known then any subgroup G of Aut defines a sub-field K^G invariant under G and L/K^G an extension having G as Galois group.

The functional composition $P_1 \cdot P_2$ creates an extensions for which the Galois group of P_2 is a normal subgroup if one has $P_i(0) = 0$ and $P_1 \circ P_2$ has also the roots of P_2.

Polynomial rings $K(t_1, ..., t_n)$ of several variables give rise to extensions via the roots $P(t) = 0$.

1. The permutation group S_n acts as automorphisms of $K(t_1, ...t_n)$. Especially interesting sub-field is the invariant field of S_n generated by polynomials generated by symmetric functions. Any finite group G is sub-group of some S_n and defines a G-invariant field of $K(t_1, ...t_n)$ as G-invariant rational functions containing the field generated by symmetric functions.
2. The completion of a number field is algebraically complete. For instance, for Q this completion $\overline{Q}$ consists of algebraic numbers. The Galois group of $\overline{Q}/Q$ is profinite, which means that it is infinite but effectively finite, and can be constructed by inverse limit construction for a sequence of extensions leading to algebraic numbers.
   The extensions of completions are necessarily transcendental. In the case of $\overline{Q}$ they involve addition of transcendentals the extensions.

11.2.2 Some results about Galois groups over rationals

It is good to start by listing some basic results related to the Galois groups [A180].

1. Galois theorem states that polynomials are solvable for degree $d \leq 5$. In these cases the Galois group is solvable meaning EQ is extensions of extension .... of rationals and that Galois groups for EQ has a descending decomposition by normal groups H_i which are commutator groups of the normal group H_{i+1} at the previous level. Equivalently, the Galois group for an extension of an extension at level i is Abelian. For $d < 5$ Galois group is $A_5 = S_5/Z_2$ with 60 elements. This is the smallest non-Abelian simple group. By its definition, a simple group does not have a decomposition to normal groups.
2. Kronecker-Webber theorem states that any abelian group appears as a Galois group for Q. This was found by studying cyclic extensions.
3. Shafarevich proved that every solvable group appears as a Galois group for an EQ.
4. Scholz and Reichardt proved that for an odd prime p, every finite p-group occurs as a Galois group over Q. The order of each element of a simple group is a power of p. 2-groups which also appear as a Galois group over Q are of special interest since for given n, most groups with order smaller than n are 2-groups. This result is of special interest from the point of view of p-adic length scale hypothesis.
5. It has been conjectured that almost all finite groups can act as a Galois group over Q. (`https://cutt.ly/imjwDKC`).
6. Simple groups are primes for finite groups. Simple groups appear in the decomposition of EQ to a sequence of extensions with a simple Galois group represented by a hierarchy of polynomials. In TGD inspired theory of cognition simple groups are analogs of elementary particles [L95, L111, L144, L104] so that they are of special interest.
   Any finite group, in particular, any simple group, appears as a Galois group over Q. The open question is whether a given simple group can appear as a Galois group over Q.
   Many simple groups appear as Galois groups over Q. The theorem of Malle and Matzat states that if p is an odd prime such that either 2, 3 or 7 is a quadratic non-residue modulo p (q is quadratic residue if $x^2 = q \ mod \ p$ has a solution) then $PSL(2,p)$ occurs as a Galois group of EQ.
   Four of the Mathieu groups, namely M_{11}, M_{12}, M_{22} and M_{24}, occur as Galois groups of EQ. For M_{23} the situation is open. The theorem of Thompson states that the Monster group, the largest sporadic simple group, appears as a Galois group over Q.

11.2.3 Various problems related to inverse Galois problem

In [A180] various problems related to inverse Galois problem are listed.

1. The general existence problem can be formulated as the following question. Given number field K and finite group G, can G act as a Galois group for some extension of K?
2. If the answer to the general existence problem is affirmative for given K and G, the explicit construction polynomials $P(t)$ is the next challenge.
   One can also consider polynomials $P(t_1, ..., t_n)$ of several arguments which could be regarded as parametric representation for a large number of polynomials. $P(t_1, ..., t_n)$ must be irreducible. If the restriction of arguments appearing as parameters to a specific value produces irreducible polynomial, one can hope that the Galois group over Q is same as that of the polynomial which is permutation group S_n of the arguments.
   This process is called specification, and Hilbert's irreducibility theorem states the conditions for when the irreducibility is preserved in the process so that Galois group is inherited. The conditions mean that the Galois group is almost independent on the parameter values and the loci were irreducibility fails are the places where this happens. Obviously they correspond to the occurrence of multiple roots. Hilbert proved that for S_n the conditions are satisfied so that they appear as Galois groups of some EQ.
   Also the subgroups $G \subset S_n$ act in $Q(t_1, ...t_n)$ and one can ask under what conditions one can find EQ for which G acts as a Galois group. It turns out that one can construct explictly an EQ, whose extension allows G as a Galois group and specify explicitly the conditions under which this EQ reduces to Q.

3. What is the smallest number of parameters for a generic polynomial $P(t_1, ...t_n)$?

Two special results are mentioned in [A180]. G can be any finite group in the following cases involving only one parameter.

1. For $K = C(t)$ any finite group G appears as Galois group of some Galois extension (defined by polynomial) of $K = C(t)$. This is true also for Galois extensions of generalizes to $K = R(t)$ and $K = \overline{Q}(t)$.
   The result for $K = C(t)$ follows from Riemann existence theorem `https://cutt.ly/FmjsnPA`, which in its original form states that the space of functions on Riemann sphere having singularities at punctures can be regarded as space of analytic functions at Riemann surface obtained as a finite branched covering of S^2 with branchings at punctures.
   The absolute Galois group over $C(t)$ corresponding to an infinite covering of the puncture sphere is identifiable as homotopy group of infinite covering and is free profinite (infinite but effectively finite) group with infinite number of generators. This true when K is closed and therefore holds true for $\overline{Q}$. Extension of $K(t)$ is obtained by adding a parameter and finding the roots.
   The homotopy group of a given finite covering corresponds to a braid group as a finite covering of S_n. This raises the idea that a given finite group having always a representation as subgroup of S_n could allow a construction giving G as Galois group of over Q.
2. If K is p-adic field Q_p, any finite group can act as a Galois group over $K = Q_p(t)$.

11.3 Methods

In this section various methods to answer to the question whether a given finite group can act as a Galois group for given number field K are briefly summarize. The discussion follows the discussion in [**?**]

11.3.1 Regular Inverse Galois Problem

Regular inverse Galois problem starts from an extension L/K, which is regular.

Regularity requires that K is algebraically closed in L - or L is purely transcendental extension of K. This means that the elements of K cannot be expressed as solutions of algebraic extensions in L. One example of a purely transcendental extension is extension of rationals by adding some transcendental numbers. If K is algebraically closed - this is the case for C and algebraic numbers - the condition is satisfied.

A further condition is that L is separable over K. For a physicist, this rather technical looking condition states that the number field $L \otimes_K \overline{K}$ is an integral domain meaning that it has no divisors of zero.

Transcendental extensions are regular. The so-called transcendence basis S consists of elements of L, which do not satisfy any algebraic equation in L. One can construct the field $K(S)$ by forming the product of basis for K and S and $L/K(S))$ is algebraic extension of $K(S)$.

Extensions of algebraic completions are regular/transcendental. The field defined by rational functions formed from S_n invariant polynomials in $K(t_1, ..., , t_n)$ define field K^G of symmetric rational functions which cannot be regarded as an algebraic extension of K. By Noether's theorem stating that K^G is isomorphic with $K(t_1, ..., t_n)$, $K(t_1, ..., t_n)/K^G$ defines an extension of K with the same Galois group.

Whenever one has a Galois extension $M/Q(t)$ (regular or not), it is an easy consequence of the Hilbert Irreducibility Theorem that there is a specialisation M/Q with the same Galois group. If $M/Q(t)$ is regular, one obtains such specialized extensions M/K over any Hilbertian field in $char = 0$, in particular over all algebraic number fields. Char denotes the integer n for which $nx = 0$ is true for all elements of the field. Finite fields F_p have $char = p$.

$C(t)$ and $\overline{Q}$(t) are algebraically closed and transcendental and any finite group defines a Galois group for the extensions of these fields are obtained from a polynomial of n variables by specification. The problem is how to get down to $Q(t)$ from $\overline{Q}$. One must restrict the coefficients of polynomials to a sub-field and it is not clear what happens to the Galois group in this process. This is one case of specialization: one starts from a parametrized set of polynomials $C(t_1, ...t_n)$ or $\overline{Q}(t_1, ...t_n)$ and restricts the parameters $t_1, ..., t_{n-1}$ to say Q.

11.3.2 Hilbert's irreducibility theorem

Consider polynomials $f(t,x)$ with parameters $t = (t_1,...t_r)$ and indeterminates $x = (x_1,...x_n)$. Assume that f irreducible polynomial. Define Hilbert f-set H_f/K as the set of parameter values $t \in K^r$ for which the restriction is well-defined and irreducible. Define Hilbert g-set as a subset of the set of the parameter values $t \in K^r$ for which $g(t)$ is non-vanishing so that g has no zeros in the set defined by the points $(x_1,..x_n)$: this is possible since K^r is a subset of all parameter values t. Define Hilbert set as an intersection of finitely many f-sets and finitely many g-sets.

The field K is said to be Hilbertian if Hilbert sets are nonempty for all r.

The above condition is rather abstract but the following characterization of Hilberianity is more concrete. For a field K with $char = 0$, K is Hilbertian if and only if the following condition holds true. If $f(t,X)$ has no roots in $K(t)$ then $f(a,X)$ has no roots in K.

For $K = Q$, the first condition means that there are no roots which are rational functions and the second condition means that $f(a,X)$ has no rational roots. Rational roots emerge when two or more algebraic roots coincide. In this situation, the irreducibility is preserved in the specification and the Galois group is inherited.

Hilbert also proved that for S_n acting as a Galois group for $Q(t_1,...,t_n)$, it is possible to find an extension of rationals with the same Galois group by specification. The polynomials in question are invariant under S_n and generated by symmetric functions. If the specification has a root, the action of S_n to a root corresponds to the action of a permutation in the parameter space and creates a new root so that the Galois group is S_n.

Under the conditions stated by Noether, this generalizes to subgroups $G \subset S_n$.

11.3.3 Noether's problem

Algebraic numbers are algebraically complete and can have only transcendental extensions, say by addition of transcendental numbers.

On the case of polynomial algebra $Q(t_1,...,t_n)$ the field of invariants Q^G, $G = S_n$ is generated by polynomials symmetric under permutations of n arguments acting as Galois group in the polynomial algebra $C(t_1,...t_n) \equiv C(t)$. Q^{S_n} is transcendental in the sense that the generators do not satisfy polynomial conditions with coefficients in $C(t)$. This algebra has $C(t)$ as extension with Galois group S_n, which obvious commutes with the field operations.

One can consider also sub-groups of $G \subset S_n$ and analogous extensions. In this case it is not obvious that the algebra Q^G is rational which meas that $C(t)$ is purely transcendental extension of Q^G.

The theorem by Emmy Noether states the following: If G is finite and $Q(X)^G/Q$ is rational (purely transcendental), then there is a Galois field extension K/Q with group G.

The proof of the theorem involves Hilbert's irreducibility theorem, rationality property implying that Q^G is isomorphic with $Q(t_1,...t_n)$, and primitive element theorem stating that the extensions of Q are generated by powers of primitive element. How G becomes the Galois group for Q has been already explained.

11.3.4 Rigidity method

Riemann existence theorem is an essential part of the rigidity method. One considers compactified plane with punctures allowing interpretation as a punctured sphere with origin as a marked point. Rieman proved that meromorphic functions singular at punctures functions can be regarded as regular functions in cover of S^2 branched at the punctures defining a Riemann surface.

The homotopy group of the sphere with n punctures has n generators $g_1,...,g_n$ satisfying the relation $g_1...g_n = 1$, since the complement of the regions containing punctures contains no punctures.

There is an infinite number of coverings characterized by the number n of sheets. Intuitively they are analogous to functions $z^{1/n}$. The homotopy group gives rise to the homotopy group of n-fold covering acting also as a Galois group for extension of meromorphic functions induced by the cover. The Galois group serves also as a braid group defining n-fold covering group for the permutation group of the punctures.

Absolute Galois group is associated with the covering with $n = \infty$ and is pro-finite group (infinite but effectively finite). Also this group satisfies the analog of the relation $g_1...g_n = 1$. The absolute Galois group is obtained as an inverse limit of the groups associated with a sequence of extensions (`https://cutt.ly/3RuXEyC`). The map h_{jk} from k:th level to $j \leq k$:th level is homomorphism but not isomorphism for $k > j$ because it is many-to-one. h_{ii} is identity homomorphism. Compatibility condition $h_{ik} = hijhjk$ is satisfied.

Since the braid group B_n is a covering of S_n, any finite group is sub-group of some B_n. Therefore subgroups of B_n could act as Galois group for Q under suitable conditions. Note that braid groups as coverings of S_n also relate to quantum groups and are therefore physically highly interesting.

According to [A180], a considerable progress has been made in the realization of simple groups as Galois groups of regular extensions over $C(t)$ and, and by Hilbert's irreducibility theorem, over every number field.

The basic idea of the rigidity method is that every finite group is a Galois group of some covering of a polynomial field with coefficient in $C(t)$ and $\overline{Q}(t)$. What covering means that one has effectively many-valued polynomials (recall the analog with function $z^{1/n}$) and Galois group permutes the values at a given point. One must only identify the conditions, which ensure that the polynomial can be defined over $Q(t)$.

Rigidity method helps to get down to Q. It is shown that there exists an extension with a given finite Galois group G over some EQ. EQ is generated by the values of G characters for $r \geq 3$ classes of G. Already this is an interesting result. However, if the characters are rational valued, EQ reduces to that for Q and has G as Galois group permuting the copies of the many-valued function.

11.4 Connections with TGD

How Galois groups emerge in TGD framework, was discussed in the introduction. In this section the connections of the inverse Galois problem with TGD are discussed.

11.4.1 Why the inverse Galois problem is so relevant for TGD?

The formulation of TGD relies on $M^8 - H$ duality in which space-time surface in complexified M^8 with octonionic interpretation are coded by polynomials with rational coefficients involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The condition allowing to answer this question is that adelic physics [L43, L42] must be able to represent all finite groups as Galois groups over Q or some EQ as physical symmetry group. More generally, TGD Universe is able to physically represent all internally consistent mathematics.

1. If any finite groups can serve as a Galois group over Q, it is enough to have polynomials with coefficients in Q.
2. It this is not possible, then also EQs are needed as coefficient fields for polynomials to have all possible finite groups as Galois groups. The answer to this question is positive. One studies rational functions of a complex variable in S^2 having singularities at n punctures. The Galois group is identified as a braid group for n braids identifiable as a covering group of S_n. G is identified as a subgroup of the braid group.

   One constructs first an extension of certain EQ with Galois group G. As already explained, EQ can be explicitly constructed in terms of the characters of G assignable to $r \geq 3$ conjugacy classes of G and defined as traces of the matrices representing the group element. G acts as a conjugation. This extension of EQ has G as a Galois group. If the characters are rational, the extension is trivial and G acts as a Galois group over Q.

Needless to say, the first option would be the more elegant one.

The folklore is that the inverse Galois hypothesis is true for very many simple groups (this is true at least in the sense that almost all simple groups are 2-groups). Simple groups do not have a non-trivial normal subgroup decomposition so that the polynomial defining the extensions is not representable as a functional composite of polynomials. If all simple Galois groups can appear as Galois groups over rationals then extensions with non-simple Galois group could correspond to composite polynomials.

Note that the functional composition of polynomials yields a fractal structure at space-time level. The polynomial P_{n_1} resp. P_{n_2} corresponds to n_1- resp. n_2-sheeted and $P_{n_1} \cdot P_{n_2}$ corresponds to n_1-sheeted structure with each sheet consisting of n_2 sheets. The question whether functional iteration of a polynomial P_n could define an analog for the approach to chaos at space-time level in the sense of Mandelbrot and Julia fractals is discussed in [L86]. Also the functional composites involving different polynomials P_{n_i} should lead to fractal-like structures at the space-time level.

For $P(0) = 0$, the roots of polynomial P are possessed also by its iterate and one could glue regions defined by m:th iterate and time reversal of n:th iterate at values $t = t_n$ corresponding to the roots of P to get a a sequence of iterates with various values of n [L114]. In this case the roots are conserved and this brings in mind the notion of conserved genes. It is difficult to avoid the idea that genes could at the level of the magnetic body of the gene actually correspond to functional composites of polynomials P_i satisfying $P_i(0) = 0$.

11.4.2 Galois invariance as a physical symmetry in TGD

Adelic physics [L41, L43] is a proposal for the physics of both sensory experience having real physics as correlate and cognition having various p-adic physics as correlates. Adele is a book-like structure formed by real numbers and the extensions of p-adic number fields induced by a given extension of rationals with the pages of the book glued together along its back consisting of numbers belonging to the extension of rationals. This picture generalizes to space-time level. Adelic physics relies on the notion of cognitive representation as a unique number theoretic discretization of the space-time surface. This discretization has also fermionic analog in terms of spinor structure associated with the group algebra of the Galois group over Q.

Adelic physics very briefly

Number theoretic vision leading to adelic physics [L42] provides a general formulation of TGD complementary to the vision [K76] (`http://tinyurl.com/sh42dc2`) about physics as geometry of "world of classical words" (WCW).

1. p-Adic number fields and p-adic space-time sheets serve as correlates of cognition. Adele is a Cartesian product of reals and extensions of all p-adic number fields induced by given extension of rationals. Adeles are thus labelled by extensions of rationals, and one has an evolutionary hierarchy labelled by these extensions. The large the extension, the more complex the extension which can be regarded as $n-D$ space in K sense, that is with K-valued coordinates.
2. Evolution is assigned with the increase of algebraic complexity occurring in statistical sense in BSFRs, and possibly also during the time evolution by unitary evolutions and SSFRs following them. Indeed, in [L86] (`http://tinyurl.com/quofttl`) I considered the possibility that the time evolution of self in this manner could be induced by an iteration of polynomials - at least in approximate sense. Iteration is a universal manner to produce fractals as Julia sets and this would lead to the emergence of Mandelbrot and Julia fractals and their 4-D generalizations. In the sequel will represent and argument that the evolution as iterations could hold true in exact sense.
   Cognitive representations are identified as intersection of reality and various p-adicities (cognition). At space-time level they consist of points of embedding space $H = M^4 \times CP_2$ or M^8 ($M^8 - H$ duality [L37, L38, L39] allows to consider both as embedding space) having preferred coordinates - M^8 indeed has almost unique linear M^8 coordinates for a given octonion structure.
3. Given extension of given number field K (rationals or extension of rationals) is characterized by its Galois group leaving K - say rationals - invariant and mapping products to products and sums to sums. Given extension E of rationals decomposes to extension E_N of extension E_{N-1} of ... of extension E_1 - denote it by $E \equiv H_N = E_N \circ E_{N-1} ... \circ E_1$. It is represented at the level of classical space-time dynamics in M^8 (`http://tinyurl.com/quofttl`) by a polynomial P which is functional composite $P = P_N \circ P_{N-1} \circ ... \circ P_1$. with $P_i(0) = 0$. The

Galois group of $G(E)$ has the Galois group $H_{N-1} = G(E_{N-1} \circ ... \circ E_1)$ as a normal subgroup so that $G(E)/H_{N-1}$ is group.
The elements of $G(E)$ allow a decomposition to a product $g = h_{N-1} \times h_{N-1} \times ...$ and the order of $G(E)$ is given as the product of orders of H_k: $n = n_0 \times .. \times n_{N-1}$. This factorization of prime importance also from quantum point of view. Galois groups with prime order do not allow this decomposition and the maximal decomposition and are actually cyclic groups Z_p of prime order so that primes appear also in this manner.
Second manner for primes to appear is as ramified primes p_{ram} of extension for which the p-adic dynamics is critical in a well-defined sense since the irreducible polynomial with rational coefficients defining the extension becomes reducible (decomposes into a product) in order $O(p) = 0$. The p-adic primes assigned to elementary particles in p-adic calculation have been identified as ramified primes but also the primes labelling prime extensions possess properties making them candidates for p-adic primes.
Iterations correspond to the sequence $H_k = G_0^{\circ k}$ of powers of generating Galois groups for the extension of K serving as a starting point. The order of H_k is the power n_0^k of integer $n_0 = \prod p_{0i}^{k_i}$. Now new primes emerges in the decomposition of n_0. Evolution by iteration is analogous to a unitary evolution as ex^{iHt} power of Hamiltonian, where t parameter takes the role of k .

4. The complexity of extension is characterized by the orders n and the orders n_k as also the number N of the factors. In the case of iterations of extension the limit of large N gives fractal.
5. At space-time level, Galois group acts in the space of cognitive representations and for Galois extensions for which Galois group has same order as extensions, it is natural do consider quantum states as wave functions in $G(E)$ forming n-D group algebra. Therefore Galois groups becomes physical symmetry groups.
One can assign to the group algebra also spinor structure giving rise to $D = 2^{M/2}$ fermionic states where one has $N = 2M$ or $N = 2M + 1$). One can also consider chirality constraints reducing D by a power of 2. An attractive idea is that this spinor structure represents many-fermion states consisting of $M/2$ fermion modes and providing representation of the fermionic Fock space in finite measurement resolution.

Adelic physics [L42], $M^8 - H$ duality [L82, L83, L104, L114], and zero energy ontology lead (ZEO) [L72, L108, L105] to a proposal [L95] that the dynamics involved with "small" state function reductions (SSFRs) as counterparts of weak measurements could be basically number theoretical dynamics with SSFRs identified as reduction cascades leading to completely un-entangled state in the space of wave functions in Galois group of EQ identifiable as wave functions in the space of cognitive representations. As a side product a prime factorization of the Galois group to simple factors as normal subgroups is obtained.

The result looks even more fascinating if the cognitive dynamics is a representation for the dynamics in real degrees of freedom in finite resolution characterized by the extension of rationals. If cognitive representations represent reality approximately, this indeed looks very natural and would provide an analog for adele formula expressing the norm of a rational as the inverse of the product of is p-adic norms. The results can be appplied to the TGD inspired model of genetic code.

The notions of invariant field and Galois confinement

The physical meaning of the invariant field is interesting from the TGD point of view. I have proposed the notion of Galois confinement as a generalization of color confinement stating that physical states are invariant under Galois group. Color confinement would force hadrons to behave like single unit so that one cannot observe free quarks and the same would happen now.

For instance, in living systems the states of magnetic bodies could be Galois singlets with respect to appropriate Galois group. This would guarantee their stability. For istance, units consisting of 3 dark protons ($h_{eff} = nh_0 > h$) and of 3 dark phonons would represent genetic codons. Galois confinement would force them to behave like single unit. In double DNA strand the codon and is conjugate would form this kind of pair. Dark $3N$-protons and $3N$-photons would in turn represent genes [L143, L141, L142, L92, L103].

Galois invariance means invariance under permutations of space-time sheets by the action of Gal and also the invariance of many-fermion states proposed to correspond to the $2^{[n/2]}$-D spinor space for the spinors assignable to the $n-D$ extension having interpretation as fermionic Fock states.

Galois confinement cannot be permanent. In transitions changing the value of h_{eff} it could be lost. For instance, gene can decay to codons and DNA strand could split during replications and transcription.

11.4.3 The physical interpretation of multi-variable polynomial rings in TGD

Consider the polynomial ring $Q(t_1,..,t_n)$ over Q defining a field of rational functions. This set could define a parametrized set of space-time surfaces. By solving the roots with respect to t_n by keeping $t_1,...t_{n-1}$ as parameters, one obtains some number of roots. This gives rise to an extension of $Q(t_1,...,t_{n-1})$ involving algebraic functions of $t_1,...,t_{n-1}$. One can actually solve the roots of the polynomial equation with respect to any variable t_k. The degrees of the polynomial with respect to t_i are in general different and this would mean that the orders of the Galois group are different.

Here the Noether's theorem comes to the rescue. Instead of polynomials in $Q(t_1,...t_n)$, one can consider symmetrized polynomials invariant under S_n for which the degree is same for all variables t_i and Galois groups have the same order. Therefore the specification with respect to any t_i can give rise to an irreducible polynomial with the same Galois group. In this case, the extensions defined by the roots of polynomials give also an extension of Q with the same Galois group except at points, where the restriction fails to be irreducible.

Noether's theorem considers a situation for a subgroup of $G \subset S_n$. If the G-invariant field Q^G is purely transcendental and therefore isometric with K, the Galois group of extension of Q^G defines a Galois group over Q. Hilbert's theorem states that this is the case for $G = S_n$.

The specification obtained by fixing the values of $t_1,...,t_{n-1}$ must be irreducible. The Galois group for the restriction of the polynomials in Q^G is the same for all parameter values at which irreducibility is true.

Could multi-variable polynomials define sub-spaces of WCW with a given Galois group

Space-time surfaces are determined by polynomials of a complex variable with rational coefficients by algebraically continuing them to polynomials of a complexified octonion.

Polynomials with several variables play a central role in the theory of Galois groups. In the TGD framework the parameter type variables would give a parameterized set of space-time surfaces with the same Galois group except at the points at which the irreducibility fails. The order of parameters matters unless one considers only polynomials invariant under S_n or in some cases its sub-group G and inverse Galois theorem does not hold true as is clear from the fact that the dimension of the local Galois group depends on what variable one regards as the variable which is solved.

This kind of parameter sets would naturally define sub-WCW) (WCW is shorthand for "the world of classical worlds") and allow to define WCW spinor fields defining quantum superpositions of space-time surfaces with the same Galois group except at a the points at which the irreducibility of the restriction to a polynomial of a single variable fails.

At the level of sub-WCW, Galois invariance would mean a restriction to the S_n invariant field of a polynomial ring defined by symmetrized multi-variable polynomials gives a parametrized set of extensions of rationals for the "behavior" variable as a complex coordinate continued to complexified octonionic coordinate of M^8. One can also consider the restricted symmetry defined by $G \subset S_n$ encountered in Noether's theorem.

1. If the Galois group is S_n, a possible physical interpretation of Galois confinement would be as a realization of Bose-Einstein statistics in the bosonic degrees of freedom of WCW. Also fermionic statistics could allow a similar interpretation.
2. Could the Galois confinement with respect to a subgroup $G \subset S_n$ have an interpretation in terms of anyonic statistics and charge fractionalization? Could the condition that the invariant field defines a transcendental extension and hence G acts as a Galois group over Q serve as a physical constraint.
3. On the other hand, the fact that anyonic statistics is essentially a 2-D phenomenon associated with the braid group suggests that it could be assigned to the function field $Q(z)$ at a partonic

2-surface containing fermions as punctures. In M^8-H duality, the positions of fermions would have interpretation as singularities and their position would represent WCW coordinates of the space-time surface. If the strongest form of holography holds true, the position of these punctures could code for the space-time surface (real polynomials are determined by their values at a finite number of points).

The failure of specification, catastrophe theory, and quantum criticality

In Hilbert's irreducibility theorem the notion of specification is essential. Specification fails when it produces as a restriction a reducible polynomial decomposing into a product of polynomials.

1. From the factorization in terms of roots it follows that this occurs when 2 or more roots coincide. For instance, if the roots correspond to a conjugate pair of real or complex roots, they become degenerate and rational. In this case the order of the Galois group decreases.
2. The polynomial can decompose to a more general product meaning a decomposition of the Galois group to a product. One can imagine that there is a small term added to a product of polynomials which vanishes at the criticality. The corresponding space-time region decomposes to distinct regions which can intersect at discrete points. Note for rational polynomials the critical situation is not achieved by a smooth change of the parameters.
   Geometrically criticality means that the space-time surface decomposes to disjoint surfaces corresponding to the roots of the factors which define lower-D extensions of rationals with smaller Galois groups. The decay of the space-time surface occurs. Particle reactions in the geometric sense could correspond to this kind of critical situation.

For the first alternative, catastrophe theoretic analogy [A129] (`https://cutt.ly/9mEG8gn`) helps to gain some physical intuition. In the simplest situations such as cusp catastrophe, one has one behavior variable x and some number of control parameters t_i. The roots are those of the gradient of the dV/dx. The equation $dV/dx = 0$ for equilibrium states gives rise to a catastrophe graph in the space defined by x and control parameters. One restricts to real roots so that the map decomposes to regions characterized by different numbers of real roots.

The cusp catastrophe is a simple example. In this case dV/dx has degree $d = 3$ allowing 3 real roots or 1 real root and complex conjugate pair or roots. By restricting x to be real, these correspond to regions which are 3 sheeted and 1-sheeted covers of the 2-D parameter space. At the different sides of the boundary two real *resp.* complex roots become degenerate and irreducibility fails.

This situation corresponds to a criticality at which sudden catastrophic changes can occur. Therefore also the lower-dimensional regions where irreducibility fails are physically highly interesting.

Self-organized criticality (SOC) (`https://cutt.ly/xmEHgcN`) is a real phenomenon but very difficult to understand in thermodynamics with a single arrow of time and should also have a number theoretical interpretation. Zero energy ontology (ZEO) [L72, L108] is crucial for the formulation of quantum measurement theory in the TGD framework. This theory extends to a theory of consciousness and leads to a model of self-organized quantum criticality (SOQC) [L143, L140, L105] .

One of the key predictions is that the arrow of time changes in the TGD counterparts of ordinary state function reductions (SFRs) - "big" SFRs (BSFRs). For the non-standard arrow of time, dissipative processes look like self-organization processes. This leads to an understanding of (SQOC). The state of the critical sub-system S_1 is unstable but in a time direction opposite to the arrow of time for the system (S). Hence the S_1 tends to criticality when viewed by S. The critical surfaces of the parameter space correspond to the analogs of catastrophes as a failure of reducibility: life seems to love catastrophes!

Hierarchies of parametrized polynomials and of infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy obtained by a repeated second quantization of arithmetic quantum field theory gave a strong boost for the speculations about TGD as a generalized number theory [K84].

1. At the lowest level, ordinary primes p label bosonic and fermionic states of an arithmetic super-symmetric quantum field theory (QFT). The product $X_1 = \prod_p p$ of all finite primes is

infinite as a real number but finite as a p-adic number for all primes p (the norm is $1/p$) and can be regarded as an analog of the Dirac sea.
Infinite primes (having unit p-adic norm for every p) are created by kicking from the Dirac sea a set of negative energy fermions represented by the product $n = \prod_{k\in U} p_k$ primes p_k. This gives rise to an object $P = X_1/n + n$. It is easy to check that P is prime. One can also add bosons by the replacement $P \to kX_1/n + ln$ such that k does not divide n and l has a decomposition to primes appearing as prime factors of n. Altogher m+l bosons have been added.
The simplest infinite primes are linear in the formal variable X_1 as analogs of roots of monomials with rational coefficients and analogous to Fock states of free bosons and fermions. Besides analogs of Fock states, also analogs of bound states are obtained as infinite primes. They correspond to irreducible polynomials $P(X_1)$ of a single variable obtained. One can decompose P just like an ordinary polynomial to factors corresponding to the roots of $P(X_1) = 0$.
Infinite primes have therefore an interpretation as many-particle states of a supersymmetric QFT with bound states included and represented in terms of extensions of rationals. There is no need to emphasize that bound states represent a basic problem of QFTs.

2. At the next level of the hierarchy infinite primes X_1 is replaced with X_2 as a product of infinite primes at the first level of the hierarchy and the construction can be continued. The formal variables X_i characterizing various levels of the hierarchy correspond to the infinite (in real sense) numbers defined by the products of all primes at the previous level.
It is possible to decompose the polynomials at level n to products of monomials defined by the roots of the polynomial equation with X_n as an independent variable to be solved. The infinite primes at the first level become single particle states and second quantization is repeated. At n:th level, one can also construct irreducible polynomials of $X_1, X_2, ..X_n$ and obtains analogs of bound states. One can decompose these polynomials just like one decomposes polynomials of ordinary variables x_i. This gives rise to algebraic extension associated with the rational field defined by polynomials of $X_1, ..., X_n$.

3. The polynomials associated with infinite primes are ordered. The polynomial at n_{th} level has polynomials at $n-1$:th level as coefficients. The nature of the construction as a hierarchy of second quantizations gives rise to states formed from states formed from ...
This suggests that the symmetrization with respect to variables X_i leading to S_n invariant field realized in terms of symmetric functions does not make sense physically. Notice that the polynomial obtained by the symmetrization does not represent an infinite prime. Physically the different levels in the quantization could correspond to space-time surfaces, whose size scales increase with n. Space-time surfaces at level $n-1$ would be glued by wormhole contacts to the space-time surfaces at the level n.

4. In M^8 picture, infinite primes mapped to polynomials of several variables could be interpreted as representations for a parametrized set of space-time possibly representable as space-time surfaces in complexified M^8 (complexified octonions) and mappable to H by $M^8 - H$ duality. They would define a sub-WCW in the "world of classical worlds" (WCW) consisting of space-time surfaces with the same Galois group defined by the roots of X_n. WCW spinor fields would be restricted to this sub-WCW as fermionic Fock states associated with the corresponding space-time surfaces.
Symmetric polynomials do not correspond to infinite primes but one can wonder whether one could construct WCW spinor fields invariant with respect to S_n or its subgroup. A possible interpretation of S_n in terms of Bose-Einstein statistics and of $G \subset S_n$ in terms of anyonic statistics was already mentioned.

This picture strengthens the hope that TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of rational complexified quaternions and complexified octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

11.4.4 About possible physical implications

The order of Galois group equals to the dimension of extension

The order of the Galois group is equal to the dimension of the extension. For S_n the order is $n!$ and for the simple group $A_n = S_n/Z_2$, is $n!/2$. The dimension $n! \simeq \sqrt{2\pi n}(n/e)^n$ as the number of space-time sheets increases roughly exponentially with n (`https://cutt.ly/Omk8K6E`). For instance, the order of A_{34} is $34!/2 \simeq 1.4710^{38}$ whereas Mersenne prime M_{127} is $M_{127}2^{127}-1 \simeq 1.7\times 10^{38}$. There might be a correlation between p-adic length scales and dark scales proportional to n reflecting resonant coupling between phases with different values of n when dark scale and p-adic length scale are nearly identical.

The Galois group need not act in CP_2 direction and the orbits of the Galois group can quite well be in M^4 direction. Coherent structures of parallel flux tubes with a very large number of flux tubes are suggestive.

The gravitational Planck constant $h_{eff} = h_{gr} = GMm/v_0$, where v_0 is a parameter with dimensions of velocity, has very large values, and extensions with very large dimensions of extension could be assigned with gravitational flux tube bundles.

Galois group acts on the Fock states of fermions. A natural expectation is that this space is effectively finite-dimensional and can be regarded as the $2^{n/2}$-dimensional space of spinors for an n-dimensional extension.

The hierarchy of infinite primes, which correspond to a hierarchy of polynomials in n variables with a natural action of S_n. n has a logarithmic dependence on the order of the Galois group. The Galois groups representable as sub-groups of S_n are representable also as sub-groups of S_{n+1} so that there is an inclusion hierarchy.

Most groups of order at most n are 2-groups

.

All p-groups can act as Galois groups of EQ. Most groups of order at most n are 2-groups (`https://cutt.ly/gRuXQM8`). This result is highly interesting from the TGD point of view.

p-Adic length scale hypothesis $p \simeq 2^k$ if the order $n = 2^k$ for Galois group correlates with p-adic prime $p \simeq 2^k$. There is also support for a more general form of hypothesis for primes near a power of 3 are involved. Could p-adic length scale hypothesis relate directly to the fact that most Galois groups are 2-groups?

The order n of the Galois group corresponds to effective Planck constant $h_{eff} = nh_0$ proportional to the number of sheets of the covering defined by Galois extension. Dark Compton length scales are proportional to n.

I have proposed that dark scales for h_{eff} and p-adic length scales for $h_{eff} = h$ interact in the sense that the transitions between states with $h_{eff} = nh_0$ and $h_{eff} = h = n_0h_0$ occur resonantly when the p-adic length scale defined for $h_{eff} = h$ is equal to the dark scale [K48]. A kind of frequency or wavelength resonance would take place.

If this proposal is correct, the p-adic length scale hypothesis could be understood as a poorly group theoretical fact. A given element of a 2-group has order 2^m, where m depends on the element, and the order 2-group is 2^k. Could these facts have some interpretation in terms of Boolean algebra of 2^k elements? Could group multiplication and collections of elements coming as powers of a generating element have some interpretation in terms of Boolean algebra and provide it with an additional group structure.

Interestingly, the spinor space of the extension would be 2^{2^k}-dimensional if the Galois group is 2-group. This might relate to the proposal that the Combinatorial Hierarchy, which consists of Mersenne primes M(k+1)=$M_{M_k} - 1$, $M^k = 2^k - 1$. One has $M(1) = 3$, $M(2) = 2^3 - 1 = 7$, $M(3) = 2^7 - 1 = 127$, $M(4) = 2^{127} - 1$. It is not known whether the subsequent Mersenne numbers are primes. M^3 is assigned to genetic code and $M(4)$ to possible memetic code. The number of genetic codons is $2^{M_3 - 1} = 2^6$ and the number of memetic codons would be $2^{M_7 - 1} = 2^{126}$.

Braids, knots, and Galois groups

The polynomials defining the space-time surface are polynomials of a complex variable. Could one also consider rational functions with n punctures as poles having Taylor or Laurent expansion.

This would bring in the braid group B_n as a covering group of S_n and make it possible to find extensions of Q or some EQ in terms of the values of the characters of $G \subset N_n$.

Braid groups also emerge in another way. Partonic 2-surfaces contain fermions as punctures, which suggests that this approach applies to partonic 2-surfaces and that there is a connection with fermionic braids at the light-like orbits of partonic 2-surfaces and that the Galois group represents the braid group.

11.4.5 Galois groups and genetic code

Abelian groups Z_p, p prime, are simple and the alternating group A_5 with order 60 is the smallest non-Abelian simple group. All groups A_n, $n \geq 5$ are simple and have $n!/2$ elements. A_5 corresponds to the icosahedral group isomorphic with the symmetry group of the dodecahedron.

The TGD based model of genetic code [L16, L92, L103] involves in an essential manner the groups A_5 (icosahedron) and A_4 (tetrahedron). Simple groups play a fundamental role in the TGD view of cognition. Could this mean that genetic code represents the lowest level of an infinite cognitive hierarchy?

The TGD inspired model model of genetic code, cognition, and Galois groups

TGD based model of bioharmony [L16, L92, L103] provides a model of genetic code as a fusion of 3 icosahedral Hamiltonian cycles and the unique tetrahedral Hamiltonian cycle (what "fusion" precisely means is far from clear and I have considered several options).

Icosahedral Hamiltonian cycles is a non-self-intersecting path at icosahedron connecting nearest points if icosahedron going through all 12 points of the icosahedron. It is interpreted as a representation of a 12-note scale with a scaling by quint assigned to a given step along the cycle. For a given Hamiltonian cycle, the allowed 3-chords of icosahedral harmony are identified as chords defined by the triangular faces of the icosahedron.

Remark: In the sequel I will use the shorthands IH, OH, and TH for icosahedral, octahedral, and tetrahedral harmonies. Also the notation $I/O/T$ will be used for icosahedron/octahedron/tetrahedron unless there is a danger of confusing them with their symmetry groups with identical shorthand notations.

Galois groups are essential for cognition in the TGD framework. In particular, simple groups as primes for groups are also primes for cognition [L95]. Genes represent information and Galois groups are crucial for cognition in the TGD framework. Genes would correspond to sequences of 3-chords of bioharmony. This raises several questions.

Could genetic code relate to Galois group A_5 as the smallest simple non-abelian Galois group (and also to the fact that the only polynomials of order smaller than 5 are generically solvable)? Could genetic code correspond to the lowest level in a hierarchy of cognition and of analogs of genetic code?

The order $n = 60$ for A_5 suggests a fusion of 3 icosahedral codes to give 20+20+20 = 60 codons.

1. 3 Platonic solids, - icosahedron (I), tetrahedron (T) , and octahedron (O) - which have triangles as faces so that one can consider the possibility of constructing a lattice like structure by gluing these Platonic solids together along their faces. Hyperbolic space H^3 indeed allows isosa-tetrahedral tessellation, which also involves O:s. I have proposed that this allows a realization of genetic code and also of genes [L103]. The notion of gene generalizes so that genes can also be 2- or 3-D lattice-like structures.
2. A_5 has $A_3 = Z_3$ as a subgroup and I(cosahedron) corresponds to A_5/Z_3. I has several Hamiltonian cycles having as a symmetry group Z_6, Z_4 or Z_2. Z_2 can act either as rotations or reflections.
   Q: Could A_5 as a Galois group as 3-fold covering of I make it possible to understand why the fusion of just 3 icosahedral codes is possible?
3. Tetrahedral group T corresponds to the alternating group $A_4 = S_4/Z_2 = Z_4 \times Z_3$ with 12 elements and tetrahedron identification as A_4/Z_3. The tetrahedral Hamiltonian cycle (4-scale) is unique and has 4 3-chords. The 3-fold copy would correspond to A_4. Information about the unique Hamiltonian cycles of O and T can be found in [A113] (`https://cutt.ly/9mlMiV8`).

Q: Could the factor that there is only one tetrahedral cycle explain why only a single tetrahedron contributes?

4. Octahedral group O has 24 elements and is the wreath product of Z_3 and Z_2^3 and has also the decomposition $O = S_2 \times S_4$. Octahedron can be identified as O/Z_3. Also octahedral Hamiltonian cycle representing 8-scale with 8 chords is unique.
   Q: Why don't octahedral codons contribute?

A model of the genetic code based on icosa-tetrahedral tessellation of hyperbolic 3-space

TGD leads to a proposal for a geometric representation of the genetic code in terms of icosa-tetrahedral tessellation of the hyperbolic 3-space H^3 (mass shell or light-cone proper time $a = constant$ hyperboloids of M^4) [L103]. Both I, O, and T having triangular faces appear in the tessellation. Recall that the corresponding harmonies are denoted by IH, OH and TH.

I do not completely understand the details of the icosa-tetrahedral tessellation. The following picture satisfies the constraints coming from the notion of harmony but I have not proven that it is correct. Here the help of a professional geometrician knowing about tessellations of H^3 would be needed.

1. The analog of the discrete translational symmetry for lattices can be assumed: all I:s , O:s and T:s are equivalent as far as common faces with neighboring Platonic solids are considered.
2. The term icosa-tetrahedral tessellation suggests that all octahedral faces are glued to tetrahedral and icosahedral faces so that octahedral chords reduce to either icosahedral or tetrahedral chords. OH would not be an independent harmony. This requires that the number of common faces between two O:s vanishes: $n_O^O = 0$.
3. T shares at least 1 face with a given I so that the number of tetrahedral chords is reduced to at most 3 for given T. 4 purely tetrahedral faces (not shared with I) are needed. I would have $n_{IT} \leq 4$ purely tetrahedral faces in such a way that the total number of purely tetrahedral 3-chords is 4.
   The simplest possibility is that I shares a common face with 2 T:s. Each T shares 2 faces with O providing 2 purely tetrahedral 3-chords and shares the remaining 2 faces with distinct I:s. One would have $n_T^I = 2$, $n_T^O = 2$, $n_T^T = 0$.
   Since each I defines independently 20 chords, 2 I:s cannot have common faces. One would have $n_I^T = 2$, $n_I^I = 0$ and $n_I^O = 18$ to give $n_I^T + n_I^O + n_I^I = 2 + 18 + 0 = 20$.
4. What remains to be fixed are the numbers n_O^I and n_O^T satisfying $n_O^I + n_O^T = 8$. The conditions $n_O^T \geq 1$ and $n_O^I \geq 1$ must be satisfied since both T and I share faces with Os.
   Music comes to rescue here. The 8 3-chords of OH could define OH sub-harmony of IH. Analogously, the 4 3-chords of TH could define TH as a sub-harmony of OH.
   Could IH sharing 18 3-chords with OH contain 2 transposed copies of OH plus 2 chords of TH? IH cannot of course contain the entire TH as a sub-harmony.
   Could OH contain one copy of TH? This would give $n_O^I = n_O^T = 4$. Could the IH part of OH actually be TH as a sub-harmony of IH so that OH would reduce to 2 copies of TH?

To sum up, if the answers to the questions are positive, the incidence matrix n_i^j, $i, j \in \{I, T, O\}$, telling how many faces i shares with j would be given by

$$\begin{bmatrix} n_I^I & n_I^O & n_I^T \\ n_O^I & N_O^O & n_O^T \\ n_T^I & N_T^O & n_T^T \end{bmatrix} = \begin{bmatrix} 0 & 18 & 2 \\ 4 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix} . \qquad (11.4.1)$$

3-fold cover of the icosa-tetrahedral tessellation

The proposed model does not yet explain the fusion of 3 icosahedral Hamiltomnian cycles. A 3-fold cover of the icosa-tetrahedral tessellation which replaces Platonic solids with their symmetry groups is highly suggestive. This raises a series of questions.

1. How could this representation relate to a possible interpretation in terms of the Galois groups $I = A_5$ and $O = S_2 \times S_4$ and $T = A_4$? Z_3 appears as a sub-group of all these groups and these Platonic solids are coset spaces I/Z_3, O/Z_3, and T/Z_3.

2. Could one lift the icosa-tetrahedral tessellation to a 3-sheeted structure formed by the geometric representations of the Galois groups of this structure acting as symmetry groups? Platonic solids would be replaced with their symmetry groups acting as Galois groups.
3. Could the 3 different icosahedral Hamiltonian cycles correspond to different space-time sheets - roughly CP_2 coordinates as 3-valued functions of M^4 coordinates whereas 20 regions representing icosahedral vertices would correspond to different loci of $E^3 \subset M^4$ just as one intuitively expects?
4. Same should apply to the tetrahedral and octahedral parts of the tessellation. But don't the 3 identical copies of the tetrahedral Hamiltonian cycle give 64+8=72 codons? How can one overcome this problem?

The following is a possible answer to these questions.

1. $h_{eff} = 60h_0$ corresponds to 60-sheeted space-time (here also $60k$-sheeted space-time is possible if 60-D extension of k-dimensional extension is in question). For T and O an analogous picture would apply. One could say that the projections of I and O and T are in M^4. At each sheet one would have icosa-tetrahedral tessellation.
2. I has 3 types of Hamiltonian cycles with symmetry groups Z_6, Z_4, and Z_2 and can give 3 different copies. However, only a single copy of tetrahedral harmony appears in the model: otherwise the number of codons would be larger than 64. Could the 3 identical Hamiltonian cycles for T and O effectively correspond to a single Hamiltonian cycle?
3. The fusion of Hamiltonian cycles is analogous to a formation of many-boson states. For T and O all Hamiltonian cycles would be identical: one would have only one Hamiltonian cycle effectively. The 3-chords associated with the 3 octahedral and tetrahedral cycles are identical so that only single tetrahedral harmony would be present.

To sum up, the lift of the icosa-tetrahedral complex to that defined by the respective Galois groups could explain why just 3 icosahedral Hamiltomian cycles and effectively only 1 tetrahedral cycle.

11.5 Does the notion of polynomial of infinite order make sense?

TGD motivates the question whether the notion of a polynomial of infinite degree could make sense. In the following I consider this question from the point of view of a physicist and start from the vision about physics as generalized number theory.

11.5.1 Background and motivations for the idea

M^8-H-duality ($H = M^4 \times CP_2$) states that space-time surfaces defined as 4-D roots of complexified octonionic polynomials so that they have quaternionic normal space, can be mapped to 4-surfaces in H [L82, L83, L104].

The octonionic polynomials are obtained by algebraic continuation of ordinary real polynomials with rational coefficients although one can also consider algebraic coefficients.

This construction makes sense also for analytic functions with rational (or algebraic) coefficients. For the twistor lift of TGD, cosmological constant Λ emerges via the coefficient of a volume term of the action containing also Kähler action. This leads to an action consisting of Kähler action with both CP_2 and M^4 terms having very interesting and physically attractive properties, such as spin glass degeneracy. $\Lambda = 0$ would correspond to an infinite volume limit making the QFT description possible as an approximate description. Also the thermodynamic limit could correspond to this limit.

Irreducible polynomials of rational coefficients give rise to algebraic extensions characterized by the Galois group and these notions are central in adelic vision.

I do not know of any deep reason preventing analytic functions with rational Taylor coefficients. These would make possible transcendental extensions. For instance, the product $\prod_p (e^x - p))$ for some subset of primes p would give as roots transcendental numbers $log(p)$. The Galois group would be however trivial although the extension is infinite. Second example is provided by trigonometric functions $sin(x)$ and $cos(x)$ with roots coming as multiples of $n\pi$ and $(2n + 1)\pi/2$. This might be necessary in order to have Fourier analysis. The translations by a multiple of π for x act permuted roots but do not leave rational numbers rational so that the interpretation as a Galois group is not possible so that also now Galois group would be trivial.

A long standing question has been whether there exist analytic functions which could be regarded as polynomials of infinite order by posing some conditions to the Taylor coefficients. If so, one might hope that the notion of Galois group could make sense also now, and one might perhaps obtain a unified view about transcendental extensions of rationals.

1. For polynomials as roots of octonionic polynomials space-time surfaces are finite and located inside finite-sized causal diamond (CD).
   In the TGD Universe cosmological constant Λ depends on the p-adic length scale and approaches zero at infinite length scale. At the $\Lambda = 0$ limit, which corresponds also to QFT and thermodynamical limits, space-time surfaces would have infinite size. Only Kähler action with M^4 and CP_2 parts and having ground state degeneracy analogous to spin glass degeneracy would be present.
2. The octonionic algebraic continuations of analytic functions with rational coefficients and subject to restrictions guaranteeing that the notion of prime function makes sense, would define space-time surfaces as their roots.
3. Prime analytic functions defining space-time surfaces would in some sense be polynomials of infinite degree and could be even characterized by the Galois group. For real polynomials complex conjugations for the roots is certainly this kind of symmetry.
   These functions should have Taylor series at origin, which is a special point for octonionic polynomials with rational (or perhaps even algebraic) coefficients. The selection of origin as a preferred point relates directly to the condition eliminating possible problems due to the loss of associativity and commutativity.
   The prime property is possible only if the set of these polynomials fails to have a field property (so that the inverse of any element would be well-defined) since for fields one does not have the notion of prime. The field property is lost if the allowed functions vanish at origin so that one cannot have a Taylor series at origin and the inverse diverges at origin.
   The vanishing at origin guarantees that the functional composite $f \circ g$ of f and g has the roots of g. Roots are inherited as algebraical complexity as a kind of evolution increases. In TGD inspired biology, the roots of polynomials are analogous to genes and the conservation of roots in the function composition would be analogous to the conservation of genes.

11.5.2 Attractor basin of fractal as set of roots

It turns out that if the polynomials of infinite degree exist, they must correspond to composites for an infinite number of polynomials. This follows from the fact that both finite and infinite Galois groups must be profinite so that an infinite Galois group is a Galois group of ...extensions of extensions.....of rationals.

The example in which the polynomials of form $P = P \circ R$ where Q is an infinite composite of a single polynomial Q vanishing at origin and having it as a critical point has as a basin of attraction a set having Julia set as boundary. All points in the basin of attraction for origin are roots at the limit.All points in the basin of attraction for origin are roots at the limit so that algebraic completion of rationals to complex numbers would result.

Profiniteness suggests an interpretation of this set in terms of p-adic topology or a product of a subset of p-adic number fields somehow determined by the number theoretic properties of Q. Algebraic completions of p-adic topology could also be in question. p-Adic number fields are indeed profinite and as additive groups can act as infinite Galois groups permuting the zeros. The action of p-adic translations could leave the basin of attraction invariant.

11.6 What is it to be a polynomial of infinite degree?

In the following the conditions on the notion of a polynomial of infinite degree are discussed.

11.6.1 Conditions for the prime analytic function

Could one make anything concrete about this idea? What kind of functions f could serve as analogs of polynomials of infinite degree with transcendental roots. The question whether any analytic function with rational coefficients vanishing at origin can have a possibly unique decomposition to prime analytic functions will not be discussed in the sequel?

1. Suppose that the analytic prime decomposes to a product over monomials $x-x_i$ with transcendental roots x_i such that the Taylor series has rational coefficients. This requires an infinite Taylor series.
2. One obtains an infinite number of conditions. Each power x^n in f has a rational coefficient f_n equal to the sum over all possible products $\prod_{k=1}^n x_{i_k}$ of n transcendental roots x_{i_k}. This gives an infinite number of conditions and each condition involves an infinite number of roots. If the number N of transcendental roots is finite as it is for polynomials, each term involves a finite number of products and the conditions imply that the roots are algebraic. The number of transcendental roots must therefore be infinite. At least formally, these conditions make sense.
3. The sums of products are generalized symmetric functions of transcendental roots and should have rational values equal to x_n. This generalizes the corresponding condition for ordinary polynomials. Symmetric functions for S_n have S_n as a group of symmetries. For a Galois extension of a polynomial of order n, the Galois group is a subgroup of S_n. This suggests that the Galois group is a subgroup of S_∞. S_∞ as the simple A_∞ as a subgroup of even permutations. The simple groups are analogs of primes for finite groups and one can hope that this is true for infinite and discrete groups [L101].

There are infinitely many ways to represent an algebraic extension in terms of a polynomial and the same is true for transcendental extensions with the rationality condition.

1. Consider a general decomposition of the polynomial of an infinite order to a product of monomials with roots spanning the transcendental extension. Could a suitable representation of extension as an infinite polynomial allow rational coefficients f_n for the function $\sum f_n x^n$ defined by the infinite product?
2. f_n is the sum over all possible products of roots obtained by dropping n different roots from the product of all roots which should be finite and equal to one for the generalization of monic polynomials. Therefore there is an infinite sum of terms, which are inverses of finite products and therefore transcendental but one can hope that the infinite number of the summands allows the rationality condition to be satisfied.

11.6.2 Profinite groups and Galois extensions as inverse limits

Infinite groups indeed appear as Galois groups of infinite extensions. Absolute Galois groups, say Galois groups of algebraic numbers, provide the basic example.

1. There exists a natural topology, known as Krull topology, which turns Galois group to a profinite group (totally disconnected, Hausdorff topological group) (`https://en.wikipedia.org/wiki/Profinite_group`), which is also Stone space (`https://en.wikipedia.org/wiki/Stone_space`).
2. Profinite groups are not countably infinite but are effectively finite just as hyper-finite factors of type II_1 are finite-dimensional: they appear naturally in the TGD framework [K99, K33]. Profinite groups have Haar measure giving them a finite volume. Profinite groups behave in many respects like finite groups (compact groups also behave in this manner as far representations are considered). Profiniteness is possessed by products, closed subgroups, and the coset groups associated with the closed normal subgroups.
3. Every profinite infinite group is a Galois group for an infinite extension for some field K but one cannot control which field K is realized for a given profinite group [A194]. Additive p-adics groups and their products appear as Galois groups of an infinite extension for some field K. The Galois theory of infinite field extensions involves profinite groups obtained as Galois groups for the inverse limits of finite field extensions $..F_n \rightarrow F_{n+1} \rightarrow$.
4. This kind of iterated extensions are of special interest in the TGD framework and an infinite extension would be obtained at the limit [L95]. The naive expectation is that the polynomial

of infinite degree is a limit of a composite $...P_n \circ P_{n-1}.. \circ P_1$ of rational polynomials. The number of infinite extensions obtained in this manner would be infinite.
An interesting question is under what conditions the limiting infinite polynomial exists as an analytic function and whether the Taylor coefficients are rational or in some extension of rationals. The naive intuition is that the inverse limit preserves rationality.

5. The identification as the iterate $...P_n \circ P_{n-1} \circ P_1$ is indeed suggestive. Infinite cyclic extension defined at the limit by the polynomial x^N, $N = \infty$, to be discussed below, has this kind of interpretation. The Galois group of this kind of extension is however not simple.
Remark: The polynomials in question are not irreducible: the composite of N polynomials has x^N as a factor.
6. Is the infinite-D extension obtained as an inverse limit transcendental or algebraic? In the TGD framework the condition that the polynomial $P_1 \circ P_2$ has the roots of P_1 as roots implies the loss of the field property of analytic functions making the notion of analytic prime possible. The roots of the infinite polynomial contain all roots of finite polynomials appearing in the sequence. This would suggest that the extension is not transcendental. Giving up the property $P_i(0) = 0$ also leads to a loss of root inheritance.

For finite-dimensional Galois extensions, there exists an infinite number of polynomials generating the extension and one can consider families of extensions parametrized by a set of rational parameters. The Galois group does not change under small variations of parameters [L101]. If the inverse limit based on an infinite composite of polynomials makes sense, the situation could be the same for possibly existing rational polynomials of infinite order? The study of infinite Galois groups could provide insights on the problem.

11.6.3 Could infinite extensions of rationals with a simple Galois group exist?

Simple Galois groups have no normal subgroups and are of special interest as the building bricks of extensions by functional composition of polynomials. The infinite Galois groups obtained as inverse limit have however an infinite hierarchy of normal subgroups and simple argument suggests that the extensions are algebraic. Could infinite-D transcendental extensions defined by an analytic function with rational coefficients and with a simple infinite Galois group, exist?

If inverse limit is essential for profiniteness for infinite groups, then simple infinite groups are excluded as Galois groups. Indeed, the topology of an infinite simple group G cannot be profinite. The Krull topology has as a basis for open sets all cosets of normal subgroups H of finite index (the number of cosets gH is finite). Simple group has no normal subgroups except a trivial group consisting of a unit element and the group itself. The only open sets would be the empty set and G itself.

In fact, there is also a theorem stating that every Galois group is profinite (see `https://cutt.ly/wQ2W1Of`). All finite groups are profinite in discrete topology. This theorem however excludes infinite simple Galois groups. If one allows only polynomials with P(0)=0, the conservation of algebraic roots suggests that infinite polynomials with transcendental roots are not possible.

The condition for the failure of the field property however leaves the iterates of polynomials for which only the highest polynomial in the infinite sequence of functional compositions vanishes at origin. These infinite polynomials could have transcendental roots.

11.6.4 Two examples

In the following two examples are considered to test whether the notion of a polynomial of infinite order might work.

Infinite cyclic extensions

The natural question is whether the transcendental roots be regarded as limits of roots for a polynomial with rational coefficients at the limit when the degree N approaches infinity. The above arguments suggest that the limits involve an infinite function composition.

Consider as an example cyclic extension defined by a polynomial X^N, which can be regarded as a composite of polynomials x^{p_i} for $\prod p_i = N$. This is perhaps the simplest possible extension than one can imagine.

1. The roots are now powers of roots of unity. The notion of the root of unity as $e^{i2\pi/N}$ does not make sense at the limit $N \rightarrow \infty$. One can however consider the roots $e^{i2\pi M/N}$ and its powers such that the limit $M/N \rightarrow \alpha$ is irrational. The powers of $exp(in\alpha)$ give a dense subset of the circle S^1 consisting of irrational points. Note that one obtains an infinite number of extensions labelled by irrational values of α.
2. The polynomial should correspond to the limit $P_N(x) = x^N - 1$, $N \rightarrow \infty$. For each finite value of N, one has $P_N(x) = \prod_{n=1}^{N}(x - U^n) - 1$, $U = e^{i2\pi/N}$. The reduction to $P = x^N - 1$ follows from the vanishing of all terms involving lower powers of x than x^N.
3. If these conditions hold true at the limit $N \rightarrow \infty$, one obtains the same result. The coefficient of x^N equals to 1 trivially. The coefficient of x^{N-1} is the sum over all roots and should vanish. This is also assumed in Fourier analysis $\sum_n e^{i\alpha n} = 0$ for irrational α. For $\alpha = 0$ the sum equals to $N = \infty$ identified as Dirac delta function. The lower terms give conditions expected to reduce to this condition. This can be explicitly checked for f_1
4. The Galois group is in this case the cyclic group $U_{\infty,\alpha}$ defined by the powers of U_α.

Infinite iteration yields continuum or roots

The iterations of polynomials define an $N \rightarrow \infty$ limit, which can be handled mathematically whereas for an arbitrary sequence of polynomials in the functional composition it is difficult to say anything about the possible emergence of transcendental roots. Note however that the $lim_{N\rightarrow\infty}(1 + 1/N)^N = e$ shows that transcendentals can appear as limits of rationals. I have considered iterations of polynomials and approach to chaos from the point of view of $M^8 - H$ duality in [L86].

Consider polynomials $P_N = Q_N \circ R$, where R with $Q(0) = 0$ is fixed polynomial and $Q_N = Q^{\circ N}$ is the N:th integrate of some irreducible polynomial Q with $Q(0) = 0$ and $dQ/dz(0) = 0$. Origin is a fixed critical point of Q and the attractor towards which the points in the attractor basin of origin end up in the iteration and become roots of P_∞ and are roots at this limit. For the real points in the intersection of the positive real axis and attractor basin are roots at this limit so that one has a continuum of roots. The set of roots consists of a continuous segment $[0, T)$ and a discrete set coming from the Julia set defining the boundary of the attractor basin.

Profiniteness suggests an interpretation of this set in terms of p-adic topology or a product of a subset of p-adic topologies somehow determined by the number theoretic properties of Q. p-Adic number fields are indeed profinite and as additive groups can act as infinite Galois groups permuting the zeros. The action of p-adic translations could indeed leave the basin of attraction invariant.

In the TGD framework these roots correspond to values of M^4 time (or energy!) in M^8 mapped to the actual time values in H by $M^8 - H$ duality. I have referred to them as "very special moments in the life of self" with a motivation coming from TGD inspired theory of consciousness [L82, L83]. One might perhaps say that at this limit subjective time consisting of these moments becomes continuous in the interval $[0, T]$.

Chapter 12

Some questions about coupling constant evolution

12.1 Introduction

In this article questions related to the notions of the p-adic CCE and hierarchy of Planck constants will be considered.

12.1.1 How p-adic primes are determined?

p-Adic length scale (PLS) hypothesis plays a central role in TGD in all length scales. For instance, it makes it possible to use simple scaling arguments to deduce quantitative predictions for the masses of new particles predicted by TGD.

PLS hypothesis states that the size scales of space-time surfaces correspond to PLSs $L_p = \sqrt{p}R(CP_2)$. The additional hypothesis is $p \simeq m^k$, $m = 2, 3, ...$ a small prime. The success of p-adic mass calculations [K50] supports $p \simeq 2^k$ hypothesis [K63] seriously. There also exists empirical evidence for a possible generalization to small primes, in particular $m = 3$, in biology [I1, I2].

The physical and mathematical identification of the origin of the p-adic prime p defining the PLS is however a problem.

The proposal has been that the p-adic prime p defining the PLS corresponds to a ramified prime of the extension of rationals (EQ) associated with the polynomial defining space-time region in M^8 picture. Ramified primes appear as factors of the discriminant of the polynomial defining EQ. I have not been able to find any really convincing explanation for why p should correspond to a ramified prime so that p-dic prime might emerge in some other way.

In p-adic thermodynamics Boltzmann weights $exp(-E/T)$ must be replaced with $p^{L_0/T}$, where L_0 is scaling generator. The exponent $\Omega = exp(K)$ of the Kähler function K of WCW defines vacuum functional. Could Ω be number-theoretically universal and thus exist as a p-adic number for some prime p determining naturally the PLS. This is the case if one has $\Omega = p^n$, n integer.

As such, this idea does not make sense but one consider a subsystem defined by sub-CD defining self in zero energy ontology (ZEO) based theory of consciousness [L72, L108] [K103]?

p-Adic prime defines naturally the scale of CD for trivial extension of rationals and this scale is scaled up by factor n for an extension of dimension n. This also conforms with the assumption that p-adic CCE and "dark" CCE are independent.

12.1.2 Trying to understand p-adic CCE

TGD leads to a number theoretic vision about CCE [L54]. Number theoretic universality plays a key role in this picture. CCE certainly involves the hierarchy of extensions of rationals (EQs) possibly involving non-rational extensions by roots of e, which induce finite extensions of p-adics. It would be nice if the EQ alone would determine the values of the coupling constants.

1. The starting point is that the continuous CCE with respect to length scale reduces to a discrete PLS evolution with respect to L_p, $p \simeq 2^k$. There is also dark evolution with respect to $n = h_{eff}/h_0$. These evolutions are separate since the scaling of the roots of the polynomial do not affect the purely algebraic properties of the extension. The natural assumption is that these evolutions factorize so that one has $\alpha_K = g_K^2(p)/2h_{eff}$.
2. p-Adic CCE would be roughly logarithmic with respect to L_p. The observation that α is near $\alpha = 1/137$ for p-adic length scale $L(137)$ suggests that for α_K defining the fundamental coupling strength one has

$$\alpha_K = \frac{g_K^2(max)}{2kh_{eff}} \ .$$

Since $1/\alpha_K(137) = 137$ is prime for ordinary matter with $h_{eff} = h$, one must have

$$\frac{g_K^2(max)}{2h} = 1 \ .$$

giving $h = g_K^2(max)/2$. The value h need not however be the minimal value h_0 of h_{eff} since one can have $h = n_0 h_0$ $\alpha_K(max) = 2n_0$ so that one can write

$$\alpha_K = \frac{1}{knn_0} \ . \quad (12.1.1)$$

$n_0 > 1$ would mean that the ordinary matter would be actually dark in the sense that the order of the extension of rationals associated with the ground state would be n_0.
For h_0 that value of α_K could be so large that the perturbation series does not converge except in very long length scales for which k is expected to be large. Exotic phases with $h_{eff} < h$ could become possible in these scales.

12.1.3 How p-adic prime is defined at the level of WCW geometry?

The p-adic prime p should emerge from the dynamics defined by Kähler function.

1. The Kähler function K of the "world of classical worlds" (WCW), or more generally the generalization of $exp(K)$ to a vacuum functional possibly involving also a genuine state dependent part is a central quantity concerning scattering amplitudes. Suppose that one can consider a subsystem defined by CD and the contribution ΔK from CD to K.
Number theoretical universality suggests that the exponential $exp(\Delta K)$ or its appropriate generalization exists in all p-adic number fields or at least in an extension of the p-adic number field corresponding to the p-adic prime p. Could this condition fix p dynamically?
2. Suppose that for some prime p one can write

$$e^{\Delta K} = p^{\frac{\Delta K}{log(p)}}$$

such that $\Delta K/log(p)$ is integer. The exponential would be a power of p just as the p-adic analog of Boltzmann weight in p-adic thermodynamics [K50]. This would select a unique p-adic prime p defining the PLS and this prime need not be a ramified prime. In p-adic thermodynamics [K50] $X = \Delta K/log(p)$ has interpretation as an eigenvalue of the scaling generator L_0 of conformal algebra and one can even consider the possibility that there is a connection.

12.1.4 What about the evolution of the gravitational fine structure constant?

Nottale hypothesis [?] predicts gravitational Planck constant $\hbar_{gr} = GMm/\beta_0$ ($\beta_0 = v_0/c$ is velocity parameter), which has gigantic values so that the above picture fails. Gravitational fine structure constant is given by $\alpha_{gr} = \beta_0/4\pi$.

Kepler's law $\beta^2 = GM/r = r_S/2r$ suggests length scale evolution $\beta^2 = xr_S/2L_N = \beta_{0,max}^2/N^2$, where x is proportionality constant, which can be fixed. Phase transitions changing β_0 are possible at $L_N/a_{gr} = N^2$ and these scales correspond to radii for the gravitational analogs of the Bohr orbits of gravitational Bohr atom. PLS hierarchy is replaced by that for the radii of Bohr orbits.

What could be the interpretation of N? The safest assumption is that the CCE of β_0 is analogous to that of the other coupling constants and induced from that of α_K.

12.1.5 What is the minimal value of h_{eff}?

The formula $h_{eff} = nh_0$ involves the minimal value h_0 of h_{eff}. The simplest explanation for the findings of Randell Mills [D2] is that one has $h = 6h_0$. h_0 could be also smaller [L22].

What is the value of h_0? A possible answer to this question came from the observation made already during the first 10 years of TGD. The observation was that the imbeddings of spherically symmetric stationary metrics (see the Appendix) suggest that CP_2 radius R is of order Planck length l_P rather than by factor about $10^{7.5}$ longer. Could one have $h = n_0h_0$, $n_0 \sim 10^{7.5}$ so that the ordinary matter would be actually dark?

CP_2 radius would be Planck length apart from numerical constant not far from unity. The p-adic mass calculations would give correct results for $h_{eff} = h_0$. R could be interpreted as $R^2 = n_0 l_P^2$. The perturbative expansion for $h_{eff} < h$ would not converge except in long p-adic length scales, where the p-adic evolution reduces the value of α_K.

Gauge coupling strengths are predicted to be practically zero at gravitational flux tubes with very large h_{eff} so that only gravitational interaction is effectively present. This conforms with the view about dark matter.

12.2 Number theoretical universality of vacuum functional and p-adic CCE

The Kähler geometry of WCW is defined by a Kähler function $K(X^4(X^3))$ identified as the action of preferred extremal consisting of volume term and Kähler action. The vacuum functional is of form $\Omega = exp(K + iS)$. Here K is the real Kähler function and S is the counterpart of real action in the path integral of QFTs.

$exp(iS)$ could be interpreted as a dynamical part of vacuum functional, which depends on state rather than being "God-given". The reason why this would be the case would be that it is possible. For $exp(K)$ there is no choice since the Kähler geometry of WCW is expected to be unique merely from its existence as already in the case of loop spaces [A121].

Number theoretical universality is a challenge for this general picture.

1. In the p-adic context the notion of WCW geometry is highly questionable. The integration associated with definition of volume term and Kähler action is the tough problem.
   This has inspired the proposal that the exponent of the action completely disappears from the scattering amplitudes. This indeed happens in quantum field theory based on path integral around stationary point.
2. The classical nondeterminism suggests a weaker formulation. The sum over the contributions of stationary points would be replaced by integral over preferred extremals consisting of 3-surfaces at PB plus sum over the paths of the tree resulting from classical non-determinism.
   The sum over the paths of the tree-like structure remains in the superposition of amplitudes for sub-CD and it might be possible to define the deviation $\Delta K + i\Delta S$ of the action for each of them and separate $exp(\Delta K + i\Delta S)$ from the entire exponent of action, which would therefore disappear from the expression of the scattering amplitudes for given X^3 and given CD. Otherwise, the knowledge of the entire WCW Kähler function would be needed.
   A possible interpretation is in terms of a decomposition to an unentangled tensor product corresponding to sub-CD and its environment so that one can separate the physics inside sub-CD from that of environment and code it by $exp(\Delta K + i\Delta S)$.
3. The simplest option, very probably too simple, would be that one has $\Delta K = 0$. Kähler function would be same for all paths of the tree and one would obtain a discretized analog of path integral. This would require that all the branches of the tree have same value of action. This does not however require the same value of volume and Kähler action separately.

It will be foud that $\Delta K \neq 0$ assuming that $exp(\Delta K)$ reduces to an integer power of p for some prime identifiable as p-adic prime defining the PLS, is more interesting option since it would reduce p-adic thermodynamics to the level of WCW and also allow to the understand of PLS evolution of couping constants.
The number theoretical existence of the phases $exp(i\Delta S)$ would require that they belong to the EQ defining the space-time region inside CD.

4. This picture suggests that the number theoretically universal part is associated with the sub-CDs and with the discrete physics of the tree-like structure whereas the Kähler function for 3-surfaces would be defined only in real framework. This would neatly separate the physics of sensory and Boolean cognition as something number theoretically universal from the physics proper, so to say.
Since conscious experience gives all information about physics, one can ask whether the adelic physics associated with various sub-selves could together be enough to represent all that is representable from the physics proper. This could result as somekind of limiting case (EQ approaches algebraic numbers).

If this view is correct, then one expects that various notions shared by QFTs and TGD, in particular CCE, could have number theoretic descriptions as indeed suggested [L54]. In the sequel I will discuss some speculations in this framework.

12.2.1 The recent view about zero energy ontology

Zero energy ontology (ZEO) [K103] [L72, L108] plays a key role in the formulation of TGD based quantum measurement theory.

1. The concept of causal diamond (CD) is central. CD serves as a correlate for the perceptive field of conscious entity: this in the case that one has sub-CD so that the space-time surfaces inside CD continue outside it.
The scale size scale of the CD identifiable as the temporal distance T between its tips could be proportional the p-adic prime p at the lowest level of dark matter hierarchy and to np at dark sectors. p-Adic length scales L_p characterizing the sizes scale of 3-surfaces are proportional to $\sqrt{p}$ and the proposal is that the relation between T and L_p is same as the relationship between diffusion time T and the root mean square distance R travelled by diffusion.
2. The twistor lift of TGD predicts that the action principle defining space-time surfaces is the sum of a volume term characterized by length scale dependent cosmological constant Λ and Kähler action and induced from 6-D Kähler action whose existence fixes the embedding space uniquely to $M^4 \times CP_2$. The reason is that the required Kähler structure exists only for the twistor spaces of M^4, E^4, and CP_2 [A150].
3. The recent progress in the understanding of zero energy ontology (ZEO) [L108] leads to rather detailed view about the dynamics of the space-time surfaces inside sub-CD.
Space-time surfaces are analogs of soap films spanned by a frame having the 3-surfaces at its ends located at the boundary of CD as fixed part of frame and the dynamically generated parts of frame in the interior of CD. Outside the frame preferred extremal is an analog of a complex surface and a simultaneous extremal of both volume term and Kähler action since the field equations reduce to conditions expressing the analogy of holomorphy [K79, L56]. The field equations reduce to contractions of tensors of type (1,1) with tensors of type (2,0)+(0,2) and are therefore trivially true.
The minimal surface property fails at the frame, and only the full field equations are true. The divergences of isometry currents associated with volume term have delta function singularities which however cancel each other to guarantee field equations and conservation laws. This is expected to give rise to a failure of determinism, which is however finite in the sense that the space-time surfaces associated with given 3-surface X^3 at the passive boundary of CD (PB) form a finite set which is a tree-like structure (for a full determinism only single space-time surface as analog of Bohr orbit would be realized). Therefore the non-determinism of classical dynamics for a fixed X^3 is extremely simple and quantum dynamics and classical dynamics are very closely related since quantum states are superpositions of the paths of the tree.
4. One also ends up to quite precise identification sub-CD or space-time surface inside sub-CD as a correlate of perceptive field of a conscious entity. The essential element of the picture that

for sub-CD the 3-surface X^3 at PB is fixed but due to the non-determinism the end at active boundary (AB) is not completely fixed and there is finite non-determinism in the state space defined by superpositions of the paths of the tree.
For the highest level in the hierarchy of CDs associated with self, the space-time surfaces inside CD do not continue outside it and this CD God-like entity, whose dynamics is not restricted by the boundary conditions.

This view provides additional perspectives on discreteness of adelic physics unifying the physics of sensory experience and cognition [L43, L42].

1. Discreteness is essential in the number theoretic universality since in these case real structures and their p-adic counterparts correspond naturally to each other. This has led to the notion of cognitive representation as a set of points of space-time surface with preferred embedding space coordinates having values in the EQ defined by the polynomial defining the space-time surface in complexified M^8 and mapped to H by M^8-H duality [L82, L83]. The finite-dimensionality of the state space associated with the tree structure conforms with this vision.
2. Discreteness is natural for the dynamics of concious experience and cognition. Mental images as sub-selves correspond to the sub-CDs inside CD. Sub-CDs are naturally located at the loci of non-determinism defined by the fixed part of the frame dynamically and generated frames in the interior and at AB.
Attention would fix the 3-surfaces at the PB of a sub-CD as a perceptive sub-field and all CDs in the hierarchy would be fixed in this manner. The loci of non-determinism would serve as targets of attention. Sensory perception, memory recall, and other functions would reduce to directed attention inside CD.
Fermionic degrees of freedom at boundaries of CD are are additional discrete degrees of freedom and responsible for Boolean cognition whereas the discrete dynamics of frame would correspond to sensory experience and sensory aspects of cognition.
3. This picture inspires the question whether the number theoretically universal parts of adelic physics might relate to the physics due to the non-determinism in the interior of sub-CD. This physics would be basically the physics that can be observed. This would mean enormous simplification.
This idea is not new. The amazing success of p-adic thermodynamics based mass calculations [K50] could be understood if p-adic physics is seen as a physics of cognitive representation of real number based physics.

In the sequel some speculations are discussed by taking the above picture as a basis.

12.2.2 Number theoretical constraints on $exp(\Delta K)$

Number theoretical universality suggests that the exponents $exp(\Delta K + i\Delta S)$ for X^4 inside sub-CD is well-defined at least for some p-adic number fields or their extensions.

It has been already found that number theoretical universality requires that the phases $exp(i\Delta S)$ belong to the EQ associated with the space-time surfaces considered.

The condition that the phase is a root of unity is more general than the condition of semi-classical approximation of wave mechanics stating that the action is quantized as a multiple of Planck constant h. The analog of this condition would imply $exp(i\Delta S) = 1$. This quantization condition would make S obsolete.

What about the number theoretical universality of $exp(\Delta K)$? One can consider three options.

1. p-Adic exponent function $exp(x)$ exists if the p-adic norm of x is smaller than 1. The problem is that the p-adic exponent function and its real counterpart behave very differently [K60]. In particular, $exp(x)$ is not periodic. Integer powers of e^p are however ordinary p-adic number by its Taylor series and roots of e define finite-D extensions of p-adic number fields. Therefore $exp(\Delta K)$ could make sense as an integer power for a root of e.
If ΔK is integer, $exp(\Delta K)$ exists p-adically for primes p dividing ΔK.
2. Also $p^{\Delta K/log(p)}$ could exist p-adically if $\Delta K/log(p)$ is integer. This implies strong conditions. ΔK must be of form $\Delta K = log(p)m$, m integer. If ΔK corresponds to Kähler function of WCW, p is fixed and would define the sought-for preferred p-adic prime p defining the PLS.

3. Since the powers p^n converge to zero for $n \to \infty$, one can formally replace $exp(\Delta K)$ with $exp(\Delta K) = p^{\Delta K)/log(p)}$ and require that the exponent is an integer. The replacement of the ordinary Boltzman weights with powers of p is indeed carried out in p-adic thermodynamics [K50]. This suggests that the Boltzman factors of p-adic thermodynamics reduce to exponents $p^{\Delta K}$ at the level of WCW.

12.3 Hierarchy of Planck constants, Nottale's hypothesis, and TGD

12.3.1 Nottale's hypothesis

Nottale's hypothesis [?] and its generalization to TGD [K80, K10] has non-relativistic and relativistic forms.

1. The non-relativistic formula for $\hbar_{gr}$ as given by the Nottale's formula

$$\begin{aligned} \hbar_{gr} &= \frac{GMm}{\beta_0} \ , \\ \alpha_{gr} &= \frac{GMm}{4\pi\hbar_{gr}} = \frac{\beta_0}{4\pi} \ . \end{aligned} \tag{12.3.1}$$

   The formula makes sense only $\hbar_{gr}/\hbar > 1$.

2. The relativistically invariant formula for $\hbar_{gr}$ reads for four momenta $P = (M, 0)$ $p = (E, p_3)$ as:

$$\hbar_{gr} = \frac{GP \cdot p}{\beta_0} = \frac{GME}{\beta_0} = \frac{r_s E}{2\beta_0} \ , \tag{12.3.2}$$

   where r_s is Schwartschild radius. Adelic physics implies that momentum components belong to an extension of rationals defining the adele so that the spectrum of E and of $\hbar_{gr}$ are discretized.

Nottale's hypothesis and biology

Nottale's hypothesis involves a lot of uncertainties also at the conceptual level. Hence it is important to see whether basic facts from TGD inspired biology support the Nottale's hypothesis.

1. The cyclotron frequencies in an "endogenous" magnetic field $B_{end} = 2B_E/5$, where $B_E = .5$ Gauss is the nominal value of the Earth's magnetic emerge in the explanation of the findings of Blackman and other [J2] showing that ELF photons have effects on vertebrate brain. B_{end} is assigned with the monopole flux tubes of B_E. Also lower and higher values of B_{end} can be considered and the models of hearing [K74] and genetic code [L16] suggests that the values of B_{end} correspond to the notes of 12-note scale. This suggests that also the Z^0 magnetic field might be involved.
2. Biophoton energies are in visible and UV range and in the TGD based model they are assumed to result in the transformations of dark photons with much smaller frequency but same energy to ordinary photons. For instance, photons with 10 Hz frequency can transform to biophotons. By $E = h_{eff}f$, requires $h_{eff} = h_{gr}$. The implication is that cyclotron energies do not depend on particle mass. Furthermore, Schwartschild radius $r_S = .9$ cm of Earth defines universal gravitational Compton length for $\beta_0 = 1/2$.

Assume that $\hbar_{gr}$ corresponds to Earth mass and $\beta_0 = 1/2$ and consider cyclotron states in $B_{end} = .2$ Gauss.

1. The value of $r = \hbar_{gr}/\hbar$ for proton is given as the ratio r_s/L_p, where L_p is the Compton radius of proton. This gives $r = .833 \times 10^{13}$. For ions with mass number A the value of r is scaled to Ar.
2. What is the cyclotron energy associated with the 10 Hz frequency in this case? The energy of a photon with frequency f is for $\hbar_{gr}(m_p)$ given by $E_c/eV = r \times 1.24 \times (f/(3 \times 10^{14}Hz)$. Proton's cyclotron frequency is $f_c = 300$ Hz in B_{end} and corresponds to 10 eV, which is in the UV region and rather large.
3. All cyclotron frequencies of charged particles correspond to $E_c = 10$ eV cyclotron energy, which seems rather large. If $\hbar_{gr}$ is reduced by factor 1/4 as required to explain the findings

of Mills at least partially, the cyclotron energy becomes 2.5 eV, which is in the visible range. Scaling by factor 1/2 gives cyclotron energy 5 eV in UV.

4. Smaller values of E_c would require smaller fields. The Z_0 charge of proton is roughly a fraction 1/50 of its em charge and since Kähler field contributes also to Z^0 field one would obtain energy about .2 eV in the IR region.

10 Hz alpha frequency which is of special interest concerning understanding of conscious experience and it is interesting to look for concrete numbers.

1. $f = 10$ Hz is alpha frequency and the cyclotron frequency $f_c = 10$ eV Fe^{2+} ion with mass number $A = 56$. Fe^{2+} ions play a central role in biology.
2. For $f = 10$ Hz the energy $\hbar_{gr}(m_p)$ (proton) is .333 eV to be compared with the metabolic energy currency $\sim$ .5 eV and is below the visible range.
3. In the TGD inspired biology, 3 proton units represent dark genetic codons and for $\hbar_{gr}(3m_p)$ the energy corresponds to $E \times 1$ eV, which is still slightly below the visible range [L143, L92, L103]. In the dark variant of double DNA strand parallel to the ordinary double strand, the 2 dark codons form a pair by the dark variant of the base pairing so that one has effective $A06$ and $E = 2$ eV, which corresponds to red light.
4. The energy $E = 2$ eV of the codon pair for $f = 10$ Hz corresponds formally to $A = 6$ and would characterize ^6Li. Litium's cyclotron frequency is around $f_c = 50$ Hz is known to have biological significance. Li is used in the treatment of depression [K105]. One might imagine that the coupling of Lithium to dark codon pairs might be involved.
5. For higher mass numbers, the energies for 10 Hz and $\hbar_{gr}(Am_p)$ belong to the UV region. For oxygen one with $A = 16$ has $E = 5.3$ eV, which could correspond to some important molecular transition energy. Molecular bond dissociation energies (`https://cutt.ly/3QoZxY9`) vary in the range .03 -10 eV. O-H, O=O ad O=CO bond energies are somewhat above 5 eV. The idea indeed is that the transformation of dark photons to ordinary bio-photons allows a control of molecular biochemistry.
6. DNA codons have charge proportional to mass and in a good approximation one has $f_c(DNA) =$ 1 Hz independently of the length of the DNA strand. For $\hbar_{gr}(Fe^{++})$ $f_c(DNA)$ would correspond to $E = 1.86$ eV in the range of visible energies.

12.3.2 Trying to understand $\hbar_{eff}$ and $\hbar_{gr}$

Although $\hbar_{eff}$and $\hbar_{gr}$ have become an essential part of quantum TGD, there are still many poorly understood aspects related to them.

Should one introduce a hierarchy of poly-local Planck constants?

The ordinary Planck constant is a universal constant and single-particle entity and serves as a quantization unit for local charges. $\hbar_{gr}$ depends on the masses of the members of the interacting systems and a bi-local character. This suggests that one should not mix these notions.

Both $\hbar_{gr}$ and its possible generalization to gauge interactions such as $\hbar_{em}$, would depend on the charges of the interacting particles. If they serve as charge units, the charges must be bilocal.

Should one introduce a hierarchy of poly-local Planck constants? Later a possible interpretation in terms of Yangian symmetries [A97] [B33, B18], which involve poly-local charges, will be considered. Each multi-local contribution to charge would involve its own Planck constant determined number theoretically.

Standard quantization rules for observables use $\hbar$ as a basic unit. Should one modify these rules by replacing $\hbar$ with (say) $\hbar_{em} = q_1q_2e^2/\alpha$ for $q_1q_2\alpha \geq 1$? Could these rules hold true at magnetic flux tubes characterized by $\hbar_{em}$? Could the charge units for the matter in the non-perturbative phase be q_1q_2-multiples of the ordinary basic units? Could one find empirical evidence for the scaling up of the quantization unit in non-perturbative phases?

In order to avoid total confusion, one must distinguish clearly between the single particle Planck constant and its 2-particle and n-particle variants as Yangian picture suggests. One must also distinguish between p-adic CCE as a discrete counterpart of ordinary CCE and dark coupling constant evolution.

The counterpart of $\hbar_{gr}$ for gauge interactions

The gauge couplings g_i for various interactions disappear completely from the basic formulation of TGD since they are automatically absorbed into the definition of the induced gauge potentials. Hence $\beta_0/4\pi \equiv \alpha_K$ appears as a coupling parameter in the perturbative expansion based on the exponent of Kähler function. $\hbar$ or $\hbar_{eff}$ appear as charge unit only in the definition of conserved charges as Noether charges but not in the action exponential.

The generalization of the Nottale formula to other interactions is not quite obvious. Two-particle Planck constant $h_{eff}(2)$ is in question and GMm would be replaced with the product $q_1q_2g_i^2$. Since α_K determines all other coupling strengths so that it is enough to consider it.

The parameter $\beta_0/4\pi$ is analogous to fine structure constant since gravitational perturbative expansion is in powers of it [K80] [L107].

β_0 is the gravitatoinal counterpart of the dimensionless coupling strength α_K defined in the QFT framework as a derived quantity $\alpha_K = g_K^2/4\pi$ but identified in the TGD context as the fundamental parameter appearing in Kähler action.

In TGD e does not appear as gauge coupling at the fundamental level (as opposed to QFT limit) but one can *define* e^2 as $e^2 = 4\pi\alpha\hbar$. α would obey p-adic CCE and $\hbar$ would be universal constant at single particle level. For dark phases, for which one has $h_{eff} > h$, $\alpha(1) \propto 2/n$, n dimension of the extension would hold true.

Consider the analog of the Nottale formula for em interactions. The coupling strength would be $q_1q_2e^2$ and for $q_1q_2e^2\alpha > 1$, one would have

$$\hbar_{em}(2) = \frac{q_1q_2e^2}{\beta_0} \ . \tag{12.3.3}$$

This would give $\alpha_{em}(2) = \beta_0$. For $\beta_0 = \alpha$, one would obtain a coupling parameter α instead of $q_1q_2\alpha$ and the interpretation would be in terms of a transition to non-perturbative phase.

Does this phase transition correspond to a transition to dark phase? Could one interpret the phase transition by saying the dimension of extension is scaled by $n = \hbar_{em}(2)/\hbar$ identified as scaling of the dimension of extension of rationals?

Number theoretic vision predicts that in the dark evolution h_{eff} scales as n, the dimension of extension of rationals for all values of particle number in the definition of $h_{eff}(h)$ so that the single particle coupling constant strength would behave like $1/n$.

Charge fractionalization and the value of $\hbar_{eff}$

$\hbar_{eff} < \hbar$ implies charge fractionalization at the level of embedding space. This inspires the question whether an analog of fractional quantum Hall effect could be in question. This is not the case.

1. The TGD based model for anyons [K69] relies on the observation that the unit for the fractional quantization of transverse conductance in fractional quantum Hall effect (FQHE) as

$$\begin{aligned} \sigma &= \nu \times \frac{e^2}{h} \ , \\ \nu &= \frac{n}{m} \ . \end{aligned} \tag{12.3.4}$$

   The proposal is that FQHE could be understood as integer quantum Hall effect corresponding to $n \to kn$ for $h_{eff} = km\hbar$. $k = 1$ is the simplest possibility. Interestingly, the observed values of m are primes [D1]: they would correspond to simple Galois groups Z_p in the TGD framework.

2. The fractionalization of charges could be understood at space-time level by noticing that n-sheetedness can be realized as analog of analytic function $z^{1/n}$. n full 2π turns are needed to return to the original point at space-time level so that it is possible to have fractional spin as multiples of $\hbar/n$. The many-particle states however have half-integer spin always since they correspond to representations of the Lorenz group as a symmetry group of $M^4 \times CP_2$. The action of rotations by multiples of 2π would correspond to the action of the Galois group.

   These two apparently conflicting mechanisms of charge fractionization correspond to two views about symmetries: either their act at the level of the embedding space or of space-time.

3. For $GMm/v_0 < \hbar$ one would have formally $\hbar_{eff} < \hbar$. Could this option make sense and give rise to a charge fractionalization? One can argue that for $\hbar_{gr} < \hbar_0$, $\hbar_0$ serves as the quantization unit and holds at the level of ordinary matter. This would give a condition $GMm \leq \beta_0$ to the product Mm of the masses involved.
A stronger condition would hold true at single particle level and state $M/M_{Pl} \geq \sqrt{\beta_0}$ (or $M/M(CP_2) \geq \sqrt{\beta_0}$) for both masses involved. Dark quantum gravity would hold true only above Planck masses. In applications to elementary particle level this would require quantum coherent states of particles with total mass not smaller than Planck mass. Interestingly, a water blob with the size of a large cell has this size for $\beta_0 = 1/2$ [L80].

What does the dependence of $\hbar_{gr}$ on particle masses mean?

$\hbar_{gr}$ depends on two masses. How could one interpret this geometrically?

1. The interpretation has been that a particle with energy E (and mass m) experiences the gravitational field of mass M via gravitational flux tubes characterized by $h_{gr} = GME/v_0$ so that every particle has its specific gravitational flux tubes.
2. Could the thickness of the gravitational flux tubes correspond to the ordinary Compton length λ_c or gravitational Compton length $\lambda_{gr} = GM/v_0$? λ_c decreases with mass and λ_{gr} looks a more reasonable option concerning gravitational interaction.
3. At least static gravitational fields are analogous to static electric fields and in many-sheeted space-time the voltages as analogs of gravitational potential difference are the same along different space-time sheets. The same should hold for gravitational potential.
Could one assume that gravitational potential has almost copies at all parallel sheets of the many-sheeted space-time (parallel with respect to M^4). Could these sheets correspond to different particle masses so that a particle with a given mass would have its own space-time sheet to represent its interactions with the central mass M.
4. These classical fields would be somehow represented by Kähler magnetic flux tubes carrying generalized Beltrami fields [B9, B44, B32, B36] having also an electric part.
Could these flux tubes somehow also represent the classical gravitational field? Could the electric part for the induced M^4 Kähler form predicted by the twistor lift of TGD [K79, L56] giving rise to CP breaking, give a representation for the gravitational potential? Could this concretely realize the analogy between gravitation and electromagnetism?
5. A possible realization of this picture would be a fractal structure consisting of flux tubes within flux tubes emanating from the central mass. The radii of the flux tubes would decrease with m as long as $GMm/\beta > \hbar$ holds true. For smaller masses, the flux tube radius would correspond to Compton length.
Fractal structures known as fractons (`https://cutt.ly/WRuXnrC`) are the recent hot topic of condensed matter physics (`https://cutt.ly/YQjqyjJ`. The explanation requires the replacement of the time evolution as a time translation with a scaling and condensed matter lattice would be replaced with fractal. These phases have exotic properties: in particular, thermal equilibrium need not be possible. There are also long range correlations due to fractality, which makes these phases ideal for quantum computation.
In the TGD framework, the time evolutions between SSFRs are indeed generated by the scaling operator L_0 of super-conformal algebra and many-sheeted space-time is both p-adic and dark fractal. The hierarchy of Planck constants makes possible quantum coherence in all scales.

Yangian symmetry and poly-local Planck constants

The product structure of $\hbar_{gr}$ and $\hbar_{em}$ has remained a mystery since it suggests that it characterizes the interaction of 2 space-time sheets whereas the ordinary Planck constant serves as a quantization unit for single particle states. Instead of a Galois group for a single space-time sheet, one would have a product of Galois groups for the two space-time sheets determined as roots for the polynomials in the product. Therefore one should write $\hbar_{gr} = \hbar_{eff,2}$ to distinguish it from a single particle Planck constant $\hbar_{eff}(1) \equiv \hbar_{eff}$.

1. In the TGD framework, wormhole contacts connecting two space-time sheets with Minkowskian signature are indeed building bricks of elementary particles and fundamental fermions appear-

ing as building bricks of elementary particles would be associated with the throats of the wormhole contact.

Could the two Minkowskian sheets be microscopically k-sheeted entities with sheets parallel to M^4 and perhaps determined as roots of a polynomial of degree k and having Galois group with order m? The maximal Galois group would be S_k with $m = k!$.

The scaling of $\hbar_0 \rightarrow \hbar(2)$ would mean that the pairs of these space-time surface sheets decompose to $\hbar(2)/\hbar_0$ pairs as orbit of $Gal \times Gal$ contributing to various quantum numbers a contribution proportional $\hbar_{gr}(2)/\hbar_0 = n_1 n_2 = k_1 k_2 m^2$.

The quantization unit would be $\hbar_0(2)$ for 2-particle quantities such as relative angular momentum. Spin is however thought to be single particle observable. The ordinary phase has a single-particle Planck constant as $\hbar(1)/\hbar_0 = m$.

2. There is no obvious reason for excluding the values of single particle $h_{eff}(1)/h_0$, which are considerably smaller than m or even equal to the minimal value $h_{eff}(1) = h_0$: they would correspond to Galois groups with smaller orders than $m = k!$ of say S_k.

   These exotic particles would have charge and spin units considerably smaller than $\hbar = mh_0$. Why have they not been observed (the findings of Mills are a possible exception and anyonic charge fractionization seems to be a different phenomenon)? Are these space-time sheets somehow unstable? Does gravitation somehow select the Galois group of stable ground state space-time surface so that R as a fundamental length scale is replaced with l_P as effective fundamental length?

3. Yangian algebras [A97] [B33, B18] involve besides single particle observables also $n > 1$-particle observables. Conserved charges have poly-local components which depend on n particles. Note that interaction energy represented as a potential energy is the simplest example about non-local 2-particle contribution to conserved energy.

   Yangian algebras are proposed to be central for TGD [L32] and would reflect the replacement of the space-time locality with locality at the level of "world of classical worlds" (WCW) due to the replacement of a point like with a 3-surface, which can also consist of disjoint parts. Yangian picture suggests that single-particle $\hbar$ has n-particle generalization. The possible number theoretical rule could be

$$\frac{\hbar_{gr,n}}{\hbar_0} = \prod_k n_k \ , \tag{12.3.5}$$

   where n_k correspond to the orders of Galois groups associated with the space-time sheets involved.

12.3.3 Do Yangians and Galois confinement provide $M^8 - H$ dual approaches to the construction of the many-particle states?

The construction of many-particle states as zero energy states defining scattering amplitudes and S-matrix is one of the basic challenges of TGD. TGD suggests two approaches implied by physics as geometry and physics as number theory views to TGD. Geometric vision suggests Yangians of the symmetry algebras of the "world of classical worlds" (WCW) at the level of $H = M^4 \times CP_2$. Number theoretic vision suggests Galois confinement at the level of complexified M^8. Could these approaches be $M^8 - H$ duals of each other?

Yangian approach

The states would be constructed from fermions and antifermions as modes of WCW spinor field. An idea taking the notion of symmetry to extreme is that this could be done purely algebraically using generators of symmetries.

Consider first the construction of TGD analogs of single particle states as representations of symmetries.

1. For a given vacuum state assignable to a partonic 2-surface and identifiable as a ground state of Kac-Moody type representation, the states would be generated by Kac-Moody algebra. Also super-Kac-Moody algebra could be used to construct states with nonvanishing fermion and antifermion numbers. In the case of super symplectic algebra the generators would correspond

to super Noether charges form the isometries of WCW and would have both fermionic and might also have bosonic parts.

2. The spaces of states assignable to partonic 2-surfaces or to a connected 3-surface is however still rather restricted since it assumes in the spirit of reductionism that the symmetries are local single particle symmetries. The first guess for many-particle states in this approach is as free states and one must introduce interactions in an ad hoc manner and the problems of quantum field theories are well-known.
3. In the TGD framework there is a classical description of interactions in terms of Bohr-orbit like preferred extremals and one should generalize this to the quantum context using zero energy ontology (ZEO). Classical interactions have as space-time correlates flux tubes and "massless extremals" connecting 3-surfaces as particle and topological vertices for the partonic 2-surfaces.
4. The construction recipe of many-particle states should code automatically for the interactions and they should follow from the symmetries as a polylocal extension of single particle symmetries. They should be coded by the modification of the usual tensor product giving only free many-particle states. One would like to have interacting many-particle states assignable to disjoint connected 3-surfaces or many-parton states assignable to single connected space-time surfaces inside causal diamond (CD).

Yangian algebras are especially interesting in this respect.

1. Yangian algebras have a co-algebra structure allowing to construct multi fermion representations for the generators using comultiplication operation, which is analogous to the time reversal of a Lie-algebra commutator (super algebra anticommutator) regarded as interaction vertex with two incoming and one outgoing particle. The co-product is analogous to tensor product and assignable to a decay of a particle to two outgoing particles.
2. What is new is that the generators of Yangian are poly-local. The infinitesimal symmetry acts on several points simultaneously. For instance, they could allow a more advanced mathematical formulation for n-local interaction energy lacking from quantum field theories, in particular potential energy. The interacting state could be created by a bi-local generator of Yangian. The generators of Yangian can be generated by applying coproducts and starting from the basic algebra. There is a general formula expressing the relations of the Yangian.
3. Yangian algebras have a grading by a non-negative integer, which could count the number of 3-surfaces (say all connected 3-surfaces appearing at the ends of the space-time surface at the boundaries of causal diamond (CD)), or the number of partonic 2-surfaces for a given 3-surface. There would also be gradings with respect to fermion and antifermion numbers.

There are indications that Yangians could be important in TGD.

1. In TGD, the notion of Yangian generalizes since point-like particles correspond to disjoint 3-surfaces, for a given 3-surface to partonic 2-surfaces, and for a partonic 2-surface to point-like fermions and antifermions. In the TGD inspired biology, the notion of dark genes involves communications by n-resonance. Two dark genes with N identical codons can exchange cyclotron 3N-photon in 3N-resonance. Could genes as dark N-codons allow a description in terms of Yangian algebra with N-local vertex? Could one speak of 3N-propagators for 3N cyclotron-photons emitted by dark codons.
2. In quantum theory, Planck constant plays a central role in the representations of the Lie algebras of symmetries. Its generalization assignable to n-local Lie algebra generators could make sense for Yangians. The key physical idea is that Nature is theoretician friendly. When the coupling strength proportional to a product of total charges or masses becomes so large that perturbation series fails to converge, a phase transition increasing the value of h_{eff} takes place. Could this transition mean a formation of bound states describable in terms of poly-local generators of Yangian and corresponding poly-Planck constant?

   For instance, the gravitational Planck constant $\hbar_{gr}$, which is bilocal and proportional to two masses to which monopole flux tube is associated, could allow an interpretation in terms of Yangian symmetries and be assignable to a bi-local gravitational contribution to energy momentum. Also other interaction momenta could have similar Yangian contributions and characterized by corresponding Planck constants.

   It is not clear whether $\hbar_{gr}$ and its generalization can be seen as a special case of the proposal $h_{eff} = nh_0$ generalizing the ordinary single particle Planck constant or whether it is something different. If so, the hierarchy of Planck constant would correspond to a hierarchy of polylocal generators of Yangian.

Galois confinement

The above discussion was at the level of $H = M^4 \times CP_2$ and "world of classical worlds" (WCW). $M^8 - H$ duality predicts that this description has a counterpart at the level of M^8. The number theoretic vision predicting the hierarchy of Planck constants strongly suggests Galois confinement as a universal mechanism for the formation of bound states of particles as Galois singlets.

1. The simplest formulation of Galois confinement states that the four-momenta of particles have components which are algebraic integers in the extension of rationals characterizing a polynomial defining a 4-surface in complexified M^8, which in turn is mapped to a space-time surface in $H = M^4 \times CP_2$, when the momentum unit is determined by the size of causal diamond (CD).
   The total momentum for the bound state would be Galois singlet so that its components would be ordinary integers: this would be analogous to the particle in box quantization. Each momentum component "lives" in n-dimensional discrete extension of rationals with coefficient group, which consists of integers.
   In principle one has a wave function in this discrete space for all momentum components as a superposition of Galois singlet states. The condition that total momentum is Galois singlet forces an entanglement between these states so that one does not have a mere product state.
2. Galois confinement poses strong conditions on many-particle states and forces entanglement. Could Galois confinement be $M^8 - H$ dual of the Yangian approach?

12.3.4 h/h_0 as the ratio of Planck mass and CP_2 mass?

Could one understand and perhaps even predict the value of h_0? Here number theory and the notion of n-particle Planck constant $h_{eff}(n)$ suggested by Yangian symmetry could serve as a guidelines.

1. Hitherto I have found no convincing empirical argument fixing the value of $r = \hbar/\hbar_0$: this is true for both single particle and 2-particle case.
   The value $h_0 = \hbar/6$ [L22] as a maximal value of $\hbar_0$ is suggested by the findings of Randell Mills [D2] and by the idea that spin and color must be representable as Galois symmetries so that the Galois group must contain $Z_6 = Z_2 \times Z_3$. Smaller values of h_0 cannot be however excluded.
2. A possible manner to understand the value r geometrically would be following. It has been assumed that CP_2 radius R defines a fundamental length scale in TGD and Planck length squared $l_P^2 = \hbar G = x^{-2} \times 10^{-6} R^2$ defines a secondary length scale. For Planck mass squared one has $m_{Pl}^2 = m(CP_2, \hbar)^2 \times 10^6 x^2$, $m(CP_2, \hbar)^2 = \hbar/R^2$. The estimate for x from p-adic mass calculations gives $x \simeq 4.2$. It is assumed that CP_2 length is fundamental and Planck length is a derived quantity.
   But what if one assumes that Planck length identifiable as CP_2 radius is fundamental and CP_2 mass corresponds the minimal value h_0 of $h_{eff}(2)$? That the mass formula is quadratic and mass is assignable to wormhole contact connecting two space-time sheets suggests in the Yangian framework that $h_{eff}(2)$ is the correct Planck constant to consider.

One can indeed imagine an alternative interpretation. CP_2 length scale is deduced indirectly from p-adic mass calculation for electron mass assuming $h_{eff} = h$ and using Uncertainty Principle. This obviously leaves the possibility that $R = l_P$ apart from a numerical constant near unity, if the value of h_{eff} to be used in the mass calculations is actually $h_0 = (l_P/R)^2\hbar$. This would fix the value of $\hbar_0$ uniquely.

The earlier interpretation makes sense if $R(CP_2)$ is interpreted as a dark length scale obtained scaling up l_P by $\hbar/\hbar_0$. Also the ordinary particles would be dark.

h_0 would be very small and $\alpha_K(\hbar_0) = (\hbar/\hbar_0)\alpha_K$ would be very large so that the perturbation theory for it would not converge. This would be the reason for why $\hbar$ and in some cases some smaller values of h_{eff} such as $\hbar/2$ and $\hbar/4$ [D2] [L22] seem to be realized.

For $R = l_P$ Nottale formula remains unchanged for the identification $M_P = \hbar/R$.

For $R = l_P$ Nottale formula remains unchanged for the identification $M_P = \hbar/R$ (note that one could consider also $\hbar_0/R^2$ as natural unit of mass squared in the p-adic mass calculations).

Various options

Number theoretical arguments allow to deduce precise value for the ratio $\hbar/\hbar_0$. Accepting the Yangian inspired picture, one can consider two options for what one means with $\hbar$.

1. $\hbar$ refers to the single particle Planck constant $\hbar_{eff}(1)$ natural for point-like particles.
2. $\hbar$ refers to $h_{eff}(2)$. This option is suggested by the proportionality $M^2 \propto \hbar$ in string models due to the proportionality $M^2 \propto \hbar/G$ in string models. At a deeper level, one has $M^2 \propto L_0$, where L_0 is a scaling generator and its spectrum has scale given by $\hbar$.
   Since M^2 is a p-adic thermal expectation of L_0 in the TGD framework, the situation is the same. This also due the fact that one has In TGD framework, the basic building bricks of particles are indeed pairs of wormhole throats.

   One can consider two options for what happens in the scaling $h_{eff} \to kh_{eff}$.

   Option 1: Masses are scaled by k and Compton lengths are unaffected.

   Option 2: Compton lengths are scaled by k and masses are unaffected.

   The interpretation of $M_P^2 = (\hbar/\hbar_0)M^2(CP_2)$ assumes Option 1 whereas the new proposal would correspond to Option 2 actually assumed in various applications.

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For Option 1 $m_{Pl}^2 = (\hbar_{eff}/\hbar)M^2(CP_2)$. The value of $M^2(CP_2) = \hbar/R^2$ is deduced from the p-adic mass calculation for electron mass. One would have $R^2 \simeq (\hbar_{eff}/\hbar)l_P^2$ with $\hbar_{eff}/\hbar = 2.54 \times 10^7$. One could say that the real Planck length corresponds to R.

Quantum-classical correspondence favours Option 2)

In an attempt to select between these two options, one can take space-time picture as a guideline. The study of the embeddings of the space-time surfaces with spherically symmetric metric carried out for almost 4 decades ago suggested that CP_2 radius R could naturally correspond to Planck length l_P. The argument is described in detail in Appendix and shows that the $l_P = R$ option with $h_{eff} = h$ used in the classical theory to determine α_K appearing in the mass formula is the most natural.

Deduction of the value of $\hbar/\hbar_0$

Assuming Option 2), the questions are following.

1. Could $l_P = R$ be true apart from some numerical constant so that CP_2 mass $M(CP_2)$ would be given by $M(CP_2)^2 = \hbar_0/l_P^2$, where $\hbar_0 \simeq 2.4 \times 10^{-7}\hbar$ ($\hbar$ corresponds to $\hbar_{eff}(2)$) is the minimal value of $\hbar_{eff}(2)$. The value of h_0 would be fixed by the requirement that classical theory is consistent with quantum theory! It will be assumed that $\hbar_0$ is also the minimal value of $\hbar_{eff}(1)$ both $\hbar_{eff}(2)$.
2. Could $\hbar(2)/\hbar_0(2) = n_0$ correspond to the order of the product of identical Galois groups for two Minkowskian space-time sheets connected by the wormhole contact serving as a building brick of elementary particles and be therefore be given as $n_0 = m^2$?

   Assume that one has $n_0 = m^2$.

1. The natural assumption is that Galois symmetry of the ground state is maximal so that m corresponds to the order a maximal Galois group - that is permutation group S_k, where k is the degree of polynomial.
   This condition fixes the value k to $k = 7$ and gives $m = k! = 7! = 5040$ and gives $n_0 = (k!)^2 = 25401600 = 2.5401600 \times 10^7$. The value of $\hbar_0(2)/\hbar(2) = m^{-2}$ would be rather small as also the value of $\hbar_0(1)\hbar(1)$. p-Adic mass calculations lead to the estimate $m_{Pl}/m(CP_2) = \sqrt{m}m(CP_2) = 4.2 \times 10^3$, which is not far from $m = 5040$.
2. The interpretation of the product structure $S_7 \times S_7$ would be as a failure of irreducibility so that the polynomial decomposes into a product of polynomials - most naturally defined

for causally isolated Minkowskian space-time sheets connected by a wormhole contact with Euclidian signature of metric representing a basic building brick of elementary particles.
Each sheet would decompose to 7 sheets. $\hbar_{gr}$ would be 2-particle Planck constant $h_{eff}(2)$ to be distinguished from the ordinary Planck constant, which is single particle Planck constant and could be denoted by $h_{eff}(1)$.
The normal subgroups of $S_7 \times S_7$ $S_7 \times A_7$ and $A_7 \times A_7$, S_7, A_7 and trivial group. A_7 is simple group and therefore does not have any normal subgroups expect the trivial one. S_7 and A_7 could be regarded as the Galois group of a single space-time sheet assignable to elementary particles. One can consider the possibility that in the gravitational sector all EQs are extensions of this extension so that $\hbar$ becomes effectively the unit of quantization and m_{Pl} the fundamental mass unit. Note however that for very small values of α_K in long p-adic length scales also the values of $h_{eff} < h$, even h_0, are in principle possible.
The large value of $\alpha_K \propto 1/\hbar_{eff}$ for Galois groups with order not considerably smaller than $m = (7!)^2$ suggests that very few values of $h_{eff}(2) < h$ are realized. Perhaps only $S_7 \times S_7$ $S_7 \times A_7$ and $A_7 \times A_7$ are allow by perturbation theory. Now however that in the "stringy phase" for which super-conformal invariance holds true, h_0 might be realized as required by p-adic mass calculations. The alternative interpretation is that ordinary particles correspond to dark phase with R identified dark scale.

3. A_7 is the only normal subgroup of S_7 and also a simple group and one has $S_7/A_7 = Z_2$. $S_7 \times S^7$ has $S_7 \times S^7/A_7 \times A_7 = Z_2 \times Z_2$ with $n = n_0/4$ and $S_7 \times S^7/A_7 \times S_7 = Z_2$ with $n = n_0/2$. This would allow the values $\hbar/2$ and $\hbar/4$ as exotic values of Planck constant.
The atomic energy levels scale like $1/\hbar^2$ and would be scaled up by factor 4 or 16 for these two options. It is not clear whether $\hbar \to \hbar/2$ option can explain all findings of Randel Mills [D2] in TGD framework [L22], which effectively scale down the principal quantum number n from n to $n/2$.
4. The product structure of the Nottale formula suggests

$$n = n_1 \times n_2 = k_1 k_2 m^2 \quad . \tag{12.3.6}$$

Equivalently, n_i would be a multiple of m. One could say that $M_{Pl} = \sqrt{\hbar/\hbar_0} M(CP_2)$ effectively replaces $M(CP_2)$ as a mass unit. At the level of polynomials this would mean that polynomials are composites $P \circ P_0$ where P_0 is ground state polynomial and has a Galois group with degree n_0. Perhaps S_7 could be called the gravitational or ground state Galois group.

12.3.5 Connection with adelic physics and infinite primes

The structure of $\hbar_{gr}$ and its electromagnetic counterpart $\hbar_{em}$ characterize 2-particle states whereas $\hbar$ characterizes single particle state. Yangian picture suggests that the notion of $\hbar_{eff}(n)$, $n = 1, 2, ..$ makes sense.

One can decompose a state consisting of N particles in several ways to partitions consisting of m subsets with n_i, $i = 1, ..., n$ in a given subset of particles. Could these subsets correspond to gravitationally bound states so that one can take these sets as basic entities characterized by masses and assume that gravitational interactions reduce to gravitational interactions between them and are quantal for $GM_iM_j/v_0 \geq \hbar$. Same question applied to electromagnetic, weak and color interactions.

Connection with adelic physics

This picture would have analog at the level of adelic physics [L82, L83, L104].

1. In the M^8 picture space-time surfaces correspond to "roots" of complexified octonionic polynomials obtained from irreducible real polynomials with rational (or perhaps even algebraic) coefficients. The dynamics realizes associativity of the normal space of the complexified space-time surface having 4-D space-time surface as real part mapped from M^8 to $H = M^4 \times CP_2$ by $M^8 - H$ correspondence.
2. One can consider irreducible polynomials of several variables such that the additional variables are interpreted as parameters [L101]. The parametrized set of polynomials defines a parametrized set of space-time surfaces and one can have a superposition of quantum states

corresponding to irreducible polynomial of degree n and products of irreducible polynomials with sum of degrees n_i equal to n. This kind of parametrized set could define sub-spaces of the "world of classical worlds" (WCW).

3. Irreducibility fails for some parameter values forming lower-dimensional manifolds of the parameter space. The failure of the irreducibility means decomposition to a product of polynomials in which the set of roots decomposes to subsets R_i, which are roots of a rational polynomial with a lower degree n_i. Spacetime surface as a coherent structure decomposes to uncorrelated space-time surfaces with a discrete set of points as intersections. In this manner one obtains a decomposition of the parameter space to subsets of decreasing dimension. The generic situation has maximal dimension and dimension equal to that of the parameter space.
4. The catastrophe theory [A129] founded by Rene Thom studies these situations. In catastrophe theory, the failure of the irreducibility is of very special nature and means that some roots of the polynomial co-incide and become multiple roots. For polynomials with rational coefficients, they would become multiple rational roots so that the degree of the polynomial determining the extension would be reduced by two units. This is discussed in detail from TGD point of view in [L101]. For polynomials with rational coefficients, typically complex conjugate roots become rational and the dimension of the algebraic extension is reduced.
5. The quantum state defined by the polynomial of several variables would be a superposition of space-time surfaces labelled by the points of the parameter space. It would decompose to subsets defining what is known as a stratification. The subsets for which the polynomial fails to be irreducible would have lower dimension. For polynomials with rational coefficients these sets would be discrete and it is not clear whether the lower-dimensional sets are non-empty in the generic case.
6. The decomposition to k irreducible polynomials with degrees n_i, $i = 1,...,k$ would correspond to a decomposition of the space-time surface to separate space-time surfaces with $h_{gr,i} = n_i h_0 = GM_i m/v_0$ (same applies to h_{em}) satisfying $\sum n_i = n$. These would correspond to different decompositions of the total energy to a sum of energies E_i: $E = \sum E_i$. The irreducible polynomials with degree n_i could be interpreted as bound states for a subset of basic units. Maximal decomposition would correspond to $n_i = 1$ and have interpretation as a set of elementary particles with $h_{eff} = h_0$ (note that $h = 6h_0$ in the proposal inspired by the findings of Randel Mills [L22]).

Connection with infinite primes

The notion of infinite prime [K84] resonates with this picture.

1. The hierarchy of infinite primes has an interpretation as a repeated second quantization of supersymmetric arithmetic QFT. Polynomial primes of variable polynomials of single variable with rational coefficients follow ordinary primes in the hierarchy. Higher levels correspond to polynomial primes for polynomials of several variables and second quantization corresponds to the formation of polynomials of single variable with coefficients as polynomials of $n-1$ variables.
   Irreducible polynomials of higher than first order have interpretation as bound states whereas polynomials reducing to products of monomials correspond to Fock states of free particles.
2. The beatiful feature would be a number theoretic description of also bound states. The description of the particle decays as a failure of the irreducibility of the polynomials corresponding to infinite primes would extend this picture to the dynamics.
3. Second beautiful feature is the number theoretic description of particle reactions. Particle reactions with unentagled final states would naturally correspond to a situation in which the initial (prepared) and final (state function reduced) states are products of polynomials. Interaction period would correspond to an irreducible polynomial.
   This picture conforms with the proposal inspired originally by a model of "cold fusion". unnelling phenomenon crucial for nuclear reactions would correspond to a formation of dark phase in which the value of h_{eff} increases [L71, L31, L87]. This picture generalizes to all particle reactions.

12.4 How to understand coupling constant evolution?

In this section, the evolutions of Kähler coupling strength α_K and gravitational fine structure constant α_{gr} are discussed. The reason for restricting to α_K is that it is expected to induce the evolution of various gauge couplings, and could also induce the evolution of α_{gr}.

12.4.1 Evolution of Kähler coupling strength

The evolution of Kähler coupling strength $\alpha_K = g_K^2/2h_{eff}$ gives the evolution of α_K as a function of dimension n of EQ: $\alpha_K = g_K^2/2nh_0$. If g_K^2 corresponds to electroweak U(1) coupling, it is expected to evolve also with respect to PLS so that the evolutions would factorize.

Note that the original proposal that g_K^2 is renormalization group invariant was later replaced with a piecewise constancy: α_K has indeed interpretation as piecewise constant critical temperature

1. In the TGD framework, coupling constant as a continuous function of the continuous length scale is replaced with a function of PLS so that coupling constant is a piecewise constant function of the continuous length scale.
   PLSs correspond to p-adic primes p, and a hitherto unanswered question is whether the extension determines p and whether p-adic primes possible for a given extension could correspond to ramified primes of the extension appearing as factors of the moduli square for the differences of the roots defining the space-time surface.
   In the M^8 picture the moduli squared for differences $r_i - r_j$ of the roots of the real polynomial with rational coefficients associated with the space-time surfaces correspond to energy squared and mass squared. This is the case of p-adic prime corresponds to the size scale of the CD.
   The scaling of the roots by constant factor however leaves the number theoretic properties of the extension unaffected, which suggests that PLS evolution and dark evolution factorize in the sense that PLS reduces to the evolution of a power of a scaling factor multiplying all roots.
2. If the exponent $\Delta K/log(p)$ appearing in $p^{\Delta K/log(p))} = exp(\Delta K)$ is an integer, $exp(\Delta K)$ reduces to an integer power of p and exists p-adically. If ΔK corresponds to a deviation from the Kähler function of WCW for a particular path in the tree inside CD, p is fixed and $exp(\Delta K)$ is integer. This would provide the long-sought-for identification of the preferred p-adic prime. Note that p must be same for all paths of the tree. p need not be a ramified prime so that the trouble-some correlation between n and ramified prime defining padic prime p is not required.
3. This picture makes it possible to understand also PLS evolution if ΔK is identified as a deviation from the Kähler function. $p^{\Delta K/log(p))} = exp(\Delta K)$ implies that ΔK is proportional to $log(p)$. Since ΔK as 6-D Kähler action is proportional to $1/\alpha_K$, $log(p)$-proportionality of ΔK could be interpreted as a logarithmic renormalization factor of $\alpha_K \propto 1/log(p)$.
4. The universal CCE for α_K inside CDs would induce other CCEs, perhaps according to the scenario based on Möbius transformations [L54].

Dark and p-adic length scale evolutions of Kähler coupling strength

The original hypothesis for dark CCE was that $h_{eff} = nh$ is satisfied. Here n would be the dimension of EQ defined by the polynomial defining the space-time surface $X^4 \subset M_c^8$ mapped to H by $M^8 - H$ correspondence. n would also define the order of the Galois group and in general larger than the degree of the irreducible polynomial.

Remark: The number of roots of the extension is in general smaller and equal to n for cyclic extensions only. Therefore the number of sheets of the complexified space-time surface in M_c^8 as the number of roots identifiable as the degree d of the irreducible polynomial would in general be smaller than n. n would be equal to the number of roots only for cyclic extensions (unfortunately, some former articles contain the obviously wrong statement $d = n$).

Later the findings of Randell Mills [D2], suggesting that h is not a minimal value of h_{eff}, forced to consider the formula $h_{eff} = nh_0$, $h_0 = h/6$, as the simplest formula consistent with the findings of Mills [L22]. h_0 could however be a multiple of even smaller value of h_{eff}, call if h_0 and the formula $h_0 = h/6$ could be replaced by an approximate formula.

The value of $h_{eff} = nh_0$ can be understood by noticing that Galois symmetry permutes "fundamental regions" of the space-time surface so that action is n times the action for this kind of region. Effectively this means the replacement of α_K with α_K/n and implies the convergence of the perturbation theory. This was actually one of the basic physical motivations for the hierarchy of Planck constants. In the previous section, it was argued that $\hbar/h_0$ is given by the ratio R^2/l_P^2 with R identified as dark scale equals to $n_0 = (7!)^2$.

The basic challenge is to understand p-adic length scale evolutions of the basic gauge couplings. The coupling strengths should have a roughly logarithmic dependence on the p-adic length scale $p \simeq 2^{k/2}$ and this provides a strong number theoretic constraint in the adelic physics framework.

Since Kähler coupling strength α_K induces the other CCEs it is enough to consider the evolution of α_K.

p-Adic CCE of α from its value at atomic length scale?

If one combines the observation that fine structure constant is rather near to the inverse of the prime $p = 137$ with PLS, one ends up with a number theoretic idea leading to a formula for α_K as a function of p-adic length scale.

1. The fine structure constant in atomic length scale $L(k = 137)$ is given $\alpha(k) = e^2/2h \simeq 1/137$. This finding has created a lot of speculative numerology.
2. The PLS $L(k) = 2^{k/2}R(CP_2)$ assignable to atomic length scale $p \simeq 2^k$ corresponds to $k = 137$ and in this scale α is rather near to 1/137. The notion of fine structure constant emerged in atomic physics. Is this just an accident, cosmic joke, or does this tell something very deep about CCE?

   Could the formula

   $$\alpha(k) = \frac{e^2(k)}{2h} = \frac{1}{k}$$

   hold true?

There are obvious objections against the proposal.

1. α is length scale dependent and the formula in the electron length scale is only approximate. In the weak boson scale one has $\alpha \simeq 1/127$ rather than $\alpha = 1/89$.
2. There are also other interactions and one can assign to them coupling constant strengths. Why electromagnetic interactions in electron Compton scale or atomic length scales would be so special?

The idea is however plausible since beta functions satisfy first order differential equation with respect to the scale parameter so that single value of coupling strength determines the entire evolution.

p-Adic CCE from the condition $\alpha_K(k = 137) = 1/137$

In the TGD framework, Kähler coupling strength α_K serves as the fundamental coupling strength. All other coupling strengths are expressible in terms of α_K, and in [L54] it is proposed that Möbius transformations relate other coupling strengths to α_K. If α_K is identified as electroweak $U(1)$ coupling strength, its value in atomic scale $L(k = 137)$ cannot be far from 1/137.

The factorization of dark and p-adic CCEs means that the effective Planck constant $h_{eff}(n, h, p)$ satisfies

$$h_{eff}(n, h, p) = h_{eff}(n, h) = nh \ . \tag{12.4.1}$$

and is independent of the p-adic length scale. Here n would be the dimension of the extension of rationals involved. $h_{eff}(1, h, p)$ corresponding to trivial extension would correspond to the p-adic CCE as the TGD counterpart of the ordinary evolution.

The value of h need not be the minimal one as already the findings of Randel Mills [D2] suggest so that one would have $h = n_0h_0$.

$$h_{eff} = nn_0h \ , \quad \alpha_{K,0} = \frac{g^2_{K,max}}{2h_0} = n_0 \ . \tag{12.4.2}$$

This would mean that the ordinary coupling constant would be associated with the non-trivial extension of rationals.

Consider now this picture in more detail.

1. Since dark and p-adic length scale evolutions factorize, one has

$$\alpha_K(n) = \frac{g_K^2(k)}{2h_{eff}} \ , \quad h_{eff} = nh_0 \ . \tag{12.4.3}$$

$U(1)$ coupling indeed evolves with the p-adic length scale, and if one assumes that $g_K^2(k, n_0)$ ($h = n_0 h_0$) is inversely proportional to the logarithm of p-adic length scale, one obtains

$$\begin{aligned} g_K^2(k, n_0) &= \frac{g_K^2(max)}{k} \ , \\ \alpha_K &= \frac{g_K^2(max)}{2kh_{eff}} \ . \end{aligned} \tag{12.4.4}$$

2. Since $k = 137$ is prime (here number theoretical physics shows its power!), the condition $\alpha_K(k = 137, h_0) = 1/137$ gives

$$\frac{g_K^2(max)}{2h_0} = \alpha_K(max) = (7!)^2 \ . \tag{12.4.5}$$

The number theoretical miracle would fix the value of $\alpha_K(max)$ to the ratio of Planck mass and CP_2 mass $n_0 = M_P^2/M^2(CP_2) = (7!)^2$ if one takes the argument of the previous section seriously.

The convergence of perturbation theory could be possible also for $h_{eff} = h_0$ if the p-adic length scale $L(k)$ is long enough to make $\alpha_K = n_0/k$ small enough.

3. The outcome is a very simple formula for α_K

$$\alpha_K(n, k) = \frac{n_0}{kn} \ , \tag{12.4.6}$$

$$\tag{12.4.7}$$

which is a testable prediction if one assumes that it corresponds to electroweak $U(1)$ coupling strength at QFT limit of TGD. This formula would give a practically vanishing value of α_K for very large values of n associated with h_{gr}. Here one must have $n > n_0$.

For $h_{eff} = nn_0h$ characterizing extensions of extension with $h_{eff} = h$ one can write

$$\alpha_K(nn_0, k) = \frac{1}{kn} \ . \tag{12.4.8}$$

4. The almost vanishing of α_K for the very large values of n associated with $\hbar_{gr}$ would practically eliminate the gauge interactions of the dark matter at gravitational flux tubes but leave gravitational interactions, whose coupling strength would be $\beta_0/4\pi$. The dark matter at gravitational flux tubes would be highly analogous to ordinary dark matter.

12.4.2 The evolution of the gravitational fine structure constant

Nottale [?] introduced the notion of gravitational Planck constant $\hbar_{gr} = GMm/\beta_0$ ($\beta_0 = v_0/c$ is velocity parameter), which has gigantic values so that the original proposal $h_{gr} = nh_0$ would predict very large values for n. If p-adic and dark evolutions are independent this is not a problem since p-adic length scales need not be gigantic.

Evolution of the parameter β_0

Gravitational fine structure constant is given by $\alpha_{gr} = GMm/4\pi\hbar_{gr} = \beta_0/4\pi$. The basic challenge is to understand the value spectrum of β_0.

1. Kepler's law $\beta^2 = GM/r = r_S/2r$ suggests length scale evolution of form

$$\beta_N = \sqrt{\frac{r_S}{2L(N)x}} = \frac{\beta_{0,max}}{N} \ . \tag{12.4.9}$$

The coefficient x has been included in the formula because otherwise a conflict with Bohr model for planetary orbits results.

2. How to identify N?
   (a) $N = n = h_{gr}/h_0$ would give a gigantic value of N and this would give extremely small value for β_0. Actually $N = n$ for n in $h_{gr} = nh_0$ is impossible as is clear from the defining equation.
   (b) It is not clear whether N be identified as a dimension for some factor in the composition of extension to simple factors rather than as n. This would conform with the vision that there are evolutionary hierarchies of extensions of extensions of... for which the dimension is product of dimensions of the extensions involved.
   (c) The simplest option is that p-adic length scale evolution determines N as in case of the gauge interactions, and it corresponds to k in $p \simeq 2^k$. $log_2(p)$ exists also for a general prime p in real sense. In p-adic sense it exists for all primes except $p = 2$ as integer valued function. $p = 2$ could be chosen to be the exceptional prime.
   This would conform with the idea that gravitational sector and gauge interaction sector correspond to different factors in the decomposition of extension of rationals. Perhaps the gravitational part of EQ extends its gauge part. This would conform with the idea that gravitation does not differentiate between states with different gauge quantum numbers.

What can one say about the value of $\beta_{0,max}$ and its length scale evolution?

1. The value of $\beta_{0,max} = 1/2$ would give for the length scale $L = GM/\beta_{0,max} = r_S$. If one requires that the scale L is not smaller than Scwartschild radius, $\beta_{0,max} \leq 1/2$ follows. $\beta_{0,max} = 1/2$ is the first guess but it turns that number theoretical constraintss exclude it and suggest $\beta_{0,max} = \pi/6$ as the simplest guess.
2. Gravitational Bohr radius a_{gr} given by
$$a_{gr} = \frac{\hbar_{gr}}{\alpha_{gr} m} per. \tag{12.4.10}$$
defines a good candidate for the minimal value of L_n as $L_1 = a_{gr}$.
3. The analogs of p-adic length scales would be equal to the radii of gravitational Bohr atom as n^2- multiples of the gravitational Bohr radius a_{gr}:
$$L_n = n^2 a_{gr} \ , \quad a_{gr} = \frac{4\pi GM}{\beta_0^2} \ . \tag{12.4.11}$$
This expression realizes the condition $\beta_0^2 = xGM/r$ inspired by the Kepler's law with $x = 4\pi$.
4. One must fix a_{gr} as a multiple $a_{gr} = kr_S$ of r_S. Substitution to the above equation gives
$$\beta_{0,max} = \sqrt{\frac{2\pi}{k}} \ .$$
The condition $\beta_{0,max} = 1/2$ would give $k = 8\pi$ and $a_{gr} = 8\pi r_S$ as a minimal radius for a Bohr orbit. The condition $\beta_{0,max} < 1$ gives $k \geq 2\pi$ and $a_{gr} \geq 2\pi r_S$.
Just as in the case of hydrogen atom, the falling of the orbiting system to the blackhole like entity (in TGD frameworkd blackholes are replaced with what might be called flux tube spaghettis [L60, L55]) is prevented. This should have obviously consequences for the view about the dynamics around blackhole like objects. The circular orbits have as analogs s-waves and of these are realized, the falling to blackhole like entity is possible.
5. The proposed formula does not force the condition $\beta_0 < 1$ and it is not clear whether it holds true at the relativistic limit. The replacement $\beta_0 \rightarrow sinh(\eta) = \beta_0/\sqrt{1-\beta_0^2}$, where η is the hyperbolic angle, forces the condition $\beta_0 < 1$, and would give
$$\beta_0 \rightarrow \frac{\beta_0}{\sqrt{1-\beta_0^2}} = \sqrt{\frac{2\pi}{k}} \ .$$
The condition $\beta_{0,max} = 1/2$ gives $k/2\pi = 3$. This would correspond to the minimal Bohr radius $a_{gr} = 6\pi r_S \simeq 18.84 r_S$.

Number theoretical universality as a constraint

Also number theoretical universality could be also used as a constraint. The condition would be that only finite-dimensional extensions are allowed. π defines an infinite-D transcendental extension so that it should disappear in central formulas.

1. The appearance of 4π in the formula $a_{gr} = 4\pi G/\beta_{0,max}^2$ creates number-theoretical worries. Suppose that a_{gr} is a rational number.
2. I have proposed that G is dynamically determined and relates to the CP_2 radius via the formula $G = R^2/\hbar_{grav} = 2\pi R^2/h_{grav}$, where $h_{grav}/h_0 \sim 10^7$ holds true [K10].
   This gives

$$a_{gr} = \frac{4\pi G}{\beta_{0,max}^2} = \frac{8\pi^2 R^2}{h_{grav}\beta_{0,max}^2} \tag{12.4.12}$$

.

3. Since $\beta_0/4\pi$ appears as coupling strength in the perturbation theory, it should also be rational. $\beta_{0,max} = \pi/6$ would realize the condition $\beta_{0,max} = 1/2$ approximately.
4. With this assumption the rationality of a_{gr} requires that h_{gr} is proportional to π so that also G would be rational. This implies that $\hbar_{eff} = h_{eff}/2\pi$ is rational. Also α_K would be rational if g_K^2 is rational. This would be true also for the other coupling constants.
5. $\beta_0 = \pi/6$ would realize the condition $\beta_0 = 1/2$ approximately. This also implies that α_{gr} is rational. The condition $k/2\pi = 1/\beta_{0,max}^2$ implies $k \propto 1/\pi$. $a_{gr} = kr_s = kGM$ is rational, and this requires $M \propto \pi$. This guarantees the rationality of GM/β_0. Gravitational fine structure constant α_{gr} would be an inverse integer multiple of $\alpha_{gr}(max) = 1/24$. It would seem that the system is consistent.
   The alternative condition $\beta_0^2/(1-\beta_0^2) = 2\pi/k$ is excluded because it implies that k is a rather complex transcendental.

What makes this interesting is that 24 is one of the magic numbers of mathematics (`https://cutt.ly/RnOxOTr`) and it appears in the bosonic string model as the number of space-like dimensions.

1. Euclidian string world sheet with torus topology has a conformal equivalence class defined by the ratio ω_2/ω_1 of the complex vectors spanning the parallelogram defining torus as an analog of a unit cell. String theory must be invariant under modular group $SL(2,Z)$ leaving the periods and thus the conformal equivalence class of torus invariant. Same applies to higher genera. In TGD these surfaces correspond to partonic 2-surfaces.
2. Modular invariance raises elliptic functions (doubly periodic analytic functions in complex plane) in a special role. In particular, Weierstrass function, which satisfies the differential equation $(d\mathcal{P}/dz)^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$ has a key role in the theory of elliptic functions (`https://cutt.ly/BnOxrMS`).
   The discriminant $\Delta = g_2^3 - 3g_2^3$ of the polynomial at the r.h.s can be locally regarded as a function of the ratio of $\tau = \omega_2/\omega_1$ of the periods of $\mathcal{P}$ defining the conformal equivalence class of torus.
   $\Delta(\tau)$ is not a genuine modular invariant function of τ. Rather, Δ defines a modular form of weight 12 transforming as $\Delta(a\tau + b/(c\tau + d)) \to (c\tau + d)^{12}\Delta(\tau)$ under $SL(2,Z)$. The number 24 comes from the fact that one can express Δ as 24^{th} power of the Dedekind η function: $\Delta = (2\pi)^{12}\eta^{24}$.
3. In dimension $D = 24$ there are 24 even positive definite unimodular lattices, called the Niemeier lattices, and the so-called Leech lattice is one of them. Interestingly, in dimension 4 there exists a 24-cell analogous to Platonic solid having 24 octahedrons as its 3-D "faces".

This encourages the question whether there might be a connection between TGD and string theory based views of quantum gravitation.

Test cases for the proposal

Phase transitions changing β_0 are possible at $r_n/a_{gr} = n^2$ at the Bohr orbits. For instance, in the Bohr orbit model the orbit of Earth is such an orbit. It can be regarded as $n = 5$ orbital with $\beta_0 \simeq 2^{-11}$ and is nearly circular so that the phase transition with $n = 1$ orbital with $\beta_0 \to \beta_0/5$ is possible. The outer planets indeed have $\beta_0/5$.

p-Adic length scale hierarchy is replaced union of hierarchies with $\beta_0 = \beta_{0,max}/n = 1/2n$, each of which is a subset of the set of Bohr orbits for $\beta_0 = \beta_{0,max}$. One can test this hypothesis for the proposed applications [L112].

1. In the Bohr orbit model the 4 inner planets Mercury, Venus, and Earth, and Mars identifiable correspond to $n = 3, 4, 5, 6$ orbitals for $\beta_0 \simeq 2^{-11}$. Solar radius is $R_{Sun} \simeq .7$ Gm. The orbital radius of Mercury is $R_M \simeq 58$ Gm $= 82.9 \times R_{Sun}$. This gives $a_{gr} = R_M/9 \simeq 9.2 R_{sun}$. This gives $\beta_0 = \sqrt{2\pi R_S/a_{gr}} \simeq 17.1 * 10^{-4}$.
   The approximation used hitherto has been $\beta_0 = 2^{-11)} \simeq 5 \times 10^{-4}$ and is by a factor about 1/3 smaller. Using $a_{gr} = R_M$ instead of $a_{gr} = R_M/9$ would give roughly correct value.
   One could indeed regard Mercury as $n = 1$ orbit for $v_0 = v_0/3$ in which case one would have $a_{gr} = R_M$ and one would obtain $\beta_0 = .57$ which is not far from the valued used. Mercury would therefore correspond to $n = 3$ dark matter gravitationally whereas Venus must correspond to $n = 1, 2$ or $n = 4$.
2. The transition $\beta_0 \rightarrow \beta_0/5$ possible for Earth and required for outer planets could be interpreted as the increase of n having interpretation as increase of dimension of extension of rationals $n \rightarrow 5n$.

For the Earth one has $R_E = 6.371 \times 10^6$ m and $r_S = 10^{-2}$ m. The model of the superfluid fountain effect [K28] [L112] suggests $\beta_0 = 1/2$ for which one would have $GM/v_0 = 1/2$. The value of $a_{gr} = 6\pi r_S$ for the relativistic form of the Nottale condition. The principal quantum number n for the Bohr orbit of the super-fluid would be $n \simeq R_E/a_{gr} = R_E/6\pi r_S \simeq 3.4 \times 10^7$. This would correspond to the large quantum number limit. The difference of radii between nearby Bohr orbits would be $\Delta r = 2R_E/n \simeq 19$ cm, which makes sense.

The levels in the hierarchy of gravitationally dark matters are labelled by $h_{gr} = GMm/\beta_0$ with $\beta_0 = \beta_{0,max}/n$, where n is the dimension of EQ, and each level defines a hierarchy of atomic orbitals. The sets of orbital radii at various levels form a nested hierarchy and phase transitions can occur at least between the states with the same angular momentum and orbital radius.

The quantum variant of the similar picture is expected to apply in the case of the hydrogen atom and the fact that there is evidence for dark valence electrons suggests that these phase transitions indeed take place.

What about long cosmic strings thickened to flux tubes explaining galactic dark matter in the TGD framework? In this case the Kepler law gives $\beta^2 = TG$ so that the all orbiting stars would correspond to the same value of β_0 and n.

12.5 Appendix: Embedding of spherically symmetric stationary symmetric metric as a guideline

There are two basic questions to be answered.

1. Is $R = l_P$ or $R = m^2 l_P$, $m = 7!$ realized?
2. Should one assume that $g_K^2 \propto \hbar_{eff}$ or $\alpha_K \propto 1/\hbar_{eff}$?

For the first option α_K is the same for dark phases but would be subject to p-adic CCE. This would conform with the notion of gravitational Planck constant predicting that the parameter. The *effective* value of α_K would be however given by α_K/n for dark phases since the Galois symmetry is n-fold multiple of the action for a "fundamental region" for the Galois group.

Second option would predict that α_K behaves like $1/n$ so that effective α_K would behave like $1/n^2$. It seems that this option is excluded and one can concentrate on the first question. The increase of g_K^2 with n is not a problem since it does not appear as a parameter of perturbative expansion since g_K is automatically absorbed to a scaling of the induced gauge potentials.

Quantum-classical correspondence suggests that classical theory theory, in particular spherically symmetric stationary embeddings, could help to answer the first question. Even the extremal property is not absolutely necessary.

The action is a sum of Kähler action and volume term proportional to length scale dependent cosmological constant approaching zero in long length scale and in equilibrium both give contributions of the same order of magnitude. This suggests that Kähler action corresponding to $\Lambda = 0$ could serve as a guideline.

I studied the embedding of a stationary spherically symmetric metric as a space-time surface during the first 10 years of TGD and the results suggested that the $R = l_P$ option looks more realistic. p-Adic mass calculations based on the definition of the Compton length as $\hbar/M$ however led to the conclusion that the one must have $r \sim 10^{7.6} l_P$. If one replaces $\hbar$ with $\hbar_0$, $R = l_P$ is natural.

The spherically symmetric ansatz assumes that space-time surfaces has a projection to a geodesic sphere S^2 of CP_2 which can be either homologically trivial or non-trivial. Using spherical coordinates (Θ, Φ) for S^2 and spherical Minkowski (t, r, θ, ϕ) coordinates for M^4, the ansatz reads

$$\begin{aligned} s &\equiv sin(\Theta) = f(r) \ , \quad \Phi = \omega t \ , \\ g_{tt} &= 1 - k^2 s^2 \ , \qquad k^2 = R^2 \omega^2 \ . \end{aligned} \tag{12.5.1}$$

In far-away region one can approximate s as

$$s = s_0 + \frac{r_1}{r}, \quad s_0 = sin(\Theta_0) \ . \tag{12.5.2}$$

The induced metric has component g_{tt} given by

$$g_{tt} = 1 - k^2 s_0^2 - 2k^2 s_0 \frac{r_1}{r} \ , \tag{12.5.3}$$

by taking $u = t\sqrt{2 - k^2 s_0^2}$ as a new time coordinate can expresses g_{tt} in terms of the parameters of Scwartshild metric

$$\begin{aligned} g_{uu} &= 1 - 2k^2 s_0 \tfrac{r_1}{r} \equiv 1 - \tfrac{r_s}{r} \ , \\ r_s &= 2GM = \tfrac{2k^2 s_0 r_1}{1 - k^2 s_0^2} \ , \\ r_1 &= \tfrac{1 - k^2 s_0^2}{2k^2 s_0} r_s \equiv k_1 r_s \ . \end{aligned} \tag{12.5.4}$$

The approximation makes sense for $s \leq 1$, which gives the condition

$$r \geq r_{min} = (1 - s_0) r_1 = (1 - s_0) k_1 r_s = (1 - s_0) \frac{1 - k^2 s_0^2}{2k^2 s_0} r_s \equiv y_1 r_s \ . \tag{12.5.5}$$

Remark: The radial component of the metric goes to zero much faster than for Schwartschild metric. The shift of time coordinates depending on the radial coordinate allows to correct this problem. This is however not essential for the recent argument. Schwartchild metric however implies that $\sqrt{g}$ in the calculation of mass gives just the volume element of the flat metric since $g_{tt} g_{rr} = 1$ is true. This is assumed in the following.

One can estimate the mass of the system as Kähler electric energy. Assume that the contribution to the mass comes only from the region $r > y_1 r_s$. The Kähler electric mass $M = r_s/2G$ is given by the expression

$$\begin{aligned} M &= \tfrac{r_s}{2G} \\ &= \tfrac{\hbar_{eff}}{2\alpha_K} \tfrac{s_0^2}{1 - s_0^2} r_1^2 \omega^2 \int_{r_{min}}^{\infty} \tfrac{dr}{r^2} = \tfrac{\hbar_{eff}}{2\alpha_K} \tfrac{(1 - k^2 s_0^2) s_0}{2(1 - s_0)} r_s \tfrac{1}{R^2} \ . \end{aligned} \tag{12.5.6}$$

This gives a consistency condition relating R and l_P

$$\begin{aligned} R^2 &= \tfrac{\hbar_{eff}}{\hbar} X l_P^2 \ , \\ X &= \tfrac{(1 - k^2 s_0^2) s_0}{\alpha_K (1 - s_0)} \ . \end{aligned} \tag{12.5.7}$$

One can consider two cases.

1. For $\hbar_{eff} = \hbar$ the condition reduces to

$$R^2 = X l_P^2 \ . \tag{12.5.8}$$

   $l_P = R$ gives $X = (1-k^2 s_0^2) s_0/\alpha_K(1-s_0) = 1$. One should have $s_0 \simeq \alpha_K$ so that the value of $1/\alpha_K$ as an analog of critical temperature would be coded to the geometry of the space-time surface.
   $R = (7!)^2) l_P$ would require $X = \hbar/\hbar_0$, one should have $1 - s_0 \sim 10^{-5}$ for $\alpha_K \sim 10^{-2}$.
2. For $\hbar_{eff} = \hbar_0$ the condition reduces to

$$R^2 = X \frac{\hbar_0}{\hbar} \times l_P^2 \ . \tag{12.5.9}$$

   $l_P = R$ gives $X = \hbar/\hbar_0$. One might of course argue that α_K decreases in long scales in the discrete p-adic length scale evolution but this option does not look plausible.

To sum up, intuitively $\hbar$ option with $R = l_P$ looks the most reasonable option.

Part III

MISCELLANEOUS TOPICS

Chapter 13

What p-adic icosahedron could mean? And what about p-adic manifold?

13.1 Introduction

This chapter was originally meant to be a summary of what I understoond about the article "The p-Adic Icosahedron" in Notices of AMS (see `http://tinyurl.com/ya9p9bda`) [A116]. The original purpose was to summarize the basic ideas and discuss my own view about more technical aspects - in particular the generalization of Riemann sphere to p-adic context which is rather technical and leads to the notion of Bruhat Tits tree and Berkovich space.

About Bruhat-Tits tree there is a nice web article titled "p-Adic numbers and Bruhat-Tits tree" [A63] (see `http://tinyurl.com/ycjf6rv9`) describing also basics of p-adic numbers in a very concise form. The Wikipedia article about space (see `http://tinyurl.com/ydgt3gfr`) is written with a jargon giving no idea about what is involved. There are video lectures (see `http://tinyurl.com/y847v7xu`) [A91] about Berkovich spaces. The web article about Berkovich spaces (see `http://tinyurl.com/q39ezjg`) by Temkin [A172] seems too technical for a non-specialist. The slides (see `http://tinyurl.com/y8ftfs6z`) [A102] however give a concise bird's eye of view about the basic idea behind Berkovich spaces.

The notion of p-adic icosahedron leads to the challenge of constructing p-adic sphere, and more generally p-adic manifolds and this extended the intended scope of the chapter and led to consider the fundamental questions related to the construction of TGD.

Quite generally, there are two approaches to the construction of manifolds based on algebra *resp.* topology.

1. In algebraic geometry manifolds - or rather, algebraic varieties - correspond to solutions of algebraic equations. Algebraic approach allows even a generalization of notions of real topology such as the notion of genus.
2. Second approach relies on topology and works nicely in the real context. The basic building brick is n-ball. More complex manifolds are obtained by gluing n-balls together. Here inequalities enter the game. Since p-adic numbers are not well-ordered they do not make sense in purely p-adic context unless expressed using p-adic norm and thus for real numbers. The notion of boundary is also one of the problematic notions since in purely p-adic context there are no boundaries.

13.1.1 The Attempt To Construct P-Adic Manifolds By Mimicking Topological Construction Of Real Manifolds Meets Difficulties

The basic problem in the application of topological method to manifold construction is that p-adic disks are either disjoint or nested so that the standard construction of real manifolds using partially overlapping n-balls does not generalize to the p-adic context. The notions of Bruhat-Tits tree [A63], building, and Berkovich disk [A102] and Berkovich space [A172] represent attempts to overcome this problem. Berkovich disk is a generalization of the p-adic disk obtained by adding additional points so that the p-adic disk is a dense subset of it. Berkovich disk allows path connected topology which is path connected. The generalization of this construction is used to construct p-adic manifolds using the modification of the topological construction in the real case. This construction provides also insights about p-adic integration.

The construction is highly technical and complex and pragmatic physicist could argue that it contains several un-natural features due to the forcing of the real picture to p-adic context. In particular, one must give up the p-adic topology whose ultra-metricity has a nice interpretation in the applications to both p-adic mass calculations and to consciousness theory.

I do not know whether the construction of Bruhat-Tits tree, which works for projective spaces but not for Q_p^n (!) is a special feature of projective spaces, whether Bruhat-Tits tree is enough so that no completion would be needed, and whether Bruhat-Tits tree can be deduced from Berkovich approach. What is however remarkable that for $M^4\times CP_2$ p-adic S^2 and CP_2 are projective spaces and allow Bruhat-Tits tree. This not true for the spheres associated with the light-cone boundary of $D\neq 4$-dimensional Minkowski spaces.

13.1.2 Two Basic Philosophies Concerning The Construction Of P-Adic Manifolds

There exists two basic philosophies concerning the construction of p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying common rationals. Finite pinary cutoff is however required to avoid totally wild behavior and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous ways but is not consistent with field equations without pinary cutoff.

1. One can try to generalize the theory of real manifolds to p-adic context. Since p-adic balls are either disjoint or nested, the usual construction by gluing partially overlapping balls fails. This leads to the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultra-metric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold defining its boundary path connected. The notion makes sense for the p-adic counterparts of projective spaces, which suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.
2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach is very natural in TGD framework, where preferred extremals of Kähler action can be characterized purely algebraically - even in a way independent of the action principle - so that they make sense also p-adically.

   At the level of WCW algebraic approach combined with symmetries works: the mere existence of Kähler geometry implies infinite-D group of isometries and fixes the geometry uniquely. One can say that infinite-D geometries are the final victory of Erlangen program. At space-time level it however seems that one must have correspondence between real and p-adic worlds since real topology is the “lab topology”.

13.1.3 Number Theoretical Universality And The Construction Of P-Adic Manifolds

Construction of p-adic counterparts of manifolds is also one of the basic challenges of TGD. Here the basic vision is that one must take a wider perspective. One must unify real and various p-adic physics to single coherent whole and to relate them. At the level of mathematics this requires fusion of real and p-adic number fields along common rationals and the notion of algebraic continuation between number fields becomes a basic tool.

The number theoretic approach is essentially algebraic and based on the gluing of reals and various p-adic number fields to a larger structure along rationals and also along common algebraic numbers. A strong motivation for the algebraic approach comes from the fact that preferred extremals [K14, K100] are characterized by a generalization of the complex structure to 4-D case both in Euclidian and Minkowskian signature. This generalization is independent of the action principle. This allows a straightforward identification of the p-adic counterparts of preferred extremals. The algebraic extensions of p-adic numbers play a key role and make it possible to realize the symmetries in the same way as they are realized in the construction of p-adic icosahedron.

The lack of well-ordering of p-adic numbers poses strong constraints on the formulation of number theoretical universality.

1. The notion of set theoretic boundary does not make sense in purely p-adic context. Quite generally, everything involving inequalities can lead to problems in p-adic context unless one is able to define effective Archimedean topology in some natural way. Canonical identification inducing real topology to p-adic context would allow to achieve this.
2. The question arises about whether real topological invariants such as genus of partonic 2-surface make sense in the p-adic sector: for algebraic varieties this is the case. One would however like to have a more general definition and again Archimedean effective topology is suggestive.
3. Integration poses problems in p-adic context and algebraic continuation from reals to p-adic number fields seems to be the only possible option making sense. The continuation is however not possible for all p-adic number fields for given surface. This has however a beautiful interpretation explaining why real space-time sheets (and elementary particles) are characterized by some p-adic prime or primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!). A more direct approach to integration could rely on canonical integration as a chart map allowing to define integral on the real side.
4. Only those discrete subgroups of real symmetries, which correspond matrices with elements in algebraic extension of p-adic numbers can be realized so that a symmetry breaking to discrete subgroup consistent with the notion of finite measurement resolution and quantum measurement theory takes place. p-Adic symmetry groups can be identified as unions of elements of discrete subgroup of the symmetry group (making sense also in real context) multiplied by a p-adic variant of the continuous Lie group. These genuinely p-adic Lie groups are labelled by powers of p telling the maximum norm of the Lie-algebra parameter. Remarkably, effective values of Planck constant come as powers of p. Whether this interpretation for the hierarchy of effective Planck constants is consistent with the interpretation in terms of n-furcations of space-time sheet remains an open question.

13.1.4 How To Achieve Path Connectedness?

The basic problem in the construction of p-adic manifolds is the total disconnectedness of the p-adic topology implied by ultra-metricity. This leads also to problems with the notion of p-adic integration. Physically it seems clear that the notion of path connectedness should have some physical counterpart.

The notion of open set makes possible path connectedness possible in the real context. In p-adic context Bruhat-Tits tree [A63] and completion of p-adic disk to Berkovich disk [A102] are introduced to achieve the same goal. One can ask whether Berkovich space could allow to achieve a more rigorous formulation for the p-adic counterparts of CP_2, of partonic 2-surfaces, their light-like orbits, preferred extremals of Kähler action, and even the "world of classical worlds" (WCW) [K42, K24]. To me this construction does not look promising in TGD framework but I could of course be wrong.

TGD suggests two alternative approaches to the problem of path connectedness. They should be equivalent.

p-Adic manifold concept based on canonical identification

The TGD inspired solution to the construction of path connected p-adic topology relies on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous way.

1. Canonical identification is used to map the values of p-adic mass squared predicted by p-adic mass calculations to their real counterparts [K50]. It makes also sense to map p-adic probabilities to their real counterparts by canonical identification. In TGD inspired theory of consciousness canonical identification is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?
2. The trivial but striking observation was that it satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals (rather than p-adics!) and vice versa and preferred extremal property allows to complete the discrete image to a space-time surface unique apart from finite measurement resolution so that topological and algebraic approach are combined. Without preferred extremal property one can complete to smooth real manifold (say) but the completion is much less unique.
3. Also the notion of integration can be defined. If the integral for - say- real curve at the map leaf exists, its value on the p-adic side for its pre-image can be defined by algebraic continuation in the case that it exists. Therefore one can speak about lengths, volumes, action integrals, and similar things in p-adic context. One can also generalize the notion of differential form and its holomomorphic variant and their integrals to the p-adic context. These generalizations allow a generalization of integral calculus required by TGD and also provide a justification for some basic assumptions of p-adic mass calculations.

Could path connectedness have a quantal description?

The physical content of path connectedness might also allow a formulation as a quantum physical rather than primarily topological notion, and could boil down to the non-triviality of correlation functions for second quantized induced spinor fields essential for the formulation of WCW spinor structure. Fermion fields and their n-point functions could become part of a number theoretically universal definition of manifold in accordance with the TGD inspired vision that WCW geometry - and perhaps even space-time geometry - allow a formulation in terms of fermions.

The natural question of physicist is whether quantum theory could provide a fresh number theoretically universal approach to the problem. The basic underlying vision in TGD framework is that second quantized fermion fields might allow to formulate the geometry of "world of classical worlds" (WCW) (for instance, Kähler action for preferred extremals and thus Kähler geometry of WCW would reduce to Dirac determinant). Maybe even the geometry of space-time surfaces could be expressed in terms of fermionic correlation functions.

This inspires the idea that second quantized fermionic fields replace the K-valued (K is algebraic extension of p-adic numbers) functions defined on p-adic disk in the construction of Berkovich. The ultra-metric norm for the functions defined in p-adic disk would be replaced by the fermionic correlation functions and different Berkovich norms correspond to different measurement resolutions so that one obtains also a connection with hyper-finite factors of type II_1. The existence of non-trivial fermionic correlation functions would be the counterpart for the path connectedness at space-time level. The 3-surfaces defining boundaries of a connected preferred extremal are also in a natural way "path connected" with "path" being defined by the 4-surface. At the level of WCW and in zero energy ontology (ZEO) [K58] WCW spinor fields are analogous to correlation functions having collections of these disjoint 3-surfaces as arguments. There would be no need to complete p-adic topology to a path connected topology in this approach.

This approach is much more speculative that the first option and should be consistent with it.

13.1.5 Topics Of The Chapter

The chapter was originally meant to discuss p-adic icosahedron. Although the focus was re-directed to the notion of p-adic manifold - especially in TGD framework - I decided to keep the original starting point since it provides a concrete way to end up with the deep problems of p-adic manifold theory and illustrates the group theoretical ideas.

- In the first section icosahedron is described in the real context. In the second section the ideas related to its generalization to the p-adic context are introduced. After that I discuss how to define sphere in p-adic context.
- In the section about algebraic universality I consider the problems related to the challenge of defining p-adic manifolds TGD point of view, which is algebraic and involves the fusion of various number fields and number theoretical universality as additional elements.
- The key section of the chapter describes the construction of p-adic space-time topology relying on chart maps of p-adic preferred extremals defined by canonical identification in finite measurement resolution and on the completion of discrete chart maps to real preferred extremals of Kähler action. The needed path-connected topology is the topology induced by canonical identification defining real chart maps for p-adic space-time surface. Canonical identification allows also the definition of p-adic valued integrals and definition of p-adic differential forms crucial in quantum TGD.
- Last section discusses in rather speculative spirit the possibility of defining space-time surfaces in terms of correlation functions of induced fermion fields.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

13.2 Real Icosahedron And Its Generalization To P-Adic Context

I summarize first the description of icosahedron in real context allowing a generalization to the p-adic context and consider the the problems related to the precise definition of p-adic icosahedron.

13.2.1 What Does One Mean With Icosahedron In Real Context?

The notion of icosahedron (see `http://tinyurl.com/ns9aa`) [A44] is a geometric concept involving the notion of distance. In p-adic context this notion does not make sense since one cannot calculated distances, between points using standard formulas. Same applies to areas and volumes. The reason is that Riemann integral does not generalize and this is due to the fact that p-adic numbers are not well-ordered: one cannot say whether for two p-adic numbers of same norm $a < b$ or $b < a$ holds true.

Platonic solids (see `http://tinyurl.com/5bd5aa`) [A64] are however characterized by their isometry groups and group theory makes sense also in p-adic context. The idea is therefore to characterize the icosahedron or any Platonic solid solely by its isometry group.

In practice this means following. Platonic solid is described as a collection of points. Vertices, midpoints of edges, and barycenters of faces. These points are fixed points for discrete subgroups of the Platonic solid. In the case of icosahedron the isometry group is A_5 the group of even permutations of 5 letters. There are are 6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges. Thus icosahedron becomes a collection of points with a label telling which is the cyclic subgroup associated with the point. This is something which might be able to generalize to p-adic context since there would be no need to talk about distances. One should however describe also the "solid" aspect of icosahedron.

13.2.2 What Does One Mean With Ordinary 2-Sphere?

In order to construct p-adic analog of icosahedron one must construct a space in which the isometry group A_5 of icosahedron acts and is imbedded to a group defining the analog of rotation group.

One could consider two options. The first option would be 3-D Euclidian space $E^3 \equiv R^3$ replaced with its p-adic counterpart Q_p^3. The action of SO(3) however leaves the distance from origin invariant and one can restrict the consideration to 2-sphere. The challenge is to define the counterpart of 2-sphere p-adically.

Before one can say anything about p-adic 2-sphere, one must understand what means with the ordinary 2-sphere identified now as sphere in metric sphere.

1. Riemann sphere is compactification of complex plane and can be regarded as complex projective space $CP_1 = P^1(C)$ is taken as starting point. This space is obtained from C^2 by identified points (z_1, z_2) which differ by a complex scaling: $(z_1, z_2) = \lambda(z_1, z_2)$. One can say that points of $P^1(C)$ are complex lines, which are nothing but Riemann spheres. This manifold requires two coordinate patches corresponding to patch containing North *resp.* South pole but not South *resp.* North pole. The coordinates in a patch containing Northern hemisphere can be taken to be $(u = z_1/z_2, 1)$ by projective equivalence allowing to select point $(z_1/z_2, 1)$ from the projective line with $z_2 \neq 0$. In the region containing Southern hemisphere one can take $v = z_2/z_1$). In the overlap region around equator the coordinates are related by $v = 1/u$. One can think also $P^1(C)$ as plane with single point ∞ (south pole) added.
2. The group $PGL(2, C)$ and also the Lorentz group $SL(2, C)$ acts at Riemann sphere as Möbius transformations. The complex matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is represented as a Möbius transformation

$$u \rightarrow \frac{au + b}{cu + d} \quad .$$

Note that the matrix elements are complex: what this means in p-adic context is not at all clear!

One can regard the coordinates z_1 and z_2 as spinor components and the action of SO(3) is lifted to the action of covering group SU(2) for which 2π rotation is represented by -1. The group A_5 can be lifted to its covering group have twice as many elements as the original one but the action of SU(2) *resp.* overing of A_5 reduces to that of SO(3) *resp.* A_5 since one considers the action on the ratio z_1/z_2 of the spinor components.
3. $S^2 = P^1(C)$ is a good structure to generalize to p-adic context since one can define it purely algebraically, and one realize the action of isometries in it.

13.2.3 Icosahedron In P-Adic Context

What does one mean with p-Adic numbers?

The article about p-icosahedron [A116] gives also a concise summary of p-adic numbers. p-Adic number fields define a hierarchy of number fields Q_p labeled by prime $p = 2, 3, 5, ...$ They are completions of rationals so that rationals can be said to be common to reals and p-adics. Each Q_p allows an infinite number of algebraic extensions whereas reals allow only one - complex numbers.

Local topology of p-adic numbers is what distinguishes them from reals. Two points of Q_p are near to each other if they differ by a very large positive power of p. As real numbers these numbers would differ very much. Most p-adic numbers have infinite number pinary digits in the pinary expansion and are infinite as real numbers.

The p-adic norm defining the p-adic topology is defined by p-adic number fixed completely by the lowest pinary digit in the expansion and is therefore very rough and obtains only values p^n for Q_p. The resulting topology is very rough. Indeed all p-adic points define open sets: one says that p-adic topology is totally disconnected. p-Adic norm is non-Archimedean. It satisfies $|x - y| \leq Max\{x, y\}$ whereas real norm satisfies $|x| - |y| \leq |x - y| \leq |x| + |y|$. This property of p-adic topology is known as ultra-metricity.

p-Adic differential calculus exists and differentiation rules are same as for the real calculus. It is however not at all clear whether given real Taylor series with rational coefficients generalizes to its p-adic counterpart since the series need not converge p-adically. Exponential and trigonometric functions have p-adic counterparts but they do not have the properties of their real counterparts: for instance, p-adic trigonometric functions are not periodic. This is a problem when one tries to generalize Fourier analysis.

p-Adic integral calculus is problematic. The reason is that p-adic numbers are not well-ordered. As a consequence, the ordering crucial for Riemann integral does not exist. In fact, formal definition of Riemann integral gives as a limit vanishing integral. The generalization of Fourier analysis based on the integration of plane wave factors $exp(ikx)$ as roots of unity appearing in algebraic extension of p-adic numbers seems to be the only manner to overcome the problem. Algebraic continuation of integrals depending on parameters (such as integration limits) from real to p-adic context is in a central role in TGD framework but requires the fusion of reals and various p-adic number fields to bigger structure along common rationals: each number field would be like one page in a big book (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or bf Fig. **??** in the appendix of this book).

What does one mean with p-adic complex projective space?

The question is what one should do for the projective space $P^1(C)$ to get its p-adic counterpart? The basic condition is that A_5 acts transitively in the p-adic analog of $P^1(C)$.

1. The first guess would be the replacement of $P^1(C)$ with $P^1(Q_p)$. This is however the p-adic analog of real projective line, not complex projective line and one cannot imbed the complex matrices representing the action of the covering group of A_5 of $PGL(2, Q_p)$.
2. What one should do? The basic observation is that complex numbers C define the only possible algebraic extension of real numbers. Generalizing this, one should consider algebraic extension of Q_p. There is infinite number of these extensions and one must choose that of minimal algebraic dimensions. This means that the phases $exp(i\pi/5)$ (10: th root of unity), $exp(i\pi/3)$ (6: th root of unity), and $exp(i\pi/2) = i$ (4: th root of unity) must be contained by the extension. The reason why one must have $exp(i\pi/5)$ rather than $exp(2\pi/5)$ representing rotation of $2\pi/5$ generating the cyclic group Z_5 is due the fact that one has two fold covering. Same applies to other roots of unity. The solutions of equation $x^{60} = 1$ give the needed roots of unity since $60 = 6 \times 10 = 4 \times 3 \times 5$ contains all the needed roots of unity needed in the representation matrices.
   The extension of Q_p containing those roots of unity which do not reduce to -1 (existing p-adically) would define the extension used. One can calculate the algebraic dimension of this extension but certainly it is much larger than 2 as in the case of complex numbers. The extension - call it K - is not unique but is minimal. There is infinite number of extensions containing this extension.
   To define things precisely one must replace the notions of p-adic integer, prime, and rational p applying in K but this is a technicality. This means that p - the only prime in Q_p - is replaced with π, the only prime in K.

I will leave the detailed construction of the projective space $P^1(Q_p)$ later because it is rather technical procedure. Some comments are however in order:

1. For $p \ mod \ 4 = 1$ (say $p = 5$ or 17) $i \equiv \sqrt{-1}$ belongs to the p-adic number field. Therefore the dimension of algebraic extension is considerably smaller than for $p \ mod \ 4 = 3$ (say $p = 3$ or 7).
2. The naïve question is whether for $p \ mod \ 4 = 3$ a considerably simpler approach could make sense. Use 2-D algebraic extension of p-adic numbers consisting of numbers $x + iy$: call this space C_p. naïve non-specialist might think that in this case the rather intricate complex construction of the projective space $P^1(Q_p)$ based on Bruhat-Tits tree might not be needed. This simpler construction however fails for $p \ mod \ 4 = 1$. It fails also more generally. The reason is that the $exp(i\pi/n)$, $n = 3, 5$ are algebraic numbers and do not belong to C_p. Therefore one must extend C_p to included also the phase factors and it seems that one ends up to the same situation as in general case.
3. Side track to TGD.
   (a) In TGD one encounters the problem "What could be the p-adic counterpart of S^2 and $CP_2 = P^2(C)$?". The above general recipe applies to this problem: replace C with an algebraic extension K of Q_p allowing the embedding of some discrete subgroup of $SU(2)$ *resp.* $SU(3)$ represented as matrices in $PGL(2, K)$ *resp.* $PGL(3, K)$. The interpretation would be that due to finite measurement resolution the Lie group $SU(2)$ *resp.* $SU(3)$ is replaced with its discrete counterpart.

This has a direct connection to the inclusions of hyperfinite factors of type II_1 (HFF) [K99], where all discrete subgroups of $SU(2)$ appear also those of SU(3), whose interpretation is in terms of finite measurement resolution with included HFF creating states which cannot be distinguished from the original state in the resolution used. General inclusions correspond to discrete subgroups of rotation group and by McKay correspondence [A170] to Lie groups of ADE type. The isometry groups of Platonic solids are the only simple groups in this hierarchy and correspond to exceptional Lie groups E_6, E_7, E_8.

(b) One could criticize the approach since the algebraic extension K containing the isometry group is not unique. In TGD framework one however interprets the algebraic extensions in terms of finite measurement resolution. One cannot measure all possible angles p-adically-actually one cannot measure angles at all but only discrete set of phase factors coming as roots $exp(ik2\pi/n)$ of unity. The large the value of n, the better the measurement resolution.

What does one mean with p-adic icosahedron?

Once the projective space $P^1(K)$ generalizing $P^1(C) = S^2$ is constructed such that it allows the action of A_5 (it does not allow the action of entire rotation group!) one can identify the points which remain fixed by the action of various subgroups of A_5 (6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges). This is a purely algebraic procedure and there is no need to define what edges and faces are.

To obtain a more concrete picture about the situation one must define precisely what $P^1(Q)$ means and here the notion of Bruhat-Tits tree [A63] seems to be unavoidable.

13.3 Trying To Explain What $P^1(Q_P)$ Could Mean Technically

The naïve approach to the construction of $P^1(Q_p)$ would be following. Do the same things as in the case of $P^1(C)$ or $P^1(R)$. The point pairs (q_1, q_2) in Q_p^2 are identified with pairs $\lambda \times (q_1, q_2)$ where $\lambda \neq 0$ is p-adic number. For some reason this simple approach is not adopted in the article [A116]. The reason is that one cannot introduce the notion of Bruhat-Tits tree [A63] in this approach. Bruhat-Trits tree is needed to obtain path-connectedness - that is connect the fixed points of icosahedron to form a "solid" and to give a more geometric meaning to the notion of icosahedron. One can regard $P^1(Q_p)$ as boundary of Bruhat-Tits tree somewhat like sphere is a boundary of ball in real context.

I am not not sure whether this approach on $P^1(Q_p)$ is equivalent with that of Berkovich [A102] based on the idea of adding some points to $P^1(Q_p)$ to make it path connected space containing $P^1(Q_p)$ as a dense subset. The outcome has rather frightening complexity.

The alternative approach would be purely algebraic. I will discuss later the problem of introducing the counterpart of path connectedness without giving up p-adic topology and by introducing induced real topology as effective topology having the desired path-connectedness.

13.3.1 Generalization Of $P^1(C)$ Making Possible To Introduce Bruhat-Tits Tree

The following construction looks somewhat artificial but its purpose is to make possible the introduction of Bruhat-Tits tree allowing to realize path-connectedness.

1. The point pairs (q_1, q_2) Q_p^2 are replaced with Z_p lattices in Q_p^2. For given lattices the points are of form (n_1u, n_2v), where u and v are linearly independent (in Q_p) vectors of Q_p^2. Note that the p-adic integers $n_i = \sum_{k\geq 0} n_{i,k}p^k$ can be and typically are infinite as real integers. This is how the lattice differs from the real lattice. Also the p-adic distances between lattices points for which n_i differ by a large power of p are very small.
   Note: Q_p^2 is the p-adic analog of space of 2-spinors. The pairs (u, v) are indeed in 1-1 correspondence with pairs (q_1, q_2).

2. Projective equivalence is realized as for point pairs (q_1, q_2). This means that lattices for which base vectors (u, v) differ by a p-adic scaling are equivalent $(u, v) \equiv (\lambda u, \lambda v)$. Only the ratio u/v defining the "direction" of point of Q_p^2 matters.
Note: In the complex case one would have two complex vectors and their ratio defines the conformal equivalence class of the plane compactified to torus by identifying the opposite edges of the polygon defined by u/v.
Note: In the article one speaks about homothety classes: homothety means scaling which in p-adic context need not change p-adic norm.

This is not quite enough yet. Real icosahedron is in a well defined sense a connected coherent structure. Not just a collection of points. p-Adic topological is however totally disconnected. This suggests that one must introduce additional structure making possible to speak about icosahedron as "solid". Bruhat-Tits tree is one possible manner to achieve this. Also TGD inspired view about p-adic manifolds makes this possible.

13.3.2 Why Bruhat-Tits Tree?

One introduces Bruhat-Tits tree [A63] as an additional structure having $P^1(Q_p)$ as its boundary in a well-defined sense (one needs its counterpart also in $P^1(K)$). In [A116] it is stated that this relates to a proper *global* definition of p-adic analytic structure in terms of Berkovich disks. As already explained, the basic problem for introducing analytic manifold structure is the total disconnectedness of p-adic topology. In p-adic topology each point is open set and all p-adic open sets are also compact. Moreover, two p-adic balls are either disjoint or nested. Therefore one cannot have partially overlapping p-adic spheres and the basic construction recipe for real manifolds fails. One can overcome this problem for algebraic varieties defined by algebraic equations but they are much less general objects than manifolds in real context.

1. There are no problems in defining p-adic differential calculus (a *local* aspect of the analytic structure) and field equations associated with action principles make sense although the definition of action as integral is problematic. p-Adic differential equations are non-deterministic: integration constants are replaced by piecewise constant functions depending on finite number of pinary digits. This has a nice interpretation in TGD inspired consciousness, where this nondeterminism would be correlate for non-determinism of imagination - one aspect of cognition. Therefore I am not at all sure whether the reinforcement of real number based notions to p-adic context is a good idea.
2. p-Adic integration (a *global* aspect of the analytic structure) is the problem in p-adic calculus and the total disconnectedness relates to the absence of well-ordering. An obvious guess is that Bruhat-Tits tree could help in the definition of p-adic integral by defining the allowed integration paths.
Note: TGD approach on integration relies on algebraic continuation from real context and is based on what might ge regarded fusion of reals and p-adics along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book).
3. Intuitively the Bruhat-Tits tree builds up a "skeleton" connecting points by edges and thus curing the total disconnectedness. This requires some non-locality and the replacement of point pairs (q_1, q_2) with integer lattices spanned by q_1 and q_2 would introduce this non-locality.
4. In any case, what one obtains is a graph with vertices and edges. Vertices are identified as homothety classes $[M]$ of the lattices and are just the points of $P^1(Q_p)$. Two vertices $[M]$ and $[N]$ are connected by an edge iff one can find representatives M and N such that $pM \subset N \subset M$. The representative N is in some sense between pM and M. Note that one has $pM \equiv M$ by homothety so that the use of representatives in the definition is necessary.
The resulting graph is also a regular $p+1$-valent tree, the number of F_p-rational points of $P^1(F_p)$, which is projective space associated with finite field. One can check this in case of $p = 2$. The points (f_1, f_2) are $(1, 0), (1, 1), (0, 1), (1, 1)$ and by projective equivalence one has just $p = 1 + 2 = 3$ points in corresponding projective space. The transitive action of $Gl(2, K)$ means that all vertices are $p + 1$-valent and this fixes the structure of the graph completely. I will consider this point in more detail later on basis of the web article (see `http://tinyurl.com/ycjf6rv9`) [A63].

Bruhat-Tits tree can be seen as a skeleton of the "full" $P^1(K)$ containing also the additional points making it a path connected Berkovich space. The "naïve" $P^1(K)$ can be regarded as boundary of the Bruhat-Tits tree.

Bruhat-Tits tree looks very nice notion but there is objection against its construction in the proposed manner. Ordinary p-adic numbers- the simplest possible situation - are not in 1-1 correspondence with the Z_p lattices as will be demonstrated later but with powers of p. Same applies to Q_p^2 where the lattices correspond to $Sl(2,Z_p)$ equivalence classes of elements of Q_p^2. One can of course ask whether projective spaces are p-adically and maybe also physically very special for this reason.

13.3.3 Berkovich Disk

Bruhat-Tits tree is not enough for p-adicizing real topologist. Also Berkovic disk is required as the analog of open ball in real context. The slides (see `http://tinyurl.com/y8ftfs6z`) of Emmy Noether Lecture by Annette Werner [A102] give a concise representation of the basic idea behind Berkovich disk serving as a basic building brick of p-adic manifolds just like real n-disk does in the case of real n-manifolds and also also explains its construction. I must admit that I do not understand well enough the connection between Berkovich disk and Bruhat-Tits tree.

One can motivate the construction with the completion of rationals to reals. By adding all irrationals (algebraic numbers and transcendentals) one obtains reals and these additional numbers glue the rationals to form a continuum so that one can defined calculus and many other nice things. The idea is to mimic this construction.

1. In the example one restricts to the unit disk for an non-archimedean field assumed for simplicity be algebraically closed, which means algebraic completion containing all algebraic numbers considered also by Khrennikov. This notion is very formal and unpractical. The idea is to form a completion of the unit disk for a non-archimedean field K (algebraic extension of Q_p) containing thus K as a dense subset with the property that the resulting topology is path connected and not anymore ultra-metric (somewhat artificial!).
   For this purpose one constructs what is called the space of bounded multiplicative non-Archimedean norms for formal K-valued power series defined in the unit disk reducing to the norm of K for constant functions. It is possible to characterize rather explicitly this space and with topology defined by a pointwise convergence (point is now the K-valued function) of the norm one obtains uniquely path connected topology. The additional points can be said to glue the points of the K-disk to a continuum as its dense subset just as the addition of irrationals glues rationals to form a continuum.
2. The construction generalizes to the construction of the counterparts of p-adic projective spaces and symmetric spaces. Berkovich has also proposed an approach to p-adic integration and harmonic analysis relying on the notion of Berkovic space.
   Note: In TGD framework integration is defined by algebraic continuation in the structure defined by the fusion of real and various p-adic numbers fields and their extensions to form a book like structure. One could perhaps say that this fusion defines a kind of "super-completion": all possible completions of rationals are fused to single book like structure and rationals indeed defined a dense subset of this structure.

The construction is rather technical. From unit disk to a function space defined in it to the space of multiplicative seminorms defined in this function space! For the simple brain of physicist desperately crying for some concreteness this looks hopelessly complicated. Physicists would be happy in finding some concrete physical interpretation for all this.

13.3.4 Bruhat-Tits Tree Allows To "Connect" The Points Of P-Adic Icosahedron As A Point Set Of $P^1(K)$

The notion of p-adic icosahedron can be defined also in terms of Bruhat-Tits tree since the $PLG(2,K)$ acts transitively on the homothety class so that one obtains all homothety classes from the one associated with $(u,v)=(1,1)$ and one can speak about orbit of this basic homothety class. This means that one can connect the vertices, mid-points of edges, and barycenters of faces to common origin by edge paths in Bruhat-Tits tree and therefore to each other. This is what path-connectedness means.

How Bruhat-Tits tree allows to build from a set of totally disconnected fixed points a "solid" ? One answer is that the addition points of completion make this possible.

1. Bruhat-Tits tree allows to define what is called an end of the Bruhat-Tits tree as an equivalence class of infinite half line with two half lines identified if they differ by a finite number of edges. These ends are in one-one corresponds with the K-rational points of $P^1(K)$ (these are not the only points of $P^1(K)$). One can say that $P^1(K)$ represents the boundary of Bruhat-Tits tree as a p-adic manifold.
   Note: Could this finite number of different edges corresponds to a finite number of pinary digits appearing in p-adic integration "constants")? The identification could mean that all choices of pseudo constants in p-adic differential equations are regarded as equivalent. Physicist might speak about the analog of gauge invariance: the values of pseudo constants do not matter.
2. For a finite set of points of totally disconnected $P^1(K)$ there exists a unique minimal subtree of the entire Bruhat-Tits tree containing the points of this set as its ends [A116]. This subtree is what connects the points of this point set to a coherent structure in the set that one can construct paths connecting the points to single point. There are of course several ways to achieve this but one can define even the analog of the geodesic line as a path with a minimal number of edges so that it becomes possible to speak also about the edges of icosahedron. The length of the geodesic could be simply the number of edges for this minimal edge path.
3. The p-adic counterpart of Platonic solid must be also "solid". This is achieved if the fixed points for the subgroups of the isometry group of Platonic solid (in particular for those of the A_5) defining the Platonic are identified as ends of a unique minimal subtree of Bruhat-Tits tree.

For higher-dimensional projective spaces $P^n(K)$ Bruhat-Tits tree generalizes from 1-D discrete homogenous space $PGl(2,K)/Gl2,Z_K)$ to n-dimensional discrete homogenous space. The reason is that the edges of tree develop higher-dimensional cycles having interpretation as simplexes. One can also define homology groups for this structure. Also now $P^n(K)$ can be regarded as a boundary of the resulting structure.

13.4 Algebraic Universality In TGD Framework

In TGD framework the algebraic approach looks very promising one - at the first glance perhaps even the only possible one - since under some assumptions the field equations for preferred extremals [K14, K100] reduce to purely algebraic ones and do not even refer to action principle explicitly. The point is that the preferred extremal property means a generalization of complex structure to 4-D situation and is a notion independent of action and the preferred extremals are solutions to field equations of very many general coordinate invariant variational principles (Einstein-Maxwell equations with cosmological term and minimal surface equations hold true). p-Adic variants of these conditions are purely algebraic and make sense so that one can hope that even space-time surfaces might have p-adic counterparts. The underlying assumptons can be questioned and I have indeed done this but it seems that this idea is not promising.

As already noticed, one can consider a compromise between topological and algebraic approach to the definition of p-adic manifolds by using a variant of canonical identification to map rational points of the p-adic preferred extremal to rational points of its real counterpart and completing this skeleton to a preferred extremal in the real context [K101]. This mapping need not be one-to-one. In the intersection of real and p-adic worlds the expression for real preferred extremal makes sense also in p-adic number field, and a direct identification makes sense and is unique.

In the real sector the preferred extremal property would boil down to to the existence of complex structure in Euclidian regions and what I call Hamilton-Jacobi structure in Minkowskian regions. Also the conjecture that preferred extremals are quaternionic surfaces in certain sense [K86] implies independence on action principle. The challenge is to prove that these two algebraic characterizations of preferred extremals are equivalent. These two purely algebraic conditions might make sense also in p-adic context with complex and hypercomplex numbers replaced with appropriate algebraic extensions of p-adic numbers.

The p-adicization program based on the notion of algebraic continuation involves many open questions to be discussed first.

13.4.1 Should One P-Adicize Entire Space-Time Surfaces Or Restrict The P-Adicization To Partonic 2-Surfaces And Boundaries Of String World Sheets?

One of the many open questions concerns the objects for which one should be able to find p-adic counterparts. The arguments based on canonical identification and universality of the preferred extremal property support the view that p-adicization can be carried out at 4-D level for space-time surfaces and also at the level of WCW . Later a detailed proposal for how p-adic preferred extremals can be mapped to real preferred extremals with the uniqueness of this correspondence restricted by the finite measurement resolution realized as pinary cutoff will be described.

One can however consider also an alternative approach in which one restricts the p-adicization to 3- or even 2-dimensional objects of some special classes of these objects and this possibility is discussed below.

1. Should one p-adicize only boundaries?

A grave objection against p-adicizing only partonic 2-surfaces and braid strands is that one loses the very powerful constraints provided by the preferred extremal property and coordinate maps defined by the canonical identification in preferred coordinates. Therefore the algebraic continuation of the partonic 2-surface can become highly non-unique ($x^n + y^n = z^n$, $n > 2$, is the basic counter example: in higher dimensions one expects that this kind of situations are very rare!). Furthermore, the restriction to partonic 2-surfaces and braid strands is artificial since embedding space must be p-adicized in any case. The replacement of the p-adicization of the partonic surface plus 4-D tangent space data with that of the preferred extremal containing it increases the number of constraints dramatically so that holography might even make the p-adicization unique.

Despite this objection one can try to invent arguments for restricting the p-adicization to some subset of objects since this would simplify the situation enormously.

1. The basic underlying idea of homology theory is that the boundary of a boundary is empty. p-Adic manifolds in turn have no boundaries because of the properties of p-adic topology. Should p-adicization in TGD framework be carried only for boundaries? Light-like 3-surfaces define boundaries between Minkowskian and Euclidian regions of space-time surface. The space-like 3-surfaces defining the ends of space-time surfaces at the boundaries of CD are boundaries. Also 2-D partonic surfaces and boundaries of string world sheets can be considered. One must consider also the boundaries of string world sheets as this kind of objects.
2. Strong form of General Coordinate Invariance implies strong form of holography. Either the data at light-like 3-surfaces (at which the signature of induced metric changes) or space-like 3-surfaces at the ends of CD codes for physics, which implies that partonic 2-surfaces and 4-D tangent space data at them code for physics.
   What 2-D tangent space data could include? The tangent space data are dictated partially by the weak form of electric magnetic duality [K23] stating that the electric component of the induced Kähler field component is proportional to its magnetic component at light-like 3-surfaces. Also the boundaries of string world sheets contribute to 4-D tangent space data and at the end of braid strands at partonic 2-surfaces both light-like and space-like direction are involved.

If space-time interior is not p-adicized (somewhat un-natural option), the p-adicization reduces to the algebraic continuation of Kähler function and Morse function to p-adic sectors of WCW . Both functions reduce to 3-D Chern-Simons terms for selected 3-surfaces. p-Adicization should reduce to algebraic continuation of various geometric parameters appearing as arguments of Kähler action.

In the minimal situation only partonic 2-surfaces and the boundaries of string world sheets - briefly braid strands - need to be p-adicized and the existing results - such as the results of Mumford derived from the existence of p-adic uniformization - could give powerful contraints. One can also ask whether the p-adic string world sheet in some sense is equivalent with the generalization of Bruhat-Tits tree allowing also loops.

Besides the string world sheet boundary and partonic 2-surface also for "4-D tangent space data" fixed at least partially by weak form of electric magnetic duality and string world sheets is needed. There are several open questions.

1. Does weak form of electric-magnetic duality have any meaning if one cannot speak about space-time interior in p-adic sense? This condition would apply only at partonic 2-surfaces. Same question applies in the case of braid strands. Can one effectively reduce space-time interior and string world sheet to their tangent spaces at partonic 2-surface/braid strands.
2. It is not even clear whether the dynamics of light-like 3-surfaces and space-like 3-surfaces is deterministic. Strong form of holography requires either determinism or non-determinism realized as gauge invariance, which could correspond to Kac-Moody type symmetries. Kac-Moody symmetry would favor the idea that p-adicization takes place only for partonic 2-surfaces and for the braid strands. Gauge symmetry would also give hopes that the integral of Chern-Simons term depends only on the data at the end points of braid strands at partonic 2-surfaces and maybe on data at braid strands: this would however require p-adic integration not possible in purely p-adic context. These data should remain invariant under Kac-Moody symmetries.
3. Should one p-adicize the weak form of electric magnetic duality? The duality involves the dual of Kähler form of the partonic surface with respect to the induced four-metric: the normal component of Kähler electric field at partonic surface and/or at string world sheet boundary equals to Kähler magnetic form at the partonic surface at particular point of its orbit (most naturally light-like curve). The induced 4-metric becomes degenerate at the light-like 4-surface and the component of electric field is finite only if weak form of electric-magnetic duality can be satisfied. Should the duality hold true for entire 3-surfaces, for partonic 2-surfaces, or perhaps only for for the braid strands? The purpose of the condition is to guarantee that Kähler electric charge as eletric flux is proportional to Kähler magnetic charge: therefore it should hold along entire 3-surfaces and if these are regarded as real surfaces there are no problems with the p-adicization of the condition.

2. What kind of algebraic 2-surfaces can have p-adic counterparts?

There is no need for a generic algebraic surface to have direct algebric p-adic counterpart for all p-adic primes. If one uses as preferred coordinates a subset of preferred coordinates of the embedding space and accepts only embedding space isometries as general coordinate transformations, the algebraic surfaces in the intersection of real and p-adic worlds must satisfy very strong conditions. For instance, a representation in terms of polynomials cannot involve real transcendentals. Even rational coefficients can force algebraic extension of Q_p, when the remaining embedding space coordinates are expressed in terms of the coordinates of the partonic two-surface.

Mumford (see `http://tinyurl.com/yab92ab2`) is one of the pioneers of p-adicization of the algebraic geometry and has demonstrated that only a restricted set of p-adic algebraic surfaces allow interpretation as p-adic Riemann surfaces if one requires that a generalization of so called uniformization theorem (see `http://tinyurl.com/yba58b64`) holds true for them [A89]. This theorem says that Riemann surfaces are constructible as factor spaces of either sphere, complex plane, or complex upper plane (hyperbolic space H^2 with the subgroup Γ identified as the finitely generated free subgroup of the isometries of the space in question. The construction does not work for all algebraic surfaces but only for the surfaces satisfying certain additional conditions. This is not a problem in TGD framework in the intersection of real and p-adic worlds since the p-adicization is not expected to be possible always but only in the intersection of real and p-adic worlds.

According to the article "Multiloop Calculations in p-Adic String Theory and Bruhat-Tits Trees" (see `http://tinyurl.com/ycwdx4n3`) by Chekhov *et al* [A136] the construction of higher genus Riemann surfaces as so called Mumford surfaces takes place by starting from Bruhat-Tits tree representing $g = 0$ surface and by taking subgraphs having interpretation as representations for an orbit of so called Schottky group (see `http://tinyurl.com/ydexpexl`) characterizing the higher genus Riemann surface and gluing these graphs together by transversal connections. This indeed represents the genus homologically as a loop of the resulting tree.

Note: The article of Chekhov *et al* describes a proposal for the construction of complex scattering amplitudes for p-adic strings in real embedding space so that the situation is not relevant for TGD as such. The amplitudes are constructed in terms of p-adic characteristics and this means that the amplitudes can be interpreted also as numbers in p-adic number fields extended by roots of unity. The characteristics $q = exp(i2pi\tau)$ exist only for the values of q which are of form $q = p^n exp(x)exp(i2\pi/m)$, $|x| < 1$ so that discretization of the p-adic norm and phase of τ is necessary.

3. Should one really restrict the p-adicization to algebraic surfaces?

One could also consider the possibility of restricting p-adicization to algebraic surfaces (they could be also 4-D). Practicing physicist would argue that the restriction of p-adicization to algebraic surfaces is quite too heavy an idealization. In the real world spheres are topological rather than algebraic.

Luckily, if the construction recipe for p-adic manifolds to be discussed later really works, canonical identification with pinary cutoff allows to generalize p-adic algebraic surfaces to p-adic manifolds, and to achieve very close correspondence with the real manifold theory. Given real preferred extremal can correspond to not necessarily unique p-adic preferred extremal for some values of p. Also two p-adic preferred extremals with different values of p-adic prime which correspond to the same real preferred extremal correspond to each other. This provides an elegant solution to all problems discussed hitherto and there is not need to restrict the p-adicization in any manner.

Finite measurement resolution would be a prerequisite for algebraic continuation in the sense that subset of rational and algebraic points defined by pinary cutoff and algebraic extension would be common to the real and p-adic preferred extremals. Therefore finite measurement resolution would make it possible to realize both number theoretical universality and p-adic manifold topology.

13.4.2 Should One P-Adicize At The Level Of WCW ?

One can of course challenge the idea about p-adicization at the level of WCW and WCW spinor fields and ask what this procedure gives. One motivation for the p-adicization would be p-adic thermodynamics. p-Adic thermodynamics should emerge at the level of M-matrix which indeed can be regarded as a "complex square root" of hermitian density matrix in zero energy ontology and therefore expressible as a product of hermitian square root of density matrix and unitary S-matrix. Hence it would seem that the p-adicization at the level of WCW is natural and the representability as a union of symmetric spaces constructible as factor groups of symplectic group of $\delta M^4_\pm \times CP_2$ gives hopes that algebraic approach works also in infinite-dimensional case. Finite measurement resolution and the properties of hyper-finite factors of type II_1 are expecetd to reduce the situation to finite-dimensional case effectively.

13.4.3 Possible Problems Of P-Adicization

The best manner to clarify one's thoughts is to invent all possible objections and in the following I do my best in this respect. The basic point is following. If one accepts the purely algebraic approach without no reference to canonical identification, one must check that everything in TGD - as I recently understand it - can be expressed without inequalities! Boundaries are defined by inequalities and one must check that they can be avoided. If this is not the case, the notion of p-adic manifold relying on the notion of canonical identification seems to remain the only manner to avoid problems.

Wormhole throats are causal rather than topological boundaries

The notion of boundary does not have any counterpart in purely p-adic context since its definition involves inequalities. The original vision was that space-time sheets possess boundaries and the boundaries carry quantum numbers - in particular family replication phenomenon for fermions would have explanation in terms of the genus of 2-dimensional boundary component of 3-surface [K21]. It however turned out that boundary conditions require that the space-time sheet approaches vacuum extremals at boundary and this does not seem to make sense. This led to the view that one must allow only closed space-time "sheets" which can be thought of as being obtained by gluing real space-time sheets together along boundaries.

Also the notion of elementary particle involves preferred extremals - massless extremals in the simplified model [?] connected by wormhole contact structure defining the elementary particle. These preferred extremals must combine to form a closed space-time surface and this is quite possible: the minimal situation corresponds to two space-time sheets glued together as in the model of elementary particles.

Genuine boundaries are replaced by the light-like 3-surfaces -orbits of wormhole throats - at which the signature of the induced metric changes from Minkowskian to Euclidian and four-metric degenerates effectively to 3-D metric locally. These can be defined by purely algebraic conditions and there is no need for inequalities.

Partonic 2-surfaces are identified as intersections of the space-like 3-surfaces at the ends of CD: the ends of CD are defined by purely algebraic equation $t^2 - r^2 = 0$ and $(t-T)^2 - r^2 = 0$ and once the equations of space-time surface are known one can solve the equations for space-like 3-surfaces. The equations defining what light-like 3-surfaces at which the induced four-metric is degenerate are algebraic and express just the degeneracy of the induced four-metric. The condition that algebraic equations for light-like 3-surfaces and space-like 3-surfaces hold true simultaneously define partonic 2-surfaces. Hence it seems that the surfaces can be expressed algebraically.

This approach might look a little bit artificial. Also the idea that only boundaries should be p-adicized should be p-adicized looks artificial. The best looking option is the use of canonical identification to define p-adic manifolds since it allows to transfer real topological notions to the p-adic context. In particular, the well-ordering of reals induces that of p-adics so that inequalities cease to be a problem and boundaries can be defined.

What about the notion of causal diamond and Minkowski causality?

A possible problem for purely p-adic approach allowing no in-equalities is caused by the notion of causal diamond (CD) defined as intersection of future and past directed light cones (as a matter fact, CP_2 is included to CD as Cartesian factor but I do not bother to mention it again and again). CD has light-like boundaries.

It is not quite clear whether space-time surface must be always localized inside CD. The notion of generalized Feynman diagram indeed suggests that the space-time surfaces can continue also outside the CDs and that CD could be seen as an embedding space correlate for what might be called spot-light of consciousness. If this were the case quite generally, the p-adicization of space-time sheets would not produce problems even if one does not use canonical identification.

In purely p-adic context, one should however give some meaning for the statement that space-time surface is contained inside CD and this seems to require the notion of boundary for CD. Does this notion of CD make sense in the p-adic context or is the fusion of real and p-adic number fields along common rationals required? The resolution of the problem seems to require the fusion. In the case of algebraic extensions also common algebraics are present.

The first questions concern the notion of Minkowski causality, which relies on light-cone and its complement expressed in terms of inequalities.

1. The first reason of worry is that in purely p-adic context also the equation $t^2 + r^2 = 0$ has a lot of solutions! The reason is that the notion of positive and negative do not make sense for p-adic numbers without some constraints. If one restricts the p-adic numbers to those having finite number of pinary digits - this happens always when one has finite pinary resolution - all p-adic numbers included rationals reduces to finite positive integers as real numbers. Therefore in finite pinary resolution the problems disappear. The condition that rationals points of Minkowski space are common with its p-adic variant, makes finite pinary resolution natural, and one could say that all p-adic numbers - including negatives of finite integers - can be said to be infinitely large positive integers in real sense. Here one must of course be very cautious.
2. The condition $s = t^2 - r^2 < 0$ for the complement of future light-cone has no meaning in the p-adic context for general p-adic numbers. If rational values of Minkowski coordinates correspond to same point in real and p-adic sense, finite pinary resolution means that all pinary cutoffs have $s \geq 0$ and $t \geq 0$ in real sense. This is also true for $a = \sqrt{t^2 - r^2}$ so that one remains inside future light-cone unavoidably. Anything outside future light-cone is unexpressible in finite measurement resolution p-adically.

Finite temporal and spatial resolution suggest integer quantization of t and r in suitable units and one could say whether s has finite of infinite number of pinary digits - that is are positive or negative. Finite real integer values of t and r have finite number of pinary digits. Their negatives have infinite number of pinary digits and one could argue they correspond to infinite future if they are interpreted as real numbers. The values of s in future light-cone have finite number of pinary digits and correspond to finite real values. Outsider the future light cone the values of s are negative in real sense and have infinite number of pinary digits and thus interpreted as real numbers are in future infinity.
One can consider also rational values of t and s if one keeps also p-adically track that rational is in question. Rationality means that pinary expansion is periodic after some pinary digit. Therefore it would seem to be possible to distinguish between $s \geq 0$ and $s \leq 0$ also p-adically for finite measurement resolution purely algebraically.

3. Causal diamond is defined as the intersection of future and past directed light cones. The lower light-cone in the intersection decomposes to pieces of hyperplanes $t \geq 0$ with $r \leq t$ and upper light-cone to pieces $T - t \geq 0$, $r \leq T - t$. If these variables are quantized as integer multiples of suitable unit and if these integer multiples can be interpreted in both real and p-adic sense, there is no need for inequalities in p-adic context. Also now rational values can be allowed.

If only boundaries are p-adicized, p-adicization would apply only to the light-like boundaries of CDs, and one would avoid possible problems related the sign of $s = t^2 - r^2$. This would conform with the strong form of holography and allow p-adicization of WCW .

Again one might argue that the number theoretical game above is artificial. The safest alternative seems to be canonical identification with pinary cutoff used to map real preferred extremal to its p-adic counterpart.

Definition of integrals as the basic technical problem

Physicist wants to perform integrals, and the problems related to the notion of integral is what any novice of p-adic physics is doomed to encounter sooner or later. As will be described the definition of p-adic manifold based on canonical identification solves these problems by inducing real integration to the p-adic realm by algebraic continuation.

Before continuing about integration it is however good to summarize the general TGD based view about the relationship between real and p-adic worlds.

1. Intersection of real and p-adic worlds as key concept

In TGD framework the basic notion is the intersection of real and p-adic worlds generalizing the idea that rationals are common to reals and p-adics. Algebraic continuation between real and p-adic worlds takes place through this intersection, in which real formulas allow interpretation as p-adic ones. The notions of intersection and algebraic continuation apply both at space-time level and WCW level.

1. At the space-time level rational (and even some algebraic) points of real surfaces are contained by p-adic surfaces. One can identify these rationals and say that real and p-adic surfaces intersect at these points and define discrete cognitive representation. Among other things this would explain why numerics is necessarily discrete and possible only using rationals with cutoff.
2. One can abstract this idea to the level of WCW . Instead of number fields one considers surfaces (partonic 2-surfaces, 3-surfaces, or space-time surfaces) in various number fields. If the representation of the surface (say in terms of rational functions) makes sense both for reals and p-adic number field in question, one can identify the real and p-adic variants of surfaces. These surfaces can be said to belong to the intersection of real and p-adic worlds (worlds of classical worlds, to be more precise). In TGD inspired theory of consciousness one would say that they belong to the intersection of material/sensory world and the world of cognition. In TGD inspired quantum biology life is identified as something residing in the intersection of realities and p-adicities.

2. Algebraic continuation as a basic tool

With this philosophical background one an consider the algebraic continuation of real integrals from the intersection of real and p-adic worlds defined by surfaces, whose representations in preferred coordinates make sense in real number field and in the p-adic number field to which one wants to continue.

1. Harmonic analysis in coset spaces with discretization defined by the algebraic extension of Q_p might make possible to avoid the problems by reducing the integrals to sums over the discrete points of the coset space. Algebraic continuation is of course central element in the program.
2. The recent progress in the calculation of planar scattering amplitudes in $\mathcal{N}=4$ SYMs gives hopes that M-matrix could be defined in number theoretically universal manner. The reason is that in TGD framework the fermions defining building bricks of elementary particles are massless - a basic prerequisite for the twistor approach - also when they appear as virtual particles. This gives enormously powerful kinematical constraints reducing the number of diagrams dramatically, and allows to express amplitude in terms of on-mass shell amplitudes just as one does in the twistor Grassmannian approach.
   For $\mathcal{N}=4$ SYM (and also more general theories) planar Feynman diagrams boil down to integrals over Grassmannians, which are coset spaces associated with $Gl(n,C)/Gl(n-m,C)\times Gl(m,C)$ allowing the already described generalization to p-adic context. The integrals reduce to multiple residue integrals, which could make sense also in the p-adic context because of the very weak dependence on integration region. The algebraic continuation of the resulting amplitudes to p-adic context replacing C with an appropriate extension of p-adic numbers might well make sense.

3. Two problems as solutions of each other

Unfortunately, the algebraic continuation of integrals is not free of technical problems. Even in the case of rational functions the algebraic continuation of the real integrals is susceptible to p-adic existence problems.

1. The basic problem with definition of ordinary 1-D integrals of rational functions is that the integral function of $1/x$ is $log(x)$ rather than rational function as for other powers. Unless the limits are very special (of form $x=1+O(p)$), the algebraic continuation requires infinite-dimensional extension of p-adic numbers containing all powers of $log(x)$ for some $1\leq x<p$. Can one allow infinite-D extensions, which are not algebraic?
2. The appearance of 2π in residue integral formulas which could otherwise make sense in p-adic context provides a second reason for worries: should one also transcendental extension containing powers of 2π?

Often two quite unrelated looking problems turn out to have a common solution. Now the second problem is purely physical: why a given particle should correspond to a particular p-adic prime? At this moment one must be satisfied with the p-adic length scale hypothesis stating that these primes are near powers of 2 and Mersenne primes are favored. I have not been able to identify any convincing dynamical principle explaining why primes near powers of two seem to be favored. It deserves however to be mentioned that the preferred p-adic length scale as a fixed point of p-adic coupling constant evolution (discrete) is one possible explanation meaning vanishing of beta functions, something very natural taking into account the quantum criticality of TGD Universe.

Could this problem define the solution of the first problem and vice versa! Maybe one must just accept that algebraic continuation to given p-adic number field is not always possible!

1. This criterion could strongly constrain the p-adic primes assignable to a given elementary particle. Consider as an example Kähler function defined as Kähler action for Euclidian portion of space-time (generalized Feynman graph) and Morse function defined as Kähler action for Minkowskian portion of space-time. The existence of the p-adic variant of Kähler function (or its real exponent) and Morse function (or its imaginary exponent) would allow to assign to a given space-time surface a highly restricted set of p-adic primes, and the allowed quantum superpositions of space-time surfaces could contain only those for which at least one of the allowed primes is same.
2. For massless particles Kähler action would vanish and algebraic continuation of Kähler action would be possible to all p-adic primes in accordance with the scale invariance of massless particles. Also the breaking of scale invariance and conformal invariance meaning selection

of a particular p-adic length scale could be basically a number theoretical phenomenon. This would provide a totally new approach to the mystery of mass scales which in standard model framework requires fine tuning of Higgs mass with a totally unrealistic accuracy (one must avoid both the Landau pole meaning infinite self-coupling of Higgs and vacuum instability preventing massivation by Higgs vacuum expectation).

3. For instance, a function of form $log(m/n)$ can be algebraically continued only to those p-adic number fields for which m and n have form $m = k + O(p)$, and $n = k + O(p)$, $0 < k < p$ so that one has $m/n = 1 + O(p)$. The exponent of Kähler function in turn can be continued to Q_p if it is proportional to power of corresponding prime p. The exponential decay of Kähler function would have p-adic counterpart as decay of p-adic norm (just like Boltzmann weight $exp(-E/T)$ corresponds to p^n in thermodynamics). This could partially answer the question why the space-time surfaces assignable to electron seem correspond to Mersenne prime $M_{127} = 2^{127} - 1$ as suggested by p-adic mass calculations.
4. Number theoretic criterion might also mean that the p-adic prime characterizing particle state is extremely sensitive to the details of the particle state in real sense. The point is that a small modification of rational number in real sense changes its prime decomposition dramatically! Number theoretic anatomy is not continuous in real sense! An extremely small symmetry breaking in real sense modifying the value of Kähler function as function of quantum numbers might modify the value of the p-adic prime dramatically by affecting profoundly the number theoretic anatomy of some rational parameter appearing in the formula for Kähler function. For instance, in the standard framework it is very difficult to imagine any breaking for the SUSY assignable to right-handed neutrinos since they interact only gravitationally. The addition of right handed neutrino transforming particle to sparticle might however modify the p-adic prime (and thus mass scale) assigned to the particle dramatically.

4. What should one achieve?

It is a long way from this heuristic number theoretic vision to the calculation of p-adic valued integrals at space-time level, say to a formula for the p-adic action integral defined by Kähler action density (if needed at all).

1. The reduction to integral of Abelian Chern-Simons form over preferred 3-surfaces would be the first step and the definition of p-adic integral of Chern-Simons form second step. The special properties of preferred extremals give hopes about the reduction of the value of the Kähler action to local data given at discrete points at partonic 2-surfaces. The braid picture for many-fermion states forced by the Kähler-Dirac equation [K100] and motivated by the notion of finite measurement resolution having discretization as a space-time correlate, suggests a reduction of real action integral to a sum of contributions from the ends of braid strands defining the boundaries of string world sheets. The optimistic hope would be that this data allows a continuation to the p-adic realm.
   Note: This kind of reduction might be quite too strong a conditon. All that is required in the approach based on canonical identification is that the values of Kähler function and Morse function exist in the given p-adic number field or its algebraic extension.
2. p-Adic valued functional integral is unavoidable at the level of WCW .
   (a) Algebraic continuation in the framework provided by the fusion of reals and various p-adic number fields looks the only reasonable approach to the p-adic functional integral.
   (b) Second element is Fourier/harmonic analysis in symmetric spaces: WCW is indeed a union of infinite-dimensional symmetric spaces over zero modes which do not contribute to WCW metric. One can hope that one can define the symmetric spaces algebraically in terms of their maximal symmetries since the metric reduces to that in single point of the symmetric space.
   (c) Canonical identification is the third element: p-adic functional integral for given p should be real functional integral restricted to preferred extremals allowing canonical identification map to the p-adic preferred extremal for that value of p. This would mean that real functional integral decomposes into a sum of contributions labelled by p-adic number fields and their algebraic extensions. This decomposition would be analogous to the formula obtained as a logarithm of the adelic formula for the rational as the inverse of the product of its p-adic norms.

Do the topological invariants of real topology make sense in the p-adic context?

In p-adic context the direct construction of topological invariants is not possible. For instance, the homology theory formulated in terms of simplexes fails since the very notion of simplex requires inequalities and well-ordering of the number system to define orientation for the simplex.

Also the notion of boundary is lacking since p-adic sets do not possess boundaries in topological sense. There however exists refined theories of p-adic homology allowing to circumvent this difficulty and the problem is that there are too many theories of this kind. A single universal theory would be needed and this was the dream of Grothendieck.

p-Adic mass calculations assume that the genus of the partonic 2-surface makes sense also in the p-adic context. For algebraic varieties the genus can be defined algebraically. There should be no problems if the partonic 2-surfaces are defined by algebraic equations which make sense for both reals and p-adic numbers. This is true for polynomial equations with rational coefficients and for algebraic extensions with coefficients in algebraic extension. By continuity algebraic continuation should allow to extend the notion of genus to surfaces for which rational coefficients are replaced with general p-adic numbers.

One expects that also more refined topological invariants making sense in the real context make sense also p-adically for algebraic varieties. A possible objection is that in the case of 3-manifolds allowing hyperbolic geometry (constant sectional curvatures) the volume of 3-manifold serves as a topological invariant. Volume is defined as an integral but in purely p-adic context volume integral is ill-defined. Is this a reason for worries? Hyperbolic n-manifolds have purely group theoretic formulation as coset spaces H^n/Γ, where Γ is discrete subgroup of the isometry group $SO(1,n)$ of n-dimensional hyperboloid H^n of $n+1$-D Minkowski space satisfying some additional conditions. Maybe this could allow to overcome the problem.

If canonical identification is used to map real preferred extremals to p-adic ones, boundaries and real topological invariants are mapped to p-adic ones both by algebraic continuation and in topological sense within finite measurement resolution. This even in the case that the real surface is not algebraic surfaces. This applies also to conformal moduli of the partonic 2-surfaces, whose p-adic variants play a key role in p-adic mass calculations.

What about p-adic symmetries?

A further objection relates to symmetries. It has become already clear that discrete subgroups of Lie-groups of symmetries cannot be realized p-adically without introducing algebraic extensions of p-adics making it possible to represent the p-adic counterparts of real group elements. Therefore symmetry breaking is unavoidable in p-adic context: one can speak only about realization of discrete sub-groups for the direct generalizations of real symmetry groups. The interpretation for the symmetry breaking is in terms of discretization serving as a correlate for finite measurement resolution reflecting itself also at the level of symmetries.

1. Definition of p-adic Lie groups

The above observation has led to TGD inspired proposal for the realization of the p-adic counterparts symmetric spaces resembling the construction of $P^1(K)$ in many respects but also differing from it.

1. For TGD option one considers a discrete subgroup G_0 of the isometry group G making sense both in real context and for extension of p-adic numbers. One combines G_0 with a p-adic counterpart of Lie group G_p obtained by exponentiating the Lie algebra by using p-adic parameters t_i in the exponentiation $exp(t_iT_i)$.
2. One obtains actually an inclusion hierarchy of p-adic Lie groups. The levels of the hierarchy are labelled by the maximum p-adic norms $|t_i|_p = p^{-n_i}$, $n_i \geq 1$ and in the special case $n_i = n$ - strongly suggested by group invariance - one can write $G_{p,1} \supset G_{p,2} \subset ...G_{p,n}....$ $G_{p,i}$ defines the p-adic counterpart of the continuous group which gets the smaller the larger the value of n is. The discrete group cannot be obtained as a p-adic exponential (although it can be obtained as real exponential), and one can say that group decomposes to a union of disconnected parts corresponding to the products of discrete group elements with $G_{p,n}$.
   This decomposition to totally uncorrelated disjoint parts is of course worrying from the point of view of algebraic continuation. The construction of p-adic manifolds by using canonical

identification to define coordinate charts as real ones allows a correspondence between p-adic and real groups and also allows to glue together the images of the disjoint regions at real side: this induces gluing at p-adic side. The procedure will be discussed later in more detail.

3. A little technicality is needed. The usual Lie-algebra exponential in the matrix representation contains an imaginary unit. For $p\ mod\ 4 = 3$ this imaginary unit can be introduced as a unit in the algebraic extension. For $p\ mod\ 4 = 1$ it can be realized as an algebraic number. It however seems that imaginary unit or its p-adic analog should belong to an algebraic extension of p-adic numbers. The group parameters for algebraic extension of p-adic numbers belong to the algebraic extension. If the algebraic extension contains non-trivial roots of unity $U_{m,n} = exp(i2\pi m/n)$, the differences $U_{m,n} - U^*_{m,n}$ are proportional to imaginary unit as real numbers and one can replace imaginary unit in the exponential with $U_{m,n} - U^*_{m,n}$. In real context this means only a rescaling of the Lie algebra generator and Planck constant by a factor $(2sin(2\pi m/n))^{-1}$. A natural imaginary unit is defined in terms of U_{1,p^n}.

4. This construction is expected to generalize to the case of coset spaces and give rise to a coset space G/H identified as the union of discrete coset spaces associated with the elements of the coset G_0/H_0 making sense also in the real context. These are obtained by multiplying the element of $G_/H_0$ by the p-adic factor space $G_{p,n}/H_{p,n}$.

One has two hierarchies corresponding to the hierarchy of discrete subgroups of G_0 requiring each some minimal algebraic extension of p-adic numbers and to the hierarchy of G_p: s defined by the powers of p. These two hierarchies can be assigned to angles (actually phases coming as roots of unity) and p-adic length scales in the space of group parameters.

2. Does the hierarchy of Planck constants emerge p-adically?

The Lie algebra of the rotation group spanned by the generators L_x, L_y, L_z provides a good example of the situation and leads to the question whether the hierarchy of Planck constants [K32] could be understood p-adically.

1. Ordinary commutation relations are $[L_x, L_y] = i\hbar L_z$. For the hierarchy of Lie groups it is convenient to extend the algebra by introducing the generators $L_i^{n)} = p^n L_i$ and one obtains $[L_x^{m)}, L_y^{n)}] = i\hbar L_z^{m+n)}$. This resembles the commutation relations of Kac-Moody algebra structurally. Since p-adic integers one the replacement of $\hbar = p^k \rightarrow np^k$, $n\ mod\ p =\neq 0$ produces same Lie-algebra.

2. For the generators of Lie-algebra generated by $L_i^{m)}$ one has $[L_x^{m)}, L_y^{m)}] = ip^m\hbar L_z^{m)}$. One can say that Planck constant is scaled from $\hbar$ to $p^m\hbar$. It is important to realize that $\hbar_{eff} = mp^k\hbar$ for $m\ mod\ p\ \neq 0$ (p-adic unit property) is equivalent with $\hbar_{eff} = p^k\hbar$ in the sense that p-adically the resulting Lie-algebras are same.

3. The earlier proposal assigns the origin of the effective hierarchy of Planck constants $\hbar_{eff} = n\hbar$ to n-furcations of space-time sheets. Recall that n-furcations are assigned with the non-determinism of Kähler action. In n-furcation the solution becomes n-valued meaning the presence of n alternative branches in the usual interpretation. The proposal is that a space-time counterpart of second quantization occurs. Single branch is in the role of single particle state and "classically" the only possible one. "Quantally" also m-branch states, $1 \leq m \leq n$, are allowed. This makes sense in zero energy ontology if the branching occurs either at the space-like ends of the space-time surface inside CD or at light-like wormhole throats. Otherwise one has problem with conservation laws allowing only single branch. The Kähler action for m-branch state would be roughly m times that for single branch states as a sum of the Kähler actions for branches so that one would have $\hbar_{eff} = m\hbar$. This prediction is inconsistent with p-adic Lie-algebra prediction unless $m = p^k$ holds true.

Can these two views about the effective hierarchy of Planck constants be consistent with each other? The connection between p-adic length scale hierarchy and hierarchy of Planck constants has been conjectured already earlier but the recent form of the conjecture is the most quantitive one found hitherto.

1. It a connection exists, it could be due to a relationship between the inherent non-determinism of Kähler action and the generic p-adic non-determinism of differential equations. Skeptic could of course counter-argue that in p-adic context both non-determinisms are present. One

can however argue that by the condition that p-adic space-time sheets are maps of real ones and vice versa, these non-determinisms must be equivalent for preferred extremals.

2. Also p-adic non-determinism induces multi-furcations of preferred extremals. These two kinds of multi-furcations should be consistent with each other. Also in p-adic context one can consider "second quantization" allowing simultaneously several branches of multi-furcation. Suppose that the p-adic non-determinism is characterized by integration pseudo-constants (functions with vanishing derivatives), and that the first p^k digits for these functions can be chosen freely. For each integration pseudo-constant involved one would have p^k branches so that for m independent variables there would be p^{mk} branches altogether.
   (a) The argument based on the sum of Kähler actions for n-branch states would suggests $\hbar_{eff} = n\hbar$, $1 \leq n \leq p^{km}$ not consistent with $\hbar_{eff} = p^{mk}\hbar$. Consistency between the two pictures is achieved if all p^{mk} branches are realized simultaneously so that the state is analogous to a full Fermi sphere. This option looks admittedly artificial.
   (b) An alternative possibility is following. Suppose that the p-adic Planck constant is $p^r\hbar$, $r \leq km$, and thus equivalent with $kp^r\hbar$ for all $k \ mod \ p \neq 0$, and that the allowed numbers for branches satisfy $n = n_1p^r \leq p^{mk}$, $n_1 \ mod \ p \neq 0$ so that Planck constant in p-adic sense is equivalent with $p^r\hbar$. This would realize a correspondence between the number of branches of multofurcation and the Planck constant associated with p-adic Lie algebras.
3. Note that also n-adic and even $q = m/n$-adic topology is possible with norms given by powers of integer or rational. Number field is however obtained only for primes. This suggests that if also integer - and perhaps even rational valued scales are allowed for causal diamonds, they correspond to effective n-adic or q-adic topologies and that powers of p are favored.

3. Integration as the problem again

The difficult questions concern again integration. The integrals reduce to sums over the discrete subgroup of G multiplied with an integral over the p-adic variant $G_{p,n}$ of the continuous Lie group. The first integral - that is summation - is number theoretically universal. The latter integral is the problematic one.

1. The easy way to solve the problem is to interpret the hierarchy of continuous p-adic Lie groups $G_{p,n}$ as analogs of gauge groups. But if the wave functions are invariant under $G_{p,n}$, what is the situation with respect to $G_{p,m}$ for $m < n$? Infinitesimally one obtains that the commutator algebras $[G_{p,k}, G_{p,l}] \subset G_p, k+l$ must annihilate the functions for $k + l \geq n$. Does also $G_{p,m}$, $m < n$ annihilate the functions for as a direct calculation demonstrates in the real case. If this is the case also p-adically the hierarchy of groups $G_{p,n}$ would have no physical implications. This would be disappointing.
2. One must however be very cautious here. Lie algebra consists of first order differential operators and in p-adic context the functions annihilated by these operators are pseudo-constants. It could be that the wave functions annihilated by $G_{p,n}$ are pseudo-constants depending on finite number of pinary digits only so that one can imagine of defining an integral as a sum. In the recent case the digits would naturally correspond to powers p^m, $m < n$. The presence of these functions could be purely p-adic phenomenon having no real counterpart and emerge when one leaves the intersections of real and p-adic worlds. This would be just the non-determinism of imagination assigned to p-adic physics in TGD inspired theory of consciousness.

Is there any hope that one could define harmonic analysis in $G_{p,n}$ in a number theoretically universal manner? Could one think of identifying discrete subgroups of $G_{p,n}$ allowing also an interpretation as real groups?

1. Exponentiation implies that in matrix representation the elements of $G_{p,n}$ are of form $g = Id + p^ng_1$: here Id represents real unit matrix. For compact groups like $SU(2)$ or CP_2 the group elements in real context are bounded above by unity so that this kind of sub-groups do not exist as real groups. For non-compact groups like $SL(2,C)$ and T^4 this kind of subgroups make sense also in real context.
2. Zero energy ontology suggests that discrete but infinite sub-groups Γ of $SL(2,C)$ satisfying certain additional conditions define hyperbolic spaces as factor spaces H^3/Γ (H^3 is hyperboloid of M^4 light-cone). These spaces have constant sectional curvature and very many 3-manifolds allow a hyperbolic metric with hyperbolic volume defining a topological invariant. The moduli

space of CDs contains the groups Γ defining lattices of H^3 replacing it in finite measurement resolution. One could imagine hierarchies of wave functions restricted to these subgroups or H^3 lattices associated with them. These wave functions would have the same form in both real and p-adic context so that number theoretical universality would make sense and one could perhaps define the inner products in terms of "integrals" reducing to sums.

3. The inclusion hierarchy $G_{p,n} \supset G_{p,n+1}$ would in the case of $SL(2,C)$ have interpretation in terms of finite measurement resolution for four-momentum. If $G_{p,n}$ annihilate the physical states or creates zero norm states, this inclusion hierarchy corresponds to increasing IR cutoff (note that short length scale in p-adic sense corresponds to long scale in real sense!). The hierarchy of groups $G_{p,n}$ makes sense also in the case of translation group T^4 and also now the interpretation in terms of increasing IR cutoff makes sense. This picture would provide a group theoretic realization for with the vision that p-adic length scale hierarchies correspond to hierarchies of length scale measurement resolutions in M^4 degrees of freedom.

What about general coordinate invariance?

In purely algebraic approach one must introduce some preferred coordinate system in which the action of various symmetry transformations is simple: typically induced from linear transformations as in the case of projective spaces. This requires physically preferred coordinate system if one hopes to avoid problems with general coordinate invariance. This approach applies also to more general space-time surfaces. A more general approach would assume general coordinate invariance only modulo finite measurement resolution.

For $H = M^4 \times CP_2$ preferred coordinate systems indeed exist but are determined only apart from the isometries of H. For M^4 the preferred coordinates correspond most naturally to linear Minkowsksi coordinates having simple behavior under isometries. Spherical coordinates are not favored since angles cannot be represented p-adically without infinite-dimensional algebraic extension. For CP_2 complex coordinates in which $U(2) \subset SU(3)$ is represented linearly are preferred. The great virtue of sub-manifold gravity is that preferred space-time coordinates can be chosen as a suitable subset of these coordinates depending on the region of the space-time surface. This reduces the general coordinate transformations to the isometries of the embedding space but does yet not mean breaking of general coordinate invariance.

Suppose that one accepts the notion of preferred coordinates and assumes that partonic two-surfaces (at least) can be expressed in terms of rational equations (for algebraic extensions rationals are generalized rationals). General coordinate transformations must preserve this state of affairs. GCI must therefore preserve the property of being a ratio of polynomials with rational coefficients. Only those isometries of H are allowed, which respect the algebraic extensions of p-adic numbers used. This means that only a discrete subgroup of isometries can induce general coordinate transformations in p-adic context.

There is however a continuum of choices of preferred coordinates induced by isometries of H so that one obtains a continuum of choices not equivalent under allowed general coordinate transformations. It would seem that general coordinate invariance is broken. The world containing a conscious observer who has chosen coordinate system M_1 differs from the world in which this coordinate system is M_2!

TGD inspired quantum measurement theory leads to this kind of symmetry breaking also in real sector induced by a selection of quantization axis. In TGD framework this choice has a correlate at the level of moduli space of CDs. For instance, the choice of a preferred rest frame forced also by number theoretical vision and construction of preferred extremals would reflect itself in the properties of the interior of the space-time surface even if it need not affect partonic 2-surfaces.

One can argue that it must be possible to realize general coordinate invariance in more general manner than defining physics using preferred coordinates and simple cubic lattice structures for the embedding space. Maybe also general coordinate invariance should be defined in finite measurement resolution. The lattice structures defining the discretization for embedding space with non-preferred coordinates would look deformed lattice structures in the preferred coordinates but difference would be vanishing in the pinary resolution used.

13.5 How To Define P-Adic Manifolds?

What p-adic manifolds are? This is the basic question also in TGD. What p-adic CP_2 could mean, and can one speak about p-adic space-time sheets and about solutions of p-adic field equations in p-adic $M^4 \times CP_2$? Does WCW have p-adic counterpart?

The TGD inspired vision about p-adic space-time sheets as correlates for cognition suggests an approach based on the identification of cognitive representations mapping real preferred extremal to its p-adic counterpart and vice versa in finite pinary resolution so that one would map discrete set of rational points to rational points (rational in algebraic extension of p-adic numbers). One would have real chart leafs for p-adic preferred extremals instead of p-adic ones.

13.5.1 Algebraic And Topological Approaches To The Notion Of Manifold

There are two approaches to the notion of manifold and they correspond to the division of mathematics to algebra and topology: some-one has talked about the devil of algebra and angel of topology. In the case of infinite-D WCW geometry and p-adic manifolds the roles of devil and angle seem to however change.

1. In the algebraic approach manifolds are regarded as purely algebraic objects - algebraic varieties - and thus number theoretically universal: only algebraic equations are allowed. Inequalities are not accepted. This notion of manifold is not so general as the topological notion and symmetries play a crucial role. The homogenous spaces associated with pairs of groups and subgroups for which all points are metrically equivalent is a good example about the power of the algebraic approach made possible by maximal symmetries formulated by Klein as Erlangen program. In the construction of WCW geometry this approach seems to be the only possible one, and gives hopes that infinite-D geometric existence - and thus physics - is unique [K24]. Standard sphere is this approach defined by condition $x^2 + y^2 + z^2 = R^2$ and makes sense in all number fields for rational values of R. Purely algebraic definition is especially suited for defining sub-varieties. Linear spaces and projective spaces are however definable as manifolds purely algebraically. The natural topology for algebraic varieties is so called Zariski topology (see `http://tinyurl.com/ksbzjsg`) [A98] in which closed sets correspond to lower-dimensional sub-varieties. TGD can be seen as sub-manifold gravity in $M^4 \times CP_2$ with space-time surfaces identified as preferred extremals characterized purely algebraically: this strongly favors algebraic approach. Algebraic definition of the embedding space as a manifold and induction of space-time manifold structure from that for embedding space is also necessary if one wants to define TGD so that it makes sense in all number fields (p-adic space-time sheets are interpreted as correlates for cognition, "thought bubbles").
   A correspondence between p-adics and reals is however required and this suggests that purely algebraic approach is not enough.

2. Second - extremely general - approach is topological but works as such nicely only in the real context. Manifolds are constructed by gluing together open n-balls. Here the inequality so dangerous in p-adic context enters the game: open ball consists of points with distance smaller than R from center. Real sphere in this approach is obtained by gluing two disks having overlap around equator.
   In p-adic context this approach fails since p-adic balls are either disjoint or nested. In fact, single point is open ball p-adically so that one can decompose a candidate for a p-adic manifold with p-adic coordinate charts to dust. It turns out that the replacement of p-adic norm with canonical identification resolves the problem and one can induce real topology to p-adic context by using canonical identification to define coordinate charts of the p-adic space-time surface as regions of real space-time surface. The essentially new elements are the use of real coordinate charts instead of p-adic ones and the notion of finite measurement resolution characterized by pinary cutoffs.

13.5.2 Could Canonical Identification Allow Construction Of Path Connected Topologies For P-Adic Manifolds?

The Berkovich approach [A172, A102] is an attempt to overcome the difficulty caused by the weird properties of p-adic balls by adding some points to p-adic balls so that its topology becomes path connected and the original p-adic ball is dense set in the Berkovich ball. Idea is same as in the completion of rationals to reals: new points make rationals a continuum and one can build calculus. I do not understand how Berkovich disks can be glued to manifolds - presumably the path connected topology implies that they can have overlaps without being identical or nested: the overlaps should be through the added points.

The problem of the Berkovich construction is that from physics point of view it looks rather complex: it is difficult to imagine physical realizations for the auxiliary spaces involved with the construction. Also giving up the p-adic topology seems strange since non-Archimedean topology has - to my opinion - a nice interpretation if one considers it as a correlate for cognition.

The Bruhat-Tits tree working for projective spaces does not seem to require completion. Path connectedness is implied by the tree having in well-defined sense projective space as boundary. Points of the p-adic projective space are represented by projective equivalence classes of lattices: this allows to connect the points of p-adic manifold by edge paths and even the notion of geodesic line can be defined.

In the following TGD inspired topological approach to the construction of p-adic manifolds is discussed. The proposal relies on the notion of canonical identification playing central role in TGD and means that one makes maps about p-adic preferred extremal using - not p-adic but *real coordinate charts* defined using canonical identification obeying the crucial triangle inequality. This approach allows also to make p-adic chart maps about real preferred extremals for some values of p-adic prime. The ultra-metric norms of Berkovich for formal power series are replaced by Archimedean norms defining coordinate functions and their information content is huge as compared to the Berkovich norms. The hierarchy of length scale resolutions gives rise to a hierarchy of canonical identifications in finite pinary resolution and preferred extremal property allows to complete the discrete image set consisting of rational points to a continuous surface. One can say that path-connectedness at the p-adic side is realized by using discretized paths using induced real topology defined by the canonical identification. This gives a resemblance with Bruhat-Tits tree.

Basic facts about canonical identification

In TGD framework one of the basic physical problems has been the connection between p-adic numbers and reals. Algebraic and topological approaches have been competing also here. The notion of canonical identification solves the conflict between algebra (in particular symmetries) and continuity. Canonical identification combined with the identification of common rationals in finite pinary resolution suggests also a natural replacement of p-adic topology with a path connected effective topology defined as real topology induced to p-adic context by canonical identification used to build real chart leafs.

1. In TGD inspired theory of consciousness canonical identification or some of its variants is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?
2. In its basic form canonical identification I maps p-adic numbers $\sum x_n p^n$ to reals and is defined by the formula $I(x) = \sum x_n p^{-n}$. I is a continuous map from p-adic numbers to reals. Its inverse is also continuous but two-valued for a finite number of pinary digits since the pinary expansion of real number is not unique ($1 = .999999..$ is example of this in 10-adic case). For a real number with a finite number of pinary digits one can always choose the p-adic representative with a finite number of pinary digits.
3. Canonical identification has several variants. Assume that p-adic integers x are represented as expansion of powers of p^k as $x = p^{rk} \sum x_n p^{kn}$ with $x_0 \neq 0$. One can map p-adic rational number $p^{rk} m/n$ with m and n satisfying the analog of $x_0 \neq 0$ regarded as a p-adic number to a real number using $I^Q_{k,l}$: $I^Q_{k,l}(p^{rk} m/n) \equiv p^{-rk} I_{k,l}(m)/I_{k,l}(n)$.

In this case canonical identification respects rationality but is ill-defined for p-adic irrationals. This is not a catastrophe if one has finite measurement resolution meaning that only rationals for which $m < p^l, n < p^l$ are mapped to the reals (real rationals actually). One can say that $I^Q_{k,l}$ identifies p-adic and real numbers along common rationals for p-adic numbers with a pinary cutoff defined by k and maps them to rationals for pinary cutoff defined by l. Discrete subset of rational points on p-adic side is mapped to a discrete subset of rational points on real side by this hybrid of canonical identification and identification along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig.** bf **??** in the appendix of this book). This form of canonical identification is the one needed in TGD framework.

4. Canonical identification does not commute with rational symmetries unless one uses the map $I^Q_{k,l}(p^{rk}m/n) = p^{-rk}I_{k,l}(m)/I_{k,l}(n)$ and also now only in finite resolution defined by k. For the large p-adic primes associated with elementary particles this is not a practical problem (electron corresponds to $M_{127} = 2^{127} - 1$). The generalization to algebraic extensions makes also sense. Canonical identification breaks general coordinate invariance unless one uses group theoretically preferred coordinates for M^4 and CP_2 and subset of these for the space-time region considered.

The resolution of the conflict between symmetries and continuity

Consider now the resolution of the conflict between algebra and topology in more detail.

1. Algebraic approach suggests the identification of reals and various p-adic numbers along common rationals defined by $I^Q_{\infty,\infty}$ but this correspondence is completely dis-continuous. Therefore one must introduce a finite pinary cutoff p^k so that one maps only integers smaller than p^k to themselves. Since $I^Q_{k,l}$ does not make sense for p-adic irrationals, one must introduce also second pinary cutoff p^l and use $I^Q_{k,l}$ so that only a finite subset of rational points is mapped to their real counterparts.
2. Topological approach relies on canonical identification and its variants mapping p-adic numbers to reals in a continuous manner. $I_{k,\infty}$ applied to p-adics expressed as $x = p^k u$, $u = \sum x_n p^n$, where u has unit norm, defines such a correspondence. This correspondence does not however commute with the basic symmetries as correspondence along common rationals would do for subgroups of the symmetries represented in terms of rational matrices. Canonical identification fails also to commute with the field equations and the real image fails to be differentiable. Finite pinary cutoff ($I^Q_{k,\infty} \to I^Q_{k,l}$) saves the situation. Below the lower pinary cutoff p^k the pseudo-constants of p-adic differential equations would naturally relate to the identification of p-adics and reals along common rationals (plus common algebraics in the case of algebraic extensions).

The notion of finite measurement resolution allows therefore to find a compromise between the symmetries and continuity (that is, algebra and topology). $I^Q_{k,l}$ maps rationals to themselves only up to k pinary digits and the remaining points up to l digits are mapped to rationals but not to themselves. Canonical identification thus maps only a skeleton of manifold formed by discrete point set from real to p-adic context and the preferred extremals on both sides would contain this skeleton. There are many ways to select this rational skeleton, which can also define a decomposition of the real manifold to simplices or more general objects allowing to define homology theory in real context and to induce it to p-adic context so that real homology would be inherited to p-adic context.

Definition of p-adic manifold in terms of canonical identification with pinary cutoff

What is remarkable is that canonical identification can be seen as a continuous generalization of the p-adic norm defined as $N_p(x) \equiv I_{k,l}(x)$ having the highly desired Archimedean property. $I_{k,l}$ is the most natural variant of canonical identification for defining the chart maps from regions p-adic manifold to regions of corresponding real mani-fold (in particular, p-adic preferred extremals to their real counterparts).

1. As already mentioned, one must restrict the p-adic points mapped to real rationals since $I^Q_{k,l}(x)$ is not well-defined for p-adic irrationals having non-unique expression as ratios of p-adic integers. For the restriction to finite rationals the chart image on the real side would

consist of rational points. The cutoff means that these rationals are not dense in the set of reals. Preferred extremal property could however allow to identify the chart leaf as a piece of preferred extremal containing the rational points in the measurement resolution used. This would realize the dream of mapping p-adic p-adic preferred extremals to real ones playing a key role in number theoretical universality. When one cannot use preferred extremal property some other constraint would restrict the number of different chart leafs.

2. Canonical identification for the various coordinates defines a chart map mapping regions of p-adic manifold to R^n_+. That each coordinate is mapped to a norm $N_p(x)$ means that the real coordinates are always non-negative. If real spaces R^n_+ would provide only chart maps, it is not necessary to require approximate commutativity with symmetries. Also Berkovich considers norms but for a space of formal power series assigned with the p-adic disk: in this case however the norms have extremely low information content.
3. $I^Q_{k,l}$ indeed defines the analog of Archimedean norm in the sense that one has $N^{k,l}_p(x+y) \leq N^{k,l}_p(x) + N^{k,l}_p(y)$. This follows immediately from the fact that the sum of pinary digits can vanish modulo p. The triangle inequality holds true also for the rational variant of I. $N^{k,l}_p(x)$ is however not multiplicative: only a milder condition $N^{k,l}_p(p^{nk}x) = N^{k,l}_p(p^{nk})N^{k,l}_p((x) = p^{-nk}N^{k,l}_p(x)$ holds true.
4. Archimedean property gives excellent hopes that p-adic space provided with chart maps for the coordinates defined by canonical identification inherits within pinary resolutions real topology and its path connectedness as a discretized version. In purely topological approach forgetting algebra and symmetries, a hierarchy of induced real topologies would be obtained as induced real topologies and characterized by various norms defined by $I_{k,\infty}$. When symmetries and algebra are brought in, $I^Q_{k,l}$ gives a correspondence discretizing the connecting paths. This would give a very close connection with physics.
5. The mapping of p-adic manifolds to real manifolds would make the construction of p-adic manifolds very concrete. For instance, one can map real preferred subset of rational points of a real preferred extremal to a p-adic one by the inverse of canonical identification by mapping the real points with finite number of pinary digits to p-adic points with a finite number of pinary digits. This does not of course guarantee that the p-adic preferred extremal is unique. One could however say that p-adic preferred extremals possesses the topological invariants of corresponding real preferred extremal.
6. The maps between different real charts would be induced by the p-adically analytic maps between the inverse images of these charts. At the real side the maps would be consistent with the p-adic maps only in the discretization below pinary cutoff and could be also smooth.
7. An objection against this approach is the loss of general coordinate invariance. One can however argue that one can require this only within the limits of finite measurement resolution. In TGD framework the symmetries of embedding space provide a very narrow set of preferred coordinates.

The idea that the discretized version of preferred extremal could lead to preferred extremal by adding new points in iterative manner is not new. I have proposed assuming that preferred extremals can be also regarded as quaternionic surfaces (tangent spaces are in well-defined sense hyper-quaterionic sub-space of complexified octonionic space containing hyper-complex octonions as a preferred sub-space) [K100].

What about p-adic coordinate charts for a real preferred extremal and for p-adic extremal in different p-adic number field?

What is remarkable that one can also build p-adic coordinate charts about real preferred extremal using the inverse of the canonical identification assuming that finite rationals are mapped to finite rationals. There are actually good reasons to expect that coordinate charts make sense in both directions.

Furthermore, if real preferred extremal can be mapped to p-adic extremals corresponding to two different primes p_1 and p_2, then p_1-adic preferred extremals serves as a chart for p_2-adic preferred extremal and vice versa (one can compose canonical identifications and their inverses to construct the chart maps).

Clearly, real and p-adic extremals define in this manner a category. Preferred extremals are the objects. The arrows are the composites of canonical identification and its inverses mapping to each other preferred extremals belonging to different number fields. This category would be very natural and have profound physical meaning: usually the notion of category tends to be quite too general for the needs of physicist. Category theoretical thinking suggests that full picture of physics is obtained only through this category: this is certainly the case if physics is extended to include physical correlates of cognition and intentionality.

Algebraic continuation from real to p-adic context is one good reason for p-adic chart maps. At the real side one can calculate the values of various integrals like Kähler action. This would favor p-adic regions as map leafs. One can require that Kähler action for Minkowskian and Euclidian regions (or their appropriate exponents) make sense p-adically and define the values of these functions for the p-adic preferred extremals by algebraic continuation. This could be very powerful criterion allowing to assign only very few p-adic primes to a given real space-time surface. This would also allow to define p-adic boundaries as images of real boundaries in finite measurement resolution. p-Adic path connectedness would be induced from real path-connectedness.

In the intersection of real and p-adic worlds the correspondence is certainly unique and means that one interprets the equations defining the p-adic space-time surface as real equations. The number of rational points (with cutoff) for the p-adic preferred extremal becomes a measure for how unique the chart map in the general case can be. For instance, for 2-D surfaces the surfaces $x^n + y^n = z^n$ allow no nontrivial rational solutions for $n > 2$ for finite real integers. This criterion does not distinguish between different p-adic primes and algebraic continuation is needed to make this distinction. The basic condition selecting preferred p-adic primes is that the value of real Kähler/Morse function or its real/imaginary exponent (or both) makes sense also p-adically in some finite-dimensional extension of p-adic numbers.

Some examples about chart maps of p-adic manifolds

The real map leafs must be mutually consistent so that there must be maps relating coordinates used in the overlapping regions of coordinate charts on both real and p-adic side. On p-adic side chart maps between real map leafs are naturally induced by identifying the canonical image points of identified p-adic points on the real side. For discrete chart maps $I_{k,l}^{Q}$ with finite pinary cutoffs one one must complete the real chart map to - say diffeomorphism. That this completion is not unique reflects the finite measurement resolution.

In TGD framework the situation is dramatically simpler. For sub-manifolds the manifold structure is induced from that of embedding space and it is enough to construct the manifold structure $M^4 \times CP_2$ in a given measurement resolution (k, l). Due to the isometries of the factors of the embedding space, the chart maps in both real and p-adic case are known in preferred embedding space coordinates. As already discussed, this allows to achieve an almost complete general coordinate invariance by using subset of embedding space coordinates for the space-time surface. The breaking of GCI has interpretation in terms of presence of cognition and selection of quantization axes.

For instance, in the case of Riemann sphere S^2 the holomorphism relating the complex coordinates in which rotations act as Möbius transformations and rotations around preferred axis act as phase multiplications - the coordinates z and w at Northern and Southern hemispheres are identified as $w = 1/z$ restricted to rational points at both side. For CP_2 one has three poles instead of two but the situation is otherwise essentially the same.

13.5.3 Could Canonical Identification Make Possible Definition OfIntegrals In P-Adic Context?

The notion of p-adic manifold using using real chart maps instead of p-adic ones allows an attractive approach also to p-adic integration and to the problem of defining p-adic version of differential forms and their integrals.

1. If one accepts the simplest form of canonical identification $I(x) : \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$, the image of the p-adic surface is continuous but not differentiable and only integers $n < p$ are mapped to themselves. One can define integrals of real functions along images of the p-adically

analytic curves and define the values of their p-adic counterparts as their algebraic continuation when it exists.

In TGD framework this does not however work. If one wants to define induced quantities - such as metric and Kähler form - on the real side one encounters a problem since the image surface is not smooth and the presence of edges implies that these quantities containing derivatives of embedding space coordinates possess delta function singularities. These singularities could be even dense in the integration region so that one would have no-where differentiable continuous functions and the real integrals would reduce to a sum which do not make sense.

2. In TGD framework finite measurement resolution realized in terms of pinary cutoffs saves the situation. $I^Q_{k,l}$ is a compromise between the direct identification along common rationals favored by algebra and symmetries but being totally discontinuous without the cutoff l. This cutoff breaks symmetries slightly but guarantees continuity in finite measurement resolution defined by the pinary cutoff l. Symmetry breaking can be made arbitrarily small and has interpretation in terms of finite measurement resolution. Due to the pinary cutoff the chart map applied to various p-adic coordinates takes discrete set of rationals to discrete set of rationals and preferred extremal property can be used to make a completion to a real space-time surface. Uniqueness is achieved only in finite measurement resolution and is indeed just what is needed. Also general coordinate invariance is broken in finite measurement resolution. In TGD framework it is however possible to find preferred coordinates in order to minimize this symmetry breaking.
3. The completion of the discrete image of p-adic preferred extremal under $I^Q_{k,l}$ to a real preferred extremal is very natural. This preferred extremal can be said to be unique apart from a finite measurement resolution represented by the pinary cutoffs k and l. All induced quantities are well defined on both sides.

   p-Adic integrals can be defined as pullbacks of real integrals by algebraic continuation when this is possible. The inverse image of the real integration region in canonical identification defines the p-adic integration region.
4. The integrals of p-adic differential forms can be defined as pullbacks of the real integrals. The integrals of closed forms, which are typically integers, would be the same integers but interpreted as p-adic integers.

It is interesting to study the algebraic continuation of Kähler action from real sector to p-adic sectors.

1. Kähler action for both Euclidian and Minkowskian regions reduces to the algebraic continuation of the integral of Chern-Simons-Kähler form over preferred 3-surfaces. The contributions from Euclidian and Minkowskian regions reduce to integrals of Chern-Simons form over 3-surfaces.

   The contribution from Euclidian regions defines Kähler function of WCW and the contribution from Minkowskian regions giving imaginary exponential of Kähler action has interpretation as Morse function, whose stationary points are expected to select special preferred extremals. One would expect that both functions have a continuous spectrum of values. In the case of Kähler function this is necessary since Kähler function defines the Kähler metric of WCW via its second derivatives in complex coordinates by the well-known formula.
2. The algebraic continuation of the exponent of Kähler function for a given p-adic prime is expected to require the proportionality to p^n so that not all preferred extremals are expected to allow a continuation to a given p-adic number field. This kind of assumption has been indeed made in the case of deformations of CP_2 type extremals in order to derive formula for the gravitational constant in terms of basic parameters of TGD but without real justification [K63].
3. The condition that the action exponential in the Minkowskian regions is a genuine phase factor implies that it reduces to a root of unity (one must have an algebraic extension of p-adic numbers). Therefore the contribution to the imaginary exponent Kähler action from these regions for the p-adicizable preferred extremals should be of form $2\pi(k + m/n)$.

   If all preferred real extremals allow p-adic counterpart, the value spectrum of the Morse function on the real side is discrete and could be forced by the preferred extremal property. If this were the case the stationary phase approximation around extrema of Kähler function on the real side would be replaced by sum with varying phase factors weighted by Kähler function.

An alternative conclusion is that the algebraic continuation of Kähler action to any p-adic number field is possible only for a subset of preferred extremals with a quantized spectrum of Morse function. One the real side stationary phase approximation would make sense. It however seems that the stationary phases must obey the above discussed quantization rule.

Also holomorphic forms allow algebraic continuation and one can require that also their integrals over cycles do so. An important example is provided by the holomorphic one-forms integrals over cycles of partonic 2-surface defining the Teichmueller parameters characterizing the conformal equivalence class of the partonic 2-surfaces as Riemann surface. The p-adic variants of these parameters exist if they allow an algebraic continuation to a p-adic number. The algebraic continuation from the real side to the p-adic side would be possible on for certain p-adic primes p if any: this would allow to assign p-adic prime or primes to a given real preferred extremal. This justifies the assumptions of p-adic mass calculations concerning the contribution of conformal modular degrees of freedom to mass squared [K21].

13.5.4 Canonical Identification And The Definition Of P-AdicCounterparts Of Lie Groups

For Lie groups for which matrix elements satisfy algebraic equations, algebraic subgroups with rational matrix elements could be regarded as belonging to the intersection of real and p-adic worlds, and algebraic continuation by replacing rationals by reals or p-adics defines the real and p-adic counterparts of these algebraic groups. The challenge is to construct the canonical identification map between these groups: this map would identify the common rationals and possible common algebraic points on both sides and could be seen also a projection induced by finite measurement resolution.

A proposal for a construction of the p-adic variants of Lie groups was discussed in previous section. It was found that the p-adic variant of Lie group decomposes to a union of disjoint sets defined by a discrete subgroup G_0 multiplied by the p-adic counterpart $G_{p,n}$ of the continuous Lie group G. The representability of the discrete group requires an algebraic extension of p-adic numbers. The disturbing feature of the construction is that the p-adic cosets are disjoint. Canonical identification $I_{k,l}$ suggests a natural solution to the problem. The following is a rough sketch leaving a lot of details open.

1. Discrete p-adic subgroup G_0 corresponds as such to its real counterpart represented by matrices in algebraic extension of rationals. $G_{p,n}$ can be coordinatized separately by Lie algebra parameters for each element of G_0 and canonical identification maps each $G_{p,n}$ to a subset of real G. These subsets intersect and the chart-to-chart identification maps between Lie algebra coordinates associated with different elements of G_0 are defined by these intersections. This correspondence induces the correspondence in p-adic context by the inverse of canonical identification.
2. One should map the p-adic exponentials of Lie-group elements of $G_{p,n}$ to their real counterparts by some form of canonical identification.
   (a) Consider first the basic form $I = I_{1,\infty}$ of canonical identification mapping all p-adics to their real counterparts and maps only the p-adic integers $0 \leq k < p$ to themselves.
   The gluing maps between groups $G_{p,n}$ associated with elements g_m and g_n of G_0 would be defined by the condition $g_m I(exp(it_a T^a) = g_n I(exp(iv_a T^a)$. Here t_a and v_a are Lie-algebra coordinates for the groups at g_m and g_n. The delicacies related to the identification of p-adic analog of imaginary unit have been discussed in the previous section. It is important that Lie-algebra coordinates belong to the algebraic extension of p-adic numbers containing also the roots of unity needed to represent g_n. This condition allows to solve v_a in terms of t_a and $v_a = v_a(t_b)$ defines the chart map relating the two coordinate patches on the real side. The inverse of the canonical identification in turn defines the p-adic variant of the chart map in p-adic context. For I this map is not p-adically analytic as one might have guessed.
   (b) The use of $I_{k,l}^Q$ instead of $I = I_{1,\infty}$ gives hopes about analytic chart-to chart maps on both sides. One must however restrict $I_{k,l}^Q$ to a subset of rational points (or generalized points in algebraic extension with generalized rational defined as ratio of generalized integers in the extension). Canonical identification respects group multiplication only if the integers

defining the rationals m/n appearing in the matrix elements of group representation are below the cutoff p^k. The points satisfying this condition do not in general form a rational subgroup. The real images of rational points however generate a rational sub-group of the full Lie-group having a manifold completion to the real Lie-group.

One can define the real chart-to chart maps between the real images of $G_{p,k}$ at different points of G_0 using $I^Q_{k,l}(exp(iv_aT^a) = g_n^{-1}g_m \times I^Q_{k,l}(exp(it_aT^a)$. When real charts intersect, this correspondence should allow solutions v_a, t_b belonging to the algebraic extension and satisfying the cutoff condition. If the rational point at the other side does not correspond to a rational point it might be possible to perform pinary cutoff at the other side.

Real chart-to-chart maps induce via common rational points discrete p-adic chart-to-chart maps between $G_{p,k}$. This discrete correspondence should allow extension to a unique chart-to-chart map the p-adic side. The idea about algebraic continuation suggests that an analytic form for real chart-to-chart maps using rational functions makes sense also in the p-adic context.

3. p-Adic Lie-groups $G_{p,k}$ for an inclusion hierarchy with size characterized by p^{-k}. For large values of k the canonical image of $G_{p,k}$ for given point of G_0 can therefore intersect its copies only for a small number of neighboring points in G_0, whose size correlates with the size of the algebraic extension. If the algebraic extension has small dimension or if k becomes large for a given algebraic extension, the number of intersection points can vanish. Therefore it seems that in the situations, where chart-to-chart maps are possible, the power p^k and the dimension of algebraic extension must correlate. Very roughly, the order of magnitude for the minimum distance between elements of G_0 cannot be larger than p^{-k+1}. The interesting outcome is that the dimension of algebraic extension would correlate with the pinary cutoff analogous to the IR cutoff defining measurement resolution for four-momenta.

13.5.5 Cut And Project Construction Of Quasicrystals From TGD PointOf View

Cut and project (see http://tinyurl.com/ybdbvjoa) [A120] method is used to construct quasicrystals (QCs) in sub-spaces of a higher-dimensional linear space containing an ordinary space filling lattice, say cubic lattice. For instance, 2-D Penrose tiling is obtained as a projection of part of 5-D cubic lattice - known as Voronyi cell - around 2-D sub-space imbedded in five-dimensional space. The orientation of the 2-D sub-space must be chosen properly to get Penrose tiling. The nice feature of the construction is that it gives the entire 2-D QC. Using local matching rules the construction typically stops.

Sub-manifold gravity and generalization of cut and project method

The representation of space-time surfaces as sub-manifolds of 8-D $H = M^4 \times CP_2$ can be seen as a generalization of cut and project method.

1. The space-time surface is not anymore a linear 4-D sub-space as it would be in cut and project method but becomes curved and can have arbitrary topology. The embedding space ceases to be linear $M^8 = M^4 \times E^4$ since E^4 is compactified to CP_2. Space-time surface is not a lattice but continuum.
2. The induction procedure geometrizing metric and gauge fields is nothing but projection for H metric and spinor connection at the continuum limit. Killing vectors for CP_2 isometries can be identified as classical gluon fields. The projections of the gamma matrices of H define induced gamma matrices at space-time surface. The spinors of H contain additional components allowing interpretation in terms of electroweak spin and hyper-charge.

Finite measurement resolution and construction of p-adic counterparts of preferred extremals forces "cut and project" via discretization

In finite measurement resolution realized as discretization by finite pinary cutoff one can expect to obtain the analog of cut and project since 8-D embedding space is replaced with a lattice structure.

1. The p-adic/real manifold structure for space-time is induced from that for H so that the construction of p-adic manifold reduces to that for H.

2. The definition of the manifold structure for H in number theoretically universal manner requires for H discretization in terms of rational points in some finite region of M^4. Pinary cutoffs- two of them - imply that the manifold structures are parametrized by these cutoffs charactering measurement resolution. Second cutoff means that the lattice structure is piece of an infinite lattice. First cutoff means that only part of this piece is a direct imagine of real/p-adic lattice on p-adic/real side obtained by identifying common rationals (now integers) of real and p-adic number fields. The mapping of this kind lattice from real/p-adic side to p-adic/real side defines the discrete coordinate chart and the completion of this discrete structure to a preferred extremal gives a smooth space-time surface also in p-adic side if it is known on real side (and vice versa).
3. Cubic lattice structures with integer points are of course the simplest ones for the purposes of discretization and the most natural choice for M^4. For CP_2 the lattice is completely analogous to the finite lattices at sphere defined by orbits of discrete subgroups of rotation group and the analogs of Platonic solids emerge. Probably some mathematician has listed the Platonic solids in CP_2.
4. The important point is that this lattice like structure is defined at the level of the 8-D embedding space rather than in space-time and the lattice structure at space-time level contains those points of the 8-D lattice like structure, which belong to the space-time surface. Finite measurement resolution suggests that all points of lattice, whose distance from space-time surface is below the measurement resolution for distance are projected to the space-time surface. Since space-time surface is curved, the lattice like structure at space-time level obtained by projection is more general than QC.

The lattice like structure results as a manifestation of finite measurement resolution both at real and p-adic sides and can be formally interpreted in terms of a generalization of cut and project but for a curved space-time surface rather than 4-D linear space, and for H rather than 8-D Minkowski space. It is of course far from clear whether one can obtain anything looking like say 3-D or 4-D version of Penrose tiling.

1. The size scale of CP_2 is so small (10^4 Planck lengths) that space-time surfaces with 4-D M^4 projection look like M^4 in an excellent first approximation and using M^4 coordinates the projected lattice looks like cubic lattice in M^4 except that the distances between points are not quite the M^4 distances but scaled by an amount determined by the difference between induced metric and M^4 metric. The effect is however very small if one believes on the general relativistic intuition.
   In TGD framework one however can have so called warped embeddings of M^4 for which the component of the induced metric in some direction is scaled but curvature tensor and thus gravitational field vanishes. In time direction this scaling would imply anomalous time dilation in absence of gravitational fields. This would however cause only a the compression or expansion of M^4 lattice in some direction.
2. For Euclidian regions of space-time surface having interpretation as lines of generalized Feynman diagrams M^4 projection is 3-dimensional and at elementary particle level the scale associated with M^4 degrees of freedom is roughly the same as CP_2 scale. If CP_2 coordinates are used (very natural) one obtains deformation of a finite lattice-like structure in CP_2 analogous to a deformation of Platonic solid regarded as point set at sphere. Whether this lattice like structure could be seen as a subset of infinite lattice is not clear.
3. One can consider also string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ with 2-D M^4 projection and their deformations. In this case the projection of M^4 lattice to X^2 - having subset of two M^4 coordinates as coordinates - can differ considerably from a regular lattice since X^2 can be locally tilted with respect to M^4 lattice. This cannot however give rise to Penrose tiling requiring 5-D flat embedding space. This argument applies also to 2-D string world sheets carrying spinor modes. In the idealized situation that string world sheet is plane in M^4 one might obtain an analog of Penrose tiling but with 4-D embedding space.

The above quasi lattice like structures (QLs) are defined by a gravitational deformation of the cubic lattice of M^4. Is there any hope about the 4-D QLs in M^4 so that gravitation would give rise to the analogs of phason waves deforming them? Could cut and project method be generalized to give QL in M^4 as projection of 8-D cubic lattice in M^8?

$M^8 - H$ duality

Before considering an explicit proposal I try to describe what I call M^8-H duality ($H = M^4 \times CP_2$).

1. What I have christened $M^8 - H$ duality is a conjecture stating that TGD can be equivalently defined in M^8 or $M^4 \times CP_2$. This is the number theoretic counterpart of spontaneous compactification of string models but has nothing to do with dynamics: only two equivalent representations of dynamics would be in question.
2. Space-time surfaces (preferred extremals) in M^8 are postulated to be quaternionic sub-manifolds of M^8 possessing a fixed $M^2 \subset M^4 \subset M^8$ as sub-space of tangent space. "Quaternionic" means that the tangent space of M^4 is quaternionic and thus associative. Associativity conditions would thus determine classical dynamics. More generally, these subspaces $M^2 \subset M^8$ can form integrable distribution and they define tangent spaces of a 2-D sub-manifold of M^4. If this duality really holds true, space-time surfaces would define a lattice like structure projected from a cubic M^8 lattice. This of course does not guarantee anything: $M^8 - H$ duality itself suggests that these lattice like structures differ from regular M^4 crystals only by small gravitational effects.
3. The crucial point is that quaternionic sub-spaces are parametrized by CP_2. Quaternionic 4-surfaces of $M^8 = M^4 \times CP_2$ containing the fixed $M^2 \subset M^8$ can be mapped to those of $M^4 \times CP_2$ by defining M^4 coordinates as projections to preferred $M^4 \subset M^8$ and CP_2 coordinates as those specifying the tangent space of 4-surface at given point.
4. A second crucial point is that the preferred subspace $M^4 \subset M^8$ can be chosen in very many ways. This embedding is a complete analog of the embedding of lower-D subspace to higher-D one in cut and project method. M^4 can be identified as any 4-D subspace imbedded in M^4 and the group $SO(1,7)$ of 8-D Lorentz transformations defines different embeddings of M^4 to M^8. The moduli space of different embeddings of M^4 is the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and has dimension $D = 28 - 6 - 6 = 16$.
   When one fixes two coordinate axes as the real and one imaginary direction (physical interpretation is as an identification of rest system and spin quantization axes), one obtains $SO(1,7)/SO(2) \times SO(4)$ with higher dimension $D = 28 - 1 - 6 = 21$. When one requires also quaternionic structure one obtains the space $SO(1,7)/SU(1) \times SU(2)$ with dimension $D = 28 - 4 = 24$. Amusingly, this happens to be the number of physical degrees of freedom in bosonic string model.

How to obtain quasilattices and quasi-crystals in M^4?

Can one obtain quasi-lattice like structures (QLs) at space-time level in this framework? Consider first the space-time QLs possibly associated with the standard cubic lattice L^4_{st} of M^4 resulting as projections of the cubic lattice structure L^8_{st} of M^8.

1. Suppose that one fixes a cubic crystal lattice in M^8, call it L^8_{st}. Standard M^4 cubic lattice L^4_{st} is obtained as a projection to some M^4 sub-space of M^8 by simply putting 4 Euclidian coordinates for lattice points o constant. These sub-spaces are analogous to 2-D coordinate planes of E^3 in fixed Cartesian coordinates. There are $7!/3!4! = 35$ choices of this kind.
   One can consider also E_8 lattice (see `http://tinyurl.com/y9x7vevr`) is an interesting identification for the lattice of M^8 since E_8 is self-dual and defines the root lattice of the exceptional group E_8. E_8 is union of Z^8 and $(Z + 1/2)^8$ with the condition that the sum of all coordinates is an even integer. Therefore all lattice coordinates are either integers or half-integers. E_8 is a sub-lattice of 8-D cubic lattice with 8 generating vectors $e_i/2$, with e_i unit vector. Integral octonions are obtained from E_8 by scaling with factor 2. For this option one can imbed L^4_{st} as a sub-lattice to Z^8 or $(Z + 1/2)^8$.
2. Although $SO(1,3)$ leaves the imbedded 4-plane M^4 invariant, it transforms the 4-D crystal lattice non-trivially so that all 4-D Lorentz transforms are obtained and define different discretizations of M^4. These are however cubic lattices in the Lorentz transformed M^4 coordinates so that this brings nothing new. The QLs at space-time surface should be obtained as gravitational deformations of cubic lattice in M^4.
3. L^4_{st} indeed defines 4-D lattice at space-time surface apart from small gravitational effects in Minkowskian space-time regions. Elementary particles are identified in TGD a Euclidian

space-time regions - deformed CP_2 type vacuum extremals. Also black-hole interiors are replaced with Euclidian regions: black-hole is like a line of a generalized Feynman diagram, elementary particle in some sense in the size scale of the black-hole. More generally, all physical objects, even in everyday scales, could possess a space-time sheet with Euclidian metric signature characterizing their size (AdS^5/CFT correspondence could inspire this idea). At these Euclidian space-time sheets gravitational fields are strong since even the signature of the induced metric is changed at their light-like boundary. Could it be that in this kind of situation lattice like structures, even QCs, could be formed purely gravitationally? Probably not: an interpretation as lattice vibrations for these deformations would be more natural.

It seems that QLs are needed *already at the level of* M^4. M^8-H duality indeed provides a natural manner to obtain them.

1. The point is that the projections of L^8_{str} to sub-spaces M^4 defined as the $SO(1,7)$ Lorentz transforms of L^4_{st} define generalized QLs parametrized by 16-D moduli space $SO(1,7)/SO(1,3)\times SO(4)$. These QLs include also QCs. Presumably QC is a QL possessing a non-trivial point group just like Penrose tiling has the isometry group of dodecagon as point group and 3-D analog of Penrose tiling has the isometries of icosahedron as point group.
   This would allow to conclude that the discretization at the level of M^8 required by the definition of p-adic variants of preferred extremals as cognitive representations of their real counterparts would make possible 4-D QCs. M^8 formulation of TGD would explain naturally the QL lattices as discretizations forced by finite measurement resolution and cognitive resolution.
   A strong number theoretical constraint on these discretizations come from the condition that the 4-D lattice like structure corresponds to an algebraic extension of rationals. Even more, if this algebraic extension is 8-D (perhaps un-necessarily strong condition), there are extremely strong constraints on the 22-parameters of the embedding. Note that in p-adic context the algebraic extension dictates the maximal isometry group identified as subgroup of $SO(1,7)$ assignable to the embedding as the discussion of p-adic icosahedron demonstrates.
2. What about the physical interpretation of these QLs/QCs? As such QLs define only natural discretizations rather than physical lattices. It is of course quite possible to have also physical QLs/QCs such that the points - rather time like edge paths - of the discretization contain real particles. What about a "particle" localized to a point of 4-D lattice? In positive energy ontology there is no obvious answer to the question. In zero energy ontology the lattice point could correspond to a small causal diamond containing a zero energy state. In QFT context one would speak of quantum fluctuation. In p-adic context it would correspond to "though bubble" lasting for a finite time.
3. It is also possible to identify physical particles as edge paths of the 4-D QC, and one can consider time= constant snapshots as candidates for 3-D QCs. It is quite conceivable that the non-trivial point group of QCs favors them as physical QLs.

Expanding hyperbolic tessellations and quasi-tessellations obtained by embedding $H^3 \subset M^4$ to $H^7 \subset M^8$

M^8-$M^4 \times CP_2$ duality and the discretization required by the notion of p-adic manifold relates in an interesting manner to expanding hyperbolic tessellations and quasi tessellations in $H^7 \subset M^8$, and possible expanding quasi-tessellations in obtained by embedding $H^3 \subset M^4$ to $H^7 \subset M^8$

1. Euclidian lattices E_8, E_7, E_6

I have already considered E_8 lattice in M^8. The background space has however Minkowskian rather than Euclidian metric natural for the carrier space of the E_8 lattice. If one assigns some discrete subgroup of isometries to it, it is naturally subgroup of $SO(8)$ rather than $SO(1,7)$. Both these groups have $SO(7)$ as a subgroup meaning that preferred time direction is chosen as that associated with the real unit and considers a lattice formed from imaginary octonions.

E_8 lattice scaled up by a factor 2 to integer lattice allows octonionic integer multiplication besides sums of points so that the automorphism group of octonions: discrete subgroups of $G_2 \subset SO(7)$ would be the natural candidates for point groups crystals or lattice like structures.

If one assumes also fixed spatial direction identified as a preferred imaginary unit, G_2 reduces to $SU(3) \subset SO(6) = SU(4)$ identifiable physically as color group in TGD framework. From this one ends up with the idea about $M^8 - M^4 \times CP_2$ duality. Different embeddings of $M^4 \subset M^8$ are quaternionic sub-spaces containing fixed M^2 are labelled by points of CP_2.

All this suggests that E_7 lattice in time=constant section of even E^6 lattice is a more natural object lattice to consider. Kind of symmetry breaking scenario $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow G_2 \rightarrow SU(3)$ is suggestive. This Euclidian lattice would be completely analogous to a slicing of 4-D space-time by 3-D lattices labelled by the value of time coordinate and is of course just what physical considerations suggest.

2. Hyperbolic tessellations

Besides crystals defined by a cubic lattice or associated with E_6 or E_7, one obtains an infinite number of hyperbolic tessellations in the case of M^8. These are much more natural in Minkowskian signature and could be also cosmologically very interesting. Quite generally, one can say that hyperbolic space is ideal for space-filling packings defined by hyperbolic manifolds H^n/Γ: they are completely analogous to space-filling packings of E^3 defined by discrete subgroups of translation group producing packings of E^3 by rhombohedra. One only replaces discrete translations with discrete Lorentz transformations. This is what makes these highly interesting from the point of view of quantum gravity.

1. In M^{n+1} one has tessellations of n-dimensional hyperboloid H^n defined by $t^2 - x1^2 - ... - x_n^2 = a^2 > 0$, where a defines Lorentz invariant which for $n = 4$ has interpretation as cosmic time in TGD framework. Any discrete subgroup Γ of the Lorentz group $SO(1, n)$ of M^{n+1} with suitable additional conditions (finite number of generators at least) allows a tessellation of H^n by basic unit H^n/Γ. These tessellations come as 1-parameter families labelled by the cosmic time parameter a. These 3-D tessellations participate cosmic expansion. Of course, also ordinary crystals are crystals only in spatial directions. One can of course discretize the values of a or some function of a in integer multiples of basic unit and assign to each copy of H^n/Γ a "center point" to obtain discretization of M^{n+1} needed for p-adicization.
2. For $n = 3$ one has M^4 and H^3, and this is very relevant in TGD cosmology. The parameter a defines a Lorentz invariant cosmic time for the embeddings of Robertson-Walker cosmologies to $M^4 \times CP_2$. The tessellations realized as physical lattices would have natural interpretation as expanding 3-D lattice like structures in cosmic scales. What is new is that discrete translations are replaced by discrete Lorentz boosts, which correspond to discrete velocities and observationally to discrete red shifts for distant objecst. Interestingly, it has been found that red shift is quantized along straight lines [?]: "God's fingers" is the term used. I proposed for roughly two decades ago an explanation based on closed orbits of photons around cosmic strings [K25]. but explanation in terms of tessellations would also give rise to periodicity. A fascinating possibility is that these tessellation have defined macroscopically quantum coherent structures during the very early cosmology the size scale of H^3/Γ was very small. One can also ask whether the macroscopic quantum coherence could still be there.
   Hyperbolic manifold property has purely local signatures such as angle surplus: the very fact that there are infinite number of hyperbolic tessellations is in conflict with the fact that we have Euclidian 3-geometry in every day length scales. In fact, for critical cosmologies, which allow a one-parameter family of embeddings to $M^4 \times CP_2$ (parameter characterizes the duration of the cosmology) one obtains flat 3-space in cosmological scales. Also overcritical cosmologies for which $a = constant$ section is 3-sphere are possible but only with a finite duration. Many-sheeted space-time picture also leads to the view that astrophysical objects co-move but do not co-expand so that the geometry of time=constant snapshot is Euclidian in a good approximation.

3. Does the notion of hyperbolic quasi-tessellation make sense?

Can one construct something deserving to be called quasi tessellations (QTs)? For QCs translational invariance is broken but in some sense very weakly: given lattice point has still an infinite number of translated copies. In the recent case translations are replaced by Lorentz transformations and discrete Lorentz invariance should be broken in similar weak manner.

If cut and project generalizes, QTs would be obtained using suitably chosen non-standard embedding $M^4 \subset M^8$. Depending on what one wants to assume, M^4 is now image of M^4_{st} by an element of $SO(1, 7)$, $SO(7), SO(6)$ or G_2. The projection - call it P - must take place to M^4

sliced by scaled copies of H^3 from M^8_{st} sliced by scaled copies of H^7/Γ tessellation. The natural option is that P is directly from H^7 to $H^3 \subset H^7$ and is defined by a projecting along geodesic lines orthogonal to H^3. One can choose always the coordinates of M^4 and M^8 in such a way that the coordinates of points of M^4 are $(t,x,y,z,0,0,0,0)$ with $t^2-r^2=a_4^2$ whereas for a general point of H^7 the coordinates are $(t,x,y,z,x_4,...x_7)$ with $t^2-r^2-r_4^2=a_8^2$ for $H^3 \subset H^7$. The projection is in this case simply $(t,x,y,z,x_4,...,x_7) \rightarrow (t,x,y,z,0,...,0)$. The projection is non-empty only if one has $a_4^2-a_8^2 \geq 0$ and the 3-sphere S^3 with radius $r_4=\sqrt{a_4^2-a_8^2}$ is projected to single point. The images of points from different copies of H^7/Γ are identical if S^3 intersects both copies. For r_4 much larger than the size of the projection $P(H^7/\Gamma)$ of single copy overlaps certainly occurs. This brings strongly in mind the overlaps of the dodecagons of Penrose tiling and icosahedrons of 3-D icosahedral QC. The point group of tessellation would be Γ.

4. Does one obtain ordinary H^3 tessellations as limits of quasi tessellations?

Could one construct expanding 3-D hyperbolic tessellations H_3/Γ_3 from expanding 7-D hyperbolic tessellations having H^7/Γ_7 as a basic building brick? This seems indeed to be the outcome at at the limit $r_4 \rightarrow 0$. The only projected points are the points of H^3 itself in this case. The counterpart of the group $\Gamma_7 \subset SO(1,7)$ is the group obtained as the intersection $\Gamma_3 = \Gamma_7 \cap SO(1,3)$: this tells that the allowed discrete symmetries do not lead out from H^3. This seems to mean that the 3-D hyperbolic manifold is H^3/Γ_3, and one obtains a space-filling 3-tessellation in complete analogy for what one obtains by projecting cubic lattice of E^7 to E^3 imbedded in standard manner. Note that $\Gamma_3 = \Gamma_7 \cap SO(1,3)$, where $SO(1,3) \subset SO(1,7)$, depends on embedding so that one obtains an infinite family of tessellations also from different embeddings parametrized by the coset space $SO(1,7)/SO(1,3)$. Note that if Γ_3 contains only unit element $H^3 \subset H^7/\Gamma_7$ holds true and tessellation trivializes.

p-Adic variant of the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and Bruhat-Tits tree

p-Adicization requires also to consider the p-adic variants of the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$. Grassmannians define a generalization of projective spaces and appear in twistor Grassmannian program. According to the article [A116], the construction of the Bruhat-Tits tree generalizes for them. This gives excellent hopes for generalizing the twistor Grassmannian program to p-adic context. Bruhat-Tits tree for $S^2 = SO(3)/SO(2) = P^1(C)$ generalized to $P^1(K)$ (K is any algebraic extension of Q_p) is constructed in terms of projective equivalence classes of integer lattices in K^2 with inclusion relation defining the notion of edge path making possible path connectedness.

In the recent p-adic manifold structure forces 8-D lattices in K^8 and they seem to take the role of the 2-D lattices K^2. Therefore TGD view about p-adic manifold structure might well be equivalent to the standard view in the case of Grassmannians. For the Grassmannian in question the projective equivalence is replaced with equivalence under $SO(1,3) \times SO(4)$. Therefore one expects that the generalization of Bruhat-Tits tree in 8-D case and its projections to sub-spaces assignable to algebraic extensions K of Q_p appear and correspond to discrete subgroups Γ of $SO(1,7)$. With some additional restrictions on Γ the spaces H^7/Γ define hyperbolic manifolds (H^7 is 7-D hyperboloid in M^8).

This argument makes sense if the counterpart of projective space $P^1(K)$ can be defined also as the analog of $SO(3)/SO(2)$. What looks like a problem is that the "Cartesian" dimension of this space is 2 whereas $P^1(K)$ is 1-D in this sense. The analog of $SO(3)/SO(2)$ can be indeed defined as the Grassmannian $SO(1,2)/Z_p^1 \times SO(2)$ with dimension 1. Z_p^1 denotes the group of p-adic integers with unit norm defining p-adic units analogous to complex phases: they have their inverse as conjugate. What this says that p-adic unit vector 1 is equivalent to any element of Z_p^1. In real context the group of units contains only the real unit so that one obtains Cartesian dimension 2.

13.6 What The Notion Of Path Connectedness Could Mean From Quantum Point Of View?

The notions of open set and path connectedness express something physical but perhaps in a highly idealized form. Canonical identification for preferred extremals provides one promising approach to the challenge of defining path connected topology and at the same time achieving a compromise with symmetries and approximate correspondence via common rationals. The variant I_k^Q for the canonical identification with pinary cutoff can be used to map rational points of the real/p-adic preferred extremal to p-adic/real space-time points to define a skeleton completed to a preferred extremal, which of course need not be unique. In particular, real paths are mapped to p-adic paths in finite pinary cutoff so that the images are always discrete paths consisting of rational points so that the notion of finite pinary resolution is un-avoidable.

One could also try to formulate path connectedness more microscopically and physically using the tools of quantum physics.

1. The basic point is that there are correlations between different points or physical events associated with different points of manifold. Manifold is more like liquid than dust: one cannot pick up single point from it. In the idealistic description based on real topology one can pick up only open ball. This relates also to finite measurement resolution for lengths: it is not possible to specify single point.
2. Quantum physicist would formulate this in terms of physical correlations. The correlation functions for two fields defined in the manifold are non-vanishing even when the two fields are evaluated at different points.

If one takes the suggestion of quantum physicist seriously, one should reformulate the notion of manifold by bringing in quantum fields and their correlation functions. This approach is alternative to the formulation of p-adic (real) manifold based on real (p-adic) coordinate charts defined by canonical identification.

13.6.1 Could Correlation Functions For Fermion Fields Code Data About Geometric Objects?

Quantum TGD suggests another approach to the notion of path connectedness. What could the quantum fields needed to formulate the notion of manifold be in TGD framework? In TGD framework there are only very few choices to consider. Only the induced second quantized fermion fields can be considered in both real and p-adic context. Their correlation functions defined as vacuum expectations of bi-local bilinears are indeed well-defined in both real and p-adic context.

One can define classical bosonic correlation functions for the invariants formed from induce bosonic field but this requires integration over the space-time surface and this might be problematic in p-adic context unless one is able to algebraically continue the real correlation functions to p-adic context. Quantum ergodicity states that these correlation functions characterizing sub-manifold geometry statistically are identical for the space-time surfaces which can appear in the quantum superposition defining WCW spinor field.

1. One could perhaps say:
   Two points are "connected by path!" / have "edge connecting them" as Bruhat and Tits would say / belong to same space-time sheet/partonic 2-surface / belog to two distinct 3-surfaces forming part of a boundary of the same connected space-time surface $\leftrightarrow$ there are non-vanishing fermion-anti-fermion correlation functions for the point pair in question.
2. Note that one must consider separately pure right-handed neutrino modes and the remaining spinor modes. For the Kähler-Dirac equation pure right-handed neutrino fields are covariantly constant in CP_2 degrees of freedom and de-localized along entire space-time sheet. In space-time interior the correlation functions for right-handed neutrinos should code for the geometry of the space-time sheet.
   The modes which do not represent pure right-handed neutrinos are restricted to 2-D string world sheets. The conformal correlation functions for the spinor fields restricted to string world sheets should code for the geometry of string world sheets.
3. Everything would reduce to fermionic correlation functions, which in principle are measurable in particle physics experiments. This is in accordance with the general vision of TGD that fermion fields provide all possible information about geometric objects. This would generalize the idea that one can hear the shape of the drum that is deduce the geometry of drum from the correlation functions for sound waves.

4. Real space-time topology would be only a highly idealized description of this physical connectedness, in more physical approach it would be described in terms of fermionic correlation functions allowing to decide whether two points belong to same geometric object or not.

13.6.2 P-Adic Variant Of WCW And M-Matrix

In zero energy ontology (ZEO) the unitary U-matrix having non-unitary M-matrices are rows and allowing interpretation as "complex" square roots of hermitian density matrices are in key role. The unitary S-matrix appears as a "phase factor" of the "complex" square root and its modulus corresponds to Hermitian square roots of density matrix. What is essential is that M-matrices are multi-local functionals of 3-surfaces defining boundary components of connected space-time surface at the light-like boundaries of causal diamond.

By strong form of holography the information about 3-surfaces reduces to data given at partonic 2-surfaces (and their tangent space data). The 3-D boundary components of space-time surface at the boundaries of CD define a coherent unit. The space-time surface takes the role of the path connecting two disjoint 3-surfaces in zero energy ontology and WCW is more like a space formed by multi-points (unions of several disjoint 3-surfaces). Hence the basic difficulty of p-adic manifold theory is circumvented.

Although WCW spinor fields are formally purely classical, the analogs of correlation functions as n-point functions in WCW make sense since the notion of 3-surface is generalized in the manner described above. M-matrix elements serve as building bricks of WCW spinor fields and they are are functionals about the data at partonic 2-surfaces at the boundaries of CD and could have an interpretation as correlation function in WCW giving rise to "path connectedness" in WCW in a number theoretically universal manner.

13.6.3 A Possible Analog For The Space Of Berkovich Norms In The Approach Based On Correlation Functions

The idea about real preferred extremal as a coordinate chart for p-adic preferred extremal (and vice versa) suggest that canonical identification with cutoff could define naturally p-adic preferred extremal as a path connected space. It would also allow to map preferred real preferred extremals to their p-adic counterparts for some preferred primes and at the same time algebraically continue various quantities such as Kähler action. The hierarchies of pinary cutoffs and resolutions in phase degrees of freedom define a hierarchy of resolutions and the resulting Archimedean norms defined by the the hierarchy of canonical identifications define the analog of the norm space of Berkovich.

Also the idea about correlation functions as counterpart for path connectedness suggests that the ultra-metric norm of K-valued field needed to defined Berkovich disk might be replaced with fermionic correlation functions. Could the space of the Berkovich norms have as an analog in this more general approach? The notion of finite measurement resolution seems to lead naturally to this analog also for this option.

One can define the correlation functions in various resolutions. This means varying angle resolution and length scale resolution. Angle resolution -or rather phase resolution in p-adic context - means a hierarchy of algebraic extensions for p-adic number fields bringing in roots of unity $exp(i2\pi/n)$ with increasing values of n. Length scale resolution means increasing number of p-adic primes and CDs with scales given by integer multiples of CP_2 scale.

Fermionic Fock space defines a canonical example about hyper-finite factor of type II_1 (HFF) [K99] and the inclusions of HFFs having interpretation in terms of finite measurement resolution should be involved in the construction. The space of Berkovich norms is replaced with the correlation functions assignable to HFF having fractal structure containing infinite inclusion hierarchies of HFFs.

13.7 Appendix: Technical Aspects Of Bruhat-Tits Tree And Berkovich Disk

In the following more technical aspects of Bruhat-Tits tree and Berkovich disk are discussed.

13.7.1 Why Notions Like Bruhat-Tits Tree And Berkovich Disk?

The constructions like Bruhat-Tits tree and Berkovich disk remain totally incomprehensible unless one understands the underlying motivations. If I have understood correctly, the motivation behind all these strange and complicated looking structures is the attempt to generalize the notion of real manifold to p-adic context using topological approach based on p-adic coordinate maps to p-adic disks which must be completed to Berkovich disks ("disk" could quite well be replaced with "ball").

In the real context manifolds have open balls of R^n defining real topology as building bricks. One glues these balls together along their intersection suitably and obtains global differential structures with various topologies and manifold structures. For instance, sphere can be obtained by gluing two disks having overlap around equator.

In the p-adic context the topology is however totally disconnected meaning that single point is the smallest open set. One cannot build anything coherent from points: they are disjoint or identical unlike the open balls in the real case. More generally: two p-adic balls are either disjoint or either one is contained by another one! No gluing by overlap is possible!

This difficulty has stimulated various theories and Bruhat-Tits tree relates to the theory of Berkovich generalizing the notion of open ball to Berkovich disk [A172, A102] serving as a building brick of p-adic manifolds. The naïve p-adic disk is contained as a dense subset to Berkovich disk so that this is like replacing rationals with reals and in this manner gluing them to continuum. Pragmatic physicist is not too enthusiastic about this kind of completions, especially so because the original p-adic topology is replaced with a new one in the completion.

13.7.2 Technical Aspects Of Bruhat-Tits Tree

The construction of Bruhat-Tits tree for $P^1(Q_p)$ and its generalizations to algebraic extensions can be understood as follows.

1. One must be able to connect any pair of points of $P^1(Q_p)$ by and edge path. The basic building brick of edge path is single edge connecting nearby points of $P^1(Q_p)$. One can start from a simpler situation first by considering Q_p^2 consisting of points (a, b). If one treats these points just as pairs of p-adic numbers, one cannot do anything. One must represent these pairs as geometric objects in order to define the notion of edge purely set theoretically. The Z_p lattice generated having the pair (a, b) as basis vectors is indeed an object labelled by the pair (a, b). If one wants projective space one must assume that the lattices different by scaling of (a, b) by a non-vanishing p-adic number are equivalent but this is not absolutely essential for the argument.
   Note: Also in TGD one has a space whose points are geometric objects. The geometric object is now 3-surface and the space is the "world of classical worlds" - the space formed by these 3-surfaces.
2. The projective space $P^1(C) = S^2$ has a representation as a coset space $PGl(2, C)/PGl(1, C) \times PGl(2, Z)$. This algebraic relation must generalize by replacing C with Q_p. This means that $PGl(2, Q_p)$ must act transitively in the set of the geometric objects associated with pairs (a, b). The action on lattices is indeed well-defined and transitive and one can generate all lattices from single lattice defined by the lattice characterized by $(a, b) = (1, 1)$. One has a discrete analog of homogeneous space in the sense that its all points are geometrically equivalent because of the transitive action of $Gl(2, Q_p)$. This reduces the construction to single point, which is an enormous simplification.
   Note: Also the construction of the geometry of WCW [K24] in TGD relies on symmetric/homogeneous space property (actually the property of being a union of infinite-dimensional symmetric spaces) making the hopeless task managable by reducing the construction to that at single point of WCW and forcing infinite-dimensional symmetries (symplectic invariance imherited from the boundary of $CD \times CP_2$ and generalization of conformal invariance for light-like 3-surfaces and light-like boundaries of CD). Already in the case of loop spaces (see `http://tinyurl.com/y9zvjm9b`) [A54] Kähler geometry exists only because of these infinite-dimensional symmetries and is also unique [A121]. One can say that infinite-dimensional Kähler geometric existence is unique.

3. The really important idea is that the internal structure of the point pairs (a, b) allows to define what the existence of "edge" between two nearby points of $P^1(Q_p)$ could mean. The definition is following. Two projective lattices $[M]$ and $[N]$ (projective equivalence classes of lattices) are connected by an edge if there exist representatives M and N such that $M \supset N \subset pM$. Note that this relation holds true only for some representatives, not all. It is also purely set-theoretic.
4. By reducing the situation to the simplest possible case $M \leftrightarrow (a, b) = (1, 1)$ one can easily find the lattices N connected to M. The calculations reduce to the finite field F_p since the inclusion condition implies that $M/pM \supset N/pM \supset pM/pM = \{0\}$ and M/pM is just F_p^2. The allowed N correspond are in one-one correspondence with the F_p subspaces of F_p^2 and there are $p+1$ of them corresponding to space generated by F_p multiples of $(a, 1)$, $a = 0, ...p-1$ and $(1, 0)$. Therefore the point $(a, b) = (1, 1)$ is connected to $p + 1$ neighbours by single edge. By symmetric space property this is true for all points of $P^1(Q_p)$. The conclusion is that edge paths correspond to a regular tree with valence $p + 1$.
5. $P^1(Q_p)$ is still totally disconnected in p-adic topology. The edge paths however provide $P^1(Q_p)$ with a path-connected topology. The example of Berkovich disk would suggest that one must add to $P^1(Q_p)$ something so that $P^1(Q_p)$ remains a dense subset of this larger structure. The situation would be same as for rationals: rationals become a path connected continuum if one adds all irrational numbers to obtain reals. Rationals define a dense subset of reals and numerics uses only them. In particular, integration becomes possible when irrationals are added. It is however not clear to me whether this kind of completion is needed.
   One can wonder what must be added to the set of Z_p lattices in Q_p^2 or to the set of their projective equivalence classes to build the global differentiable structure. The answer perhaps comes from the observation that the ends of Bruhat-Tits tree correspond to *K-rationals* expressible as ratios of two K-integers - something that numerics can catch at least in real case. Could the completion mean adding also the ends which are K-irrationals? If so then the situation would be very similar to that in TGD inspired definition of p-adic manifolds.
6. Every pair of points in the completion $P^1(Q_p)$ is connected by an edge path consisting of some minimal number n_{min} of edges and this edge path defines the analog of geodesic with length n_{min}. This number is p-adic integer and could be infinite as a real integer for the completion of the p-adic manifold to a path connected manifold. Here the canonical identification $\sum x_n p^n \rightarrow z_n p^{-n}$ mapping p-adic integers to real numbers and playing a key role in p-adic mass calculations could come into play and allow to obtain a real valued finite distance measure. Real distances have continuous spectrum in the interval $[0, p)$. The objection is that this definition is not consistent with the idea of algebraic continuation of integrals from real context.

This construction generalizes to algebraic extensions K of Q_p and also to higher-dimensional projective spaces and symmetric spaces. In particular, the construction of the p-adic counterpart of CP_2 becomes possible. Now one replaces Q_p^2 with Q_p^3 or K^3 allowing the action of some discrete subgroup of the isometry group $SU(3)$ of CP_2. Lattices in K^3 replace the points of Q_p^3 and defines the counterpart of Bruhat-Tits tree in exactly the same manner as for $P^1(K)$.

Physically the highly interesting point is that only a discrete subgroup of CP_2 can be represented in the algebraic extension so that symmetry breaking to discrete subgroup is un-avoidable. In TGD framework the interpretation is in terms of finite measurement resolution forcing discretization and therefore also symmetry breaking. This symmetry breaking is quite different from that defined by Higgs mechanism or symmetry breaking taking place for the solutions of field equations for a variational principle characterized by the unbroken symmetry group.

13.7.3 The Lattice Construction Of Bruhat-Tits Tree Fails For K^N But Works For $P^N(K)$: Something Deep?

The naïve expectation is that the construction of Bruhat-Tits tree should work also in the simplest possible case that one can imagine: for p-adic numbers Q_p themselves. The naïve guess is that the tree for p-adic numbers with norm bounded by p^n the tree is just the p+1-valent tree with trunk and representing all possible pinary expansions of these p-adic numbers. The lattice construction does not however give this correspondence.

Z_p lattices M in Q_p are parameterized by non-vanishing elements a of Q_p in this case. The multiplication by p-adic integer n of unit norm does not affect a given lattice M a since one has $nka = k_1a$ where n, k, k_1 are p-adic integers. Therefore these lattices are not in one-one correspondence with Q_p but with powers p^n: $|q|_p \leq p^n$ for a given lattice. Therefore the lattice construction fails. It is essential that one considers projective space $P^1(Q)$ instead of Q_p. For Q_p^2 the construction however seems to work.

Note: The condition $M \supset N \supset pM$ for the existence of an edge between two lattices allows only two solutions: the trivial solution $N = M$ and the solution $N = pM$. The counterpart of Bruhat-Tits tree is now 1-valent tree with edges labelled by powers of p.

Also in the case of Q_p^n the correspondence between lattices and points of Q_p^n is 1-to-many since the multiplication by an element of Z_p with unit norm does not affect the lattice. As a matter fact, all elements of Qp^n related by $Sl(n, Q_p)$ correspond to same lattice. Hence the replacement of points with lattices must be restricted to the case of projective spaces.

Physicist might argue that the use of lattices is un-natural and quite too complicated from the point of view of practical physics. I am not sure: it might be that the lattices have some nice physical interpretation and perhaps the outcome - the tree - is more important than the lattices used to achieve it. The fact is that p-adic projective spaces have this kind of "skeleton", and one might well argue that there is no need for the ugly looking completion to a bigger space with path connected and non-ultra-metric topology.

In TGD framework the p-adic variants of S^2 and CP_2 are central and the existence of the "skeleton" might be of fundamental significance from the point of view of p-adic TGD and number theoretical universality. Note that S^2 emerges naturally for the light-cone boundary in the case of M^4 ($\delta M^4_+ = S^2 \times R_+$, where R_+ represents light-like radial direction). For M^n, $n \neq 4$, one obtains $S^{k=n-2}$, $k \neq 2$, and this space is not projective space. Also in twistor Grassmannian approach to scattering amplitudes utilizing residue integrals in projective spaces $Gl(n, C)/Gl(n-m, C) \times Gl(m, C)$ this property for the p-adic counterparts of these spaces might be of primary importance.

13.7.4 Some Technicalities About Berkovich Disk

Berkovich disk is a p-adic generalization of open ball and meant to serve as a building brick of p-adic manifolds in the same manner as open ball is the building brick of real manifolds. The first guess is that ordinary open ball for p-adic numbers defined by $|x - a| < r$ could work. As a matter fact, p-adic distance is quantized: $|x - a| = p^n$ holds true. The basic outcome of total disconnectedness of the ultrametic topology is that two p-adic balls are either disjoint of the other one is contained by another one. One cannot build manifolds by taking p-adic balls and allowing them to partially overlap to get global differentiable structures and various topologies.

The construction of Berkovich disk - call it B - is motivated by the need to generalize the standard approach to the construction of real manifolds. I do not know whether it is equivalent with the approach based on Bruhat-Tits tree. The explicit realization of Berkovich disk as a completion of ultra-metric unit disk is something which one cannot guess easily but when one has understood that the basic premises are satisfied for it, it begins to look less artificial.

I try to explain this construction described briefly in the lecture notes "Buildings and Berkovich spaces" (see `http://tinyurl.com/y8ftfs6z`) [A102] by Annette Werner. I neglect all technical issues (I do not even understand them properly!). The basic idea is to imbed ultra-metric unit disk as a dense subset to some space possessing path connected topology. The challenge is to guess what this space is.

1. One starts from p-adic unit disk D: $|x|_p \leq 1$, which one wants to complete to Berkovich disk B containing D as a dense subset and possessing path connected topology. One could also replace Q_p with Q_p^n or K^n, where K is any algebraic extension Q_p. In the explanation provided in the lecture notes one considers for simplicity K, which is algebraically complete: this requires an algebraic extension allowing containing all algebraic numbers. This is unrealistic but the construction is possible also for general K but involves more technicalities.
2. One introduces the space of formal K-valued power series $f(z) = \sum f_n z^n$ in $D(0,1) \equiv D$. One can define for the an ultra-metric norm as $||f|| = Max\{|f_n|_K\}$. This is actually the supremum of p-adic norm $|f(x)|_K$ in $D(0,1)$. The p-adically largest coefficient f_n defines the

norm known as Gauss norm. This norm is multiplicative. For constant functions, which are in one-one correspondence with points of K, this norm reduces to K-norm.

3. One considers also more general norms. In fact, the space of norms with attributes ultra-metric, bounded, and multiplicative and reducing for constant functions to K-norm $||_K$ defines the Berkovich unit disk B, which turns out to be a completion of the unit disk D containing D as a dense subset. Furthermore, B turns out to have have path connected topology as required making possible global differentiable structure and even hopes about p-adic integration.
4. Berkovich manages to construct these norms explicitly. The simplest norms of this kind are defined by points a of D. The norm is simply $|f(a)|_K$. These norms are in one-one correspondence with points of D and should define a dense subset of the entire space of norms. The points of K are therefore mapped to subspace of the space of norms: this is absolutely essential.
5. There are also other multiplicative, ultra-metric norms reducing to $||_K$ for constant functions in D. They are defined in terms of disks $|x-a|_K \leq r \leq 1$. The Gauss norm corresponds to $r=1$ and the norm described in previous item to $r=0$. These norms are analogous to irrationals numbers in the case of completion of rationals to reals. The Berkovich disk B contains points of four different types.
   - Points of type 1: $|f_a| = |f(a)|_K$ (embedding of D to Berkovich disk B.
   - Points of type 2: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \in |K*|$, the value spectrum of K-norms (powers of p for Q_p). The Gauss norm corresponds to $r=1$.
   - Points of type 3: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \notin |K*|$. There is a delicate difference between types 2 and 3 which I fail to understand.
   - Points of type 4: $|f|_{a,r} = lim_{n\rightarrow\infty}|f|_{a_n,r_n}$ for a nested sequence $D(a_1,r_1) \supset D(a_2,r_2)$.... of closed disks in $D(0,1)$.
6. The topology in Berkovich disk is defined by a pointwise convergence of the norm in the space of functions f in D. This topology makes Berkovich disk path connected.

The above construction is rather complicated although and also assumes algebraic completeness. For finite-dimensional algebraic extensions the construction is expected to be even more complicated. I do not understand the possible connection between Bruhat-Tits tree and Berkovich construction: does Bruhat-Tits tree follow from Berkovich construction or not?

13.7.5 Could The Construction Of Berkovich Disk Have A Physical Meaning?

For the physicist the obvious question is whether the function space associated with the K-disk D could have some some physical interpretation? And what about the interpretation of the space of bounded multiplicative ultra-metric norms for this function space? Could these norms have some physical interpretation?

Consider first basic criticism what might be represented by a physicist.

1. The ultra-metric multiplicative norms in the function space carry extremely scarce information about the functions. Just the norm of the value of the function at single point. If one wants information in several points on must have a manifold consisting of large minimal number of Berkovich disks. An alternative manner to get information about the function space is to combine the information about all norms.
2. Physicists could also wonder what these K-valued functions are physically. Are they physical fields perhaps? If so, why not consider p-adic variants of correlation functions instead of p-adic norms scalars formed from these fields at single point. This forces however to ask whether the non-vanishing of these physical correlation functions for these fields could code for the existence of "connections" between points of the p-adic manifold so that there would be no need for the completion to Berkovich disk after all. Could the solution of the problem be achieved by bringing quantum physics a part of the definition of the manifold structure.
   It seems that in TGD framework there is no natural counterpart for the K-valued formal power series and their norms. One must perform a stronger generalization and this leads to the use of canonical identification mapping p-adic coordinate variables to their Archimedean norms defined by canonical identification and serving as real coordinates. Another, very speculative

approach would be based on correlation functions of fermion fields as a possible manner to code the physical counterpart of path connectedness.

Chapter 14

TGD and Non-Standard Numbers

14.1 Introduction

This chapter represents some comments on articles of Elemer E. Rosinger as a physicist from the point of view of Topological Geometrodynamics. To a large extent a comparison of two possible generalizations of reals is in question: the surreal numbers introduced originally by Robinson [?]nd infinite primes and corresponding generalization of reals inspired by TGD approach [?] The articles which have inspired the comments below are following:

"How Far Should the Principle of Relativity Go?" (see `http://tinyurl.com/ya76yv3t`)

"Quantum Foundations: Is Probability Ontological?" (see `http://tinyurl.com/y767ftxn`)

"Group Invariant Entanglements in Generalized Tensor Products" (see `http://tinyurl.com/yc8xzmp2`)

"Heisenberg Uncertainty in Reduced Power Algebras" (see `http://tinyurl.com/y8yzkmlt`)

"Surprising Properties of Non-Archimedean Field Extensions of the Real Numbers" (see `http://tinyurl.com/ycy4hex7`)

"No-Cloning in Reduced Power Algebras" (see `http://tinyurl.com/yd7bebuy`)

I have a rather rudimentary knowledge about non-standard numbers and my comments are very subjective and TGD centered. I however hope that they might tell also something about Rosinger's work [?] My interpretation of the message of articles relies on associations with my own physics inspired ideas related to the notion of number. I divide the articles to physics related and purely mathematical ones. About the latter aspects I am not able to say much.

The construction of ultrapower fields (generalized scalars) is explained using concepts familar to physicist using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields. Some questions related to the physical applications of non-standard numbers are discussed including interpretational problems and the problems related to the notion of definite integral. The non-Archimedean character of generalized scalars is discussed and compared with that of p-adic numbers. Rosinger considers several physical ideas inspired by ultrapower fields including the generalization of general covariance to include the independence of the formulation of physics on the choice of generalized scalars, the question whether generalized scalars might allow to understand the infinities of quantum field theories, and the question whether the notion of measurement precision could realized in terms of scale hierarchy with levels related by infinite scalings. These ideas are commented in the article by comparison to p-adic variants of these ideas.

Non-standard numbers are compared with the numbers generated by infinite primes. It is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1 represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra. The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

With motivations coming from quantum computation, Rosinger discusses also a possible generalization of the notion of entanglement [?]llowing to define it also for what could be regarded as classical systems. Entanglement is also number theoretically very interesting notion. For instance, for infinite primes and integers the notion of number theoretical entanglement emerges and relates to the physical interpretation of infinite primes as many particles states of second quantized supersymmetry arithmetic QFT. What is intriguing that the algebraic extension of rationals induces de-entanglement. The de-entanglement corresponds directly to the replacement of a polynomial with rational coefficients with a product of the monomials with algebraic roots in general.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

14.2 Could The Generalized Scalars Be Useful In Physics?

The basic question is whether the generalized scalars could replace reals in theoretical physics. It is best to proceed by making questions.

14.2.1 Are Reals Somehow Special And Where To Stop?

The following questions relate to the interpretation of generalized scalars.

1. Why reals should be so special? The possible answer is that reals, complex numbers and quaternions form associative continua. Classical number fields are indeed in central role in TGD [K86], [L7]. Already p-adic number fields consist of disconnected pieces in the sense that one cannot connect two arbitrary points by a continuous curve (p-adic norm of point must change discontinuously at some point of curve is the norms of end points are different).
2. What -if anything physical- it means to replace temperature at space-time point with a function of a natural number? Doesn't this mean the replacement of real numbers with $R \times N$ and replacement of Minkowski space with $M^4 \times N^4$?
3. What is the physical meaning of generalized scalar understood as an equivalence class of real functions of natural number modulo functions vanishing in some set belonging to a filter (possibly ultrafilter)? What could be the physical meaning of filter? Could the quotient construction be interpreted as some sort of gauge invariance or could it just realize the idea "almost-everywhere is everywhere physically" ?
4. Can one stop if the step replacing reals with generalized scalars is taken? Recall that quantization means replacement of the WCW with the function space associated with it. Second quantization brings in function space associated with this space and so on. This hierarchy of quantizations is involved with the construction of infinite primes (and rationals) in TGD framework [K84], [L9] and in this case one has a concrete physical interpretation in terms of many-sheeted space-time.

   Should one replace natural numbers with the power set of natural numbers consisting of finite subsets of natural numbers (dual of the Frechet filter for $\mathbb{N}$) at the next step and perform similar construction. This could be continued ad infinitum. Does one obtain an infinite hierarchy of increasingly surreal numbers in this manner? One can imagine also other kinds of constructions but it is this construction with would be analogous to that for the hierarchy of infinite primes.

14.2.2 Can One Generalize Calculus?

The obvious question of physicist is whether one can generalize differential and integral calculus - necessary for physics as we know it. Surreals (see `http://tinyurl.com/3yacx24`) were actually introduced to justify the notion of infinitesimal so that differential calculus should not be a problem. The notion of integral function is neither a problem but definite integral might be due to the loss of Archimedean property. One could try to define the notion of integral in terms of the embedding of real numbers as constant functions and define definite integral algebraically as a substitution of the integral function between real limits. For arbitrarily limits one cannot order the limits and it seems that one should restrict the considerations to real limits.

What might also pose a problem is the definition of numerical integration - in terms of Riemann sum in its simplest form. One should divide the integration range to short ordered pieces and approximate the integral with sum. But there exists infinite number of paths connecting two functions to each other and one cannot order the pieces in general. Should one generalize complex analyticity so that functions of surreals would be expressible as power series of function and the integrals would not depend on integration path unless the surreal analytic function has singularities such as poles? Does this mean that one can choose one particular path which corresponds a path restricted to real axis so that the integral would reduce to the ordinary real integral.

In p-adic context non-Archimedean property implies that the notion of definite integral is indeed problematic [K60]. The basic problem is that one cannot in general tell which one of the two p-adic numbers with the same norm is the larger one and therefore one cannot define the notion boundary essential in variational calculus. One could use algebraic definition of definite integral as a substitution of integral function and in complex case residue calculus could help. One could use the ordering of rational numbers imbedded to p-adic numbers fields to induce the ordering of p-adic rationals. The p-adic existence of the integral function poses additional conditions encountered already for the integrals of rational functions which can give logarithms of rationals leading out from the realm of rationals. These difficulties have served as a key guiding principle in the attempts to fuse real and p-adic physics to a larger structure.

14.2.3 Generalizing General Covariance

What happens to the notion general covariance (or Principle of Relativity in the terminology used by Rosinger, see the article How Far Should the Principle of Relativity Go? (see `http://tinyurl.com/ya76yv3t`) [A130])? Here I would like to do some nitpicking by distinguishing between Principle of Relativity which refers to the isometries of Minkowski space and General Coordinate Invariance analogous to gauge symmetry. Various symmetry groups make sense also in the surreal context since they are defined algebraically. A generalization of General Coordinate Invariance meaning that the formulation of physics becomes independent of the choice of generalized scalars is proposed by Rosinger. This notion could be interpreted as a form invariance or as the condition that the physics is indeed the same irrespective of what number field is used in which case the introduction of generalize scalars would not bring in anything new.

Rosinger chooses the non-trivial option which means that the formulation of the laws of physics should make sense irrespective of the number field chosen and considers various examples as applications of the generalized view. He shows that no-cloning theorem (see `http://tinyurl.com/yd7bebuy`) of quantum computation holds true also for generalized scalars because the theorem depends on the linearity of quantum theory alone (cloning would map state to two of its copies, something essentially nonlinear).

In TGD framework the notion Number Theoretical Universality interpreted as number field independent formulation of physics seems to relate closely to this principle.

1. All constructions making sense in real context should makes sense also in the p-adic context [K85], [L8]. Real and p-adic physics meet in the intersection of real and p-adic worlds and result from each other by a kind of algebraic continuation. Simplifying somewhat, at the level of space-time surfaces the intersection would correspond to rational points in some preferred coordinates shared by real and p-adic surfaces and at the level of "world of classical worlds" (WCW) to surfaces expressible in terms of rational functions expressible using polynomials with rational coefficients so that real and p-adic variants of this kind of surfaces are can be identified.
2. Number Theoretic Universality leads to extremely powerful conditions on the geometry of WCW since both its real and p-adic sectors should exist and integrate to a larger structure [K100]. Rationals defining the intersection of reals and various p-adics play a key role and one ends up with a generalization of number concept obtained by gluing reals and p-adics as well as their algebraic extensions to single book like structure [K85], [L8].
3. One is also forced to adopt a more refined view about General Coordinate Invariance since the coordinate transformations must respect the algebraic extensions of p-adic numbers used. This

brings also non-uniqueness: there are several choices of coordinate frames not transformable to each other. The interpretation would be that they serve as correlates of cognition. Mathematician is not an outsider and the choice of coordinate system affects the reality albeit in very delicate manner.

This allows to see a relationship between TGD inspired fusion of real and p-adic physics and Rosingers's proposal as roughly following correspondence.

Reals and p-adic number fields resp. rationals defining the intersection of reals and p-adic worlds ↔ various generalized scalars resp. reals defining the intersection of various surreals worlds.

The independence on the choice of generalized scalars might give powerful constraints on the formulation of the theory.

If surreal number fields are important for theoretical physics, physical systems must be characterized by the generalized scalars. What determines this number field or algebra? Can one speak about some kind of quantal evolution in which physical systems evolve more and more complex number theoretically. Could the field of generalized scalars be replaced with a new one in quantum jump taking place via reals common to different generalized scalars?

The attempt to fuse real physics as physics of matter and p-adic physics as physics of cognition one ends up with this kind of picture and one can say that the prime characterizing p-adic number field and the algebraic numbers defining its extension (say roots of unity) characterize its evolutionary level. During evolution the algebraic complexity of the systems steadily increases.

14.2.4 The Notion Of Precision And Generalized Scalars

Rosinger proposes [A133] that the notion of precision of experiment could be assigned to the self-similar structure of the generalized scalars meaning a hierarchy of scales which differ from each other by infinite scale factors if real norm is used as a measure for the scale. There would be infinite hierarchy of precisions and what looks infinitesimal, finite, or infinite would depend on the precision used and characterized by what generalized scalars are used. Thus one can speak about relative precision.

That one could have units of (say length) differing by infinite scaling in real sense looks rather weird idea. In TGD framework one interpretation for the hierarchy of infinite primes would be that there is infinite hierarchy of variants of Minkowski space such that at the given level of the hierarchy lower levels represent infinitesimals. This would mean fractal cosmology in which the conscious entities above us in the hierarchy would be literally God like as compared to us. No hopes about testing this at LHC!

In p-adic context similar notion emerges but the infinities at different levels are not related by infinite scalings with respect to the p-adic measure for size. Given walkable world correspond in p-adic context to p-adic numbers with fixed norm and in this operational sense p-adic primes with larger norm are infinite. p-Adic prime p indeed characterizes length scale resolution and the roots of unitary used in algebraic extension of p-adics characterize the angle resolution.

Even more, if one accepts that p-adic space-time surfaces serve as correlates for cognition one is forced to conclude that cognition cannot be localized in a finite space-time volume and that "thought bubbles" have actually the size of the entire Universe. Only cognitive representations defined by rational intersections of real and p-adic space-time surfaces would be localized to a finite real volume. Maybe the infinite hierarchy of Rosinger could be assigned to the levels of existence that we are used to assign with cognition and matter corresponds to the lowest level.

14.2.5 Further Questions About Physical Interpretation

Rosinger raises further interesting questions about physical interpretation.

1. In the article *Does Heisenberg Uncertainty Principle make sense in reduced power algebras?* (see `http://tinyurl.com/y8yzkmlt`) [A133] Rosenberg shows that the answer to the question of the title is affirmative. Rosinger asks in the same article whether the values of fundamental constants like c and $\hbar$ depend on the choice of generalized scalars. For instance, could $\hbar$ be infinitesimal for some generalized scalars? Could c have a well-defined infinite value for some generalized scalars.

In the case of c one could argue that it is just a conversion factor so that one can put $c = 1$ always by a suitable choice of units. Most physicists would argue that the same is true for $\hbar$. I have however proposed a different vision explaining some strange findings in both astrophysics and biology.

2. Could the fact that infinitesimal and infinite numbers have precise meaning for generalized scalars allow to resolve the problems caused by the infinities of local quantum field theories? Rosinger argues that this might be the case (see `http://tinyurl.com/y8yzkmlt`) [A133]. The notion of infinity is relative one for generalized scalars and one could replace reals with some other generalized scalars and this could make infinite finite. As a matter fact, in p-adic context for a given p-adic number all p-adic numbers with larger norm represent an operational infinity in the sense that they cannot be reached by walks consisting of integer valued steps. As p-adic numbers they are however finite. It seems that one must be very careful how one defines the infinite: does one use norm or does on use reachability by integer valued steps as the criterion. One can counter argue that reals can be distinguished uniquely by their topological properties just like rationals can be distinguished by their number theoretic properties uniquely. Skeptic might say that the situation would become even worse since one would had infinite number of different kind of infinities. The infinities would be completely well-defined functions with finite number of poles but what it means to replaces temperature at space-time point with a function of natural number? Doesn't this mean that space-time point is replaced with natural numbers.

   I have myself considered the possibility that p-adic mathematics for which integers infinite in real sense can make sense p-adically and have norm not larger than unity could allow to resolve the problem of infinities. In particular ultra-metric topology implies that the sum of n numbers is never larger than the maximum of the largest number involved -this is just what walkable universe expresses- raises optimism. It turned however that these ideas did not work in my hands.

14.3 How Generalized Scalars And Infinite Primes Relate?

The comparison of Rosinger's ideas with the number theoretic ideas of TGD inspires further questions.

1. Classical number fields play a key role in the formulation of quantum TGD. Do the notions of sur-complex, sur-quaternionandsur-octonion make sense as one might expect?
2. What happens if one replaces real functions define in Λ (say natural numbers) with p-adic valued functions. One obtains algebra also now and one can define ideals and use quotient construction using ultrafilter. Does the notion of sur-p-adic make sense?
3. In TGD framework one ends up with the notion of infinite prime having direct connection with repeated second quantization of super-symmetric arithmetic quantum field theory with fermions and bosons labelled by primes- finite primes at the lowest level of hierarchy. This notion of infinity is essentially number theoretical and implies that the number theoretic anatomy of numbers and space-time points becomes an essential aspect of physics. Can one assign number theoretic anatomy also to non-standard numbers or does the real topology wipe it out?
4. How does the hierarchy of infinite primes relate to the possibly existing hierarchy of reals, surreals, sursurreals, ... obtained by replacing real number valued function with surreal number valued functions replaced in turn with....?

The last question deserves a more detailed consideration since it could provide an improved understanding of infinite primes. Consider first the construction of infinite primes [K84], [L9].

1. Infinite primes at the lowest level of hierarchy can be generated from two fermionic vacuum states $P_\pm = X \pm 1$, where X is defined as a product of all finite primes having p-adic norm less than one for all finite primes p. X is analogous to Dirac sea with all negative energy states filled. Simple infinite primes are of form $mX/n + rn$, where m and n have no common divisors and r consists of same primes as n. $m = \prod p_i^{k_i}$ corresponds to many boson state with k_i bosons with "momentum" p_i. In fermionic sector the square free integer n has interpretation as many-fermion state with single fermion in the modes involved. r corresponds to many-boson states in these modes. Simple infinite primes are clearly analogous to many particle

states obtained by kicking fermions from sea to get positive energy holes and adding bosons whose number is arbitrary in a given mode labelled by finite prime. Simple infinite primes have unit p-adic norm so that "infinite" is a relative notion.

2. More complex infinite primes are infinite integers obtained as sums of products of infinite primes. The interpretation is in terms of bound many-particle states.
3. In zero energy ontology (ZEO) an attractive interpretation for infinite rationals is as zero energy states with numerator and denominator representing positive and negative energy parts of the state.
4. One can continue the construction indefinitely. At the next level X is replaced with the product of all infinite primes at the first level of the hierarchy and the process is repeated. The physical interpretation would be that at the next level many particle states of previous level take the role of single particle states and one constructs free and bound many particle states of these. The many-sheeted space-time of TGD suggests a concrete realization of this process and I have indeed proposed a concrete physical interpretation of standard model quantum numbers in terms of what I call (hyper-)octonionic primes, which would generate a structure analogous to infinite primes.

Generalized scalars define a function algebra and this inspires the question is whether one could somehow assign a function algebra also to infinite primes and in this manner to see what is common features these very different looking notions might have. Infinite primes can be indeed mapped to polynomial primes as the following argument shows.

1. Simple infinite primes are characterized by two integers which have no common divisors and can be thus mapped in a natural manner to rationals $q = rn^2/m$. They can can be also mapped to monomials $x - q$, $q = rn^2/m$, where X could be seen as a particular value of x. Complex infinite primes constructed as products of simple infinite primes can be mapped to products of these monomials and sums of their products to sums of these so that on obtains a mapping to polynomial primes at the lowest level of the hierarchy. Vacua are mapped to rationals 1 and -1. One can decompose the polynomials to products of monomials $x-r$, where r is a finite algebraic number, and the interpretation would be that one considers primes in an algebraic extension of rationals and this representation applies to infinite prime when x is substituted with X.
2. This mapping makes sense also at the next level of hierarchy at least formally. Call the product of finite and infinite primes at the first level X_1 and corresponding formal variable x_1. Infinite rationals correspond now to rational functions of x_1 and x defined as ratios of polynomials $P_k(x_1, x)$ for which the highest power of x_1 is by definition x_1^k. The roots in the product representation of polynomials are obtained by the substitution $x \rightarrow X$ in the expressions of the roots as functions of x. The roots are generalized algebraic numbers which can be infinite or vanish as real numbers. This kind of mapping makes also sense at the higher levels of hierarchy. The roots of polynomial at the n: th level of the hierarchy are obtained by substituting to their expressions as algebraic functions $x_m = X_m$, $m < n$.
3. What one obtains is a map to polynomials so that one can indeed map infinite primes and also integers and rationals to a function algebra consisting of polynomials. Ideals correspond now to polynomial ideals consisting of polynomials proportional to some polynomial prime. There are no divisors of zero so that quotient construction is not needed now.

This construction leads to intriguing observations relating the construction of infinite primes to the construction of generalized scalars and suggesting that infinite primes represent a generalization of the concept of sur-complex numbers by identifying ultrafilter in terms of complements of finite subsets of algebraic numbers (Frechet filter actually). The heuristic argument goes as follows.

1. The hierarchy of subsets of algebraic numbers defined by the infinite primes at the lowest level of hierarchy defines *complement of Frechet filter* $\mathcal{CF}$ with the following defining properties. $\mathcal{CF}$ contains empty set and all finite subsets of Λ, unions of sets of $\mathcal{CF}$ belong to $\mathcal{CF}$, and subsets of a set belonging to $\mathcal{CF}$ belong to $\mathcal{CF}$.

   Note that powers of infinite primes define the same set in $\mathcal{CF}$ as infinite prime itself so that the correspondence does not seem to be many-to-one. It is not clear whether fermionic statistics could be used as a physical excuse to exclude these powers and more generally products of infinite primes for which same finite prime appears in more than one different infinite primes. Also subsets of genuinely algebraic numbers could correspond to several infinite integers and rationals.

If one restricts the consideration to square free integers defined by the fermionic parts of infinite primes then the sets of natural numbers assignable to infinite primes correspond to finite subsets of square free natural numbers defining a Frechet filter for them.

2. $\Lambda = \mathbb{N}$ is replaced with algebraic numbers $\mathbb{A}$ so that the function space defining generalized scalars would consist of functions $f : \mathbb{A} \rightarrow \mathbb{C}$. It is not however clear what kind of functions one should consider.
   (a) The first guess is that the quantum states of supersymmetric arithmetic QFT (SAQFT) correspond to functions non-vanishing only in some finite set belonging to $\mathcal{CF}$. They would map to zero in the quotient construction of ultrapower field. The functions which do not map to zero would correspond to non-vanishing elements of the ultrapower field and would have no physical interpretation. This does not sound sensible physically.
   (b) The many-particle states of arithmetic QFT could more naturally correspond to functions having values on circle S^1 -rather than $\mathbb{C}$- identified as complex numbers with unit magnitude. The value of this kind of functions would be constant - most naturally 1 - for given infinite set of $\mathcal{U}$ and root of unity in the complement of $\mathcal{U}$ defined by infinite integer or rational.

   These functions would be analogous to plane waves having modulus equal to 1 and if they correspond to roots of unity they would make sense also for algebraic extensions of p-adic numbers. This conforms with the fact that p-adic norms of infinite primes and rationals are equal to unity. This would lead to a rather astonishing conclusion: there are no infinite numbers nor infinitesimals in the field generated by infinite primes in the sense of generalized scalars!

   Note that functions which reduce to phases in the set of algebraic numbers are also natural in the sense that there are hopes of defining for them inner product as sum over algebraic numbers. The inner product should be consistent with the inner product induced by that for Fock states and it might be better to start directly from this inner product.
   (c) It is important to realize that the complements of infinite rationals do not define support for functions but the functions themselves so that the analogy with the ultrapower construction fails.
3. The higher levels in the hierarchy of infinite primes are also present and require a further generalization of the construction. At the second level of the hierarchy algebraic numbers are replaced with the power set consisting of all finite subsets of algebraic numbers and dual of Frechet filter with that consisting of all finite subsets of this power set. Higher levels of the hierarchy would correspond a repeated replacement of the set with its power set.
4. Mathematical skeptic reader might wonder why this infinite hierarchy of constructions? Does it even lead outside the realm of algebraic numbers? What is however remarkable is that it generalizes the physics by replacing the first two quantizations with an infinite hierarchy of quantizations.

14.3.1 Explicit Realization For The Function Algebra Associated With Infinite Rationals

Consider now an explicit realizations of this algebra as a function algebra. The idea is to assigns to a given infinite rational a unique phase representing and that the algebraic structure defined by multiplication is preserved. This is like mapping rationals $q = m/n$ to phases $exp(i2\pi q)$ so that products are mapped to products. One can start from the observation that simple infinite primes can be mapped to rationals. More complex infinite primes, integers, and rationals can be mapped to collections of algebraic numbers representing the roots of corresponding polynomial primes.

1. The simplest option is that the value of the complex valued function of algebraic numbers assigned to simple infinite prime characterized by rational q is equal to $exp(i2\pi q)$ for rational q and to 1 for other algebraic numbers. The product of simple infinite integers os mapped to the product of these functions assigned to the factors. The ratio of two simple infinite integers is mapped to the ratio of corresponding functions.

2. By utilizing the decomposition the map to polynomial or rational function and its decomposition into monomials with possibly algebraic roots one could map the polynomials of rational function to factors $\prod_i exp(2\pi r_i)$ for a given infinite rational in its polynomial representation decompose to a product of monomials. This representation would map products (ratios) of infinite integers to products (ratios) but sums would not be mapped to sums but products in algebraic extension of rationals. That the images would be always non-vanishing functions would conform with the basic properties of infinite primes and with non-existence of infinitesimals and infinite numbers in the sense of the usual ultrapower construction.
3. One would have functions in the set of algebraic numbers at the first level of hierarchy. At the next level of hierarchy one would have complex complex defined in the set of generalized rationals constructed from infinite integers. These phases are actually well defined since the infinite rational appearing in the exponent can be decomposed to a sum of terms. Only those terms which are finite contribute to the phase so that one obtains a well-defined outcome. This hierarchy would continue ad infinitum. Similar hierarchy can be associated with generalized scalars.
4. Primes are replaced with prime ideals in a more abstract approach to number theory. One could also assign to the rationals assigned to simple infinite primes the prime ideal of real or complex valued functions with value equal to one for all rationals except the selected rational. The product of simple infinite primes would correspond to the ideal consisting of functions which differ from unity for the rationals appearing in the product. The sum of simple infinite primes would in turn correspond to similar functions but differing from unity also for algebraic numbers. This would give a hierarchy of ideals with particular ideal defined in terms of functions whose value is larger than integer n for most rationals and algebraic numbers.

14.3.2 Generalization Of The Notion Of Real By Bringing In Infinite Number Of Real Units

Infinite rationals lead also to a generalization of the real numbers in the sense that given real number is replaced with infinitude of numbers having the same magnitude by multiplying it by real units which differ number theoretically [K84], [L9]. There exists infinite number of rationals constructed as ratios of infinite integers at various levels of the hierarchy which as real numbers are equal to real unit but have arbitrarily complex number theoretical anatomy. Single point of real line is replaced with infinitely complex infinite-dimensional structure defined by the space of real units. This generalization applies also to other classical number fields. The role of infinitesimals would be taken by the infinitude of real units and this would extend real numbers.

This has inspired the ontological proposal that the quantum states of Universe (and even the world of classical worlds (or its sub-world defined associated with 4-surfaces inside $CD\times CP_2$) could be imbedded to this space. A less wild statement is that at least the quantum states and sub-WCW assignable to the so called causal diamond identified as the intersection of future and past directed light-cones and defining the basic structural unit in zero energy ontology can be realized in terms of the number theoretic anatomy of single space-time point.

Real units (and their generalizations to octonionic context) are analogous to quantum states. Their sum is analogous to a quantum superposition and gives a real unit by using a simple normalization. Real units are also analogous to zero energy states. By writing each infinite prime P_i at a given level of hierarchy in the form $P_i = Q_i(X_n - 1)$ (note that P_i is infinitesimal as compared to X_n), one finds that real unit condition implies that the total numbers of X_n: s in the numerator and denominator of a real unit must be same. One can apply the same procedure for the factor

$$\frac{\prod_{num} Q_i}{\prod_{den} Q_i}$$

(here "*num* " and "*den*" denote numerator and denominator of infinite prime) to conclude that it must contain same number of X_{n-1}: s in its numerator and denominator. At the lowest level one finds that one obtains ratio of integers expressed as products of powers of finite primes p_i which must be equal to unity. The interpretation in positive energy ontology is that the total number theoretic momentum coming as integer multiple of $log(p_i)$ is same for the positive and negative energy parts of the state and therefore conserved for each finite prime p_i separately (the numbers

$log(p_i)$ are algebraically independent). Conservation is indeed what one expects in arithmetic QFT.

$M^4 \times CP_2$ with structured space-time points could be able to represent all the structures of quantum theory having otherwise somewhat questionable ontological status. A given mathematical structure would "really" exist if it allows embedding to generalized $M^4 \times CP_2$, which itself has interpretation in terms of classical number fields. Accordingly, one could talk about number theoretic Brahman=Atman identity or algebraic holography.

The above considerations suggest that the hierarchy of infinite primes and hierarchy of generalized scalars cannot be identified. It is not clear clear whether could consider the fusion of these notions. Also the fusion of real and p-adic number fields to a book like structure and of generalized scalars could be considered.

14.3.3 Finding The Roots Of Polynomials Defined By Infinite Primes

Infinite primes identifiable as analogs of bound states correspond at n: th level of the hierarchy to irreducible polynomials in the variable X_n which corresponds to the product of all primes at the previous level of hierarchy. At the first level of hierarchy the roots of this polynomial are ordinary algebraic numbers but at higher levels they correspond to infinite algebraic numbers which are somewhat weird looking creatures. These numbers however exist p-adically for all primes at the previous levels because one one can develop the roots of the polynomial in question as powers series in X_{n-1} and this series converges p-adically. This of course requires that infinite-p p-adicity makes sense. Note that all higher terms in series are p-adically infinitesimal at higher levels of the hierarchy. Roots are also infinitesimal in the scale defined X_n. Power series expansion allows to construct the roots explicitly at given level of the hierarchy as the following induction argument demonstrates.

1. At the first level of the hierarchy the roots of the polynomial of X_1 are ordinary algebraic numbers and irreducible polynomials correspond to infinite primes. Induction hypothesis states that the roots can be solved at n: th level of the hierarchy.
2. At $n+1$: th level of the hierarchy infinite primes correspond to irreducible polynomials

$$P_m(X_{n+1}) = \sum_{s=0,...,m} p_s X_{n+1}^s \quad .$$

The roots R are given by the condition

$$P_m(R) = 0 \quad .$$

The ansatz for a given root R of the polynomial is as a Taylor series in X_n:

$$R = \sum r_k X_n^k \quad ,$$

which indeed converges p-adically for all primes of the previous level. Note that R is infinitesimal at $n+1$: th level. This gives

$$P_m(R) = \sum_{s=0,...,m} p_s (\sum r_k X_n^k)^s = 0 \quad .$$

(a) The polynomial contains constant term (zeroth power of X_{n+1} given by

$$P_m(r_0) = \sum_{s=0,...,m} p_r r_0^s \quad .$$

The vanishing of this term determines the value of r_0. Although r_0 is infinite number the condition makes sense by induction hypothesis.
One can indeed interpret the vanishing condition

$$P_{m\times m_1}(r_0) = 0$$

as a vanishing of a polynomial at the n: th level of hierarchy having coefficients at $n-1$: th level. Here m_1 is determined by the dependence on infinite primes of lower level expressible in terms of rational functions. One can continue the process down to the lowest level of hierarchy obtaining $m \times m_1 ... \times m_k$: th order polynomial at k: th step. At the lowest level of the hierarchy one obtains just ordinary polynomial equation having ordinary algebraic numbers as roots.

One can expand the infinite primes as a Taylor expansion in variables X_i and the resulting number differs from an ordinary algebraic number by an infinitesimal in the multi-P infinite-P p-adic topology defined by any choice of n-plet of infinite-P p-adic primes $(P_1, ..., P_n)$ from subsequent levels of the hierarchy appearing in the expansion. In this sense the resulting number is infinitely near to an ordinary algebraic number and the structure is analogous to a completion of algebraic numbers to reals. Could one regard this structure as a possible alternative view about reals remains an open question. If so, then also reals could be said to have number theoretic anatomy.

(b) If one has found the values of r_0 one can solve the coefficients r_s, $s > 0$ as linear expressions of the coefficients r_t, $t < s$ and thus in terms of r_0.

(c) The naïve expectation is that the fundamental theorem of algebra generalizes so that that the number of different roots r_0 would be equal to m in the irreducible case. This seems to be the case. Suppose that one has constructed a root R of P_m. One can write $P_m(X_{n+1})$ in the form

$$P_m(X_{n+1}) = (X_{n+1} - R) \times P_{m-1}(X_{n+1}) \ ,$$

and solve P_{m-1} by expanding P_m as Taylor polynomial with respect to $X_{n+1} - R$. This is achieved by calculating the derivatives of both sides with respect to X_{m+1}. The derivatives are completely well-defined since purely algebraic operations are in question. For instance, at the first step one obtains $P_{m-1}(R) = (dP_m/dX_{n+1})(R)$. The process stops at m: th step so that m roots are obtained.

What is remarkable that the construction of the roots at the first level of the hierarchy forces the introduction of p-adic number fields and that at higher levels also infinite-p p-adic number fields must be introduced. Therefore infinite primes provide a higher level concept implying real and p-adic number fields. If one allows all levels of the hierarchy, a new number X_n must be introduced at each level of the hierarchy. About this number one knows all of its lower level p-adic norms and infinite real norm but cannot say anything more about them. The conjectured correspondence of real units built as ratios of infinite integers and zero energy states however means that these infinite primes would be represented as building blocks of quantum states and that the points of embedding space would have infinitely complex number theoretical anatomy able to represent zero energy states and perhaps even the world of classical worlds associated with a given causal diamond.

14.4 Further Comments About Physics Related Articles

In the following I represent comments on the physics related articles of Rosinger not directly related to generalized scalars. I have not commented the purely mathematics related more technical articles since I do not have the competence to say anything interesting about them.

14.4.1 Quantum Foundations: Is Probability Ontological?

In this highly interesting article [A131] (see `http://tinyurl.com/y767ftxn`) Rosinger poses the question whether the notion of probability is ontological or only epistemic. Are probabilities basic aspect of existence or are they are "a useful construct of mind only". My own very first reaction is a counter question. Can one speak about "mere construct of mind" ? "Mind" is a part of existence and the future physics must include it to its world order. If mind is able to construct a notion like probability this notion could have some quantal correlate.

Rosinger introduces the notions of deterministic (classical typically) and non-deterministic systems and distinguishes probabilistic, fuzzy and chaotic systems as special cases of non-deterministic systems. For fuzzy and chaotic systems probability is clearly a fictive but useful notion. For probabilistic systems, in particular quantum systems the situation is not clear at all.

As a mathematician Rosinger raises purely mathematical objections against the ontological status of probability. Rosinger mentions the technical difficulties with the description of stochastic processes with continuous time and objections against axiomatizations - say in terms of Kolmogorov axioms (see `http://tinyurl.com/ybhpw7yq`). Rosinger mentions also frequency interpretation and somewhat fuzzy propensity interpretation (see `http://tinyurl.com/yafc2m2o`) of probabilities and that the notion of infinity is unavoidable also now. I cannot say much about these technical aspects and can only represent the comments based on my own physics inspired belief system.

To my very subjective view the situation is far from settled from the point of view of theoretical physics and one can consider several deformations of the notion of probability.

1. Khrennikov [A104] has formulated the notion of p-adic valued probability and also I have considered p-adic thermodynamics based model for particle masses (see the first part of [K59]) whose predictions, which are basically due to number theoretic existence constraints- are mapped to real numbers by a canonical correspondence between reals and p-adics.
2. Also the notion of quantum spinors related in TGD framework to the description of finite measurement resolution [K99] raises the possibility that the probability itself becomes observable instead of spin (by the finite precision associated with the determination of quantization axes) and has a universal spectrum.
3. The findings of Russian biologist Shnoll [K5], [C2], [C2] suggesting that the expected single peaked distributions for fluctuations of various process described by probability distributions for integer valued observable are replaced by many-peaked distributions encourage to think that the time scale of experiment is essential and the usual idea about smooth approach to probabilities as the duration of experiment increases is not correct. I have proposed an explanation of these findings in terms of the deformations of probability distributions depending on rational valued parameters so that they make sense also p-adically. This predicts precise and universal deviations which can be tested.

Rosinger relates [A131] the famous Bohr-Einstein debate to the ontological status of probability concept. The divisor line between Bohr and Einstein was the attitude towards non-determinism. Neither of them could accept the idea that the determinism of Schrödinger equation could fail temporarily. Bohr was ready to give up the notion of objective reality altogether whereas Einstein refused to accept state function reduction since it would have meant giving up also the deterministic dynamics of the space-time geometry. According to Rosinger, Copenhagenist would regard probability and probability amplitudes as a fundamental aspect of existence whereas Einstein would have given for probability only episthemic role.

To my opinion both Einstein and Bohr were both right and wrong. If one accepts the view that quantum states actually correspond to superpositions of deterministic histories (generalized Bohr orbits) -as suggested also by holography principle- the problem disappears. Quantum jump recreates the quantum state as quantum superposition of entire deterministic time evolution rather than tinkering with a particular time evolution. There is no contradiction between the determinism of field equation and non-determinism of quantum jump and genuine evolution emerges as a by-product.

In this framework one also ends up with the identification of theory as a mathematical objects with the reality itself. There is no need to assume reality behind the quantum states as mathematical objects. Reality is its mathematical description as quantum state and therefore nothing but this "construct of mind". Probability amplitudes receive a firm ontological status and in TGD framework correspond to what I call spinors fields of WCE having purely geometric interpretation. Whether probabilities defined in terms of density matrix have independent ontological status is not quite clear. In quantum theory continuous stochastic process would not really occur and could be seen as a mere idealization of a process which takes as discrete quantum jumps. The technical difficulties in their description would not represent argument against the ontological status of probability amplitudes.

Thermodynamical probability is usually regarded as having only episthemic status but in zero energy ontology - one characteristic aspect of TGD quantum - positive energy quantum states are replaced with zero energy states which can be regarded mathematically as complex square roots of density matrices -which I call M-matrices- decomposable to diagonal matrix representing square roots of probabilities and unitary S-matrix. M-matrices can be organized to orthogonal rows of unitary U-matrix defining the theory. Does this mean thermodynamical holography in the sense that single particle states are able to represent the mathematics of thermodynamical ensembles in terms of their quantum states?

14.4.2 Group Invariant Entanglements In Generalized Tensor Products

Rosinger proposes [A132] (see http://tinyurl.com/yc8xzmp2) a generalization of the notion of entanglement from Hilbert space context to much more general context. The motivation is that it might allow quantum computation like operations even in classical physics context so that the problems caused by the fragility of quantum entanglement could be circumvented.

Recall that ordinary quantization leads from Cartesian product to tensor product as one replaces the points of Cartesian factors with quantum states localized at these points and forms all possible tensor products and also their superpositions. In quantum theory entanglement would emerge at the level of the function space associated with Cartesian space. Already ordinary functions of several variables allow entanglement in this sense. Un-entangled functions of several variables correspond to products of functions of single variable and the sums of these products are in general entangled. Quite generally the special functions of mathematical physics emerges as separable/un-entangled solutions of linear partial differential equations and non-linearity typically implies entanglement in this sense.

The goal of Rosinger is to generalize this framework that is to find spaces - which he calls non-Cartesian spaces- containing Cartesian product as a sub-space with the points in the complement of Cartesian product identified entangled states. Rosinger defines what he calls group invariant entanglement for a Cartesian product and shows that group operations respect the property of being entangled. As an example sequences of point pairs of Cartesian product with algebraic operation analogous to tensor product defined by convolution are considered.

The notion of entanglement has turned out to be highly interesting and non-trivial also in TGD framework.

1. A rather abstract view about entanglement is in terms of correlations. In TGD framework quantum classical correspondence realized as holography defines a very abstract form of entanglement. In this case, the quantum states assignable to the partonic 2-surfaces plus 4-D tangent space-data correspond to classical physics in the interior of space-time surface so that one obtains entanglement through this correlation. This kind of entanglement would give rise to quantum classical correspondence.
2. For infinite primes [K84], [L9] the notion of entanglement emerges naturally from number theory. This is not so surprising because they can be interpreted in terms of Fock state basis for second quantized arithmetic quantum field theory. The point is that the sum of infinite integers cannot be done by using fingers since we do not possess infinite number of fingers. Therefore the sum of infinite integers is just as it is written: one cannot in general eliminate the plus from the expression unless one leaves the realm of rationals in which case one can decompose the infinite integer to a product of infinite primes. The sums of infinite integers are like superpositions of quantum states and one cannot indeed use reals as field multiplying the infinite primes. Since the products of infinite primes at the lowest level of hierarchy involve parts which can be organized to a polynomial in powers of the variable X defined by the product of finite primes identifiable formally as a variable of polynomial, one can find the expansion of infinite integer as sums over products of infinite primes and this representation is very much like the representation of entangled state.
   What is interesting is that a decomposition into unentangled state product state is obtained if one allows algebraic extension of rationals and the question is whether something like this could be achieved also for quantum states quite generally by some extension of state space concept.

Entanglement has also other number theoretic aspects.

1. One could speak about irreducible entanglement in a given extension of rationals or p-adic numbers in the sense that entanglement is reducible only if the diagonalization of the density matrix is possible in the number field considered.
2. Shannon entropy has also infinite number of number theoretic variants of entanglement probabilities are rational and even algebraic numbers [K53]. The number theoretic Shannon entropy

is obtained by replacing the probabilities p_i in the argument of $log(p_i)$ with their p-adic norms and changing the overall sign in the definition of Shannon entropy. The resulting entanglement negentropy can be negative and achieves negative minimum for a unique prime. This means a possibility of information carrying entanglement conjecture to characterize the difference between living and inanimate matter identified as something residing in the intersection of real and p-adic worlds. Negentropy Maximization Principle [K53] stating that state function reduction reduces entanglement entropy would indeed make this kind of entanglement stable under state function reduction.

3. The stability of entanglement could also follow from the hypothesis that physical systems are ordered with respect to the hierarchy of algebraic extensions of rationals assigned with them if one believes on number theoretically irreducib le entanglement. The hierarchy of Planck constants with arbitrarily large values of Planck constants [K32] would provide a further stabilization mechanism since quantum time scales typically scale like $\hbar$. The implications for quantum computation for which the fragility of entanglement is the basic obstacle are obvious.

4. A further aspect is related to finite measurement resolution which I have suggested to be realized in terms of inclusions of hyper-finite factors [K99]. The basic idea is that complex rays of state space are replaced with the orbits of included algebra characterizing measurement resolution. This leads to the replacement of complex numbers with non-commutative algebra as generalized scalars and generalizes the proposal of Rosinger in another direction. In this framework quantum spinors appear as finite-dimensional non-commutative spinors characterized by fractal dimension and probability becomes the observable instead of spin. One can speak also about quantum entanglement in given measurement resolution defined by the included algebra.

Chapter 15

Infinite Primes and Motives

15.1 Introduction

The construction of twistor amplitudes has led to the realization that the work of Grothendieck (see http://tinyurl.com/dbojps) related to motivic cohomology simplifies enormously the calculation of the integrals of holomorphic forms over sub-varieties of the projective spaces involved. What one obtains are integrals of multi-valued functions known as Grassmannian poly-logarithms generalizing the notion of poly-logarithm [B30] and Goncharov has given a simple formula for these integrals [B12] using methods of motivic cohomology (see http://tinyurl.com/yb9b2zme) [A59] in terms of classical polylogarithms $Li_k(x)$, $k = 1, 2, 3, ...$ This suggests that motivic cohomology might have applications in quantum physics also as a a conceptual tool. One could even hope that quantum physics could provide fresh insights algebraic geometry and topology.

Ordinary theoretical physicist probably does not encounter the notions of homotopy, homology, and cohomology in his daily work and Grothendieck's work looks to him (or at least me!) like a horrible abstraction going completely over the head. Perhaps it is after all good to at least try to understand what this all is about. The association of new ideas with TGD is for me the most effective way to gain at least the impression that I have managed to understand something and I will apply this method also now. If anything else, this strategy makes the learning of new concepts an intellectual adventure producing genuine surprises, reckless speculations, and in some cases perhaps even genuine output. I do not pretend of being a real mathematician and I present my humble apologies for all misunderstandings unavoidable in this kind enterprise. One should take the summary about the basics of cohomology theory just as a summary of a journalist. I still hope that these scribblings could stimulate mathematical imagination of a real mathematician.

While trying to understand Wikipedia summaries about the notions related to the motivic cohomology I was surprised in discovering how similar the goals and basic ideas about how to achieve them of quantum TGD and motive theory are despite the fact that we work at totally different levels of mathematical abstraction and technicality. I am however convinced that TGD as a physical theory represents similar high level of abstraction and therefore dare hope that the interaction of the these ideas might produce something useful. As a matter fact, I was also surprised that TGD indeed provides a radically new approach to the problem of constructing topological invariants for algebraic and even more general surfaces.

15.1.1 What Are The Deep Problems?

In motivic cohomology one wants to relate and unify various cohomologies defined for a given number field and its extensions and even for different number fields if I have understood correctly. In TGD one would like to fuse together real and various p-adic physics and this would suggest that one must relate also the cohomology theories defined in different number fields. Number theoretical universality [K85] allowing to relate physics in different number fields is one of the key ideas involved.

Why the generalization of homology (see http://tinyurl.com/y9443vaq) [A37] and cohomology (see http://tinyurl.com/3yvnqz8) [A16] to p-adic context is so non-trivial? Is it the failure of the notion of boundary does not allow to define homology in geometric sense in p-adic context using geometric approach. The lack of definite integral in turn does not allow to define p-adic counterparts of forms except as a purely local notion so that one cannot speak about values of forms for sub-varieties. Residue calculus provides one way out and various cohomology theories defined in finite and p-adic number fields actually define integration for forms over closed surfaces (so that the troublesome boundaries are not needed), which is however much less than genuine integration. In twistor approach to scattering amplitudes one indeed encounters integrals of forms for varieties in projective spaces.

Galois group (see `http://tinyurl.com/ydgmpudx`) [A33] is defined as the group leaving invariant the rational functions of roots of polynomial having values in the original field. A modern definition is as the automorphism group of the algebraic extension of number field generated by roots with the property that it acts trivially in the original field.

1. Some examples Galois group in the field or rationals are in order. The simplest example is second order polynomial in the field of rationals for which the group is Z_2 if roots are not rational numbers. Second example is $P(x) = x^n - 1$ for which the group is cyclic group $S(n)$ permuting the roots of unity which appear in the elementary symmetric functions of the roots which are rational. When the roots are such that all their products except the product of all roots are irrational numbers, the situation is same since all symmetric functions appearing in the polynomial must be rational valued. Group is smaller if the product for two or more subsets of roots is real. Galois group generalizes to the situation when one has a polynomial of many variables: in this case one obtains for the first variable ordinary roots but polynomials appearing as arguments. Now one must consider algebraic functions as extension of the algebra of polynomial functions with rational coefficients.
2. Galois group permutes branches of the graph $x = (P_n^{-1})(y, ...)$ of the inverse function of the polynomial analogous to the group permuting sheets of the covering space. Galois group is therefore analogous to first homotopy group. Since Galois group is subgroup of permutation group, since permutation group can be lifted to braid group acting as the first homotopy group on plane with punctures, and since the homotopies of plane can be induced by flows, this analogy can be made more precise and leads to a connection with topological quantum field theories for braid groups.
3. Galois group makes sense also in padic context and for finite fields and its abelianization by mapping commutator group to unit element gives rise to the analog of homology group and by Poincare duality to cohomology group. One can also construct p-adic and finite field representations of Galois groups.

These observations motivate the following questions. Could Galois group be generalized to so that they would give rise to the analogs of homotopy groups and homology and cohomology groups as their abelianizations? Could one find a geometric representation for boundary operation making sense also in p-adic context?

15.1.2 TGD Background

The visions about physics as geometry and physics as generalized number theory suggest that number theoretical formulation of homotopy-, homology-, and cohomology groups might be possible in terms of a generalization of the notion of Galois group, which is the unifying notion of number theory. Already the observations of Andre Weil suggesting a deep connection between topological characteristics of a variety and its number theoretic properties indicate this kind of connection and this is what seems to emerge and led to Weil cohomology formulated. The notion of motivic Galois group (see `http://tinyurl.com/yb9b2zme`) is an attempt to realize this idea.

Physics as a generalized number theory involves three threads.

1. The fusion of real and p-adic number fields to a larger structure requires number theoretical universality in some sense and leads to a generalization of the notion of number by fusion reals and p-adic number fields together along common rationals (roughly) [K85].
2. There are good hopes that the classical number fields could allow to understand standard model symmetries and there are good hopes of understanding $M^4 \times CP_2$ and the classical dynamics of space-time number theoretically [K86].
3. The construction of infinite primes having interpretation as a repeated second quantization of an supersymmetric arithmetic QFT having very direct connections with physics is the third

thread [K84]. The hierarchy has many interpretations: as a hierarchy of space-time sheets for many-sheeted space with each level of hierarchy giving rise to elementary fermions and bosons as bound states of lower level bosons and fermions, hierarchy of logics of various orders realized as statements about statements about..., or a hierarchy of polynomials of several variables with a natural ordering of the arguments.
This approach leads also to a generalization of the notion of number by giving it an infinitely complex number theoretical anatomy implied by the existence of real units defined by the ratios of infinite primes reducing to real units in real topology. Depending one one's tastes one can speak about number theoretic Brahman=Atman identity or algebraic holography. This picture generalizes to the level of quaternionic and octonionic primes and leads to the proposal that standard model quantum numbers could be understand number theoretically. The proposal is that the number theoretic anatomy could allow to represent the "world of classical worlds" (WCW) as sub-manifolds of the infinite-dimensional space of units assignable to single point of space-time and also WCW spinor fields as quantum superpositions of the units. One also ends up with he idea that there is an evolution associated with the points of the embedding space as an increase of number theoretical complexity. One could perhaps say that this space represents "Platonia".

15.1.3 Homology And Cohomology Theories Based On Groups Algebras For A Hierarchy Of Galois Groups Assigned To Polynomials Defined By Infinite Primes

The basic philosophy is that the elements of homology and cohomology should have interpretation as states of supersymmetric quantum field theory just as the infinite primes do have. Even more, TGD as almost topological QFT requires that these groups should define quantum states in the Universe predicted by quantum TGD. The basic ideas of the proposal are simple.

1. One can assign to infinite prime at n:th level of hierarchy of second quantizations a rational function and solve its polynomial roots by restricting the rational function to the planes $x_n, ...x_k = 0$. At the lowest level one obtains ordinary roots as algebraic number. At each level one can assign Galois group and to this hierarchy of Galois groups one wants to assign homology and cohomology theories. Geometrically boundary operation would correspond to the restriction to the plane $x_k = 0$. Different permutations for the restrictions would define non-equivalent sequences of Galois groups and the physical picture suggests that all these are needed to characterize the algebraic variety in question.
2. The boundary operation applied to G_k gives element in the commutator subgroup $[G_{k-2}, G_{k-2}]$. In abelianization this element goes to zero and one obtains ordinary homology theory. Therefore one has the algebraic analog of homotopy theory,
3. In order to obtain both homotopy and cohomotopy and cohomology and homology as their abelizations plus a resemblance with ordinary cohomology one must replace Galois groups by their group algebras. The elements of the group algebras have a natural interpretation as bosonic wave functions. The dual of group algebra defines naturally cohomotopy and cohomology theories. One expects that there is a large number of boundary homomorphisms and the assumption is that these homomorphisms satisfy anti-commutation relations with anti-commutator equal to an element of commutator subgroup $[G_{k-2}, G_{k-2},]$ so that in abelianization one obtains ordinary anti-commutation relations. The interpretation for the boundary and coboundary operators would be in terms of fermionic annihilation (creation) operators is suggestive so that homology and cohomology would represent quantum states of super-symmetric QFT. Poincare duality would correspond to hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. It however turns out that the analogy with Dolbeault cohomology with several exterior derivatives is more appropriate.
4. In quantum TGD states are realized as many-fermion states assignable to intersections of braids with partonic 2-surfaces. Braid picture is implied by the finite measurement resolution implying discretization at space-time level. Symplectic transformations in turn act as fundamental symmetries of quantum TGD and given sector of WCW corresponds to symplectic group as far as quantum fluctuating degrees of freedom are considered. This encourages the hypothesis that the hierarchy of Galois groups assignable to infinite prime (integer/rational)

having interpretation in terms of repeated second quantization can be mapped to a braid of braids of.... The Galois group elements lifted to braid group elements would be realized as symplectic flows and boundary homomorphism would correspond to symplectic flow induced at given level in the interior of sub-braids and inducing action of braid group. In this framework the braided Galois group cohhomology would correspond to the states of WCW spinor fields in "orbital" degrees of freedom in finite measurement resolution realized in terms of number theoretical discretization.

If this vision is correct, the construction of quantum states in finite measurement resolution would have purely number theoretic interpretation and would conform with the interpretation of quantum TGD as almost topological QFT. That the groups characterize algebraic geometry than mere topology would give a concrete content to the overall important "almost" and would be in accordance with physics as infinite-dimensional geometry vision.

15.1.4 P-Adic Integration And Cohomology

This picture leads also to a proposal how p-adic integrals could be defined in TGD framework.

1. The calculation of twistorial amplitudes reduces to multi-dimensional residue calculus. Motivic integration gives excellent hopes for the p-adic existence of this calculus and braid representation would give space-time representation for the residue integrals in terms of the braid points representing poles of the integrand: this would conform with quantum classical correspondence. The power of 2π appearing in multiple residue integral is problematic unless it disappears from scattering amplitudes. Otherwise one must allow an extension of p-adic numbers to a ring containing powers of 2π.
2. Weak form of electric-magnetic duality and the general solution ansatz for preferred extremals reduce the Kähler action defining the Kähler function for WCW to the integral of Chern-Simons 3-form. Hence the reduction to cohomology takes places at space-time level and since p-adic cohomology exists there are excellent hopes about the existence of p-adic variant of Kähler action. The existence of the exponent of Kähler gives additional powerful constraints on the value of the Kähler fuction in the intersection of real and p-adic worlds consisting of algebraic partonic 2-surfaces and allows to guess the general form of the Kähler action in p-adic context.
3. One also should define p-adic integration for vacuum functional at the level of WCW. p-Adic thermodynamics serves as a guideline leading to the condition that in p-adic sector exponent of Kähler action is of form $(m/n)^r$, where m/n is divisible by a positive power of p-adic prime p. This implies that one has sum over contributions coming as powers of p and the challenge is to calculate the integral for K= constant surfaces using the integration measure defined by an infinite power of Kähler form of WCW reducing the integral to cohomology which should make sense also p-adically. The p-adicization of the WCW integrals has been discussed already earlier using an approach based on harmonic analysis in symmetric spaces and these two approaches should be equivalent. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $n + p - 2$ powers of $e^{1/n}$ can be considered.
4. If the Galois group algebras indeed define a representation for WCW spinor fields in finite measurement resolution, also WCW integration would reduce to summations over the Galois groups involved so that integrals would be well-defined in all number fields.

15.1.5 Topics Related To TGD-String Theory Correspondence

Although M-theory has not been successful as a physical theory it has led to a creation of enormously powerful mathematics and there are all reasons to expect that this mathematics applies also in TGD framework.

Floer homology, Gromov-Witten invariants, and TGD

Floer homology defines a generalization of Morse theory allowing to deduce symplectic homology groups by studying Morse theory in loop space of the symplectic manifold. Since the symplectic transformations of the boundary of $\delta M^4_\pm \times CP_2$ define isometry group of WCW, it is very natural to expect that Kähler action defines a generalization of the Floer homology allowing to understand the symplectic aspects of quantum TGD. The hierarchy of Planck constants implied by the one-to-many correspondence between canonical momentum densities and time derivatives of the embedding space coordinates leads naturally to singular coverings of the embedding space and the resulting symplectic Morse theory could characterize the homology of these coverings.

One ends up to a more precise definition of vacuum functional: Kähler action reduces Chern-Simons terms (imaginary in Minkowskian regions and real in Euclidian regions) so that it has both phase and real exponent which makes the functional integral well-defined. Both the phase factor and its conjugate must be allowed and the resulting degeneracy of ground state could allow to understand qualitatively the delicacies of CP breaking and its sensitivity to the parameters of the system. The critical points with respect to zero modes correspond to those for Kähler function. The critical points with respect to complex coordinates associated with quantum fluctuating degrees of freedom are not allowed by the positive definiteness of Kähler metric of WCW. One can say that Kähler and Morse functions define the real and imaginary parts of the exponent of vacuum functional.

The generalization of Floer homology inspires several new insights. In particular, space-time surface as hyper-quaternionic surface could define the 4-D counterpart for pseudo-holomorphic 2-surfaces in Floer homology. Holomorphic partonic 2-surfaces could in turn correspond to the extrema of Kähler function with respect to zero modes and holomorphy would be accompanied by super-symmetry.

Gromov-Witten invariants appear in Floer homology and topological string theories and this inspires the attempt to build an overall view about their role in TGD. Generalization of topological string theories of type A and B to TGD framework is proposed. The TGD counterpart of the mirror symmetry would be the equivalence of formulations of TGD in $H = M^4 \times CP_2$ and in $CP_3 \times CP_3$ with space-time surfaces replaced with 6-D sphere bundles.

K-theory, branes, and TGD

K-theory and its generalizations play a fundamental role in super-string models and M-theory since they allow a topological classification of branes. After representing some physical objections against the notion of brane more technical problems of this approach are discussed briefly and it is proposed how TGD allows to overcome these problems. A more precise formulation of the weak form of electric-magnetic duality emerges: the original formulation was not quite correct for space-time regions with Euclidian signature of the induced metric. The question about possible TGD counterparts of R-R and NS-NS fields and S, T, and U dualities is discussed.

15.1.6 P-Adic Space-Time Sheets As Correlates For Boolean Cognition

p-Adic physics is interpreted as physical correlate for cognition. The so called Stone spaces are in one-one correspondence with Boolean algebras and have typically 2-adic topologies. A generalization to p-adic case with the interpretation of p pinary digits as physically representable Boolean statements of a Boolean algebra with $2^n > p > p^{n-1}$ statements is encouraged by p-adic length scale hypothesis. Stone spaces are synonymous with profinite spaces about which both finite and infinite Galois groups represent basic examples. This provides a strong support for the connection between Boolean cognition and p-adic space-time physics. The Stone space character of Galois groups suggests also a deep connection between number theory and cognition and some arguments providing support for this vision are discussed.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

15.2 Some Backgbround About Homology And Cohomology

Before representing layman's summary about the motivations for the motivic cohomology it is good to introduce some basic ideas of algebraic geometry [A137].

15.2.1 Basic Ideas Of Algebraic Geometry

In algebraic geometry one considers surfaces defined as common zero locus for some number $m \leq n$ of functions in n-dimensional space and therefore having dimension $n-m$ in the generic case and one wants to find homotopy invariants for these surfaces: the notion of variety is more precise concept in algebraic geometry than surface. The goal is to classify algebraic surfaces represented as zero loci of collections of polynomials.

The properties of the graph of the map $y = P(x)$ in (x, y)-plane serve as an elementary example. Physicists is basically interested on the number of roots x for a given value of y. For polynomials one can solve the roots easily using computer and the resulting numbers are in the generic case algebraic numbers. Galois group is the basic object and permutes the roots with each other. It is analogous to the first homotopy group permuting the points of the covering space of graph having various branches of the many-valued inverse function $x = P^{-1}(y)$ its sheets. Clearly, Galois group has topological meaning but the topology is that of the embedding or immersion.

There are invariants related to the internal topology of the surface as well as invariants related to the external topology such as Galois group. The generalization of the Galois group for polynomials of single variable to polynomials of several variables looks like an attractive idea. This would require an assignment of sequence of sub-varieties to a given variety. One can assign algebraic extensions also to polynomials and it would seem that these groups must be involved. For instance, the absolute Galois group (see `http://tinyurl.com/yaffmruw`) associated with the algebraic closure of polynomials in algebraically closed field is free group of rank equal to the cardinality of the field (rank is the cardinality of the minimal generating set).

Homotopy (see `http://tinyurl.com/6xbeur`) [A38], homology (see `http://tinyurl.com/y9443vaq`) [A38], and cohomology (see `http://tinyurl.com/3yvnqz8`) [A38] characterize algebraically the shape of the surface as invariant not affected by continuous transformations and by homotopies. The notion of continuity depends on context and in the most general case there is no need to restrict the consideration to rational functions or polynomials or make restrictions on the coefficient field of these functions. For algebraic surfaces one poses restrictions on coefficient field of polynomials and the ordinary real number based topology is replaced with much rougher Zariski topology for which algebraic surfaces define closed sets. Physicists might see homology and cohomology theories as linearizations of nonlinear notions of manifold and surface obtained by gluing together linear manifolds. This linearization allows to gain information about the topology of manifolds in terms of linear spaces assignable to surfaces of various dimensions.

In homology one considers formal sums for these surfaces with coefficients in some field and basically algebraizes the statement that boundary has no boundary. Cohomology is kind of dual of homology and in differential geometry based cohomology forms having values as their integrals over surfaces of various dimensions realize this notion.

Betti cohomology (see `http://tinyurl.com/ybyurgao`) or singular cohomology [A10] defined in terms of simplicial complexes is probably familiar for physicists and even more so the de Rham cohomology (see `http://tinyurl.com/pndr57e`) [A18] defined by n-forms as also the Dolbeault cohomology (see `http://tinyurl.com/y7ggezzu`) [A23] using forms characterized by m holomorphic and n antiholomorphic indices. In this case the role of continuous maps is taken by holomorphic maps. For instance, the classification of the moduli of 2-D Riemann surfaces involves in an essential manner the periods of one forms on 2-surfaces and plays important role in the TGD based explanation of family replication phenomenon [K21].

In category theoretical framework homology theory can be seen as a http://en.wikipedia.org/wiki/functorfunctor [A29] that assigns to a variety (or manifold) a sequence of homology groups characterized by the dimension of corresponding sub-manifolds. One considers formal sums of surfaces. The basic operation is that of taking boundary which has operation δ as algebraic counterpart. One identifies cycles as those sums of surfaces for which algebraic boundary vanishes. This is identically true for exact cycles defined as a boundaries of cycles since boundary of boundary is empty. Only those cycles with are not exact matter and the homology group is defines as the coset space of the kernel at n: th level with respect to the image of the $n+1$: th level two spaces. Cohomology groups can be defined in a formally similar manner and for de Rham cohomology Poincare duality maps homology group H_k to H^{n-k}. The correspondence between covariant with vanishing exterior derivative and contravariant antisymmetric tensors with vanishing divergence is the counterpart of homology-cohomology correspondence in Riemann manifolds.

The calculation of homology and cohomology groups relies on general theorems which are often raised to the status of axioms in generalizations of cohomology theory.

1. Exact sequences (see `http://tinyurl.com/68ryo2`) [A26] of Abelian groups define an important calculational tool. So called short exact sequence $0 \to B \to C \to 0$ of chain complexes gives rise to long exact sequence $H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$....
   One example of short exact sequence is $0 \to H \to G \to G/H \to 0$ holding true when H is normal subgroup so that also G/H is group. This condition allows to express the homology groups of G as direct sums of those for H and G/H. In relative cohomology inclusion and δ define exact sequences allowing to express relative cohomology groups (see `http://tinyurl.com/y7jsddw7`) [A72] $H_n(X, A \subset X)$ in terms of those for X and A. Mayer-Vietoris sequence (see `http://tinyurl.com/y7jbky8l`) relates the cohomologies of sets A, B and $X = A \cup B$.
2. Künneth theorem (see `http://tinyurl.com/yddruw4w`) [A49] allows to calculated homology groups for Cartesian product as convolution of those for the factors with respect to direct sum.

Steenrod-Eilenberg axioms (see `http://tinyurl.com/ycahsz4u`) [A78] axiomatize cohomology theory in the category of topological spaces: cohomology theory in this category is a functor to graded abelian groups, satisfying the Eilenberg-Steenrod axioms: functoriality, naturality of the boundary homomorphism, long exact sequence, homotopy invariance, and excision. In algebraic cohomology the category is much more restricted: algebraic varieties defined in terms of polynomial equations and these axioms are not enough. In this case Weil cohomology (see `http://tinyurl.com/y75d95xg`) [A92] defines a possible axiomatization consisting of finite generation, vanishing outside the range $[0, dim(X)]$, Poincare duality, Künneth product formula, a cycle class map, and the weak and strong Lefschetz axioms.

In p-adic context sets do not have boundaries since p-adic numbers are not well-ordered so that the statement that boundary has vanishing boundary should be formulated using purely algebraic language. Also cohomology is problematic since definite integral is ill-defined for the same reason. This forces to question either the notion of cohomology and homology groups or the definition of geometric boundary operation and inspires the question whether Galois groups might be a more appropriate notion.

Perhaps it is partially due to the lack of a geometric realization of the boundary operation in the case of general number field that there are very many cohomology theories: the brief summary by Andreas Holmstrom (see `http://tinyurl.com/ycupslpa`) written when he started to work with his thesis, gives some idea about how many!

15.2.2 Algebraization Of Intersections And Unions Of Varieties

There are several rather abstract notions involved with cohomology theories: categories, functoriality, sheaves, schemes, abelian rings. Abelian ring is essentially the ring of polynomial functions generated by the coordinates in the open subset of the variety.

1. The spectrum of ring consists of its proper prime ideals of this function algebra. Ideal is subset of functions s closed under sum and multiplication by any element of the algebra and proper ideal is subspace of the entire algebra. In the case of the abelian ring defined on algebraic variety maximal ideals correspond to functions vanishing at some point. Prime ideals correspond to functions vanishing in some sub-variety, which does not reduce to a union of sub-varieties (meaning that one has product of two functions of ring which can separately vanish). Thus the points in spectrum correspond to sub-varieties and product of functions correspond to a union of sub-varieties.
2. What is extremely nice that the product of functions represents in general union of disjoint surfaces: for physicist this brings in mind many boson states created by bosonic creation

operators with particles identified as surfaces. Therefore union corresponds to a product of ideals defining a non-prime ideal. The notion of ideal is needed since there is enormous gauge invariance involved in the sense that one can multiply the function defining the surface by any everywhere non-vanishing function.

3. The intersection of varieties in turn corresponds to the condition that the functions defining the varieties vanish separately. If one requires that all sums of the functions belonging to the corresponding ideals vanish one obtains the same condition so that one can say that intersection corresponds to vanishing condition for the sum for ideals. The product of cohomology elements corresponds by Poincare duality [A65] the intersection of corresponding homology elements interpreted as algebraic cycles so that a beautiful geometric interpretation is possible in real context at least.

Remark: For fermionic statistics the functions would be anti-commutative and this would prevent automatically the powers of ideals. In fact, the possibility of multiple roots for polynomials of several variables implying what is known as ramification (see `http://tinyurl.com/yatd4za3`) [A70] represents a non-generic situation and one of the technical problems of algebraic geometry. For ordinary integers ramification means that integer contains in its composition to primes a power of prime which is higher than one. For the extensions of rationals this means that rational prime is product of primes of extension with some roots having multiplicity larger than one. One can of course ask whether higher multiplicity could be interpreted in terms of many-boson state becoming possible at criticality: in quantum physics bosonic excitations (Goldstone bosons) indeed emerge at criticality and give rise to long range interactions. In fact, for infinite primes allowing interpretation in terms of quantum states of arithmetic QFT boson many particle states corresponds to powers of primes so that the analogy is precise.

15.2.3 Motivations For Motives

In the following I try to clarify for myself the motivations for the motivic cohomology which as a general theory is still only partially existent. There is of course no attempt to say anything about the horrible technicalities involved. I just try to translate the general ideas as I have understood (or misunderstood) them to the simple language of mathematically simple minded physicist.

Grothendieck has carried out a monumental work in algebraizing cohomology which only mathematician can appreciate enough. The outcome is a powerful vision and mathematical tools allowing to develop among other things the algebraic variant of de Rham cohomology, etale cohomology having values in p-adic fields different from the p-adic field defining the values of cohomology, and crystalline cohomology (see `http://tinyurl.com/y8nmg486`) [A17].

As the grand unifier of mathematics Grothendieck posed the question whether there good exists a more general theory allowing to deduce various cohomologies from single grand cohomology. These cohomology theories would be like variations of the same them having some fundamental core element -motive- in common.

Category theory (see `http://tinyurl.com/24s2hj`) [A13] and the notion of scheme (see `http://tinyurl.com/4dr5vt7`) [A76], which assigns to open sets of manifold abelian rings - roughly algebras of polynomial functions- consistent with the algebra of open sets, provide the backbone for this approach. To the mind of physicist the notion of scheme brings abelian gauge theory with non-trivial bundle structure requiring several patches and gauge transformations between them. A basic challenge is to relate to each other the cohomologies associated with algebraic varieties with given number field k manifolds. Category theory is the basic starting point: cohomology theory assigns to each category of varieties category of corresponding cohomologies and functors between these categories allow to map the cohomologies to each other and compare different cohomology theories.

One of the basic ideas underlying the motivic cohomology seems is that one should be able perform a local lifting of a scheme from characteristic p (algebraic variety in p-adic number field or its algebraic extension) to that in characteristic 0 (characteristic is the integer n for which the sum of n units is zero, for rational numbers, p-adic number fields and their extensions characteristic is zero and p for finite fields) that is real or complex algebraic variety, to calculate various cohomologies here as algebraic de Rham cohomology and using the lifting to induce the cohomology to p-adic context. One expects that the ring in which cohomology has naturally values consists of ordinary or p-adic integers or extension of p-adic integers. In the case of crystalline cohomology this is however not enough.

The lifting of the scheme is far from trivial since number fields are different and real cohomology has naturally Z or Q as coefficient ring whereas p-adic cohomology has p-adic integers as coefficient ring. This lift must bring in analytic continuation which is lacking at p-adic side since n particular in p-adic topology two spheres with same radius are either non-intersecting or identical. Analytical continuation using a net of overlapping open sets is not possible.

One could even dream of relating the cohomologies associated with different number fields. I do not know to what extend this challenge is taken or whether it is regarded as sensible at all. In TGD framework this kind of map is needed and leads ot the generalization of the number field obtained by glueing together reals and p-adic numbers among rationals and common algebraic numbers. This glueing together makes sense also for the space of surfaces by identifying the surfaces which correspond to zero loci of rational functions with rational coefficients. Similar glueing makes sense for the spaces of polynomials and rational functions.

Remarks: :

1. The possibility of p-adic pseudo-constants in the solutions of p-adic differential and p-adic differential equations reflects this difficulty. This lifting should remove this non-uniqueness in analytical continuation. One can of course ask whether the idea is good: maybe the p-adic pseudo constants have some deep meaning. A possible interpretation would be in terms of non-deterministic character of cognition for which p-adic space-time sheets would be correlates. The p-adic space-time sheets would represent intentions which can be transformed to actions in quantum jumps. If one works in the intersection of real and p-adic worlds in which one allows only rational functions with coefficients in the field or rationals or possibly in some algebraic extension of rationals situation changes and non-uniqueness disappears in the intersection of real and p-adic worlds and one might argue that it is here where the universal cohomology applies or that real and p-adic cohomologies are obtained by some kind of algebraic continuation from this cohomology.
2. The universal cohomology theory brings in mind the challenge encountered in the construction of quantum TGD. The goal is to fuse real physics and various p-adic physics to single coherent whole so that one would have kind of algebraic universality. To achieve this I have been forced to introduce a heuristic generalization of number field by fusing together reals and various p-adic number fields among rationals and common algebraic numbers. The notion of infinite primes is second key notion. The hierarchy of Planck constants involving extensions of p-adic numbers by roots of unity is closely related to p-adic length scale hierarchy and seems to be an essential part of the number theoretical vision.

15.3 Examples Of Cohomologies

In the following some examples of cohomologies are briefly discussed in hope of giving some idea about the problems involved. Probably the discussion reflects the gaps in my understanding rather than my understanding.

15.3.1 Etale Cohomology And L-Adic Cohomology

Etale cohomology (see http://tinyurl.com/meyupuc) [A25] is defined for algebraic varieties as analogues of ordinary cohomology groups of topological space. They are defined purely algebraically and make sense also for finite fields. The notion of definite integral fails in p-adic context so that also the notion of form makes sense only locally but not as a map assigning numbers to surfaces. This is cohomological counterpart for the non-existence of boundaries in p-adic realm. Etale cohomology allows to define cohomology groups also in p-adic context as l-adic cohomology groups.

In Zariski topology closed sets correspond to surfaces defined as zero loci for polynomials in given field. The number of functions is restricted only by the dimension of the space. In the real case this topology is much rougher than real topology. In etale cohomology Zariski topology is too rough. One needs more open sets but one does not want to give up Zariski topology.

The category of etale maps is the structure needed and actually generalizes the notion of topology. Instead of open sets one considers maps to the space and effectively replaces the open sets with their inverse images in another space. Etale maps -idempotent are essentially projections from coverings of the variety to variety. One can say that open sets are replaced with open sets for the covering of the space and mapping is replaced with a correspondence (for algebraic surfaces X and Y the correspondence is given by algebraic equations in $X\times Y$) which in general is multi-valued and this leads to the notion of etale topology. The etale condition is formulated in the Wikipedia article in a rather tricky manner telling not much to a physicist trying to assign some meaning to this word. Etale requirement is the condition that would allow one to apply the implicit function theorem if it were true in algebraic geometry: it is not true since the inverse of rational map is not in general rational map except in the ase of birational maps to which one assigns birational geometry (see `http://tinyurl.com/ya6yeo3t`) [A11].

Remarks:

1. In TGD framework field as a map from M^4 to some target space is replaced with a surface in space $M^4 \times CP_2$ and the roles of fields and space are permuted for the regions of space-time representing lines of generalized Feynman diagrams. Therefore the relation between M^4 and CP_2 coordinates is given by correspondence. Many-sheeted space-time is locally a many-sheeted covering of Minkowski space.
2. Also the hierarchy of Planck constant involving hierarchy of coverings defined by same values of canonical momentum densities but different values of time derivatives of embedding space coordinates. The enormous vacuum degeneracy of Kähler action is responsible for this many-valuedness.
3. Implicit function theorem indeed gives several values for time derivatives of embedding space coordinates as roots to the conditions fixing the values of canonical momentum densities.

The second heuristic idea is that certain basic cases corresponding to dimensions 0 and 1 and abelian varieties which are also algebraic groups obeying group law defined by regular (analytic and single valued) functions are special and same results should follow in these cases.

Etale cohomologies satisfy Poincare duality and Künneth formula stating that homology groups for Cartesian product are convolutions of homology groups with respect to tensor product. l-adic cohomology groups have values in the ring of l-adic integers and are acted on by the absolute Galois group of rational numbers for which no direct description is known.

15.3.2 Crystalline Cohomology

Crystalline cohomology represents such level of technicality that it is very difficult for physicists without the needed background to understand what is in question. I however make a brave attempt by comparing with analogous problems encountered in the realization of number theoretic universality in TGD framework. The problem is however something like follows.

1. For an algebraically closed field with characteristic p it is not possible to have a cohomology in the ring Z_p of p-adic integers. This relates to the fact that the equation for $x^n = x$ in finite field has only complex roots of unity as its solutions when n is not divisible by p whereas for he integers n divisible by p are exceptional due to the fact that $x^p = x$ holds true for all elements of finite field $G(p)$. This implies that $x^p = x$ has p solutions which are ordinary p-adic numbers rather than numbers in an algebraic extension by a root of unity. p-Adic numbers indeed contain n: th cyclotomic field only if n divides $p-1$. On the other hand, any finite field (see `http://tinyurl.com/376w58`) has order $q = p^n$ and can be obtained as an algebraic extension of finite field $G(p)$ with p elements. Its elements satisfy the Frobenius condition $x^{q=p^n} = x$. This condition cannot be satisfied if the extension contains p: th root of unity satisgying $u^p = 1$ since one would have $(xu)^{p^n} = x \neq xu$. Therefore finite fields do not allow an algebraic extensions allowing p :th root of unity so the extension of p-adic numbers containing p: th root of unity cannot be not induced by the extension of $G(p)$. As a consequence one cannot lift cohomology in finite field $G(p^n)$ to p-adic cohomology.
2. Also in TGD inspired vision about integration $p-1$: th and possibly also p: th roots are problematic. p-Adic cohomology is about integration of forms and the reason why integration necessitates various roots of unity can be understood as follows in TGD framework. The idea is to reduce integration to Fourier analysis which makes sense even for the p-adic variant of

the space in the case that it is symmetric space. The only reasonable definition of Fourier analysis is in terms of discrete plane waves which come as powers of n: th root of unity. This notion makes sense if n is not divisible by p. This leads to a construction of p-adic variants of symmetric spaces G/H obtained by discretizing the groups to some algebraic subgroup and replacing the discretized points by p-adic continuum. Certainly the n: th roots of unity with n dividing $p-1$ are problematic since they do not corresponds to phase factors. It seems however clear that one can construct an extension of p-adic numbers containing p: th roots of unity. If it is however necesssary to assume that the extension of p-adic numbers is induced by that for a finite field, situation changes. Only roots of unity for n not divisible by factors of $p-1$ and possibly also by p can appear in the discretizations. There is infinite number extensions and the interpretation is in terms of a varying finite measurement resolution.

3. In TGD framework one ends up with roots of unity also when one wants to realize p-adic variants of various finite group representations. The simplest case is p-adic representations of angular momentum eigenstates and plane waves. In the construction of p-adic variants of symmetric spaces one is also forced to introduce roots of unity. One obtains a hierarchy of extensions involving increasing number of roots of unity and the interpretation is in terms of number theoretic evolution of cognition involving both the increase of maximal value of n and the largest prime involved. Witt ring could be seen as an idealization in which all roots of unity possible are present.

For $l=p$ l-adic cohomology fails for characteristic p. Crystalline cohomology (see `http://tinyurl.com/y8nmg486`) fills in this gap. Roughly speaking crystalline cohomology is de Rham cohomology of a smooth lift of X over a field k with with characteristic p to a variety so called ring of Witt vectors with characteristic 0 consisting of infinite sequences of the elements of k while de Rham cohomology of X is the crystalline cohomology reduced modulo p.

The ring of Witt vectors for characteristic p is particular example of ring of Witt vectors (see `http://tinyurl.com/ybnp7xd8`) [A95] assignable to any ring as infinite sequences of elements of ring. For finite field G_p the Witt vectors define the ring of p-adic integers. For extensions of finite field one has extensions of p-adic numbres. The algebraically closed extension of finite field contains n: th roots of unity for all n not divisible by p so that one has algebraic closure of finite field with p elements. For maximal extension of the finite field G_p the Witt ring is thus a completion of the maximal unramified extension of p-adic integers and contains n: th roots of unity for n not divisible by p. "Unramified" [A70] means that p defining prime for p-adic integers splits in extension to primes in such a way that each prime of extension occurs only once: the analogy is a polynomial whose roots have multiplicity one. This ring is much larger than the ring of p-adic integers. The algebraic variety is lifted to a variety in Witt ring with characteristic 0 and one calculates de Rham cohomology using Witt ring as a coefficient field.

15.3.3 Motivic Cohomology

Motivic cohomology is a attempt to unify various cohomologies as variations of the same motive common to all of them. In motivic cohomology (see `http://tinyurl.com/yb9b2zme`) [A59] one encounters pure motives and mixed motives. Pure motives is a category associated with algebraic varieties in a given number field k with a contravariant functor from varieties to the category assigning to the variety its cohomology groups. Only smooth projective varieties are considered. For mixed motives more general varieties are allowed. For instance, the condition that projective variety meaning that one considers only homogenous polynomials is given up.

Chow motives (see `http://tinyurl.com/yav3ju2o`) [A60] is an example of this kind of cohomology theory and relies on very geometric notion of Chow ring (see `http://tinyurl.com/ybc6mkqm`) with equivalence of algebraic varieties understood as rational equivalence. One can replace rational equivalence with many variants: birational, algebraic, homological, numerical, etc...

The vision about rationals as common points of reals and p-adic number fields leads to ask whether the intersection of these cohomologies corresponds to the cohomology associated with varieties defined by rational functions with rational coefficients. In both p-adic and real cases the number of varieties is larger but the equivalences are stronger than in the intersection. For a non-professional it is impossible to say whether the idea about rational cohomology in the intersection of these cohomologies makes sense.

Homology and cohomology theories rely in an essential manner to the idea of regarding varieties with same shape equivalent. This inspires the idea that the polynomials or rational functions with rational coefficients could correspond to something analogous to a gauge choice without losing relevant information or bringing in information which is irrelevant. If this gauge choice is correct then real and p-adic cohomologies and homologies would be equivalent apart from modifications coming from the different topology for the real and p-adic integers.

15.4 Infinite Rationals Define Rational Functions Of Several Variables: A Possible Number Theoretic Generalization For The Notions Of Homotopy, Homology, And Cohomology

This section represents my modest proposal for how the generalization of number theory based on infinite integers might contribute to the construction of topological and number theoretic invariants of varieties. I can represent only the primitive formulation using the language of second year math student. The construction is motivated by the notion of infinite prime but applies to ordinary polynomials in which case however the motivation is not so obvious. The visions about TGD as almost topological QFT, about TGD as generalized number theory, and about TGD as infinite-dimensional geometry serve as the main guidelines and allow to resolve the problems that plagued the first version of the theory.

15.4.1 Infinite Rationals And Rational Functions Of Several Variables

Infinite rationals correspond in natural manner to rational functions of several variables.

1. If the number of variables is 1 one has infinite primes at the first level of the hierarchy as formal rational functions of variable X having as its value as product of all finite primes and one can decompose the polynomial to prime polynomial factors. This amounts to solving the roots of the polynomial by obtained by replacing X with formal variable x which is real variable for ordinary rationals. For Gaussian rationals one can use complex variable.
2. If the roots are not rationals one has infinite prime. Physically this state is the analog of bound state whereas first order polynomials correspond to free many-particle states of supersymmetric arithmetic QFT.
3. Galois group permuting the roots has geometric interpretation as the analog of the group of deck transformations permuting the roots of the covering of the graph of the polynomial y=f(x) at origin. Galois group is analogous to fundamental group whose abelianization obtained as a coset group by dividing with the commutator group gives first homology group. The finiteness of the Galois group does not conform with the view about cohomology and homology, which suggests that it is the group algebra of Galois group which is the correct mathematical structure to consider.

One can find the roots also at the higher levels of the hierarchy of infinite primes. One proceeds by finding the roots at the highest level as roots which are algebraic functions. In other words finds the decomposition

$$P(x_n, ...) = \prod_k (x_n - R_k(x_{n-1}, ...)$$

with R_k expanded in powers series with respect to x_{n-1}. This expansion is the only manner to make sense about the root if x_{n-1} corresponds to infinite prime. At the next step one puts $x_n = 0$ and obtains a product of R_k and performs the same procedure for x_{n-1} and continues down to $n = 1$ giving ordinary algebraic numbers as roots. One therefore obtains a sequence of sub-varieties by restricting the polynomial to various planes $x_i = 0$, $i = k, ..., n$ of dimension $k-1$. The invariants associated with the intersections with these planes define the Galois groups characterizing the polynomial and therefore also infinite prime itself.

1. The process takes place in a sequential manner. One interprets first the infinite primes at level n+1 as as polynomial function in the variable X_{n+1} with coefficients depending on X_k,

$k < n+1$. One expands the roots R in power series in the variable X_n. In p-adic topology this series converges for all primes of the previous levels and the deviation from the value at $X_n = 0$ is infinitesimal in infinite-P p-adic topology.

2. What is new as compared to the ordinary situation is that the necessity of Taylor expansion, which might not even make sense for ordinary polynomials. One can find the roots and one can assign a Galois group to them.
3. One obtains a hierarchy of Galois groups permuting the roots and at the lowest level on obtains roots as ordinary algebraic numbers and can assign ordinary Galois group to them. The Galois group assigned to the collection of roots is direct sum of the Galois groups associated with the individual roots. The roots can be regarded as a power series in the variables X and the deviation from algebraic number is infinitesimal in infinite-p p-adic topology.
4. The interesting possibility is that the infinitesimal deformations of algebraic numbers could be interpreted as a generalization of real numbers. In the construction of motivic cohomology the idea is to lift varieties defined for surfaces in field of characteristic p (finite fields and their extensions) to surfaces in characteristic 0 field (p-adic numbers) in some sense to infinitesimal thickenings of their characteristic 0 counterparts. Something analogous is encountered in the proposed scenario since the roots of the polynomials are algebraic numbers plus multi-p p-adic expansion in terms of infinite-p p-adic numbers representing infinitesimal in infinite-p p-adic topology.

15.4.2 Galois Groups As Non-Commutative Analogs Of Homotopy Groups

What one obtains is a hierarchy of Galois groups and varieties of $n+1$-dimensional space with dimensions $n, n-1, ..., 1, 0$.

1. A suggestive geometric interpretation would be as an analog of first homotopy group permuting the roots which are now surfaces of given dimension k on one hand and as a higher homotopy group π_k on the on the other hand. This and the analogy with ordinary homology groups suggests the replacement of Galois group with their group algebras. Homology groups would be obtained by abelianization of the analogs of homotopy groups with the square of the boundary homomorphism mapping the group element to commutator sub-group. Group algebra allows also definition of cohomotopy and cohomology groups by assigning them to the dual of the group algebra.
2. The boundary operation is very probably not unique and the natural proposal inspired by physical intuition is that the boundary operations form an anti-commutative algebra having interpretation in terms of fermionic creation (say) operators. Cohomology would in turn correspond to annilation operators. Poincare duality would be hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. Number theoretic vision combined with the braid representation of the infinite primes in turn suggests that the construction actually reduces the construction of quantum TGD to the construction of these homology and cohomology theories.
3. The Galois analogs of homotopy groups and their duals up to the dimension of the algebraic surface would be obtained but not the higher ones. Note that for ordinary homotopy groups all homotopy group π_n, $n > 1$ are Abelian so that the analogy is not complete. The abelianizations of these Galois groups could in turn give rise to higher homology groups. Since the rational functions involved make sense in all number fields this could provide a possible solution to the challenge of constructing universal cohomology theory.

The hierarchy of infinite primes and the hierarchy of Galois groups associated with the corresponding polynomials have as an obvious analogy the hierarchy of loop groups and corresponding homotopy groups.

1. The construction brings in mind the reduction of n-dimensional homotopy to a 1-D homotopy of n-1-D homotopy. Intuitively n-dimensional homotopy indeed looks like a 1-D homotopy of n-1-D homotopy so that everything should reduce to iterated 1-dimensional homotopies by replacing the original space with the space of maps to it.
2. The hierarchical ordering of the variables plays an essential role. The ordering brings strongly in mind loop groups. Loop group $L(X^m, G)$ defined by the maps from space X^n to group G

can be also regarded as a loop group from space X^m to the loop group $L(X^{n-m}, G)$ and one obtains $L(X^n, G) = L(X^1, L(X^{n-1})$.
The homotopy equivalence classes of these maps define homotopy groups using the spaces X^n instead of spheres. Infinite primes at level n would correspond to $L(X^n, G)$. Locally the fundamental loop group is defined by $X = S^1$ which would suggest that homotopy theory using tori might be more natural then the one using spheres. naïvely one might hope that this kind of groups could code for all homotopic information about space. As a matter fact, even more general identity $L(X \times Y, G) = L(X, L(Y, G))$ seems to hold true.

3. Note that one can consider also many variants of homotopy theories since one can replace the image of the sphere in manifold with the image of any manyfold and construct corresponding homotopy theory. Sphere and tori define only the simplest homotopy theories.

15.4.3 Generalization Of The Boundary Operation

The algebraic realization of boundary operation should have a geometric counterpart at least in real case and it would be even better if this were the case also p-adically and even for finite fields.

1. The geometric analog of the boundary operation would replace the k-dimensional variety with its intersection with $x_k = 0$ hyperplane producing a union of $k-1$-dimensional varieties. This operation would make sense in all number fields. The components in the union of the surface would be very much analogous to the lower-dimensional edges of k-simplex so that boundary operation might make sense. What comes in mind is relative homology (see `http://tinyurl.com/y7jsddw7`) $H(X, A)$ in which the intersection of X with $A \subset X$ is equivalent with boundary so that its boundary vanishes. Maybe one should interpret the homology groups as being associated with the sequence of relative homologies defined by the sequence of varieties involved as $A_0 \subset A_1 \subset ..$ and relativizing for each pair in the sequence. The ordinary geometric boundary operation is ill-defined in p-adic context but its analog defined in this manner would be number theoretically universal notion making sense also for finite fields.
2. The geometric idea about boundary of boundary as empty set should be realized somehow- at least in the real context. If the boundary operation is consistent with the ordinary homology, it should give rise to a surface which as an element of H_{n-2} is homologically trivial. In relative homology interpretation this is indeed the case. In real context the condition is satisfied if the intersection of the n-dimensional surface with the $x_{n-1} = 0$ hyper-plane consists of closed surface so that the boundary indeed vanishes. This is indeed the case as simplest visualizations in 3-D case demonstrate. Therefore the key geometric idea would be that the intersection of the surface defined by zeros of polynomial with lower dimensional plane is a closed surface in real context and that this generalizes to p-adic context as algebraic statement at the level of homology.
3. The sequence of slicings could be defined by any permutation of coordinates. The question is whether the permutations lead to identical homologies and cohomologies. The physical interpretation does not encourage this expection so that different permutation would all be needed to characterize the variety using the proposed homology groups.

15.4.4 Could Galois Groups Lead To Number TheoreticalGeneralizations Of Homology And Cohomology Groups?

My own humble proposal for a number theoretic approach to algebraic topology is motivated by the above questions. The notion of infinite primes leads to a proposal of how one might assign to a variety a sequence of Galois group [A33] algebras defining analogs of homotopy groups assignable to the algebraic extensions of polynomials of many variables obtained by putting the variables of a polynomial of n-variable polynomial one by one to zero and finding the Galois groups of the resulting lower dimensional varieties as Galois groups of corresponding extensions of polynomial fields. The construction of the roots is discussed in detail [K56], where infinite primes are compared with non-standard numbers. The earlier idea about the possibility to lift Galois groups to braid groups is also essential and implies a connection with several key notions of quantum TGD.

1. One can assign to infinite primes at the n: th level of hierarchy (n is the number of second quantizations) polynomials of n variables with variables ordered according to the level of the

hierarchy by replacing the products $X_k = \pi_i P_i$ of all primes at k: th level with formal variables x_n to obtain polynomial in x_n with coefficients which are rational functions of x_k, $k < n$. Note that X_k is finite in p-adic topologies and infinitesimal in their infinite-P variants.

2. One can construct the root decomposition of infinite prime at n: th level as the decomposition of the corresponding polynomial to a product of roots which are algebraic functions in the extensions of polynomials. One starts from highest level and derives the decomposition by expanding the roots as powers series with respect to x_n. The process can be done without ever mentioning infinite primes. After this one puts $x_n = 0$ to obtain a product of roots at $x_n = 0$ expressible as rational functions of remaining variables. One performs the decomposition with respect to x_{n-1} for all the roots and continues down to $n = 1$ to obtain ordinary algebraic numbers.
3. One obtains a collection of varieties in n-dimensional space. At the highest level one obtains $n-1$-D variety referred to as divisor in the standard terminology, $n-2$-D variety in $x_n = 0$ hyperplane, $n-3$-D surface in $(x_n, x_{n-1}) = (0, 0)$ plane and so on. To each root at given level one can assign polynomial Galois group permuting the polynomial roots at various levels of the hierarchy of infinite primes in correspondence with the branches of surfaces of a many-valued map. At the lowest level one obtains ordinary Galois group relating the roots of an ordinary polynomial. The outcome is a collection of sequences of Galois groups $\{(G_n, G_{n,i}, G_{n,i,j}...)\}$ corresponding to all sequences of roots from $k = n$ to $k = 1$.
One can also say that at given level one has just one Galois group which is Cartesian product of the Galois groups associated with the roots. Similar situation is encountered when one has a product of irreducible polynomials so that one has two independent sets of roots.

The next question is how to induce the boundary operation. The boundary operation for the analogs of homology groups should be induced in some sense by the projection map putting one of the coordinates x_k to zero. This suggests a geometric interpretation in terms of a hierarchy of relative homologies $H_k(S_k, S_{k-1})$ defined by the hierarchy of surfaces S_k. Boundary map would map S_k to is intersection at $(x_n = 0, ..., x_k = 0)$ plane. This map makes sense also p-adically. The square of boundary operation would produce an intersection of this surface in $x_{k-1} = 0$ plane and this should correspond to boundary sense for Galois groups.

Algebraic representation of boundary operations in terms of group homomorphisms

The challenge is to find algebraic realizations for the boundary operation or operations in terms of group homomorphisms $G_k \to G_{k-1}$. One can end up with the final proposal through heuristic ideas and counter arguments and relying on the idea that algebraic geometry should have interpretation in terms of quantum physics as it is described by TGD as almost topological QFT.

1. n-dimensional Galois group is somewhat like a fundamental group acting in the space of n-1-dimensional homotopies so that Grothendieck's intuition that 1-D homotopies are somehow fundamental is realized. The abelianizations of these Galois groups would define excellent candidates for homology groups and Poincare duality would give cohomology groups. The homotopy aspects becomes clearer if one interprets Galois group for n: th order polynomials as subgroup of permutation group and lifts the Galois group to a subgroup of corresponding braid group. Galois groups are also stable againt small changes of the coefficients of the polynomial so that topological invariance is guaranteed.
2. Non-abelian boundary operations $G_k \to G_{k-1}$ must reduce to their abelian counterparts in abelianization so that they their squares defining homomorphisms from level k to $k-2$ must be maps of G_k to the commutator subgroup $[G_{k-2}, G_{k-2}]$.
3. There is however a grave objection. Finite abelianized Galois groups contain only elements with finite order so that in this sense the analogy with ordinary homotopy and homology groups fails. On the other hand, if Galois group is replaced with its group algebra and group algebra is defined by (say) integer valued maps, one obtains something very much analogous to homotopy and homology groups. Also group algebras in other rings or fields can be considered. This replacement would provide the basis of the homotopy and homology groups with an additional multiplicative structure induced by group operation allowing the interpretation as representations of Galois group acting as symmetry groups. The tentative physical interpretation would in terms of quantum states defined by wave functions in groups. Coboundary operation in

the dual of group algebra would be induced by the action of boundary operation in group algebra. Homotopy and homology would be associated with the group algebraandcohomotopy and cohomology with its dual.

4. A further grave objection against the analog of homology theory is there is no reason to expect that the boundary homomorphism is unique. For instance, one can always have a trivial solution mapping G_k to unit element of G_{k-1}. Isomorphism theorem (see `http://tinyurl.com/mn6nol`) [A48] implies that the image of the group G_k in G_{k-1} under homomorphism h_k is $G_k/ker(h_k)$, where $ker(h_k)$ is a normal subgroup of G_k as is easy to see. One must have $h_{k-1}(G_k/ker(h_k)) \subset [G_{k-2}, G_{k-2}]$, which is also a normal subgroup.
The only reasonable option is to accept all boundary homomorphisms. This collection of boundary homomorphisms would satisfy anti-commutation relations inducing similar anti-commutation relations in cohomology. Putting all together, one would would obtain the analog of fermionic oscillator algebra. In particular, Poincare duality would correspond to the mapping exchanging fermionic creation and annihilation operators. It however turns out that tis interpretation fails. Rather, braided Galois homology could represent the states of WCW spinor fields in "orbital" degrees of freedom of WCW in finite measurement resolution. A better analogy for braided Galois cohomology is provided by Dolbeault cohomology which also allows complex conjugation.

If this picture makes sense, one would clearly have what category theorist would have suggested from the beginning. TGD as almost topological QFT indeed suggests strongly the interpretation of quantum states in terms of homology and cohomology theories.

Lift of Galois groups to braid groups and induction of braidings by symplectic flows

One can build a tighter connection with quantum TGD by developing the idea about the analogy between homotopy groups and Galois groups.

1. The only homotopy groups (see `http://tinyurl.com/6xbeur`) [A38], which are non-commutative are first homotopy groups π_1 and plane with punctures provides the minimal realization for them. The lift of permutation groups to braid groups (see `http://tinyurl.com/3yusbn3`) [A12] by giving up the condition that the squares of generating permutations satisfy $s_i^2 = 1$ defines a projective representation for them and should apply also now. There is also analogy with Wilson loops. This leads to topological QFTs for knots and braids [A186, A127].
2. In TGD framework light-like 3-surfaces (and also space-like at the ends of causal diamonds) carry braids beginning at partonic 2-surfaces and ending at partonic 2-surfaces at the boundaries of causal diamonds. This realization is highly suggestive now. This also conforms with the general TGD inspired vision about absolute Galois group of rationals as permutation group S_∞ lifted to braiding groups such that its representation always reduce to finite-dimensional ones [K46]. This also conforms with the view about the role of hyper-finite factors of type II_1 and the idea about finite measurement resolution and one would obtain a new connection between various mathematical structure of TGD.
3. The physical interpretation of infinite primes represented by polynomials as bound states suggests that infinite prime at level n corresponds to a braid of braids of... braids such that at given level of hierarchy braid group acting on the physical states is associated with covering group realized as subgroup of the permutation group for the objects whose number is the number of roots. This gives also a connection with the notion of operad [A62, A168, A109] which involves also a hierarchy of discrete structures with the action of permutation group inside each and appears also in quantum TGD as a natural notion [K18, K22].
4. The assumption that the braidings are induced by flows of the partonic 2-surface could glue the actions of different Galois groups to single coherent whole was originally motivated by the hope that boundary homomorphism could be made unique in this manner. This restriction is however un-necessary and the physical picture does not support it. The basic motivation for the braid representation indeed comes from TGD as an almost topological QFT vision.
5. The role of symplectic transformations in TGD suggests the identification of flows as symplectic flows induced by those of $\delta M^2 \times CP_2$. These flows should map the area enclosed by the sub-braid (of braids) to itself and corresponding Hamiltonian should be constant at the boundary of the area and induce a flow horizontal to the boundary and also continuous at the boundary. The

flow would in general be non-trivial inside the area and induce the braiding of the sub-braid of braids. One could assign "Galois spin" to the sub-braids with respect to the higher Galois group and boundary homomorphism would realize unitary action of G_k as spin rotation at k_1: th level. At k_2: th level the "Galois spin" rotation would reduce to that in commutator subgroup and in homology theory would become trivial. The interpretation of the commutator group as the analog of gauge group might make sense. This would conform with an old idea of quantum TGD that the commutator subgroup of symplectic group acts as gauge transformations.

6. It is not necessary to assign the braids at various level of the hierarchy to the same partonic 2-surface. Since the symplectic transformations act on $\delta M^4_\pm \times CP_2$, one can consider also the projections of the braids to the homologically non-trivial 2-sphere of CP_2 or to the 2-sphere at light-cone boundary: both of these spheres play important part in the formulation of quantum TGD and I have indeed assigned the braidings to these surfaces [K45].
7. The representation of the hierarchy of Galois groups acting on the braid of braids of... can be understood in terms of the replacement of symplectic group of $\delta M^4_\pm \times CP_2$ -call it G-permuting the points of the braids with its discrete subgroup obtained as a factor group G/H, where H is a normal subgroup of G leaving the endpoints of braids fixed. One must also consider subgroups of the permutation group for the points of the triangulation since Galois group for n: th order polynomial is in general subgroup of S_n One can also consider flows with these properties to get braided variant of G/H.

The braid group representation works also for ordinary polynomials with continuous coefficients in all number fields as also finite fields. One therefore achieves number theoretical universality. The values of the variables x_i appearing in the polynomials can belong to any numer field and the representation spaces of the Galois groups correspond to any number field. Since the Galois groups are stable against small perturbations of coefficients one obtains topological invariance in both real and p-adic sense. Also the representation in all number fields are possible for the Galois groups.

The construction is universal but infinite primes provide the motivation for it and can be regarded as a representation of the generalized cohomology group for surfaces which belong to the intersection of real and p-adic worlds (rational coefficients). In particular, the expansion of the roots in powers series is the only manner to make sense about the roots when x_n is identified with X_n so that convergence takes place if some of the lower level infinite primes appearing in the product defining X_n is interpreted as infinite p-adic prime. All higher powers are infinitesimal in infinite-P p-adic norm. At the lowest level one obtains expansion in X_1 for which X_1^n has norm p^{-n} with respect to any prime p. The value of the product of primes different from p is however not well-defined for given p-adic topology. If it makes sense to speak about multi-p p-adic expansion all powers X_1^n, $n > 0$ would be infinitesimal.

What can one say about the lifting to braid groups?

The generators of symmetry group are given by permutations s_i permuting i: th and $i+1$: th element of n-element set. The permutations s_i and s_j obviously commute for $|i-j| > 2$. It is also easy to see that the identity $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ holds true. Besides this the identity $s_i^2 = 1$ holds true.

Braid group B_n [A12] is obtained by dropping the condition $s_i^2 = 1$ and can be regarded as an infinite covering group of the permutation group. For instance, for the simplest non-trivial case $n = 3$ the braid group is universal central extrension of the modular group $PSL(2, Z)$. In the general case the braid group is isomorphic to the mapping class group of a punctured disk with n punctures and the realization of the braidings as a symplectic transformations would mean additional restriction to the allowed isotopies inducing the braid group action.

One can decompose any element of braid group B_n to a product of element of symmetric group S_n and of pure braid group P_n consisting of braidings which correspond to trivial permutations. P_n is a normal subgroup of braid group and the following short exact sequence $1 \rightarrow F_{n-1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow 1$ allows to decompose P_n to a product of image of free group F_{n-1} and of the image of P_n in P_{n-1}. This leads to a decomposition to a representation of P_n as an iterated semidirect product of free groups.

Concerning the lifting of Galois groups to subgroups of braid groups following observations are relevant.

1. For n: th order polynomial of single variable Galois group can be regarded as a subgroup of permutation group S_n. The identification is probably not completely unique (at least inner automorphisms make the identification non-unique) but I am unable to say whether this has significance in the recent context.
2. The natural lifting of Galois group to its braided version is as a product of corresponding subgroup of S_n with with pure braid group of n braids so that pure braidings would allow also braidings of all permutations as intermediate stages. Pure braid group is normal subgroup trivially. Whether also more restricted braidings are possible is not clear to me. Braid group has a subgroup obtained by coloring braid strands with a finite number of colors and allowing only the braidings which induce permutations of braids of same color. Clearly this group is a good candidate for the minimal group decomposable to a product of subgroups of symmetric subgroups containing braided Galois group. Different colors would correspond to the decomposition of S_n to a product of permutation groups. Note that one can have cyclic subgroups of permutation sub-groups.

One might hope that it is enough to lift the boundary homomorphisms between Galois groups G_k and G_{k-1} to homomorphisms between corresponding braided groups. Life does not look so simple.

1. The group algebra of Galois group is replaced with an infinite-dimensional group algebra of braid groups so that the number of physical states is expected to become much larger and the interpretation could be in terms of many-boson states.
2. The square of the boundary homomorphism must map braided Galois group $B(G_k)$ to $[B(G_{k-2}), B(G_{k-2})]$. The obvious question is whether this conditions reduces to corresponding conditions for Galois group and pure braided groups. In other words, does the braiding commute with the formation of commutator sub-group: $[B(G_k), B(G_k)] = B([G_k, G_k])$? In this case the decomposition of the braided Galois group to a product of Galois group and pure braid group would allow to realize the braided counterpart of boundary homomorphism as a product of Galois group homomorphism and homomorphism acting on the pure braid group. Direct calculation however shows that this is not the case so that the problem is considerably more complicated.

More detailed view about braided Galois homology

Consider next a more detailed view about the braided Galois homology.

1. One can wonder whether the application of only single boundary operator creates a state which represents gauge degree of freedom or whether boundaries correspond to "full" boundaries obtained by applying maximum number of boundary operations, which k: th level is k. "Full boundary" would correspond to what one obtains by applying at most k boundary operators to the state, and many combinations are possible if the number of boundary homomorphisms is larger than k. The physical states as elements of homology group would be analogous many-fermion states bu would differ from them in the sense that they would be annihilated by all fermionic creation operators. In particular, full Fermi spheres at k: th level would represent gauge degrees of freedom.
   Homologically non-trivial states are expected to be rather rare, especially so if already single boundary operation creates gauge degree of freedom. Certainly the existence of constraints is natural since infinite primes corresponding to irreducible polynomials of degree higher are interpreted as bound states. Homological non-triviality would most naturally express bound state property in bosonic degrees of freedom. In any case, one can argue that fermionic analogy is not complete and that a more natural interpretation is as an analog of cohomology with several exterior derivatives.
2. The analogy with fermionic oscillator algebra makes also the realization of bosonic oscillator operator algebra suggestive. Pointwise multiplication of group algebra elements regarded as functions in group looks the most plausible option since for continuous groups like $U(1)$ this implies additivity of quantum numbers. Many boson states for given mode would correspond to powers of group algebra element with respect to pointwise multiplication. If the commutator for the analogs of the bosonic oscillator operators is defined as

$$[B_1, B_2] \equiv \sum_{g_1, g_2} B_1(g_1) B_2(g_2) [g_1, g_2] \quad , [g_1, g_2] \equiv g_1 g_2 g_1^{-1} g_2^{-1} \quad ,$$

it is automatically in the commutator sub-group. This condition is not consistent with fermionic anti-commutation relations. The consistency requires that the commutator is defined as

$$[B_1, B_2] \equiv \sum_{g_1,g_2} (B_1(g_1)B_2(g_2)[[g_1, g_2] \ \ , [g_1, g_2] \equiv g_1g_2 - g_2g_1 \ \ . \tag{15.4.1}$$

The commutator must belong to the group algebra of the commutator subgroup. In this case the commutativity conditions are non-trivial. Bosonic commutation relations would put further constraints on the homology.
A delicacy related to commutation and anti-commutation relations should be noticed. One could fermionic creation (annihilation) operators as elements in the dual of group algebra. If group algebra and its dual are not identified (this might not be possible) then the anti-commutator is element of the field of ring in which group algebra elements have values. In the bosonic case the conjugate of the bosonic group algebra element should be treated in the same manner as a pointwise multiplication operator instead of an exterior derivative like operator.

3. One could perhaps interpret the commutation and anti-commutation relations modulo commutator subgroup in terms of finite measurement resolution realized by the transition to homology implying that observables commute in the standard sense. The connection of finite measurement resolution with inclusions of hyper-finite factors of type II_1 implying a connection with quantum groups and non-commutative geometry conforms also with the vision that finite measurement resolution means commutativity modulo commutator group.
4. The alert reader has probably already asked why one could not define also diagonal homology for G_k via diagonal boundary operators $\delta_k : G_k \to H_k$, where H_k is subgroup of G_k. The above argument would suggest interpretation for this cohomology in terms of finite measurement resolution. If one allows this the Galois cohomology groups would be labelled by two integers. Similar situation is encountered in (see `http://tinyurl.com/yb9b2zme`) [A59].

Some remarks

Some remarks about the proposal are in order.

1. The proposal makes as such sense if the polynomials with rational coefficients define a subset of more general function space able to catch the non-commutative homotopy and homology and their duals terms of Galois groups associated with rational functions with coefficients. One could however abstract the construction so that it applies to polynomials with coefficients in real and p-adic fields and forget infinite primes altogether. One can even consider the replacement of algebraic surfaces with more general surfaces as along as the notion of Galois group makes sense since braiding makes sense also in more general situation. This picture would conform with the idea of number theoretical universality based on algebraic continuation from rationals to various number fields. In this case infinite primes would characterize the rational sector in the intersection of real and p-adic worlds.
2. The above discussion is for the rational primes only. Each algebraic extension of rationals however gives rise to its own primes. In particular, one obtains also complex integers and Gaussian primes. Each algebraic extension gives to its own notion of infinite prime. One can also consider quaternionic and octonionic primes and their generalization to infinite primes and this generalization is indeed one of the key ideas of the number theoretic vision [K84]. Note that already for quaternions Galois group defined by the automorphisms of the arithmetics is continuous Lie group.
3. The decomposition of infinite primes to primes in extension of rational or polynomials is analogous to the decomposition of hadron to quarks in higher resolution and suggests that reduction of the quantum system to its basic building bricks could correspond number theoretically to the introduction of higher algebraic extensions of various kinds of number fields. The emergence of higher extensions would mean emergence of algebraic complexity and have interpretation as evolution of cognition in TGD inspired theory of consciousness.

This picture conforms with the basic visions of quantum TGD about physics as infinite-dimensional geometry on one hand and physics as generalized number theory on one hand implying that algebraic geometry reduces in some sense to number theory and one can also regard quantum states as representations of algebraic geometric invariants in accordance with the vision about TGD as almost topological QFT.

Infinite primes form a discrete set since all the coefficients are rational (unless one allows even algebraic extensions of infinite rationals). Physically infinite primes correspond to elementary particle like states so that elementary particle property corresponds to number theoretic primeness. Infinite integers define unions of sub-varieties identifiable physically as many particle states. Rational functions are in turn interpreted in zero energy ontology as surfaces assignable to initial and final states of physical event such that positive energy states correspond to the numerator and negative energy states to the denominator of the polynomial. One also poses the additional condition that the ratio equals to real unit in real sense so that real units in this sense are able to represent zero energy state and the number theoretic anatomy of single space-time point might be able to represent arbitrary complex quantum states.

The generalization of the notion of real point has been already mentioned as also the fact that the number theoretic anatomy could in principle allow to code for zero energy states if they correspond to infinite rationals reducing to unit in real sense. Also space-time surfaces could by quantum classical correspondence represent in terms of this anatomy as I have proposed. Single space-time point could code in its structure not only the basic algebraic structure of topology as proposed but represent Platonia. If the above arguments really maks sense then this number theoretic Brahman=Atman identify would not be a mere beautiful philosophical vision but would have also practical consequences for mathematics.

15.4.5 What Is The Physical Interpretation Of The Braided Galois Homology

The resulting cohomology suggests either the interpretation in terms of many-fermion states or as a generalization of de Rham cohomology involving several exterior derivative operators. The arguments below show that fermionic interpretation does not make sense and that the more plausible interpretation in concordance with finite measurement resolution is in terms of "orbital" WCW degrees of freedom represented by the symplectic group assignable to the product of light-cone boundary and CP_2.

What the restriction to the plane $x_k = 0$ could correspond physically?

The best manner to gain a more detailed connection between physics and homology is through an attempt to understand what operation putting $x_k = 0$ could mean physically.

1. Given infinite prime at level n corresponds to single particle state characterized by Galois group G_n. The fermionic part of the state corresponds to its small part and purely bosonic part multiplies X_{n-1} factors as powers of primes not dividing the fermionic part of the state. Therefore the finite part of the state contains information about fermions and bosons labelled by fermionic primes. When one puts $x_n = 0$, the information about the bosonic part is lost.
   One can of course divide the polynomial by a suitable infinite integer of previous level so that its highest term is just power of X_n with a unit coefficient. Bosonic part appears in this case in the denominator of the finite part of the infinite prime and does not contribute to zeros of the resulting rational function at $n-1$: th level: it of course affects the zeros at n: th level. Hence the information about bosons at $n-1$: th level is lost also now unless one considers also the Galois groups assignable to the poles of the resulting rational function at $n-1$: th level.
2. What could this loss of information about bosons correspond geometrically and physically? To answer this question must understand how the polynomial of many variables can be represented physically in TGD Universe.
   The proposal has been that a union of hierarchically ordered partonic 2-surfaces gives rise to a local representation of n-fold Cartesian power for a piece of complex plane. A more concrete realization would be in terms of wormhole throats at the end of causal diamond at 3-surfaces topologically condensed at each other. The operation $x_n = 0$ would corresponding to the basic reductionistic step destroying the bound state by removing the largest space-time sheets so that one would have many-particle state rather than elementary particle at the lower level of

the hierarchy of space-time sheet. This loss of information would be unavoidable outcome of the reductionistic analysis.

One can consider two alternative geometric interpretations depending on whether one identifies to infinite primes connected 3-surfaces or connected 2-surfaces.

1. If infinite primes correspond to connected 3-surfaces having hierarchical structure of topological condensate the disappearing bosons could correspond to the wormhole throats connecting smaller space-time sheet to the largest space-time sheet involved. Wormhole throats would carry bosonic quantum numbers and would be removed when the largest space-time sheet disappears. Many-fermion state at highest level represented by the "finite" part of the infinite prime would correspond to "half" wormhole throats- CP_2 type vacuum extremals topological condensed at smaller space-time sheets but not at the highest one.
2. If elementary particles/infinite primes correspond to connected partonic 2-surfaces (this is not quite not the case since tangent space data about partonic 2-surfaces matters), one must replace 3-D topological condensation by its 2-dimensional version. Infinite prime would correspond to single wormhole throat asa partonic 2-surface at which smaller wormhole throats would have suffered topological condensation. Topological condensation would correspond to a formation of a connection by flux tube like structure between the 2-surfaces considered. The disappearance of this highest level would mean decay to a many particle state containing several wormhole contacts. The formation of anyonic many-particle states could be interpreted in terms of build-up of higher level infinite primes.
3. What ever the interpretation is, it should be consistent with the idea that braiding as induced by symplectic flow. If the symplectic flow is defined by the inherent symplectic structure of the partonic 2-surface only the latter option works. If the symplectic flow acts at the level of the embedding space - as is natural to assume- both interpretations make sense.

The restriction to $x_k = 0$ plane cannot correspond to homological boundary operation

Can one model the restriction to $x_k = 0$ plane as boundary operation in the sense of generalized homology? There are several objections.

1. There are probably several homological boundary operations δ_i at given level whereas the restriction $x_k = 0$ is a unique operation (recall however the possibility to permute the arguments in the case of polynomial).
2. The homology is expected to contain large number of generators whereas the state defined by infinite prime is unique as are also the states resulting via restriction operations.
3. It is not possible to assign fermion number to $x_k = 0$ operation since fermion number is not affected: this would not allow to assign fermion number to the homological boundary operators.

Although the interpretation as many-fermion states does not make sense, one must notice that the structure of homology is highly analogous to the space of states of super-symmetric QFT and of the set of infinite primes. Only the infinite primes $X_n \pm 1$, where X_n is the product of all primes at level n, correspond to states containing no fermions and have interpretation as Dirac sea and vacuum state. In the same manner the elements of braided Galois homology in general are obtained by applying the analogs of fermionic annihilation (creation) operators to a full Fermi sphere (Fock vacuum). Also the identification of *all* physical states as many-fermion states in quantum TGD where all known elementary bosons are identified as fermion pairs conforms with this picture.

A more natural interpretation of the restriction operation is as an operation making possible to assign to a given state in fermionic sector the space of possible states in WCW degrees of freedom characterized in terms of Galois cohomology represented in terms of the symplectic group of acting as isometries of WCW . The transition from Lie algebra description natural for continuum situation to discrete subgroup is natural due to the discretization realizing the finite measurement resolution.

One cannot however avoid a nasty question. What about the lower level bosonic primes associated with the infinite prime? What is their interpretation if they do not correspond to WCW degrees of freedom? Maybe one could identify the bosonic parts of infinite prime as super-partners of fermions behaving like bosons. The addition of a right handed neutrino to a given quantum state could represent this supersymmetry.

Braided Galois group homology and construction of quantum states in WCW degrees of freedom in finite measurement resolution

The above arguments fix the physical interpretation of infinite primes and corresponding group cohomology to quite high degree.

1. From above it is clear that the restriction operation cannot correspond directly to homological boundary operation. Single infinite prime corresponds to an entire spectrum of states. Hence the assignment of fermion number to the boundary operators is not correct thing to do and one must interpret the coboundary operations as analogs of exterior derivatives and various states as bosonic excitations of a given state analogous to states assignable to closed forms of various degrees in topological or conformal quantum field theories.
2. The natural interpretation of Galois homology is as a homology assignable to a discrete subgroup hierarchy of the symplectic group acting as isometries of WCW and therefore as the space of wave functions in WCW degrees of freedom in finite measurement resolution. Infinite primes would code for fermionic degrees of freedom identifiable as spinor degrees of freedom at the level of WCW .
3. The connection between infinite primes and braided Galois homology would basically reflect the supersymmetry relating these degrees of freedom at the level of WCW geometry where WCW Hamiltonians correspond to bosonic generators and contractions of WCW gamma matrices with symplectic currents to the fermionic generators of the super-symmetry algebra. If this identification is correct, it would solve the problem of constructing the modes of WCW spinor fields in finite measurement resolution. An especially well-come feature would be the reduction of WCW integration to summations in braided Galois group algebra allowing an easy realization of number theoretical universality. If the picture is correct it should also have connections to the realization of finite measurement resolution in terms of inclusions of hyper-finite factors of type II_1 [K32] for which fermionic oscillator algebra provides the basic realization.
4. Of course, it is far from clear whether it is really possible to reduce spin, color and electroweak quantum numbers to number theoretic characteristics of infinite primes and it might well be that the proposed construction does not apply to center of mass degrees of freedom of the partonic 2-surface. I have considered these questions for the octonionic generalization of infinite primes and suggested how standard model quantum numbers could be understood in terms of subset of infinite octonionic primes [K84].

15.4.6 Is There A Connection With The Motivic Galois Group?

The proposed generalized of Galois group brings in mind he notion of motivic Galois group (see `http://tinyurl.com/yav3ju2o`), which is one possible generalization for the notion of zero-dimensional Galois group associated with algebraic extensions of number fields to the level of algebraic varieties.

One of the many technical challenges of the motivic cohomology theory is the non-uniqueness of the embedding of the algebraic extension as a subfield in the algebraic closure of k. The number of these embeddings is however finite and absolute Galois group associated with the algebraic closure of k acts in the set of the embeddings. Which of them one should choose?

Quantum physicist would solve this problem by saying that there is no need to choose: one could introduce quantum superpositions of different choices and “Galois spin” regarding the different embeddings as analogs of different spin components. Absolute Galois group would act on the quantum states regarded as superpositions of different embeddings by permuting them. In TGD framework this kind of representation could emerge in p-adic context raise Galois group to a role of symmetry group acting on quantum states: indeed absolute Galois group is very natural notion in TGD framework. I have proposed this kind of interpretation for some years ago in a chapter [K46] about Langlands program [A125, K46, A126, A124].

If I have understood correctly, the idea of the motivic Galois theory is to generalize this correspondence so that the varieties in field k are replaced the varieties in the extension of k imbedded to the algebraic closure of k, the number of which is finite. Whether the number of the lifts for varieties is finite seems to depends on the situation.

1. If the embedding is assumed to be same for all points of the variety the situation seems to reduce to the embeddings of k to the algebraic completion of rationals and one would have quantum superposition of varieties in the union of finite number of representatives of the algebraic extension to which the absolute Galois group acts.

2. Physicist could however ask whether the invariance under the action of Galois group could be local in some sense. The selection of separable extension could indeed be only pseudo-constant in p-adic case and thus depend on finite number of pinary digits of the k-valued coordinates of the point of the algebraic variety. Local gauge invariance would say that any pseudo constant element of local absolute Galois group acts as a symmetry. This would suggest that one can introduce Galois connection. Since Lie algebra is not defined now one should introduce the connection as parallel translations by Galois group element for paths in the algebraic variety.

One key result (see `http://tinyurl.com/yav3ju2o`) is that pure motives using numerical equivalence are equivalent with the category of representations of an algebraic group called motivic Galois group which has Lie algebra and is thus looks like a continuous group.

1. Lie algebra structure for something apparently discrete indeed makes sense for profinite groups (synonymous to Stone spaces). Spaces with p-adic topology are basic examples of this kind of spaces. For instance, 2-adic integers is a Stone space obtained as the set of all bit sequences allowed to contain infinite number of non-vanishing digits. This implies that real discreteness transforms to p-adic continuity and the notion of Lie algebra makes sense. For polynomials this would correspond to polynomials with strictly infinite degree unless one considers the absolute Galois group associated with the algebraic extension of rationals associated with an ordinary polynomial. For infinite primes this would correspond to many-fermion states containing infinite number of fermions kicked out from the Dirac sea and from the point of view of physics would look like an idealization.

2. Motivic Galois group does not obviously correspond to the Galois groups as they are introduced above. Absolute Galois group for the extension of say rationals however emerges if one performs the lift to the algebraic completion and this might be how one ends up with motivic Galois group and also with p-adic physics. One can perhaos say that the Galois groups as introduced above make sense in the intersection of real and p-adic worlds.

3. The choice of algebraic extension might be encountered also in the construction of roots for the polynomials associated with infinite primes and since this choice is not unique it seems that one must use quantum superposition of the different choices and must introduce the action of an appropriate absolute Galois group. This group would be absolute Galois group for algebraic extension of polynomials of n variables at n: th level and ordinary Galois group at the lowest level of hierarchy which should be or less the same as the Galois group introduced above. This could bring in additional spin like degrees of freedom in which the absoltey Galois group acts. The fascinating question is whether one could regard not only the degrees of freedom associated with the finite Galois groups but even those associated with the absolute Galois group as physical. Physically the analogs of color quantum numbers whose net values vanish for confined states would be in question. To sum up, it seems that number theory could contain implicitly an incredible rich spectrum of physics.

15.5 Motives And Twistor Approach Applied To TGD

Motivic cohomology has turned out to pop up in the calculations of the twistorial amplitudes using Grassmannian approach [B30, B12]. The amplitudes reduce to multiple residue integrals over smooth projective sub-varieties of projective spaces. Therefore they represent the simplest kind of algebraic geometry for which cohomology theory exists. Also in Grothendieck's vision about motivic cohomology (see `http://tinyurl.com/h9bp68p`) [A161] projective spaces are fundamental as spaces to which more general spaces can be mapped in the construction of the cohomology groups (factorization).

15.5.1 Number Theoretic Universality, Residue Integrals, AndSymplectic Symmetry

A key challenge in the realization of the number theoretic universality is the definition of p-adic definite integral. In twistor approach integration reduces to the calculation of multiple residue integrals over closed varieties. These could exist also for p-adic number fields. Even more general integrals identifiable as integrals of forms can be defined in terms of motivic cohomology.

Yangian symmetry [A97], [B19] is the symmetry behind the successes of twistor Grassmannian approach [B28] and has a very natural realization in zero energy ontology [K91]. Also the basic prerequisites for twistorialization are satisfied. Even more, it is possible to have massive states as bound states of massless ones and one can circumvent the IR difficulties of massless gauge theories. Even UV divergences are tamed since virtual particles consist of massless wormhole throats without bound state condition on masses. Space-like momentum exchanges correspond to pairs of throats with opposite sign of energy.

Algebraic universality could be realized if the calculation of the scattering amplitudes reduces to multiple residue integrals just as in twistor Grassmannian approach. This is because also p-adic integrals could be defined as residue integrals. For rational functions with rational coefficients field the outcome would be an algebraic number apart from power of 2π, which in p-adic framework is a nuisance unless it is possible to get rid of it by a proper normalization or unless one can accepts the infinite-dimensional transcendental extension defined by 2π. It could also happen that physical predictions do not contain the power of 2π.

Motivic cohomology defines much more general approach allowing to calculate analogs of integrals of forms over closed varieties for arbitrary number fields. In motivic integration [A189] - to be discussed below - the basic idea is to replace integrals as real numbers with elements of so called scissor group whose elements are geometric objects. In the recent case one could consider the possibility that $(2\pi)^n$ is interpreted as torus $(S^1)^n$ regarded as an element of scissor group which is free group formed by formal sums of varieties modulo certain natural relations meaning.

Motivic cohomology allows to realize integrals of forms over cycles also in p-adic context. Symplectic transformations are transformation leaving areas invariant. Symplectic form and its exterior powers define natural volume measures as elements of cohomology and p-adic variant of integrals over closed and even surfaces with boundary might make sense. In TGD framework symplectic transformations indeed define a fundamental symmetry and quantum fluctuating degrees of freedom reduce to a symplectic group assignable to $\delta M^4 \pm \times CP_2$ in well-defined sense [K24]. One might hope that they could allow to define scissor group with very simple canonical representatives-perhaps even polygons- so that integrals could be defined purely algebraically using elementary area (volume) formulas and allowing continuation to real and p-adic number fields. The basic argument could be that varieties with rational symplectic volumes form a dense set of all varieties involved.

15.5.2 How To Define The P-Adic Variant For The Exponent Of Kähler Action?

The exponent of Kähler function defined by the Kähler action (integral of Maxwell action for induced Kähler form) is central for quantum at least in the real sector of WCW . The question is whether this exponent could have p-adic counterpart and if so, how it should be defined.

In the real context the replacement of the exponent with power of p changes nothing but in the p-adic context the interpretation is affected in a dramatic manner. Physical intuition provided by p-adic thermodynamics [K50] suggest that the exponent of Kähler function is analogous to Bolzmann weight replaced in the p-adic context with non-negative power of p in order to achieve convergence of the series defining the partition function not possible for the exponent function in p-adic context.

1. The quantization of Kähler function as $K = rlog(m/n)$, where r is integer, $m > n$ is divisible by a positive power of p and n is indivisible by a power of p, implies that the exponent of Kähler function is of form $(m/n)^r$ and therefore exists also p-adically. This would guarantee the p-adic existence of the vacuum functional for any prime dividing m and for a given prime p would select a restricted set of p-adic space-time sheets (or partonic 2-surfaces) in the intersection of

real and p-adic worlds. It would be possible to assign several p-adic primes to a given space-time sheet (or partonic 2-surface). In elementary particle physics a possible interpretation is that elementary particle can correspond to several p-adic mass scales differing by a power of two [K54]. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $p + n - 2$ powers of $e^{1/n}$ can be considered.

2. The natural continuation to p-adic sector would be the replacement of integer coefficient r with a p-adic integer. For p-adic integers not reducing to finite integers the p-adic norm of the vacuum functional would however vanish and their contribution to the transition amplitude vanish unless the number of these space-time sheets increases with an exponential rate making the net contribution proportional to a finite positive power of p. This situation would correspond to a critical situation analogous to that encountered in string models as the temperature approaches Hagedorn temperature [B23] and the number states with given energy increases as fast as the Boltzmann weight. Hagedorn temperature is essentially due to the extended nature of particles identified as strings. Therefore this kind of non-perturbative situation might be encountered also now.
3. Rational numbers m/n with n not divisible by p are also infinite as real integers. They are somewhat problematic. Does it make sense to speak about algebraic extensions of p-adic numbers generated by $p^{1/n}$ and giving $n - 1$ fractional powers of p in the extension or does this extension reduce to something equivalent with the original p-adic number field when one redefines the p-adic norm as $|x|_p \rightarrow |x|^{1/n}$? Physically this kind of extension could have a well defined meaning. If this does not make sense, it seems that one must treat p-adic rationals as infinite real integers so that the exponent would vanish p-adically.
4. If one wants that Kähler action exists p-adically a transcendental extension of rational numbers allowing all powers of $log(p)$ and $log(k)$, where $k < p$ is primitive $p-1$: th root of unity in $G(p)$. A weaker condition would be an extension to a ring with containing only $log(p)$ and $log(k)$ but not their powers. That only single $k < p$ is needed is clear from the identity $log(k^r) = rlog(k)$, from primitive root property, and from the possibility to expand $log(k^r + pn)$, where n is p-adic integer, to powers series with respect to p. If the exponent of Kähler function is the quantity coding for physics and naturally required to be ordinary p-adic number, one could allow $log(p)$ and $log(k)$ to exists only in symbolic sense or in the extension of p-adic numbers to a ring with minimal dimension.
Remark: One can get rid of the extension by $log(p)$ and $log(k)$ if one accepts the definition of p-adic logarithm (see `http://tinyurl.com/y97ezkro`) as $log(x) = log(p^{-k}x/x_0)$ for $x = p^k(x_0 + py)$, $|y|_p < 1$. To me this definition looks somewhat artificial since this function is not strictly speaking the inverse of exponent function but might have a deeper justification.
5. What happens in the real sector? The quantization of Kähler action cannot take place for all real surfaces since a discrete value set for Kähler function would mean that WCW metric is not defined. Hence the most natural interpretation is that the quantization takes place only in the intersection of real and p-adic worlds, that is for surfaces which are algebraic surfaces in some sense. What this actually means is not quite clear. Are partonic 2-surfaces and their tangent space data algebraic in some preferred coordinates? Can one find a universal identification for the preferred coordinates- say as subset of embedding space coordinates selected by isometries?

If this picture inspired by p-adic thermodynamics holds true, p-adic integration at the level of WCW would give analog of partition function with Boltzmann weight replaced by a power of p reducing a sum over contributions corresponding to different powers of p with WCW integra.l over space-time sheets with this value of Kähler action defining the analog for the degeneracy of states with a given value of energy. The integral over space-time sheets corresponding to fixed value of Kähler action should allow definition in terms of a symplectic form defined in the p-adic variant of WCW . In finite-dimensional case one could worry about odd dimension of this sub-manifold but in infinite-dimensional case this need not be a problem. Kähler function could defines one particular zero mode of WCW Kähler metric possessing an infinite number of zero modes.

One should also give a meaning to the p-adic integral of Kähler action over space-time surface assumed to be quantized as multiples of $log(m/n)$.

1. The key observation is that Kähler action for preferred extremals reduces to 3-D Chern-Simons

form by the weak form of electric-magnetic duality. Therefore the reduction to cohomology takes place and the existing p-adic cohomology gives excellent hopes about the existence of the p-adic variant of Kähler action. Therefore the reduction of TGD to almost topological QFT would be an essential aspect of number theoretical universality.

2. This integral should have a clear meaning also in the intersection of real and p-adic world. Why the integrals in the intersection would be quantized as multiple of $log(m/n)$, m/n divisible by a positive power of p? Could $log(m/n)$ relate to the integral of $\int_1^p dx/x$, which brings in mind $\oint dz/z$ in residue calculus. Could the integration range $[1, m/n]$ be analogous to the integration range $[0, 2\pi]$. Both multiples of 2π and logarithms of rationals indeed emerge from definite integrals of rational functions with rational coefficients and allowing rational valued limits and in both cases $1/z$ is the rational function responsible for this.
3. $log(m/n)$ would play a role similar to 2π in the approach based on motivic integration where integral has geometric objects as its values. In the case of 2π the value would be circle. In the case of $log(m/n)$ the value could be the arc between the points $r = m/n > 1$ and $r = 1$ with r identified the radial coordinate of light-cone boundary with conformally invariant length measures dr/r. One can also consider the idea that $log(m/n)$ is the hyperbolic angle analogous to 2π so that these two integrals could correspond to hyper-complex and complex residue calculus respectively.
4. TGD as almost topological QFT means that for preferred extremals the Kähler action reduces to 3-D Chern-Simons action, which is indeed 3-form as cohomology interpretation requires, and one could consider the possibility that the integration giving $log(m/n)$ factor to Kähler action is associated with the integral of Chern-Simons action density in time direction along light-like 3-surface and that the integral over the transversal degrees of freedom could be reduced to the flux of the induced CP_2 Kähler form. The logarithmic quantization of the effective distance between the braid end points the in metric defined by Kähler-Dirac gamma matrices has been proposed earlier [K100].

Since p-adic objects do not possess boundaries, one could argue that only the integrals over closed varieties make sense. Hence the basic premise of cohomology would fail when one has p-adic integral over braid strand since it does not represent closed curve. The question is whether one could identify the end points of braid in some sense so that one would have a closed curve effectively or alternatively relative cohomology. Periodic boundary conditions is certainly one prerequisite for this kind of identification.

1. In one of the many cohomologies known as quantum cohomology (see `http://tinyurl.com/yaov8g2s`) [A68, A100] one indeed assumes that the intersection of varieties is fuzzy in the sense that two surfaces for which points are connected by what is called pseudo-holomorphic curve can be said to intersect at these points. As a special case pseudo-holomorphic curve reduce to holomorphic curve defined by a holomorphic map of 2-D Kähler manifold to complex manifold with Kähler structure. The question arises what “pseudoholomorphic curve connects points” really means. In the recent case a natural analog would be 2-D string world sheets or partonic 2-surfaces so that complex numbers are replaced by hyper-complex numbers effectively. The boundaries of string world sheets would be 1-D braid strands at wormhole throats and at the end of space-time sheet at boundaries of CD. In spirit of algebraic geometry one could also call the 1-D braid strands holomorphic curves connecting points of the partonic 2-surfaces at the two light-like boundaries of CD. In the similar manner space-like braid strands would connect points of partonic 2-surface at same end of CD.
2. In the construction of the solutions of the Kähler-Dirac equation one assumes periodic boundary conditions so that in physical sense these points are identified [K100]. This assumption actually reduces the locus of solutions of the Kähler-Dirac equation to a union of braids at light-like 3-surfaces so that finite measurement resolution for which discretization defines space-time correlates becomes an inherent property of the dynamics. The coordinate varying along the braid strands is light-like so that the distance in the induced metric vanishes between its end points (unlike the distance in the effective metric defined by the Kähler-Dirac gamma matrices): therefore also in metric sense the end points represent intersection point. Also the effective 2-dimensionality means are effectively one and same point.
3. The effective metric 2-dimensionality of the light-like 2-surfaces implies the counterpart of conformal invariance with the light-like coordinate varying along braid strands so that it

might make sense to say that braid strands are pseudo-holomorphic curves. Note also that the end points of a braid along light-like 3-surface are not causally independent: this is why M-matrix in zero energy ontology is non-trivial. Maybe the causal dependence together with periodic boundary conditions, light-likeness, and pseudo-holomorphy could imply a variant of quantum cohomology and justify the p-adic integration over the braid strands.

15.5.3 Motivic Integration

While doing web searches related to motivic cohomology I encountered also the notion of motivic measure (see `http://tinyurl.com/y73r7el8`) [A189] proposed first by Kontsevich. Motivic integration is a purely algebraic procedure in the sense that assigns to the symbol defining the variety for which one wants to calculate measure. The measure is not real valued but takes values in so called scissor group, which is a free group with group operation defined by a formal sum of varieties subject to relations. Motivic measure is number theoretical universal in the sense that it is independent of number field but can be given a value in particular number field via a homomorphism of motivic group to the number field with respect to sum operation.

Some examples are in order.

1. A simple example about scissor group is scissor group consisting operations needed in the algorithm transforming plane polygon to a rectangle with unit edge. Polygon is triangulated; triangles are transformed to rectangle using scissors; long rectangles are folded in one half; rectangles are rescaled to give an unit edge (say in horizontal direction); finally the resulting rectangles with unit edge are stacked over each other so that the height of the stack gives the area of the polygon. Polygons which can be transformed to each other using the basic area preserving building bricks of this algorithm are said to be congruent.
   The basic object is the free abelian group of polygons subject to two relations analogous to second homology group. If P is polygon which can be cut to two polygons P_1 and P_2 one has $[P] = [P_1] + [P_2]$. If P and P' are congruent polygons, one has $[P] = [P']$. For plane polygons the scissor group turns out to be the group of real numbers and the area of polygon is the area of the resulting rectangle. The value of the integral is obtained by mapping the element of scissor group to a real number by group homomorphism.
2. One can also consider symplectic transformations leaving areas invariant as allowed congruences besides the slicing to pieces as congruences appearing as parts of the algorithm leading to a standard representation. In this framework polygons would be replaced by a much larger space of varieties so that the outcome of the integral is variety and integration means finding a simple representative for this variety using the relations of the scissor group. One might hope that a symplectic transformations singular at the vertices of polygon combined with with scissor transformations could reduce arbitrary area bounded by a curve into polygon.
3. One can identify also for discrete sets the analog of scissor group. In this case the integral could be simply the number of points. Even more abstractly: one can consider algebraic formulas defining algebraic varieties and define scissor operations defining scissor congruences and scissor group as sums of the formulas modulo scissor relations. This would obviously abstract the analytic calculation algorithm for integral. Integration would mean that transformation of the formula to a formula stating the outcome of the integral. Free group for formulas with disjunction of formulas is the additive operation [A161]. Congruence must correspond to equivalence of some kind. For finite fields it could be bijection between solutions of the formulas. The outcome of the integration is the scissor group element associated with the formula defining the variety.
4. For residue integrals the free group would be generated as formal sums of even-dimensional complex integration contours. Two contours would be equivalent if they can be deformed to each other without going through poles. The standard form of variety consists of arbitrary small circles surrounding the poles of the integrand multiplied by the residues which are algebraic numbers for rational functions. This generalizes to rational functions with both real and p-adic coefficients if one accepts the identification of integral as a variety modulo the described equivalence so that $(2\pi)^n$ corresponds to torus $(S^1)^n$. One can replace torus with 2π if one accepts an infinite-dimensional algebraic extension of p-adic numbers by powers of 2π. A weaker condition is that one allows ring containing only the positive powers of 2π.

5. The Grassmannian twistor approach for two-loop hexagon Wilson gives dilogarithm functions $L_k(s)$ [B12]. General polylogarithm is defined by obey the recursion formula:

$$Li_{s+1}(z) = \int_0^z Li_s(t)\frac{dt}{t} \quad .$$

   Ordinary logarithm $Li_1(z) = -log(1-z)$ exists p-adically and generates a hierarchy containing dilogarithm, trilogarithm, and so on, which each exist p-adically for $|x| < 1$as is easy to see. If one accepts the general definition of logarithhs one finds that the entire function series exists p-adically for integer values of s. An interesting question is how strong constraints p-adic existence gives to the twistor loop integrals and to the underlying QFT.

6. The ring having p-adic numbers as coefficients and spanned by transcendentals $log(k)$ and $log(p)$, where k is primitive root of unity in $G(p)$ emerges in the proposed p-adicization of vacuum functional as exponent of Kähler action. The action for the preferred extremals reducing to 3-D Chern-Simons action for space-time surfaces in the intersection of real and p-adic worlds would be expressible p-adically as a linear combination of $log(p)$ and $log(k)$. $log(m/n)$ expressible in this manner p-adically would be the symbolic outcome of p-adic integral $\int dx/x$ between rational points. x could be identified as a preferred coordinate along braid strand. A possible identification for x earlier would be as the length in the effective metric defined by Kähler-Dirac gamma matrices appearing in the Kähler-Dirac equation [K100].

15.5.4 How Could One Calculate P-Adic Integrals Numerically?

Riemann sum gives the simplest numerical approach to the calculation of real integrals. Also p-adic integrals should allow a numerical approach and very probably such approaches already exist and "motivic integration" presumably is the proper word to google. The attempts of an average physicist to dig out this kind of wisdom from the vastness of mathematical literature however lead to a depression and deep feeling of inferiority. The only manner to avoid the painful question "To whom should I blame for ever imagining that I could become a real mathematical physicist some day?" is a humble attempt to extrapolate real common sense to p-adic realm. One must believe that the almost trivial Riemann integral must have an almost trivial p-adic generalization although this looks far from obvious.

A proposal for p-adic numerical integration

The physical picture provided by quantum TGD gives strong constraints on the notion of p-adic integral.

1. The most important integrals should be over partonic 2-surfaces. Also p-adic variants of 3-surfaces and 4-surfaces can be considered. The p-adic variant of Kähler action would be an especially interesting integral and reduces to Chern-Simons terms over 3-surfaces for preferred extremals. One should use this definition also in the p-adic context since the reduction of a total divergence to boundary term is not expected to take place in numerical approach if one begins from a 4-dimensional Kähler action since in p-adic context topological boundaries do not exist. The reduction to Chern-Simons term means also a reduction to cohomology and p-adic cohomology indeed exists.
   At the first step one could restrict the consideration to algebraic varieties - in other words zero loci for a set of polynomials $P_i(x)$ at the boundary of causal diamond consisting of pieces of $\delta M^4_\pm \times CP_2$. 5 equations are needed. The simplest integral would be the p-adic volume of the partonic 2-surface.

2. The numerics must somehow rely on the p-adic topology meaning that very large powers p^n are very small in p-adic sense. In the p-adic context Riemann sum makes no sense since the sum never has p-adic norm larger than the maximum p-adic norm for summands so that the limit would give just zero. Finite measurement resolution suggests that the analog for the limit $\Delta x \rightarrow 0$ is pinary cutoff $O(p^n) = 0$, $n \rightarrow \infty$, for the function f to be integrated. In the spirit of algebraic geometry one must asume at least power series expansion if not even the representability as a polynomial or rational function with rational or p-adic coefficients.

3. Number theoretic approach suggests that the calculation of the volume $vol(V)$ of a p-adic algebraic variety V as integral should reduce to the counting of numbers for the solutions for the equations $f_i(x) = 0$ defining the variety. Together with the finite pinary cutoff this would mean counting of numbers for the solutions of equations $f_i(x)\ mod\ p^n = 0$. The p-adic volume $Vol(V, n)$ of the variety in the measurement resolution $O(p^n) = 0$ would be simply the number of p-adic solutions to the equations $f_i(x)\ mod\ p^n = 0$. Although this number is expected to become infinite as a real number at the limit $n \to \infty$, its p-adic norm is never larger than one. In the case that the limit is a well-defined as p-adic integer, one can say that the variety has a well-defined p-adic valued volume at the limit of infinite measurement resolution. The volume $Vol(V, n)$ could behave like n_p^n and exist as a well defined p-adic number only if n_p is divisible by p.
4. The generalization of the formula for the volume to an integral of a function over the volume is straightforward. Let f be the function to be integrated. One considers solutions to the conditions $f(x) = y$, where y is p-adic number in resolution $O(p^n) = 0$, and therefore has only a finite number of values. The condition $f(x) - y = 0$ defines a codimension 1 sub-variety V_y of the original variety and the integral is defined as the weighted sum $\sum_y y \times vol(V_y)$, where y denotes the point in the finite set of allowed values of $f(x)$ so that calculation reduces to the calculation of volumes also now.

General coordinate invariance

From the point of view of physics general coordinate invariance of the volume integral and more general integrals is of utmost importance.

1. The general coordinate invariance with respect to the internal coordinates of surface is achieved by using a subset of embedding space-coordinates as preferred coordinates for the surface. This is of also required if one works in algebraic geometric setting. In the case of projective spaces and similar standard embedding spaces of algebraic varieties natural preferred coordinates exist. In TGD framework the isometries of $M^4 \times CP_2$ define natural preferred coordinate systems.
2. The question whether the formula can give rise to a something proportional to the volume in the induced metric in the intersection of real and rational worlds interesting. One could argue that one must include the square root of the determinant of the induced metric to the definition of volume in preferred coordinates but this might not be necessary. In fact, p-adic integration is genuine summation whereas the determinant of metric corresponds density of volume and need not make no sense in p-adic context. Could the fact that the preferred coordinates transform in simple manner under isometries of the embedding space (linearly under maximal subgroup) alone guarantee that the information about the embedding space metric is conveyed to the formula?
3. Indeed, since the volume is defined as the number of p-adic points, the proposed formula should be invariant at least under coordinate transformations mediated by bijections of the preferred coordinates expressible in terms of rational functions. In fact, even more general bijections mapping p-adic numbers to p-adic numbers could be allowed since they effectively mean the introduction of new summation indices. Since the determinant of metric changes in coordinate transformations this requires that the metric determinant is not present at all. Thus summation is what allows to achieve the p-adic variant of general coordinate invariance.
4. This definition of volume and more general integrals amounts to solving the remaining coordinates of embedding space as (in general) many-valued functions of these coordinates. In the integral those branches contribute to the integral for which the solution is p-adic number or belongs to the extension of p-adic numbers in question. By p-adic continuity the number of p-adic value solutions is locally constant. In the case that one integrates function over the surface one obtains effectively many-valued function of the preferred coordinates and can perform separate integrals over the branches.

Numerical iteration procedure

A convenient iteration procedure is based on the representation of integrand f as sum $\sum_k f_k$ of functions associated with different p-adic valued branches $z_k = z_k(x)$ for the surface in the coordinates chosen and identified as a subset of preferred embedding space coordinates. The number of branches z_k contributing is by p-adic continuity locally constant.

The function f_k -call it g for simplicity - can in turn be decomposed into a sum of piecewise constant functions by introducing first the piecewise constant pinary cutoffs $g_n(x)$ obtained in the approximation $O(p^{n+1}) = 0$. One can write g as

$$g(x) = \sum h_n(x) \ , \ h_0(x) = g_0(x) \ , \ h_n = g_n(x) - g_{n-1}(x) \ \text{for } n > 0 \ .$$

Note that $h_n(x)$ is of form $g_n(x) = a_n(x)p^n$, $a_n(x) \in \{0, p-1\}$ so that the representation for integral as a sum of integrals for piecewise constant functions h_n converge rapidly. The technical problem is the determination of the boundaries of the regions inside which these functions contribute.

The integral reduces to the calculation of the number of points for given value of $h_n(x)$ and by the local constancy for the number of p-adic valued roots $z_k(x)$ the number of points for $N_0 \sum_{k\geq 0} p^k = N_0/(1-p)$, where N_0 is the number of points x with the property that not all points $y = x(1 + O(p))$ represent p-adic points $z(x)$. Hence a finite number of calculational steps is enough to determine completely the contribution of given value to the integral and the only approximation comes from the cutoff in n for $h_n(x)$.

Number theoretical universality

This picture looks nice but it is far from clear whether the resulting integral is that what physicist wants. It is not clear whether the limit $Vol(V, n)$, $n \to \infty$, exists or even should exist always.

1. In TGD Universe a rather natural condition is algebraic universality requiring that the p-adic integral is proportional to a real integral in the intersection of real and p-adic worlds defined by varieties identified as loci of polynomials with integer/rational coefficients. Number theoretical universality would require that the value of the p-adic integral is p-adic rational (or algebraic number for extensions of p-adic numbers) equal to the value of the real integral and in algebraic sense independent of the number field. In the eyes of physicist this condition looks highly non-trivial. For a mathematician it should be extremely easy to show that this condition cannot hold true. If true the equality would represent extremely profound number theoretic truth.
   The basic idea of the motivic approach to integration is to generalize integral formulas so that the same formula applies in any number field: the specialization of the formula to given number field would give the integral in that particular number field. This is of course nothing but number theoretical universality. Note that the existence of this kind of formula requires that in the intersection of the real and p-adic worlds real and p-adic integrals reduce to same rational or transcendentals (such as $log(1 + x)$ and polylogarithms).
2. If number theoretical universality holds true one can imagine that one just takes the real integral, expresses it as a function of the rational number valued parameters (continuable to real numbers) characterizing the integrand and the variety and algebraically continues this expression to p-adic number fields. This would give the universal formula which can be specified to any number field. But it is not at all clear whether this definition is consistent with the proposed numerical definition.
3. There is also an intuitive expectation in an apparent conflict with the number theoretic universality. The existence of the limit for a finite number p-adic primes could be interpreted as mathematical realization of the physical intuition suggesting that one can assign to a given partonic 2-surface only a finite number of p-adic primes [K100]. Indeed, quantum classical correspondence combined with the p-adic mass calculations suggests that the partonic 2-surfaces assignable to a given elementary particle in the intersection of real and p-adic worlds corresponds to a finite number of p-adic primes somehow coded by the geometry of the partonic 2-surface.
   One way out of the difficulty is that the functions - say polynomials - defining the surface have as coefficients powers of e^n. For given prime p only the powers of e^p exist p-adically so that only the primes p dividing n would be allowed. The transcendentals of form $log(1 + px)$ and their polylogarithmic generalizations resulting from integrals in the intersection of real and p-adic worlds would have the same effect. Second way out of the difficulty would be based on the condition that the functional integral over WCW ("world of classical worlds") converges.

There is a good argument stating that the exponent of Kähler action reduces to an exponent of integer n and since all powers of n appear the convergence is achieved only for p-adic primes dividing n.

Can number theoretical universality be consistent with the proposed numerical definition of the p-adic integral?

The equivalence of the proposed numerical integral with the algebraic definition of p-adic integral motivated by the algebraic formula in the real context expressed in terms of various parameters defining the variety and the integrand and continued to all number fields would be such a number theoretical miracle that it deserves italics around it:

For algebraic surfaces the real volume of the variety equals apart from constant C to the number of p-adic points of the variety in the case that the volume is expressible as p-adic integer.

The proportionality constant C can depend on p-adic number field, and the previous numerical argument suggests that the constant could be simply the factor $1/(1-p)$ resulting from the sum of p-adic points in p-adic scales so short that the number of the p-adic branches $z_k(x)$ is locally constant. This constant is indeed needed: without it the real integrals in the intersection of real and p-adic worlds giving integer valued result $I = m$ would correspond to functions for which the number of p-adic valued points is finite.

The statement generalizes also to the integrals of rational and perhaps even more general functions. The equivalence should be considered in a weak form by allowing the transcendentals contained by the formulas have different meanings in real and p-adic number fields. Already the integrals of rational functions contain this kind of transcendentals.

The basic objection that number of p-adic points without cannot give something proportional to real volume with an appropriate interpretation cannot hold true since real integral contains the determinant of the induced metric. As already noticed the preferred coordinates for the embedding space are fixed by the isometries of the embedding space and therefore the information about metric is actually present. For constant function the correspondence holds true and since the recipe for performing of the integral reduce to that for an infinite sum of constant functions, it might be that the miracle indeed happens.

The proposal can be tested in a very simple manner. The simplest possible algebraic variety is unit circle defined by the condition $x^2 + y^2 = 1$.

1. In the real context the circumference is 2π and p-adic transcendental requiring an infinite-dimensional algebraic extension defined in terms of powers of 2π. Does this mean that the number of p-adic points of circle at the limit $n \to \infty$ for the pinary cutoff $O(p^n) = 0$ is ill-defined? Should one define 2π as this integral and say that the motivic integral calculus based on manipulation of formulas reduces the integrals to a combination of p-adically existing numbers and 2π? In motivic integration the outcome of the integration is indeed formula rather than number and only a specialization gives it a value in a particular number field. Does 2π have a specialization to the original p-adic number field or should one introduce it via transcendental extension?
2. The rational points $(x, y) = (k/m, l/m)$ of the p-adic unit circle would correspond to Pythagorean triangles satisfying $k^2 + l^2 = m^2$ with the general solution $k = r^2 - s^2$, $l = 2rs$, $m = r^2 + s^2$. Besides this there is an infinite number of p-adic points satisfying the same equation: some of the integers k, l, m would be however infinite as real integers. These points can be solved by starting from $O(p) = 0$ approximation $(k, l, m) \to (k, l, m)\ mod\ p \equiv (k_0, l_0, m_0)$. One must assume that the equations are satisfied only modulo p so that Pythagorean triangles modulo p are the basic objects. Pythagorean triangles can be also degenerate modulo p so that either k_0, l_0 or even m_0 vanishes. Note that for surfaces $x^n + y^n = z^n$ no non-trivial solutions exists for $x^n, y^n, z^n < p$ for $n > 2$ and all p-adic points are infinite as real integers.
The Pythagorean condition would give a constraint between higher powers in the expressions for k, l and m. The challenge would be to calculate the number of this kind of points. If one can choose the integers $k - (k\ mod\ p)$ and $l - (l\ mod\ p)$ freely and solve $m - (m\ mod\ p)$ from the quadratic equations uniquely, the number of points of the unit circle consisting of p-adic integers must be of form $N_0/(1-p)$. At the limit $n \to \infty$ the p-adic length of the unit circle

would be in p-adic topology equal to the number of modulo p Pythagorean triangles (r, s). The p-adic counterpart of 2π would be ordinary p-adic number depending on p. This definition of the length of unit circle as number of its modulo p Pythagorean points also Pythagoras would have agreed with since in the Pythagorean world view only rational triangles were accepted.

3. One can look the situation also directly solving y as $y = \pm\sqrt{1-x^2}$. The p-adic square root exists always for $x = O(p^n)$, $n > 0$. The number of these points x is $2/(1-p)$. For $x = O(p^0)$ the square root exist for roughly one half of the integers $n \in \{0, p-1\}$. The number of integers $(x^2)_0$ is therefore roughly $(p-1)/2$. The study of $p = 5$ cae suggests that the number of integers $(1-(x^2)_0)_0 \in \{0, p-1\}$ which are squares is about $(p-1)/4$. Taking into account the $\pm$ sign the number of these points by $N_0 \simeq (p-1)/2$. In this case the higher $O(p)$ contribution to x is arbitrary and one obtains total contribution $N_0/(1-p)$. Altogether one would have $(N_0+2)/(1-p)$ so that eliminating the proportionality factor the estimate for the p-adic counterpart of 2π would be $(p+3)/2$.
4. One could also try a trick. Express the points of circle as $(x, y) = (cos(t), sin(t))$ such that t is any p-adic number with norm smaller than one in p-adic case. This unit circle is definitely not the same object as the one defined as algebraic variety in plane. One can however calculate the number of p-adic points at the limit $n \to \infty$. Besides $t = 0$, all p-adic numbers with norm larger than p^{-n} and smaller than 1 are acceptable and one obtains as a result $N(n) = 1+p^{n-1}$, where "1" comes from overall important point $t = 0$. One has $N(n) \to 1$ in p-adic sense. If $t = 0$ is not allowed the length vanishes p-adically. The circumference of circle in p-adic context would have length equal to 1 in p-adic topology so that no problems would be encountered (numbers $exp(i2\pi/n)$ would require algebraic extension of p-adic numbers and would not exist as power series).
The replacement of the coordinates (x, y) with coordinate t does not respect the rules of algebraic geometry since trigonometric functions are not algebraic functions. Should one allow also exponential and trigonometric functions and their inverses besides rational functions and define circle also in terms of these. Note that these functions are exceptional in that corresponding transcendental extensions -say that containing e and its powers- are finite-dimensional?
5. To make things more complicated, one could allow algebraic extensions of p-adic numbers containing roots $U_n = exp(i2\pi/n)$ of unity. This would affect the count too but give a well-defined answer if one accepts that the points of unit circle correspond to the Pythagorean points multiplied by the roots of unity.

p-Adic thermodynamics for measurement resolution?

The proposed definition is rather attractive number theoretically since everything would reduce to the counting of p-adic points of algebraic varieties. The approach generalizes also to algebraic extensions of p-adic numbers. Mathematicians and also physicists love partition functions, and one can indeed assign to the volume integral a partition function as p-adic valued power series in powers $Z(t) = \sum v_n t^n$ with the coefficients v_n giving the volume in $O(p^n) = 0$ cutoff. One can also define partition functions $Z_f(t) = \sum f_n t^n$, with f_n giving the integral of f in the same approximation.

Could this kind of partition functions have a physical interpretation as averages over physical measurements over different pinary cutoffs? p-Adic temperature can be identified as $t = p^{1/T}$, $T = 1/k$. For p-adically small temperatures the lowest terms corresponding to the worst measurement resolution dominate. At first this sounds counter-intuitive since usually low temperatures are thought to make possible good measurement resolution. One can however argue that one must excite p-adic short range degrees of freedom to get information about them. These degrees of freedom correspond to the higher pinary digits by p-adic length scale hypothesis and high energies by Uncertainty Principle. Hence high p-adic temperatures are needed. Also measurement resolution would be subject to p-adic thermodynamics rather than being freely fixed by the experimentalist.

15.5.5 Infinite Rationals And Multiple Residue Integrals As Galois Invariants

In TGD framework one could consider also another kind of cohomological interpretation. The basic structures are braids at light-like 3-surfaces and space-like 3-surfaces at the ends of space-time surfaces. Braids intersects have common ends points at the partonic 2-surfaces at the light-like boundaries of a causal diamond. String world sheets define braid cobordism and in more general case 2-knot [K45]. Strong form of holography with finite measurement resolution would suggest that physics is coded by the data associated with the discrete set of points at partonic 2-surfaces. Cohomological interpretation would in turn would suggest that these points could be identified as intersections of string world sheets and partonic 2-surface defining dual descriptions of physics and would represent intersection form for string world sheets and partonic 2-surfaces.

Infinite rationals define rational functions and one can assign to them residue integrals if the variables x_n are interpreted as complex variables. These rational functions could be replaced with a hierarchy of sub-varieties defined by their poles of various dimensions. Just as the zeros allow realization as braids or braids also poles would allow a realization as braids of braids. Hence the n-fold residue integral could have a representation in terms of braids. Given level of the braid hierarchy with n levels would correspond to a level in the hierarchy of complex varieties with decreasing complex dimension.

One can assign also to the poles (zeros of polynomial in the denominator of rational function) Galois group and obtains a hierarchy of Galois groups in this manner. Also the braid representation would exists for these Galois groups and define even cohomology and homology if they do so for the zeros. The intersections of braids with of the partonic 2-surfaces would represent the poles in the preferred coordinates and various residue integrals would have representation in terms of products of complex points of partonic 2-surface in preferred coordinates. The interpretation would be in terms of quantum classical correspondence.

Galois groups transform the poles to each other and one can ask how much information they give about the residue integral. One would expect that the n-fold residue integral as a sum over residues expressible in terms of the poles is invariant under Galois group. This is the case for the simplest integrals in plane with n poles and probably quite generally. Physically the invariance under the hierarchy of Galois group would mean that Galois groups act as the symmetry group of quantum physics. This conforms with the number theoretic vision and one could justify the formula for the residue integral also as a *definition* motivated by the condition of Galois invariance. Of course, all symmetric functions of roots would be Galois invariants and would be expected to appear in the expressions for scattering amplitudes.

The Galois groups associated with zeros and poles of the infinite rational seem to have a clear physical significance. This can be understood in zero energy ontology if positive (negative) physical states are indeed identifiable as infinite integers and if zero energy states can be mapped to infinite rationals which as real numbers reduce to real units. The positive/negative energy part of the zero energy state would correspond to zeros/poles in this correspondence. An interesting question is how strong correlations the real unit property poses on the two Galois groups hierarchies. The asymmetry between positive and negative energy states would have interpretation in terms of the thermodynamic arrow of geometric time [K7] implied by the condition that either positive or negative energy states correspond to state function reduced/prepared states with well defined particle numbers and minimum amount of entanglement.

15.5.6 Twistors, Hyperbolic 3-Manifolds, And Zero Energy Ontology

While performing web searches for twistors and motives I have begun to realize that Russian mathematicians have been building the mathematics needed by quantum TGD for decades by realizing the vision of Grothendieck. One of the findings was the article "Volumes of hyperbolic manifolds and mixed Tate motives" (see `http://tinyurl.com/yargy3uw`) [A101] by Goncharov- one of the great Russian mathematicians involved with the drama - about volumes of hyperbolic n-manifolds and motivic integrals.

Hyperbolic n-manifolds (see `http://tinyurl.com/y8d3udpr`) [A42] are n-manifolds equipped with complete Riemann metric having constant sectional curvature equal to -1 (with suitable choice of length unit) and therefore obeying Einstein's equations with cosmological constant. They are obtained as coset spaces on proper-time constant hyperboloids of n+1-dimensional Minkowski space by dividing by the action of discrete subgroup of SO(n, 1), whose action defines a lattice like structure on the hyperboloid. What is remarkable is that the volumes of these closed spaces are homotopy invariants in a well-define sense.

What is even more remarkable that hyperbolic 3-manifolds (see `http://tinyurl.com/`

2vyksy) [A41] are completely exceptional in that there are very many of them. The complements of knots and links in 3-sphere are often cusped hyperbolic 3-manifolds (having therefore tori as boundaries). Also Haken manifolds are hyperbolic. According to Thurston's geometrization conjecture, proved by Perelman (whom we all know!), any closed, irreducible, atoroidal 3-manifold with infinite fundamental group is hyperbolic. There is an analogous statement for 3-manifolds with boundary. One can perhaps say that very many 3-manifolds are hyperbolic.

The geometrization conjecture of Thurston (see http://tinyurl.com/y8otvjau) [A34] allows to see hyperbolic 3-manifolds in a wider framework. The theorem states that compact 3-manifolds can be decomposed canonically into sub-manifolds that have geometric structures. It was Perelman who sketched the proof of the conjecture. The prime decomposition with respect to connected sum reduces the problem to the classification of prime 3-manifolds and geometrization conjecture states that closed 3-manifold can be cut along tori such that the interior of each piece has a geometric structure with finite volume serving as a topological invariant. There are 8 possible geometric structures in dimension three and they are characterized by the isometry group of the geometry and the isotropy group of point.

Important is also the behavior under Ricci flow (see http://tinyurl.com/2cwlzh9) [A74] $\partial_t g_{ij} = -2R_{ij}$: here t is not space-time coordinate but a parameter of homotopy. If I have understood correctly, Ricci flow is a dissipative flow gradually polishing the metric for a particular region of 3-manifold to one of the 8 highly symmetric metrics defining topological invariants. This conforms with the general vision about dissipation as the source of maximal symmetries. For compact n-manifolds the normalized Ricci flow $\partial_t g_{ij} = -2R_{ij} + (2/n)Rg_{ij}$ preserving the volume makes sense. Interestingly, for $n = 4$ the right hand side is Einstein tensor so that the solutions of vacuum Einstein's equations in dimension four are fixed points of normalized Ricci flow. Ricci flow expands the negatively curved regions and contracts the positively curved regions of space-time time. Hyperbolic geometries represent one these 8 geometries and for the Ricci flow is expanding. The outcome is amazingly simple and gives also support for the idea that the preferred extremals of Kähler action could represent maximally symmetries 4-geometries defining topological or algebraic geometric invariants: the preferred extremals would be maximally symmetric representatives - kind of archetypes- for a given topology or algebraic geometry.

The volume spectrum for hyperbolic 3-manifolds forms a countable set which is however not discrete: some reader might understand what the statement that one can assign to them ordinal ω^ω could possibly mean for the man of the street. What comes into my simple mind is that p-adic integers and more generally, profinite spaces which are not finite, are something similar: one can enumerate them by infinitely long sequences of pinary digits so that they are countable (I do not know whether also infinite p-adic primes must be allowed). They are totally disconnected in real sense but do not form a discrete set since since can connect any two points by a p-adically continuous curve.

What makes twistor people excited is that the polylogarithms emerging from twistor integrals and making sense also p-adically seems to be expressible in terms of the volumes of hyperbolic manifolds. What fascinates me is that the moduli spaces for causal diamonds or rather for the double light-cones associated with their M^4 projections with second tip fixed are naturally lattices of the 3-dimensional hyperbolic space defined by all positions of the second tip and 3-dimensional hyperbolici spaces are the most interesting ones! At least in the intersection of the real and p-adic worlds number theoretic discretization requires discretization and volume could be quantized in discrete manner.

For $n = 3$ the group defining the lattice is a discrete subgroup of the group of SO(3, 1) which equals to $PSL(2,C)$ obtained by identifying $SL(2,C)$ matrices with opposite sign. The divisor group defining the lattice and hyperbolic spaces as its lattice cell is therefore a subgroup of $PSL(2,Z_c)$, where Z_c denotes complex integers. Recall that $PSL(2,Z_c)$ acts also in complex plane (and therefore on partonic 2-surfaces) as discrete Möbius transformations wheras $PSL(2,Z)$ correspond to 3-braid group. Reader is perhaps familiar with fractal like orbits of points under iterated Möbius transformations. The lattice cell of this lattice obtained by identifying symmetry related points defines hyperbolic 3-manifolds. Therefore zero energy ontology realizes directly the hyperbolic manifolds whose volumes should somehow represent the polylogarithms.

The volumes, which are topological invariants, are said to be highly transcendental. In the intersection of real and p-adic worlds only algebraic volumes are possible unless one allows extension by say finite number of roots of e (e^p is p-adic number). The p-adic existence of polylogarithms suggests that also p-adic variants of hyperbolic spaces make sense and that one can assign to them volume as topological invariant although the notion of ordinary volume integral is problematic. In fact, hyperbolic spaces are symmetric spaces and general arguments allow to imagine what the p-adic variants of real symmetric spaces could be.

15.6 Floer Homology And TGD

Floer homology (see `http://tinyurl.com/m3thlqx`) [A28] has provided considerable understanding of symplectic manifolds using physics based approach relying on 2-D variational principle called symplectic action. One variant of Floer theory has been applied also to deduce topological invariants of 3-manifolds in terms of SU(2) Chern-Simons action. The basics of Floer homology without recourse to quantum field theoretic approach are described at technical level in the lectures of Dietmar Salamon (see `http://tinyurl.com/y7spzfce`) [A117]. The notion of quantum cohomology (see `http://tinyurl.com/y94n3xd3`) closely related to Floer homology and related approaches and involving also supersymmetry is described by Alexander Givental (see `http://tinyurl.com/y94n3xd3`) in [A100].

The quantum fluctuating degrees of freedom of TGD Universe are parameterized by symplectic group acting as isometries of WCW , which can be regarded as a union of symmetric spaces assignable to the symplectic group. Hence the optimistic hunch is that Floer homology might provide new insights about quantum TGD - in particular about the problem of understanding the preferred extremals of Kähler action. Especially interesting is the relationship of Floer homology to the proposed vision about braided Galois homology. The following considerations encourage this optimism. In particular, completely new insights about the role of Minkowskian and Euclidian regions emerge.

15.6.1 Trying To Understand The Basic Ideas Of Floer Homology

I do not have competence to describe Floer's homology as a mathematician. Instead, I try just to outline the basic ideas as I have (possibly mis-)understood them as a physicist by reading the basic introduction to the theory [A28]. The motivation for the symplectic Floer homology came from Arnold's conjecture (see `http://tinyurl.com/y86scus7`) stating that for a closed symplectic manifold the number of fixed points for non-degenerate (isolated critical points) symplecto-morphisms has the sum of the Betti numbers as a lower bound. The equivalence of Floer's symplectic homology for closed symplectic manifolds with singular homology (see `http://tinyurl.com/y9d6cg8n`) proves this conjecture. This means that symplectic Floer homology as such is not interesting from TGD view point of view.

Morse function in the loop space of the symplectic manifold

Recall that Morse function is a monotonically increasing real valued function in n-manifold for which critical points are isolated. Its level surfaces induce the slicing of the manifold $n-1$-dimensional surfaces. At the extrema the topology of the slice changes as is clear from a simple example provided by torus (standing on tangent plane orthogonal to the plane defined by the torus with Morse function identified as the height function defined by the coordinate orthogonal to the plane). There is minimum and maximum and two saddle points. Quite generally, the signature of the matrix defined by the second derivatives of the Morse function -Hessian- characterizes the properties of the critical point. Hessian allows to deduce information about the topology of the manifold and Morse theorem states that the number of critical points has a lower limit given by the sum of the Betti numbers defining the dimensions of various homology groups of the manifolds in singular homology.

Floer generalizes Morse theory from the level of symplectic manifold M with a Morse function defined by Hamiltonian to the level of the free loop space LM of M. This Morse function depends on preferred Hamiltonian and its cyclic time variation defining a loop in LM. Salamon represents the approach without recourse to the methods of topological quantum field theories [A117]. A very schematic representation -even more schematic than that in [A100] - using referring to quantum about what one does is attempted in following.

1. 2-dimensional action for an orbit of string in M replaces Morse function. The extrema of the action analogous to critical points of Morse function are crucial for calculating path integral in QFT approach using saddle point approximation. In topological QFTs path integral reduces to a well-defined finite dimensional integrals over moduli spaces. One constructs action principle in the form

$$S = \int_{-\infty}^{\infty} (||\partial_u m||^2 + ||\nabla f||^2) du \quad (15.6.1)$$

where u can be seen regarded as a coordinate parallel to cylinder axes defined by the orbit of the loop of M and t could be regarded as an angle coordinate of the loop. f denotes the symplectic action functional of the loop defined by time dependent Hamiltonian H_t. ∇f is the functional gradient of f with respect to coordinates of m regarded as analogous to fields $S^1 \times R$. $||...||^2$ defines inner product in the space of maps $S^1 \rightarrow M$ involving integral over the circle parameterized by coordinate t. Note that this action introduces preferred parameterization of the cylinder meaning breaking of at least manifest general coordinate invariance.

2. Schematically the field equations read as

$$\partial_u^2 m = \nabla^2 f \ , \quad (15.6.2)$$

where ∇^2 is functional d'Alembertian reducing to its analog at the level of M but depending on preferred Hamilton H_t. This condition states that the cylinder represents a harmonic map $S^1 \times R \rightarrow M$ with respect to the almost Kähler metric of M.

3. Assuming the analog of $\mathcal{N} = 2$ supersymmetry for the solution the above equation reduces to

$$\partial_u m = \pm \nabla f \ . \quad (15.6.3)$$

This condition is just the condition saying that one has a wave packed moving to right or left and state the hyper-complex variant of holomorphy. These left and right moving solutions are in key role in string model. In Euclidian metric of $S^1 \times R$ the conditions have interpretation as the generalization of Cauchy-Riemann conditions stating that the map $S^1 \times R \rightarrow M$ commutes with complex conjugation: in other worlds the multiplication by imaginary unit in $S^1 \times R$ is equivalent with the tensor multiplication defined by the almost Kähler form in M. The tangent space of image is complex sub-space of tangent space of M. Depending on the sign on the right hand side one has pseudo-holomorphy or anti-pseudo-holomorphy.

4. The solutions with finite action become asymptotically independent of u so that one has $\nabla f = 0$. This states that the loop represents a cyclic solution of Hamilton's equations for Hamilton H. Hamilton could also depend on time in periodic manner so that for $t = 0$ and $t = 2\pi$ one has $H_t = H$.

5. One can consider also solutions which are independent of u and t asymptotically so that the circles reduce to critical points asymptotically. One can also consider solutions representing spheres with more than two critical points as marked points. Also solutions with higher genus can be considered These solutions are relate closely to the definition of Gromov-Witten invariants in quantum cohomology.

This approach generalizes also to Chern-Simons action by replacing f with Chern-Simons action for the 3-manifold X^3 and $R \times S^1$ with $R \times X^3$ to get space-time. The symplectic manifold is replaced with the space of Yang-Mills gauge potentials. In this case field equations from the variational principle are YM equations and instanton and anti-instanton equations are obtained in the super-symmetric case. Time independent solutions correspond asymptotically to static solutions describing magnetic monopoles. In this case the critical points of Morse function can be seen as points at which the topology of the slice of field space defined by the Morse function changes its topology. A good intuitive guideline is Morse function for torus.

About Witten's approach to Floer homology

Using the ideas discussed for the first time in Witten's classic work (see `http://tinyurl.com/yclbmjld`) revealing a connection between supersymmetry and Morse theory (see `http://tinyurl.com/y82ymev9`) [A157], one can extend M to a super-manifold. Witten defines $\mathcal{N} = 2$ SUSY algebra by introducing a parameter dependent deformation of the exterior algebra via $d_t = exp(-th)dexp(th)$ and its conjugate $d_t^* = exp(th)dexp(-th)$: for $t = 0$ one has $d_t = d_t^*$. h takes the role the role of Morse function. $Q_1 = d_t + d_t^*$ and $Q_2 = i(d_t - d_t^*)$ obey standard supersymmetry algebra $Q_1Q_2 + Q_2Q_1 = 0$ and $Q_1^2 = Q_2^2 \equiv H_t$. The solutions of $d_t\Psi = 0$ are differential forms of various degrees and correspond to zero energy solutions for which the supersymmetry is not broken. The deformed cohomology is equivalent with the original cohomology by $\Psi \rightarrow exp(th)\Psi$. This gives a direct connection between cohomology and supersymmetry whose existence is to be expected from the basic properties of exterior algebra.

The motivation for the deformation is that for degree p closed forms are localized around critical points of h with Hessian having p negative eigenvalues so that the correspondence between homology generators and critical points becomes manifest. There is indeed a natural mapping from de Rham cohomology to the critical points such that the degree of the form correspond to the number of negative eigenvalues of the Hessian.

Later Witten managed to expand his ideas about supersymmetric Morse theory so that it could be applied to Floer homology (1+1 case) and to the calculation of Donaldson invariants of 4-manifold (1+3 case). Recently Witten has been working with the applications to knot theory (1+2 case) for ordinary knots and for 2-knots and cobordisms of 1-knots (1+3 case) [A127, A90, A128].

Representation of loops with fixed based in terms of Hamiltonians with cyclic time dependence

As already noticed Floer - whose work preceded Witten's work - considered instead of the symplectic manifold M its free loop space LM. One begins with symplectic action identified as the sum of the symplectic area of the loop expressible as the value of the one-form defining the symplectic form over the loop and integral of the Hamiltonian H around the loop. The natural choice of the loop parameter is as the canonical conjugate of the symplectic potential so that the integrated quantity is analogous to the minimal substitution $p - eA$ of familiar from elementary quantum mechanics. The variational equations for the symplectic action are Hamiltonian equations of motion in the force field defined by the Hamiltonian H and one considers periodic orbits (recall that there is conserved energy associated with the orbits defined by the Hamiltonian). The counterparts of critical points are loops which correspond to the extrema of symplectic action.

One can also consider time dependent Hamiltonians H_t for which the initial and final value of the Hamiltonian is the same preferred Hamiltonian. This kind of Hamiltonians define via their time evolutions loops in the loop space LG of the symplectic group. At the level of LM the resulting map of M to itself is symplecto-morphism. Now however energy is not in general conserved. By periodicity the critical points of the Hamiltonian H correspond to cyclic orbits of periodically time varying Hamiltonian so that the homotopies of LM with base point defined by H are mapped to a collection of homotopies of M defined by the critical points of the Hamiltonian. For constant Hamiltonian $H_t = H$ the critical orbits reduce to a point and the need to obtain non-trivial elements of homotopy group of M explains why one needs Hamiltonians with cyclic time dependence. The homotopy group of LM is mapped to that of M by homomorphism.

One could consider also higher homotopy groups of the loop space. The first homotopy group would correspond to loops in loop space mapped to tori associated with the fixed points of the Hamiltonian. In this manner one would obtain analogs of homotopy groups defined by mappings from $(S^1)^n$ to loop space to M and also of homotopy groups. By taking the initial loop to be trivial so that initial Hamiltonian is constant Hamiltonian, one obtains the symplectic analogs of ordinary homotopy groups defined as a map from S^n to loop space to M. Also the condition that loops are contracted to points asymptotically gives rise to homotopy groups.

Representation of non-closed paths of LM as paths connecting critical points of M

In Floer homology one considers also paths of LM and M, which are not closed. These paths form the first homotopy groupoid of LM. Since the elements of $\pi_0(LM)$ (loops not deformable to each other) represented by Hamiltonians with cyclic time dependence are mapped to those of $\pi_1(M)$ at critical points, a good guess is that the elements of homotopy group $\pi_1(LM)$ can be mapped to elements of $\pi_2(M)$ connecting critical points of H. If the loops at the ends of cylinder reduce to points the images of $\pi_1(LM)$ are indeed elements of $\pi_2(M)$ containing two critical points. As noticed, the number critical points can be also higher.

To achieve the representation of first homotopy group one considers a path of LM parameterized by a parameter u defining a cylinder in M which should connect the critical points. This requires that the deformation becomes at the limit $u \to \pm\infty$ independent of u so that one obtains a cyclic deformation of H. The partial differential equations state that one has gradient flow defined by symplectic action in loop space. The equations (resulting from supersymmetry in QFT approach) pseudo-holomorphy or generalized Cauchy-Riemann conditions as

$$\partial_u m \pm L_{H_t}(m) = 0 \ ,$$

where $L_{H_t}(m) = 0$ denotes Hamiltonian equations for the coordinates m of M so that $L_{H_t}m$ is indeed the functional gradient of symplectic action. At the asymptotic limit $\partial_u m \to 0$ boundary conditions give just Hamiltonian equations.

As already found, one can assign to these equations a supersymmetric action functional defined in terms of the almost Kähler metric defining the analog of energy. As a matter fact, the existence of almost complex structure in M is enough (transitions functions between coordinate patches need not be holomorphic in this case). The condition that the energy is finite requires asymptotic u-independence and super-symmetry condition since energy density is the sum of kinetic energy densities associated with the motion in u direction and of the square of the vector $L_{H_t}m$. Since the time evolution with respect to u is not energy conserving, the cylinders can connect different critical points of H. This motivates the term "connecting cylinder". From the point of view of physicist the role of the field equations is to perform a "gauge choice" selecting particular representative for homotopy.

The orbit of the loop as a pseudo-holomorphic surface

The cylinder defined by the loop defines a pseudo-holomorphic surface. The sub-spaces connected by pseudo-holomorphic surfaces intersect in quantum cohomology and Gromow-Witten invariant counts for the number of the pseudo-holomorphic surfaces connecting/intersecting given n surfaces. Stringy interpretation for the pseudo-holomorphic curves (holomorphic for Kähler manifolds) would be as string world sheets. There is an obvious connection with the vision about branes connected by string world sheets. If the asymptotic images of S^1 contract to points, they correspond to critical points (marked points). One can consider also more general solutions of field with n asymptotic circles containing n critical points as marked points.

The statement of quantum cohomology that two surfaces intersect in fuzzy sense when they are connected by pseudo-holomorphic curve would mean that that two surfaces intersect when they both have points common with the pseudo-holomorphic curve. The 2-dimensional mapping cylinders can be filled to 3-D objects by adding the 2-dimensional pseudo-holomorphic surface. From this the connection with Chern-Simons action and possibility to apply analogous construction to 3-D manifold topology becomes obvious. Chern-Simons action in turn implies connection to 4-D manifold topology.

The correspondence with the singular homology

Symplectic Floer homology for closed symplectic manifolds is equivalent with singular homology. This means that one has one-to-one map of the space spanned by the critical points to the singular homology. Critical points are classified by the signature of the Hessian of Hamiltonian so that there is natural ordering of the critical points, which should correspond to the ordering of the homology groups since signature varies from n (maximum of Morse function) to zero (minimum of Morse function). The study of the homology of torus defined in terms of critical points of height function h serves as a guide-line when one tries to guess the idea behind the correspondence.

To each critical point one can assign a tangent plane defined as the plane of negative signature of the Hessian of h. Its value equals to 0, 1, 1, 2 for the critical points of h. The critical manifolds assigned with the negative signature tangent space at critical points can be identified as point, first homologically non-trivial circle, second homologically non-trivial circle, and the entire torus and correspond to the generators of the homology. In Floer homology the correspondence need not be as simple as this but one expect similar correspondence so that the value of grading of homology corresponds to the signature of the critical point. One must allow only the connections going to the direction of smaller energy and by a proper choices of signs the dynamics defined by the action defined gradient flow is indeed dissipative so that this condition is satisfied.

Quantum cup product and pseudo-holomorphic surfaces

As the analog of intersection product in ordinary cohomology homology, the cohomology associated with the symplectic Floer homology corresponds to the so called pair of pants product of quantum cohomology [A100] which is a deformed cup product having fuzzy intersection as its dual at the level of homology.

Ordinary cup product for two forms of degree n_1 and n_2 is a form which is characterized by its values for the elements of homology with co-dimension $n_1 + n_2$ so that $d - n_1 - n_2$ is the dimension of the intersection of the corresponding surfaces. The product is characterized by a coefficients $W(\alpha, \beta, \gamma)$ where the arguments represent homology equivalence classes identifiable as Gromov-Witten invariants assignable to sphere with three punctures. One can say that three representatives α, β, γ of homology give rise to a non-vanishing coefficient $W(\alpha, \beta, \gamma)$ if there is a pair of pants having non-empty intersections with α, β, γ. The coefficient $W(\alpha, \beta, \gamma)$ is analogous to a coupling constant associated with vertex with α, β, γ representing the particles entering to the vertex.

The factors of the cup product of quantum cohomology are associated with the two legs of the pants and the outcome of the product to the "waist". More abstractly, by conformal transformations the legs and "waist" can be reduced to 3 marked points and the number of marked points can be arbitrary and represent the intersection points for n manifolds connected by a pseudo-holomorphic surface with n marked points. One can indeed generalize the variational principle to allow besides cylinders also pseudo-holomorphic surfaces with arbitrary number holes whose boundaries are associated with loops containing critical point so that critical points would indeed represent marked points of a sphere with holes. When H_t reduces to H, loops and marked spheres reduce to point a so that ordinary cup product results.

15.6.2 Could Floer Homology Teach Something New About Quantum TGD?

The understanding of both quantum TGD and its classical counterpart is still far from from comprehensive. For instance, skeptic could argue that the understanding of the preferred extremals of Kähler action is still just a bundle of ideas without a coherent overview. Also the physical roles of Kähler actions for Euclidian and Minkowskian space-time regions is far from clear. Do they provide dual descriptions as suggested or are both needed? Kähler action for preferred extremal in Euclidian regions defines naturally Kähler function. Could Kähler action in Minkowskian regions- naturally imaginary by negative sign of metric determinant- give an imaginary contribution to the vacuum functional and define Morse function so that both Kähler and Morse would find a prominent role in the world order of TGD? One might hope that the mathematical insights from Floer homology combined with the physical picture and constraints from quantum classical correspondence could provide additional insights about the construction preferred extremals of Kähler action.

Basic picture about preferred extremals of Kähler action

It is useful to gather some basic ideas about construction of preferred extremals before the discussion of ideas inspired by Föloer homology.

1. For the preferred extremals Kähler action reduces to Chern-Simons term at the light-like surfaces defining orbits of partonic 2-surfaces and space-like 3-surfaces the ends of the space-time sheets. These 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric-magnetic duality implying that TGD does not reduce to a mere topological QFT. One has clearly two dynamics: one along light-like 3-surfaces and one along space-like 3-surfaces and their internal consistency is a powerful constraint.
2. The Chern-Simons contributions from Minkowskian region is imaginary and corresponds to almost topological QFT aspect of TGD. The argument reducing the action to Chern-Simons term has been discussed in detail only in Minkowskian regions and involves in an essential

manner the notions of local polarization and light-like momentum direction: the latter one does not make sense in Euclidian regions. Note however that Laplace equation makes sense and local polarization and momentum directions are replaced by those for color quantum numbers. It will be found that internal consistency requires holography both in Minkowskian and Euclidian regions. In any case, the Euclidian contribution would give rise to the exponent of Kähler function and Minkowskian contribution to a phase factor appearing usually in path integral defining topological QFT. Exponent of Kähler function would guarantee that integration over WCW is mathematically well-defined.

3. How could one extend the 3-surfaces to 4-surfaces using strong form of holography? One could think of having for each time=constant collection of 2-D slices of the light-like 3-surfaces a space-like Chern-Simons dynamics connecting them to each other. One would have two dynamics-one time-like and one space-like as effective 2-dimensionality required by the strong form of holography requires. These dynamics should be mutually consistent and this should give consistency conditions. The time parameters for these two dynamics would correspond to the two coordinates of string world sheets involved.
4. The idea that one could assign Hamiltonians to the marked points of the partonic 2-surfaces as carriers is physically compelling. The Hamiltonians of $\delta M^4_\pm \times CP_2$ inducing Hamiltonians of WCW play essential role in quantum theory. Also the Hamiltonians at ends of braid strands should have classical counterparts at space-time level. Could braid strand obey Hamiltonian dynamics defined by Hamiltonian attached to it? This would give a constraint to the wormhole throat making itself visible also a properties of the space-time sheet. If so then braid strands would define a kind of the skeleton for the space-time sheet. This idea could be generalized so that one would have a skeleton of space-time consisting of string world sheets and finite measurement resolution would mean the restriction of consideration to this skeleton. Also the braid strands carrying fermion number (other than right handed neutrino number) should obey their own dynamics.

Braided Galois homology as counterpart of Floer homology?

The picture suggested by braided Galois homology seems to have natural correspondences with that provided by Floer homology.

1. The quantum fluctuating degrees of freedom correspond to the symplectic group of $\delta M^4_\pm \times CP_2$. Finite measurement resolution leads to the discretization. One considers the subgroup G of symplectic group of $\delta M^4_\pm \times CP_2$ permuting a given set of n points of the partonic 2-surface defining the end points of braids. Subgroup of S_n having interpretation as Galois group is in question. The normal subgroup H of symplecto-morphisms leaving these points invariant and the factor group G/H is the target of primary interest and expected to be discrete group. The braiding of this group is intuitively equivalent with the replacement of symplectic transformations with flows and the points can be interpreted as critical points of infinite number of Hamiltonian belonging to H. In Floer's theory one makes a gauge choice selecting a generic non-degenerate Hamiltonian. This choice -or a generalization of it- should have a definite physical meaning in TGD framework in terms of classical correlates for the quantum numbers of the zero energy state.
2. Preferred Hamiltonian acting and its time dependent deformation play a key role in Floer homology and represent homotopy in symplectic group. In the recent case braided Galois homology assigns to preferred extremals subgroup of symplectic flow in Minkowskian space-time regions and the braid points are invariant under its normal subgroup. The flow defined by time dependent deformation a Hamiltonian of subgroup defines a candidate for the flow defined by preferred Hamiltonian. The connecting flows in turn would correspond to the Galois group. The condition that the flow lines of the Hamilton along 3-surfaces poses a strong condition on the choice of Hamiltonian on one hand and on the preferred extremal on the other hand. The time evolution of Hamiltonian could be realized by the slicing of embedding space by light-cone boundaries parallel to the lower or upper boundary of CD.
3. For braided Galois homology the generators d_i representing boundary homomorphisms whose square maps to commutator subgroup and to zero after abelianization define candidates for the algebra of SUSY generators. Parameter dependent deformation of these generators would

make sense also now and give rise a homology analogous to that of Witten. The generators of the cohomology would correspond to supersymmetric ground states and one would expect that cohomology is non-trivial for the critical points of Morse function. This super-symmetry, which need not have anything to do with the standard notion of supersymmetry, would be assigned to Minkowskian regions of space-time. One cannot of course exclude purely fermionic representations of braided Galois homology and number theoretic quantization of fermions would pose a powerful constraint on the spectrum of fermionic modes.

Kähler function as Kähler action in Euclidian regions and Morse function as Kähler action in Minkowskian regions?

The role of Kähler action in the Floer like aspects of TGD has been already briefly discussed.

1. Symplectic Floer homology for embedding space gives just the homology groups of $S^2 \times CP_2$. This homology is crucial for the interpretation of TGD but much more detailed information is required. The analog of Floer homology must be associated with WCW for which quantum fluctuating degrees of freedom are parametrized by symplectic group of $\delta M^4_\pm \times CP_2$ or symmetric space associated with it. In finite measurement resolution one would have discrete subgroup defined as a factor group of subgroup permuting braid points and normal subgroup leaving them invariant identifiable in terms of a hierarchy of Galois groups. Flows must be considered in order to have braiding. The flows could also correspond to parameter dependent Hamiltonians with the parameter varying along light-like wormhole throat or space-like 3-surface at the end of CD.
2. In the case of Chern-Simons action the critical points correspond to flat connections and define the generators of the homology for the space of connections. For YM action instanton solutions play similar role. In the recent case the space of 3-surfaces associated with given CD seems to be natural object of study.
   Kähler function - to be distinguished from Kähler action - would be the first guess for the Morse function in WCW and the analog of Floer homology would be formally defined by the sums of the 3-surfaces which correspond to the extrema of Kähler function. This idea fails. Kähler metric must be positive definite. Therefore the Hessian of the Kähler function in holomorphic quantum fluctuating degrees of freedom characterized by complex coordinates of WCW should have only non-negative or non-positive eigen values.
   One could try to circumvent the difficulty by assuming that the allowed extrema with varying signature of Hessian are associated with the zero modes. Therefore the analog of Floer homology based on Kähler function would not however tell anything about symplectic degrees of freedom -at least those assignable to the Euclidian regions.
   Remark: One can wonder how the Kähler function can escape the implications of Morse theorem. In the case of CP_2 the degeneracy of Kähler function - meaning that it depends on single $U(2)$ invariant CP_2 coordinate only - takes care of the problem. Also now infinite-dimensional symmetries of WCW are expected to allow to circumvent the Morse theorem.
3. The only manner to save this idea is that the Euclidian regions defined by the generalized Feynman graphs define Kähler function and Minkowskian regions the analog of the action defining path integral. The earlier proposed duality states that the formulation TGD is possible either as a functional integral or a path integral. If duality holds true, its effect would be analogous to that of Wick rotation. The alternative approach would assign physical significance to both contributions. The Kähler action in Minkowskian regions could serve as Morse function. This identification is rather natural since the determinant of the induced metric appearing in the action indeed gives imaginary unit in Minkowskian regions. If this were the case interference effects would result already at the level of action and the connection with quantum field theories would be much tighter than previously thought.
   Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. The analog of Floer homology would represent quantum superpositions of critical points identifiable a ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function.
4. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

(a) All arguments for this have been represented for Minkowskian regions [K100] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of CP_2 bounded by wormhole throats: for CP_2 itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time super-symmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the Kähler-Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.

(b) If the reduction occurs in Euclidian regions, it gives in the case of CP_2 two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for CP_2 so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.

(c) There is also a very beautiful argument stating that Dirac determinant for Kähler-Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

5. The preferred extremal of Kähler action itself would connect 3-surfaces at the opposite boundaries of CD just as the action for Floer theory connects two loops assignable to critical points. In zero energy ontology the unions of 3-surfaces at the ends of CD is the basic unit and correspond to the critical points of Morse function. The question is whether objects can be mapped to a set of critical points of the preferred Hamiltonian in a natural manner. Braided Galois homology with preferred Hamiltonian defining the braids as its flow lines gives hopes about this.

6. In Floer theory the homology of LM is mapped to homology of M. The homology of the WCW cannot be mapped to that of the embedding space. The hierarchy of Planck constants [K32] assigned to the multi-valued correspondence between canonical momentum densities of Kähler action and time derivatives of embedding space coordinates leads to the introduction of singular covering spaces of the embedding space with the number of sheets of covering depending on space-time region. The homology of WCW might be mapped homomorphically to the homology of this space.

   In the case of loop space $H_0(LM)$ is mapped to $H_1(M)$. Something similar should take place now since all odd homology groups of WCW must vanish if it is Kähler manifold whereas zeroth homology could be non-trivial. In zero energy ontology 3-surfaces having disjoint components at the ends of CD indeed correspond naturally to paths of connected 3-surface so that this condition might be realized.

On basis of these arguments it seems that the general structure of Floer homology fits rather nicely the structure of quantum TGD.

TGD counterparts for pseudo-holomorphic surfaces

If the Morse function exists as Kähler action for preferred extremal in the Minkowskian regions of the space-time, there are good hopes of obtaining the analog of Floer homology in TGD framework. Consider first pseudo-holomorphic surfaces.

1. The analogy with Floer homology would suggest that the analogs of pseudo-holomorphic surfaces assignable to the critical points of Morse function correspond to 3-surfaces at the ends of CD are 3-surface defined by the simultaneous vanishing of two holomorphic rational functions of the complex coordinates of $S^2 \subset \delta M^4_\pm$ and of CP_2 depending parametrically on the light-like radial coordinate of $\delta M^\pm$ giving $7-4=3$ conditions. The effective metric 2-dimensionality implied by the strong form of holography is expected to pose conditions on the radial dependence of these functions.

2. Pseudo-holomorphic closed string world sheets with punctures provide a beautiful geometric realization of quantum cohomology. If positive and negative energy parts of zero energy states

can be regarded as elements of homology, space-time sheets could take a similar role. In finite measurement resolution string world sheets would perform the same function so that closed strings would be replaced with open ones as connectors in TGD based quantum cohomology. Signature is not a problem: in string theories the hypercomplex variant of holomorphy is allowed. String world sheets would connect partonic two surfaces at the given end of partonic CD and also at different ends of CD. String world sheets could branch but the mechanism would be the decay of open string creating new partonic 2-surfaces meeting at TGD counterpart of Feynman vertex. Note that also in Witten's approach to Floer theory and Donaldson theory the signature of string world sheets is Minkowskian.

Remarks:

(a) One can imagine an extremely simple definition for the intersection for partonic 2-surfaces at opposite boundaries of CD proposed actually earlier. One could identify the opposite boundaries of CD given by pieces $\delta M^4_\pm \times CP_2$ by identifying δM^4_+ and δM^4_- in an obvious manner. This definition is however a natural dynamical counterpart for intersection in classical sense obtained by identifying the boundaries of CD.

(b) So called massless extremals represent one example about the analogs of right and left moving solutions in TGD framework [K14]. They distinguish sharply between classical TGD and Maxwell's hydrodynamics. There are arguments suggesting that quite generally the preferred extremals in Minkowskian regions representable as graphs of maps $M^4 \times CP_2$ decompose to regions characterized by local directions of momentum and polarization representing propagation of massless waves. This would be the classical space-time correlate for the decomposition of radiation to massless quanta.

3. Partonic 2-surfaces with particles at the ends of braid strands would define basic objects and would naturally correspond to holomorphic surfaces for the critical points of Morse function defined by the contribution of Minkowskian regions to Kähler action. The hyper-complex string world sheets and hyper-quaternionicity are however necessary for the $M^4 \times CP_2 - M^8$ correspondence suggested by physics as generalized number theory vision. The finite dimensions of the moduli spaces would not be a problem since holomorphy would characterize only the critical points. The connection between super-symmetry and cohomology plays a key role in TQFT and pseudo-holomorphy is an excellent candidate for the geometric correlate of supersymmetry of some kind.

The natural question is whether pseudo-holomorphy could generalize in 4-D context to its quaternionic analog.

1. One of the basic conjectures of TGD is that preferred extremals of Kähler action can be regarded as associative (co-associative) sub-manifolds. The tangent spaces of space-time surfaces would define hyper-quaternionic sub-spaces of complexified octonions with imaginary units of quaternions would be multiplied by commuting imaginary unit.
2. The tangent spaces of space-time surface would also contain a preferred hyper-complex plane or more generally, a hyper-complex plane which depends on position so that these planes integrate to string world sheet. This would allow to regard space-time surfaces either as surfaces in $M^4 \times CP_2$ or in hyper-octonionic subspace M^8 [K86]. Integrable distributions of the hyper-complex sub-manifolds would define string world sheets analogous with hypercomplex sub-manifolds. The physical interpretation would be in terms of local preferred planes of unphysical polarizations. The philosophical motivation of hyper-quaternionicity would be that associativity for space-time surfaces and commutativity for string world sheets could define a number theoretical variational principle.
3. The role of pseudo-holomorphy suggests that hyper-quaternionicity could characterize the critical points of Morse function defined by Kähler action in Minkowskian regions of space-time. If all preferred extremals are hyper-quaternionic, this property cannot imply holomorphy of the partonic surfaces.
4. It was already mentioned that finite measurement resolution defines a skeleton of space-time surface realized in terms of string world sheets. This skeleton would generalize a curve of complex plane at which holomorphic function defining a complex coordinate is real to hyper-complex sub-manifold of hyper-quaternionic space-time surface. Given this skeleton, the construction of space-time surface would be analogous to an analytic continuation from hyper-complex realm to hyper-quaternionic realm.

Hierarchy of Planck constants, singular coverings of the embedding space, and homology of WCW

1. As already noticed, the homology groups of embedding space are certainly too simple to be of interest from the point of physics and quantum TGD. Physically interesting analogs of homology groups could be associated with the space-time surface itself or with the singular covering of embedding space allowing to describe the many-valued correspondence between canonical momentum densities and time derivatives of embedding space coordinates. This would allow to interpret the resulting non-trivial homology as a property of either space-time surface or of effective embedding space. In any case, one should add to the homology the constraint that the elements of homology are representable as sub-varieties for the preferred extremals of Kähler action. This might allow to code physics using the formalism of homology theory. Floer like theory would also define a homomorphism mapping the homology $H_n(WCW)$ to the homology group H_{m+1} of the singular covering of the embedding space.
2. The recent interpretation for the effective hierarchy of Planck constants coming as integer multiples of ordinary Planck constants has interpretation in terms of effective coverings of space-time surface implied by the 1-to-many character of the map assigning to canonical momentum densities of Kähler action time derivatives of embedding space coordinates. The strange sounding proposal is that at partonic two surfaces branching occurs in the sense that the various branchings of the many-valued function involved with this correspondence coincide. Branching would however occur both in the direction of the light-like 3-surface and space-like 3-surface at the end of CD. Branching could occur at both ends of given CD or only at single end if the branching is taken as a space-time correlate for dissipation and arrow of time, and perhaps even for quantum superposition as will be discussed below.
3. This branching brings in mind the emergence of homologically non-trivial curves from the critical points in Floer cohomology and possibility of several curves connecting two critical points (torus serves as a good illustration also now). The analogy would be more convincing if one could assign to the branches a sign factor analogous to the sign of the eigenvalue of Hessian as physical signature. One possibility is that the sign factor tells whether the line is incoming or outgoing. Also the sign of energy in the case of virtual particles could appear in the sign factor.

How detailed quantum classical correspondence can be?

The gradient dynamics is quite essential for the super-symmetric solutions of Floer theory and typically gradient dynamics is dissipative leading to fixed points of the function function involved. Dissipative dynamics allows to order critical points in terms of the energy defined by Hamilton and also connect different critical points. Physicist would obviously ask whether this aspect of the dynamics is only an artefact of the model or whether it has a much deeper physical significance. If it does not, the following considerations can be taken only as a proposal for how the quantum correlates could be represented at space-time level and how detailed they can be.

Can the dynamics defined by preferred extremals of Kähler action be dissipative in some sense? The generation of the arrow of time has a nice realization in zero energy ontology as a choice of well-defined particle numbers and other quantum numbers at the "lower" end of CD. By quantum classical correspondence this should have a space-time correlate. Gradient dynamics is a highly phenomenological realization of the dissipative dynamics and one must try to identify a microscopic variant of dissipation in terms of entropy growth of some kind. If the arrow of time and dissipation has space-time correlate, there are hopes about the identification of this kind of correlate.

Quantum classical correspondence has been perhaps the most useful guiding principle in the construction of quantum TGD. What is says that not only quantum numbers but also quantum jump sequences should have space-time correlates: about this the failure of strict determinism of Kähler action gives good hopes. Even the quantum superposition- at least for certain situations -might have space-time correlates.

1. Measurement interaction term in the modified Dirac action at the ends of CD indeed defines a coupling of quantala dynamics to the classical dynamics [K100]. The interpretation of TGD as square root of thermodynamics suggests that measurement interaction terms are completely

analogous to the Lagrange multiplier terms fixing the values of observables in thermodynamics. Now the classical conserved charges would be fixed to their quantal values for the space-time surfaces appearing in quantum superposition. These Lagrange multiplier terms would also give to Kähler-Dirac action 3-D boundary terms. By the localization spinor modes to space-time sheets these boundary terms are effectively one-dimension and localized to stringy curves.
This kind of measurement interaction is indeed basic element of quantum TGD. Also the color and charges and angular momentum associated with the Hamiltonians at point of braids could couple to the dynamics via the boundary conditions.

2. The braid strand with a given Hamiltonian could obey Hamiltonian equations of motion: this would give rise to a skeleton of space-time defined by braid strands possibly continued to string world sheets and would provided different realization of quantum classical correspondence.
3. Quantum TGD can be regarded as a square root of thermodynamics in well-defined sense. Could it be possible to couple the Hermitian square root of density matrix appearing in M-matrix and characterizing zero energy state thermally to the geometry of space-time sheets by coupling it to the classical dynamical via boundary conditions depending on its eigenvalues? This is indeed the case if one accepts the description of the equality of classical and quantum charges in terms of Lagrange multipliers.
The necessity to choose single eigenvalue would give a representation for single measurement outcome. One can achieve a representation of the ensemble at space-time level consisting of space-time sheets representing various outcomes of measurement. This ensemble would be realized as ensemble of sub-CDs for a given CD.
4. One can as whther a quantum superposition of WCW spinor fields could have a space-time correlate in the sense that all space-time surfaces in the superposition would carry information about the superposition itself? Obviously this would mean self-referentiality via quantum-classical feedback.

The following discussion concentrates on possible space-time correlates for the quantum superposition of WCW spinor fields and for the arrow of time.

1. It seems difficult to imagine space-time correlate for the quantum superposition of final states with varying quantum numbers since these states correspond to quantum superpositions of different space-time surfaces. How could one code information about quantum superposition of space-time surfaces to the space-time surfaces appearing in the superposition? This kind of self-referentiality seems to be necessary if one requires that various quantum numbers characterizing the superposition (say momentum) couple via boundary conditions to the space-time dynamics.
2. The failure of non-determinism of quantum dynamics is behind dissipation and strict determinism fails for Kähler action. This gives hopes that the dynamics induces also arrow of time. Energy non-conservation is of course excluded and one should be able to identify a measure of entropy and the analog of second law of thermodynamics telling what happens at for preferred extremals when the situation becomes non-deterministic. The vertices of generalized Feynman graphs are natural places were non-determinism emerges as are also sub-CDs. naïve physical intuition would suggest that dissipation means generation of entropy: the vertices would favor decay of particles rather than their spontaneous assembly. The analog of blackhole entropy assignable to partonic 2-surfaces might allow to characterize this quantatively. The symplectic area of partonic 2-surface could be a symplectic invariant of this kind.
3. Could the mysterious branching of partonic 2-surfaces -obviously analogous to even more mysterious branching of quantum state in many worlds interpretation of quantum mechanics- assigned to the multi-valued character of the correspondence between canonical momentum densities and time derivatives of H coordinates allow to understand how the arrow of time is represented at space-time level?
   (a) This branching would effectively replace CD with its singular covering with number of branches depending on space-time region. The relative homology with respect to the upper boundary of CD (so that the branches of the trees would effectively meet there) could define the analog of Floer homology with various paths defined by the orbits of partonic 2-surfaces along lines of generalize Feynman diagram defining the first homology group. Typically tree like structures would be involved with the ends of the tree at the upper boundary of CD effectively identified.

(b) This branching could serve as a representation for the branching of quantum state to a superposition of eigenstates of measured quantum observables. If this is the case, the various branches to which partonic 2-surface decays at partonic 2-surface would more or less relate to quantum superposition of final states in particle reaction. The number of branches would be finite by finite measurement resolution. For a given choice of the arrow of geometric time the partonic surface would not fuse back at the upper end of CD.

(c) Rather paradoxically, the space-time correlate for the dissipation would reduce the dissipation by increasing the effective value of $\hbar$: the interpretation would be however in terms of dark matter identified in terms of large $\hbar$ phase. In the same manner dissipation would be accompanied by evolution since the increase of $\hbar$ naturally implies formation of macroscopically quantum coherent states. The space-time representation of dissipation would compensate the increase of entropy at the ensemble level.

(d) The geometric representation of quantum superposition might take place only in the intersection of real and p-adic worlds and have interpretation in terms of cognitive representations. In the intersection one can also have a generalization of second law [K53] in which the generation of genuine negentropy in some space-time regions via the build up of cognitive representation compensated by the generation of entropy at other space-time regions. The entropy generating behavior of living matter conforms with this modification of the second law. The negentropy measure in question relies on the replacement of logarithms of probabilities with logarithms of their p-adic norms and works for rational probabilities and also their algebraic variants for finite-dimensional algebraic extensions of rationals.

(e) Each state in the superposition of WCW quantum states would contain this representation as its space-time correlate realizing self-referentiality at quantum level in the intersection of real and p-adic worlds. Also the state function reduced members of ensemble could contain this cognitive representation at space-time level. Essentially quantum memory making possible self-referential linguistic representation of quantum state in terms of space-time geometry and topology would be in question. The formulas written by mathematicians would define similar map from quantum level to the space-time level making possible to "see" one's thoughts.

15.7 Could Gromov-Witten Invariants And Braided Galois Homology Together Allow To Construct WCW Spinor Fields?

The challenge of TGD is to understand the structure of WCW spinor fields both in the zero modes which correspond to symplectically invariant degrees of freedom not contributing to the WCW Kähler metric and in quantum fluctuating degrees of freedom parametrized by the symplectic group of $\delta M^4_\pm \times CP_2$. The following arguments suggest that an appropriate generalization of Gromov-Witten invariants to covariants combined with braid Galois homology could allow do construct WCW spinor fields and at the same time M-matrices and U-matrices, which in ZEO context generalize quantum theory to what might be called square root of thermodynamics.

The recent view about general structure of U- and S-matrices is discussed in [K58]. Suffice it to notice that the time translation operator of quantum field theories is represented now as scalings of causal diamonds (CDs) allowing to circumvent the problems related to the loss of manifest Lorentz invariance. U-matrix represents a semi-group of scalings rather than group. Also unitarily represented discrete Lorentz boosts for CDs are in an essential role but not visible at the limit, when the size scale of CD is large and one obtains the counterpart of usual S-matrix.

15.7.1 Gromov-Witten Invariants

Gromov-Witten invariants (see `http://tinyurl.com/y7nled63`) [A36] are rational numbers $GW^{X,A}_{g,n}$, which in a loose sense count the number of pseudo-holomorphic curves of genus g and n marked points and homology equivalence class A in symplectic space X meeting n surfaces of X with given homology equivalence classes. These invariants can distinguish between different symplectic manifolds. Since also the proposed generalized homology groups would define symplectic invariants if the realization of braided Galois groups as symplectic flows works, the attempt to understand the relation of Gromov-Witten invariants of TGD is well-motivated.

Let X be a symplectic manifold with almost complex structure J (the transition functions are not holomorphic) and C be an algebraic variety in X of genus g and with complex structure j having n marked points x_1, $...x_n$, which are points of X. Pseudo-homolomorphic maps of C to X are by definition maps, whose Jacobian map commutes with the multiplication of the tangent space vectors with the antisymmetric tensor representing imaginary unit $J \circ df = df \circ j$. If the symplectic manifold allows Kähler structure, one can say that pseudohomolomorphic maps commute with the multiplication by imaginary unit so that tangent plane of complex 2-manifold is mapped to a complex tangent plane of X.

The moduli space $M_{g,n}(X)$ of the pseudoholomorphic maps is finite-dimensional. One considers also its subspaces $M_{g,n}(X,A)$ of $M_{g,n}(X)$, where A represents a fixed homology equivalence class A for the image of C in X. The so called evaluation map from $M_{g,n}(X,A)$ to $M_{g,n}(X)) \times X^n$ defined by $(C, x_1, x_2, ...x_n, f) \rightarrow (st(C, x_1, x_2, ...x_n); f(x_1), ..., f(x_n))$. Here $st(C, x_1, x_2, ...x_n)$ denotes so called stabilization of $(C, x_1, ...x_n)$ defined in the following manner. A smooth component of Riemann surface is said to be stable if the number of automorphisms (conformal transformations) leaving the marked and nodal (double) points invariant is finite. Stabilization is obtained by dropping away the unstable components from the domain of C.

The image of the fundamental class of the moduli space $M_{g,n}(X)$ defines a homology class in $M_{g,n}(X)) \times X^n$. Since the homology groups of $M_{g,n}(X)) \times X^n$ are by Künneth theorem expressible as convolutions of homology groups of $M_{g,n}(X)$ and n copies of X, this homology class can be expressed as a sum

$$\sum_{\beta,\alpha_i} GW_{g,n}^{X,A} \beta \times \alpha_1 ... \times \alpha_n \quad .$$

The coefficients, which in the general case are rational valued, define Gromov-Witten invariants. One can roughly say that these rational numbers count the number of surfaces C intersecting the n homology classes α_i of X. n surfaces intersect when there is a surface of genus g with n marked points intersection the surfaces at marked points and Gromov-Witten invariant counts the number of homologically non-equivalent pseudo-holomorphic 2-surfaces of this kind [A100].

Branes connected by closed strings would represent a basic example about quantum intersections. Also in Floer homology [A117] and quantum cohomology [A68] this kind of fuzzy intersection is encountered. The fundamental Gromov-coefficients $W(\alpha, \beta, \gamma)$ are for three homology generators α, β, γ and connecting surface correspond to pseudo-holomorphic spheres (or higher genus surfaces) with three marked points obtained by contracting the outgoing three strings of stringy trouser vertex to point.

15.7.2 Gromov-Witten Invariants And Topological String Theory Of Type A

Gromow-Witten invariants appear in topological string theory (see `http://tinyurl.com/yaydkshu`) of type A [A86] for which the scattering amplitudes depend on Kähler structure of X only. The target space X of this theory is 6-dimensional symplectic manifold. X can correspond to 6-dimensional Calabi-Yau manifold. Twistor space is one particular example of this kind of manifold and one can indeed relate twistor amplitudes to those of topological string theory in twistor space.

Type A topological string theory contains both fundamental string orbits, which are 2-surfaces wrapping over 2-real-D holomorphic curves in X and D2 branes, whose 3-D "orbits" in X wrap over Lagrangian manifolds having by definition a vanishing induced symplectic form. There are also strings connecting the branes. C corresponds now to the world sheet of string with n marked points representing emitted particles. Gromov-Witten invariants are defined as integrals over the moduli spaces $M_{g,n}(X)$ and provide a rigorous definition for path integral and the free energy at given genus g is the generating function for Gromov-Witten invariants.

Witten introduced the formulation of the topological string theories in terms of topological sigma models [A85]. The formulation involves the analog of BRST symmetry encountered in gauge fixing meaning that one replaces target space with super-space by assigning to target space-coordinates anti-commuting partners which do not however represent genuine fermionic degrees of freedom. One also replaces string world sheet with a super-manifold $\mathcal{N} = (2,2)$ SUSY and spinors are world sheet spinors and Lorentz transformations act on string world sheet. Topological string models are characterized by continuous R-symmetries and the mixing of rotational and R-symmetries takes place. The R-symmetry associated with 2-D world sheet Lorentz transformation compensates for the spin rotation so that one indeed obtains a BRST charge Q (for elementary introduction to BRST symmetry see [B47]), which is scalar and the condition $Q^2 = 0$ is satisfied identically so that cohomology is obtained.

15.7.3 Gromov-Witten Invariants And WCW Spinor Fields In Zero Mode Degrees Of Freedom

One can ask whether Gromow-Witten invariants of something more general could emerge naturally in TGD framework.

1. Gromov-Witten invariants modified so that closed string orbits are replaced by open string world sheets with boundaries identifiable as braid strands relate to the braided Galois homology. Both the geometric interpretation these invariants in terms of fuzzy quantum intersection induced by connecting string world sheets and the discussion of the Floer homology like aspects of quantum TGD support this idea.
2. Another interpretation is that Gromov-Witten invariants or their generalizations emerge in the construction of WCW spinor fields in zero mode degrees of freedom, which do not contribute to the line element of WCW Kähler metric. Contrary to the first hopes there is no convincing support for this view.

Comparison of the basic geometric frameworks

The basic geometric frameworks are sufficiently similar to encourage the idea that Gromov-Witten type invariants might make sense in TGD framework.

1. In the standard formulation of TGD the 6-dimensional symplectic manifold is replaced with the metrically 6-dimensional manifold $\delta M^4_\pm \times CP_2$ having degenerate symplectic and Kähler structure and reducing effectively (metrically) to the symplectic manifold $S^2 \times CP_2$. Partonic 2-surfaces at the light-like boundaries of CD identifiable as wormhole throats define the counterparts of fundamental string like object of topological string theory of type A. The n marked points of Gromov-Witten theory could correspond to the ends of braid strands carrying purely bosonic quantum numbers characterized by the attached $\delta M^4_\pm \times CP_2$ Hamiltonians with well defined angular momentum and color quantum numbers. One must distinguish these braid strands from the braid strands carrying fermion quantum numbers.
2. There are also differences. One assigns 3-D surfaces to the boundaries of CD and partonic 2-surfaces at CD are connected with are interpreted as strings so that partonic 2-surfaces have also brane like character. One can identify 3-D surfaces for which induced Kähler forms of CP_2 and $\delta M^4_\pm$ vanish (any surface with 1-D projection to $\delta M^4_\pm$ and 2-D CP_2 projection with Lagrangian manifold would define counterpart of brane) but it is not natural to raise these objects to a special role.
3. I have proposed that quantum TGD is analogous to a physical analog of Turing machine in the sense that the inclusions of HFFs could allow to emulate any QFT with almost gauge group assignable to the included algebra [K32]. The representation of these gauge groups as subgroups of symplectic transformations leaving the marked points of the partonic 2-surfaces invariant gives hopes of realizing this idea mathematically. Symplectic groups (see `http://tinyurl.com/y8us9sgw`) are indeed completely exceptional because of their representative power [A81] and used already in classical mechanics and field theory to represent symmetries. An interesting question is whether the symplectic group associated with $\delta M^4_\pm \times CP_2$ could be universal in the sense that any gauge group of this kind allows a faithful homomorphism to this group.

One should understand what pseudo-holomorphy means in TGD framework. One must consider both the identification of pseudo-holomorphic surfaces as string world sheets or as partonic 2-surfaces. Consider first the interpretation of pseudo-holomorphic 2-surfaces as string world sheets assignable to the space-time sheets.

1. String world sheets would not represent closed strings and their ends would define braid strands at light-like 3-surfaces and at the space-like 3-surfaces defining the ends of space-time. This is not a problem: also the standard picture about pseudo-holomorphic surfaces as spheres with punctures is obtained by idealizing the holes of closed string with punctures [A117]. Open string world sheet be seen as a string containing holes defined by the boundary braid strands. Disjoint partonic two surfaces at the ends of braid strands would intersect in quantum sense. The interpretation for the fuzzy intersection would be in terms of causal dependence of the quantum state at the ends of CD so that the assignment of Gromov-Witten invariants to them would be natural.
2. This option looks very natural from TGD point of view since the moduli space is expected to be finite-dimensional and have interpretation in terms of the preferred extremal property. For a given partonic 2-surfaces and tangent space data at them the moduli would be fixed more or less uniquely and the variation of the tangent space data would vary the moduli.

Also the identification of pseudo-holomorphic surfaces as partonic 2-surfaces can be considered. It would apparently conform with the canonical identification of pseudo-holomorphic surfaces but the interpretation as connectors in fuzzy cup product can be challenged.

1. Since the moduli space of pseudo-holomorphic surfaces is finite-dimensional, only a very restricted set of partonic 2-surfaces satisfies pseudo-holomorphy condition. The induced metric of the partonic 2-surface defines a unique complex structure. Pseudo-holomorphy states that Jacobian takes the complex tangent place of partonic 2-surface to a comlex plane of the tangent space of $\delta M^4_\pm \times CP_2$. Pseudo-holomorphy is implied by holomorphy stating that both CP_2 coordinates and S^2 coordinates as functions of the complex coordinate of the partonic 2-surface are holomorphic functions implying that the induced metric as the standard $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Holomorphy is also implied if one can express as a variety using functions which are holomorphic functions of $\delta M^4_\pm$ and CP_2 complex coordinates and analytic functions of the radial coordinate r. These surfaces are characterized by the homology-equivalence classes of their projections in $\delta M^4_\pm$ (3-D Euclidian space with puncture at origin) and in CP_2. Both are characterized by integer. These surfaces obviously define a subset of partonic 2-surfaces and one can actually assign to the string-like objects as cartesian products of string world sheets satisfying minimal surface equations and of 2-D complex sub-manifolds of CP_2.
2. The first objection is that partonic two-surfaces do not represent time-evolution so punctures associated with them cannot be regarded as causally dependent. From physics point of view it does not make sense to speak about fuzzy intersection except in terms of finite measurement resolution implying that second quantized induced spinor fields have finite number of modes so that they do not anti-commute at partonic 2-surfaces anymore.
3. Second objection is that there is nothing physically interesting that partonic 2-surfaces could connect!
4. The third counter argument is that pseudo-holomorphy condition allows only finite-dimensional moduli space whereas the space of partonic 2-surfaces is infinite-dimensional. Two explanations suggest itself.
   (a) The finite-measurement resolution might imply an effective reduction of the space of partonic 2-surfaces to this moduli space? Finite measurement resolution could be understood also as a kind of gauge invariance when realized in terms of inclusion of hyper-finite factors of type II_1 (HFFs) with the action of sub-factor having no effect on its observable properties. Holomorphy would serve as a gauge fixing condition.
   (b) If TGD as almost topological QFT can be formulated as an analog of Floer's theory relying on action principle, the natural proposal is that holomorphic partonic 2-surfaces correspond to critical values for the Kähler action assignable to the Minkowskian regions of the preferred extremal.

It seems relatively safe to conclude that only the string world sheets have a natural interpretation as connectors the deformed interwection product in TGD framework.

Could an analog of topological string theory make sense in TGD framework

The observations of previous paragraphs motivate the question whether an analog of type A topological string theory could emerge in the construction of WCW spinor fields. The basic problem is to understand how the WCW spinor fields depend on symplectic invariants, which however need not correspond to zero modes which should be expressible in terms of symplectic fluxes alone. One might hope that topological string theory of some kind could allow to construct this kind of symplectic invariants.

1. The encouraging symptom is that the n-point functions of both A and B type topological string theories are non-trivial only in dimension $D = 6$, which is the metric dimension of $\delta M^4_\pm \times CP_2$. Since the n-point functions of type A topological string theory depend only on the Kähler structure associated now by CP_2 and $\delta M^4_\pm$ Kähler forms they could code for the physics associated with the zero modes representing non-quantum fluctuating degrees of freedom. Since type B topological string theory requires vanishing of the first Chern class implying Calabi-Yau property, this theory is not possible in the standard formulation of TGD. The emergence of the topological string theory of type A seems to be in conflict with what twistorialization suggests. Witten suggested in his classic article [B21] boosting the twistor revolution, that the Fourier transforms of the sattering amplitudes from momentum space to twistor space scattering amplitudes for perturbative $\mathcal{N} = 4$ SUSY could be interpreted in terms of D-instanton expansion of topological string theory of type B defined in twistor space CP_3.
2. One can identify the marked points as the end points of both space-like and time-like braids but it is not natural to assign them fermionic quantum numbers except those of covariantly constant right-handed neutrino spinor with the points of symplectic triangulation. This is well-motivated since symplectic algebra extends to super-symplectic algebra with covariantly constant right handed neutrino spinor defining the super-symmetry. One can assign the values of Hamiltonians of $\delta M^4_\pm \times CP_2$ to the marked points belonging to the irreducible representations of rotation group and color group such that the total quantum numbers vanish by the symplectic invariance. n-point functions would be correlation functions for Hamiltonians. In a well-defined sense one would have color and angular momentum confinement in WCW degres of freedom.

   The vanishing of net quantum numbers need not hold true for single connected partonic 2-surface. Also it could hold true only for a collection of partonic 2-surfaces associated with same 3-surface at either end of CD. The most general condition would be that the total color and spin numbers of positive and negative energy parts of the state sum up to zero in symplectic degrees of freedom.
3. The generating function for Gromov-Witten invariants is defined for a connected pseudo-holomorphic 2-surface with a fixed genus g as such is not general enough if one allows partonic 2-surfaces with several components. The generalization would provide information about the preferred extremal of Kähler action and about the topology of space-time surface. The generalization of the Gromov-Witten partition function would define as its inverse the normalization factor for zero energy state identifiable as M-matrix defined as a positive diagonal square root of density matrix multiplied by S-matrix for which initial partons possess fixed genus and which contains superposition over braids with arbitrary number of strands. The intuition from ordinary thermodynamics suggests that this partition function is in a reasonable approximation expressible as convolution for n-points functions for individual partonic 2-surfaces allowing the set of marked points to carry net $\delta M^4_\pm$ angular momentum and color quantum numbers.

Description of super-symmetries in TGD framework

It is interesting to see whether the formulation of super-symmetries in the framework of topological sigma models giving rise to Gromov-Witten invariants [A85] has any reasonable relation to TGD where the notion of super-space does not look natural as a fundamental notion although it might be very useful as a formal tool in the formulation of SUSY QFT limit [?] and even quantum TGD itself.

1. Almost topological QFT property means that Kähler action for the preferred extremals reduces to Chern-Simons action assuming the weak form of electric magnetic duality. In the fermionic sector one must use Kähler-Dirac gamma matrices defined as contractions of the canonical momentum densities for Kähler action (Kähler-Chern-Simon action) with embedding space gamma matrices in the counterpart of Dirac action in the interior of space-time sheet and at

3-D wormhole throats. The Kähler-Dirac gamma matrices define effective metric quadratic in canonical momentum densities which is typically highly degenerate. It contains information about the induced metric. Therefore one cannot expect that topological sigma model approach could work as such in TGD framework.

2. In TGD framework supersymmetries are generated by right-handed covariantly constant neutrinos and antineutrinos with both spin directions. These spinors are embedding space spinors rather than world sheet spinors but one can say that the induction of the spinor structure makes them world sheet spinors. Since the momentum of the spinors is vanishing, one can assign all possible spin directions to the neutrinos.

3. Covariantly constant right-handed neutrino and antineutrino can have all possible spin directions and for fixed choice of quantization axes two spin directions are possible. Therefore one could say that rotation group acts as non-Abelian group of R-symmetries. TGD formulation need not be based on sigma model so that it is not all clear whether a twisted Lorenz transformations are needed. If so, the most obvious guess is that space-time rotations are accompanied by R-symmetry rotation of right-handed neutrino spinors compensating the ordinary rotation it as in the case of topological sigma model originally introduced by Witten.

It is interesting to look the situation also from the point of view of the breaking of SUSY for supergravity defined in dimension 8 by using the table listing super-gravities (see `http://tinyurl.com/y7lzxu47`) in various dimensions [B5].

1. One can assign to the causal diamond a fixed direction as a WCW correlate for the fixing of spin quantization axis and this direction corresponds to a particular modulus. The preferred time directiond defined by the line connecting the tips of CD and this direction define a plane of non-physical polarizations having in number theoretical approach as a preferred hypercomplex plane of hyper-octonions [K86]. Hence it would seem that by the symmetry breaking by the choice of quantization axes allows only two spin directions the right handed neutrino and antineutrino and that different choices of the quantization axes correspond to different values for the moduli space of CDs.

2. Since embedding space spinors are involved, the sugra counterpart of TGD is $\mathcal{N} = 2$ super gravity in dimension 8 for which super charges are Dirac spinors and their hermitian conjugates with $U(2)$ acting as R-symmetries. Note that the supersymmetry does not require Majorana spinors unlike $N = 1$ supersymmetry does in string model and fixes the target space dimension to $D = 10$ or $D = 11$. Just like $D = 11$ of M-theory is the unique maximal dimension if one requires fundamental Majorana spinors (for which there is no empirical support), $D = 8$ of TGD is the unique maximal dimension if one allows only Dirac spinors.

3. In dimensional reduction to $D = 6$, which is the metric dimension of the boundary of δCD a breaking of $\mathcal{N} = 8$ sugra $\mathcal{N} = (2, 2)$ sugra occurs, and one obtains decomposition into pseudoreal representations with supercharges in representations (4, 0) and (0, 4) of $R = Sp(2) \times Sp(2)$ ($Sp(2) = Sl(2, R)$ corresponds to 2-D symplectic transformations identifiable also as Lorentz group SO(1, 2)). (4, 0) and (0, 4) could correspond to left and right handed neutrinos with both directions of helicities and thus potentially massive. CP_2 geometry breaks this supersymmetry.

4. The reduction to the level of right handed neutrinos requires a further symmetry breaking and $D = 5$ sugra indeed contains supercharges Q and their conjugates in 4-D pseudoreal representation of $R = Sp(4)$. Note that this group corresponds to 2×2 quaternionic matrices. A possible interpretation would be as a reduction in CP_2 degrees freedom to $U(2) \times U(1)$ invariant sphere.

5. The R-symmetries mixing neutrinos and antineutrinos are pysically questionable so that a breaking of R-symmetry to $Sp(2) \times Sp(2)$ to $SU(2) \times SU(2)$ or even $SU(2)$ should take place. A further reduction to homologically non-trivial geodesic sphere of CP_2 might reduce the action of $CP_2(2)$ holonomies to those generated by electric charge and weak isospin and thus leaving right-handed neutrinos invariant. Fixing the quantization axis of spin would reduce R-symmetry to $U(1)$. The inverse imaged of this geodesic sphere is identified as string world sheet [K45].

How braided Galois homology and Gromov-Witten type homology and WCW spinor fields could relate?

One can distinguish between WCW "orbital" degrees of freedom and fermionic degrees of freedom and in the case of WCW degrees of freedom also between zero modes expressible in terms of Kähler fluxes and quantum fluctuating degrees of freedom expressible using wave functions in symplectic group.

1. Quantum fluctuating degrees of freedom

As far as quantum number are considered, quantum fluctuating degrees of freedom correspond to the symplectic algebra in the basis defined by Hamiltonians belonging to the irreps of rotation group and color group.

1. At the level of partonic 2-surfaces finite measurement resolution leads to discretization in terms of braid ends and symplectic triangulation. At the level of WCW discretization replaces symplectic group with its discrete subgroup. This discrete subgroup must result as a coset space defined by the subgroup of symplectic group acting as Galois group in the set of braid points and its normal subgroup leaving them invariant. The group algebra of this discrete subgroup of symplectic group would have interpretation in terms of braided Galois cohomology. This picture provides an elegant realization for finite measurement resolutions and there is also a connection with the realization of finite measurement resolution using categorification [A155], [K18].
2. The proposed generalized homology theory involving braided Galois group and symplectic group of $\delta M^4_\pm \times CP_2$ would realize the "almost" in TGD as almost topological QFT in finite measurement resolution replacing symplectic group with its discretized version. This algebra would relate to the quantum fluctuating degrees of freedom. The braids would carry only fermion number and there would be no Hamiltonians attached with them. The braided Galois homology could define in the more general situation invariants of symplectic isotopies.
3. The generalization of Gromov-Witten invariants to n-point functions defined by Hamiltonians of $\delta M^4_\pm \times CP_2$ are symplectic invariants if net $\delta M^4_\pm \times CP_2$ quantum numbers vanish. As As a special case one obtains Gromove-Witten invariants. The most general definition assumes that the vanishing of quantum numbers occurs only for zero energy states having disjoint unions of partonic 2-surfaces at the boundaries of CDs as geometric correlate. Since Hamiltonians correspond to quantum fluctuating degrees of freedom the interpretation in terms of zero modes is not not possible. The comparison of Floer homology with quantum TGD encourages to think that the generalizations of Gromov-Witten invariants can be assigned to the braided Galois homology.
4. One should also add four-momenta and twistors to this picture. The separation of dynamical fermionic and sup-symplectic degrees of freedom suggests that the Fourier transforms for amplitudes containing the fermionic braid end points as arguments define twistorial amplitudes. The representations of light-like momenta using twistors would lead to a generalization of the twistor formalism. At zero momentum limit one would obtain symplectic QFT with states characterized by collections of Hamiltonians and their super-counterparts.

2. Zero modes

WCW spinor field depends also on zero modes and the challenge is to identify the appropriate variables coding for this information in accordance with quantum classical correspondence. The best that one could achieve would be a basis for the parts of WCW spinor fields in these degrees of freedom. Zero modes correspond essentially to the non-local symplectic invariants assignable to the projections of the $\delta M^4_\pm$ and CP_2 symplectic forms to the space-time surface and expressible in terms of symplectic fluxes only. The appropriate symplectic fluxes should be determined by the information about the quantum state in quantum fluctuating degrees of freedom by quantum classical correspondence.

1. The exponent of Kähler action for preferred extremal- by above proposal real in Euclidian regions and imaginary in Minkowskian regions and reducing to Chern-Simons action at both sides - contains also information about zero modes and would code implicitly the vacuum functional in zero modes. What would be needed is an explicit representation for this part of vacuum

functional. The identification of zero modes as classical variables requires entanglement between zero modes and quantum fluctuating degrees of freedom and one-one correspondence analogous to that between the states of the measurement apparatus and the outcome of quantum measurement is expected. This duality would express quantum holography and quantum classical correspondence crucial for quantum measurement theory.

2. Could the generating function for appropriately generalized Gromow-Witten invariants define a candidate for what might be regarded as a vacuum functional in zero modes separating into a factor in WCW spinor field? The first thing to notice is that symplectic invariance is not equivalent with zero mode property. In Floer homology there is a preferred Hamiltonian interpreted in TGD framework in terms of the braiding defining braided Galois homology. Neither Floer homology, Gromov-Witten invariants nor braided Galois homology do depend on the details of the Hamiltonian. Does this mean that the TGD counterparts of Gromov-Witten invariants might could be interpreted as zero modes and generating function for these invariants as vacuum functional in zero modes? Or does the fact that Hamiltonian flow is involved mean that information about quantum fluctuating degrees of freedom is present?

Symplectic QFT [K18] provides a more promising approach to the description of zero modes in terms of symplectic fluxes.

1. The earlier proposal [K18] for symplectic QFT defined as a generalization of conformal QFT coding for these degrees of freedom assigns to the partonic 2-surface collections of marked points defining its division to 2-polygons carrying Kähler magnetic flux together with the signed area defined by R^3_+ symplectic form (essentially solid angle assignable to partonic 2-surface or its portion with respect to the tip of light-cone). A given assignment of marked points defines symplectic fusion algebra and these algebras integrate to an operad with a product defined by the product of fusion algebras.
2. Symplectic triangulation would define symplectic invariants. The nodes of the symplectic triangulation could be identified as the ends of braid strands assignable to string world sheets. If the information about quantum state can be used to fix the edges of the triangulation, the phases defined by the fluxes associated with the triangles define physically interesting symplectic invariants. If one assumes that each Hamiltonian assignable to the partonic 2-surface defines its own symplectic triangulation, the Hamiltonian equations associated with the Hamiltonian would naturally define the edges of the triangulation. Symplectic triangulation would characterize a Bose-Einstein condensate like state assignable to single Hamiltonian. The total magnetic flux for the triangulation would characterize the Hamiltonian. If only single Hamiltonian is involved the orbit should be a closed orbit connecting the node to itself and also now could assign to it a symplectic area.
3. Symplectic triangulation would add additional pieces to the proposed skeleton of the space-time surface. If the symplectic triangulation can be continued from partonic 2-surfaces to the interior of space-time in both time and spatial direction it would provide space-time with a web string world sheets connected by sheets assignable to the edges of the symplectic triangulation.

15.8 K-Theory, Branes, And TGD

K-theory has played important role in brane classification in super string models and M-theory. The excellent lectures by Harah Evslin with title *What doesn't K-theory classify?* (see `http://tinyurl.com/y9og83ut`) [B37] make it possible to learn the basic motivations for the classification, what kind of classifications are possible, and what are the failures. Also the Wikipedia article (see `http://tinyurl.com/ycuuh7j4`) [B3] gives a bird's eye of view about problems. As a by-product one learns something about the basic ideas of K-theory - at least I hope so - and about possible mathematical and physical problems of string theories and M-theory.

In the sequel I will discuss critically the basic assumptions of brane world scenario, sum up my meager understanding about the problems related to the topological classification of branes and also to the notion itself, ask what could go wrong with branes and demonstrate how the problems could be avoided in TGD framework, and just to irritate colleauges conclude with a proposal for a natural generalization of K-theory to include also the division of bundles inspired by the generalization of Feynman diagrammatics in quantum TGD, by zero energy ontology, and by the notion of finite measurement resolution.

15.8.1 Brane World Scenario

The brane world scenario looks attractive from the mathematical point of view one is able to get accustomed with the idea that basic geometric objects have varying dimensions. Even accepting the varying dimensions, the basic physical assumptions behind this scenario are vulnerable to criticism.

1. Branes (see http://tinyurl.com/665osee) are geometric objects of varying dimension in the 10-/11-dimensional space-time -call it M- of superstring theory/M-theory. In M-theory the fundamental strings are replaced with M-branes, which are 2-D membranes with 3-dimensional orbit having as its magnetic dual 6-D M5-brane. Branes are thought to emerge non-perturbatively from fundamental 2-branes but what this really means is not understood. One has D-p-branes (see http://tinyurl.com/y7tdcmbp) with Dirichlet boundary conditions fixing a $p+1$-dimensional surface of M as brane orbit: one of the dimensions corresponds to time. Also S-branes localized in time have been proposed.
2. In the description of the classical limit branes interact with the classical fields of the target space by the generalization of the minimal coupling of charged point-like particle to electromagnetic gauge potential. The coupling is simply the integral of the gauge potential over the world-line - the value of 1-form for the word-line. Point like particle represents 0-brane and in the case of p-brane the generalization is obtained by replacing the gauge potential represented by a 1-from with $p+1$-form. The exterior derivative of this $p+1$-form is $p+2$-form representing the analog of electromagnetic field. Complete dimensional democracy strongly suggests that string world sheets should be regarded as 1-branes.
3. From TGD point of view the introduction of branes looks a rather ad hoc trick. By generalizing the coupling of electromagnetic gauge potential to the word line of point like particle one could introduce extended objects of various dimensions also in the ordinary 4-D Maxwell theory but they would be always interpreted as idealizations for the carriers of 4- currents. Therefore the crucial step leading to branes involves classical idealization in conflict with Uncertainty Principle and the genuine quantal description in terms of fields coupled to gauge potentials.

   My view is that the most natural interpretation for what is behind branes is in terms of currents in D=10 or D= 11 space-time. In this scheme branes have role only as semi-classical idealizations making sense only above some scale. Both the reduction of string theories to quantum field theories by holography and the dynamical character of the metric of the target space conforms with super-gravity interpretation. Internal consistency requires also the identification of strings as branes so that superstring theories and M-theory would reduce to an idealization to 10-/11-dimensional quantum gravity.

In this framework the brave brane world episode would have been a very useful Odysseia. The possibility to interpret various geometric objects physically has proved to be an extremely powerful tool for building provable mathematical conjectures and has produced lots of immensely beautiful mathematics. As a fundamental theory this kind of approach does not look convincing to me.

15.8.2 The Basic Challenge: Classify The Conserved Brane Charges Associated With Branes

One can of course forget these critical arguments and look whether this general picture works. The first thing that one can do is to classify the branes topologically. I made the same question about 32 years ago in TGD framework: I thought that cobordism for 3-manifolds might give highly interesting topological conservation laws. I was disappointed. The results of Thom's classical article about manifold cobordism demonstrated that there is no hope for really interesting conservation laws. The assumption of Lorentz cobordism meaning the existence of global time-like vector field would make the situation more interesting but this condition looked too strong and I could not see a real justification for it. In generalized Feynman diagrammatics there is no need for this kind of condition.

There are many alternative approaches to the classification problem. One can use homotopy, homology, cohomology and their relative and other variants, topological or algebraic K-theory, twisted K-theory, and variants of K-theory not yet existing but to be proposed within next years. The list is probably endless unless something like motivic cohomology brings in enlightenment.

1. First of all one must decide whether one classifies p-dimensional time=constant sections of p-branes or their $p+1$-dimensional orbits. Both approaches have been applied although the first one is natural in the standard view about spontaneous compactification. For the first option topological invariants could be seen as conserved charges: homotopy invariants and homological and cohomological characteristics of branes provide this kind of invariants. For the latter option the invariants would be analogous to instanton number characterizing the change of magnetic charge.
2. Purely topological invariants come first in mind. Homotopy groups of the brane are invariants inherent to the brane (the brane topology can however change). Homological and cohomological characteristics of branes in singular homology characterize the embedding to the target space. There are also more delicate differential topological invariants such as de Rham cohomology defining invariants analogous to magnetic charges. Dolbeault cohomology emerges naturally for even-dimensional branes with complex structure.
3. Gauge theories - both abelian and non-Abelian - define a standard approach to the construction of brane charges for the bundle structures assigned with branes. Chern-Simons classes are fundamental invariants of this kind. Also more delicate invariants associated with gauge potentials can be considered. Chern-Simons theory with vanishing field strengths for solutions of field equations provides a basic example about this. For instance, SU(2) Chern-Simons theory provides 3-D topological invariants and knot invariants.
4. More refined approaches involve K-theory -closely related to motivic cohomology - and its twisted version. The idea is to reduce the classification of branes to the classification of the bundle structures associated with them. This approach has had remarkable successes but has also its short-comings.

The challenge is to find the mathematical classification which suits best the physical intuitions (, which might be fatally wrong as already proposed) but is universal at the same time. This challenge has turned out to be tough. The Ramond-Ramond (RR) p-form fields (see http://tinyurl.com/y9kmbxoy) of type II superstring theory are rather delicate objects and a source of most of the problems. The difficulties emerge also by the presence of Neveu-Schwartz 3-form $H = dB$ defining classical background field.

K-theory has emerged as a good candidate for the classification of branes. It leaves the confines of homology and uses bundle structures associated with branes and classifies these. There are many K-theories. In topological K-theory bundles form an algebraic structure with sum, difference, and multiplication. Sum is simply the direct sum for the fibers of the bundle with common base space. Product reduces to a tensor product for the fibers. The difference of bundles represents a more abstract notion. It is obtained by replacing bundles with pairs in much the same way as rationals can be thought of as pairs of integers with equivalence $(m, n) = (km, kn)$, k integer. Pairs $(n, 1)$ representing integers and pairs $(1, n)$ their inverses. In the recent case one replaces multiplication with sum and regards bundle pairs and (E, F) and $(E + G, F + G)$ equivalent. Although the pair as such remains a formal notion, each pair must have also a real world representativs. Therefore the sign for the bundle must have meaning and corresponds to the sign of the charges assigned to the bundle. The charges are analogous to winding of the brane and one can call brane with negative winding antibrane. The interpretation in terms of orientation looks rather natural. Later a TGD inspired concrete interpretation for the bundle sum, difference, product and also division will be proposed.

15.8.3 Problems

The classification of brane structures has some problems and some of them could be argued to be not only technical but reflect the fact that the physical picture is wrong.

Problems related to the existence of spinor structure

Many problems in the classification of brane charges relate to the existence of spinor structure. The existence of spinor structure is a problem already in general general relativity since ordinary spinor structure exists only if the second Stiefel- Whitney class (see http://tinyurl.com/y7m9ksq7) [A79] of the manifold is non-vanishing: if the third Stiefel-Whitney class vanishes one can introduce so called spin^c structure. This kind of problems are encountered already in lattice QCD, where periodic boundary conditions imply non-uniqueness having interpretation in terms of 16 different spinor structures with no obvious physical interpretation. One the strengths of TGD is that the notion of induced spinor structure eliminates all problems of this kind completely. One can therefore find direct support for TGD based notion of spinor structure from the basic inconsistency of QCD lattice calculations!

1. Freed-Witten anomaly (see `http://tinyurl.com/y77znbqr`) [B34] appearing in type II string theories represents one of the problems. Freed and Witten show that in the case of 2-branes for which the generalized gauge potential is 3-form so called spin^c structure is needed and exists if the third Stiefel-Whitney class w_3 related to second Stiefel Whitney class whose vanishing guarantees the existence of ordinary spin structure (in TGD framework spin^c structure for CP_2 is absolutely essential for obtaining standard model symmetries).
   It can however happen that w_3 is non-vanishing. In this case it is possible to modify the spin^c structure if the condition $w_3 + [H] = 0$ holds true. It can however happen that there is an obstruction for having this structure - in other words $w_3 + [H]$ does not vanish - known as Freed-Witten anomaly. In this case K-theory classification fails. Witten and Freed argue that physically the wrapping of cycle with non-vanishing $w_3 + [H]$ by a Dp-brane requires the presence of $D(p-2)$ brane cancelling the anomaly. If $D(p-2)$ brane ends to anti-Dp in which case charge conservation is lost. If there is not place for it to end one has semi-infinite brane with infinite mass, which is also problematic physically. Witten calls these branes baryons: these physically very dubious objects are not classified by K-theory.
2. The non-vanishing of $w_3 + [H] = 0$ forces to generalize K-theory to twisted K-theory (see `http://tinyurl.com/ya2awfuk`) [A87]. This means a modification of the exterior derivative to get twisted de Rham cohomology and twisted K-theory and the condition of closedness in this cohomology for certain form becomes the condition guaranteeing the existence of the modified spin^c structure. D-branes act as sources of these fields and the coupling is completely analogous to that in electrodynamics. In the presence of classical Neveu-Schwartz (NS-NS) 3-form field H associated with the back-ground geometry the field strength $G^{p+1} = dC_p$ is not gauge invariant anymore. One must replace the exterior derivative with its twisted version to get twisted de Rham cohomology:

$$d \rightarrow d + H\wedge \quad .$$

   There is a coupling between p- and p+2-forms together and gauge symmetries must be modified accordingly. The fluxes of twisted field strengths are not quantized but one can return to original p-forms which are quantized. The coupling to external sources also becomes more complicated and in the case of magnetic charges one obtains magnetically charged Dp-branes. Dp-brane serves as a source for $D(p-2)$- branes.
   This kind of twisted cohomology is known by mathematicians as Deligne cohomology. At the level of homology this means that if branes with dimension of p are presented then also branes with dimension $p+2$ are there and serve as source of Dp-branes emanating from them or perhaps identifiable as their sub-manifolds. Ordinary homology fails in this kind of situation and the proposal is that so called twisted K-theory could allow to classify the brane charges.
3. A Lagrangian formulation of brane dynamics based on the notion of p-brane democracy (see `http://tinyurl.com/yb462wn9`) [B45] due to Peter Townsend has been developed by various authors.

Ashoke Sen (see `http://tinyurl.com/yannv4q2`) has proposed a grand vision for understanding the brane classification in terms of tachyon condensation in absence of NS-NS field H [B10]. The basic observation is that stacks of space-filling D- and anti D-branes are unstable against process called tachyon condensation which however means fusion of $p+1$-D brane orbits rather than p-dimensional time slices of branes. These branes are however accompanied by lower-dimensional branes and the decay process cannot destroy these. Therefore the idea arises that suitable stacks of D9 branes and anti-D9-branes could code for all lower-dimensional brane configurations as the end products of the decay process.

This leads to a creation of lower-dimensional branes. All decay products of branes resulting in the decay cascade would be by definition equivalent. The basic step of the decay process is the fusion of D-branes in stack to single brane. In bundle theoretic language one can say that the

D-branes and anti-D branes in the stack fuse together to single brane with bundle fiber which is direct sum of the fibers on the stack. This fusion process for the branes of stack would correspond in topological K-theory. The fusion of D-branes and anti-D branes would give rise to nothing since the fibers would have opposite sign. The classification would reduce to that for stacks of D9-branes and anti D9-branes.

Problems with Hodge duality and S-duality

The K-theory classification is plagued by problems all of which need not be only technical.

1. R-R fields are self dual and since metric is involved with the mapping taking forms to their duals one encounters a problem. Chern characters appearing in K-theory are rational valued but the presence of metric implies that the Chern characters for the duals need not be rational valued. Hence K-theory must be replaced with something less demanding.
   The geometric quantization inspired proposal of Diaconescu, Moore and Witten [B17] is based on the polarization using only one half of the forms to get rid of the proboem. This is like thinking the 10-D space-time as phase space and reducing it effectively to 5-D space: this brings strongly in mind the identification of space-time surfaces as hyper-quaternionic (associative) sub-manifolds of embedding space with octonionic structure and one can ask whether the basic objects also in M-theory should be taken 5-dimensional if this line of thought is taken seriously. An alternative approach uses K-theory to classify the intersections of branes with 9-D space-time slice as has been porposed by Maldacena, Moore and Seiberg (see `http://tinyurl.com/ycm3l9nt`) [B42].
2. There another problem related to classification of the brane charges. Witten, Moore and Diaconescu (see `http://tinyurl.com/y8kdz6wm`) [B17] have shown that there are also homology cycles which are unstable against decay and this means that twisted K-theory is inconsistent with the S-duality of type IIB string theory. Also these cycles should be eliminated in an improved classification if one takes charge conservation as the basic condition and an hitherto un-known modification of cohomology theory is needed.
3. There is also the problem that K-theory for time slices classifies only the R-R field strengths. Also R-R gauge potentials carry information just as ordinary gauge potentials and this information is crucial in Chern-Simons type topological QFTs. K-theory for entire target space classifies D-branes as $p+1$-dimensional objects but in this case the classification of R-R field strengths is lost.

The existence of non-representable 7-D homology classes for targent space dimension $D > 9$

There is a further nasty problem which destroys the hopes that twisted K-theory could provide a satisfactory classification. Even worse, something might be wrong with the superstring theory itself. The problem is that not all homology classes allow a representation as non-singular manifolds. The first dimension in which this happens is $D = 10$, the dimension of super-string models! Situation is of course the same in M-theory. The existence of the non-representables was demonstrated by Thom - the creator of catastrophe theory and of cobordism theory for manifolds- for a long time ago.

What happens is that there can exist 7-D cycles which allow only singular embeddings. A good example would be the embedding of twistor space CP_3, whose orbit would have conical singularity for which CP_3 would contract to a point at the "moment of big bang". Therefore homological classification not only allows but demands branes which are orbifolds. Should orbifolds be excluded as unphysical? If so then homology gives too many branes and the singular branes must be excluded by replacing the homology with something else. Could twisted K-theory exclude non-representable branes as unstable ones by having non-vanishing $w_3 + [H]$? The answer to the question is negative: D6-branes with $w_3 + [H] = 0$ exist for which K-theory charges can be both vanishing or non-vanishing.

One can argue that non-representability is not a problem in superstring models (M-theory) since spontaneous compactification leads to $M \times X_6$ ($M \times X_7$). On the other hand, Cartesian product topology is an approximation which is expected to fail in high enough length scale resolution and near big bang so that one could encounter the problem. Most importantly, if M-theory is theory of everything it cannot contain this kind of beauty spots.

15.8.4 What Could Go Wrong With Super String Theory And How TGD Circumvents The Problems?

As a proponent of TGD I cannot avoid the temptation to suggest that at least two things could go wrong in the fundamental physical assumptions of superstrings and M-theory.

1. The basic failure would be the construction of quantum theory starting from semiclassical approximation assuming localization of currents of 10 - or 11-dimensional theory to lower-dimensional sub-manifolds. What should have been a generalization of QFT by replacing point-like particles with higher-dimensional objects would reduce to an approximation of 10- or 11-dimensional supergravity.
   This argument does not bite in TGD. 4-D space-time surfaces are indeed fundamental objects in TGD as also partonic 2-surfaces and braids. This role emerges purely number theoretically inspiring the conjecture that space-time surfaces are associative sub-manifolds of octonionic embedding spaces, from the requirement of extended conformal invariance, and from the non-dynamical character of the embedding space.
2. The condition that all homology equivalence classes are representable as manifolds excludes all dimensions $D > 9$ and thus super-strings and M-theory as a physical theory. This would be the case since branes are unavoidable in M-theory as is also the landscape of compactifications. In semiclassical supergravity interpretation this would not be catastrophe but if branes are fundamental objects this shortcoming is serious. If the condition of homological representability is accepted then target space must have dimension $D < 10$ and the arguments sequence leading to D=8 and TGD is rather short. The number theoretical vision provides the mathematical justification for TGD as the unique outcome.
3. The existence of spin structure is clearly the source of many problems related to R-R form. In TGD framework the induction of spin^c structure of the embedding space resolves all problems associated with sub-manifold spin structures. For some reason the notion of induced spinor structure has not gained attention in super string approach.
4. Conservative experimental physicist might criticize the emergence of branes of various dimensions as something rather weird. In TGD framework electric-magnetic duality can be understood in terms of general coordinate invariance and holography and branes and their duals have dimension 2, 3, and 4 organize to sub-manifolds of space-time sheets.
   The TGD counterpart for the fundamental D-2-brane is light-like 3-surface. Its magnetic dual has dimension given by the general formula (see `http://tinyurl.com/y9aueyup`) $p_{dual} = D - p - 4$, where D is the dimension of the target space [B24]. In TGD one has $D = 8$ giving $p_{dual} = 2$. The first interpretation is in terms of self-duality. A more plausible interpretation relies on the identification of the duals of light-like 3-surfaces as space-like 3-surfaces at the light-like boundaries of CD. General Coordinate Invariance in strong sense implies this duality. For partonic 2-surface and string world sheets carrying spinor modes one would have $p = 1$ and $p_{dual} = 3$. The identification of the dual would be as 4-D space-time surface: does this correspond to strong form of holography?. The crucial distinction to M-theory would be that branes of different dimension would be sub-manifolds of space-time surface.
5. For $p = 0$ one would have $p_{dual} = 4$ assigning five-dimensional surface to orbits of point-like particles identifiable most naturally as braid strands. One cannot assign to it any direct physical meaning in TGD framework and gauge invariance for the analogs of brane gauge potentials indeed excludes even-dimensional branes in TGD since corresponding forms are proportional to Kähler gauge potential (so that they would be analogous to odd-dimensional branes allowed by type II_B superstrings).
   4-branes might be however mathematically useful by allowing to define Morse theory for the critical points of the Minkowskian part of Kähler action. While writing this I learned that Witten (see `http://tinyurl.com/y8ganhrz`) has proposed a 4-D gauge theory approach with $\mathcal{N} = 4$ SUSY to the classification of knots. Witten also ends up with a Morse theory using 5-D space-times in the category-theoretical formulation of the theory [A144]. For some time ago I also proposed that TGD as almost topological QFT defines a theory of knots, knot braidings,

and of 2-knots in terms of string world sheets [K45]. Maybe the 4-branes could be useful for understanding of the extrema of TGD of the Minkowskian part of Kähler action which would take take the same role as Hamiltonian in Floer homology: the extrema of 5-D brane action would connect these extrema.

6. Light-like 3-surfaces could be seen as the analogs von Neuman branes for which the boundary conditions state that the ends of space-like 3-brane defined by the partonic 2-surfaces move with light-velocity. The interpretation of partonic 2-surfaces as space-like branes at the ends of CD would in turn make them D-branes so that one would have a duality between D-branes and N-brane interpretations. T-duality (see `http://tinyurl.com/ycvp7rnq`) exchanges von Neumann and Dirichlet boundary conditions so that strong from of general coordinate invariance would correspond to both electric-magnetic and T-duality in TGD framework. Note that T-duality exchanges type II_A and type II_B super-strings with each other.
7. What about causal diamonds and their 7-D light-like boundaries? Could one regard the light-like boundaries of CDs as analogs of 6-branes with light-like direction defining time-like direction so that space-time surfaces would be seen as 3-branes connecting them? This brane would not have magnetic dual since the formula for the dimensions of brane and its magnetic dual allows positive brane dimension p only in the range (1, 3).

15.8.5 Can One Identify The Counterparts Of R-R And NS-NS Fields In TGD?

R-R and NS-NS 3-forms are clearly in fundamental role in M-theory. Since in TGD partonic 2-surfaces define the analogs of fundamental D-2-branes, one can wonder whether these 3-forms could have TGD counterparts.

1. In TGD framework the 3-forms $G_{3,A} = dC_{2,A}$ defined as the exterior derivatives of the two-forms $C_{2,A}$ identified as products $C_{2,A} = H_A J$ of Hamiltonians H_A of $\delta M^4_\pm \times CP_2$ with Kähler forms of factors of $\delta M^4_\pm \times CP_2$ define an infinite family of closed 3-forms belonging to various irreducible representations of rotation group and color group. One can consider also the algebra generated by products $H_A A$, $H_A J$, $H_A A \wedge J$, $H_A J \wedge J$, where A *resp.* J denotes the Kähler gauge potential *resp.* Kähler form or either $\delta M^4_\pm$ or CP_2. A *resp.* Also the sum of Kähler potentials *resp.* forms of $\delta M^4_\pm$ and CP_2 can be considered.
2. One can define the counterparts of the fluxes $\int Adx$ as fluxes of $H_A A$ over braid strands, $H_A J$ over partonic 2-surfaces and string world sheets, $H_A A \wedge J$ over 3-surfaces, and $H_A J \wedge J$ over space-time sheets.Gauge invariance however suggests that for non-constant Hamiltonians one must exclude the fluxes assigned to odd dimensional surfaces so that only odd-dimensional branes would be allowed. This would exclude 0-branes and the problematic 4-branes. These fluxes should be quantized for the critical values of the Minkowskian contributions and for the maxima with respect to zero modes for the Euclidian contributions to Kähler action. The interpretation would be in terms of Morse function and Kähler function if the proposed conjecture holds true. One could even hope that the charges in Cartan algebra are quantized for all preferred extremals and define charges in these irreducible representations for the isometry algebra of WCW . The quantization of electric fluxes for string world sheets would give rise to the familiar quantization of the rotation $\int E \cdot dl$ of electric field over a loop in time direction taking place in superconductivity.
3. Should one interpret these fluxes as the analogs of NS-NS-fluxes or R-R fluxes? The exterior derivatives of the forms G_3 vanish which is the analog for the vanishing of magnetic charge densities (it is however possible to have the analogs of homological magnetic charge). The self-duality of Ramond p-forms could be posed formally ($G_p =^* G_{8-p}$) but does not have any implications for $p < 4$ since the space-time projections vanish in this case identically for $p > 3$. For $p = 4$ the dual of the instanton density $J \wedge J$ is proportional to volume form if M^4 and is not of topological interest. The approach of Witten eliminating one half of self dual R-R-fluxes would mean that only the above discussed series of fluxes need to be considered so that one would have no troubles with non-rational values of the fluxes nor with the lack of higher dimensional objects assignable to them. An interesting question is whether the fluxes could define some kind of K-theory invariants.

4. In TGD embedding space is non-dynamical and there seems to be no counterpart for the NS 3-form field $H = dB$. The only natural candidate would correspond to Hamiltonian $B = J$ giving $H = dB = 0$. At quantum level this might be understood in terms of bosonic emergence meaning that only Ramond representations for fermions are needed in the theory since bosons correspond to wormhole contacts with fermion and anti-fermions at opposite throats. Therefore twisted cohomology is not needed and there is no need to introduce the analogy of brane democracy and 4-D space-time surfaces containing the analogs of lower-dimensional brains as sub-manifolds are enough. The fluxes of these forms over partonic 2-surfaces and string world sheets defined non-abelian analogs of ordinary gauge fluxes reducing to rotations of vector potentials and suggested be crucial for understanding braidings of knots and 2-knots in TGD framework. [K45]. Note also that the unique dimension D=4 for space-time makes 4-D space-time surfaces homologically self-dual so that only they are needed.

15.8.6 What About Counterparts Of S And U Dualities In TGD Framework?

The natural question is what could be the TGD counterparts of $S-$, $T-$ and U-dualities. If one accepts the identification of U-duality as product $U = ST$ and the proposed counterpart of T duality as a strong form of general coordinate invariance, it remains to understand the TGD counterpart of S-duality - in other words electric-magnetic duality - relating the theories with gauge couplings g and $1/g$.

Quantum criticality selects the preferred value of g_K: Kähler coupling strength is very near to fine structure constant at electron length scale and can be equal to it. Note that the hierarchy of Planck constants (dark matters) could be understood in terms of a spectrumfor $\alpha_K = g_K^2/4\pi h_{eff}$, $h_{eff} = n \times h$: in thermodynamical analogy one would have accumulation of critical points at zero temperature.

If there is no coupling constant evolution associated with α_K, it does not make sense to say that g_K becomes strong and is replaced with its inverse at some point. One should be able to formulate the counterpart of S-duality as an identity following from the weak form of electric-magnetic duality and the reduction of TGD to almost topological QFT. This might be the case.

1. For preferred extremals the interior parts of Kähler action reduces to a boundary term if the term $j^\mu A_\mu$ from them vanishes. The weak form of electric-magnetic duality requires that Kähler electric charge is proportional to Kähler magnetic charge, which implies reduction to abelian Chern-Simons term: the Kähler coupling strength does not appear at all in Chern-Simons term. The proportionality constant beween the electric and magnetic parts J_E and J_B of Kähler form however enters into the dynamics through the boundary conditions stating the weak form of electric-magnetic duality. At the Minkowskian side the proportionality constant must be proportional to g_K^2 to guarantee a correct value for the unit of Kähler electric charge - equal to that for electric charge in electron length scale- from the assumption that electric charge is proportional to the topologically quantized magnetic charge. It has been assumed that

$$J_E = \alpha_K J_B$$

holds true at *both sides* of the wormhole throat but this is an un-necessarily strong assumption at the Euclidian side. In fact, the self-duality of CP_2 Kähler form stating

$$J_E = J_B$$

favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for CP_2 type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of CP_2 radius and α_K the effective replacement $g_K^2 \to 1$ would spoil the argument.

2. Minkowskian and Euclidian regions should correspond to a strongly/weakly interacting phase in which Kähler magnetic/electric charges provide the proper description. In Euclidian regions

associated with CP_2 type extremals there is a natural interpretation of interactions between magnetic monopoles associated with the light-like throats: for CP_2 type vacuum extremal itself magnetic and electric charges are actually identical and cannot be distinguished from each other. Therefore the duality between strong and weak coupling phases seems to be trivially true in Euclidian regions if one has $J_B = J_E$ at Euclidian side of the wormhole throat. This is however an un-necessarily strong condition as the following argument shows.

3. In Minkowskian regions the interaction is via Kähler electric charges and elementary particles have vanishing total Kähler magnetic charge consisting of pairs of Kähler magnetic monopoles so that one has confinement characteristic for strongly interacting phase. Therefore Minkowskian regions naturally correspond to a weakly interacting phase for Kähler electric charges. One can write the action density at the Minkowskian side of the wormhole throat as

$$\frac{(J_E^2 - J_B^2)}{\alpha_K} = \alpha_K J_B^2 - \frac{J_B^2}{\alpha_K} \quad .$$

The exchange $J_E \leftrightarrow J_B$ accompanied by $\alpha_K \rightarrow -1/\alpha_K$ leaves the action density invariant. Since only the behavior of the vacuum functional infinitesimally near to the wormhole throat matters by almost topological QFT property, the duality is realized. Note that the argument goes through also in Euclidian regions so that it does not allow to decide which is the correct form of weak form of electric-magnetic duality.

4. S-duality could correspond geometrically to the duality between partonic 2-surfaces responsible for magnetic fluxes and string worlds sheets responsible for electric fluxes as rotations of Kähler gauge potentials around them and would be very closely related with the counterpart of T-duality implied by the strong form of general coordinate invariance and saying that space-like 3-surfaces at the ends of space-time sheets are equivalent with light-like 3-surfaces connecting them.

The boundary condition $J_E = J_B$ at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded CP_2 is such that in CP_2 coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for CP_2 type vacuum extremals since by the light-likeness of M^4 projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole throat. Self-duality is indeed an un-necessarily strong condition.

Comparison with standard view about dualities

One can compare the proposed realization of T, S and U to the more general dualities defined by the modular group $SL(2, Z)$, which in QFT framework can hold true for the path integral over all possible gauge field configurations. In the resent case the dualities hold true for every preferred extremal separately and the functional integral is only over the space-time projections of fixed Kähler form of CP_2. Modular invariance for Maxwell action was discussed by E. Verlinde for Maxwell action with θ term (see `http://tinyurl.com/ycx6lve3`) for a general 4-D compact manifold with Euclidian signature of metric in [B20]. In this case one has path integral giving sum over infinite number of extrema characterized by the cohomological equivalence class of the Maxwell field the action exponential to a high degree. Modular invariance is broken for CP_2: one obtains invariance only for $\tau \rightarrow \tau + 2$ whereas S induces a phase factor to the path integral.

1. In the recent case these homology equivalence classes would correspond to homology equivalence classes of holomorphic partonic 2-surfaces associated with the critical points of Kähler function with respect to zero modes.

2. In the case that the Euclidian contribution to the Kähler action is expressible solely in terms of wormhole throat Chern-Simons terms, and one can neglect the measurement interaction terms fixing the values of some classical conserved quantities to be equal with their quantal counterparts for the space-time surfaces allowed in quantum superposition, the exponent of Kähler action can be expressed in terms of Chern-Simons action density as

$$\begin{aligned} L &= \tau L_{C-S} \ , \\ L_{C-S} &= J \wedge A \ , \\ \tau &= \frac{1}{g_K^2} + i\frac{k}{4\pi} \ , \ \ k=1 \ . \end{aligned} \tag{15.8.1}$$

Here the parameter τ transforms under full $SL(2,Z)$ group as

$$\tau \ \rightarrow \ \frac{a\tau+b}{c\tau+d} \ . \tag{15.8.2}$$

The generators of $SL(2,Z)$ transformations are $T:\tau \rightarrow \tau+1$, $S:\tau \rightarrow -1/\tau$. The imaginary part in the exponents corresponds to Kac-Moody central extension $k=1$.
This form corresponds also to the general form of Maxwell action with CP breaking θ term given by

$$L \ = \ \frac{1}{g_K^2} J \wedge^* J + i\frac{\theta}{8\pi^2} J \wedge J \ , \ \ \theta = 2\pi \ . \tag{15.8.3}$$

Hence the Minkowskian part mimics the θ term but with a value of θ for which the term does not give rise to CP breaking in the case that the action is full action for CP_2 type vacuum extremal so that the phase equals to 2π and phase factor case is trivial. It would seem that the deviation from the full action for CP_2 due to the presence of wormhole throats reducing the value of the full Kähler action for CP_2 type vacuum extremal could give rise to CP breaking. One can visualize the excluded volume as homologically non-trivial geodesic spheres with some thickness in two transverse dimensions. At the limit of infinitely thin geodesic spheres CP breaking would vanish. The effect is exponentially sensitive to the volume deficit.

CP breaking and ground state degeneracy

Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define 2×2 matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full CP_2 type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.

2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K-\overline{K}$ and of CKM matrix should reduce to this mixing. K^0 mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of CP_2 type extremals representing wormhole throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for B^0 mesons.

3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and short-lived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral mesons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only K^0 but not

for the environment in the construction of states. One can probably imagine also alternative interpretations.

Remark: The proportionality of Minkowskian and Euclidian contributions to the same Chern-Simons term implies that the critical points with respect to zero modes appear for both the phase and modulus of vacuum functional. The Kähler function property does not allow extrema for vacuum functional as a function of complex coordinates of WCW since this would mean Kähler metric with non-Euclidian signature. If this were not the case. the stationary values of phase factor and extrema of modulus of the vacuum functional would correspond to different configurations.

15.8.7 Could One Divide Bundles?

TGD differs from string models in one important aspects: stringy diagrams do not have interpretation as analogs of vertices of Feynman diagrams: the stringy decay of partonic 2-surface to two pieces does not represent particle decay but a propagation along different paths for incoming particle. Particle reactions in turn are described by the vertices of generalized Feynman diagrams in which the ends of incoming and outgoing particles meet along partonic 2-surface. This suggests a generalization of K-theory for bundles assignable to the partonic 2-surfaces. It is good to start with a guess for the concrete geometric realization of the sum and product of bundles in TGD framework.

1. The analogs of string diagrams could represent the analog for direct sum. Difference between bundles could be defined geometrically in terms of trouser vertex $A + B \to C$. B would by definition represent $C - A$. Direct sum could make sense for single particle states and have as space-time correlate the conservation of braid strands.
2. A possible concretization in TGD framework for the tensor product is in terms of the vertices of generalized Feynman diagrams at which incoming light-like 3-D orbits of partons meet along their ends. The tensor product of incoming state spaces defined by fermionic oscillator algebras is naturally formed. Tensor product would have also now as a space-time correlate conservation of braid strands. This does not mean that the number of braid strands is conserved in reactions if also particular exchanges can carry the braid strands of particles coming to the vertex.

Why not define also division of bundles in terms of the division for tensor product? In terms of the 3-vertex for generalized Feynman diagrams $A \otimes B = C$ representing tensor product B would be by definition C/A. Therefore TGD would extend the K-theory algebra by introducing also division as a natural operation necessitated by the presence of the join along ends vertices not present in string theory. I would be surprised if some mathematician would not have published the idea in some exotic journal. Below I represent an argument that this notion could be also applied in the mathematical description of finite measurement resolution in TGD framework using inclusions of hyper-finite factor. Division could make possible a rigorous definition for for non-commutative quantum spaces.

Tensor division could have also other natural applications in TGD framework.

1. One could assign bundles M_+ and M_- to the upper and lower light-like boundaries of CD. The bundle M_+/M_- would be obtained by formally identifying the upper and lower light-like boundaries. More generally, one could assign to the boundaries of CD positive and negative energy parts of WCW spinor fields and corresponding bundle structures in "half WCW ". Zero energy states could be seen as sections of the unit bundle just like infinite rationals reducing to real units as real numbers would represent zero energy states.
2. Finite measurement resolution would encourage tensor division since finite measurement resolution means essentially the loss of information about everything below measurement resolution represented as a tensor product factor. The notion of coset space formed by hyper-finite factor and included factor could be understood in terms of tensor division and give rise to quantum group like space with fractional quantum dimension in the case of Jones inclusions [K99]. Finite measurement resolution would therefore define infinite hierarchy of finite dimensional non-commutative spaces characterized by fractional quantum dimension. In this case the notion of tensor product would be somewhat more delicate since complex numbers are effectively replaced by the included algebra whose action creates states not distinguishable from each other [K99]. The action of algebra elements to the state $|B\rangle$ in the inner product $\langle A|B\rangle$ must be equivalent with the action of its hermitian conjugate to the state $\langle A|$. Note that zero

energy states are in question so that the included algebra generates always modifications of states which keep it as a zero energy state.

15.9 A Connection Between Cognition, Number Theory, Algebraic Geometry, Topology, And Quantum Physics

I have had some discussions with Stephen King and Hitoshi Kitada in a closed discussion group about the idea that the duality between Boolean algebras and Stone spaces could be important for the understanding of consciousness, at least cognition. In this vision Boolean algebras would represent conscious mind and Stone spaces would represent the matter: space-time would emerge.

I am personally somewhat skeptic because I see consciousness and matter as totally different levels of existence. Consciousness (and information) is about something, matter just is. Consciousness involves always a change as we no from basic laws about perception. There is of course also the experience of free will and the associated non-determinism. Boolean algebra is a model for logic, not for conscious logical reasoning. There are also many other aspects of consciousness making it very difficult to take this kind of duality seriously.

I am also skeptic about the emergence of space-time say in the extremely foggy form as it used in entropic gravity arguments. Recent day physics poses really strong constraints on our view about space-time and one must take them very seriously.

This does not however mean that Stone spaces could not serve as geometrical correlates for Boolean consciousness. In fact, p-adic integers can be seen as a Stone space naturally assignable to Boolean algebra with infinite number of bits.

15.9.1 Innocent Questions

I ende up with the innocent questions, as I was asked to act as some kind of mathematical consultant and explain what Stone spaces actually are and whether they could have a connection to p-adic numbers. Anyone can of course go to Wikipedia and read the article "Stone's representation theorem for Boolean algebras" (see `http://tinyurl.com/ybyf56e3`). For a layman this article does not however tell too much.

Intuitively the content of the representation theorem looks rather obvious, at least at the first sight. As a matter fact, the connection looks so obvious that physicists often identify the Boolean algebra and its geometric representation without even realizing that two different things are in question. The subsets of given space- say Euclidian 3-space- with union and intersection as basic algebraic operations and inclusion of sets as ordering relation defined a Boolean algebra for the purposes of physicist. One can assign to each point of space a bit. The points for which the value of bit equals to one define the subset. Union of subsets corresponds to logical OR and intersection to AND. Logical implication B→ A corresponds to A contains B.

When one goes to details problems begin to appear. One would like to have some non-trivial form of continuity.

1. For instance, if the sets are form open sets in real topology their complements representing negations of statements are closed, not open. This breaks the symmetry between statement and it negation unless the topology is such that closed sets are open. Stone's view about Boolean algebra assumes this. This would lead to discrete topology for which all sets would be open sets and one would lose connection with physics where continuity and differential structure are in key role.
2. Could one dare to disagree with Stone and allow both closed and open sets of E^3 in real topology and thus give up clopen assumption? Or could one tolerate the asymmetry between statements and their negations and give some special meaning for open or closet sets- say as kind of axiomatic statements holding true automatically. If so, one an also consider algebraic varieties of lower dimension as collections of bits which are equal to one. In Zariski topology used in algebraic geometry these sets are closed. Again the complements would be open. Could one regard the lower dimensional varieties as identically true statements so that the set of identically true statements would be rather scarce as compared to falsities? If one tolerates some quantum TGD, one could ask whether the 4-D quaternionic/associative varieties defining

classical space-times and thus classical physics could be identified as the axiomatic truths. Associativity would be the basic truth inducing the identically true collections of bits.

15.9.2 Stone Theorem And Stone Spaces

For reasons which should be clear it is perhaps a good idea to consider in more detail what Stone duality says. Stone theorem states that Boolean algebras are dual with their Stone spaces. Logic and certain kind of geometry are dual. More precisely, any Boolean algebra is isomorphic to closed open subsets of some Stone space and vice versa. Stone theorem respects category theory. The homomorphisms between Boolean algebras A and B corresponds to homomorphism between Stone spaces $S(B)$ and $S(A)$: one has contravariant functor between categories of Boolean algebras and Stone spaces. In the following set theoretic realization of Boolean algebra provides the intuitive guidelines but one can of course forget the set theoretic picture altogether and consider just abstract Boolean algebra.

1. Stone space is defined as the space of homomorphisms from Boolean algebra to 2-element Boolean algebra. More general spaces are spaces of homomorphisms between two Boolean algebras. The analogy in the category of linear spaces would be the space of linear maps between two linear spaces. Homomorphism is in this case truth preserving map: $h(A\ AND\ B) = h(a)\ AND\ h(B)$, $h(\ OR\ B) = h(a)\ OR\ h(B)$ and so on.
2. For any Boolean algebra Stone space is compact, totally disconnected Hausdorff space. Conversely, for any topological space, the subsets, which are both closed and open define Boolean algebra. Note that for a real line this would give 2-element Boolean algebra. Set is closed and open simultaneously only if its boundary is empty and in p-adic context there are no boundaries. Therefore for p-adic numbers closed sets are open and the sets of p-adic numbers with p-adic norm above some lower bound and having some set of fixed pinary digits, define closed-open subsets.
3. Stone space dual to the Boolean algebra does not conform with the physicist's ideas about space-time. Stone space is a compact totally disconnected Hausdorff space. Disconnected space is representable as a union of two or more disjoint open sets. For totally disconnected space this is true for every subset. Path connectedness is stronger notion than connected and says that two points of the space can be always connected by a curve defined as a mapping of *real* unit interval to the space. Our physical space-time seems to be however connected in this sense.
4. The points of the Stone space $S(B)$ can be identified ultrafilters. Ultrafilter defines homomorphism of B to 2-element of Boolean algebra Boolean algebra. Set theoretic realization allows to understand what this means. Ultrafilter is a set of subsets with the property that intersections belong to it and if set belongs to it also sets containing it belong to it: this corresponds to the fact that set inclusion $A \supset B$ corresponds to logical implication. Either set or its complement belongs to the ultrafilter (either statement or its negation is true). Empty set does not. Ultrafilter obviously corresponds to a collection of statements which are simultaneously true without contradictions. The sets of ultrafilter correspond to the statements interpreted as collections of bits for which each bit equals to 1.
5. The subsets of B containing a fixed point b of Boolean algebra define an ultrafilter and embedding of b to the Stone space by assigning to it this particular principal ultrafilter. b represents a statement which is always true, kind of axiom for this principal ultrafilter and ultrafilter is the set of all statements consistent with b.
   Actually any finite set in the Boolean algebra consisting of a collection of fixed bits b_i defines an ultrafilter as the set all subsets of Boolean algebra containing this subset. Therefore the space of all ultra-filters is in one-one correspondence with the space of subsets of Boolean statements. This set corresponds to the set of statements consistent with the truthness of b_i analogous to axioms.

15.9.3 2-Adic Integers And 2-Adic Numbers As Stone Spaces

I was surprised to find that p-adic numbers are regarded as a totally disconnected space. The intuitive notion of connected is that one can have a continuous curve connecting two points and this is certainly true for p-adic numbers with curve parameter which is p-adic number but not for curves with real parameter which became obvious when I started to work with p-adic numbers and invented the notion of p-adic fractal. In other words, p-adic integers form a continuum in p-adic but not in real sense. This example shows how careful one must be with definitions. In any case, to my opinion the notion of path based on p-adic parameter is much more natural in p-adic case. For given p-adic integers one can find p-adic integers arbitrary near to it since at the limit $n \to \infty$ the p-adic norm of p^n approaches zero. Note also that most p-adic integers are infinite as real integers.

Disconnectedness in real sense means that 2-adic integers define an excellent candidate for a Stone space and the inverse of the Stone theorem allows indeed to realize this expectation. Also 2-adic numbers define this kind of candidate since 2-adic numbers with norm smaller than 2^n for any n can be mapped to 2-adic integers. One would have union of Boolean algebras labelled by the 2-adic norm of the 2-adic number. p-Adic integers for a general prime p define obviously a generalization of Stone space making sense for effectively p-valued logic: the interpretation will be discussed below.

Consider now a Boolean algebra consisting of all possible infinitely long bit sequences. This algebra corresponds naturally to 2-adic integers. The generating Boolean statements correspond to sequences with single non-vanishing bit: by taking the unions of these points one obtains all sets. The natural topology is that for which the lowest bits are the most significant. 2-adic topology realizes this idea since n: th bit has norm 2^{-n}. 2-adic integers as an p-adic integers are as spaces totally disconnected.

That 2-adic integers and more generally, 2-adic variants of n-dimensional p-adic manifolds would define Stone bases assignable to Boolean algebras is consistent with the identification of p-adic space-time sheets as correlates of cognition. Each point of 2-adic space-time sheet would represent 8 bits as a point of 8-D embedding space. In TGD framework WCW ("world of classical worlds") spinors correspond to Fock space for fermions and fermionic Fock space has natural identification as a Boolean algebra. Fermion present/not present in given mode would correspond to true/false. Spinors decompose to a tensor product of 2-spinors so that the labels for Boolean statements form a Boolean algebra too in this case. A possible interpretation is as statements about statements.

In TGD Universe life and thus cognition reside in the intersection of real and p-adic worlds. Therefore the intersections of real and p-adic partonic 2-surfaces represent the intersection of real and p-adic worlds, those Boolean statements which are expected to be accessible for conscious cognition. They correspond to rational numbers or possibly numbers in an algebraic extension of rationals. For rationals pinary expansion starts to repeat itself so that the number of bits is finite. This intersection is also always discrete and for finite real space-time regions finite so that the identification looks a very natural since our cognitive abilities seem to be rather limited. In TGD inspired physics magnetic bodies are the key players and have much larger size than the biological body so that their intersection with their p-adic counterparts can contain much more bits. This conforms with the interpretation that the evolution of cognition means the emergence of increasingly longer time scales. Dark matter hierarchy realized in terms of hierarchy of Planck constants realizes this.

15.9.4 What About P-Adic Integers With $P > 2$?

The natural generalization of Stone space would be to a geometric counterpart of p-adic logic which I discussed for some years ago. The representation of the statements of p-valued logic as sequences of pinary digits makes the correspondence trivial if one accepts the above represented arguments. The generalization of Stone space would consist of p-adic integers and embedding of a p-valued analog of Boolean algebra would map the number with only n: th digit equal to $1, ..., p-1$ to corresponding p-adic number.

One should however understand what p-valued statements mean and why p-adic numbers near powers of 2 are important. What is clear that p-valued logic is too romantic to survive. At least our every-day cognition is firmly anchored to a reality where everything is experience to be true or false.

1. The most natural explanation for $p > 2$ adic logic is that all Boolean statements do not allow a physical representation and that this forces reduction of 2^n valued logic to $p < 2^n$-

valued one. For instance, empty set in the set theoretical representation of Boolean logic has no physical representation. In the same manner, the state containing no fermions fails to represent anything physically. One can represent physically at most $2^n - 1$ one statements of n-bit Boolean algebra and one must be happy with $n-1$ completely represented digits. The remaining statements containing at least one non-vanishing digit would have some meaning, perhaps the last digit allowed could serve as a kind of parity check.

2. If this is accepted then p-adic primes near to power 2^n of 2 but below it and larger than the previous power 2^{n-1} can be accepted and provide a natural topology for the Boolean statements grouping the binary digits to p-valued digit which represents the allowed statements in 2^n valued Boolean algebra. Bit sequence as a unit would be represented as a sequence of physically realizable bits. This would represent evolution of cognition in which simple yes or not statements are replaced with sequences of this kind of statements just as working computer programs are fused as modules to give larger computer programs. Note that also for computers similar evolution is taking place: the earliest processors used byte length 8 and now 32, 64 and maybe even 128 are used.
3. Mersenne primes $M_n = 2^n - 1$ would be ideal for logic purposes and they indeed play a key role in quantum TGD. Mersenne primes define p-adic length scales characterize many elementary particles and also hadron physics. There is also evidence for p-adically scaled up variants of hadron physics (also lepto-hadron physics allowed by the TGD based notion of color predicting colored excitations of leptons). LHC will certainly show whether M_{89} hadron physics at TeV energy scale is realized and whether also leptons might have scaled up variants.
4. For instance, M_{127} assignable to electron secondary p-adic time scale is.1 seconds, the fundamental time scale of sensory perception. Thus cognition in.1 second time scale single pinary statement would contain 126 digits as I have proposed in the model of memetic code. Memetic codons would correspond to 126 digit patterns with duration of.1 seconds giving 126 bits of information about percept.

If this picture is correct, the interpretation of p-adic space-time sheets- or rather their intersections with real ones- would represent space-time correlates for Boolean algebra represented at quantum level by fermionic many particle states. In quantum TGD one assigns with these intersections braids- or number theoretic braids- and this would give a connection with topological quantum field theories (TGD can be regarded as almost topological quantum field theory).

15.9.5 One More Road To TGD

The following arguments suggests one more manner to end up with TGD by requiring that fermionic Fock states identified as a Boolean algebra have their Stone space as space-time correlate required by quantum classical correspondence. Second idea is that space-time surfaces define the collections of binary digits which can be equal to one: kind of eternal truths. In number theoretical vision associativity condition in some sense would define these divine truths. Standard model symmetries are a must- at least as their p-adic variants -and simple arguments forces the completion of discrete lattice counterpart of M^4 to a continuum.

1. If one wants Poincare symmetries at least in p-adic sense then a 4-D lattice in M^4 with $SL(2,Z) \times T^4$, where T^4 is discrete translation group is a natural choice. $SL(2,Z)$ acts in discrete Minkowski space T^4 which is lattice. Poincare invariance would be discretized. Angles and relative velocities would be discretized, etc..
2. The p-adic variant of this group is obtained by replacing Z and T^4 by their p-adic counterparts: in other words Z is replaced with the group Z_p of p-adic integers. This group is p-adically continuous group and acts continuously in T^4 defining a p-adic variant of Minkowski space consisting of all bit sequences consisting of 4-tuples of bits. Only in real sense one would have discreteness: note also that most points would be at infinity in real sense. Therefore it is possible to speak about analytic functions, differential calculus, and to write partial differential equations and to solve them. One can construct group representations and talk about angular momentum, spin and 4-momentum as labels of quantum states.
3. If one wants standard model symmetries p-adically one must replace T^4 with $T^4 \times CP_2$. CP_2 would be now discrete version of CP_2 obtained from discrete complex space C^3 by identifying

points different by a scaling by complex integer. Discrete versions of color and electroweak groups would be obtained.

The next step is to ask what are the laws of physics. TGD fan would answer immediately: they are of course logical statements which can be true identified as subsets of $T^4 \times CP_2$ just as subset in Boolean algebra of sets corresponds to bits which are true.

1. The collections of 8-bit sequences consisting of only 1: s would define define 4-D surfaces in discrete $T^4 \times CP_2$. Number theoretic vision would suggest that they are quaternionic surfaces so that one associativity be the physical law at geometric level. The conjecture is that preferred extremals of Kähler action are associative surfaces using the definition of associativity as that assignable to a 4-plane defined by Kähler-Dirac gamma matrices at given point of space-time surface.
2. Induced gauge field and metric make sense for p-adic integers. p-Adically the field equations for Kähler action make also sense. These p-adic surfaces would represent the analog of Boolean algebra. They would be however something more general than Stone assumes since they are not closed-open in the 8-D p-adic topology.

One however encounters a problem.

1. Although the field equations associated with Kähler action make sense, Kähler action itself does not exists as integral nor does the genuine minimization make sense since p-adically numbers are not well ordered and one cannot in general say which of two numbers is the larger one. This is a real problem and suggests that p-adic field equations are not enough and must be accompanied by real ones. Of course, also the metric properties of p-adic space-time are in complete conflict with what we believe about them.
2. One could argue that for preferred extremals the integral defining Kähler action is expressible as an integral of 4-form whose value could be well-defined since integrals of forms for closed algebraic surfaces make sense in p-adic cohomology theory pioneered by Grothendieck. The idea would be to use the definition of Kähler action making sense for preferred extremals as its definition in p-adic context. I have indeed proposed that space-time surfaces define representatives for homology with inspiration coming from TGD as almost topological QFT. This would give powerful constraints on the theory in accordance with the interpretation as a generalized Bohr orbit.
3. This argument together with what we know about the topology of space-time on basis of everyday experience however more or less forces the conclusion that also real variant of $M^4 \times CP_2$ is there and defines the proper variational principle. The finite points (on real sense) of $T^4 \times CP_2$ (in discrete sense) would represent points common to real and p-adic worlds and the identification in terms of braid points makes sense if one accepts holography and restricts the consideration to partonic 2-surfaces at boundaries of causal diamond. These discrete common points would represent the intersection of cognition and matter and living systems and provide a representation for Boolean cognition.
4. Finite measurement resolution enters into the picture naturally. The proper time distance between the tips would be quantized in multiples of CP_2 length. There would be several choices for the discretized embedding space corresponding to different distance between lattice points: the interpretation is in terms of finite measurement resolution.

It should be added that discretized variant of Minkowski space and its p-adic variant emerge in TGD also in different manner in zero energy ontology.

1. The discrete space $SL(2,Z) \times T^4$ would have also interpretation as acting in the moduli space for causal diamonds identified as intersections of future and past directed light-cones. T^4 would represent lattice for possible positions of the lower tip of CDand$SL(2,Z)$ leaving lower tip invariant would act on hyperboloid defined by the position of the upper tip obtained by discrete Lorentz transformations. This leads to cosmological predictions (quantization of red shifts). CP_2 length defines a fundamental time scale and the number theoretically motivated assumption is that the proper time distances between the tips of CDs come as integer multiples of this distance.
2. The stronger condition explaining p-adic length scale hypothesis would be that only octaves of the basic scale are allowed. This option is not consistent with zero energy ontology. The reason is that for more general hypothesis the M-matrices of the theory for Kac-Moody type algebra

with finite-dimensional Lie algebra replaced with an infinite-dimensional algebra representing hermitian square roots of density matrices and powers of the phase factor replaced with powers of S-matrix. All integer powers must be allowed to obtain generalized Kac-Moody structure, not only those which are powers of 2 and correspond naturally to integer valued proper time distance between the tips of CD. Zero energy states would define the symmetry Lie-algebra of S-matrix with generalized Yangian structure.

3. p-Adic length scale hypothesis would be an outcome of physics and it would not be surprising that primes near power of two are favored because they are optimal for Boolean cognition.

The outcome is TGD. Reader can of course imagine alternatives but remember the potential difficulties due to the fact that minimization in p-adic sense does not make sense and action defined as integral does not exist p-adically. Also the standard model symmetries and quantum classical correspondence are to my opinion "must": s.

15.9.6 A Connection Between Cognition And Algebraic Geometry

Stone space is synonym for profinite space. The Galois groups associated with algebraic extensions of number fields represent an extremely general class of profinite group (see `http://tinyurl.com/y92ms8f3`) [A66]. Every profinite group appears in Galois theory of some field K. The most most interesting ones are inverse limits of $Gal(F_1/K)$ where F_1 varies over all intermediate fields. Profinite groups appear also as fundamental groups in algebraic geometry. In algebraic topology fundamental groups are in general not profinite. Profiniteness means that p-adic representations are especially natural for profinite groups.

There is a fascinating connection between infinite primes and algebraic geometry discussed above leads to the proposal that Galois groups - or rather their projective variants- can be represented as braid groups acting on 2-dimensional surfaces. These findings suggest a deep connection between space-time correlates of Boolean cognition, number theory, algebraic geometry, and quantum physics and TGD based vision about representations of Galois groups as groups lifted to braiding groups acting on the intersection of real and p-adic variants of partonic 2-surface conforms with this.

Fermat theorem serves as a good illustration between the connection between cognitive representations and algebraic geometry. A very general problem of algebraic geometry is to find rational points of an algebraic surface. These can be identified as common rational points of the real and p-adic variant of the surface. The interpretation in terms of consciousness theory would be as points defining cognitive representation as rational points common to real partonic 2-surface and and its p-adic variants. The mapping to polynomials given by their representation in terms of infinite primes to braids of braids of braids.... at partonic 2-surfaces would provide the mapping of n-dimensional problem to 2-dimensional one.

One considers the question whether there are integer solutions to the equation $x^n+y^n+z^n = 1$. This equation defines 2-surfaces in both real and p-adic spaces. In p-adic context it is easy to construct solutions but they usually represent infinite integers in real sense. Only if the expansion in powers of p contains finite number of powers of p, one obtains real solution as finite integers.

The question is whether there are any real solutions at all. If they exist they correspond to the intersections of the real and p-adic variants of these surfaces. In other words p-adic surface contains cognitively representable points. For $n > 2$ Fermat's theorem says that only single point $x = y = z = 0$ exists so that only single p-adic multi-bit sequence $(0, 0, 0, ...)$ would be cognitively representable.

This relates directly to our mathematical cognition. Linear and quadratic equations we can solve and in these cases the number in the intersection of p-adic and real surfaces is indeed very large. We learn the recipes already in school! For $n > 2$ difficulties begin and there are no general recipes and it requires mathematician to discover the special cases: a direct reflection of the fact that the number of intersection points for real and p-adic surfaces involved contains very few points.

15.9.7 Quantum Mathematics

To my view the self referentiality of consciousness is the real "hard problem" of consciousness theories. The "hard problem" as it is usually understood is only a problem of dualistic approach.

My own belief is that the understanding of self-referentiality requires completely new mathematics with explicitly built-in self-referentiality. One possible view about this new mathematics is described in [K65]: here I provide only a brief summary in a form of recipe. The basic idea could have been abstracted from algebraic holography: replace numbers by Hilbert spaces and basic arithmetic operations with their counterparts for Hilbert spaces. Repeat this procedure by replacing the points of Hilbert spaces with Hilbert spaces and continue this procedure ad infinitum. It is quite possible that this procedure analogous to second quantization is more or less equivalent with the construction of infinite primes [K84].

Construction recipe

The construction recipe is following.

1. The idea is to start from arithmetics: + and $\times$ for natural numbers and generalize it.
   (a) The key observation is that + and $\times$ have direct sum and tensor product for Hilbert spaces as complete analogs and natural number n has interpretation as Hilbert space dimension and can be mapped to n-dimensional Hilbert space.
   Replace natural numbers n with n-dimensional Hilbert spaces at the first abstraction step. $n + m$ and $n \times m$ go to direct sum $n \oplus m$ and tensor product $n \otimes m$ of Hilbert spaces. One would calculate with Hilbert spaces rather than numbers. This induces calculation also for Hilbert space states and sum and product are like 3-particle vertices.
   (b) At second step construct integers (also negative) as pairs of Hilbert spaces (m, n) identifying $(m \oplus r, n \oplus r)$ and (m, n). This gives what might be called negative dimensional Hilbert spaces! Then take these pairs and define rationals as Hilbert space pairs (m, n) of this kind with (m, n) equivalent to $(k \otimes m, k \otimes n)$. This gives rise to what might be called m/n-dimensional Hilbert spaces!
   (c) At the third step construct Hilbert space variants of algebraic extensions of rationals. Hilbert space with dimension $\sqrt{2}$ say: this is a really nice trick [K65]. The idea is to consider for n-dimensional extension n-tuples of Hilbert spaces and induce tensor product for them from the product for the numbers of extension. After that one can continue with p-adic number fields and even reals: one can indeed understand even what π-dimensional Hilbert space could be! These spaces could also have interpretation in term of hyper-finite factors for which Hilbert spaces which otherwise would have infinite-dimension have finite and continuous dimension [K99]. If Hilbert space infinite-dimensional in the usual sense has dimension 1 (say) in the sense that identity operator has trace equal to 1 then subspaces in general have continuous range of dimensions smaller than one.

   The direct sum decompositions and tensor products would have genuine meaning Hilbert spaces associated with transcendentals are finite-dimensional in the sense as it is defined here but infinite-dimensional in ordinary sense. These Hilbert spaces would have different decompositions and would not be equivalent. Also in quantum physics decompositions to tensor products and direct sums (say representations of symmetry group) have phyiscal meaning: abstract Hilbert space of infinite dimension is too rough a concept.
   A direct connection with the ideas about complexity emerges. Rationals correspond to pairs of pairs of finite-dimensional Hilbert spaces corresponding to integers. Algebraic numbers correspond to n-tuples of finite-dimensional Hilbert spaces. Transcendentals correspond to infinite-dimensional Hilbert spaces decomposing to direct sums of tensor products: for instance, pinary expansion could define this decomposition. This decomposition matters so that abstract infinite-dimensional Hilbert spaces are not in question. The additional structure due to tensor product and direct sum is present also in physical applications: for instance the decomposition to irreducible representations defines this kind of direct sum decomposition.
2. Do the same for complex numbers, quaternions, and octonions, embedding space $M^4 \times CP_2$, etc.. The objection is that the construction is not general coordinate invariant. In coordinates in which point corresponds to integer valued coordinate one has finite-D Hilbert space and in coordinates in which coordinates of point correspond to transcendentals one has infinite-D Hilbert space. This makes sense only if one interprets the situation in terms of cognitive representations for points. π is very difficult to represent cognitively since it has infinite number of digits for which one cannot give a formula. "2" in turn is very simple to represent. This

suggests interpretation in terms of self-referentiality. The two worlds with different coordinatizations are not equivalent since they correspond to different cognitive contents.

3. Replace also the coordinates of points of Hilbert spaces with Hilbert spaces again and again! The second key observation is that one can do all this again but at new level. Replace the numbers defining vectors of the Hilbert spaces (number sequences) assigned to numbers with Hilbert spaces! Continue ad infinitum by replacing points with Hilbert spaces again and again. One obtains a sequence of abstractions, which would be analogous to a hierarchy of n: th order logics. At lowest levels would be just predicate calculus: statements like $4 = 2^2$. At second level abstractions like $y = x^2$. At next level collections of algebraic equations, etc....
This construction is structurally very similar to - if not equivalent with - the construction of infinite primes which corresponds to repeated second quantization in quantum physics. There is also a close relationship to - maybe equivalence with - what I have called algebraic holography or number theoretic Brahman=Atman identity [K84]. Numbers have infinitely complex anatomy not visible for physicist but necessary for understanding the self referentiality of consciousness and allowing mathematical objects to be holograms coding for mathematics. Hilbert spaces would be the DNA of mathematics from which all mathematical structures would be built!

Generalized Feynman diagrams as mathematical formulas?

One can assign to direct sum and tensor product their co-operations [K65, K12] and sequences of mathematical operations are very much like generalized Feynman diagrams. Co-product for instance would assign to integer m superposition of all its factorizations to a product of two integers with some amplitude for each factorization. Same applies to co-sum. Operation and co-operation would together give meaning to number theoretical 3-particle vertices. The amplitudes for the different factorizations must satisfy consistency conditions: associativity and distributivity could give constraints to the couplings to different channels- as particle physicist might express it.

The proposal is that quantum TGD is indeed quantum arithmetics with product and sum and their co-operations. Perhaps even something more general since also quantum logics and quantum set theory could be included! Generalized Feynman diagrams would correspond to formulas and sequences of mathematical operations with stringy 3-vertex as fusion of 3 -surfaces corresponding to $\oplus$ and Feynmanian 3-vertex as gluing of 3-surfaces along their ends, which is partonic 2-surface, corresponding to $\otimes$! One implication is that all generalized Feynman diagrams would reduce to a canonical form without loops and incoming/outgoing legs could be permuted. This is actually a generalization of old fashioned string model duality symmetry that I proposed years ago but gave it up as too “romantic” [K12].

15.10 Boolean algebras, Stone spaces and p-adic physics

The Facebook discussion with Stephen King about Stone spaces (see `http://tinyurl.com/ze2o4o5`) led to a highly interesting development of ideas concerning Boolean, algebras, Stone spaces, and p-adic physics. I have discussed these ideas already earlier but the improved understanding of the notion of Stone space helped to make the ideas more concrete. The following piece of text emerged from the attempt to clarify thoughts and to summarize what I think (just now).

15.10.1 Boolean algebras

The most familiar representation of Boolean algebras (see `http://tinyurl.com/cwhw8kd` and `http://tinyurl.com/jznz7kq`) is in terms of set theory. Intersection $\cap$ and union $\cup$ for subsets of given set are the basic commutative and associative set theoretic operations having logical meaning as $\wedge$ (AND) and $\vee$. Negation $\neg$ corresponds to complement of set and is reflection like operation. $\wedge$ ($\cap$) is distributive over $\vee$ ($\cup$) just like product is distributive over sum in arithmetics ($a(b + c) = ab + ac$). $\wedge$ ($\cap$) has unit element 1 (entire set) acting as annihilator for $\vee$ ($\cup$). $\vee$ ($\cup$) has unit element 0 (empty set) acting as annihilator for $\wedge$ ($\cap$). Both $\wedge$ ($\cap$) and $\vee$ ($\cup$) are idempotent and are thus analogous to projection operations. The law of absorbtion states $x \wedge (x \vee y) = x \vee (x \wedge y) = x$. Only distribution law breaks the symmetry between $\wedge$ and $\vee$.

For sets the Boolean algebra B of sets can be realized algebraically as maps from set to Boolean algebra Z_2. Given set is defined as points for which the value of map is 1 and its complement as points for which it is zero: the points of the entire set are colored with black or white, and white points form the subset. Boolean operations correspond to simple operations for these Z_2 valued functions in the set representable as bit sequences with one bit for each element of set. AND as intersection of sets corresponds to bit-wise product

$$f_1 \wedge f_2 = f_1 \times f_2 \quad .$$

OR as union of sets to

$$f_1 \vee f_2 = f_1 + f_2 + f_1 \times f_2 \quad .$$

Negation corresponds to the addition of bit 1 to each bit:

$$\neg f = f + 1 \quad .$$

For finite sets Boolean algebra is identical to its power set consisting of its subsets and having 2^N elements if the set has N elements: each element of set corresponds to a bit telling whether it is present in the subset or not.

For infinite sets situation is not at all so obvious. For instance, for subsets of real line the condition that sets are open is in conflict with the existence of negation. The complement of open set is closed (containing its boundaries).

Stone spaces (see `http://tinyurl.com/ze2o4o5`) could be seen as a formulation of Boolean logic in which one gets rid of the difficulty. One does not try to make the topology of set consistent with Boolean algebra (by assuming that open sets correspond to all elements of Boolean algebra: this would produce discrete topology, which is totally trivial). Instead, one topologizes the Boolean algebra and the outcome is so called Stone space (or pro-finite space) in honour of Marshal Stone who discovered the notion. Stone spaces have compact-open topology meaning that open sets are also compact sets. This means that points of space - if they belong to the Boolean algebra - are open sets. If I have understood correctly the idea of Stone space is to give up the points of continuum as elements of Boolean algebra and replace Boolean algebra with the space of ultrafilters defining Stone space.

What makes Stone spaces so interesting from the point of view of TGD is that also p-adic numbers are Stone spaces. My first misunderstanding was that *all* Stone spaces are associated with Boolean algebras. This is not the case. The obvious guess is however that 2-adic numbers as sequences of possibility infinite bits ordered by their significance correspond to some Boolean algebra. A slight generalization would suggest that p-adic numbers correspond to p-valued logics and "p-Boolean" algebra for some set. Some-one has said that God created the natural numbers and humans did the rest so that the first guess is that this set consists of natural numbers. In the following also these innocent guesses are considered in more detail.

15.10.2 Stone spaces

First some basic notions about Boolean algebras relevant to the notion of Stone space.

1. The notion of filter is important in the theory of Boolean algebras and Stone spaces (see `http://tinyurl.com/hhvvpe4`). Non-empty subset of F of Boolean algebra B is a filter if
   (a) for any pair x, y elements of F there exists $z \leq x, z \leq y$,
   (b) for any x in F and $x \leq y$, also y belongs to F.

   It is easy to see that filter does not contain mutually inconsistent statements. It is like the set of all theorems of axiomatic system with some basic axioms from which theorems are deduced.
2. Ultrafilter is a filter not contained in any filter. Ultrafilter has an important property that for every element x in Boolean algebra either x or its negation $\neg x$ but not both belongs to ultrafilter.
   Ultrafilters on a Boolean algebra can be related to prime ideals, maximal ideals, and homomorphisms to the 2-element Boolean algebra Z_2. For given homomorphism of this kind the inverse image of "true" is ultrafilter. The inverse image of false is a maximal ideal. Given a

maximal ideal, its complement is an ultrafilter and there is unique homomorphism taking the maximal ideal to "false". The dual of this statement holds for given ultrafilter.

Prime ideals of Boolean algebra are maximal and have the property that if $x \wedge y$ belongs to the ideal, then either x or y does so. In finite case maximal ultrafilter the number of elements in maximal ultrafilter is one half of that for the entire Boolean algebra. Maximal ideal and ultrafilter correspond to subset and its complement in Boolean algebra ideal contains empty set and ultrafilter entire set.

3. Stone space (see `http://tinyurl.com/jsapyeq`) for a set S is defined as the set of ultrafilters for the Boolean algebra associated with it. It is contained by the power set of S consisting of its subsets but not equal to it if the set is infinite. Ultrafilters are equivalent with Z_2 homomorphisms from the set. The realization of ultrafilters as inverse images of "true" for Z_2 valued homomorphisms allow to understand Stone space as the set of true statements about fundamental statements defined by the points of the set.

   Homomorphism property tells that these statements about fundamental statements are logically consistent: either given element of Boolean algebra or its negation belongs to the ultrafilter. From Wikipedia (see `http://tinyurl.com/ofysow5`) one learns that for a finite set Boolean algebra equals to its power set. The Boolean algebra of infinite set is a subset of power set. One can intuit that at least points and discrete subsets are excluded.

4. Category theory is an additional aspect. Homomorphisms between Boolean algebras correspond to homeomorphisms between their Stone spaces.

A couple of additional remarks relevant for TGD point of view about Stone spaces are in order.

1. Given Stone space is not necessarily associated with any Boolean algebra as the space of its ultrafilters.
2. What is important is the "statements about statements" structure and ultrafilter as set of true statements about statements. Stone space represents higher level of abstraction hierarchy.

   Around 1990 or so I discussed for the first time a model of genetic code inspired by so called Combinatorial Hierarchy [K37] [L21]. Mersenne prime $M_7 = 2^{127} - 1$ corresponds to the number of elements a Boolean algebra of 7 bits with the statement corresponding to physically non-realizable empty set thrown away. One can however choose 64 statements representable using 6 bits and identified in terms of genetic code as statements, which correspond to subsets represented as bit sequences with bit 1 for the elements of sub-set and 0 for the rest. These subsets form an inclusion hierarchy which corresponds to implication hierarchy in opposite direction. They correspond also to all statements consistent with atomic statement (1 bit fixed). I talked about axioms but the correct interpretation is perhaps as theorems deducible from axioms. This selection of 64 bit sequences is nothing but selection of an ultrafilter, which I did not realize because I could not go to Wikipedia and check what it says about Boolean algebras.

15.10.3 Stone spaces and TGD

p-Adic number fields define Stone space and one expect that 2-adic numbers correspond to Boolean algebra. p-Adic numbers would most naturally correspond to p-valued logic. What could be the interpretation of p-valued logic? The difficult quesetion concerns the Stone spaces associated various classical number fields? Could TGD allow to speculate about them?

p-Adic numbers and Stone spaces

Some examples might make the notion of Stone space more concrete and clarify the connection to p-adic physics as physics of cognition and therefore also physics of Boolean mind.

1. 2-adic integers define Stone space for natural numbers very naturally. The '1':s in the bit sequence specify the elements of the subset.
2. Also p-adic integers are a Stone space but defined in terms of Z_p valued homomorphisms from natural numbers to Z_p defining p-valued logic and analogs of its ultrafilters. In this case the set is decomposed to p subsets with different colors and generalized union and intersection can

be defined for these decompositions using exactly the same algebraic formulas as in the case of Boolean algebra.
What is important is that these operations are not anymore operations for a pair of subsets but for two decompositions of the set to p subsets. Cyclic transformations in Z_p are natural operations analogous to negation. Now however p:th power represents identify transformation. The operation $x \rightarrow -x$ is possible since Z_p is finite field but is trivial for $p = 2$.
For natural numbers p-valued logic gives p-adic integers as decomposition of natural numbers to p sub-sets. The homomorphisms generalize also to positive rationals and one expects that Stone space consists of all p-adic numbers. There are good reasons to expect that one can extend this notion also to algebraic extensions of rationals and corresponding integers so that algebraic extensions of p-adic numbers have interpretation as Stone space for corresponding algebraic extension of rationals.

3. Also n-valued logic is possible and correspond to expansions of natural numbers in powers of n. Z_n is not however finite field unless n is a power of prime - for $n = p^k$ one obtains finite field $G(p, k)$ reducing to Z_p for $k = 1$. It also makes sense to speak about n-adic topology but n-adic numbers numbers form only ring rather than number field unless n is prime. For general n the operation $x \rightarrow -x$ does not exist by the loss of field property anymore but other operations are well-defined.
4. In TGD framework adelic picture strongly suggests that 2-valued logic is only the lowest one in the hierarchy of p-valued logics. A possible interpretation for $p < 2^k$-valued logics is in terms of error correction and will be discussed below. One selects p statements from k-bit Boolean algebra and error correction routine checks whether the k-bit sequence belongs to this sub-space. The classical analog of error correction in quantum computation. p-Adic evolution would have interpretation in terms of evolving error correction mechanisms.

Could the generation of elements of n-valued logic (n-Boolean algebra) consisting of n subsets of set be reduced to Boolean measurements decomposing set to subset and its complement?

1. A natural operation yielding decomposition of a set to n-subsets is as a sequence of Boolean measurements. Decompose first the set to set x and its complement by Boolean measurement, decompose then x to set y and its complement, etc... n-valued logic would require $n - 2$ Boolean measurements for independent observables. The problem is how one selects the set to be decomposed at given step and there are $n-2$ choices meaning 2^{n-2} ways to do the operation labelled by elements of $n-2$-bit Boolean algebra. One possibility is that at each step the next set to be decomposed corresponds to "true" for the previous Boolean measurement. This construction might allow to express elements of n-Boolean algebra as sequences of elements for Boolean algebra and sub-algebras associated with subsets.
2. Physically this process could correspond to a sequence of Boolean measurements. Measure first the Boolean variable P_1 for the elements of set. After than measure whether Boolean variable P_2 is true for the subset for which P_1 is true and false. This gives decomposition of this set to n subsets defining a sequence of truth values (P_1=false, P_1=true and P_2=false,..., P_i=true, $i = 1, .., n - 3$ and P_{n-2}= true/false). The sets of decomposition are ordered with respect to the number of measured properties P_i and thus amount of information. It is not clear how unique this decomposition process is.

One can consider several physical realizations of the n-valued logics. An attractive idea is that all discrete quantum numbers could provide a realization for these logics.

1. 2-valued logic allows a natural fermionic realization. In positive energy ontology super-selection rule for fermion numbers makes this realization problematic but in zero energy ontology (ZEO) the problem is avoided. In ZEO one can interpret zero energy states as assigning to a quantum superposition of quantum Boolean statements represented by positive energy state similar similar superposition represented by negative energy state. Physical laws correspond to the conservation laws for various fermionic quantum numbers.
2. Pairs of space-time sheets connected by wormhole contact are fundamental in TGD: for instance, elementary particles correspond to this kind of structures. An interesting question is whether they could give rise to a geometric realization of Boolean logic.
3. n-valued logics could allow alternative realization realization in terms of algebraic extensions of rationals defined by roots of unity. In p-adic context 2-valued logic does not require extensions ($exp(i\pi) = -1$) and this could exclude their realization in this manner.

4. The inclusions of hyperfinite factors are labelled by n:th roots of unity with $n > 2$ and one can assign to this hierarchy Kac-Moody groups defined by simply laced Lie-groups which are excellent candidates for dynamical symmetries in TGD Universe.
   The hierarchy of Planck constants realized in terms of n-sheeted covering spaces could provide a realization of n-valued logic. The internal quantum numbers assignable to the internal dynamical symmetries would emerge as remnants of the huge super-symplectic symmetry [K23]. For preferred extremals a sub-algebra isomorphic to super-symmetric algebra and its commutator with super-symmetric algebra annihilate physical states and what is obtained is presumably Kac-Moody algebra fof simply laced Lie-group. For this algebra also classical Noether charges are non-vanishing. This would give additional spin like degrees of freedom and could allow to realize n-valued logics in terms of quantum states.

p-valued logic and hierarchy of partition algebras

As found, one can formally generalize Boolean logic to a logic in finite field $G(p)$ with p elements. p-Logics have very nice features. For a given set the p-Boolean algebra can be represented as maps having values in finite field $G(p)$. The subsets with a given value $0 \leq k < p$ define subsets of a partition and one indeed obtains p subsets some of which are empty unless the map is surjection.

The basic challenges are following: generalize logical negation and generalize Boolean operationsandOR. I have considered several options but the one based on category theoretical thinking seems to be the most promising one. One can imbed p_1-Boolean algebras to p-Boolean algebra by considering functions which have values in $G(p_1) \subset G(p)$. One can also project $G(p)$ valued functions to $G(p_1)$ by mod p_1 operation. The operations should respect the logical negation and p-Boolean operations if possible.

1. The basic question is how to define logical negation. Since 2-Boolean algebra is imbeddable to any p-Boolean algebra, it is natural to require that also in p-Boolean case the operation permute 0 and 1. These elements are also preferred elements algebraically since they are neutral elements for sum and product. This condition could be satisfied by simply defining negation as an operation leaving other elements of $G(p)$ un-affected. An alternative definition would be as shift $k \rightarrow k-1$. This is an attractive option since it corresponds to a cyclic symmetry.For $G(p)$ also higher powers of this operation would define analogs of negation in accordance with p-valuedness.
   I have considered also the possibility that for $p > 2$ the analog of logical negation could be defined as an additive inverse $k \rightarrow p-k$ in $G(p)$ and $k = p-1$ would be mapped to $k = 1$ as one might expect. The non-allowed value $k = 0$ is mapped to $k = p = 0$. $k = 0$ would be its own negation. This would suggest that $k = 0$ corresponds to an ill-defined truth value for $p > 2$. For $p = 2$ $k = 0$ must however correspond to false. This option is not however consistent with category theory inspired thinking.
2. For $G(p)$-valued functions f, one can define the p-analogs of both XOR (excluded or [(A OR B) but not (A AND B)] and using local sum and product for the everywhere-non-vanishing $G(p)$-valued functions. One can also define the analog of OR in terms of $f_1 + f_2 - f_1 f_2$ for arbitrary $G(p)$-valued functions. Note that minus sign is essential as one can see by considering $p = 3$ case ($1 + 1 - 1 \times 1 = 1$ and $1 + 1 + 1 \times 1 = 0$). For $p = 2$ this would give ordinary OR and it would be obviously non-vanishing unless both functions are identically zero. For $p > 2$ $AORB$ defined in this manner $f_1 + f_2 - f_1 f_2$ for functions having no zeros can however have zeros. The mod p_1 projection from $G(p) \rightarrow G(p_1)$ indeed commutes with these operations.
   Could 3-logic with 0 interpreted as ill-defined logical value serve as a representation of Boolean logic? This is not the case: $1 \times 2 = 2$ would correspond to $1 \times 0 = 0$ but $2 \times 2 = 1$ does not correspond to $0 \times 0 = 0$.
3. It would be nice to have well-defined inverse of Boolean function giving additional algebra structure for the partitions. For non-vanishing values of $f(x)$ one would have $(1/f)(x) = 1/f(x)$. How to define $(1/f)(x)$ for $f(x) = 0$? One can consider three options.
   (a) Option I: If 0 is interpreted as ill-defined value of p-Boolean function, there is a temptation to argue that the value of $1/f$ is also ill defined: $(1/f)(x) = 0$ for $f(x) = 0$. That function values would be replaced with their inverses only at points, where they are no-vanishing would conform with how ill-defined Boolean values are treated in computation. This leads

to a well-defined algebra structure but the inverse defined in this manner is only local inverse. One has $f \circ f^{-1}(x) = 1$ only for $f(x) \neq 0$. One has algebra but not a field.

(b) Option II: One could consider the extension of $G(p)$ by the inverse of 0, call it ∞, satisfying $0 \times \infty = 1$ ("false" AND ∞ = "true"!). Arithmetic intuition would suggest $k \times \infty = \infty$ for $k > 0$ and $k + \infty = \infty$ for all k.
On the other hand, the interpretation of + as XOR would suggest that $k + \infty$ corresponds to [(k OR ∞) but not (k AND ∞)=∞] suggesting $k + \infty = k$ so that 0 and ∞ would be in completely symmetrical position with respect to product and sum ($k\infty = k$ and $k+0 = k$; $k \times \infty = \infty$ and $k \times 0 = 0$). It would be nice to have a logical interpretation for the inverse and for the element ∞. Especially so in 2-Boolean case. A plausible looking interpretation of ∞ would be as "ill-defined" implying that [$k \sum \infty$] and [k AND ∞] is also "ill-defined". ["false" AND "ill-defined"]="true" sounds however strange.
For a set with N elements this would give a genuine field with $(p+1)^N$ elements. For the more convincing arithmetic option the outcome is completely analogous to the addition of point ∞ to real or complex numbers.

(c) Option III: One could also consider functions, which are non-vanishing at all points of the set are allowed. This function space is not however closed under summation.

4. For these three options one would have $K(N) = p^N$, $K(N) = (p+1)^N$ and $K(N) = (p-1)^N$ different maps of this kind having additive and multiplicative inverses. This hierarchy of statements about statements continues ad infinitum with $K(n) = K(K(n-1))$. For Option II this gives $M(n) = (p+1)^{M(n-1)}$ so that one does not obtain finite field $G(p, N)$ with p^N elements but function field.

5. One can also consider maps for which values are in the range $0 < k < p$. This set of maps would be however closed with respect to OR and would not obtain hierarchy of finite fields. In this case the interpretation of 0 would be is un-determined and for $p = 2$ this option would be trivial. For $p = 3$ one would have effectively two well-defined logic values but the algebra would not be equivalent with ordinary Boolean algebra.

The outcome for Option II would be a very nice algebraic structure having also geometric interpretation possibly interesting from the point of view of logic. p-Boolean algebra provides p-partitions with generalizations of XOR, OR, AND, negation, and finite field structure at each level of the hierarchy: kind of calculus for p-partitions.

The lowest level of the algebraic structure generalizes as such also to p-adic-valued functions in discrete or even continuous set. The negation fails to have an obvious generalization and the second level of the hierarchy would require defining functions in the infinite-D space of p-adic-valued functions.

p-Valued logics and error correction

Can one imagine any interpretation for the p-valued - and more generally - n-valued logics?

1. Error correction suggests a possible interpretation of p-valued logic. In quantum computation error correction poses conditions on the quantum states so that sub-space of all possible quantum states is realized. The idea is to check whether the state belongs to this space: if not, error has occurred and must be corrected.
In the same manner one could perhaps choose a n-element subset in n-bit Boolean algebra having $2^k > p$ elements by some constraints. Error correction algorithm would check whether the bit sequence belongs to this subset. The elements elements of k-bit Boolean algebra are labelled by integers $0, ...2^k - 1$ in a natural manner. Could the map $x \to x \mod n$ project these elements to elements of n-Boolean algebra? The elements $x \geq p$ would be mapped to same elements as x mod n or that only bit sequences $x < p$ are used. This would have a natural interpretation as pinary cutoff in p-adic topology. For some prime values of k dropping just the empty set gives Mersenne prime $M_k = 2^k - 1$ and M_k-valued logic would have a natural realization.

2. It seems that the error correction using n-valued logic does not allow a description in terms of Boolean ultrafilters and ideals for the full set. By studying the illustration of the Wikipedia article (see `http://tinyurl.com/hhvvpe4`) one can indeed get convinced that the number of elements for filters is power of two as one might expect from the logical consistency condition.

What about Stone spaces of reals, p-adic numbers, etc.?

Can one speculate anything interesting about the Boolean algebra and Stone spaces of *real line* , *complex numbers*, or *p-adic numbers*? TGD suggests two very interesting structures. Adeles and hierarchy of infinite primes (, integers and rationals). It however seems that adeles provide as coherent description of Stone space for the product of all p-valued logics so that only infinite primes [K84] are left under consideration.

1. Real numbers are in a well-defined sense at the same hierarchy level as p-adic number fields as extensions of rationals. This is suggested also by canonical identification mapping p-adics to reals.
2. In the case of real/p-adic numbers one would have possibily infinite sequences of real/p-adic numbers and one would map each such sequence to Z_p (a map from real line to Z_p). The map cannot be continuous in real topology.
   In the case of p-adic numbers one would have Stone space of Stone space. In the general p-adic case one would have q-valued statements about p-valued statements about natural numbers realized as collections of q subsets of p-adic numbers. A priori it is not necessary to have $q = p$ although internal consistency might demand this. This might help to get some grasp about the complexity involved.
   The set of Z_q valued maps forming q-ultra-filter is extremely large and expected to have naturally q-adic topology. What this monster could be? The "world of classical worlds" (WCW) and the generalization of the notion of real and p-adic number using the notion of algebraic holography suggested by the hierarchy of infinite primes is what comes in mind in TGD framework [K84].
   If it is possible to continue to make statements about statements indefinitely (we would represent rather low level in this hierarchy!), a hierarchical structure should be in question given p_n-Boolean algebras of p_{n-1}-Boolean algebras of.... At given level one has statements about statements of previous level that is Z_{p_n} valued maps from p_{n-1}-Boolean algebra having interpretation as subsets of p_{n-1}-Boolean Stone space /p_{n-1}-Boolean algebra. The first task is to try to identify a hierarchical abstraction structure and TGD Universe is indeed full of them.
3. *Infinite primes* (integers, and rationals) could define this kind of hierarchical structure [K84]. They are obtained by a repeated second quantization of an arithmetic QFT with supersymmetry. The single particle states at the lowest level are labelled by primes and are both bosons and fermions. Infinite primes correspond to both Fock states of free bosons and fermions and to analogs of bound states. These many-particle states define single particle states at the next level of hierarchy. Infinite primes are infinite only with respect to real norm. With respect to p-adic norms they have unit norm.
   By repeated second quantization infinite primes themselves form an infinite hierarchy mappable to polynomial primes at the first level of hierarchy: these irreducible polynomials depend on single variable only. At higher levels of hierarchy one has primes, which correspond to functions of $n > 1$ variables. There is resemblance with the statements about statements hierarchy of Boolean algebras but the correspondence is not so obvious. What is common that new level is constructed using primes of previous level as building bricks.
   The interpretation of finite fermionic part of infinite prime is as a Boolean statement with true assignable to a finite number of primes of the previous level. Besides this infinite primes contain analogs of n-boson Bose Einstein condensates in various modes labelled by primes serving as analogs of momenta. Their interpretation is open.
   The physical correlate for the hierarchy infinite primes could be the hierarchy of space-time sheets and would define a hierarchy of WCWs. At the level of logic one expects also a hierarchy. The attempt to identify somehow the Stone space as the space of infinite primes does not however look a promising idea. Could it be better to try to guess the hierarchy of Stone spaces?
4. Infinite primes lead to what might be called algebraic holography or algebraic Atman= Brahman identity [K84]. There exists a huge number of infinite integers, whose ratio equals to one as real number and has lower level p-adic norms equal to 1. These pairs of integers have also interpretation as analogs of zero energy states. Conservation of quantum numbers implying the vanishing of total quantum numbers for zero energy states would correspond to the fact that incoming and outgoing infinite integer have unit ratio in real topology although they have

different number theoretic anatomies.
The first thing to come in mind is to proceed using analogy. If p-adic number fields give Stone spaces for p-Boolean algebras of natural numbers then one might expect that the analogs of p-adic number fields for infinite primes - call them P - could give rise to Stone space for reals and p-adics. The pinary expansion of P-adic integer in power of infinite prime P however contains effectively only the lowest term for p-adic integers since already $O(P)$ term has p-adic norm $1/P = 0$. The second problem is how to make sense of the generalization of the condition $0 \leq k < p$ for the coefficients of the powers of p for infinite primes. On the other hand, infinite rationals with finite real norm make sense. This would suggest that infinite-P P-adic numbers are are just infinite-rationals of finite real norm.
Infinite rationals of unit norm can be interpreted in both real and p-adic senses and would be number theoretically universal. Finiteness condition and ZEO suggests that one could restrict the consideration to those infinite rationals for which the real norm and p-adic norms for lower level primes equals to one. Thus one would have huge space of real units.
One could replace both reals and p-adics and even adelics with the bundle with fiber formed by the huge infinite-D space of these units. This generalizes also to higher dimensional spaces. Could these bundles or their fiber spaces of be identifiable as Stone spaces for reals, p-adics, and adeles in a number theoretically universal manner? There would be infinite hierarchy of these spaces.
I have proposed earlier that this extension of embedding space and its p-adic and adelic variants could allow to realize WCW as fiber bundle with embedding space as base space. Could this hierarchy correspond to the hierarchy of Stone spaces assignable to reals, p-adics and adeles? The only new thing would be the replacement of space-time points with a space of real units, whose structure would not be visible in real number based space-time geometry and visible only via the number theoretical anatomy and via our ability to think mathematically. Single point of space-time would represent - if not entire WCW - at least some hierarchy levels of WCW. This opens up rather wild vision about what might be behind mathematical consciousness.

5. To make this really complicated, one can of course ask whether also infinite primes could contribute to adeles at higher hierarchy levels! The definition of p-adic number fields for infinite primes is problematic unless it is possible to make the p-adic norm finite.

Chapter 16

Langlands Program and TGD

16.1 Introduction

Langlands program [?]s an attempt to unify number theory and representation theory of groups and as it seems all mathematics. About related topics I know frustratingly little at technical level. Zeta functions and theta functions [?, ?] and more generally modular forms [?]re the connecting notion appearing both in number theory and in the theory of automorphic representations of reductive Lie groups. The fact that zeta functions have a key role in TGD has been one of the reasons for my personal interest.

The vision about TGD as a generalized number theory [?]ives good motivations to learn the basic ideas of Langlands program. I hasten to admit that I am just a novice with no hope becoming a master of the horrible technicalities involved. I just try to find whether the TGD framework could allow new physics inspired insights to Langlands program and whether the more abstract number theory relying heavily on the representations of Galois groups could have a direct physical counterpart in TGD Universe and help to develop TGD as a generalized number theory vision. After these apologies I however dare to raise my head a little bit and say aloud that mathematicians might get inspiration from physics inspired new insights.

The basic vision is that Langlands program could relate very closely to the unification of physics as proposed in TGD framework [?] TGD can indeed be seen both as infinite-dimensional geometry, as a generalized number theory involving several generalizations of the number concept, and as an algebraic approach to physics relying on the unique properties of hyper finite factors of type II_1 so that unification of mathematics would obviously fit nicely into this framework. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and sub-factors, and the notion of infinite prime, inspired a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

16.1.1 Langlands Program Very Briefly

Langlands program [?]tates that there exists a connection between number theory and automorphic representations of a very general class of Lie groups known as reductive groups (groups whose all representations are fully reducible). At the number theoretic side there are Galois groups characterizing extensions of number fields, say rationals or finite fields. Number theory involves also so called automorphic functions to which zeta functions carrying arithmetic information via their coefficients relate via so called Mellin transform $\sum_n a_n n^s \rightarrow \sum_n a_n z^n$ [?]

Automorphic functions, invariant under modular group $SL(2,Z)$ or subgroup $\Gamma_0(N) \subset SL(2,Z)$ consisting of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ c \ mod \ N = 0 \ .$$

emerge also via the representations of groups $GL(2,R)$. This generalizes also to higher dimensional groups $GL(n,R)$. The dream is that all number theoretic zeta functions could be understood in terms of representation theory of reductive groups. The highly non-trivial outcome would be possibility to deduce very intricate number theoretical information from the Taylor coefficients of these functions.

Langlands program relates also to Riemann hypothesis and its generalizations. For instance, the zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [?] in this case Riemann hypothesis has been proven.

It must be emphasized that algebraic 1-dimensionality is responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [?] In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

One might also conjecture that Langlands duality for Lie groups reflects some deep duality on physical side. For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in the framework of YM theories and string models. In particular, Witten proposes that electric-magnetic duality which indeed relates gauge group and its dual, provides a physical correlate for the Langlands duality for Lie groups and could be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [?] Interestingly, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT. In this chapter it will be proposed that super-symmetry might correspond to the Langlands duality in TGD framework.

16.1.2 Questions

Before representing in more detail the TGD based ideas related to Langlands correspondence it is good to summarize the basic questions which Langlands program stimulates.

Could one give more concrete content to the notion of Galois group of algebraic closure of rationals?

The notion of Galois group for algebraic closure of rationals $Gal(\overline{Q}/Q)$ is immensely abstract and one can wonder how to make it more explicit? Langlands program adopts the philosophy that this group could be defined only via its representations. The so called automorphic representations constructed in terms of adeles. The motivation comes from the observation that the subset of adeles consisting of Cartesian product of invertible p-adic integers is a structure isomorphic with the maximal abelian subgroup of $Gal(\overline{Q}/Q)$ obtained by dividing $Gal(\overline{Q}/Q)$ with its commutator subgroup. Representations of finite abelian Galois groups are obtained as homomorphisms mapping infinite abelian Galois group to its finite factor group. In this approach the group $Gal(\overline{Q}/Q)$ remains rather abstract and adeles seem to define a mere auxiliary technical tool although it is clear that so called l-adic representations for Galois groups are are natural also in TGD framework.

This raises some questions.

1. Could one make $Gal(\overline{Q}/Q)$ more concrete? For instance, could one identify it as an infinite symmetric group S_∞ consisting of finite permutations of infinite number of objects? Could one imagine some universal polynomial of infinite degree or a universal rational function resulting as ratio of polynomials of infinite degree giving as its roots the closure of rationals?
2. S_∞ has only single normal subgroup consisting of even permutations and corresponding factor group is maximal abelian group. Therefore finite non-abelian Galois groups cannot be represented via homomorphisms to factor groups. Furthermore, S_{infty} has only infinite-dimensional non-abelian irreducible unitary representations as a simple argument to be discussed later shows.

   What is highly non-trivial is that the group algebras of S_∞ and closely related braid group B_∞ define hyper-finite factors of type II_1 (HFF). Could sub-factors characterized by finite groups G allow to realize the representations of finite Galois groups as automorphisms p HFF? The interpretation would be in terms of "spontaneous symmetry breaking" $Gal(\overline{Q}/Q) \to G$. Could it be possible to get rid of adeles in this way?
3. Could one find a concrete physical realization for the action of S_∞? Could the permuted objects be identified as strands of braid so that a braiding of Galois group to infinite braid

group B_∞ would result? Could the outer automorphism action of Galois group on number theoretic braids defining the basic structure of quantum TGD allow to realize Galois groups physically as Galois groups of number theoretic braids associated with subset of algebraic points defined by the intersection of real and p-adic partonic 2-surface? The requirement that mathematics is able to represent itself physically would provide the reason for the fact that reality and various p-adicities intersect along subsets of rational and algebraic points only.

Could one understand the correspondences between the representations of finite Galois groups and reductive Lie groups?

Langlands correspondence involves a connection between the representations of finite-dimensional Galois groups and reductive Lie groups.

1. Could this correspondence result via an extension of the representations of finite groups in infinite dimensional Clifford algebra to those of reductive Lie groups identified for instance as groups defining sub-factors (any compact group can define a unique sub-factor)? If Galois groups and reductive groups indeed have a common representation space, it might be easier to understand Langlands correspondence.
2. Is there some deep difference between between general Langlands correspondence and that for $GL(2,F)$ and could this relate to the fact that subgroups of $SU(2)$ define sub-factors with quantized index $\mathcal{M}:\mathcal{N}\leq 4$.
3. McKay correspondence [**?**]elates finite subgroups of compact Lie groups to compact Lie group (say finite sub-groups of SU(2) to ADE type Lie-algebras or Kac-Moody algebras). TGD approach leads to a general heuristic explanation of this correspondence in terms of Jones inclusions and Connes tensor product. Could sub-factors allow to understand Langlands correspondence for general reductive Lie groups as both the fact that any compact Lie group can define a unique sub-factor and an argument inspired by McKay correspondence suggest.

Could one unify geometric and number theoretic Langlands programs?

There are two Langlands programs: algebraic Langlands program and geometric one [**?**]ne corresponding to ordinary number fields and function fields. The natural question is whether and how these approaches could be unified.

1. Could the discretization based on the notion of number theoretic braids induce the number theoretic Langlands from geometric Langlands so that the two programs could be unified by the generalization of the notion of number field obtained by gluing together reals with union of reals and various p-adic numbers fields and their extensions along common rationals and algebraics. Certainly the fusion of p-adics and reals to a generalized notion of number should be essential for the unification of mathematics.
2. Could the distinction between number fields and function fields correspond to two kinds of sub-factors corresponding to finite subgroups $G\subset SU(2)$ and SU(2) itself leaving invariant the elements of imbedded algebra? This would obviously generalize to imbeddings of Galois groups to arbitrary compact Lie group. Could gauge group algebras contra Kac Moody algebras be a possible physical interpretation for this. Could the two Langlands programs correspond to two kinds of ADE type hierarchies defined by Jones inclusions? Could minimal conformal field theories with finite number of primary fields correspond to algebraic Langlands and full string theory like conformal field theories with infinite number of primary fields to geometric Langlands? Could this difference correspond to sub-factors defined by disrete groups and Lie groups?
3. Could the notion of infinite rational [**?**]e involved with this unification? Infinite rationals are indeed mapped to elements of rational function fields (also algebraic extensions of them) so that their interpretation as quantum states of a repeatedly second quantized arithmetic super-symmetric quantum field theory might provide totally new mathematical insights.

Is it really necessary to replace groups $GL(n,F)$ with their adelic counterparts?

If the group of invertible adeles is not needed or allowed then a definite deviation from Langlands program is implied. It would seem that multiplicative adeles (ideles) are not favored by TGD view about the role of p-adic number fields. The l-adic representations of p-adic Galois groups corresponding to single p-adic prime l emerge however naturally in TGD framework.

1. The 2×2 Clifford algebra could be easily replaced with its adelic version. A generalization of Clifford algebra would be in question and very much analogous to $GL(2, A)$ in fact. The interpretation would be that real numbers are replaced with adeles also at the level of imbedding space and space-time. This interpretation does not conform with the TGD based view about the relationship between real and p-adic degrees of freedom. The physical picture is that H is 8-D but has different kind of local topologies and that spinors are in some sense universal and independent of number field.
2. WCW spinors define a hyper-finite factor of type II_1. It is not clear if this interpretation continues to make sense if configuration space spinors (fermionic Fock space) are replaced with adelic spinors. Note that this generalization would require the replacement of the group algebra of S_{infty} with its adelic counterpart.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L13]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

16.2 Basic Concepts And Ideas Related To The Number Theoretic Langlands Program

The basic ideas of Langlands program are following.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL(2, A)$ and more generally $GL(n, A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A58], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL(n, A)$.
2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL(n, A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n, R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1, .., n$ acting in group algebra of $Gl(n, R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.
3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_L(F)$ of $G(F)$.

16.2.1 Correspondence Between N-Dimensional Representations Of $Gal(\overline{F}/F)$ And Representations Of $Gl(N, A_F)$ In The Space Of Functions In $Gl(N, F)\backslash Gl(N, A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^\times$, where $Z_p^\times$ consists of invertible p-adic integers [A126].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p \in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^\times \backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^\times \backslash A_F^\times$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of* $Gal(\overline{F}/F)$ *correspond to representations of* $GL(1, A_F)$ *in the space of functions defined in* $GL(1, F)\backslash GL(1, A_F)$.

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of* $Gal(\overline{F}/F)$ *correspond to representations of* $GL(n, A_F)$ *in the space of functions defined in* $GL(n, F)\backslash GL(n, A_F)$.

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.
2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal embedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the embedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.
3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.
4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [A73]. In the case of elliptic curves [A126] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL(n, E)$.
   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative ways to state the same thing.

Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \to Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A126]. The representations can be also l-adic being defined in $GL(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A126] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p,1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.
2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^\times$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^\times|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A126] for the route from automorphic adelic representations of $GL(2,R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL(2,Q)$ are constructed in the space of smooth bounded functions $GL(2,Q)\backslash GL(2,A) \rightarrow C$ or equivalently in the space of $GL(2,Q)$ left-invariant functions in $GL(2,A)$. A denotes adeles and $GL(2,A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL(2,A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.
   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL(2,Z_p)$. For ramified primes K_p consists of $SL(2,Z_p)$ matrices with $c \in p^{n_p}Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.
2. The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$

   where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \ , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) \ .$$

3. The center $A^\times$ of $GL(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \rightarrow C$ is a character providing a multiplicative representation of $A^\times$.
4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g)du = 0$$

   is satisfied [A126]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the

modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.
The groups $gl(2, R)$, $O(2)$ and $GL(2, Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL(2, A_F) \times gl(2, R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL(2, R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL(2, Q)\backslash GL(2, A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2, R)$, where N is conductor [A126]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.
2. The group $\Gamma_0(N) \subset SL(2, Z)$ consists of matrices
$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ , \ c \ mod \ N = 0.$$
$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL(2, Z)$ so that also $SL(2, Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.
3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL(2, R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL(2, R)$ *to upper half-plane* $H_u = SL(2, R)/SO(2)$

The representations of $(gl(2, C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A126]. For the discrete series representation π giving square integrable representation in $SL(2, R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ , \ X_+ v_\infty = 0 \ .$$

The latter module is in turn generated by the lowest weight vector

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) v_\infty \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL(2, R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL(2, R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL(2, R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci + d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci + d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition X_+v_∞ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n \ . \tag{16.2.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL(2, Z_p)$ bi-invariant functions on $GL(2, Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A126].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g, Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

16.2.2 Some Remarks About The Representations Of $Gl(N)$ And Of More General Reductive Groups

The simplest representations of $Gl(n, R)$ have the property that the Borel group B of upper diagonal matrices is mapped to diagonal matrices consisting of character ξ which decomposes to a product of characters χ_k associated with diagonal elements b_k of B defining homomorphism

$$b_k \rightarrow sgn(b)^{m(k)}|b_k|^{ia_k}$$

to unit circle if a_k is real. Also more general, non-unitary, characters can be allowed. The representation itself satisfies the condition $f(bg) = \chi(b)f(g)$. Thus n complex parameters a_k defining a reducible representation of $C^\times$ characterize the irreducible representation.

In the case of $GL(2, R)$ one can consider also genuinely two-dimensional discrete series representations characterized by only single continuous parameter and the previous example represented just this case. These representations are square integrable in the subgroup $SL(2, R)$. Their origin is related to the fact that the algebraic closure of R is 2-dimensional. The so called Weil group W_R which is semi-direct product of complex conjugation operation with $C^\times$ codes for this number theoretically. The 2-dimensional representations correspond to irreducible 2-dimensional representations of W_R in terms of diagonal matrices of $Gl(2, C)$.

In the case of $GL(n,R)$ the representation is characterized by integers n_k: $\sum n_k = n$ characterizing the dimensions $n_k = k = 1,2$ of the representations of W_R. For $Gl(n,C)$ one has $n_k = 1$ since Weil group W_C is obviously trivial in this case.

In the case of a general reductive Lie group G the homomorphisms of W_R to the Langlands dual G_L of G defined by replacing the roots of the root lattice with their duals characterize the automorphic representations of G.

The notion of Weil group allows also to understand the general structure of the representations of $GL(n,F)$ in $GL(n,K)$, where F is p-adic number field and K its extension. In this case Weil group is a semi-direct product of Galois group of $Gal(K/F)$ and multiplicative group $K^\times$. A very rich structure results since an infinite number of extensions exists and the dimensions of discrete series representations.

The deep property of the characterization of representations in terms of Weil group is functoriality. If one knows the homomorphisms $W_F \rightarrow G$ and $G \rightarrow H$ then the composite homomorphism defines an automorphic representation of H. This means that irreps of G can be passed to those of H by homomorphism [A125] .

16.3 TGD Inspired View About Langlands Program

In this section a general TGD inspired vision about Langlands program is described. If is of course just a bundle of physics inspired ideas represented in the hope that real professionals might find some inspiration. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and their sub-factors, and the notion of infinite prime, lead to a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

16.3.1 What Is The Galois Group Of Algebraic Closure Of Rationals?

Galois group is essentially the permutation group for the roots of an irreducible polynomial. It is a subgroup of symmetric group S_n, where n is the degree of polynomial. One can also imagine the notion of Galois group $Gal(\overline{Q}/Q)$ for the algebraic closure of rationals but the concretization of this notion is not easy.

$Gal(\overline{Q}/Q)$ as infinite permutation group?

The maximal abelian subgroup of $Gal(\overline{Q}/Q)$, which is obtained by dividing with the normal subgroup of even permutations, is identifiable as a product of multiplicative groups $Z_p^\times$ of invertible p-adic integers $n = n_0 + pZ$, $n_0 \in \{1,..p-1\}$ for all p-adic primes and can be understood reasonably via its isomorphism to the product $\hat{Z} = \prod_p Z_p$ of multiplicative groups Z_p of invertible p-adic integers, one factor for each prime p [A125, K46, A126].

Adeles [A4] are identified as the subring of $(\hat{Z} \otimes_Z Q) \times R$ containing only elements for which the elements of Q_p belong to Z_p except for a finite number of primes so that the number obtained can be always represented as a product of element of $\hat{Z}$ and point of circle R/Z: $A = \hat{Z} \times R/Z$. Adeles define a multiplicative group $A^\times$ of ideles and $GL(1,A)$ allow to construct representations $Gal(Q^{ab}/Q)$.

It is much more difficult to get grasp on $Gal(\overline{Q}/Q)$. The basic idea of Langlands program is that one should try to understand $Gal(\overline{Q}/Q)$ through its representations rather than directly. The natural hope is that n-dimensional representations of $Gal(\overline{Q}/Q)$ could be realized in $GL(n,A)$.

1. $Gal(\overline{Q}/Q)$ as infinite symmetric group?

One could however be stubborn and try a different approach based on the direct identification $Gal(\overline{Q}/Q)$. The naïve idea is that $Gal(\overline{Q}/Q)$ could in some sense be the Galois group of a polynomial of infinite degree. Of course, for mathematical reasons also a rational function defined as a ratio of this kind of polynomials could be considered so that the Galois group could be assigned to both zeros and poles of this function. In the generic case this group would be an infinite symmetric group S_∞ for an infinite number of objects containing only permutations for subsets containing a finite number of objects. This group could be seen as the first guess for $Gal(\overline{Q}/Q)$.

S_∞ can be defined by generators e_m representing permutation of $m^{th}$and $(m+1)^{th}$ object satisfying the conditions

$$\begin{aligned} e_m e_m &= e_n e_m \text{ for } |m-n|>1 , \\ e_n e_{n+1} e_n &= e_n e_{n+1} e_n e_{n+1} \text{ for } n = 1,...,n-2 , \\ e_n^2 &= 1 . \end{aligned} \quad (16.3.1)$$

By the definition S_∞ can be expected to possess the basic properties of finite-dimensional permutation groups. Conjugacy classes, and thus also irreducible unitary representations, should be in one-one correspondence with partitions of n objects at the limit $n \to \infty$. Group algebra defined by complex functions in S_∞ gives rise to the unitary complex number based representations and the smallest dimensions of the irreducible representations are of order n and are thus infinite for S_∞. For representations based on real and p-adic number based variants of group algebra situation is not so simple but it is not clear whether finite dimensional representations are possible.

S_n and obviously also S_∞ allows an endless number of realizations since it can act as permutations of all kinds of objects. Factors of a Cartesian and tensor power are the most obvious possibilities for the objects in question. For instance, S_n allows a representation as elements of rotation group $SO(n)$ permuting orthonormalized unit vectors e_i with components $(e_i)^k = \delta_i^k$. This induces also a realization as spinor rotations in spinor space of dimension $D = 2^{d/2}$.

2. Group algebra of S_∞ as HFF

The highly non-trivial fact that the group algebra of S_∞ is hyper-finite factor of type II_1 (HFF) [A43] suggests a representation of permutations as permutations of tensor factors of HFF interpreted as an infinite power of finite-dimensional Clifford algebra. The minimal choice for the finite-dimensional Clifford algebra is $M^2(C)$. In fermionic Fock space representation of infinite-dimensional Clifford algebra e_i would induce the transformation $(b^\dagger_{m,i}, b^\dagger_{m,i+1}) \to (b^\dagger_{m,i+1}, b^\dagger_{m,i})$. If the index m is lacking, the representation would reduce to the exchange of fermions and representation would be abelian.

3. Projective representations of S_∞ as representations of braid group B_∞

S_n can be extended to braid group B_n by giving up the condition $e_i^2 = 1$ for the generating permutations of the symmetric group. Generating permutations are represented now as homotopies exchanging the neighboring strands of braid so that repeated exchange of neighboring strands induces a sequence of twists by π. Projective representations of S_∞ could be interpreted as representations of B_∞. Note that odd and even generators commute mutually and for unitary representations either of them can be diagonalized and are represented as phases $exp(i\phi)$ for braid group. If $exp(i\phi)$ is not a root of unity this gives effectively a polynomial algebra and the polynomials subalgebras of these phases might provide representations for the Hecke operators also forming commutative polynomial algebras.

The additional flexibility brought in by braiding would transform Galois group to a group analogous to homotopy group and could provide a connection with knot and link theory [A127] and topological quantum field theories in general [A186]. Finite quantum Galois groups would generate braidings and a connection with the geometric Langlands program where Galois groups are replaced with homotopy groups becomes suggestive [A126, A124].

4. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is an entire infinity of infinities in number theoretical sense. After all, any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

The group algebra of Galois group of algebraic closure of rationals as hyper-finite factor of type II_1

The most natural framework for constructing unitary irreducible representations of Galois group is its group algebra. In the recent case this group algebra would be that for S_∞ or B_∞ if braids are allowed. What puts bells ringing is that the group algebra of S_∞ is a hyper-finite factor of type II_1 isomorphic as a von Neumann algebra to the infinite-dimensional Clifford algebra [A43], which in turn is the basic structures of quantum TGD whose localized version might imply entire quantum TGD. The very close relationship with the braid group makes it obvious that same holds true for corresponding braid group B_∞. Indeed, the group algebra of an infinite discrete group defines under very general conditions HFF. One of these conditions is so called amenability [A6]. This correspondence gives hopes of understanding the Langlands correspondence between representations of discrete Galois groups and the representations of $GL(n, F)$ (more generally representations of reductive groups).

Thus it seems that WCW spinor s (fermionic Fock space) could naturally define a finite-dimensional spinor representation of finite-dimensional Galois groups associated with the number theoretical braids. Inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realize the notion of finite measurement resolution and give rise to finite dimensional representations of finite groups G leaving elements of $\mathcal{N}$ invariant. An attractive idea is that these groups are identifiable as Galois groups.

The identification of the action of G on $\mathcal{M}$ as homomorphism $G \rightarrow Aut(\mathcal{M})$ poses strong conditions on it. This is discussed in the thesis of Jones [C3] which introduces three algebraic invariants for the actions of finite group in hyperfinite-factors of type II_1, denoted by $\mathcal{M}$ in the sequel. In general the action reduces to inner automorphism of $\mathcal{M}$ for some normal subgroup $H \subset G$: this group is one of the three invariants of G action. In general one has projective representation for H so that one has $u_{h_1}u_{h_2} = \mu(h_1, h_2)u_{h_1h_2}$, where $\mu(h_1)$ is a phase factor which satisfies cocycle conditions coming from associativity.

1. The simplest action is just a unitary group representation for which $g \in G$ is mapped to a unitary operator u_g in $\mathcal{M}$ acting in $\mathcal{M}$ via adjoint action $m \rightarrow u_g m u_g^\dagger = Ad(u_g)m$. In this case one has $H = G$. In this case the fixed point algebra does not however define a factor and there is no natural reduction of the representations of $Gal(\overline{Q}/Q)$ to a finite subgroup.
2. The exact opposite of this situation outer action of G mean $H = \{e\}$. All these actions are conjugate to each other. This gives gives rise to two kinds of sub-factors and two kinds of representations of G. Both actions of Galois group could be realized either in the group or braid algebra of $Gal(\overline{Q}/Q)$ or in infinite dimensional Clifford algebra. In neither case the action be inner automorphic action $u \rightarrow gug^\dagger$ as one might have naïvely expected. This is crucial for circumventing the difficulty caused by the fact that $Gal(\overline{Q}/Q)$ identified as S_∞ allows no finite-dimensional complex representation.
3. The first sub-factor is $\mathcal{M}^G \subset \mathcal{M}$ corresponding, where the action of G on $\mathcal{M}$ is outer. Outer action defines a fixed point algebra for all finite groups G. For $D = \mathcal{M} : \mathcal{N} < 4$ only finite subgroups $G \subset SU(2)$ would be represented in this manner. The index identifiable as the fractal dimension of quantum Clifford algebra having $\mathcal{N}$ as non-abelian coefficients is $D = 4cos^2(\pi/n)$. One can speak about quantal representation of Galois group. The image of Galois group would be a finite subgroup of $SU(2)$ acting as spinor rotations of quantum Clifford algebra (and quantum spinors) regarded as a module with respect to the included algebra invariant under inner automorphisms. These representations would naturally correspond to 2-dimensional representations having very special role for the simple reason that the algebraic closure of reals is 2-dimensional.
4. Second sub-factor is isomorphic to $\mathcal{M}^G \subset (\mathcal{M} \otimes L(H))^G$. Here $L(H)$ is the space of linear operators acting in a finite-dimensional representation space H of a unitary irreducible representation of G. The action of G is a tensor product of outer action and adjoint action. The index of the inclusion is $dim(H)^2 \geq 1$ [A177] so that the representation of Galois group can be said to be classical (non-fractal).
5. The obvious question is whether and in what sense the outer automorphisms represent Galois subgroups. According to [C3] the automorphisms belong to the completion of the group of inner automorphisms of HFF. Identifying HFF as group algebra of S_∞, the interpretation would be that outer automorphisms are obtained as diagonal embeddings of Galois group to

$S_n \times S_n \times ....$ If one includes only a finite number of these factors the outcome is an inner automorphisms so that for all finite approximations inner automorphisms are in question. At the limit one obtains an automorphisms which does not belong to S_∞ since it contains only finite permutations. This identification is consistent with the identification of the outer automorphisms as diagonal embedding of G to an infinite tensor power of sub-Clifford algebra of Cl_∞.

This picture is physically very appealing since it means that the ordering of the strands of braid does not matter in this picture. Also the reduction of the braid to a finite number theoretical braid at space-time level could be interpreted in terms of the periodicity at quantum level. From the point of view of physicist this symmetry breaking would be analogous to a spontaneous symmetry breaking above some length scale L. The cutoff length scale L would correspond to the number N of braids to which finite Galois group G acts and corresponds also to some p-adic length scale.

One might hope that the emergence of finite groups in the inclusions of hyper-finite factors could throw light into the mysterious looking finding that the representations of finite Galois groups and unitary infinite-dimensional automorphic representations of $GL(n, R)$ are correlated by the connection between the eigenvalues of Frobenius element Fr_p on Galois side and eigenvalues of commuting Hecke operators on automorphic side. The challenge would be to show that the action of Fr_p as outer automorphism of group algebra of S_∞ or B_∞ corresponds to Hecke algebra action on configuration space spinor fields or in modular degrees of freedom associated with partonic 2-surface.

Could there exist a universal rational function having $Gal(\overline{Q}/Q)$ as the Galois group of its zeros/poles?

The reader who is not fascinated by the rather speculative idea about a universal rational function having $Gal(\overline{Q}/Q)$ as a permutation group of its zeros and poles can safely skip this subsection since it will not be needed anywhere else in this chapter.

1. Taking the idea about permutation group of roots of a polynomial of infinite order seriously, one could require that the analytic function defining the Galois group should behave like a polynomial or a rational function with rational coefficients in the sense that the function should have an everywhere converging expansion in terms of products over an infinite number of factors $z - z_i$ corresponding to the zeros of the numerator and possible denominator of a rational function. The roots z_i would define an extension of rationals giving rise to the entire algebraic closure of rationals. This is a tall order and the function in question should be number theoretically very special.
2. One can speculate even further. TGD has inspired the conjecture that the non-trivial zeros $s_n = 1/2 + iy_n$ of Riemann zeta [A99] (assuming Riemann hypothesis) are algebraic numbers and that also the numbers p^{s_n}, where p is any prime, and thus local zeta functions serving as multiplicative building blocks of ζ have the same property [K77]. The story would be perfect if these algebraic numbers would span the algebraic closure of rationals.
   The symmetrized version of Riemann zeta defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfying the functional equation $\xi(s) = \xi(1-s)$ and having only the trivial zeros could appear as a building block of the rational function in question. The function

   $$f(s) = \frac{\xi(s)}{\xi(s+1)} \times \frac{s-1}{s}$$

   has non-trivial zeros s_n of ζ as zeros and their negatives as $-s_n$ as poles. There are no other zeros since trivial zeros as well as the zeros at $s = 0$ and $s = 1$ are eliminated. Using Stirling formula one finds that $\xi(s)$ grows as s^s for real values of $s \rightarrow \infty$. The growths of the numerator and denominator compensate each other at this limit so that the function approaches constant equal to one for $Re(s) \rightarrow \infty$.
   If $f(s)$ indeed behaves as a rational function whose product expansion converges everywhere it can be expressed in terms of its zeros and poles as

$$f(s) = \prod_{n>0} A_n(s) \ ,$$
$$A_n = \frac{(s-s_n)(s-\overline{s}_n)}{(1+s-s_n)(1+s-\overline{s}_n)} \ . \quad (16.3.2)$$

The product expansion seems to converge for any finite value of s since the terms A_n approach unity for large values of $|s_n| = |1/2+iy_n|$. $f(s)$ has $s_n = 1/2+iy_n$ indeed has zeros and $s_n = -1/2+iy_n$ as poles.

3. This proposal might of course be quite too simplistic. For instance, one might argue that the phase factors p^{iy} associated with the non-trivial zeros give only roots of unity multiplied by Gaussian integers. One can however imagine more complex functions obtained by forming products of $f(s)$ with its shifted variants $f(s+\Delta)$ with algebraic shift Δ in, say, the interval $[-1/2, 1/2]$. Some kind of limiting procedure using a product of this kind of functions might give the desired universal function.

16.3.2 Physical Representations Of Galois Groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by discretization of continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [K23, K22, K85] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [K23, K22]. The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire WCW of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

1. It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian

subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [A108] the non-trivial normal subgroups are fintary alternating subgroup A_∞ and finitary symmetric group consisting if finitary permutations. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

2. The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [A96] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal embedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

About the detailed definition of number theoretic braids

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [K32]. The hierarchy is realized via a generalization of the notion of embedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of embedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_\pm \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

1. What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [K60] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations

to Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_\pm$).
3. Another manner to deal with CP_2 is to apply number $M^8 - H$ duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

2. The identification of number theoretic braids

The identification of number theoretic braids is not by no means a trivial task [K100, K69]. As a matter fact, there are several alternative identifications and it seems that all of them are needed. Consider first just braids without the attribute "number theoretical".

1. Braids could be identified as lifts of the projections of X^3_l to the quantum critical sub-manifolds M^2 or S^2_I, $i = I, II$, and in the generic case consist of 1-dimensional strands in X^3_l These sub-manifolds are obviously in the same role as the plane to which the braid is projected to obtain a braid diagram. This requires that a unique identification of the slicing of space-time surfaces by 3-surfaces.
2. Braid points are always quantum critical against the change of Planck constant so that TQFT like theory characterizes the freedom remaining intact at quantum criticality. Quantum criticality in this sense need not have anything to do with the quantum criticality in the sense that the second variation of Kähler action vanishes -at least for the variations representing dynamical symmetries in the sense that only the inner product $\int (\partial L_D/\partial h^k_\alpha)\delta h^k d^4x$ (L_D denotes Kähler-Dirac Lagrangian) without the vanishing of the integrand. This criticality leads to a generalization of the conceptual framework of Thom's catastrophe theory [K100].
The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$.
3. It is not clear whether these three braids form some kind of trinity so that one of them is enough to formulate the theory or whether all of them are needed. Note also that one has quantum superposition over CDs corresponding to different choices of M^2 and the pair formed by S^2_I and S^2_{II} (note that the spheres are not independent if both appear). Quantum measurement however selects one of these choices since it defines the choice of quantization axes.
4. One can consider also more general definition. The extrema of Kähler magnetic field strength defined as coordinate invariant $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define in natural manner a discrete set of points defining the nodes of symplectic triangulation: note that this involves division with metric determinant in preferred coordinates. This set of extremals is same for all deformations of X^3_l allowed in the functional integral over symplectic group although the positions of points change. For preferred symplectically invariant light-like coordinate of X^3_l braid results. Also now geodesic spheres and M^2 would define the counterpart of the plane to which the braids are projected.
5. A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K45] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of CDs and wormhole throats would define space-like and time-like braids uniquely.
The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A128] to TGD framework. It leads to the identification of slicing by 3-surfaces as that induced by the inverse images of $r = constant$

surfaces of CP_2, where r is $U(2)$ invariant radial coordinate of CP_2 playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing CD and would have the choice of braids and string world sheets as a space-time correlate. $r = \infty$ points correspond to three homologically non-trivial geodesic spheres S^2 analogous to North and South poles of CP_2 and the projections to M^4 and S^2 define braid projections. Braid strands could be interpreted as orbits of Kähler charged particle in Kähler magnetic field and enclosing fractional Kähler flux.
The beauty of this identification is that one starts from braids at the ends of space-time surface partonic 2-surfaces at boundaries of CD and from intersection of braid points and determines space-time surface and string world sheets from this data in accordance with holography and quantum classical correspondence. This picture conforms also with the recent view about Kähler-Dirac equation for which the construction of solutions leads to the notion of braid too.

Number theoretic braids would be braids which are number theoretically critical. This means that the points of braid in preferred coordinates are algebraic points so that they can be regarded as being shared by real partonic 2-surface and its p-adic counterpart obeying same algebraic equations.

Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyper-finite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [A138]. Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [A138] !

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [A6], group defines a unique sub-factor by outer action [A138]. It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands [B46] [A126, A163], whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. Number theoretic braids belong to the intersection of real and p-adic partonic surface

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [K23]).

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r,z,w) = 0$. By substituting w as a polynomial $w = Q(z,r)$ of z and r this gives polynomial equation $P(r,z,Q(z,r)) = 0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and WCW spinor s.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y - x^2 = 0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Kähler-Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the Kähler-Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [K23, L54]. One can solve the roots of the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

1. The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.

2. For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X^2_k of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X^2_k. This means separation of modular degrees of freedom to those assignable to X^2_k and to "center of mass" modular degrees of freedom assignable to the boundaries between X^2_k. Only the piece X^2_k associated with the k^{th} braid would be affected non-trivially by the Galois group of braid. The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [A58] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [A125, A126, L77]. This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [L77]. Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \rightarrow (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a way that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [K21] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

Hierarchy of Planck constants and dark matter and generalization of embedding space

Second hierarchy of candidates for Galois groups is based on the generalization of the notion of the embedding space $H = M^4 \times CP_2$, or rather the spaces $H_\pm = M^4_\pm \times CP_2$ defining future and past light-cones inside H [K32]. This generalization is inspired by the quantization of Planck constant explaining dark matter as a hierarchy of macroscopically quantum coherent phases and by the requirement that sub-factors have a geometric representation at the level of the embedding space and space-time (quantum-classical correspondence).

Galois groups could also correspond to finite groups $G_a \times G_b \subset SU(2) \times SU(2) \subset$SL(2, C)$\times SU(3)$. These groups act as covering symmetries for the sectors of the embedding space, which can be regarded as singular $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm/G_a \times G_b$ bundles containing orbifold points (fixed points of $G_a \times G_b$ or either of them. The copies of H with same G_a or G_b are glued together along $M^4_\pm$ or CP_2 factor and along common orbifold points left fixed by G_b or G_a. The group $G_a \times G_b$ plays both the role of both Galois group and homotopy group.

There are good reasons to expect that both these Galois groups and those associated with number theoretic braids play a profound role in quantum TGD based description of dark matter as macroscopically quantum coherent phases. For instance, G_a would appear as symmetry group of dark matter part of bio-molecules in TGD inspired biology [L4].

Question about representations of finite groups

John Baez made an interesting question in n-Category-Cafe [A105]. The question reads as follows:

Is every representation of every finite group definable on the field Q^{ab} obtained by taking the field Q of rational numbers and by adding all possible roots of unity?

Since every finite group can appear as Galois group the question translates to the question whether one can represent all possible Galois groups using matrices with elements in Q^{ab}.

This form of question has an interesting relation to Langlands program. By Langlands conjecture the representations of the Galois group of algebraic closure of rationals can be realized in the space of functions defined in $GL(n,F)\backslash GL(n,Gal(Q^{ab}/Q))$, where $Gal(Q^{ab}/Q)$ is the maximal Abelian subgroup of the Galois group of the algebraic closure of rationals. Thus one has group algebra associated with the matrix group for which matrix elements have values in $Gal(Q^{ab}/Q)$. Something by several orders of more complex than matrices having values in Q^{ab}.

Suppose that Galois group of algebraic numbers can be regarded as the permutation group S_∞ of infinite number of objects generated by permutations for finite numbers of objects and that its physically interesting representations reduce to the representations of finite Galois groups G with element $g \in G$ represented as infinite product $g \times g \times ...$ belonging to the completion of S_∞ and thus to the completion of its group algebra identifiable as hyper-finite factor of type II_1. This would mean number theoretic local gauge invariance in the sense that all elements of S_∞ would leave physical states invariant whereas G would correspond to global gauge transformations. These tensor factors would have as space-time correlates number theoretical braids allowing to represent the action of G.

What this has then to do with John's question and Langlands program? S_∞ contains any finite group G as a subgroup. If all the representations of finite-dimensional Galois groups could be realized as representations in $Gl(n,Q^{ab})$, same would hold true also for the proposed symmetry breaking representations of the completion of S_∞ reducing to the representations of finite Galois groups. There would be an obvious analogy with Langlands program using functions defined in the space $Gl(n,Q)\backslash Gl(n,Gal(Q^{ab}/Q))$. Be as it may, mathematicians are able to work with incredibly abstract objects! A highly respectful sigh is in order!

16.3.3 What Could Be The TGD Counterpart For The Automorphic Representations?

The key question in the following is whether quantum TGD could act as a general math machine allowing to realize any finite-dimensional manifold and corresponding function space in terms of configuration space spinor fields and whether also braided representations of Galois groups accompanying the braiding could be associated naturally with this kind of representations.

Some general remarks

Before getting to the basic idea some general remarks are in order.

1. WCW spinor fields would certainly transform according to a finite-dimensional and therefore non-unitary representation of $SL(2,C)$ which is certainly the most natural group involved and should relate to the fact that Galois groups representable as subgroups of SU(2) acting as rotations of 3-dimensional space correspond to sub-factors with $\mathcal{M}:\mathcal{N} \leq 4$.
2. Also larger Lie groups can be considered and diagonal embeddings of Galois groups would be naturally accompanied by diagonal embeddings of compact and also non-compact groups acting on the decomposition of infinite-dimensional Clifford algebra Cl_∞ to an infinite tensor power of finite-dimensional sub-Clifford algebra of form $M(2,C)^n$.
3. The basic difference between Galois group representation and corresponding Lie group representations is that the automorphisms in the case of discrete groups are automorphisms of S_∞ or B_∞ whereas for Lie groups the automorphisms are in general automorphisms of group algebra of S_∞ or B_∞. This could allow to understand the correspondence between discrete groups and Lie groups naturally.
4. Unitary automorphic representations are infinite-dimensional and require group algebra of $GL(n,F)$. Therefore WCW spinors - to be distinguished from WCW spinor fields- cannot realize them. WCW spinor field might allow the realization of these infinite-dimensional representations if groups themselves allow a finite-dimensional geometric realization of groups. Are this kind of realizations possible? This is the key question.

Could TGD Universe act as a universal math machine?

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of WCW spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X_l^3? Note that this would assign to the representations of closed paths in the group manifold a representation of braid group and Galois group of the braid and might make it easier to understand the Langlands correspondence.

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H = \delta M^4_\pm \times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but $E^{2n}/S_n = C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4 = M^2 \times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.
2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of embedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.
3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the embedding space one can realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.
4. WCW spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If WCW spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the WCW spinor fields in $SU(3)$ or as C^3/Z by requiring that WCW spinor field is scale invariant. Projective spaces might be also realized more concretely as embeddings to $(CP_2)^n$.
5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X_l^3 taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.
6. Braiding in X_l^3 would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.
7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X_l^3 correspond to zero modes assignable to Kac-Moody symmetries [K24, K85]. Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X_l^3 correlates with the quantum numbers assigned with X^2 so that Boolean statements

represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \to B_i$ representing various instances of the general rule $A \to B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \to B)$ would produce from a given state a zero energy state representing statement $A \to B$. If the measurement of observable $O(C \to D)$ affects this state then the statement $(A \to B) \to (C \to D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \to B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

16.3.4 Super-Conformal Invariance, Modular Invariance, And Langlands Program

The geometric Langlands program [A126, A124] deals with function fields, in particular the field of complex rational analytic functions on 2-dimensional surfaces. The sheaves in the moduli spaces of conformal blocks characterizing the n-point functions of conformal field theory replaces automorphic functions coding both arithmetic data and characterizing the modular representations of $GL(n)$ in number theoretic Langlands program [A126]. These moduli spaces are labelled both by moduli characterizing the conformal equivalence class of 2-surface, in particular the positions of punctures, in TGD framework the positions of strands of number theoretic braids, as well as the moduli related to the Kac-Moody group involved.

Transition to function fields in TGD framework

According to [A126] conformal field theories provide a very promising framework for understanding geometric Langlands correspondence.

1. That the function fields on 2-D complex surfaces would be in a completely unique role mathematically fits nicely with the 2-dimensionality of partons and well-defined stringy character of anti-commutation relations for induced spinor fields. According to [A126] there are not even conjectures about higher dimensional function fields.
2. There are very direct connections between hyper-finite factors of type II_1 and topological QFTs [A186, A127], and conformal field theories. For instance, according to the review [H1] [A138] Ocneacnu has show that Jones inclusions correspond in one-one manner to topological quantum field theories and TGD can indeed be regarded as almost topological quantum field theory (metric is brought in by the light-likeness of partonic 3-surfaces). Furthermore, Connes has shown that the decomposition of the hierarchies of tensor powers $\mathcal{M} \otimes_{\mathcal{N}} .... \otimes_{\mathcal{N}} M$ as left and right modules to representations of lower tensor powers directly to fusion rules expressible in terms of 4-point functions of conformal field theories [A138].

In TGD framework the transition from number fields to function fields would not be very dramatic.

1. Suppose that the representations of $SL(n, R)$ occurring in number theoretic Langlands program can indeed be realized in the moduli space for conformal equivalence classes of partonic 2-surface (or, by previous arguments, moduli space for regions of them with fixed boundaries). This means that representations of local Galois groups associated with number theoretic braids would involve global data about entire partonic 2-surface. This is physically very important since it otherwise discretization would lead to a loss of the information about dimension of partonic 2-surfaces.
2. In the case of geometric Langlands program this moduli space would be extended to the moduli space for n-point functions of conformal field theory defined at these 2-surfaces containing the original moduli space as a subspace. Of course, the extension could be present also in the number theoretic case. Thus it seems that number theoretic and geometric Langlands

programs would utilize basic structures and would differ only in the sense that single braid would be replaced by several braids in the geometric case.

3. In TGD Kac-Moody algebras would be also present as well as the so called super-symplectic algebra [K23] related to the isometries of "the world of classical worlds" (the space of light-like 3-surfaces) with generators transforming according to the irreducible representations of rotation group $SO(3)$ and color group $SU(3)$. It must be emphasized that TGD view about conformal symmetry generalizes that of string models since light-like 3-surfaces (orbits of partons) are the basic dynamical objects [K23].

What about more general reductive groups?

Langlands correspondence is conjectured to apply to all reductive Lie groups. The question is whether there is room for them in TGD Universe. There are good hopes.

1. Pairs formed by finite Galois groups and Lie groups containing them and defining sub-factors

Any amenable (in particular compact Lie) group acting as outer automorphism of $\mathcal{M}$ defines a unique sub-factor $\mathcal{N} \subset \mathcal{M}$ as a group leaving the elements of $\mathcal{N}$ invariant. The representations of discrete subgroups of compact groups extended to representations of the latter would define natural candidates for Langlands correspondence and would expand the repertoire of the Galois groups representable in terms of unique factors. If one gives up the uniqueness condition for the sub-factor, one can expect that almost any Lie group can define a sub-factor.

2. McKay correspondences and inclusions

The so called McKay correspondence assigns to the finite subgroups of SU(2) extended Dynkin diagrams of ADE type Kac-Moody algebras. McKay correspondence also generalizes to the discrete subgroups of other compact Lie groups q [A190]. The obvious question is how closely this correspondence between finite groups and Lie groups relates with Langlands correspondence.

The principal graphs representing concisely the fusion rules for Connes tensor products of $\mathcal{M}$ regarded as $\mathcal{N}$ bi-module are represented by the Dynkin diagrams of ADE type Lie groups for $\mathcal{M} : \mathcal{N} < 4$ (not all of them appear). For index $\mathcal{M} : \mathcal{N} = 4$ extended ADE type Dynkin diagrams labelling Kac-Moody algebras are assigned with these representations.

I have proposed that TGD Universe is able to emulate almost any ADE type gauge theory and conformal field theory involving ADE type Kac-Moody symmetry and represented somewhat misty ideas about how to construct representations of ADE type gauge groups and Kac-Moody groups using many particle states at the sheets of multiple coverings $H \rightarrow H/G_a \times G_b$ realizing the idea about hierarchy of dark matters already mentioned. Also vertex operator construction also distinguishes ADE type Kac-Moody algebras in a special position.

It is possible to considerably refine this conjecture picture by starting from the observation that the set of generating elements for Lie algebra corresponds to a union of triplets $\{J_i^{\pm}, J_i^3\}$, $i = 1, ..., n$ generating SU(2) sub-algebras. Here n is the dimension of the Cartan sub-algebra. The non-commutativity of quantum Clifford algebra suggests that Connes tensor product can induce deformations of algebraic structures so that ADE Lie algebra could result as a kind of deformation of a direct sum of commuting SU(2) Lie (Kac-Moody) algebras associated with a Connes tensor product. The physical interpretation might in terms of a formation of a bound state. The finite depth of $\mathcal{N}$ would mean that this mechanism leads to ADE Lie algebra for an n-fold tensor power, which then becomes a repetitive structure in tensor powers. The repetitive structure would conform with the diagonal embedding of Galois groups giving rise to a representation in terms of outer automorphisms.

This picture encourages the guess that it is possible to represent the action of Galois groups on number theoretic braids as action of subgroups of dynamically generated ADE type groups on configuration space spinors. The connection between the representations of finite groups and reductive Lie groups would result from the natural extension of the representations of finite groups to those of Lie groups.

3. What about Langlands correspondence for Kac-Moody groups? vm

The appearance of also Kac-Moody algebras raises the question whether Langlands correspondence could generalize also to the level of Kac-Moody groups or algebras and whether it could be easier to understand the Langlands correspondence for function fields in terms of Kac-Moody groups as the transition from global to local occurring in both cases suggests.

Could Langlands duality for groups reduce to super-symmetry?

Langlands program involves dualities and the general structure of TGD suggests that there is a wide spectrum of these dualities.

1. A very fundamental duality would be between infinite-dimensional Clifford algebra and group algebra of S_∞ or of braid group B_∞. For instance, one can ask could it be possible to map this group algebra to the union of the moduli spaces of conformal equivalence classes of partonic 2-surfaces. HFFs consists of bounded operators of a separable Hilbert space. Therefore they are expected to have very many avatars: for instance there is an infinite number sub-factors isomorphic to the factor. This seems to mean infinite number of ways to represent Galois groups reflected as dualities.
2. Langlands program involves the duality between reducible Lie groups G and its Langlands dual having dual root lattices. The interpretation for this duality in terms of electric-magnetic duality is suggested by Witten [A163]. TGD suggests an alternative interpretation. The super symmetry aspect of super-conformal symmetry suggests that bosonic and fermionic representations of Galois groups could be very closely related. In particular, the representations in terms of WCW spinor s and in terms of modular degrees of freedom of partonic 2-surface could be in some sense dual to each other. Rotation groups have a natural action on WCW spinor s whereas symplectic groups have a natural action in the moduli spaces of partonic 2-surfaces of given genus possessing symplectic and Kähler structure. Langlands correspondence indeed relates $SO(2g+1,R)$ realized as rotations of WCW spinor s and $Sp(2g,C)$ realized as transformations in modular degrees of freedom. Hence one might indeed wonder whether super-symmetry could be behind the Langlands correspondence.

16.3.5 What Is The Role Of Infinite Primes?

Infinite primes primes at the lowest level of the hierarchy can be represented as polynomials and as rational functions at higher levels. These in turn define rational function fields. Physical states correspond in general to infinite rationals which reduce to unit in real sense but have arbitrarily complex number theoretical anatomy [K84], [L3, L6].

Does infinite prime characterize the l-adic representation of Galois group associated with given partonic 2-surface

Consider first the lowest level of hierarchy of infinite primes [K84]. Infinite primes at the lowest level of hierarchy are in a well-defined sense composites of finite primes and correspond to states of super-symmetric arithmetic quantum field theory. The physical interpretation of primes appearing as composites of infinite prime is as characterizing of the p-adic prime p assigned by the Kähler-Dirac action to partonic 2-surfaces associated with a given 3-surface [K100, K23].

This p-adic prime could naturally correspond to the possible prime associated with so called l-adic representations of the Galois group(s) associated with the p-adic counterpart of the partonic 2-surface. Also the Galois groups associated with the real partonic 2-surface could be represented in this manner. The generalization of moduli space of conformal equivalence classes must be generalized to its p-adic variant. I have proposed this generalization in context of p-adic mass calculations [K21].

It should be possible to identify WCW spinor s associated with real and p-adic sectors if anti-commutations relations for the fermionic oscillator operators make sense in any number field (that is involve only rational or algebraic numbers). Physically this seems to be the only sensible option.

Could one assign Galois groups to the extensions of infinite rationals?

A natural question is whether one could generalize the intuitions from finite number theory to the level of infinite primes, integers, and rationals and construct Galois groups and there representations for them. This might allow alternative very number theoretical approach to the geometric Langlands duality.

1. The notion of infinite prime suggests that there is entire hierarchy of infinite permutation groups such that the N_∞ at given level is defined as the product of all infinite integers at that level. Any group is a permutation group in formal sense. Could this mean that the hierarchy of infinite primes could allow to interpret the infinite algebraic sub-groups of Lie groups as Galois groups? If so one would have a unification of group theory and number theory.
2. An interesting question concerns the interpretation of the counterpart of hyper-finite factors of type II_1 at the higher levels of hierarchy of infinite primes. Could they relate to a hierarchy of local algebras defined by HFF? Could these local algebras be interpreted in terms of direct integrals of HFFs so that nothing essentially new would result from von Neumann algebra point of view? Would this be a correlate for the fact that finite primes would be the irreducible building block of all infinite primes at the higher levels of the hierarchy?
3. The transition from number fields to function fields is very much analogous to the replacement of group with a local gauge group or algebra with local algebra. I have proposed that this kind of local variant based on multiplication by of HFF by hyper-octonion algebra could be the fundamental algebraic structure from which quantum TGD emerges. The connection with infinite primes would suggest that there is infinite hierarchy of localizations corresponding to the hierarchy of space-time sheets.
4. Perhaps it is worth of mentioning that the order of S_∞ is formally $N_\infty = \lim_{n\rightarrow\infty} n!$. This integer is very large in real sense but zero in p-adic sense for all primes. Interestingly, the numbers $N_\infty/n + n$ behave like normal integers in p-adic sense and also number theoretically whereas the numbers $N_\infty/n + 1$ behave as primes for all values of n. Could this have some deeper meaning?

Could infinite rationals allow representations of Galois groups?

One can also ask whether infinite primes could provide representations for Galois groups. For instance, the decomposition of infinite prime to primes (or prime ideals) assignable to the extension of rationals is expected to make sense and would have clear physical interpretation. Also (hyper-)quaternionic and (hyper-)octonionic primes can be considered and I have proposed explicit number theoretic interpretation of the symmetries of standard model in terms of these primes. The decomposition of partonic primes to hyper-octonionic primes could relate to the decomposition of parton to regions, one for each number theoretic braid.

There are arguments supporting the view that infinite primes label the ground states of super-conformal representations [K23, K84]. The question is whether infinite primes could allow to realize the action of Galois groups. Rationality of infinite primes would imply that the invariance of ground states of super-conformal representations under the braid realization of $Gal(\overline{Q}/Q)$ of finite Galois groups. The infinite prime as a whole could indeed be invariant but the primes in the decomposition to a product of primes in algebraic extension of rationals need not be so. This kind of decompositions of infinite prime characterizing parton could correspond to the above described decomposition of partonic 2-surface to regions X^2_k at which Galois groups act non-trivially. It could also be that only infinite integers are rational whereas the infinite primes decomposing them are hyper-octonionic. This would physically correspond to the decomposition of color singlet hadron to colored partons [K84].

16.3.6 Could Langlands Correspondence, Mckay Correspondence And Jones Inclusions Relate To Each Other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having

Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [A138]. For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence q [A190] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

About McKay correspondence

McKay correspondence q [A190] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows q [A190].

1. Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.
2. Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j - 1/2$, and $j + 1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j + 1/2, j + 1, ...$

   One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n ($SO(2n + 1)$), C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M} : \mathcal{N} < 4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [A84].

1. The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2
2. The classification of simply laced semisimple Lie groups.
3. The classification of finite sub-groups of the 3-dimensional rotation group.
4. The classification of simple singularities. In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.
5. The classification of tame quivers.

Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

1. The so called principal graphs characterizing $\mathcal{M} : \mathcal{N} = 4$ Jones inclusions for $G = SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n \geq 4$.

2. $\mathcal{M} : \mathcal{N} < 4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n ($SU(n)$), D_{2n} ($SO(2n)$), and E_6 and E_8. Thus D_{2n+1} ($SO(2n+2)$) and E_7 are not allowed. For instance, for $G = S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

1. The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N} \subset \mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has $\mathcal{N}' \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}^1 \subset ...$ and $C = \mathcal{M}' \cap \mathcal{M} \subset \mathcal{M}' \cap \mathcal{M}^1 \subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}' \cap \mathcal{M}^k \subset \mathcal{N}' \cap \mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras Templieb assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [A111].
2. The information provided by infinite Bratteli diagram can be coded by a much simpler bipartite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

1. It is natural to decompose the Connes tensor powers q [A190] $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ to irreducible $\mathcal{M}-\mathcal{M}$, $\mathcal{N}-\mathcal{M}$, $\mathcal{M}-\mathcal{N}$, or $\mathcal{N}-\mathcal{N}$ bi-modules. If $\mathcal{M} : \mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.
2. If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M} - \mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N} - \mathcal{N}$ representations resulting in the decomposition of $\mathcal{M} - \mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [K32] is that in $\mathcal{M} : \mathcal{N} < 4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M} : \mathcal{N} = 4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \to H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

1. McKay correspondence for subgroups of G ($\mathcal{M} : \mathcal{N} = 4$) *resp.* its variants ($\mathcal{M} : \mathcal{N} < 4$) and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.
2. Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

1. The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2) \otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$ module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.
2. This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of WCW spinor s extend to the representations of SU(2) naturally.
3. Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \to H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

WCW spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between $(\mathcal{M} : \mathcal{N} = 4)$ and $(\mathcal{M} : \mathcal{N} < 4)$ cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in $(\mathcal{M} : \mathcal{N} = 4)$ case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \to G_a \times G_b$ bundle structure would be $H \to H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups q [A190]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

7. Flavor groups of hadron physics as a support for HFF?

The deformation assigning to an n-fold tensor power of representations of Lie group G with k-dimensional Cartan algebra a representation of a Lie group with nk-dimensional Cartan algebra could be also seen as a dynamically generated symmetry. If quantum measurement is characterized by the choice of Lie group G defining measured quantum numbers and defining Jones inclusion characterizing the measurement resolution, the measurement process itself would generate these dynamical symmetries. Interestingly, the flavor symmetry groups of hadron physics cannot be justified from the structure of the standard model having only electro-weak and color group as fundamental symmetries. In TGD framework flavor group $SU(n)$ could emerge naturally as a fusion of n quark doublets to form a representation of $SU(n)$.

Conformal representations of braid group and a possible further generalization of McKay correspondence

Physically especially interesting representations of braid group and associated Temperley-Lieb-Jones algebras (TLJ) are representations provided by the n-point functions of conformal field theories studied in [A183]. The action of the generator of braid group on n-point function corresponds to a duality transformation of old-fashioned string model (or crossing) represented as a monodromy relating corresponding conformal blocks. This effect can be calculated. Since the index $r = \mathcal{M} : \mathcal{N}$ appears as a parameter in TLJ algebra, the formulas expressing the behavior of n-point functions under the duality transformation reveal also the value of index which might not be easy to calculate otherwise.

Note that in TGD framework the arguments of n-point function would correspond to the strands of the number theoretic braid and thus to the points of the geodesic sphere S^2 associated with the light-cone boundary $\delta M^4_\pm$. The projection to the geodesic sphere of CP_2 projection would be same for all these strands.

WZW model for group G and Kac-Moody central charge k quantum phase is discussed in [A183]. The non-triviality of braiding boils to the fact that quantum group G_q defines the effect of braiding operation. Quantum phase is given as $q = exp(i\pi/(k+C(G))$, where $C(G)$ is the value of Casimir operator in adjoint representation. The action of the braid group generator reduces to the unitary matrix relating the basis defined by the tensor product of representations of G_q to the basis obtained by application of a generator of the braid group. For n-point functions of primary fields belonging to a representation D of G, index is the square of the quantum dimension $d_q(D)$ of the corresponding representation of G_q. Hence each primary field correspond to its own inclusion of HFF, which corresponds to $n \rightarrow \infty$-point function.

The result could have been guessed as the dimension of quantum Clifford algebra emerging naturally in inclusion when HFF is represented as an infinite tensor power of $M(d(D), C)$. For $j = 1/2$ representation of $SU(2)$ standard Jones inclusions with $r < 4$ are obtained. The resulting inclusion is irreducible ($\mathcal{N}' \cap \mathcal{M} = C$, where $\mathcal{N}'$ is the commutator of $\mathcal{N}'$). Using the representation of HFF as infinite tensor power of $M(2, C)$ the result would not be so easy to understand.

The mathematical challenge would be to understand how the representations HFF as an infinite tensor power of $M(n, C)$ relate to each other for different values of n. It might be possible to understand the relationship between different infinite tensor power representations of HFF by representing $M(n_1, C)$ as a sub-algebra of a tensor power of a finite tensor power of $M(n_2, C)$. Perhaps a detailed construction of the maps between representations of HFF as infinite tensor power of $M(n, C)$ for various values of n could reveal further generalizations of McKay correspondence.

16.3.7 Technical Questions Related To Hecke Algebra And Frobenius Element

Frobenius elements

Frobenius element Fr_p is mapped to a conjugacy class of Galois group using the decomposition of prime p to prime ideals in the algebraic extension K/F.

1. At the level of braid group Frobenius element Fr_p corresponds to some conjugacy class of Galois group acting imbedded to S_n (only the conjugacy equivalence class is fixed) and thus can be mapped to an element of the braid group. Hence it seems possible to assign to Fr_p an element of infinitely cyclic subgroup of the braid group.
2. One can always reduce in given representation the element of given conjugacy class to a diagonal matrix so that it is possible to chose the representatives of Fr_p to be commuting operators. These operators would act as a spinor rotation on quantum Clifford algebra elements defined by Jones inclusion and identifiable as element of some cyclic group of the group G defining the sub-factor via the diagonal embedding.
3. Fr_p for a given finite Galois group G should have representation as an element of braid group to which G is imbedded as a subgroup. It is possible to chose the representatives of Fr_p so that they commute. Could one chose them in such a way that they belong to the commuting subgroup defined by even (odd) generators e_i? The choice of representatives for Fr_p for various Galois groups must be also consistent with the hierarchies of intermediate extensions of rationals associated with given extension and characterized by subgroups of Galois group for the extension.

How the action of commutative Hecke algebra is realized in hyper-finite factor and braid group?

One can also ask how to imbed Hecke algebra to the braid algebra. Hecke algebra for a given value of prime p and group $GL(n, R)$ is a polynomial algebra in Hecke algebra generators. There is a fundamental difference between Hecke algebra and Frobenius element Fr_p in the sense that Fr_p has finite order as an element of finite Galois group whereas Hecke algebra elements do not except possibly for representations. This means that Hecke algebra cannot have a representation in a finite Galois groups.

Situation is different for braid algebra generators since they do not satisfy the condition $e_i^2 = 1$ and odd and even generators of braid algebra commute. The powers of Hecke algebra generators would correspond to the powers of basic braiding operation identified as a π twist of neighboring strands. For unitary representations eigenvalues of e_i are phase factors. Therefore Hecke algebra might be realized using odd or even commuting sub-algebra of braid algebra and this could allow to deduce the Frobenius-Hecke correspondence directly from the representations of braid group. The basic questions are following.

1. Is it possible to represent Hecke algebra as a subalgebra of braid group algebra in some natural manner? Could the infinite cyclic group generated by braid group image of Fr_p belong represent element of Hecke algebra fixed by the Langlands correspondence? If this were the case then the eigenvalues of Frobenius element Fr_p of Galois group would correspond to the eigen values of Hecke algebra generators in the manner dictated by Langlands correspondence.
2. Hecke operators $H_{p,i}$, $i = 1, .., n$ commute and expressible as two-side cosets in group $GL(n, Q_p)$. This group acts in $\mathcal{M}$ and the action could be made rather explicit by using a proper representations of $\mathcal{M}$ (note however that physical situation can quite well distinguish between various representations). Does the action of the Hecke sub-algebra fixed by Hecke-Frobenius correspondence co-incide with the action of Frobenius element Fr_p identified as an element of braid sub-group associated with some cyclic subgroup of the Galois group identified as a group defining the sub-factor?

16.4 Langlands Conjectures And The Most Recent View About TGD

Langlands program (see http://tinyurl.com/q7x85j9) [K46, A126, A124] relies on very general conjectures about a connection between number theory and harmonic analysis relating the representations of Galois groups with the representations of certain kinds of Lie groups to each other. Langlands conjecture has many forms and it is indeed a conjecture and many of them are imprecise since the notions involved are not sharply defined.

Peter Woit (see http://tinyurl.com/3j6c98k) noticed that Edward Frenkel had given a talk with rather interesting title "What do Fermat's Last Theorem and Electro-magnetic Duality Have in Common?" (see http://tinyurl.com/ydeq4zue) [A94] ? I listened the talk and found it very inspiring. The talk provides bird's eye of view about some basic aspects of Langlands program using the language understood by physicist. Also the ideas concerting the connection between Langlands duality and electric-magnetic duality generalized to S-duality in the context of non-Abelian gauge theories and string theory context and developed by Witten and Kapustin [A163] and followers are summarized. In this context $D = 4$ and twisted version of $\mathcal{N} = 4$ SYM familiar from twistor program and defining a topological QFT appears.

For some years ago I made my first attempt to understand what Langlands program is about and tried to relate it to TGD framework [K46]. At that time I did not really understand the motivations for many of the mathematical structures introduced. In particular, I did not really understand the motivations for introducing the gigantic Galois group of algebraic numbers regarded as algebraic extension of rationals.

1. Why not restrict the consideration to finite Galois groups [A33] or their braided counterparts (as I indeed effectively did [K46])? At that time I concentrated on the question what the enormous Galois group of algebraic numbers regarded as algebraic extension of rationals could mean, and proposed that it could be identified as a symmetric group consisting of permutations of infinitely many objects. The definition of this group is however far from trivial. Should one allow as generators of the group only the permutations affecting only finite number of objects or permutations of even infinite number of objects?
The analogous situation for the sequences of binary digits would lead to a countable set of sequence of binary digits forming a discrete set of finite integers in real sense or to 2-adic integers forming a 2-adic continuum. Something similar could be expected now. The physical constraints coming the condition that the elements of symmetric group allow lifting to braidings suggested that the permutations permuting infinitely many objects should be periodic meaning that the infinite braid decomposes to an infinite number of identical N-braids and braiding is same for all of them. The p-adic analog would be p-adic integers, which correspond to rationals having periodical expansion in powers of p. Braids would be therefore like pinary digits. I regarded this choice as the most realistic one at that time. I failed to realize the possibility of having analogs of p-adic integers by general permutations. In any case, this observation makes clear that the unrestricted Galois group is analogous to a Lie group in topology analogous to p-adic topology rather than to discrete group. Neither did I realize that the Galois groups could be finite and be associated with some other field than rationals, say a Galois group associated with the field of polynomials of n-variable with rational coefficients and with its completion with coefficients replaced by algebraic numbers.
2. The ring of adeles (see http://tinyurl.com/64pgerm) [A4] can be seen as a Cartesian product of non-vanishing real numbers $R_\times$ with the infinite Cartesian product $\prod Z_p$ having as factors p-adic integers Z_p for all values of prime p. Rational adeles are obtained by replacing R with rationals Q and requiring that multiplication of rational by integers is equivalent with multiplication of any Z_p with rational. Finite number of factors in Z_p can correspond to Q_p: this is required in to have finite adelic norm defined as the product of p-adic norms. This definition implicitly regards rationals as common to all number fields involved. At the first encounter with adeles I did not realize that this definition is in spirit with the basic vision of TGD.
The motivation for the introduction of adele is that one can elegantly combine the algebraic groups assignable to rationals (or their extensions) and all p-adic number fields or even more general function fields such as polynomials with some number of argument at the same time

as a Cartesian product of these groups as well as to finite fields. This is indeed needed if one wants to realize number theoretic universality which is basic vision behind physics as generalized number theory vision. This approach obviously means enormous economy of thought irrespective of whether one takes adeles seriously as a physicist.

In the following I will discuss Taniyama-Shimura-Weil theorem and Langlands program from TGD point view.

16.4.1 Taniyama-Shimura-Weil Conjecture From The Perspective Of TGD

Taniyama-Shimura-Weil theorem

It is good to consider first the Taniyama-Shimura-Weil conjecture (see http://tinyurl.com/y8n9czrm) [A82] from the perspective provided by TGD since this shows that number theoretic Langlands conjecture could be extremely useful for practical calculations in TGD framework.

1. Number theoretical universality requires that physics in real number field and various p-adic number fields should be unified to a coherent hole by a generalization of the notion of number: different number fields would be like pages of book intersecting along common rationals. This would hold true also for space-time surfaces and embedding space but would require some preferred coordinates for which rational points would determined the intersection of real and p-adic worlds. There are good reasons for the hypothesis that life resides in the intersection of real and p-adic worlds.
   The intersection would correspond at the level of partonic 2-surfaces rational points of these surfaces in some preferred coordinates, for which a finite-dimensional family can be identified on basis of the fundamental symmetries of the theory. Allowing algebraic extensions one can also consider also some algebraic as common points. In any case the first question is to count the number of rational points for a partonic 2-surface.
   2-dimensional Riemann surfaces serve also as a starting point of number theoretic Langlands problem and the same is true for the geometric Langlands program concentrating on Riemann surfaces and function fields defined by holomorphic functions.
2. The number theoretic side of Taniyama-Shimura-Weil (TSW briefly) theorem for elliptic surfaces, which is essential for the proof of Fermat's last theorem, is about counting the integer (or equivalently rational) points of the elliptic surfaces

$$y^2 = x^3 + ax + b \ , \quad a, b \in Z \ .$$

   The theorem relates number theoretical problem to a problem of harmonic analysis, which is about group representations. What one does is to consider the above Diophantine equation modulo p for all primes p. Any solution with finite integers smaller than p defines a solution in real sense if $mod\ p$ operation does not affect the equations. Therefore the existence of a finite number of solutions involving finite integers in real sense means that for large enough p the number a_p of solutions becomes constant.
3. On harmonic analysis one studies so called modular forms $f(\tau)$, where τ is a complex coordinate for upper half plane defining moduli space for the conformal structures on torus. Modular forms have well defined transformation properties under group $Gl_2(R)$: the action is defined by the formula $\tau \rightarrow (a\tau+b)/(c\tau+d)$. The action of $Gl_2(Z)$ or its appropriate subgroup is such that the modular form experiences a mere multiplication by a phase factor: $D(hk) = c(h, k)D(h)D(k)$. The phase factors obey cocycle conditions $D(h, k)D(g, hk) = D(gh, k)D(g, h)$ guaranteeing the associativity of the projective representation.
   Modular transformations are clearly symmetries represented projectively as quantum theory indeed allows to do. The geometric interpretation is that one has projective representations in the fundamental domain of upper plane defined by the identification of the points differing by modular transformations. In conformally symmetric theories this symmetry is essential. Fundamental domain is analogous to lattice cell. One often speaks of cusp forms: cusp forms vanish at the boundary of the fundamental domain defined as the quotient of the upper half plane by a subgroup -call it Γ of the modular group $Sl_2(Z)$. The boundary corresponds to $Im(\tau) \rightarrow \infty$ or equivalently $q = exp(i2\pi\tau) \rightarrow 0$.

Remark: In TGD framework modular symmetry says that elementary particle vacuum functionals are modular invariants. For torus one has the above symmetry but for Riemann surface with higher genus modular symmetries correspond to a subgroup of $Sl_{2g}(Z)$.

4. One can expand the modular form as Fourier expansion using the variable $q = exp(i2\pi\tau)$ as

$$f(\tau) = \sum_{n>0} b_n q^n \quad .$$

$b_1 = 1$ fixes the normalization. $n > 0$ in the sum means that the form vanishes at the boundary of the fundamental domain associated with the group Γ. The TSW theorem says that for prime values $n = p$ one has $b_p = a_p$, where a_p is the number of mod p integer solutions to the equations defining the elliptic curve. At the limit $p \to \infty$ one obtains the number of real actual rational points of the curve if this number is finite. This number can be also infinite. The other coefficients b_n can be deduced from their values for primes since b_n defines what is known as a multiplicative character in the ring of integers implying $b_{mn} = b_m b_n$ meaning that b_n obeys a decomposition analogous to the decomposition of integer into a product of primes. The definition of the multiplicative character is extremely general: for instance it is possible to define quantum counterparts of multiplicative characters and of various modular forms by replacing integers with quantum integers defined as products of quantum primes for all primes except one -call it p_0, which is replaced with its inverse: this definition of quantum integer appears in the deformation of distributions of integer valued random variable characterized by rational valued parameters and is motivated by strange findings of Shnoll [K5]. The interpretation could be in terms of TGD based view about finite measurement resolution bringing in quantum groups and also preferred p-adic prime naturally.

5. TSW theorem allows to prove Fermat's last theorem: if the latter theorem were wrong also TSW theorem would be wrong. What also makes TSW theorem so wonderful is that it would allow to count the number of rational points of elliptic surfaces just by looking the properties of the automorphic forms in $Gl_2(R)$ or more general group. A horrible looking problem of number theory is transformed to a problem of complex analysis which can be handled by using the magic power of symmetry arguments. This kind of virtue does not matter much in standard physics but in quantum TGD relying heavily on number theoretic universality situation is totally different. If TGD is applied some day the counting of rational points of partonic surfaces is everyday practice of theoretician.

How to generalize TSW conjecture?

The physical picture of TGD encourages to imagine a generalization of the Tanyama-Shimura-Weil conjecture.

1. The natural expectation is that the conjecture should make sense for Riemann surfaces of arbitrary genus g instead of $g = 1$ only (elliptic surfaces are tori). This suggests that one should one replace the upper half plane representing the moduli space of conformal equivalence classes of toric geometries with the $2g$-dimensional (in the real sense) moduli space of genus g conformal geometries identifiable as Teichmüller space (see `http://tinyurl.com/bzxdqlz`) . This moduli space has symplectic structure analogous to that of $g+g$-dimensional phase space and this structure relates closely to the cohomology defined in terms of integrals of holomorphic forms over the $g+g$ cycles which each handle carrying two cycles. The moduli are defined by the values of the holomorphic one-forms over the cycles and define a symmetric matrix Ω_{ij} (modular parameters), which is modular invariant [K21]. The modular parameters related $Sp_2g(Z)$ transformation correspond to same conformal equivalence class.
If Galois group and effective symmetry group G are representable as symplectic flows at the light-like boundary of $CD(\times CP_2)$, their action automatically defines an action in the moduli space. The action can be realized also as a symplectic flow defining a braiding for space-like braids assignable to the ends of the space-time surface at boundaries of CD or for time-like braids assignable to light-like 3-surfaces at which the signature of the induced metric changes and identified as orbits of partonic 2-surfaces analogous to black hole horizons.

2. It is possible to define modular forms also in this case. Most naturally they correspond to theta functions used in the construction of elementary particle functionals in this space [K21].

Siegel modular forms (see http://tinyurl.com/ybatvjul) transform naturally under the symplectic group $Sp_{2g}(R)$ and are projectively invariant $Sp_{2g}(Z)$. More general moduli spaces are obtained by allowing also punctures having interpretation as the ends of braid strands and very naturally identified as the rational points of the partonic 2-surface. The modular forms defined in this extended moduli space could carry also information about the number of rational points in the same manner as the automorphic representations of $Gl_2(R)$ carry information about the number of rational points of elliptic curves.

3. How Tanyama-Shimura-Weil conjecture should be generalized? Also now one can consider power series of modular forms with coefficients b_n defining multiplicative characters for the integers of field in question. Also now the coefficients a_p could give the number of integer/rational points of the partonic 2-surface in mod p approximation and at the limit $p \to \infty$ the number of points a_p would approach to a constant if the number of points is finite.
4. The only sensical interpretation is that the analogs of elementary particle vacuum functionals [K21] identified as modular forms must be always restricted to partonic 2-surfaces having the same number of marked points identifiable as the end points of braid strands rational points. It also seems necessary to assume that the modular forms factorize to a products of two parts depending on Teichmüller parameters and positions of punctures. The assignment of fermionic and bosonic quantum numbers with these points conforms with this interpretation. As a special case these points would be rational. The surface with given number or marked points would have varying moduli defined by the conformal moduli plus the positions of the marked points. This kind of restriction would be physically very natural since it would mean that only braids with a given number of braid strands ending at fixed number of marked points at partonic 2-surfaces are considered in given quantum state. Of course, superpositions of these basis states with varying braid number would be allowed.

16.4.2 Unified Treatment Of Number Theoretic And Geometric Langlands Conjectures In TGD Framework

One can already now wonder what the relationship of the TGD view about number theoretic Langlands conjecture to the geometric Langlands conjecture could be?

1. The generalization of Taniyama-Shinamure-Weil theorem to arbitrary genus would allow to deduce the number of rational points already for finite but large enough values of p from the Taylor coefficients of an appropriate modular form. Is this enough for the needs of TGD? The answer is "No". One must be able to count also numbers of "rational 2-surfaces" in the space of 2-surfaces and the mere generalization of TSW conjecture does not allow this. Geometric Langlands replacing rational points with "rational" surfaces is needed.
   If the geometric Langlands conjecture holds true in the spirit with TGD, it must allow to deduce the number of rational variants of of partonic 2-surfaces assignable to given quantum state defined to be a state with fixed number of braid strands for each partonic 2-surface of the collection. What is new is that collections of partonic 2-surfaces regarded as sub-manifolds of $M^4 \times CP_2$ are considered.
2. Finite measurement resolution conjectured to be definable in terms of effective symmetry group G defined by the inclusion of hyper-finite factors of type II_1 [K99] (HFFs in the sequel) effectively replaces partonic 2-surfaces with collections of braid ends and the natural idea is that the orbits of these collections under finite algebraic subgroup of symmetry group defining finite measurement resolution gives rise to orbit with finite number of points (point understood now as collection of rational points). The TGD variant of the geometric Langlands conjecture would allow to deduce the number of different collections of rational braid ends for the quantum state considered (one particular WCW spinor field) from the properties of automorphic form.
3. Quantum group structure is associated with the inclusions of HFFs, with braid group representations, integrable QFTs, and also with the quantum Yangian symmetry [A173, A147] suggested strongly by twistor approach to TGD. In zero energy ontology physical states define Lie-algebra and the multi-locality of the scattering amplitudes with respect to the partonic 2-surfaces (that is at level of WCW) suggests also quantum Yangian symmetry. Therefore the Yangian of the Kac-Moody type algebra defining measurement resolution is a natural candidate for the symmetry considered. What is important is that the group structure is associated

with a finite-dimensional Lie group.

This picture motivates the question whether number theoretic and geometric Langlands conjecture could be realized in the same framework? Could electric-magnetic duality generalized to S-duality imply these dualities and bring in the TGD counterpart of effective symmetry group G in some manner. This framework would be considerably more general than the 4-D QFT framework suggested by Witten and Kapustin (see `http://tinyurl.com/y9duma5u`) [A163] and having very close analogies with TGD view about space-time.

The following arguments support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The basic notions are following.

1. Zero energy ontology and the related notion of causal diamond CD (CD is short hand for the cartesian product of causal diamond of M^4 and of CP_2). This notion leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet.
2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.
3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [K99]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations ("symmetry group" hitherto). At space-time level the finite measurement resolution would be represented in terms of braids at space-time level which come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.
   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multilocality and therefore quantum Yangian algebra with multilocal generators is unavoidable.
   In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.
4. The natural identification of the associate of G is as quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized.
5. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation

as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

6. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [K24] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.
7. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program [A163] is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

Number theoretic Langlands conjecture in TGD framework

Number theoretic Langlands conjecture generalizes TSW conjecture to a duality between two kinds of groups.

1. At the number theoretic side of the duality one has an n-dimensional representation of Galois group for the algebraic numbers regarded as algebraic extension of rationals. In the more general case one can consider arbitrary number field identified as algebraic extension of rationals. One can assign to the number field its rational adele. In the case of rationals this brings in both real numbers and p-adic numbers so that huge amount of information can be packed to the formulas. For anyone who has not really worked concretely with number theory it is difficult to get grasp of the enormous generality of the resulting theory.
2. At the harmonic analysis side of the conjecture one has n-dimensional representation of possibly non-compact Lie group G and its Langlands dual (see `http://tinyurl.com/yclcloaj`) ^LG appearing also in the non-Abelian form of electric-magnetic duality. The idea that electric-magnetic duality generalized to S-duality could provide a physical interpretation of Langlands duality is suggestive. $U(n)$ is self dual in Langlands sense but already for $G = SU(3)$ one has $^LG = SU(3)/Z_3$. For most Lie groups the Lie algebras of G and ^LG are identical but even the Lie algebras can be different. $Gl_2(R)$ is replaced with any reductive algebraic group and in the matrix representation of the group the elements of the group are replaced by adeles of the discrete number field considered.
3. Langlands duality relates the representations of the Galois group in question to the automorphic representations of G. The action of the Lie group is on the argument of the modular form so that one obtains infinite-dimensional representation of G for non-compact G analogous to a unitary representation of Lorentz group. The automorphic forms are eigenstates of the Casimir operator of G. Automorphy means that a subgroup Γ of the modular group leaves the automorphic form invariant modulo phase factor.
4. The action of the modular transformation $\tau \rightarrow -1/\tau$ in the case of $Gl_2(R)$ replaces G with ^LG. In the more general case (for the moduli space of Riemann surfaces of genus g possessing n punctures) the definition of the modular transformation induce the change $G \rightarrow^L G$ does not look obvious. Even the idea that one has only two groups related by modular transformation is not obvious. For electromagnetic duality with τ interpreted in terms of complexified gauge coupling strength this interpretational problem is not encountered.

Geometric Langlands conjecture in TGD framework

Consider next the geometric Langlands conjecture from TGD view point.

1. The geometric variant of Langlands conjecture replaces the discrete number field F (rationals and their algebraic extensions say) with function number field- say rational function with rational coefficients- for which algebraic completion defines the gigantic Galois group. Witten and Kapustin [A163] proposed a concrete vision about how electric-magnetic duality generalized to S-duality could allow to understand geometric Langlands conjecture.
2. By strong form of general coordinate invariance implying holography the partonic 2-surfaces and their 4-D tangent space data (not completely free probably) define the basic objects so that WCW reduces to that for partonic 2-surfaces so that the formulation of geometric Langlands conjecture for the local field defined by holomorphic rational functions with rational coefficients at partonic 2-surface might make sense.
3. What geometric Langlands conjecture could mean in TGD framework? The transition from space-time level to the level of world of classical worlds suggests that polynomials with rational functions with rational coefficients define the analog of rational numbers which can be regarded to be in the intersection real and p-adic WCW s. Instead of counting rational points of partonic 2-surface one might think of counting the numbers of points in the intersection of real and p-adic WCW s in which life is suggested to reside. One might well consider the possibility that a kind of volume like measure for the number of these point is needed. Therefore the conjecture would be of extreme importance in quantum TGD. Especially so if the intersection of real and p-adic worlds is dense subset of WCW just as rationals form a dense subset of reals and p-adic numbers.

Electric-magnetic duality in TGD framework

Consider first the ideas of Witten and Kapustin in TGD framework.

1. Witten and Kapustin suggest that electric-magnetic duality and its generalization to S-duality in non-abelian is the physical counterpart of $G \leftrightarrow^L G$ duality in geometric Langlands. The model is essentially a modification $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 2$ SUSY allowing this duality with Minkowski space replaced with a Cartesian product of two Riemann surfaces. In TGD framework M^4 would correspond naturally to space-time sheet allowing a slicing to string world sheets and partonic 2-surfaces. Witten and Kapustin call these 2-dimensional surfaces branes of type A and B with motivation coming from M-theory. The generalization of the basic dimensional formulas of S-duality to TGD framework implies that light-like 3-surfaces at which the signature of the induced metric changes and space-like 3-surfaces at the boundaries of CDs are analogs of brane orbits. Branes in turn would be partonic 2-surfaces. S-duality would be nothing but strong form of general coordinate invariance.
2. Witten and Kapustin introduce the notions of electric and magnetic eigen branes and formulate the duality as a transformation permuting these branes with each other. In TGD framework the obvious identification of the electric eigen branes are as string world sheets and these can be indeed identified essentially uniquely. Magnetic eigen branes would correspond to partonic 2-surfaces.
3. Witten and Kapustin introduce gauge theory with given gauge group. In TGD framework there is no need to introduce gauge theory description since the symmetry group emerges as the effective symmetry group defining measurement resolution. Gauge theory is expected to be only an approximation to TGD itself. In fact, it seems that the interpretation of G as Lie-group associated with Kac-Moody symmetry is more appropriate in TGD framework. This would mean generalization of 2-D sigma model to string model in moduli space. The action of G would not be visible in the resolution used.
4. Edward Frenkel represents the conjecture that there is mysterious 6-dimensional theory behind the geometric Langlands duality. In TGD framework this theory might correspond to twistorial formulation of quantum TGD using instead of $M^4 \times CP_2$ the product of twistor spaces M^4 and CP_2 with space-time surfaces replaced by 6-D sphere bundles.

Finite measurement resolution realized group theoretically

The notion of finite measurement resolution allows to identify the effective symmetry groups G and ${}^LG$ in TGD framework. The most plausible interpretation of G is as Lie group giving rise to

Kac-Moody type symmetry and assignable to a string model defined in moduli space of partonic 2-surfaces. By electric magnetic duality the roles of the string world sheet and partonic 2-surface can be exchanged provided the replacement $G \rightarrow G_L$ is performed. The duality means a duality of closed Euclidian strings and Minkowskian open strings.

1. The vision is that finite measurement resolution realized in terms of inclusions of HFFs corresponds to effective which is gauge or Kac-Moody type local invariance extended to quantum Yangian symmetry. A given finite measurement resolution would correspond to effective symmetry G giving rise to confinement so that the effective symmetry indeed remains invisible as finite measurement resolution requires. The finite measurement resolution should allow to emulate almost any gauge theory or string model type theory. This theory might allow super-symmetrization reducing to broken super-symmetries of quantum TGD generated by the fermionic oscillator operators at partonic 2-surfaces and string world sheets.
2. Finite measurement resolution implies that the orbit of the partonic 2-surface reduces effectively to a braid. There are two kinds of braids. Time-like braids have their ends at the boundaries of CD consisting of rational points in the intersection of real and p-adic worlds. Space-like braids are assignable to the space-like 3-surfaces at the boundaries of CD and their ends co-incide with the ends of time-like braids. The electric-magnetic duality says that the descriptions based using either kind of braids is all that is needed and that the descriptions are equivalent.
   The counterpart of $\tau \rightarrow -1/\tau$ should relate these descriptions. This need not involve transformation of effective complex Kähler coupling strength although this option cannot be excluded. If this view is correct the descriptions in terms of string world sheets and partonic 2-surfaces would correspond to electric and magnetic descriptions, which is indeed a very natural interpretation. This geometric transformation should replace G with ${}^LG$.
3. Finite measurement resolution effectively replaces partonic 2-surface with a discrete set of points and space-time surface with string world sheets or partonic 2-surfaces. The natural question is whether finite measurement resolution also replaces geometric Langlands and the "rational" intersection of real and p-adic worlds with number theoretic Langlands and rational points of the partonic 2-surface. Notice that the rational points would be common to the string world sheets and partonic 2-surfaces so that the duality of stringy and partonic descriptions would be very natural for fintie measurement resolution.

The basic question is how the symmetry group G emerges from finite measurement resolution. Are all Lie groups possible? Here the theory of Witten and Kapustin suggests guidelines.

1. What Witten and Kapustin achieve is a transformation of a twisted $\mathcal{N} = 4$ SUSY in $M^4 = \Sigma \times C$, where Σ is "large" as compared to Riemann surface C SUSY to a sigma model in Σ with values of fields in the moduli space of Higgs bundle defined in C. If one accepts the basic conjecture that at least regions of space-time sheets allow a slicing by string world sheets and partonic 2-surfaces one indeed obtains $M^4 = \Sigma \times C$ type structure such that Σ corresponds to string world sheet and C to partonic 2-surface.
   The sigma model -or more generally string theory- would have as a natural target space the moduli space of the partonic 2-surfaces. This moduli space woudl have as coordinates its conformal moduli and the positions of the punctures expressible in terms of the embedding space coordinates. For M^4 coordinates only the part transversal to Σ would represent physical degree of freedom and define complex coordinate. Each puncture would give rise to two complex E^2 coordinates and 2 pairs of complex CP_2 coordinates. If one identifies the string world sheets as an inverse image of a homologically non-trivial geodesic sphere as suggested in [K45]. This would eliminate CP_2 coordinates as dynamical variables and one would have just n complex valued coordinates.
2. How to construct the Lie algebra of the effective symmetry group G defining the measurement resolution? If G is gauge group there is no obvious guess for the recipe. If G defines Kac-Moody algebra (see `http://tinyurl.com/yavow9wd`) the situation is much better. There exists an extremely general construction allowing a stringy construction of Kac-Moody algebra using only the elements of its Cartan algebra with central extension defined by integer valued central extension parameter k. The vertex operators (see `http://tinyurl.com/y97kteeq`) defining the elements of the complement of the Cartan algebra of complexified Kac-Moody algebra are ordered exponentials of linear combinations of the Cartan algebra generators with coefficient

given by the weights of the generators, which are essentially the quantum numbers assignable to them as eigenvalues of Cartan algebra generators acting in adjoint representations.
The explicit expression for the Kac-Moody generator as function of complex coordinate of Riemann sphere S^2 is

$$J_\alpha(z) =: exp(\alpha \cdot \phi(z))) : \quad .$$

$J_\alpha(z)$ represents a generator in the complement of Cartan algebra in standard Cartan basis having quantum numbers α and $\phi(z)$ represents the Cartan algebra generator allowing decomposition into positive and negative frequency parts. The weights α must have the same length ($(\alpha,\alpha) = 2$) meaning that the Lie group is simply laced. This representation corresponds to central extension parameter $k = 1$. In bosonic string models these operators are problematic since they represent tachyons but in the recent context this not a problem. The central extension parameter c for the associated Virasoro representation is also non-vanishing but this should not be a problem now.

3. What is remarkable that depending on the choice of the weights α one obtains a large number of Lie algebras with same dimension of Cartan algebra. This gives excellent hopes of realizing in finite measurement resolution in terms of Kac-Moody type algebras obtained as ordered exponentials of the operators representing quantized complex E^2 coordinates. Any complexified simply laced Lie group would define a Kac-Moody group as a chacterizer of finite measurement resolution. Simply laced groups correspond by MacKay correspondence finite subgroups of $SU(2)$, which suggests that only Galois groups representable as subgroups of $SU(2)$ can be realized using this representation. It however seems that free field representations can be defined for an arbitrary affine algebra (see `http://tinyurl.com/y9lkeelk`): these representations are discussed by Edward Frenkel [A122].

4. The conformal moduli of the partonic 2-surface define part of the target space. Also they could play the role of conformal fields on string world sheet. The strong form of holography poses heavy constraints on these fields and the evolution of the conformal moduli could be completely fixed once their values at the ends of string world sheets at partonic 2-suraces are known. Are also the orbits of punctures fixed completely by holography from initial values for "velocities" at partonic 2-surfaces corresponding to wormhole throats at which the signature of the metric changes? If this were the case, stringy dynamics would reduce to that for point like particles defined by the punctures. This cannot be true and the natural expectation is that just the finite spatial measurement resolution allows a non-trivial stringy dynamics as quantum fluctuations below the measurement resolution.
The assumption that electromagnetic charge is well-defined for the modes of the induced spinor field implies in the generic case that the modes are localized to 2-D surfaces carrying vanishing induced W fields and above weak scale also vanishing induced Z^0 field. This makes sense inside the Minkowskian regions at least. The boundaries of the string world sheets carrying fundamental fermions would define uniquely braids and their intersections with partonic 2-surfaces would define the braid points. The embedding space coordinates of these points in preferred coordinates should be rational in the intersection of realities and p-adicities.
Finite measurement resolution would pose upper limit of the number of the string world sheets and thus to the fermion number of wormhole throat.

5. One can assign to the lightlike parton orbits at which the signature of the induced metric changes a 1-D Dirac action and its bosonic counterpart. The outcome spectrum of light-like 8-momenta and light-like geodesics with the direction of the 8.momenta. Since spinor modes are localized at string world sheets - at least in Minkowskian space-time regions - this term is actually localized at their 1-D boundaries. Finite measurement resolution would mean IR and UV cutoffs to the spectrum of p^k. IR cutoff would be due to the finite size of causal diamond (CD) and UV cutoff to the lower bound for the size of sub-CDS involved.
Note that Kähler action contains also measurement interaction terms at the space-like ends of the space-time surface. They fix the values of some classical conserved quantities to be equal to their quantum counterparts for the space-time surfaces allowed in quantum superpositions [K100]. Also here finite measurement resolution is expected.

6. The electric-magnetic duality induces S-duality permuting G and ${}^LG$ and the roles of string world sheet as 2-D space-time and partonic 2-surface defining defining the target manifold of

string model. The moduli spaces of string world sheets and partonic 2-surfaces are in very close correspondence as implied by the strong form of holography.

How Langlands duality relates to quantum Yangian symmetry of twistor approach?

The are obvious objections against the heuristic considerations represented above.

1. One cannot restrict the attention on single partonic 2-surface or string world sheet. It is the collection of partonic 2-surfaces at the two light-like boundaries of CD and the string world sheets which define the geometric structure to which one should assign both the representations of the Galois group and the collection of world sheets as well as the groups G and ${}^LG$. Therefore also the group G defining the measurement resolution should be assigned to the entire structure and this leaves only single option: G defines the quantum Yangian defining the symmetry of the theory. If this were not complicated enough, note that one should be also able to take into account the possibility that there are CDs within CDs.
2. The finite measurement resolution should correspond to the replacement of ordinary Lie group with something analogous to quantum group. In the simplest situation the components of quantum spinors cease to commute: as a consequence the components correlate and the dimension of the system is reduced to quantum dimension smaller than the algebraic dimension $d = 2$. Ordinary (p, q) wave mechanics is a good example about this: now the dimension of the system is reduced by a factor two from the dimension of phase space to that of configuration space.
3. Quantum Yangian algebra is indeed an algebra analogous to quantum group and according to MacKay did not receive the attention that it received as a symmetry of integrable systems because quantum groups became the industry [A173]. What can one conclude about the quantum Yangian in finite measurement resolution. One can make only guesses and which can be defended only by their internal consistency.
   (a) Since the basic objects are 2-dimensional, the group G should be actually span Kac-Moody type symplectic algebra and Kac-Moody algebra associated with the isometries of the embedding space: this conforms with the proposed picture. Frenkel has discussed the relations between affine algebras, Langlands duality, and Bethe ansatz already at previous millenium [A123].
   (b) Finite measurement resolution reduces the partonic 2-surfaces to collections of braid ends. Does this mean that Lie group defining quantum Yangian group effectively reduces to something finite-dimensional? Or does the quantum Yangian property already characterize the measurement resolution as one might conclude from the previous argument? The simplest guess is that one obtains quantum Yangian containing as a factor the quantum Yangian associated with a Kac-Moody group defined by a finite-D Lie group with a Cartan algebra for which dimension equals to the total number of ends of braid strands involved. Zero energy states would be singlets for this group. This identification conforms with the general picture.
   (c) There is however an objection against the proposal. Yangian algebra contains a formal complex deformation parameter h but all deformations are equivalent to $h = 1$ deformation by a simple re-scaling of the generators labelled by non-negative integers trivial for $n = 0$ generators. Is Yangian after all unable to describe the finite measurement resolution. This problem could be circumvented by replacing Yangian with so called (twisted) quantum Yangian characterized by a complex quantum deformation parameter q. The representations of twisted quantum Yangians are discussed in [A147].
   (d) The quantum Yangian group should have also as a factor the quantum Yangian assigned to the symplectic group and Kac-Moody group for isometries of H with M^4 isometries extended to the conformal group of M^4. Finite measurement resolution would be realized as a q-deformation also in these degrees of freedom.
   (e) The proposed identification looks consistent with the general picture but one can also consider a reduction of continuous Kac-Moody type algebra to its discrete version obtained by replacing partonic 2-surfaces with the ends of braid strands as an alternative.
4. The appearance of quantum deformation is not new in the context of Langlands conjecture.

Frenkel has proposed Langlands correspondence for both quantum groups [A141], and finite-dimensional representations of quantum affine algebras [A142].

The representation of Galois group and effective symmetry group as symplectic flow

Langlands duality involves both the Galois group and effective gauge or Kac-Moody groups G and ${}^LG$ extended to quantum Yangian and defining the automorphic forms and one should understand how these groups emerge in TGD framework.

1. What is the counterpart of Galois group in TGD? It need not be the gigantic Galois group of algebraic numbers regarded as an extension of rationals or algebraic extension of rationals. Here the proposal that infinite primes, integers and rationals are accompanied by collections of partonic 2-surfaces is very natural. Infinite primes can be mapped to irreducible polynomials of n variables and one can construct a procedure which assigns to infinite primes a collection of Galois groups. This collection of Galois groups characterizes a collection of partonic 2-surfaces.
2. How the Galois group is realized and how the symmetry group G realization finite measurement resolution is realized. How the finite-dimensional representations of Galois group lift to the finite-dimensional representations of G. The proposal is that Galois group is lifted to its braided counterpart just like braid group generalizes the symmetric group. One can speak about space-like and time-like braidings so that one would have two different kind of braidings corresponding to stringy and partonic pictures and it might be possible to understand the emergence of G and ${}^LG$. The symplectic group for the boundary of CD define the isometries of WCW and by its infinite-dimensionality it is unique candidate for realizing representation of any group as its subgroup. he braidings are induced by symplectic flows.
3. Obviously also the symmetry groups G and ${}^LG$ should be realized as symplectic flows in appropriate moduli spaces. There are two different symplectic flows corresponding to space-like and time-like braids so that G and ${}^LG$ can be different and might differ even at the level of Lie algebra. The common realization of Galois group and symmetry group defining measurement resolution would imply Langlands duality automatically. The electric magnetic duality would in turn correspond to the possibility of two kinds of braidings. It must be emphasized that Langlands duality would be something independent of electric-magnetic duality and basically due to the realization of group representations as projective representations realized in terms of braidings. Note that also the automorphic forms define projective representations of G.

Why should the finite Galois group (possibly so!) correspond to Lie group G as it does in number theoretic Langlands correspondence?

1. The dimension of the representation of Galois group is finite and this dimension would correspond to the finite dimension for the representation of G defined by the finite-dimensional space in which G acts. This space is very naturally the moduli space of partonic 2-surfaces with n punctures corresponding to the n braid ends. A possible additional restriction is that the end points of braids are only permuted under the action of G. If the representations of the Galois group indeed automatically lift to the representations of the group defining finite measurement resolution, then Langlands duality would follow automatically.
2. The group G would correspond to the Galois group in very much the same manner as finite subgroups of $SU(2)$ correspond to simply laced Lie groups in MacKay correspondence (see `http://tinyurl.com/yb9hosn7`) [A55]. This would generalize Mc Kay correspondence to much more general theorem holding true for the inclusions of HFFs.

An interesting open question is whether one should consider representations of the collection of Galois groups assignable to the construction of zeros for polynomials associated with infinite prime or the gigantic Galois group assignable to algebraic numbers. The latter group could allow naturally p-adic topology. The notion of finite measurement resolution would strongly suggest that one should consider the braided counterpart of the finite Galois group. This would give also a direct connection with the physics in TGD Universe. Langlands correspondence would be basic physics of TGD Universe.

The practical meaning of the geometric Langlands conjecture

This picture seems to lead naturally to number theoretic Langlands conjecture. What geometric Langlands conjecture means in TGD Universe?

1. What it means to replace the braids with entire partonic 2-surfaces. Should one keep the number of braid strands constant and allow also non-rational braid ends? What does the number of rational points correspond at WCW level? How the automorphic forms code the information about the number of rational surfaces in the intersection?
2. Quantum classical correspondence suggests that this information is represented at space-time level. Braid ends characterize partonic 2-surfaces in finite measurement resolution. The quantum state involves a quantum super position of partonic 2-surfaces with the same number of rational braid strands. Different collections of rational points are of course possible. These collections of braid ends should be transformed to each other by a discrete algebraic subgroup of the effective symmetry group G. Suppose that the orbit for a collection of n braid end points contains N different collections of braid points.
   One can construct irreps of a discrete subgroup of the symmetry group G at the orbit. Could the number N of points at the orbit define the number which could be identified as the number of rational surfaces in the intersection in the domain of definition of a given WCW spinor field defined in terms of finite measurement resolution. This would look rather natural definition and would nicely integrate number theoretic and geometric Langlands conjectures together. For infinite primes which correspond to polynomials also the Galois groups of local number fields would also entire the picture naturally.
3. One can of course consider the possibility of replacing them with light-like 3-D surfaces or space-like 3- surfaces at the ends of causal diamonds but this is not perhaps not essential since holography implies the equivalence of these identifications. The possible motivation would come from the observations that vanishing of two holomoprhic functions at the boundary of CD defines a 3-D surface.

How TGD approach differs from Witten-Kapustin approach?

The basic difference as compared to Witten-Kapustin approach (see `http://tinyurl.com/y8xmxzx7`) [A163] is that the moduli space for partonic 2-surfaces replaces in TGD framework the moduli space for Higgs field configurations. Higgs bundle (see `http://tinyurl.com/dytahre`) defined as a holomorphic bundle together with Higgs field is the basic concept. In the simplest situations Higgs field is not a scalar but holomorphic 1-form at Riemann surface Y (analog of partonic 2-surface) related closely to the gauge potential of $M^4 = C \times Y$ whose components become scalars in spontaneous compactification to C. This is in complete analogy with the fact that the values of 1-forms defining the basis of cohomology group for partonic 1-surface for cycles defining the basis of 1-homology define conformal moduli.

A possible interpretation is in terms of geometrization of all gauge fields and Higgs field in TGD framework. Color and electroweak gauge fields are geometrized in terms of projections of color Killing vectors and induced spinor connection. Conformal moduli space for the partonic 2-surface would define the geometrization for the vacuum expectation value of the Higgs field.

One can even argue that dynamical Higgs is not consistent with the notion that the modulus characterizes entire 2-surfaces. Maybe the introducing of the quantum fluctuating part of Higgs field is not appropriate. Also the fact, that for Higgs bundle Higgs is actually 1-form suggests that something might be wrong with the notion of Higgs field. Concerning Higgs the recent experimental situation at LHC is critical: it might well turn out that Higgs boson does not exist. In TGD framework the most natural option is that Higgs like particles exist but all of them are "eaten" by gauge bosons meaning that also photon, gluons possess a small mass. Something analogous to the space of Higgs vacuum expectation values might be however needed and this something could correspond to the conformal moduli space. In TGD framework the particle massivation is described in terms of p-adic thermodynamics and the dominant contribution to the mass squared comes from conformal moduli. It might be possible to interpret this contribution as an average of the contribution coming from geometrized Higgs field.

One challenge is to understand whether the moduli spaces assignable to partonic 2-surfaces and with string world sheets are so closely related that they allow the analog of mirror symmetry of the super-string models relating 6-dimensional Calabi-Yau manifolds. For Calabi-Yau: s the mirror symmetry exchanges complex and Kähler structures. Could also now something analogous make sense.

1. Strong form of general coordinate invariance and the notion of preferred extremal implies that the collection of partonic 2-surfaces fixes the collection of string world sheets (these might define single connected sheet as a connected sum). This alone suggests that there is a close correspondence between moduli spaces of the string world sheets and of partonic 2-surfaces.
2. One problem is that space-time sheets in the Minkowskian regions have hyper-complex rather than complex structure. The analog of Kähler form must represent hypercomplex imaginary unit and must be an antisymmetric form multiplied by the complex imaginary unit so that its square equals to the induced metric representing real unit.
3. How the moduli defined by integrals of complex 1-forms over cycles generalize? What one means with cycles now? How the handle numbers g_i of handles for partonic 2-surfaces reveal themselves in the homology and cohomology of the string world sheet? Do the ends of the string world sheets at the orbits of a given partonic 2-surface define curves which rotate around the handles and is the string would sheet a connected structure obtained as topological sum of this kind of string world sheets. Does the dynamics for preferred extremals of Kähler dictate this?

In the simplest situation (abelian gauge theory) the Higgs bundle corresponds to the upper half plane defined by the possible values of the inverse of the complexified coupling strength

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \quad .$$

Does the transformation for τ defined in this manner make sense?

1. The vacuum functional is the product of exponent of imaginary Kähler action from Minkowskian regions and exponent of real Kähler action from Euclidian regions appears as an exponent proportional to this kind of parameter. The weak form of electric-magnetic duality reduces Kähler action to 3-D Chern-Simons terms at light-like wormhole throats plus possible contributions not assignable to wormhole throats. This realizes the almost topological QFT property of quantum TGD and also holography and means an enormous calculational simplification. The complexified Kähler coupling strength emerges naturally as the multiplier of Chern-Simons term if the latter contributions are not present.
2. There is however no good reason to believe that string world sheets and partonic two-surface should correspond to the values of τ and $-1/\tau$ for a moduli space somehow obtained by gluing the moduli spaces of string worlds sheets and partonic 2-surfaces. More general modular symmetries for τ seem also implausible in TGD framework. The weak form of electric magnetic duality leads to the effective complexification of gauge coupling but there is no reason to give up the idea about the quantum criticality implying quantization of Kähler coupling strength.
3. From the foregoing it is clear that the identification of G as a Kac-Moody type group extended to quantum Yangian and assignable to string model in conformal moduli space is strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.
4. The notion of finite measurement resolution suggesting strongly quantum group like structure is what distinguishes TGD approach from Witten's approach and from the foregoing it is clear that the identification of G as a group defining Kac-Moody type group assignable to string model in conformal moduli space and further extended to quantum Yangian is the strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.
5. In his lecture Edward Frenkel explains that the recent vision about the conformal moduli is as parameters analogous to gauge coupling constants. It might well be that the moduli could take the role of gauge couplings. This might allow to have a fresh view to the conjecture that the lowest three genera are in special role physically because all these Riemann surfaces are hyper-elliptic (this means global Z_2 conformal symmetry) and because for higher genera elementary particle vacuum functionals vanish for hyper-elliptic Riemann surfaces [K21].

To sum up, the basic differences seem to be due to zero energy ontology, finite measurement resolution, and the identification of space-time as a 4-surface implying strong form of general coordinate invariance implying electric-magnetic and S-dualities implying also the replacement of Higgs bundle with the conformal moduli space.

16.4.3 About The Structure Of The Yangian Algebra

The attempt to understand Langlands conjecture in TGD framework led to a completely unexpected progress in the understanding of the Yangian symmetry expected to be the basic symmetry of quantum TGD and the following vision suggesting how conformal field theory could be generalized to four-dimensional context is a fruit of this work.

The structure of the Yangian algebra is quite intricate and in order to minimize confusion easily caused by my own restricted mathematical skills it is best to try to build a physical interpretation for what Yangian really is and leave the details for the mathematicians.

1. The first thing to notice is that Yangian and quantum affine algebra are two different quantum deformations of a given Lie algebra. Both rely on the notion of R-matrix inducing a swap of braid strands. R-matrix represents the projective representations of the permutation group for braid strands and possible in 2-dimensional case due to the non-commutativity of the first homotopy group for 2-dimensional spaces with punctures. The R-matrix $R_q(u,v)$ depends on complex parameter q and two complex coordinates u,v. In integrable quantum field theories in M^2 the coordinates u,v are real numbers having identification as exponentials representing Lorenz boosts. In 2-D integrable conformal field theory the coordinates u,v have interpretation as complex phases representing points of a circle. The assumption that the coordinate parameters are complex numbers is the safest one.
2. For Yangian the R-matrix is rational whereas for quantum affine algebra it is trigonometric. For the Yangian of a linear group quantum deformation parameter can be taken to be equal to one by a suitable rescaling of the generators labelled by integer by a power of the complex quantum deformation parameter q. I do not know whether this true in the general case. For the quantum affine algebra this is not possible and in TGD framework the most interesting values of the deformation parameter correspond to roots of unity.

Slicing of space-time sheets to partonic 2-surfaces and string world sheets

The proposal is that the preferred extremals of Kähler action are involved in an essential manner the slicing of the space-time sheets by partonic 2-surfaces and string world sheets. Also an analogous slicing of Minkowski space is assumed and there are infinite number of this kind of slicings defining what I have called Hamilton-Jaboci coordinates [K14]. What is really involved is far from clear. For instance, I do not really understand whether the slicings of the space-time surfaces are purely dynamical or induced by special coordinatizations of the space-time sheets using projections to special kind of sub-manifolds of the embedding space, or are these two type of slicings equivalent by the very property of being a preferred extremal. Therefore I can represent only what I think I understand about the situation.

1. What is needed is the slicing of space-time sheets by partonic 2-surfaces and string world sheets. The existence of this slicing is assumed for the preferred extremals of Kähler action [K14]. Physically the slicing corresponds to an integrable decomposition of the tangent space of space-time surface to 2-D space representing non-physical polarizations and 2-D space representing physical polarizations and has also number theoretical meaning.
2. In zero energy ontology the complex coordinate parameters appearing in the generalized conformal fields should correspond to coordinates of the embedding space serving also as local coordinates of the space-time surface. Problems seem to be caused by the fact that for string world sheets hyper-complex coordinate is more natural than complex coordinate. Pair of hyper-complex and complex coordinate emerge naturally as Hamilton-Jacobi coordinates for Minkowski space encountered in the attempts to understand the construction of the preferred extremals of Kähler action.

   Also the condition that the flow lines of conserved isometry currents define global coordinates lead to the to the analog of Hamilton-Jacobi coordinates for space-time sheets [K14]. The physical interpretation is in terms of local polarization plane and momentum plane defined by local light-like direction. What is so nice that these coordinates are highly unique and determined dynamically.

3. Is it really necessary to use two complex coordinates in the definition of Yangian-affine conformal fields? Why not to use hyper-complex coordinate for string world sheets? Since the inverse of hyper-complex number does not exist when the hyper-complex number is light-like, hyper-complex coordinate should appear in the expansions for the Yangian generalization of conformal field as positive powers only. Intriguingly, the Yangian algebra is "one half" of the affine algebra so that only positive powers appear in the expansion. Maybe the hyper-complex expansion works and forces Yangian-affine instead of doubly affine structure. The appearance of only positive conformal weights in Yangian sector could also relate to the fact that also in conformal theories this restriction must be made.
4. It seems indeed essential that the space-time coordinates used can be regarded as embedding space coordinates which can be fixed to a high degree by symmetries: otherwise problems with general coordinate invariance and with number theoretical universality would be encountered.
5. The slicing by partonic 2-surfaces could (but need not) be induced by the slicing of CD by parallel translates of either upper or lower boundary of CD in time direction in the rest frame of CD (time coordinate varying in the direction of the line connecting the tips of CD). These slicings are not global. Upper and lower boundaries of CD would definitely define analogs of different coordinate patches.

Physical interpretation of the Yangian of quantum affine algebra

What the Yangian of quantum affine algebra or more generally, its super counterpart could mean in TGD framework? The key idea is that this algebra would define a generalization of super conformal algebras of super conformal field theories as well as the generalization of super Virasoro algebra. Optimist could hope that the constructions associated with conformal algebras generalize: this includes the representation theory of super conformal and super Virasoro algebras, coset construction, and vertex operator construction in terms of free fields. One could also hope that the classification of extended conformal theories defined in this manner might be possible.

1. The Yangian of a quantum affine algebra is in question. The heuristic idea is that the two R-matrices - trigonometric and rational- are assignable to the swaps defined by space-like braidings associated with the braids at 3-D space-like ends of space-time sheets at light-like boundaries of CD and time like braidings associated with the braids at 3-D light-like surfaces connecting partonic 2-surfaces at opposite light-like boundaries of CD. Electric-magnetic duality and S-duality implied by the strong form of General Coordinate Invariance should be closely related to the presence of two R-matrices. The first guess is that rational R-matrix is assignable with the time-like braidings and trigonometric R-matrix with the space-like braidings. Here one must or course be very cautious.
2. The representation of the collection of Galois groups associated with infinite primes in terms of braided symplectic flows for braid of braids of.... braids implies that there is a hierarchy of swaps: swaps can also exchange braids of...braids. This would suggest that at the lowest level of the braiding hierarchy the R-matrix associated with a Kac-Moody algebra permutes two braid strands which decompose to braids. There would be two different braided variants of Galois groups.
3. The Yangian of the affine Kac-Moody algebra could be seen as a 4-D generalization of the 2-D Kac-Moody algebra- that is a local algebra having representation as a power series of complex coordinates defined by the projections of the point of the space-time sheet to geodesic spheres of light-cone boundary and geodesic sphere of CP_2.
4. For the Yangian the generators would correspond to polynomials of the complex coordinate of string world sheet and for quantum affine algebra to Laurent series for the complex coordinate of partonic 2-surface. What the restriction to polynomials means is not quite clear. Witten sees Yangian as one half of Kac-Moody algebra containing only the generators having $n \geq 0$. This might mean that the positivity of conformal weight for physical states essential for the construction of the representations of Virasoro algebra would be replaced with automatic positivity of the conformal weight assignable to the Yangian coordinate.

5. Also Virasoro algebra should be replaced with the Yangian of Virasoro algebra or its quantum counterpart. This construction should generalize also to Super Virasoro algebra. A generalization of conformal field theory to a theory defined at 4-D space-time surfaces using two preferred complex coordinates made possible by surface property is highly suggestive. The generalization of conformal field theory in question would have two complex coordinates and conformal invariance associated with both of them. This would therefore reduce the situation to effectively 2-dimensional one rather than 3-dimensional: this would be nothing but the effective 2-dimensionality of quantum TGD implied by the strong form of General Coordinate Invariance.
6. This picture conforms with what the generalization of $D = 4$ $\mathcal{N} = 4$ SYM by replacing point like particles with partonic 2-surfaces would suggest: Yangian is replaced with Yangian of quantum affine algebra rather than quantum group. Note that it is the finite measurement resolution alone which brings in the quantum parameters q_1 and q_2. The finite measurement resolution might be relevant for the elimination of IR divergences.

How to construct the Yangian of quantum affine algebra?

The next step is to try to understand the construction of the Yangian of quantum affine algebra.

1. One starts with a given Lie group G. It could be the group of isometries of the embedding space or subgroup of it or even the symplectic group of the light-like boundary of $CD \times CP_2$ and thus infinite-dimensional. It could be also the Lie group defining finite measurement resolution with the dimension of Cartan algebra determined by the number of braid strands.
2. The next step is to construct the affine algebra (Kac-Moody type algebra with central extension). For the group defining the measurement resolution the scalar fields assigned with the ends of braid strands could define the Cartan algebra of Kac-Moody type algebra of this group. The ordered exponentials of these generators would define the charged generators of the affine algebra.
   For the embedding space isometries and symplectic transformations the algebra would be obtained by localizing with respect to the internal coordinates of the partonic 2-surface. Note that also a localization with respect to the light-like coordinate of light-cone boundary or light-like orbit of partonic 2-surface is possible and is strongly suggested by the effective 2-dimensionality of light-like 3-surfaces allowing extension of conformal algebra by the dependence on second real coordinate. This second coordinate should obviously correspond to the restriction of second complex coordinate to light-like 3-surface. If the space-time sheets allow slicing by partonic 2-surfaces and string world sheets this localization is possible for all 2-D partonic slices of space-time surface.
3. The next step is quantum deformation to quantum affine algebra with trigonometric R-matrix $R_{q_1}(u,v)$ associated with space-like braidings along space-like 3-surfaces along the ends of CD. u and v could correspond to the values of a preferred complex coordinate of the geodesic sphere of light-cone boundary defined by rotational symmetry. It choice would fix a preferred quantization axes for spin.
4. The last step is the construction of Yangian using rational R-matrix $R_{q_2}(u,v)$. In this case the braiding is along the light-like orbit between ends of CD. u and v would correspond to the complex coordinates of the geodesic sphere of CP_2. Now the preferred complex coordinate would fix the quantization axis of color isospin.

These arguments are of course heuristic and do not satisfy any criteria of mathematical rigor and the details could of course change under closer scrutinity. The whole point is in the attempt to understand the situation physically in all its generality.

How 4-D generalization of conformal invariance relates to strong form of general coordinate invariance?

The basic objections that one can rise to the extension of conformal field theory to 4-D context come from the successes of p-adic mass calculations. p-Adic thermodynamics relies heavily on the properties of partition functions for super-conformal representations. What happens when one replaces affine algebra with (quantum) Yangian of affine algebra? Ordinary Yangian involves the original algebra and its dual and from these higher multi-local generators are constructed. In the recent case the obvious interpretation for this would be that one has Kac-Moody type algebra with expansion with respect to complex coordinate w for partonic 2-surfaces and its dual algebra with expansion with respect to hyper-complex coordinate of string world sheet.

p-Adic mass calculations suggest that the use of either algebra is enough to construct single particle states. Or more precisely, local generators are enough. I have indeed proposed that the multi-local generators are relevant for the construction of bound states. Also the strong form of general coordinate invariance implying strong form of holography, effective 2-dimensionality, electric-magnetic duality and S-duality suggests the same. If one could construct the states representing elementary particles solely in terms of either algebra, there would be no danger that the results of p-adic mass calculations are lost. Note that also the necessity to restrict the conformal weights of conformal representations to be non-negative would have nice interpretation in terms of the duality.

16.4.4 Summary And Outlook

It is good to try to see the relationship between Langlands program and TGD from a wider perspective and relate it to other TGD inspired views about problems of what I would call recent day physical mathematics. I try also to become (and remain!) conscious about possible sources of inconsistencies to see what might go wrong.

I see the attempt to understand the relation between Langlands program and TGD as a part of a bigger project the goal of which is to relate TGD to physical mathematics. The basic motivations come from the mathematical challenges of TGD and from the almost-belief that the beautiful mathematical structures of the contemporary physical mathematics must be realized in Nature somehow.

The notion of infinite prime is becoming more and more important concept of quantum TGD and also a common denominator. The infinite-dimensional symplectic group acting as the isometry group of WCW geometry and symplectic flows seems to be another common denominator. Zero energy ontology together with the notion of causal diamond is also a central concept. A further common denominator seems to be the notion of finite measurement resolution allowing discretization. Strings and super-symmetry so beautiful notions that it is difficult to imagine physics without them although super string theory has turned out to be a disappointment in this respect. In the following I mention just some examples of problems that I have discussed during this year.

Infinite primes are certainly something genuinely TGD inspired and it is reasonable to consider their possible role in physical mathematics.

1. The set theoretic view about the fundamentals of mathematics is inspired by classical physics. Cantor's view about infinite ordinals relies on set theoretic representation of ordinals and is plagued by difficulties (say Russel's paradox) [K84]. Infinite primes provide an alternative to Cantor's view about infinity based on divisility alone and allowing to avoid these problems. Infinite primes are obtained by a repeated second quantization of an arithmetic quantum field theory and can be seen as a notion inspired by quantum physics. The conjecture is that quantum states in TGD Universe can be labelled by infinite primes and that standard model symmetries can be understood in terms of octonionic infinite primes defined in appropriate manner.

   The replacement of ordinals with infinite primes would mean a modification of the fundamentals of physical mathematics. The physicists's view about the notion set is also much more restricted than the set theoretic view. Subsets are typically manifolds or even algebraic varieties and they allow description in terms of partial differential equations or algebraic equations.

   Boolean algebra is the quintessence of mathematical logic and TGD suggests that quantum Boolean algebra should replace Boolean algebra [K84]. The representation would be in terms of fermionic Fock states and in zero energy ontology fermionic parts of the state would define Boolean states of form $A \to B$. This notion might be useful for understanding the physical correlates of Boolean cognition and might also provide insights about fundamentals of physical mathematics itself. Boolean cognition must have space-time correlates and this leads to a space-time description of logical OR *resp.* AND as a generalization of trouser diagram of string models *resp.* fusion along ends of partonic 2-surfaces generalizing the 3-vertex of Feyman diagrammatics. These diagrams would give rise to fundamental logic gates.

2. Infinite primes can be represented using polynomials of several variables with rational coefficients [K84]. One can solve the zeros of these polynomials iteratively. At each step one can identify a finite Galois group permuting the roots of the polynomial (algebraic function in general). The resulting Galois groups can be arranged into a hierarchy of Galois groups and the natural idea is that the Galois groups at the upper levels act as homomorphisms of Galois groups at lower levels. A generalization of homology and cohomology theories to their non-Abelian counterparts emerges [K52]: the square of the boundary operation yields unit element in normal homology but now an element in commutator group so that abelianization yields ordinary homology. The proposal is that the roots are represented as punctures of the partonic 2-surfaces and that braids represent symplectic flows representing the braided counterpars of the Galois groups. Braids of braids of.... braids structrure of braids is inherited from the hierarchical structure of infinite primes.
   That braided Galois groups would have a representation as symplectic flows is exactly what physics as generalized number theory vision suggests and is applied also to understand Langlands conjectures. Langlands program would be modified in TGD framework to the study of the complexes of Galois groups associated with infinite primes and integers and have direct physical meaning.

The notion of finite measurement resolution realized at quantum level as inclusions of hyperfinite factors and at space-time level in terms of braids replacing the orbits of partonic 2-surfaces - is also a purely TGD inspired notion and gives good hopes about calculable theory.

1. The notion of finite measurement resolution leads to a rational discretization needed by both the number theoretic and geometric Langlands conjecture. The simplest manner to understand the discretization is in terms of extrema of Chern-Simons action if they correspond to "rational" surfaces. The guess that the rational surfaces are dense in the WCW just as rationals are dense in various number fields is probably quite too optimistic physically. Algebraic partonic 2-surfaces containing typically finite number of rational points having interpretation in terms of finite measurement resolution. Same might apply to algebraic surfaces as points of WCW in given quantum state.

2. The charged generators of the Kac-Moody algebra associated with the Lie group G defining measurement resolution correspond to tachyonic momenta in free field representation using ordered exponentials. This raises unpleasant question. One should have also a realization for the coset construction in which Kac-Moody variant of the symplectic group of $\delta M^4_\pm$ and Kac-Moody algebra of isometry group of H assignable to the light-like 3-surfaces (isometries at the level of WCW *resp.* H) define a coset representation. The actions of corresponding super Virasoro algebras are identical. Now the momenta are however non-tachyonic.
   How these Kac-Moody type algebras relate? From p-adic mass calculations it is clear that the ground states of super-conformal representations have tachyonic conformal weights. Does this mean that the ground states can be organized into representations of the Kac-Moody algebra representing finite measurement resolution? Or are the two Kac-Moody algebra like structures completely independent. This would mean that the positions of punctures cannot correspond to the H-coordinates appearing as arguments of symplectic and Kac-Moody algebra. The fact that the groups associated with algebras are different would allow this.

TGD is a generalization of string models obtained by replacing strings with 3-surfaces. Therefore it is not surprising that stringy structures should appear also in TGD Universe and the strong form of general coordinate invariance indeed implies this. As a matter fact, string like objects appear also in various applications of TGD: consider only the notions of cosmic string [K25] and nuclear string [K57]. Magnetic flux tubes central in TGD inspired quantum biology making possible topological quantum computation [K3] represent a further example.

1. What distinguishes TGD approach from Witten's approach is that twisted SUSY is replaced by string model like theory with strings moving in the moduli space for partonic 2-surfaces or string world sheets related by electric-magnetic duality. Higgs bundle is replaced with the moduli space for punctured partonic 2-surfaces and its electric dual for string world sheets. The new element is the possibility of trouser vertices and generalization of 3-vertex if Feynman diagrams having interpretation in terms of quantum Boolean algebra.

2. Stringy view means that all topologies of partonic 2-surfaces are allowed and that also quantum superpositions of different topologies are allowed. The restriction to single topology and fixed moduli would mean sigma model. Stringy picture requires quantum superposition of different moduli and genera and this is what one expects on physical grounds. The model for CKM mixing indeed assumes that CKM mixing results from different topological mixings for U and D type quarks [K61] and leads to the notion of elementary particle vacuum functional identifiable as a particular automorphic form [K21].
3. The twisted variant of $\mathcal{N} = 4$ SUSY appears as TQFT in many mathematical applications proposed by Witten and is replaced in TGD framework by the stringy picture. Supersymmetry would naturally correspond to the fermionic oscillator operator algebra assignable to the partonic 2-surfaces or string world sheet and SUSY would be broken.

When I look what I have written about various topics during this year I find that symplectic invariance and symplectic flows appear repeatedly.

1. Khovanov homology (see `http://tinyurl.com/5dgksb`) provides very general knot invariants. In [?] rephrased Witten's formulation about Khovanov homology as TQFT in TGD framework. Witten's TQFT is obtained by twisting a 4-dimensional $\mathcal{N} = 4$ SYM. This approach generalizes the original 3-D Chern-Simons approach of Witten. Witten applies twisted 4-D $\mathcal{N} = 4$ SYM also to geometric Langlands program and to Floer homology.
   TGD is an almost topological QFT so that the natural expectation is that it yields as a side product knot invariants, invariants for braiding of knots, and perhaps even invariants for 2-knots: here the dimension $D = 4$ for space-time surface is crucial. One outcome is a generalization of the notion of Wilson loop to its 2-D variant defined by string world sheet and a unique identification of string world sheet for a given space-time surface. The duality between the descriptions based on string world sheets and partonic 2-surfaces is central. I have not yet discussed the implications of the conjectures inspired by Langlands program for the TGD inspired view about knots.
2. Floer homology (see `http://tinyurl.com/m3thlqx`) generalizes the usual Morse theory and is one of the applications of topological QFTs discussed by Witten using twisted SYM. One studies symplectic flows and the basic objects are what might regarded as string world sheets referred to as pseudo-holomorphic surfaces. It is now wonder that also here TGD as almost topological QFT view leads to a generalization of the QFT vision about Floer homology [K52]. The new result from TGD point of view was the realization that the naïvest possible interpretation for Kähler action for a preferred extremal is correct. The contribution to Kähler action from Minkowskian regions of space-time surface is imaginary and has identification as Morse function whereas Euclidian regions give the real contribution having interpretation as Kähler function. Both contributions reduce to 3-D Chern-Simons terms and under certain additional assumptions only the wormhole throats at which the signature of the induced metric changes from Minkowskian to Euclidian contribute besides the space-like regions at the ends of the space-time surface at the light-like boundaries of CD.
3. Gromov- Witten invariants (see `http://tinyurl.com/y7nled63`) are closely related to Floer homology and their definition involves quantum cohomology in which the notion of intersection for two varieties is more general taking into account "quantum fuzziness". The stringy trouser vertex represent the basic diagram: the incoming string world sheets intersect because they can fuse to single string world sheet. Amazingly, this is just that OR in quantum Boolean algebra suggested by TGD. Another diagram would be *and* responsible for genuine particle reactions in TGD framework. There would be a direct connection with quantum Boolean algebra.

Number theoretical universality is one of the corner stones of the vision about physics as generalized number theory. One might perhaps say that a similar vision has guided Grothendieck and his followers.

1. The realization of this vision involves several challenges. One of them is definition of p-adic integration. At least integration in the sense of cohomology is needed and one might also hope that numerical approach to integration exists. It came as a surprise to me that something very similar to number theoretical universality has inspired also mathematicians and that there exist refined theories inspired by the notion of motive introduced by Groethendieck to to define universal cohomology applying in all number fields. One application and also motivation for

taking motives very seriously is motivic integration which has found applications in in physics as a manner to calculate twistor space integrals defining scattering amplitudes in twistor approach to $\mathcal{N} = 4$ SUSY. The essence of motivic integral is that integral is an algebraic operation rather than defined by a measure. One ends up with notions like scissor group and integration as processing of symbols. This is of course in spirit with number theoretical approach where integral as measure is replaced with algebraic operation. The problem is that numerics made possible by measure seems to be lost.

2. The TGD inspired proposal for the definition of p-adic integral relies on number theoretical universality reducing the integral essentially to integral in the rational intersection of real and p-adic worlds. An essential role is played at the level of WCW by the decomposition of WCW to a union of symmetric spaces allowing to define what the p-adic variant of WCW is. Also this would conform with the vision that infinite-dimensional geometric existence is unique just from the requirement that it exists. One can consider also the possibility of having p-adic variant of numerical integration [K52].

Twistor approach has led to the emergence of motives to physics and twistor approach is also what gives hopes that some day quantum TGD could be formulated in terms of explicit Feynman rules or their twistorial generalization [K91].

1. The Yangian symmetry and its quantum counterpart were discovered first in integrable quantum theories is responsible for the success fo the twistorial approach. What distinguishes Yangian symmetry from standard symmetries is that the generators of Lie algebra are multi-local. Yangian symmetry is generalized in TGD framework since point like particles are replaced by partonic 2-surfaces meaning that Lie group is replaced with Kac-Moody group or its generalization. Finite measurement resolution however replaces them with discrete set of points defining braid strands so that a close connection with twistor approach and ordinary Yangian symmetry is suggestive in finite measurement resolution. Also the fact that Yangian symmetry relates closely to topological string models supports the expectation that the proposed stringy view about quantum TGD could allow to formulate twistorial approach to TGD.
2. The vision about finite measurement resolution represented in terms of effective Kac-Moody algebra defined by a group with dimension of Cartan algebra given by the number of braid strands must be consistent with the twistorial picture based on Yangians and this requires extension to Yangian algebra- as a matter to quantum Yangian. In this picture one cannot speak about single partonic 2-surface alone and the same is true about the TGD based generalization of Langlands program. Collections of two-surfaces and possibly also string world sheets are always involved. Multi-locality is also required by the basic properties of quantum states in zero energy ontology.
3. The Kac-Moody group extended to quantum Yangian and defining finite measurement resolution would naturally correspond to the gauge group of $\mathcal{N} = \triangle$ SUSY and braid points to the arguments of N-point functions. The new element would be representation of massive particles as bound states of massless particles giving hopes about cancellation of IR divergences and about exact Yangian symmetry. Second new element would be that virtual particles correspond to wormholes for which throats are massless but can have different momenta and opposite signs of energies. This implies that absence of UV divergences and gives hopes that the number of Feynman diagrams is effectively finite and that there is simple expression of twistorial diagrams in terms of Feynman diagrams [K91].

16.5 Appendix

16.5.1 Hecke Algebra And Temperley-Lieb Algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{16.5.1}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and embedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M} : \mathcal{N} < 4$ is given by the relations

$$\begin{aligned} e_{n+1}e_n en+1 &= e_{n+1} \\ e_n e_{n+1} e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ ,t = -\sqrt{\mathcal{M} : \mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \quad (16.5.2)$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_n en+1 &= te_{n+1} \\ e_n e_{n+1} e_n &= te_n \ , \\ e_n^2 &= e_n = e_n^* \ , \ \ ,t = -\sqrt{\mathcal{M} : \mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \quad (16.5.3)$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

16.5.2 Some Examples Of Bi-Algebras And Quantum Groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n) \to A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a,b,c,d)$ to $M_2^{\otimes 2} = k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \left(\begin{array}{cc} a'' & b'' \\ c'' & d'' \end{array} \right) \ .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det - 1 = ad - bc - 1 = 0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a, b, c, d of the algebra. For instance, one has $\Delta(a) = a \to a \otimes a + b \otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a, b, c, d by the relations

$$\begin{array}{rlrl} ba = qab & , & db = qbd & , \\ ca = qac & , & dc = QCD & , \\ bc = cb & , & ad - da = (q^{-1}-1)bc & , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{16.5.4}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned} &\Delta(J) = J \otimes 1 + 1 \otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_\pm, H \ , \\ &S(1) = 1 \ , \qquad\qquad \epsilon(1) = 1 \ . \end{aligned} \tag{16.5.5}$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \to H^\star$ reads as

$$
\begin{array}{l}
a \rightarrow a^{\star} \ , \quad ab = (ab)^{\star} = b^{\star}a^{\star} \ , \\
1 \rightarrow 1 \ , \quad H \rightarrow H^{\star} = hH \ , \quad X_{+} \rightarrow (X_{+})^{\star} = hX_{-} \ .
\end{array}
\tag{16.5.6}
$$

The commutation relations of $U_q(sl(2)$ read as

$$
[X_{+}, X_{-}] = \frac{q^{H} - q^{-H}}{q - q^{-1}} \ , \quad [H, X_{\pm}] = \pm 2X_{\pm} \ . \tag{16.5.7}
$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_{\pm}$:

$$
\begin{array}{l}
\Delta(X_{\pm}) = X_{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X_{\pm} \ , \\
S(X_{\pm}) = -q^{\pm 1} X_{\pm} \ .
\end{array}
\tag{16.5.8}
$$

When q is not a root of unity, the universal R-matrix is given by

$$
R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_{+}^{n} \otimes q^{-\frac{nH}{2}} X_{-}^{n} \ . \tag{16.5.9}
$$

When q is m: th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [A146].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \rightarrow \Psi(x)$ $M_q(2) \rightarrow U_q^{\star}$ defined by a bilinear form $\langle u, x \rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$
\begin{array}{ll}
\langle uv, x \rangle = \langle u \otimes v, \Delta(x) \rangle \ , & \langle u, xy \rangle = \langle \Delta(u), x \otimes y \rangle \ , \\
\langle 1, x \rangle = \epsilon(x) \ , & \langle u, 1 \rangle = \epsilon(u)
\end{array}
$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$
\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K = q^{H}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,
$$

extended to a representation

$$
\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}
$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^{\star}$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^{\star}$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^{\star}$.

General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [A146]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:
i) A is indecomposable, that is does not reduce to a direct sum of matrices.
ii) $a_{ij} \leq 0$ holds true for $i < j$.
iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.
A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.
The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{cc} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}} e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}} e_jk_i \ , & e_if_j - f_je_i = \delta_{ij}\frac{k_i-k_i^{-1}}{q_i-q_i^{-1}} \ , \end{array} \tag{16.5.10}$$

and so called Serre relations

$$\begin{array}{l} \sum_{l=0}^{1-a_{ij}} (-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \ \ i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}} (-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \ , \ \ i \neq j \ . \end{array} \tag{16.5.11}$$

Here $q_i = q^{D_i}$ where one has $D_ia_{ij} = a_{ij}D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{16.5.12}$$

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{16.5.13}$$

$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{16.5.14}$$

(16.5.15)

The action of antipode S is defined as

$$S(e_i) = -e_ik_i^{-1} \ , \ \ S(f_i) = -k_if_i \ , \ \ S(k_i) = -k_i^{-1} \ . \tag{16.5.16}$$

Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [A146].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1)\times(l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0: th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \left(\begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array} \right)$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0,1,..,l)$ satisfying the relations of Eq. 16.5.11 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0}e_i \ , \quad [d, f_i] = \delta_{i0}f_i, \quad [d, k_i] = 0 \ . \tag{16.5.17}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{16.5.18}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{16.5.19}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{16.5.20}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\begin{aligned} \Psi(a, b) &= -\Psi(b, a) \ , \\ \Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) &= 0 \ . \end{aligned} \tag{16.5.21}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x \otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{16.5.22}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{16.5.23}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{16.5.24}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{16.5.25}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{16.5.26}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{16.5.27}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

Chapter 17

Langlands Program and TGD: Years Later

17.1 Introduction

Langlands correspondence is for mathematics what unified theories are for physics. The number theoretic vision about TGD has intriguing resemblances with number theoretic Langlands program [A184, A124] (see http://tinyurl.com/z6tew2e). There is also geometric variant of Langlands program [A126, A123, A141, A163] (see https://en.wikipedia.org/wiki/Geometric_Langlands_correspondence). I am of course amateur and do not have grasp about the mathematical technicalities and can only try to understand the general ideas and related them to those behind TGD. Physics as geometry of WCW ("world of classical worlds") and physics as generalized number theory are the two visions about quantum TGD: this division brings in mind geometric and number theoretic Langlands programs. This motivates re-consideration of Langlands program from TGD point of view. I have written years ago a chapter about this [?] but TGD has evolved considerably since then so that it is time for a second attempt to understand what Langlands is about.

17.1.1 Langlands program briefly

The basic concept in number theoretical Langlands program is algebraic extension L/Q of rational numbers Q and more generally, an extension L/K of algebraic extension of Q called global number field. K can denote also other number fields. If K corresponds to reals or complex numbers or to p-adic numbers or their extension, it is called local. Also extensions of finite fields and function fields can be considered. Already gives idea about the generality of Langlands program.

1. Algebraic extension of rational numbers can be constructed by finding the roots of an irreducible n:th order monic polynomial of real argument (coefficients are integers and the coefficients of the highest power is unity so that modulo p reduction conserves the degree) see http://tinyurl.com/gwrhgat) and extending Q by them so that one obtains algebraically n-dimensional number field as an algebraic extension of Q. Denote the extension of rationals Q defined by irreducible polynomial P by L. Galois group $Gal(L/K)$ consists of the automorphisms of this structure mapping sums into sums, products into products, and rationals of K into rationals and its order is the dimension of the extension.
   One can combine several extensions of this kind by extending with corresponding roots and can construct algebraic numbers by combining all extensions of this kind. The Galois group of algebraic numbers is known as absolute Galois group and enormously complex. Absolute Galois group Gal_{abs} (see http://tinyurl.com/gvcywrs) has the Galois groups $Gal(L/K)$ of irreducible polynomials as subgroups.
2. Algebraic numbers have infinite algebraic dimension and can be regarded as an extension of any global field K and has factor groups $Gal_{abs}/Gal(K)$ as Galois group. One has restriction homomorphisms from $(Gal_{abs}/Gal(K))Gal(K)$ to $Gal(K)$ and imbedding homomorphisms of $Gal(K)$ to Gal_{abs}. One can construct representations of Galois groups in various groups such

as classical Lie groups and algebraic groups and this kind of representations give information about number theory. The distinctions between Lie groups and algebraic groups are very delicate and not of practical significance for a physicist.
The term algebraic matrix group G tells that the matrices satisfy some algebraic conditions specifying a subgroup of general linear group. One can specify the number field for matrix elements by using the notion $G(K)$. In TGD framework discrete subgroups of matrix groups with values in algebraic extension of rationals are highly interesting.

3. Langlands program extends also the ring of integers associated with global number field to the ring of adeles (see `http://tinyurl.com/gt6j9me`) associated with global number field K inducing extensions of p-adic number fields. Adeles correspond to the Cartesian product of non-vanishing positive reals R_+ and of the p-adic integers for the algebraic extensions of p-adic number fields induced by K. Adeles contain as a multiplicative subgroup the group of ideles, which apart from finite number of exceptional primes have p-adic norm equal to 1. This is essential for the existence of non-vanishing multiplicative inverse of adele.

The great vision of Langlands resting on the work carried out by number theorists during centuries is that there is a deep connection between number theory and representation theory for Lie groups and reductive algebraic groups. Originally groups $GL(n)$ were considered already by Artin as providing representations of non-Abelian Galois groups but Langlands proposed a generalization to reductive algebraic groups. To my best - not so impressive - understanding both classical Lie groups and algebraic groups are reductive.

By Langlands correspondence the representations of $G \rtimes Gal$ and G should correspond to each other. The analogy with the representations of Lorentz group suggests that the representations of G should have "spin" for some compact subgroup of G acting from left or right such that the dimension of this representation is same as the representation of non-commutative Galois group.

Automorphic functions are indeed typically functions in G, which reduce to a function invariant under left and/or right action of a compact or even discrete subgroups H_1 and H_2 or more generally, belong to a finite-dimensional unitary representation of $H_1 \times H_2$ in $H_1 \backslash G/H_2$. Therefore they can be said to have $H_1 \times H_2$ quantum numbers analogous to spin if interpreted as "field modes" in the space of double cosets $H_1 g H_2$. This would conform with the vision about physics as generalized number theory. If I have understood correctly, the question is whether a finite-dimensional representation of H_1 or H_2 could correspond to a finite-dimensional representation of Galois group at the number theory side.

Langlands formulated a correspondence between so called **a)** admissible infinite-dimensional automorphic representations for a reductive group $G(K)$ and **b)** representations of Galois groups in its Langlands dual $G_L(C)$ (complex non-compact group). Infinite-dimensionality requires non-compactness for $G(R)$ since compact groups have only finite-dimensional unitary irreducible representations. Here K is either local (archimedean (real or complex) or non-archimedean (p-adic number field or its extension) or global number field (algebraic extensions of rationals) so that the approach is extremely general.

Archimedean fields represent relatively simple situation. Non-archimedean fields are much more difficult and global fields extremely difficult and to my understanding very few proofs exist. For algebraic extension of rationals adele ring is obtained as Cartesian product of p-adic integers with extension induced by the extension of rationals. If K is itself non-Archimedean field, the notion of adele ring does not seem to make sense as such: should the extension define an extension of rationals in turn inducing an extension of other p-dic number fields?

17.1.2 A modest attempt for an overview

I try to give an overall view about Langlands conjecture.

1. G is reductive group (includes semisimple Lie groups) in given algebraic extension K of rationals, and can be extended to adelic group $G(A)$, where A denotes the adele formed by non-vanishing reals and integers for extensions of p-adic number fields induced by K. $G_L(C)$ is complex group and provides a representation of Galois group of K: one speaks of homomorphisms of Galois group to $G_L(C)$.

2. Langlands started from the representations of Galois group in group $Gl(n, K)$ and later generalized to arbitrary reductive Lie group $G(K)$. Here K is arbitrary number field, which could

be global number field (algebraic extension of rationals) or real or complex variant of G or a variant of G for p-adic number field or its extension induced by algebraic extension of rationals. The representations in real and p-adic number fields are combined to adelic representation and could be seen as infinite tensor product. For global number fields $G(K)$ (extensions of rationals) is discrete and does not allow the analytic machinery requiring Lie groups: just these are of special interest in TGD framework.

3. Since $G(K)$ is discrete for global fields K, one wants to simplify things by replacing K with what is called separable closure $\overline{K}$ of K analogous to complex numbers. This also allows to have infinite-dimensional representations . $G(\overline{K})$ allows Lie-group and Lie-algebra structure so that the machinery of Lie algebras can be used.
   One can assign Galois group $Gal(\overline{K}/K)$ to the extension of K to $\overline{K}$. If K is a finite-dimensional extension of rationals this Galois group (absolute Galois group) is extremely complex object and is known to possess topology highly reminiscent of p-adic topologies. $\overline{K}$ corresponds to complex algebraic numbers for the algebraic extensions of rationals. For p-adic number fields the fact that all polynomials effectively reduce to polynomials of degree not larger than $p-1$, $\overline{K}$ and $Gal(\overline{K}/K)$ are considerably simpler entities (see `http://tinyurl.com/mkqhp5n`). The transition to $\overline{K}$ does not delete the information about K also the adele structure keeps information about K.
4. In $G(\overline{K})$ one can speak about Lie algebra and its root system. One assigns to this root system a co-root system and in terms of it defines the connected component $G_L^0(C)$ of Langlands dual as a complex group. To keep information about the algebraic extension, one extends $G_L^0(C)$ to the semi-direct product $G_L^0(C) \rtimes Gal(K)$. The Galois group of finite-dimensional extension K acting appears and preserves information about the extension. It would seem that the representations of this group must be constructed from products of representations of $Gal(K)$ and $GL^0(C)$ so that additional discrete degrees of freedom appear. Kind of Galois covering of $G_L^0(C)$ serves as Langlands dual for $G(\overline{K})$.
5. This correspondence involves reductive algebraic group G and its Langlands dual G_L interpreted as complex group (see `http://tinyurl.com/zts4rqf`). G_L has as its roots co-roots of G:

$$\alpha \rightarrow \alpha' = 2\alpha/(\alpha,\alpha)$$

   so that the dimension of Cartan algebra and number of roots is same but the angles between some roots have changed:

$$(\alpha',\beta') = 4(\alpha,\beta)/(\alpha,\alpha)(\beta,\beta) \ .$$

   All simply laced Lie groups (ADE groups) with $(\alpha,\alpha) = 2$ are self-dual as also G_2 and F_4 and $Gl(n)$.
   The root systems B_n and C_n are mapped to each other so that $SO(2N+1)$ is dual to $Sp(N)$ whereas $SO(2n)$ is self dual as D_n type group. Connected Lie groups are dual to adjoint type Lie groups: for instance $SU(N)$ is dual to $SU(N)/Z_n$. One could try to understand the complexification of the dual from the fact that the natural representation of the roots of polynomial is as points of complex plane and Galois group therefore naturally acts in complex plane. Why the type of the group is changed looks however mysterious.
6. Information about K is not lost at the group theory side since adele group contains information about K. Also the separable closure $\overline{K}$ for p-adic number fields and their extensions is not equal to the algebraic closure since separable closure contains only separable extensions (minimal polynomial has only roots with multiplicity one).

Langlands conjecture states that the automorphic forms - so called Artin's L-functions - defined by the homomorphisms from Galois group $Gal(K)$ to $G_L^0(C)$ extended to a semi-direct product with the Galois or is modification Weil group (see `http://tinyurl.com/hk74sw7`) to be distinguished from Weyl group in Lie-algebra theory co-incide with the automorphic forms assignable to "good" representations of $G(\overline{K})$, which correspond to group theory side of the duality - group theoretic L-functions.

Connections of Langlands program with physics have been found already at the level of gauge theories and in string models. Electric-magnetic duality discovered by Montonen involves gauge group and its Langlands dual and there are reasons to expect that electric-magnetic duality - weak form of electric-magnetic self-duality in TGD framework [K23] - could have important implications for the understanding Langlands duality.

Witten, Frenkel and many other leading mathematicians and theoretical physicists have been developing geometric Langlands program [A126, A123, A141, A163]. Geometric Langlands is considerably simpler (simplicity is relative notion here!) than its number theoretical counterpart since the monstrous automorphism group of algebraic numbers (by definition mapping products to products and sums to sums) with the fundamental group of Riemann surface with punctures. Kac-Moody algebras and the monodromy groups as representations of fundamental group of Riemann surface are essentially involved.

17.1.3 Why number theoretic vision about TGD could have something to do with Langlands program?

Due to the technicalities involved it is impossible for a physicist like me to understand Langlands program at technical level. TGD is however proposed to be a unified theory of physics and it would not be surprising if some connections would exist.

1. The number theoretic universality [K98] is one of the basic principles of TGD with motivations coming from both p-adic mass calculations [K59] and mathematical description of cognition in TGD inspired theory of consciousness [K62, K7]. This principle states that physics is adelic and the physics in real and various p-adic sectors is obtained by a kind of analytic continuation from physics for algebraic extensions of rationals. The analogy with Langlands program is obvious and suggests strongly a connection with number theoretic Langlands.
2. In TGD framework Kac-Moody algebras generalize to super-symplectic algebra, which is immensely more complex than Kac-Moody algebras and has strong number theoretic flavor (for instance, conformal weights could relate closely to the zeros of Riemann zeta). Could super-symplectic algebra be for number theoretic Langlands what Kac-Moody is for geometric Langlands (see `http://tinyurl.com/j7tdho6` and `http://tinyurl.com/zj8lf2w`)?
3. Discretizations based on algebraic extension are a corner stone of TGD view about space-time relying on the notion of finite measurement resolution. Discretization means replacement of Lie group G by finite discrete subgroup assignable to algebraic extension K of rationals. The discretization are at the level of embedding space and their existence as coset spaces relies heavily on the symmetries of embedding space.
   One can perform completion for the points of discretization to what one might call monads [L23]. In real context they are analogous to the open sets defining charts of manifold. In p-adic sectors monads are disjoint and consist of p-adic integers. The field equations for Kähler action (or its modification suggested by twistorialization containing extremely small volume term) are satisfied inside monads.
   Galois group of K act as dynamical symmetry group transforming discretizations to each other so that one has kind of covering space structure at the level of WCW with sheets correspond to points of Galois group. This suggests that the counterparts of symmetries with elements in the extensions of rationals combined to semi-direct product with Galois group are crucial in TGD and that Galois groups act as symmetry groups having action very similar to that for fundamental groups.
   Note that also the isometry group G of embedding space restricted to $G(K)$ acts as discrete symmetries so that space-time surfaces (and 3-surfaces at boundaries of causal diamonds, string world sheets, and partonic 2-surfaces) provide a representation space for these groups. $G(K)$ act also on the induced spinor fields which can be assumed to have components in K (or $\overline{K}$).
4. The geometric realization for the hierarchy of Planck constants [**?**] is proposed to be in terms of coverings of space-time surfaces for which ends at the boundaries of CD correspond to singular covering with all sheets co-inciding. Could Galois group define this covering. This would require that Galois maps the discretization of 3-surface to itself at boundaries of CD The stronger condition that it maps the ends points to itself seems too strong.
   A further conjecture is that the hierarchy of Planck constants corresponds to the hierarchy of inclusions of hyperfinite factors (HFFs) having canonical representation in terms of second

quantized induced spinors needed to define WCW gamma matrices and WCW spinors. The inclusions are known to correspond discrete subgroups of $SU(2)$ and labelled by ADE diagrams, which by McKay correspondence correspond to Dynkin diagrams for ADE type Kac-Moody groups (see http://tinyurl.com/jyjplzc). The conjecture is that the Kac-Moody groups form a hierarchy of dynamical symmetries as remnants of symplectic symmetries due the infinite number of conditions stating the vanishing for a subset of symplectic Noether charges. These would be self-dual under Langlands duality.
Since the representations of $G \rtimes Gal$ and G should correspond to each other, the representations of G should have G-spin such that the dimension of this representation is same as the representation of non-commutative Galois group. This would conform with the vision about physics as generalized number theory. Could this be the really deep physical content of Langlands correspondence?

17.2 More detailed view about Langlands correpondence

Langlands correspondence [A184, A124] (see http://tinyurl.com/z6tew2e) has group theoretical and number theoretical sides and in the following I try to summarize what I have vaguely understood about these aspects.

17.2.1 Group theory side of Langlands correspondence

Consider first the group theory side. I want to confess that the following explanations are just a collection of physicist's impressions and probably too much for the patience of mathematician.

First the view of physicist about what the representations of $G(\overline{K})$ might be.

1. These groups have representations defined by functions in some complex analytic manifolds (say complex groups) and more general representations involving the analog of classical field representing particle with spin which are defined in Minkowski space and so that the action of Lorentz group $G = SO(1,3)$ on field is well-defined and spin characterizes the representation of field under rotation group $SO(3) \subset SO(1,3)$. The field corresponds to well defined mass and satisfied d'Alembert equation representing Casimir operator for $SO(1,3)$. At the level of momentum space one has representation of $SO(1,3)$ at mass shell, that is coset space $H_3 = SO(1,3)/SO(3)$, 3-D hyperbolic space.
More generally, the field can live in group manifold G or its coset space G/H and have spin in the sense that this field transforms as finite-dimensional representation of a sub-group $H \subset G$. The so called automorphic representations are in question: the action of group element $h \in H$ to the field $f(g)$ is given by $f(hg) = D_h(g)f(g)$. Here $D_h(g)$ is finite-D representation matrix which is easily found to satisfy so called co-cycle property: $D_{h_1h_2}(g) = D_{h_1}(h_2g)D_{h_2}(g)$. For 1- representations this equation holds for functions defining Abelian representation of H. Also now the analog of d'Alembert equation satisfied by free particle in field theory is assumed: one has eigenfunctions of the Casimir operator of the group: this requires that one consider Lie group. The interpretation would be that one has spinning particle in the coset space G/H.
2. The trace of the representation matrix $D_h(g)$ as function of group element is a fundamental characterizer of the representation invariant under automorphisms $h \to ghg^{-1}$ of the group and is known as character of group representation. For instance, for rotation group character depends on rotation angle only, not on the direction of the rotation axis. Now the matrix $D_h(e)$ defines a character as a function in sub-group H which can be discrete.

Automorphic forms characterize the group representations in question. The following definition from Wikipedia (see http://tinyurl.com/gquturl and http://tinyurl.com/hsy8ewf) resembles the description anticipated above except that I am not sure whether G-spin is allowed or whether only the analogs of scalar fields are considered.

Suppose f is function in complex manifold X in which group Γ acts. f is automorphic form if one has

$$f(\gamma(x)) = j_\gamma(x)f(x) \ ,$$

where $j_\gamma(x)$ is everywhere non-vanishing holomorphic function called factor of automorphy. Factor of automorphy is cocycle for the action of G meaning that one has from the definition

$$j_{\gamma_1\gamma_2}(x) = j_{\gamma_1}(\gamma_2(x))j_{\gamma_2}(x) \ .$$

Product of automorphic forms is automorphic with factor of automorphy given by the product of the factors. Automorphic forms form a vector space for a given factor of automorphy. If Γ is a lattice in Lie group then factor of automorphy for Γ corresponds to a line bundle on the quotient G/Γ. For instance, Γ can be a subgroup of $SL(2,R)$ acting on upper half complex plane. One can generalize the definition by replacing complex functions f with vector valued functions. In this case j corresponds to a representation matrix for Γ.

The complex analytic manifold X is often topological group G having Γ as its discrete subgroup. Hence automorphic form corresponds to a collection of functions $j_\gamma(g)$ of functions in G. As a special case one obtains modular forms for PSL(2,R) and Γ a modular subgroup $PSL(2,Z)$ or one of itse congruence subgroups with diagonal elements 1 modulo prime and diagonal elements zero modulo prime. In adelic approach these congruence subgroup can be treated at once using adeles.

Automorphic form could be at least formally defined also as a vector valued function f in G. Components of vector can be said to define analogs of component of a field with G-"spin". In the case of non-compact groups this representation would be by its finite dimension non-unitary but in principle this is possible (the unitary representations of Poincare group with spin are good example).

1. The vector transforms under $\gamma \in \Gamma$ according to a given factor j of automorphy which is matrix in general case. I do not know whether it is allowed to be matrix in case of non-Abelian Galois groups.
2. It is an eigenfunction of Casimir operators of G.
3. Satisfies some conditions on growth at infinity.

Automorphic functions can be defined in terms of Hecke characters (the analogy with Riemann zeta) and Hecke characters can in turn be defined for the unitary representations of group G, which is in general non-compact. The basic idea is to start from the representation of finite and compact groups in terms of group algebra endowed with sum (quantum superpositions of wave functions in group) and convolution (product induced by group product) and generalize to non-compact case. One can also require invariance under left and/or right action by some sub-group so that one obtains functions in coset spaces $H_1\backslash G/H_2$. One can consider functions in G, which are invariant under the left action of H_1 and right action of H_2. More generally, the functions could belong to irreducible unitary representations of H_1 and H_2 - physicist would perhaps say that the classical field "field" in double coset space has H_1 and H_2 "spin". Obviously the number of possibilities is endless.

In the simplest case these functions are constant in doublet cosets H_1gH_2 and one can construct them by taking a function $f(g)$ in G and forming a sum of the values $f(g_1gh_2)$ normalized suitably to give a kind of averaging. If the group in question is continuous group one can perform integration using left/right-invariant Haar measure. One can identify the action of Hecke operator as the formation of this average and identify eigen functions and eigen values of Hecke operator. One can generalize the Hecke operator to an operator producing function that belongs to a representation of $H_1 \times H_2$ and defining also now eigenfunctions. This leads an elegant mathematics. The upper complex plane identifiable as $SL(2,R)/SO(2)$ defines a coset space and posing left invariance of a complex analytic function $f(z)$ under $SL(2,Z)$ or its subgroup acting as Möbius transformations one obtains Hecke operators and Hecke characters defining examples of automorphic functions. The coefficients of the Fourier expansion of eigen function are eigen values of Hecke operator.

The group $SL(2,C)$ - double covering of Lorentz group $SO(1,3)$ is of special interested both number theoretically and geometrically. In this case the group H is typically discrete subgroup Γ of $SL(2,C)$ and the coset space $\Gamma\backslash SL(2,C)/SU(2)$. In this case the "spin" could correspond to a finite-D representation of Γ, which should be unitary. There are additional more technical conditions to be satisfied for the representation to be unitary. Often non-compact groups such as $GL(n,F)$ for an arbitrary algebraic number field is considered. Algebraic extensions of rationals, p-adic number fields, reals, complex numbers. The generality of the approach is stunning.

17.2.2 Number theoretical side of Langlands correspondence

On the number theoretic side the challenge is to find representations of Galois groups and their extensions to Weil groups. Also these lead to the notion of automorphic function. Here I can only give some notices about the historical development of the ideas leading to the vision of Langlands.

1. The story begins from the study of the simplest possible algebraic extensions defined by root of integer and characterized by this integer, call it n. These extensions are known as quadratic extensions and have Abelian Galois group consisting of 2 elements. One can generalize the notions of integer and prime to corresponding ideals for any algebraic extensions and the general phenomenon is that rational prime (ideals) can either stable, split to a product of different prime ideals of the extension, or ramify in which case higher powers of prime ideals of extension can appear. For instance, in the extension $Q(\sqrt{-1}$ number to ramifies to $(2) = (1+i)^2$ (note that 2 and $2i$ differing by unit are equivalent as representatives of ideal), primes $p\ mod\ 4 = 1$ split and primes $p\ mod\ 4 = 3$ are stable.
   The physical analogy for splitting is that proton as elementary particle is in improved resolution a bound state of 3 quarks.
2. Quadratic resiprocity (see http://tinyurl.com/njpnx69) can be seen starting point of the developments leading to Langlands conjecture. For instance, Euler, Legendre, and Gauss have made contributions here. One considers the question when prime q is square modulo prime p that is quadratic residue modulo p: $q = x^2\ mod\ p$, prime. Define Legender symbol (p/q) to be 1 if q is quadratic residue modulo p and -1 if this not the case. Quadratic resiprocity states
   $$(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{(p-1)}{2}\frac{(q-1)}{2}} .$$
   This law allows to relate (p/q) and (q/p) in the four cases corresponding to $p\ mod\ 4 \in \{1,3\}$, $q\ mod\ 4 \in \{1,3\}$. Legendre symbol is relevant for quadratic extensions of rationals since its value tells whether a given prime q ramifies in p-adic number field Q_p or not.
   Quadratic resiprocity generalizes to qubic, quadratic, quartic resiprocities and Eisenstein resiprocity (see http://tinyurl.com/huxm68w generalizes this law to higher powers. There is also reciprocity theorem for cyclotomic extensions (see http://tinyurl.com/z43cb5u and http://tinyurl.com/gm3sbzj) which are Abelian as also quadratic extensions. Artin's resiprocity (see http://tinyurl.com/j8ngckh) is a further generalization.

The next step was the emergence of class field theory applying to Abelian extensions L/K of global field K. The goal was to describe L/K in terms of arithmetics of K: this includes finite Abelian extensions of KK, realization of $Gal(L/K)$ and describe the decomposition of prime ideal from K to L (see http://tinyurl.com/z3s4kjn). Local number fields integrated into adele provide the needed tool by reducing the arithmetics to modulo p arithmetics. This can be seen as an application of Hasse principle (see http://tinyurl.com/jkh3auq).

1. A typical problem is the splitting of primes of K to primes of the extension L/K which has been already described. One would like to understand what happens for a given prime in terms of information about K. The splitting problem can be formulated also for the extensions of the local fields associated with K induced by L/K.
2. Consider what happens to a prime ideal p of K in L/K. In general p decomposes to product $p = \prod_{i=1}^{g} P_i^{e_i}$ of powers of prime ideals P_i of L. For $e_i > 1$ ramification is said to occur. The finite field K/p is naturally imbeddable to the finite field L/P_j defining its extension. The degree of the residue field extension $(L/P_i)/(K/p)$ is denoted by f_i and called inertia degree of P_i over p. The degree of L/K equals to $[L:K] = \sum e_i f_i$.
   If the extension is Galois extension (see http://tinyurl.com/zu5ey96), one has $e_i = e$ and $f_i = f$ giving $[L:K] = efg$. The subgroups of Galois group $Gal(L/K)$ known as decomposition group D_i and inertia group I_i are important. The Galois group of F_i/F equals to D_i/I_i.
   For Galois extension the Galois group $Gal(L/K)$ leaving p invariant acts transitively on the factors P_i permuting them with each other. Decomposition group D_i is defined as the subgroup of $Gal(L/K)$ taking P_i to itself.
   The subgroup of $Gal(L/K)$ inducing identity isomorphism of P_i is called inertia group I_i and is independent of i. I_i induces automorphism of $F_i = L/P_i$. $Gal(F_i/F)$ is isomorphic to D_i/I_i.

The orders of I_i and D_i are e and ef respectively. The theory of Frobenius elements identifies the element of $Gal(F_i/F) = D_i/I_i$ as generator of cyclic group $Gal(F_i/F)$ for the finite field extension F_i/F. Frobenius element can be represented and defines a character.

3. Quadratic extensions $Q(\sqrt{n})$ are simplest Abelian extensions and serve as a good starting point (see `http://tinyurl.com/zofhmb8`) the discrimant $D = n$ for $p \ mod \ 4 = 1$ and $D = 4n$ otherwise characterizes splitting and ramification. Odd prime p of the extension not dividing D splits if and only if D quadratic residue modulo p. p ramifies if D is divisible by p. Also the theorem by Kronecker and Weber stating that every Abelian extension is contained in cyclotomic extension of Q is a helpful result (cyclotonic polynomials has as it roots all n roots of unity for given n)

Even in quadratic extensions L of K the decomposition of ideal of K to a product of those of extension need not be unique so that the notion of prime generalized to that of prime ideal becomes problematic. This requires a further generalization. One ends up with the notion of ideal class group (see `http://tinyurl.com/hasyllh`): two fractional ideals I_1 and I_2 of L are equivalent if the are elements a and b such that $aI_1 = bI_2$. For instance, if given prime of K has two non-equivalent decompositions $p = \pi_1\pi_2$ and $p = \pi_3\pi_4$ of prime ideal p associated with K to prime ideals associated with L, then π_2 and π_3 are equivalent in this sense with $a = \pi_1$ and $b = \pi_4$. The classes form a group J_K with principal ideals defining the unit element with product defined in terms of the union of product of ideals in classes (some products can be identical). Factorization is non-unique if the factor J_K/P_K - ideal class group - is non-trivial group. $Q(\sqrt{-5})$ gived a representative example about non-unique factorization: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ (the norms are 4×9 and 6×6 for the two factorizations so that they cannot be equivalent.

This leads to class field theory (see `http://tinyurl.com/zdnw7j3` and `http://tinyurl.com/z3s4kjn`).

1. In class field theory one considers Abelian extensions with Abelian Galois group. The theory provides a one-to-one correspondence between finite abelian extensions of a fixed global field K and appropriate classes of ideals of K or open sub-groups of the idele class group of K. For example, the Hilbert class field, which is the maximal unramified abelian extension of K, corresponds to a very special class of ideals for K.
2. Class field theory introduces the adele formed by reals and p-adic number fields Q_p or their extensions induced by algebraic extension of rationals. The motivation is that the very tough problem for global field K (algebraic extension of rationals) defines much simpler problems for the local fields Q_p and the information given by them allows to deduce information about K. This because the polynomials of order n in K reduce effectively to polynomials of order $n \ mod \ p^k$ in Q_p if the coefficients of the polynomial are smaller than p^k. One reduces monic irreducible polynomial f characterizing extension of Q to a polynomial in finite field F_p. This allows to find the extension Q_p induced by f.
   An irreducible polynomial in global field need not be irreducible in finite field and therefore can have multiple roots: this corresponds to a ramification. One identifies the primes p for which complete splitting (splitting to first ordinary monomials) occurs as unramified primes.
3. Class field theory also includes a reciprocity homomorphism, which acts from the idele class group of a global field K, i.e. the quotient of the ideles by the multiplicative group of K, to the Galois group of the maximal abelian extension of K. Wikipedia article makes the statement "*Each open subgroup of the idele class group of K is the image with respect to the norm map from the corresponding class field extension down to K*". Unfortunately, the content of this statement is difficult to comprehend with physicist's background in number theory.

Number theoretic Langlands program is the next step in the process and could be seen as an extension of class field theory to the case of non-Abelian extensions. The following must be understood as an attempt of a physicist to understand what is involved. In my attempts to understand the formulas a valuable guideline is that they should involve only information about the number field K. Hecke character and L-function defined by Dirichlet series are basic notions besides notions of ideal generalizing of the notion of integer, and the notions of adele and idele (invertible adele). I must admit that I am still unable to understand how resiprocity theorems identifying two kinds of characters lead to the concrete form of resiprocity.

1. Hecke character (see `http://tinyurl.com/hxg6l9e`) is a generalization of Dirichlet character for Z/kZ (see `http://tinyurl.com/jqtp5cv`) giving rise to Dirichlet L-functions (see `http:`

//tinyurl.com/zsssrms) generalizing Riemann Zeta and defined as

$$L(\chi,s) = \sum_{n>0} \chi(n) n^{-s} \quad .$$

Hecke character is defined for idele class group rather than Galois group and can be seen as a character of idele group trivial in principal ideles. The conductor of Hecke character χ is defined as the largest ideal m such that χ is a Hecke character mod m.
The L-function associated with the Hecke character is an analog of Rieman zeta. There is sum over ideals not divided by m and weighted by Hecke character analogous to that over integers in Riemann zeta and its variants. The number $n > 0$ in Riemann zeta is replaced by the ideal norm $N(I)$ of ideal I, which is the finite size of the quotient ring R/I, where R is the ring of integers associated with K. One sums only over ideals not divisible by m. Hence the formula for the Dirichlet series defining L-function reads as

$$L(\chi,s) = \sum_{(I,m)=1} \chi(I) N(I)^{-s} \quad . \qquad (17.2.1)$$

Note that the character could be replaced with a character defined for the adelic extensions of group and the L-function also now carries information about ideles and therefore about K.

2. Already Artin's resiprocity (see http://tinyurl.com/j8ngckh) introduced the representations of group $GL(1,F)$, where F is global or local field. Artin proved that the L-functions associated with the characters of Galois group and with ideal class group were identical. The homomorphisms of Abelian Galois group to $GL(1,C)$ define so called Artin's L-functions in (analogous to Riemann zeta) in terms of characters of Galois group. These make sense also for non-Abelian extensions. Hecke characters defined as characters for the representations of the ideal class group give rise to the generalizations of Dirichlet L-functions analogous to Riemann zeta. Artin's resiprocity states that these two kind of L-functions are identical. For non-Abelian extensions higher-dimensional representation of Galois group are possible and this inspires the idea the introduction of $Gl(n,C)$ and is higher-D representations defining L-functions as so called automorphic forms.

3. Langlands conjecture (see http://tinyurl.com/mkqhp5n) generalizes Artin's approach to non-Abelian case. This requires non-Abelian infinite-dimensional representations possible for $Gl(n,F)$ and the theory of infinite-dimensional group representations becomes a tool of number theorist.
Langlands generalizes $Gl(n,F)$ to arbitrary reductive algebraic groups $G(F)$ and extends these groups to their adelic variants $G(A)$ bringing in ideles appearing also in Artin's L-function associated with the homomorphisms of Galois (Weil) group to the non-abelian case. These give rise to Artin's to L-functions for the semi-direct product of the dual G_L with Galois (Weil group) and the conjecture is that the automorphic forms for G for admissible representations co-incide with these.
The characters of the idele group are replaced with those for the "good" automorphic representations $G(\overline{K})$ defined by the Eq. 17.2.1. The summation over ideals of K follows automatically from the fact that the representations are for the adelic variant of G. It carries also information about Weyl group since one considers separable closures.
Langlands postulates also functoriality [A162] (see http://tinyurl.com/zts4rqf) making category theory so powerful. This allows to deduce from the existence of homomorphism between two groups G information about the relationship between representations of the dual group.

To sum up, I cannot claim of understanding much about this at the level of details. I however realize that the number theoretic vision relates in a highly interesting manner to Langlands theory and comparison might provide fresh insights to TGD and maybe even to Langlands theory by suggesting concrete physical identifications of groups associated with the Langlands correspondence and also suggesting a purely geometric action for the Galois groups via the adelic manifold concept.

17.3 TGD and Langlands correspondence

In the sequel I compare first Langlands program with TGD, which also involves both number theoretic and geometric visions and after that consider more detailed ideas.

17.3.1 Comparing the motivations

There are important similarities and also differences between the mathematical machineries used in Langlands approach and in TGD. Also motivations are different.

Motivation for number theoretical universality

In Langlands approach reductive algebraic groups are allowed with matri elements in various number fields (number theoretical universality). Classical Lie groups with matrix elements in some number field are algebraic groups. The basic motivation is generality. One studies algebraic groups over field K, which can be archimedean local field (reals or complex numbers), non-archimedean local field (finite extension of p-adic number field induced by extension of rationals), or global field (extension of rationals). One introduces also the separable closure $\overline{K}$ of K making possible to use the machinery of Lie groups and algebras. Separability means that only the roots of polynomials with different roots appear in extension. For p-adic number fields the separable closure is rather intricate notion. For algebraic extensions of rationals it correspond to algebraic numbers.

TGD view:

1. In TGD framework number theoretic universality implies that algebraic extensions of rationals define kind of intersection of reality and p-adicities. Therefore the discrete counterparts of Lie groups with matrix elements in the extensions of Q are of special importance in TGD. Langlands program includes these and are the most difficult ones.
2. If the hypothesis about ADE hierarchy assignable inclusions of HFFs [K99] holds true and has direct connection with $h_{eff}/h = n$ phases, all ADE Lie groups are allowed as dynamical symmetry groups and one achieves almost the same generality as in the case of Langlands correspondence. The maximal separable extensions for global and local fields make these fields analogous to complex numbers so that Lie-algebraic machinery can be used.
3. What is new that TGD suggest the allowance of all extensions of rationals inducing finite-dimensional extensions of p-adic number fields. In TGD context the extension of rationals can include also powers of a root of e since e^p is ordinary p-adic number and root of e induces finite-D extension of p-adic numbers (finite-dimensionality of extension is natural from the point of view of cognition). For non-compact groups the discretization of hyperbolic angles in this manner in p-adic context corresponds to the use of roots of unity for ordinary angles. One can say that the matrices with adele valued elements act in what might be called extension of the world of sensory experience to involve also cognition. That e^p is ordinary p-adic number suggests that non-compact groups are effectively compact in p-adic context.
   The Galois group of the extension by e^q the map $\sigma(e) = 1/e$ generates automorphism mapping rationals to rationals. The linear maps induced by $f(e) = e^k$, K integer are homomorphism since they map sums into sums and products into products but are not bijections except for $k = \pm 1$. One can wonder whether these maps could define analogs of automorphisms defining analog of inclusion hierarchy for hyper-finite factors (HFFs) [K99].

Motivation for p-adic number fields

In Langlands approach one motivation motivation for including p-adic number fields is Hasse principle (see `http://tinyurl.com/jkh3auq`): in the case of p-adic number fields the notion of algebraic number is not so stunningly complex as for rationals. The reason is that for p-adic units polynomials reduce effectively to polynomials with degree $n \ mod \ p<p$ with integer coefficients in the range $[0, p-1]$. This implies a huge simplification. The main reason for the mathematical applications of p-adic numbers is just this.

TGD view: From the viewpoint of TGD inspired theory of consciousness the motivation is the need to describe cognition mathematically. Cognition indeed simplifies: 2-adic cognition represents the largest possible simplification and cognitive evolution means increase of p as well as the increase of the dimension of algebraic extension of rationals (perhaps also that induced by root of e). It was however p-adic mass calculations assuming that mass squared is thermal in p-adic thermodynamics, which led to the p-adic physics [K59, K50].

Motivation for adelization

In Langlands approach adelicization means treatment of all number fields simultaneously. p-Adic number fields are combined to form kind of Cartesian product called adeles. Only p-adic integers are allowed and it is natural to pose the additional condition that apart from a finite number of exceptions these integers are p-adic units. Automorphic representations can be seen as infinite tensor products of representations associated with the number fields defining the adele.

TGD view:

1. The adelic view is used in different sense in TGD framework. Infinite tensor product of representations would create serious problems related to the physical interpretation in TGD framework since it seems that real and p-adic representations are only different views from the same number theoretically universal thing in the intersection of real and various p-adic sectors. One could say that the subgroups of algebraic groups with the matrix elements in the extension of rationals are in the intersection of real and various p-adic group theories.
2. The notion of p-adic manifold relies on the same idea. The discretization in algebraic extension of rationals is in the intersection and to each discrete point one can assign a monad which is real or p-adic and in which field equations such as those satisfied by preferred extremals of Kähler action are satisfied. One could perhaps say that these discrete algebraic points give rise to a number theoretically universal "spine" or back-bone of the space-time surface or any adelic geometry.
   Real continua around these points would give rise to the flesh around these bones (sensory representations). Also mind is needed and p-adic monads realized as p-adic integers would give it (cognitive representations). The definition of p-adic geometry works nicely for coset spaces [L23] and induction procedure allows to define adelic geometries for space-time surfacse using discretization consisting of algebraic embedding space points. The interpretation is in terms of finite measurement resolution and the hierarchy of algebraic extensions of rationals defines an infinite hierarchy of resolutions.
   This physical picture would suggest a generalization of the notion of geometry by fusing real and p-adic variants of the manifold to adelic geometry. In group theory this would mean a hierarchy of groups assignable to algebraic extensions of rationals with discrete group elements of discrete subgroups accompanied by monads defining the neighborhood of group element in archimedean or non-archimedean sense. These monads would make sense also in real context.
3. One could see the variants of group G in various number fields as completions of the number theoretically universal core part of G define in an extension of rationals common to all local number fields. Each point in the discretization would correspond to real or p-adic monad or for standard notion of manifold to an open neighborhood.
   What is new is that the system of open sets would correspond to the discretization having interpretation in terms of finite measurement resolution and the discrete subgroup could have direct physical meaning. For instance, Lorentz boosts would be quantized to velocities $\beta = tanh(n/m)$, $n \in Z$ and this velocity quantization could be seen in cosmology. There is indeed evidence for the quantization of redshifts [**?**, **?**]: possible TGD based explanations are discussed in [K81].
4. For instance, group $SO(2)$ represented by matrices

   $$\begin{pmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

   could be replaced with group for which $\theta = k2\pi/n$ so that one has roots of unity and one would have in p-adic context union of these group elements multiplied by a genuine p-adic Lie group with trigonometric functions replaced by the p-adic counterparts.
   The group $SO(1,1)$ represented by the matrices

$$\begin{pmatrix} cosh(\eta) & sinh(\eta) \\ sinh(\eta) & cosh(\eta) \end{pmatrix}$$

could be replaced with the group obtained by quantizing η in the manner already described and multiplying this group with the p-adic Lie group with hyperbolic functions replaced with their p-adic counterparts. Since e^p is ordinary p-adic number the number of discrete points of the monad would be finite and one would have analog of compactness for a group which is non-compact in real context.

Motivation for global fields

In Langlands approach the motivation for considering groups with matrix elements in global number fields is purely mathematical and Galois group is studied as number theoretical symmetry.

TGD view: In TGD framework the discretization of the embedding space in terms of points belonging to algebraic extension of rationals (or that including also the root of e) and inducing corresponding discretization of space-time surface means that Galois group of the extension acts as a physical symmetry group inducing an orbit of discretizations.

1. Does this mean that isometry group and symmetry groups with elements in K must be combined to semi-direct product with Galois group? One would have analogs of particle multiplets defined by irreducible representations of Galois group. Would this bring in a kind of number theoretic spin as additional degree of freedom? These particle like entities would emerge in number theoretical evolution as increase of the algebraic extension of rationals.
2. Or is an interpretation as a discrete orbital degree of freedom more appropriate? The singular n-fold coverings assignable to space-time surface associated with $h_{eff}/h = n$ phases identified in terms of dark matter could have natural interpretation as Galois coverings. Singularity means that the sheets of the covering co-incide at the ends of space-time surface at light-like boundaries of the causal diamond (CD). The action of Galois group becomes trivial if the points at the ends of space-time are rational. One can consider also Galois groups which are are associated with a given extension of an extensions and same picture would hold true. This identification of $h_{eff}/h = n$ would imply very strong correlation between number theory and dark matter phases.

17.3.2 TGD inspired ideas related to number theoretic Langlands correspondence

The question is whether TGD might allow to get new perspective to the Langlands duality. TGD certainly suggests number theoretical view about quantum physics as also view about quantum physics as infinite-dimensional geometry of WCW.

There are many notions which could relate to Langlands correspondence.

1. The notion of p-adic or monadic geometry [L23] emerges as a realization for finite measurement resolution at space-time level based on discretization in terms of algebraic extension of rationals and having Galois group as symmetry group. This geometry is also adelic geometry. Also the semi-direct products of various symmetry groups restricted to an extension of rationals and their semi-direct products with Galois group emerge naturally in this framework. Could this be the physical counterpart for the semi-direct product of G_L with Galois group?
   Complexification and replacement of K with separable closure are carried out for technical reasons in Landlands approach. Could automorphic functions have physical meaning in TGD framework? In principle $\overline{K}$ makes sense also in TGD framework. Could one think that one just restricts the automorphic functions from $G(\overline{K})$ to $G(K)$ and from $G_L(C)$ to $G_L(k)$ by embedding K to C as number theoretic universality suggests? Could one continue the universal automorphic functions from the discrete spine of the adelic geometry to the interior of the monads by using their form in $\overline{K}$ defining formulas?
2. Dark matter phases labelled by hierarchy of Planck constants and proposed to correspond singular coverings of space-time surface. Could the hierarchy of extensions of rationals correspond to this hierarchy. Could these coverings be Galois coverings becoming singular at the

3-D ends of space-time surface about boundaries of CD so that Galois leaves the corresponding 3-surface invariant by mapping it to itself or even leaving it invariant in point-wise manner?

3. Inclusions of hyperfinite factors (HFFs) [K99] are proposed to realize for finite measurement resolution quantum level in TGD framework. McKay correspondence (see `http://tinyurl.com/z48d92t`) suggesting that ADE Lie groups of Kac-Moody groups act as dynamical Lie groups identifiable as remnants of symplectic symmetries acting as isometries of WCW [K24, K76]. Could the hierarchy of extensions of rationals correspond to this hierarchy?
4. Weak form of electric-magnetic duality [K23] as self-duality reflecting self-duality of CP_2 and leading to ask whether Langlands duality reduces to self-duality for various symmetry groups of TGD.
5. Symplectic group defines the isometries of "world of classical worlds" (WCW) [K24, K23] and it is difficult to avoid the idea that the generalization of Kac-Moody algebra defined by symplectic group is crucial for the physical realization of Langlands correspondence in TGD framework.

Galois groups as symmetry groups in number theoretic vision

I have already earlier proposed that Galois groups could act as physical symmetries in TGD framework.

1. Number theoretic vision about TGD leads to the notion of adelic geometry involving both real, algebraic and various p-adic geometries giving meaning a generalization of manifold based on finite measurement resolution. Extensions of rationals inducing finite-dimensional extensions of p-adic numbers are central. The outcome is what might be called monadology.

   p-Adic space-time geometries make sense as induced geometries with discretizations defining points labelling the monads induced from the discretization of the embedding space. Strong form of holography allows reduction to the level string world sheets and partonic 2-surfaces serving as space-time genes (also gauge equivalence classes light-like orbits of partonic 2-surface labelled by Galois group might be involved).

   What is remarkable is that the Galois group of extension defines a symmetry group for discretizations in physical sense giving from given set of monads a new one. The roots of a polynomial behind the extension label a set of n surfaces defining a kind of covering for one of the sheets and Galois group acts in this set defining a covering space.
2. Do the sheets of the covering define disjoint space-time surfaces or do they form single connected space-time surface as in the case of Riemann surface for $z^{1/n}$? Could both options be involved? In the latter case there should be 3-regions at which the space-time sheets are glued together to give a singular covering. Either these 3-surfaces or even the points at these 3-surfacse could be fixed points of Galois group.
3. Also the hierarchy of Planck constants is associated with the emergence of coverings of space-time surface. These coverings are singular in the sense that the sheets co-incide at the eds of space-time surface at the boundaries of causal diamond (CD: there is scale hierarchy of CDs). Could these coverings be Galois coverings defined by the orbit of discretized space-time surface under Galois group of extension of rationals? Could $n = h_{eff}/h$ - tentatively identified as the number of sheets of covering - correspond to the dimension of the Galois group of the algebraic extension? More generally, if the Galois group is Galois group for an extension L of K which itself can be extension, singularity requires that the reduction must take place to K at the ends [K35]. One can imagine two options.
   - Option a): The discrete points of the 4-surface reduce to rational points (or points of K) at its 3-D ends at boundaries of CD and perhaps also at light-like orbits of partonic 2-surfaces? One variant of this option is that reduction occurs only at partonic 2-surfaces and string world sheets or strings at the ends.
   - Option b): Galois group leave only the discretized 3-surface invariant and maps its points along it?

   One can invent an objection against Option a). It is essential that the points of discretization have the same interpretation in real and p-adic senses. Hence the points should be expressible solely in terms of the algebraic numbers defining the extension: say roots of unity and powers for the roots of unity but not involving integers larger than 1 with varying p-adic norm. If integers appear then the p-adic norm of point can differ from unity. Points of unit circle or points of sphere with trigonometric functions of angles expressible solely in terms of roots of unity (Platonic solids) are representative examples. This does not allow the reduction of points of 3-surfaces at the ends of CD to rational points of H.

Option b) looks more attractive. Galois groups would act as dynamical symmetries of dark matter. Although the action on 3-surfaces at the ends of CD would be trivial, the action on the modes of induced spinor fields could be trivial also at the ends of CD.

Note that Galois covering is not the only interpretation for the covering that I have proposed: I have considered also an identification based on twistor lift of the space-time surface to its 6-D twistor space in the product of twistor spaces of M^4 and CP_2 which are twistorially unique in that they allow Kähler structure [K35].

4. The action of Galois group as a symmetry group acting geometrically on adelic geometries brings in mind the Belyi's theorem stating that Riemann surfaces describable as dessin d'enfants - "*child's drawings*" providing a combinatorial representation of Riemann surface as graph - can be defined as algebraic curves over the field of algebraic numbers.

   The mysterious absolute group has therefore has a geometric interaction on these Riemann surfaces allowing representation in terms of dessin d'enfant (see `http://tinyurl.com/zy393e3`). Now the Galois group of algebraic extension would have analogous representation on the discretization using points with coordinates in extension of rationals induced by the corresponding discretization for embedding space (actually causal diamond (CD)) defining the analog of dessin d'enfant.

This raises several questions.

1. This picture brings in mind the notion of virtual particle. At boundaries of CD the 3-surface would be on mass shell in the sense of being fixed point of Galois group and inside the CD it could be off-mass shel number theoretically although field equations for preferred extremal would be satisfied. Could this correspond to the non-determinism of Kähler action? Should one sum in the construction of scattering amplitudes over the surfaces at Galois orbit as in path integral?
2. Or should one regard the entire many-sheeted covering as the basic entity? I have indeed proposed that one can perform second quantization for the n-sheeted cover associated with $h_{eff}/h = n$ by adding fermions to different sheets of this cover and obtain this manner states with fractional quantum numbers with fractionization by factor $1/n$.
3. The interpretation as discrete gauge invariance with gauge fixing as a choice of single representative from Galois orbit does not look attractive. Note however that I have discussed the possibility of a huge generalization of M-theory dualities relating Calabi-Yau's and their mirrors as a generalization of old-fashioned string model duality [K35]: space-time surface could be seen as space-time correlates for computations connecting initial and final collections of algebraic objects with algebraic operations taking place at the vertices at which the Euclidian space-time regions representing lines of scattering diagram meet along their 3-D ends. This symmetry can be also seen as discrete analog of gauge symmetry involving the analog of gauge choice.

$h_{eff}/h = n$ hierarchy, hierarchy of inclusions of HFFs and McKay correspondence, and hierarchy of extensions of rationals

The relationship between dark matter hierarchy as a hierarchy $h_{eff}/h = n$ phases, hierarchy of inclusions of HFFs, McKay correpondence, and hierarchy of extensions of rationals and corresponding hierarchy of Galois groups is highly interesting and has been already touched.

1. I have proposed that dark matter hierarchy corresponds to a hierarchy of inclusions of HFFs [K99] giving rise to a hierarchy of ADE Lie groups or Kac-Moody as effective symmetry groups. By McKay correspondence (see `http://tinyurl.com/z48d92t`) ADE groups correspond to finite discrete sub-groups of $SU(2)$ in one-one correspondence with Dynkin diagrams assignable to ADE type Kac-Moody algebras. This leads to ask whether the inclusion hierarchy is accompanied by a hierarchy of ADE type Kac-Moody algebras or Lie algebras.
2. ADE type Lie or Kac-Moody groups self-dual under Langlands correspondence could emerge as remnant of the symplectic symmetries (a sub-algebra of full symplectic algebra *Sympl*

isomorphic to it and its commutator with *Sympl* have vanishing Noether charges). It could be assignable to string world sheets carrying the modes of induced spinor fields as dynamical symmetries. The duals of these ADE type groups are essentially identical with them and could combine with Galois groups to form semi-direct products.

3. One has a fractal hierarchy of sub-algebras of isomorphic sub-algebras of the symplectic algebra with conformal weights coming as n_1-multiples of the full algebra. Could n_1 satisfy $n_1 = h_{eff}/h = n$ with n identifiable as the dimension of algebraic extension of rationals? Or could one have $n_1 = ord(G)$, where $ord(G)$ is the order of the Galois group having n as a factor?

Weak form of electric-magnetic duality and Langlands correspondence

The first question about Langlands correpondence is why $G_L \rtimes Gal$ corresponds to G and what this precisely means.

1. One can extend Galois group and symmetry group (say Poincare or Lorentz group acting on discretized space-time surface or on 2-surface or on induced spinor field) to their semi-direct product: group multiplication law would be $(t_1, g_1)(t_2, g_2) = (t_1t_2, g_1t_1(g_2))$: this group would be the analog of $G_L \times Gal(K)$. The finite-dimensional representations of Galois group clearly give rise to what might be called number theoretic spin.
2. The innocent question of a physicists familiar with the unitary representations of Poincare group defined by fields with spin is whether the dimension of Galois representation for G_L could correspond to dimension for the representation for the spin associated with the representation of the dual G in analogy with Langlands correspondence. If the idea about hierarchy of Planck constants makes sense, strings and partonic 2-surfaces at the ends of space-time surface at boundaries of CD would correspond to G since the action of Galois would be trivial on them and G_L effectively reduces to G. String world sheets and light-like orbits of partonic 2-surfaces would correspond to G_L and Galois group would bring in additional degrees of freedom. The action of G_L on the induced spinors with components in field K would be however non-trivial. This could serve as a motivation for the introduction of n-D representations of G formed by many-fermion states.

Second basic mystery relates to the duality $G - G_L$ with G and G_L. Why the groups G ad G_L different? Or are they same in TGD framework?

1. G and G_L are essentially the same for Lorentz group, Poincare group, color group, for the holonomy group of spinor connection and for ADE groups possibly accompanying the hierarchy of inclusions of HFFs. One might also expect that the situation remains the same for Kac-Moody groups. "Essentially" means for color group $G = SU(3)$ one has $G_L = SU(3)/Z_3$.
   Whether the situation is same for the infinite-dimensional symplectic group assignable to the boundary of CD, is not clear since finite-dimensional symplectic groups are dual to odd-dimensional rotational groups. In fact, the infinite number of vanishing conditions for symplectic charges is expected to reduce it effectively to finite-dimensional Lie group or Kac-Moody group.
2. In TGD framework wormhole throats with identical electric and magnetic fluxes serves as the building bricks of elementary particles. Weak form of electric-magnetic duality is self-duality restricted to the light-like orbits of partonic 2-surfaces defining boundary conditions and inspired by the fact that electric and magnetic Kähler charge for CP_2 are identical. One can assign magnetic fluxes to partonic 2-surfaces and electric fluxes to the boundaries defined by the orbits of partonic 2-surfaces. One can define weighted fluxes for Hamiltonians of $\delta M^4_\pm \times CP_2$ as this kind of fluxes and obtain analogs of magnetic and electric representations classically. Trivial form of duality would mean that these representations are identical. Weak form of electric magnetic duality in this form suggests Langlands self-duality.
   One can assign magnetic and electric fluxes also to string world sheets. If one assumes weak form of self-duality also for them, electric and magnetic fluxes are identical also form them.
   One must be here cautious since algebraic discretization is involved and fluxes are defined only by assuming a continuation to continuous surface. This is indeed provide by the interiors of the monads assignable to the discrete points. In p-adic context the definition of flux as integral can be problematic.

$G = G_L$ does not trivialize Langlands correspondence.

1. If one considers semi-direct product of $G_L \rtimes Gal(K)$ and representation of G without the addition of Galois group as a semi-direct factor then the situation is non-trivial even for $G_L = G$. In Langlands program one must indeed use semi-direct product since the action of Galois group in $G_L(C)$ is usually trivial. For $G(\overline{K})$ the action of $Gal(K)$ is non-trivial in both global and local fields so that the inclusion of $Gal(K)$ as semi-direct factor would not be needed. Adeles contain however also positive reals and the action of $Gal(K)$ is trivial. This suggests that one must in the double coset representations in $H_1\backslash G/H_2$ an irreducible unitary representation to either H_1 or H_2 and that this representation corresponds to the higher-dimensional representation of non-Abelian Galois group. If so, the representation of $G_L \rtimes Gal(K)$ could factor to a product of a representation of G_L invariant under $Gal(K)$ with a finite-dimensional representation of non-Abelian $Gal(K)$ and would correspond to a representation of G in $H_1\backslash G/H_2$ with H_i-"spin" in analogy with representations of Lorentz group.
   The reduction of the quantum numbers assignable to Lie groups to number theory would be of course in accordance with the vision about physics as generalized number theory and could be perhaps seen as the deep physical content of Langlands correspondence.
2. The relationship to the fractionization of quantum numbers occurring for anyons is interesting. The covering analogous to that for $z^{1/n}$ gives an idea about the situation. Using single sheet with coordinate z one would obtain $1/n$ fractionization of spin at this sheet since 2π rotation leads to different sheet and only $n \times 2\pi$ rotation must leave the state unaffected. If one uses w as coordinate the range of angle coordinate is 2π - no fractionization [K69]. In TGD framework fractionization would mean that spin fractionizes for the rotation generator assignable to M^4 but does not so for the rotation generator assignable to the space-time surface X^4. Spin fractionization is associated with magnetic monopoles (maybe 2-sheeted coverings forced by the fact that monopole flux must flow to another space-time sheet through wormhole contact) so that there might be a connection.

$M^8 - M^4 \times CP_2$ duality, classical number fields, and Langlands correspondence

Quaternions and octonions seem to relate closely to the basic structure TGD [K86]: $M^4 \times CP_2$ allows octonionic structure in tangent space and space-time surfaces as preferred extremals could correspond to quaternionic/co-quaternionic surfaces with tangent space/normal space being quaternionic/associative. Also the notion of quaternion analyticity makes sense [K35]. The interesting question concerns the properties of various automorphism groups under Langlands duality. G_2 acting as automorphisms of octonions, its subgroup $SU(3)$ preserving preferred imaginary unit of octonions, and the covering group $SU(2)$ of the group $SO(3)$ of quaternionic automorphisms are self dual. $SO(3)$ has $SL(1,R)$ (I use $SL(n,R)$ to mean the same as $SL(2n,R)$ by some authors) as Langlands dual but the complexified groups are same so that one has self-duality also now.

For years ago I proposed what I called $M^8 - M^4 \times CP_2$ duality [K86, K98] and have not been able to kill this proposal. M^8 can be seen as tangent space of $M^4 \times CP_2$ and can be interpreted as subspace of complexified octonions. The idea is that 4-surfaces of M^8 with the property that tangent space at each point is associative (co-associative) or equivalently quaternionic (co-quaternionic) and containing in their tangent space $M^2 \subset M^8 = M^2 \subset E^6$ are mappable to surfaces in $M^4 \times CP_2$. The point of CP_2 would parameterize the tangent space as subspace of E^6 and transform as $3 + \overline{3}$ under $SU(3)$ automorphisms. That the coordinates for time= constant section of M^8 transform either as 7-D G_2 representation whereas the points of 7-D hyperboloid transform as $7-D$ representation of $SO(7)$ suggest some kind of duality.

The isometry group of M^8 is $SO(1,7)$ and decomposes for a fixed $M^8 = M^4 \times E^4$ decomposition to $SO(1,3) \times SO(4)$. The automorphism group of M^8 identified in terms of octonions is G_2 and $SU(3)$ is the automorphism group associated with $M^6 = M^2 \times E^6$ decomposition and acts as isometries of CP_2. There is infinite number of different octonion structures corresponding to the choices of subspaces $M^2 \times E^6$ parameterized by $SO(1,7)/SO(1,1) \times SO(6)$ having dimension $D = 28 - 1 - 15 = 12$. Note that all groups involved are self-dual in Langlands correspondence.

The notions of p-adic octonions and quaternions do not make sense: the reason is that the norm of non-vanishing quaternion/octonion can be vanishing. This can be case also for p-adic analog of complex numbers if -1 is square of p-adic number as it is for $p \ mod \ 4 = 1$. This does not allow definition of p-adic Hilbert space. This difficulty is not present if one restricts the consideration to points of algebraic extension interpreted as p-adic numbers. In this case one can construct versions of G_2 and $SU(3)$ by replacing real numbers with global field. Also the action of Galois group is well-defined on space-time surface so that one can form semi-direct sum of these groups with Galois group. G_2 and $SU(3)$ are self-dual.

Could supersymplectic algebra be for number theoretic Langlands what Kac-Moody algebra is for geometric Langlands

Super-symmetric symplectic algebra [K24, K23] and conformal algebra of light-cone boundary is much more complex structure than Kac-Moody algebras and central in TGD. The reason is that effective 2-dimensionality of the light-cone boundary of four-dimensional Minkowski space leads to huge extension of the ordinary conformal symmetries.

1. Supersymplectic algebra has the structure of conformal algebra. The analog of complex coordinate for the is the light-like radial coordinate r of light-cone boundary. Radial onformal weights can be complex numbers and numbers $s = 1/2 + iy$ are favored since they give rise to the analogs of plane waves. Light-cone boundary having the structure $S^2 \times R_+$ metrically with R_+ corresponding to null direction. Therefore there is also an extension of conformal algebra of sphere S^2. For this extension one has ordinary conformal weight assignable to S^2 and radial conformal weight assignable to R_+. The physical role of this algebra which is actually also isometry algebra has remained unclear. What is however clear that dimension for M^4 makes it mathematically completely unique.
2. I have proposed that the conformal weights for the generators of the symplectic algebra could correspond to poles of fermionic zeta function $\zeta_F(s) = \zeta(s)/\zeta(2s)$ [L18]. The number of generators of the algebra could be infinite so that it would be extremely complex as compared to the Kac-Moody algebras. Unitarity demands that for physical states the imaginary part of the total conformal weight which is essentially the sum of zeros of zeta is real. This implies conformal confinement and that physical states have integer or half-integer valued total conformal weights as for the ordinary super-conformal algebras.
3. A further conjecture is that for the zeros $s = 1/2+iy$ of Riemann zeta p^{iy} is root of unity [L18]. This conjecture is motivated by the findings suggesting that the zeros form a quasicrystal meaning that the Fourier transforms for the function located at zeros is of similar form.

Kac-Moody algebras are important for geometric Langlands based on fundamental group.

1. So called critical representations for Kac-Moody algebra are involved. For them the central extension parameter equals to $k = -c^g_\psi/2$, where c^g_ψ is Casimir operator for the adjoint representation. Negativity of k implies non-unitarity. The Virasoro generators in the associated Sugawara representation for Virasoro algebra would have infinite normalization constant $N = 1/2\beta$, $\beta = k + c^g_\psi/2 = 0$ and it would not be well-defined. Physically critical Kac-Moody representation does not seem interesting.
2. A formal generalization of Sugawara construction of representation of Virasoro algebra from that of symplectic algebra mimicking Kac-Moody case does not seem to work. The normalization factor $k+c^g_\psi/2$ dividing the quadratic expression of L_n in terms of Kac-Moody generators diverges if Casimir diverges and the outcome is ill-defined. Quadratic Casimir in adjoint representation is expressible in terms of structure constants as $f^{ABC}f_{ABC}$. Structure constants now Glebsch-Gordans for the representations of $SO(3) \times SU(3)$. Obviously the symplectic counterpart for the sum $f^{ABC}f_{ABC}$ for the Casimir operator of Lie group diverges so that Sugawara construction fails. This is of course not a real problem since Sugawara construction fails in any case for the critical weight needed in Kac-Moody algebra approach to geometric Langlands.

Could super-symplectic algebra help to understand number theoretic Langlands?

1. The conditions defining preferred extremal state vanishing of almost all symplectic Noether charges and suggest that symplectic group reduces effectively to a finite-D Lie group or Kac-Moody group. These groups form a hierarchy and could be assigned to inclusions of HFFs identifiable as ADE groups dictated by the inclusion (essentially self-dual under Langlands correspondence). These Kac-Moody groups could also have natural action at strings identified as boundaries of string world sheets. Also for these Kac-Moody groups critical representations lack physical interpretation.

2. Number theoretic discretization requires the consideration of discrete subgroup of ADE Lie group obtained by restriction to global field rather than Lie algebra. One could restrict the L-functions of automorphic representations to the subgroup of complex Lie group $G(C)$ having the group $G(K)$ associated with the global number field K.
3. This picture suggests tha the extension of Galois group in extension E/K has counterpart for the Lie groups appearing in ADE hierarchy realized at the level of Lie algebras: perhaps by adding n generators to the Cartan algebra.

17.3.3 Could geometric and number theoretic Langlands relate to each other?

One can see the analogy between Galois group and fundamental group also in the following manner (see the blog posting of Peter Woit at `http://tinyurl.com/hlgrrjk`). Primes are analogous to prime polynomials from which one can construct more complex polynomials as products. Rational numbers are analogous to rational functions defined as ratios of polynomials. This suggests an analogy between number theoretic Langlands and geometric Langlands for which rationals and their extensions are replaced by rational functions. One manner to interpret this analogy is to see ordinary rationals as kind of functions. Second manner is to see rational functions as generalizations of rationals. The latter interpretation looks more attractive to me.

There are indeed strong analogies between Galois groups and fundamental groups. Covering spaces can be assigned with fundamental groups and algebraic extensions of rationals are analogous to coverings: the orbit of a given point under Galois group is analogs to set of copies of the point at the sheets of the covering.

The problem is that fundamental group typically contains Z as a summand, which does not occur for Galois groups. For a punctured plane having Z as fundamental group one can construct infinite covering with trivial homotopy group. If one identifies k:th $k+n$:th sheet the fundamental group is $Z_n = Z/nZ$. For Q_p one expects reduction of fundamental group to Z_m, $m = n \ mod \ p$. This encourage speculative ideas related to the connection of number-theoretic and geometric Langlands.

Adelic geometries and the realization of fundamental group in terms of Galois group

Could geometric Langlands reduce to number theoretic Langlands in some cases? This would mean representation of fundamental group as Galois group of algebraic extension.

1. The notion of the adelic geometry involving algebraic discretization in both real and p-adic sectors with discretized points accompanied by locally smooth neighborhoods in which field equations for Kähler action are satisfied would suggests this. For a given discretization in terms of points of extension one obtains set of discretizations by applying Galois group and Galois group acts as symmetry group permuting the sheeets of the coverig covering space: the number of sheets as dimension of extension divides the order of Galois group.
2. If the n-fold singular coverings assigned with $h_{eff}/h = n$ corresponds to a Galois coverings, the sheets of covering reduce to single one at the singular ends of space-time surface at the lightlike boundaries of CD and one obtains a space analogous to the base space of covering and having homotopy group given by Galois group. Therefore the representations of Galois group would become representations of fundamental group for the adelic geometry. The action of this group would be non-trivial on spinors also at the ends of CD.

The analogy with fundamental group suggests that there are two ways to consider the situation. The images of the discrete adelic geometry under Galois group define the covering for which fundamental group is trivial. The restriction to single space-time sheet at the orbit under Galois group would mean the restriction to base space with non-trivial fundamental group given by Galois group. For the first option Galois group would permute the sheets of covering and define dynamical symmetry. For the second option non-trivial homotopy would correspond to these degrees of freedom. These two descriptions might define the core of number theoretic Langlands duality having interpretation also as geometric duality.

Does the hierarchy of infinite primes generalize number theoretic Langlands?

In TGD framework one can see the analogy from other direction. The construction of infinite primes leads to a repeated second quantization of arithmetic quantum field theory with bosonic and fermionic single particl estates labelled by primes [K84]. At the lowest level ordinary primes label the single particle states and at the first level one obtains infinite primes as Fock states.

Infinite primes can be mapped to monomials of single variable with zeros which are rational numbers. One obtains also infinite primes analogous of bound states as analogs of irreducible polynomials of single variables: now the zeros correspond to algebraic numbers.

One can continue the second quantization by taking these infinite primes as labels of single particle states and repeating the procedure. Now one can map the infinite primes to polynomials of two variables. This process can be continued ad infinitum.

The variables appearing in irreducible polynomials assignable to the hierarchy of infinite primes are formal variables and it is no clear it makes to sense to interpret them as coordinates for some space. If this were the case, one might consider connecting with Geometric Langlands associated with these space with generalization of number theoretic Langlands.

17.4 What generalization of Fermat's last theorem could have to do with TGD

I received a link to a popular article published in Quanta Magazine (`http://tinyurl.com/t44qv8o`) with title *'Amazing' Math Bridge Extended Beyond Fermat's Last Theorem* suggesting that Fermat's last theorem could generalize and provide a bridge between two very different pieces of mathematics suggested also by Langlands correspondence [K46, A126, A124, A184].

I would be happy to have the technical skills of real number theorist but I must proceed using physical analogies. What the theorem states is that one has two quite different mathematical systems, which have a deep relationship between each other.

1. Diophantine equations give solutions as roots of a polynomial $P_y(x)$ containing second variable y as parameter. The coefficients of $P_y(x)$ and y are integers but one can consider a generalization allowing them to be in extension of rationals.
   The general solution of $P_y(x) = 0$ for given value of n is in extension of rationals, whose dimension is determined by the degree n of $P_y(x)$. One is however interested only on the roots (x, y) of $P_y(x) = 0$ coming pairs of integers.
   Diophantine equations can be solved also in p-adic number fields labelled by primes p and in the adelic physics of TGD they are present. Also are present the extensions of p-adic number fields induced by the extensions of rational numbers. There is infinite hierarchy of them. The dimension n of extension serves as a measure for algebraic complexity and kind of "IQ" and $n = h_{eff}/h_0$ gives to effective Planck constant: the larger the value of n, the longer the scale of quantum coherence. This gives a direct connection to quantum biology.
   In p-adic number fields the p-adic integer solutions of the Diophantine equation can be infinite as real numbers. The solutions which are finite as real integeres for all primes p define real solutions as finite integers. The sequence of these solutions modulo prime p - that is in finite field - characterizes Diophantine equations. For large p these solutions would stabilize and start to repeat themselves for finite integer solutions. This picture can be generalized from simple low degree polynomials to higher degree polynomials with rational coefficients and even with coefficients in extension of rationals.
2. Second system consists of automorphic functions in lattice like systems, tessellations. They are encountered in Langlands conjecture [K46, A126, A124, A184], whose possible physical meaning I still fail to really understand physically so well that I could immediately explain what it is.
   The hyperboloid L (L for Lobatchevski space) defined as $t^2 - x^2 - y^2 - z^2 = constant$ surface of Minkowski space (particle physicist talks about mass shell) is good example about this kind of system in TGD framework. One can define in this kind of tessellation automorphic functions, which are quasi-periodic in sense that the values of function are fixed once one knows them for single cell of the lattice. Bloch waves serve as condensed matter analog.

One can assign to automorphic function what the article calls its "energy spectrum". In the case of hyperboloid it could correspond to the spectrum of d'Alembertian - this is physicist's natural guess. Automorphic function could be analogous to a partition function build from basic building bricks invariant under the sub-group of Lorentz group leaving the fundamental cell invariant. Zeta function assignable to extension of rationals as generalization of Riemann zeta is one example [L58].

What the discovery could be? I can make only humble guesses. The popular article tells that the "clock solutions" of given Diophantine equation in various finite fields F_p are in correspondence with the "energy" spectra of some automorphic form defined in some space.

The problem of finding the automorphic forms is difficult and the message is that here a great progress has occurred. So called torsion coefficients for the modular form would correspond the integer value roots of Diophantine equations for various finite fields F_p. What could this statement mean?

1. What does automorphic form mean? One has a non-compact group G and functions from G to some vector space V. For instance, spinor modes could be considered. Automorphic forms are eigenfunctions of Casimir operators of G, d'Alembert type operator is one such operator and in TGD framework $G = SO(1,3)$ is the interesting group to consider. There is also discrete infinite subgroup $\Gamma \subset G$ under which the eigenfunctions are not left invariant but transform by factor $j(\gamma)$ of automorphy acting as matrix in V - one speaks of twisted representation.
   Basic space of this kind of is upper half plane of complex plane in which $G = SL(2,C)$ acts as also does $\gamma = SL(2,Z)$ and various other discrete subgroups of $SL(2,C)$ and defines analog of lattice consisting of fundamental domains $\gamma\backslash G$ as analogs of lattice cells. 3-D hyperboloid of M^4 allows similar structures and is especially relevant from TGD point of view. When $j(\gamma)$ is non-trivial one has analogy of Bloch waves.
   Modular invariant functions is second example. They are defined in the finite-D moduli space for the conformal structures of 2-D surfaces with given genus. Automorphic forms transform by a factor $j(\gamma)$ under modular transformations which do not affect the conformal equivalence class. Modular invariants formed from the modular forms can be constructed from these and the TGD based proposal for family replication phenomenon involves this kind invariants as elementary particle vacuum functions in the space of conformal equivalence classes of partonic 2-surfaces [K21].
   One can also pose invariance under a compact group K acting on G from right so that one has automorphic forms in G/K. In the case of $SO(3,1)$ this would give automorphic forms on hyperboloid H^3 ("mass shell") and this is of special interest in TGD. One could also require invariance under discrete finite subgroup acting from the left so that $j(\gamma) = 1$ would be true for these transformations. Here especially interesting is the possibility that Galois group of extension of rationals is represented as this group. The correct prediction of Newton's constant from TGD indeed assumes this [L85].
2. What does the spectrum (`http://tinyurl.com/vakzxye`) mean? Spectrum would be defined by the eigenvalues of Casimir operators of G: simplest of them is analog of d'Alembertian for say $SO(3,1)$. The number of these operators equals to the dimension of Cartan sub-algebra of G. Additional condition is posed by the transformation properties under Γ characterized by $j(\gamma)$.

One can assign to automorphic forms so called torsion coefficients in various finite fields F_p and to the eigen functions of d'Alembertian and other Casimir operators in coset space G/K. Consider discrete but infinite subgroup Γ such that solutions are apart from the factor $j(\gamma)$ of automorphy left invariant under Γ. For trivial $j(\gamma)$ they would be defined in double coset space $\Gamma\backslash G/K$. Besides this Galois group represented as finite discrete subgroup of $SU(2)$ would leave the eigenfunctions invariant.

1. Torsion group T is for the first homotopy group Π_1 (fundamental group) a finite Abelian subgroup decomposing Z_n to direct summands Z_p, p prime. The fundamental group in the recent case would be naturally that of double coset space $\Gamma\backslash G/K$.
2. What could torsion coefficients be (`http://tinyurl.com/u3jv86t`)? Π_1 is Abelian an representable as a product $T \times Z^s$. Z_s is the dimension of Π_1 - rank - as a linear space over Z and $T = Z_{m_1} \times Z_{m_2} \times ....Z_{m_n}$ is the torsion subgroup. The torsion coefficients m_i satisfy the conditions $m_1 \perp m_2 \perp ... \perp m_n$. The torsion coefficients in F_p would be naturally $m_i \ mod \ p$.

The torsion coefficients characterize also the automorphic functions since they characterize the first homotopy group of $\Gamma\backslash G/K$. If I have understood correctly, torsion coefficients m_i for various finite fields F_p for given automorphic form correspond to a sequence of solutions of Diophantine equation in F_p. This is the bridge.

3. How are the Galois groups related to this (`http://tinyurl.com/tje4hvc`)? Representations of Galois group $Gal(F)$ for finite-D extension F of rationals could act as a discrete finite subgroup of $SO(3) \subset SO(1,3)$ and would leave eigenfunctions invariant: these ADE groups form appear in McKay correspondence and in inclusion hierarchy of hyper-finite factors of type II_1 [K99, K33].
   The invariance under $Gal(F)$ would correspond to a special case of what I call Galois confinement, a notion that I have considered in [L93, L16] with physical motivations coming partially from the TGD based model of genetic code based on dark photon triplets.
   The problem is to understand how dark photon triplets occur as asymptotic states - one would expect many-photon states with single photon as a basic unit. The explanation would be completely analogous to that for the appearance of 3-quark states as asymptotic states in hadron physics - the analog of color confinement. Dark photons would form Z_3 triplets under Z_3 subgroup of Galois group associated with corresponding space-time surface, and only Z_3 singlets realized as 3-photon states would be possible.
   Mathematicians talk also about the Galois group $Gal(\overline{Q})$ of algebraic numbers regarded as an extension of finite extension F of rationals such that the Galois group $Gal(F)$ would leave eigenfunctions invariant - this would correspond to what I have called Galois confinement.
4. There is also the idea that the torsion group could have representation as sub-group of Galois group. In TGD the correspondence between physics as geometry and cognitive physics as number theory supports this idea: in adelic physics [L42] cognition would represent number theoretically.

What could be the general vision concerning the connection between Diophantine equations and automorphic forms in TGD framework?

1. In TGD framework an obvious candidate for a space at automorphic side would be the product of $H^3 \times CP_2$ carrying the representations of $SO(1,3) \times SU(3)$. H^3 is 3-D hyperboloid H^3 of M^4 having $SO(1,3)$ as group of isometries. The infinite discrete subgroups of $SO(1,3)$ define tessellations of H^3 analogous lattices in E^3, and one can assign to these automorphic functions as analogs of Bloch waves. They would be associated with separable solutions of spinor d'Alembertian in future light-cone, which corresponds to empty Robertson-Walker cosmology. This is however not the only option: automorphic functions appear also in the description of family replication phenomenon and give rise to modular invariant elementary particle functions in the spaces of conformal moduli for partonic 2-surfaces [K21].
   $M^8 - H$ duality states that space-time can be regarded as a 4-surface in either complexified 8-D Minkowski space having interpretation as complexified octonions or $H = M^4 \times CP_2$. At the level M^8 space-time surfaces are algebraic surfaces assignable to an algebraic continuation of a polynomial with rational (or even algebraic) coefficients to M^8. In H one has minimal surfaces with 2-D algebraic singularities - string world sheets and partonic 2-surfaces. Each polynomial defines extension of rationals and the Galois group of extension acts as a symmetry group for the cognitive representations identified as the set of points of space-time surface with coordinate values in the extension of rationals considered. This is central for adelic physics fusing real physics and physics for extensions of p-adic numbers induced by that for rationals. Cognitive representations would define the number theoretic side and Langlands correspondence and generalization of Fermat's theorem would mean that there is many-to-one correspondence from the automorphic side (embedding space level) to the number theoretic side (cognitive representations). In particular, Galois group of extension would have action as a discrete finite subgroup of $SO(3) \subset SO(1,3)$.
2. In TGD framework Galois group $Gal(F)$ has natural action on the cognitive representation identified as a set of points of space-time surface for which preferred embedding space coordinates belong to given extension of rationals [L37, L38, L39, L67]. In general case the action of Galois group gives a cognitive representation related to a new space-time surface, and one can construct representations of Galois group as superpositions of space-time surfaces and they are effectively wave functions in the group algebra of $Gal(F)$. Also the action of discrete subgroup

of $SO(3) \subset SO(1,3)$ gives a new space-time surface.
There would be two actions of $Gal(F)$: one at the level of embedding spaces at H^3 and second at the level of cognitive representations. Possible applications of Langlands correspondence and generalization of Fermat's last theorem in TGD framework should relate to these two representations. Could the action of Galois group on cognitive representation be equivalent with its action as a discrete subgroup of $SO(3) \subset SO(1,3)$? This would mean concrete geometric constraint on the preferred extremals.

In the sequel I try to make this picture more concrete.

17.4.1 The analog for Diophantine equations in TGD

What could this discovery have to do with TGD?

1. In adelic physics [L42, L43] M^8-H duality is in key role. Space-time surfaces can be regarded either as algebraic 4-surfaces in complexified M^8 determined as roots of polynomial equations. Second representation is as minimal surfaces with 2-D singularities identified as preferred extremals of action principle: analogs of Bohr orbits are in question.
2. The Diophantine equations generalize in TGD framework. One considers the roots of polynomials with rational coefficients and extends them to 4-D space-time surfaces defined as roots of their continuations to octonion polynomials in the space of complexified octonions [L67, L37, L38, L39]. Associativity is the basic dynamical principle: the tangent space of these surfaces is quaternionic, and therefore associative. Each irreducible polynomial defines extension of rationals via its roots and one obtains a hierarchy of them having physical interpretation as evolutionary hierarchy. These surface can be mapped to surface in $H = M^4 \times CP_2$ by $M^8 - H$ duality.
3. So called cognitive representations for given space-time surface are identified as set of points for which points have coordinate in extension of rationals. They realize the notion of finite measurement resolution and scattering amplitudes can be expressed using the data provided by cognitive representations: this is extremely strong form of holography.
4. Cognitive representation generalizes the solutions of Diophantine equation: instead of integers one allows points in given extension of rationals. These cognitive representations determine the information that conscious entity can have about space-time surface. As the extensions approaches algebraic numbers, the information is maximal since cognitive representation defines a dense set of space-time surface.

17.4.2 The analog for automorphic forms in TGD

1. The above mentioned hyperboloids H^3 of M^4 are central in zero energy ontology (ZEO) of TGD: in TGD based cosmology they correspond to cosmological time constant surfaces. Also the tessellations of hyperboloids are expected to have a deep physical meaning - quantum coherence even in cosmological scales is possible [K80, K68] and there are pieces of evidence about the lattice like structures in cosmological scales.
2. Also the finite lattices defined by finite discrete subgroups of $SU(3)$ in CP_2 analogous to Platonic solids and and regular polygons for rotation group are expected to be important. For what this could mean in number theoretic vision about TGD see for the correct prediction of the Newton's constant in terms of CP_2 radius [L85] (`http://tgdtheory.fi/public_html/articles/Gagain.pdf`).
3. One can imagine analogs of automorphic forms for these tessellations. The spectrum would correspond to that for massless spinor d'Alembertian of $L \times CP_2$, where L denotes the hyperboloid, satisfying the boundary conditions given by tessellation. The mass eigenvalues would be determined by the CP_2 spinor Laplacian. In condensed matter physics solutions of Schrödinger equation consistent with lattice symmetries would be in question as quasi-periodic Bloch waves. The spectrum would correspond to mass squared eigenvalues and to the spectra for observables assignable to the discrete subgroup of Lorentz group defining the tessellation.
4. The theorem described in the article suggests a generalization in TGD framework based on physical motivations. The "energy" spectrum of these automorphic forms identified as mass

squared eigenvalues and other quantum numbers characterized by the subgroup of Lorentz group are at the other side of the bridge.
At the other side of bridge could be the spectrum of the roots of polynomials defining space-time surfaces: the roots indeed fix the polynomial of one argument and therefore entire space-time surface as a "root" of the octonionic counterpart of the polynomial. A more general conjecture would be that the discrete cognitive representations for space-time surfaces as "roots" of octonionic polynomial are at the other side of bridge. These two would correspond to each other.
Cognitive representations at space-time level would code for the spectrum of d'Alembertian like operator at the level of embedding space. This could be seen as example of quantum classical correspondence (QCC) , which is basic principle of TGD.

17.4.3 What is the relation to Langlands conjecture (LC)?

I understand very little about LC [K46, A126, A124, A184] at technical level but I can try to relate it to TGD via physical analogies. I have done this actually two times already earlier [K46, K47].

1. LC relates two kinds of groups.
   (a) Algebraic groups satisfying certain very general additional conditions (complex nxn matrices satisfying algebraic conditions is one example). Matrix groups such as Lorentz group are a good example.
   The Cartesian product of future light-cone and CP_2 would be the basic space. d'Alembertian inside future light-cone in the variables defined by Robertson- Walker coordinates. The separation of variables a as light-cone proper time and coordinates of H^3 for given value of a assuming eigenfunction of H^3 d'Alembertian satisfying additional symmetry conditions would be in question. The dependence on a is fixed by the separability and by the eigenvalue value of CP_2 spinor Laplacian.
   (b) So called L-groups assigned with extensions of rationals and function fields defined by algebraic surfaces as as those defined by roots of polynomials. This brings in adelic physics in TGD.
2. The physical meaning in TGD could be that the discrete the representations provided by the extensions of rationals and function fields on algebraic surfaces (space-time surfaces in TGD) determined by them. Function fields might be assigned to the modes of induce spinor fields.
   The physics at the level of embedding space (M^8 or $H = M^4 \times CP_2$) described in terms of real and complex numbers - the physics as we usually understand it - would by LC corresponds to the physics provided by discretizations of space-time surfaces as algebraic surfaces. This correspondence would not be 1-1 but many-to-one. The discretizations provided by cognitive representations would provide hierarchy of unique approximations. Langlands conjecture (or rather, its proof!) would justify this vision.
3. Galois groups of extensions are excellent examples of L-groups an indeed play central role in TGD. The proposal is that Galois groups provide a representation for the isometries of the embedding space and also for the hierarchy of dynamically generated symmetries. This is just what the Langlands conjecture motivates to say.
   Amusingly, just last week I wrote an article deducing the value of Newton's constant using the conjecture that discrete subgroup of isometries common to M^8 and $M^4 \times CP_2$ consisting of a product of icosahedral group with 3 copies of its covering corresponds to Galois group for extension of rationals. The prediction is correct. The possible connection with Langlands conjecture came into my mind while writing these comments.

To sum up, Langlands correspondence would relate two descriptions. Discrete description for cognitive representations at space-time level and continuum description at embedding space level in terms of eigenfunctions of spinor d'Alembertian.

Chapter 18

Some New Ideas Related to Langlands Program *viz.* TGD

18.1 Introduction

Langlands' program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Langlands program is described by Edward Frenkel as a kind of grand unified theory of mathematics (`https://cutt.ly/1BgbfsL`). I have a strong feeling that Langlands program is essential for TGD but every time I encounter the Langlands program, I feel myself an extremely stupid physicist, who tries to understand something, which simply goes over his head. But still I try once again.

18.1.1 About Langlands program

I am not mathematician enough to really describe Langlands program (`https://cutt.ly/ABj2G7D`) and its results. I have only a dim idea about the implications of Langlands correspondence and the following is my humble attempt to get some grasp the basic ideas of this immense topic.

Basic ideas

Wikipedia article (`https://cutt.ly/ABj2G7D`) and the references therein gives a more detailed view of Langlands program [A126, A124], discussed from the TGD perspective in [?, K47]. The following is a brief summary of this article.

1. The slogan "philosophy of cusp forms" was introduced by Harish-Chandra, expressing his idea of a kind of reverse engineering of automorphic form theory, from the point of view of representation theory. Also Israel Gelfand proposed a similar philosophy.
   The discrete completely discontinuous group Γ of $SL(2,R)$ acting in hyperbolic space H^2, fundamental to the classical theory of modular forms, loses its central role. What remains is the basic idea that representations in general are to be constructed by parabolic induction of so-called cuspidal representations.
   Cuspidal representations assignable to hyperbolic 2-manifolds and their higher-D generalizations, of which Teichmueller spaces as moduli spaces of conformal equivalence classes of Riemann surfaces represent an example, become the fundamental class of objects, from which other representations may be constructed by procedures of induction. Note that in TGD, hyperbolic 3-manifolds could replace hyperbolic 2-manifolds and one challenge is to understand how hyperbolic 2-manifolds relate to hyperbolic 3-manifolds.
   Remark: Cusps correspond geometrically to peak-like singularities of say $SL(2,R)/\Gamma$. Parabolic group (`https://cutt.ly/HBj4t4e`) is a subgroup of a linear algebraic group G in field k such that G/P is a projective algebraic variety and contains some Borel subgroup of G as a subgroup (upper diagonal matrices with units at diagonal is the standard example).
2. Functoriality as a category theoretic notion is the second key notion. Roughly, functoriality means that what holds true for a representative of a given type group, should hold generally.

This makes the statements extremely general. The statements can be formulated in adelic framework so that they hold simultaneously for both rationals, extensions of rationals and extensions of p-adic number fields induced by them.

Contents of Langlands conjectures

1. Langlands correspondence is between L-functions associated with irreps of finite Galois group analogous to zeta functions and automorphic cuspidal representations of $Gl(n, C)$ and of even more general reductive groups representable as matrix groups which are analogous to partition functions. Both partition functions and L-functions code for the numbers of objects of particular kind, typically for the degeneracies of quantum states with given quantum numbers. $SL(2, C)$ as a covering of Lorentz group is of special interest in TGD but TGD involves many other reductive groups and partition function type objects could define analogies of automorphic forms, which Langlands correspondence maps to L-functions, which are conjectured to satisfy Riemann hypothesis and functional equations analogous to that satisfied by Riemann ζ.
2. In the case of Artin function L-function is a characteristic determinant for an special element of Galois group, which is Frobenius element mapping elements of the ring of integers of L/K to their p:th power: $x \rightarrow x^p$. For finite fields, $x^p = x$ holds true.
   The Artin conjecture states that automorphic forms (`https://cutt.ly/qBgb6Fw`) as representations of reductive groups correspond to Artin L-functions (`https://cutt.ly/NBgnozT`) assigned to Galois groups and having a product representation analogous to the Euler product for ζ. Artin zeta function is a product of powers of Artin L-functions for all finite-D irreducible representations of the Galois group (see Appendix).
   Langlands pointed out that the Artin conjecture follows from strong enough results implied by the Langlands philosophy, relating to the L-functions associated to automorphic representations for $GL(n)$ for all $n \geq 1$.
3. More precisely, the Langlands correspondence associates an automorphic representation of the adelic version of an algebraic group $GL_n(A_Q)$ to every n-dimensional irreducible representation of the Galois group. The automorphic representation is a cuspidal representation (the representation functions vanish at the tips of cusps) if the Galois representation is irreducible. The Artin L-function of the Galois representation is the same as the automorphic L-function of the automorphic representation. Therefore finite-D representations of Galois group and cuspidal representations of $Gl(n, A_Q)$ correspond to each other.
   The Artin conjecture follows immediately from the known fact that the L-functions of cuspidal automorphic representations are holomorphic. This was one of the major motivations for Langlands' work.
4. Dedekind conjecture states that if L/K is an extension of number fields, then the quotient $s \mapsto \zeta_L(s)/\zeta_K(s)$ of their Dedekind zeta functions is entire function. The Aramata-Brauer theorem states that the conjecture holds if L/K is Galois.
5. There are a number of related Langlands conjectures. There are many different groups over many different fields for which they can be stated, and for each field there are several different versions of the conjectures.

   There are different types of objects for which the Langlands conjectures can be formulated.

1. Representations of reductive groups over local fields, that is archimedean local fields, p-adic local fields, and completions of function fields over complex numbers). In the case of algebraic groups over local fields, adeles allow to combine the representations in all these fields to a single adelic representation, which implies huge generality.
2. Automorphic forms on reductive groups over global fields, which are extensions of rationals or to a function field over finite field defined by rational functions.
3. Representations of reductive groups over finite fields.

18.1.2 Why Langlands program could be relevant for TGD?

It is increasingly clear that the conjectures of the Langlands program have physical analogies in the quantum TGD proposed to be a grand unification of physics.

1. In the view of TGD based on fusion number theoretical and geometric views of physics, rational polynomials determine space-time regions at the fundamental level [L82, L83]. The observations of [L127, L124] inspired the question whether L-functions as generalizations of polynomials be used to define space-time surfaces.
   Conformal confinement would favor this [L124]. The hypothesis that roots are algebraic numbers becomes an interesting possibility strongly favored by Galois confinement implying that the 4-momenta of physical states have integer components whereas virtual states have momenta with algebraic integer valued components. Momentum components would be algebraic integers in an infinite-D extension of rationals.
   What could be the interpretation of these surfaces? Could they represent a higher level of intelligence and define infinite cognitive representations as algebraic integer valued virtual momenta at the mass shells of $M^4 \subset M^8$?
2. Artin's L-functions are associated with n-D representations of Galois groups on one hand and with infinite-D unitary representations ($Gl(n,C)$ and more general Lie groups. The extensions of the representations of Galois groups would be very relevant in TGD since Galois groups become symmetry groups in the number theoretic vision of TGD.
   Quantum TGD provides several candidates for these kinds of groups [L127]. There are groups assignable to the representations of supersymplectic algebras, isometry algebras of the light-cone boundary δM^4_+, and the Kac-Moody type algebras assignable to light-like 3-surfaces defining either boundaries of Minkowskian regions or orbits of partonic 2-surfaces as boundaries between Minkowskian and Euclidean space-time regions [L126]. There are also extended conformal symmetries due to the fact that the light-cone boundary and light-like 3-surfaces are metrically 2-D.
3. The mass shells H^3 of causal diamond (CD) defined by the roots of polynomials allow a realization of $SO(1,3)$ and $SL(2,C)$ allow tessellations and hyperbolic manifolds as analogs of unit cells of lattice. They could make possible the realization of holographic continuations of modular forms associated with hyperbolic 2-manifolds defining boundaries of 3-D hyperbolic manifolds, which could be mapped to L-functions, possibly defining space-time surfaces as analogies of polynomials [L118].
4. Elementary particle vacuum functionals are analogous to partition functions and are determined as modular invariant modular forms in the Teichmueller space parameterizing the conformal equivalence classes of partonic 2-surfaces [K21]. These functions should define L-functions with several variables and they could give rise to L-functions of a single variable by multiple residue integral. For multiple-zetas this procedure gives a product expressible in terms of zetas having the desired physical properties (allowing conformal confinement and possibly even Galois confinement).

18.1.3 Quantum classical correspondence as a feedback loop between the classical space-time level and the quantal WCW level?

Quantum classical correspondence (QCC) has been one of the guidelines in the development of TGD but its precise formulation has been missing. A more precise view of QCC could be that there exists a feedback loop between classical space-time level and quantal "world of classical worlds" (WCW) level. This idea is new and akin to Jack Sarfatti's idea about feedback loop, which he assigned with the conscious experience. The difference between consciousness and cognition at the human *resp.* elementary particle level could correspond to the difference between L-functions and polynomials.

This vision inspires the question whether the generalization of the number theoretic view of TGD so that besides rational polynomials (subject to some restrictions) also L-functions, which have a nice physical interpretation if RH holds true for them, can be defined via their roots 4-surfaces in M^8_c and by M^8-H duality 4-surfaces in H. Both conformal confinement (in weak and strong form) and Galois confinement (having also weak and strong form) support the view that L-functions are Langlands duals of the partition functions defining quantum states.

If L functions indeed appear as a generalization of polynomials and define space-time surfaces, there must be a very deep reason for this.

1. The key idea of computationalism is that computers can emulate/mimic each other. Universe

should be able to emulate itself. Could WCW level and space-time level mimic each other? If this were the case, it could take place via QCC. If so, it should be possible to assign to a quantum state a space-time surface as its classical space-time correlate and vice versa.

2. There are several space-time surfaces with a given Galois group but fixing the polynomial P fixes the space-time surface. An interesting possibility is that the observed classical space-time corresponds to superposition of space-time surfaces with the same discretization defined by the extension defined by the polynomial P. If so, the superposition of space-time surfaces would be effectively absent in the measurement resolution used and the quantum world would look classical.

3. A given polynomial P fixes the mass shells $H^3 \subset M^4 \subset M^8$ but does not fix the space-time surface X^4 completely since the polynomial hypothesis says nothing about the intersections of X^4 with H^3 defining 3-surfaces. The associativity hypothesis for the normal space of $X^4 \subset M^8$ [L82, L83] implies holography, which fixes X^4 to a high degree for a given X^3. Holography is not expected to be completely deterministic: this non-determinism is proposed to serve as a correlate for intentionality.
If space-time has boundaries, the boundaries X^2 of $X^3 \subset H^3$ could be ends of light-like 3-surfaces X^3_L [L126]. An attractive idea is that they are hyperbolic manifolds or pieces of a tessellation defined by a hyperbolic manifold as the analog of a unit cell [L118]. The ends X^2 of these 3-surfaces at the boundaries of CD would define partonic 2-surfaces.
By quantum criticality of the light-like 3-surfaces satisfying $det(-g_4) = 0$ [L126], their time evolution is not expected to be completely unique. If the extended conformal invariance of 3-D light-like surfaces is broken to a subgroup with conformal weights, which are multiples of integer n the conformal algebra defines a non-compact group serving as a reductive group allowing extensions of irreps of Galois group to its representations.
One can also consider space-time surfaces without boundaries. They would define coverings of M^4 and there would be several overlapping projections to H^3, which would meet along 2-D surfaces as analogies of boundaries of 3-space. Also in this case, the idea that the X^3 is a hyperbolic 3-manifold is attractive.

4. Quantum TGD involves a general mechanism reducing the infinite-D symmetry groups to finite-D groups, which has an interpretation in terms of finite measurement resolution [L127] describable both in terms of inclusions of hyperfinite factors of type II_1 and inclusions of extensions of rationals inducing inclusions of cognitive representations. One can also consider an interpretation in terms of symmetry breaking.
This reduction means that the conformal weights of the generators of the Lie-algebras of these groups have a cutoff so that radial conformal weight associated with the light-like coordinate of δM^4_+ is below a maximal value n_{max}. The generators with conformal weight $n > n_{max}$ and their commutators with the entire algebra would act like a gauge algebra, whereas for $n \leq n_{max}$ they generate genuine symmetries. The alternative interpretation is that the gauge symmetry breaks from $n_{max} = 0$ to $n_{max} > 0$ by transforming to dynamical symmetry.
Note that the gauge conditions for the Virasoro algebra and Kac-Moody algebra are assumed to have $n_{max} = 0$ so that a breaking of conformal invariance would be in question for $n_{max} > 0$.

5. The natural expectation is that the representation of the Galois group for these space-time surfaces defines representations in various degrees of freedom in terms of the semi-direct products of the Langlands duals $^LG^0$ with the Galois group (here $^LG^0$ denotes the connected component of Langlands dual of G). Semi-direct product means that the Galois group acts on the algebraic group G assignable to algebraic extension by affecting the matrix elements of the group element.
There are several candidates for the group G [L127]. G could correspond to a conformal cutoff A_n of algebra A, which could be the super symplectic algebra SSA of $\delta M^4 \times CP_2$, the infinite-D algebra I of isometries of δM^4_+, or the algebra $Conf$ extended conformal symmetries of $\delta M4_+$. Also the extended conformal algebra and extended Kac-Moody type algebras of H isometries associated with the light-like partonic orbits can be considered.

6. One could assign to these representations modular forms interpreted as generalized partition functions, kind of complex square roots of thermodynamic partition functions. Quantum TGD can be indeed formally regarded as a complex square root of thermodynamics. This partition function could define a ground state for a space of zero energy state defined in WCW as a

superposition over different light-like 3-surfaces.

These considerations boil down to the following questions.

1. Could the quantum states at WCW level have classical space-time correlates as space-time surfaces, which would be defined by the L-functions associated with the modular forms assignable to finite-D representations of Galois group having a physical interpretation as partition functions?
2. Could this give rise to a kind of feedback loop representing increasingly higher abstractions as space-time surfaces. This sequence could continue endlessly. This picture brings in mind the hierarchy of infinite primes [L127].
   Many-sheeted space-time would represent a hierarchy of abstractions. The longer the scale of the space-time sheet the higher the level in the hierarchy.

18.1.4 TGD analogy of Langlands correspondence

Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC).

1. QCC is formulated as a correspondence between the quantum states in WCW characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via finite-dimensional representations of Galois groups partition functions. Finite-dimensionality in the case of L-functions would have an interpretation as a finite cognitive and measurement resolution. QCC would define a kind of closed loop giving rise to a hierarchy.
2. If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.
3. One should construct Riemann zeta and the associated ξ function as the simplest instances of L-functions assignable to $SL(2,R)$. The Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2 (upper complex plane). Thus the proposal that mass squared values correspond algebraic numbers generalizes: also roots of e can appear as roots.
4. One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework [L127].
5. In particular, elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters [K21]. Multiple residue integral is proposed as a way to obtain L-functions defining space-time surfaces.
6. A highly interesting feedback to the number theoretic vision emerges. The rational polynomials P defining space-time surfaces are characterized by ramified primes. Without further conditions, they do not correlate at all with the degree n of P as the physical intuition suggests. In [L127] it was proposed that P can be identified as the polynomial Q defining an infinite prime [K84]: this implies that P is irreducible.
   An additional condition is that the coefficients of P are smaller than the degree n of P. For $n = p$, P could as such be regarded as a polynomial in a finite field. This proposal is too strong to be true generally but could hold true for so-called prime polynomials of prime order having no functional decomposition to polynomials of lower degree [A103, A160]. The proposal is that all physically allowed polynomials are constructible as functional composites of these. Also finite fields would become fundamental in the TGD framework.

18.2 Langlands conjectures in the TGD framework?

M^8-H duality is a central element of TGD and states the duality of number theoretic and geometric views of physics. This duality is very analogous to Langlands duality.

18.2.1 How Langlands duality could be realized in TGD

It has become gradually more and more clear that the conjectures of the Langlands program could be an essential part of quantum TGD [L110, L127, L126, L124, L118] proposed as a candidate for a grand unification of physics.

1. Could L-functions as generalizations of polynomials be used to define space-time surfaces? The generalization of Riemann hypothesis (RH) states that the non-trivial zeros of L-functions are at critical line and trivial ones at negative real axis. This makes possible conformal confinement in both weak form (conformal weight is integer) and strong form (the sum positive and negative (tachyonic) conformal weights vanishes [L128]). The hypothesis that the roots of L-function are algebraic numbers in an infinite-D extension of rationals is the simplest conjecture and allows the realization of Galois confinement so that the 4-momenta have integer valued momenta using the unit defined by the scale of CD. The transcendental extensions by roots of e define finite-D extensions of p-adic numbers and could also be involved.
   What could be the interpretation of these surfaces? Could they represent higher level of intelligence, could they define infinite cognitive representations.
2. Artin's L-functions are associated with n-D representations of Galois groups on one hand and with infinite-D unitary representations ($Gl(n,C)$ and more general Lie groups. $GL(n,C)$ generalizes to n-dimensional reductive group of which $SL(n), SO(k,n-k)$, and $Sp(2n)$ are examples
   The general proposal [L127] is that the super-symplectic algebra assignable to $\delta M^4_+ \times CP_2$ defining the boundary of causal diamond (CD) in zero energy ontology (ZEO) acts as isometries of WCW.
   The dimension 3 of the light-cone boundary makes possible conformal transformations of $S^2 \subset \delta M^4_+$ made local with respect to the light-like radial coordinate of δM^4_+ and CP_2 as candidates for symmetries. As a special case, one has isometries of $\delta M^4_+ \times CP_2$ for which the local conformal scaling from conformal transformation of S^2 is compensated by a scaling for the radial light-like coordinate depending also on $S^2 \times CP_2$ coordinates are possible symmetries.
   The light-like partonic orbits as boundaries between Minkowskian and Euclidean regions and more general light-like boundaries of space-time surfaces are metrically 2-D and allow generalization of conformal symmetries and possibly also Kac-Moody symmetries assignable to isometries as candidates for symmetries.
   All these algebras, denote them by A, allow infinite-D Lie-algebra labelled by radial conformal weights containing as sub-algebras a hierarchy of sub-algebras A_n for which the conformal weights come as n-multiples of the conformal weights of the entire algebra.
   The states spaces annihilated algebra A_n and the commutator $[A_n, A]$ define a hierarchy of state spaces generalizing the state space for which entire algebra annihilates the states. The associated groups would allow a realization of Langlands groups.
3. $n=2$-D representations could be assigned with complex 2-D representations $SL(2,C)$ at the mass shells H^3 defined by the roots of L-function. The tessellations defined by the discrete completely discontinuous subgroups of $SL(2,C)$ would give rise to hyperbolic manifolds as analogs of unit cells for lattices [L118]. One can also associated with them modular forms which would be mapped to L-functions. Fermionic spin could provide these representations.
   The represention of Galois group sould be somehow extended to a representation of $SL(2,C)$. Could this give a connection between the number theoretical physics of TGD involving the irreps of Galois groups and spinor representations of Lorentz group at mass shells H^3?
4. Elementary particular vacuum functionals as analogs of partition functions in modular degrees of freedom of partonic 2-surface are central for TGD view of family replication phenomenon and can be regarded as modular invariants. They could be mapped to the analogs of L-functions of m arguments. Symmetrized multiple-zetas decompose to a sum over products of ordinary zetas. $m-1$-fold residue integral would give something proportional to ordinary ζ

satisfying RH and could define space-time surface as a correlate of the corresponding quantum states.

18.2.2 Could quantum classical correspondence define an infinite hierarchy of abstractions?

The realization of QCC between WCW and classical levels, proposed in the introduction, gives rise to a hierarchy of space-time sheets with increasing algebraic complexity possibly related also to the hierarchy of infinite primes. Schematically one has the following hierarchy.

Polynomial $P\ rightarrow$ space-time surface with Galois group $\rightarrow$ partition function $Z \rightarrow$ L-function $\rightarrow$ space-time surface with Galois group $Gal_L \rightarrow$ ... There are however strong bounds to the complexity which is representable at quantum level.

It is not easy to imagine the complexity at the higher levels of the hierarchy.

1. If one can speak of a Galois group Gal_L of L-function, it is infinite but profinite and has an ultrametric topology, presumably consisting of p-adic sectors p (there is an analogy with the energy landscape of spin glasses in the TGD view of them [L109]).
   It is not enough that the space-time surface defined by L-function contains information of quantum state, this information must be also represented as quantum state and this requires a new partition function assigned with Gal_L. This suggests a connection with the hierarchy of infinite primes [K84] analogous to a hierarchy of second quantizations of a supersymmetric arithmetic QFT [L127]. The assumption that the representations of Gal_L are finite-dimensional would pose a strong constraint to the complexity.
2. For composite polynomials $P_n \circ .... \circ P_1$, the Galois group Gal has a decomposition to a hierarchy of normal subgroups such that a normal subgroup H is Galois group for an extension rationals. The group representation reduced to that for H if Gal/H is represented trivially. If also Gal_L has finite normal subgroup H, one obtains finite-D representations by requiring that Gal/H is represented trivially. This would mean a huge loss of information.
   Can Gal_L have finite normal subgroups? If the L-function is determined by a partition function associated with a representation of Gal, Gal itself is a good guess for H so that Gal_L would reduce to Gal in this particular case! This would conform with the idea that the higher levels of the hierarchy contain all the lower levels.

What one can say about the Galois group Gal_L having variants for rationals and various p-adic number fields.

1. The Absolute Galois group (`https://cutt.ly/nBgndkY`) assignable to algebraic numbers acts as automorphisms of algebraic numbers leaving rationals invariant. This definition could apply also in the case of L-function even in the case that the extension of rationals assigned to L-function involves transcendentals.
   For rationals Absolute Galois group is infinite but profinite, which says that it is in some sense composed of finite groups. Profinite topology is totally discontinuous as also p-adic topology (`hthttps://cutt.ly/MBxdFg8`). A system of finite groups and homomorphisms between them is needed and implies that finite approximations are excellent. Profiniteness is analogous to hyperfiniteness for the factors of von Neumann algebras, which are central in quantum TGD [L127], and are indeed assumed to be closely related to the hierarchies of extensions of rationals.
2. The absolute Galois groups for finite-D extensions K of p-adic number field Q_p have a finite number of elements given by $N = K/Q_p + 3$ so that in p-adic sectors the situation simplifies dramatically, and this reduction would naturally be behind the profiniteness. This must be essential also in the case of the absolute Galois group of rationals and its extensions.
3. Galois groups of infinite-D extensions, say those possibly associated with L-functions, are also profinite.

Suppose that one can speak of the Galois group Gal_L of an L-function associated with a finite Galois group Gal. Suppose Gal_L has finite subgroups, such as Gal.

1. Could this kind of finite-D representation for Gal_L be assigned with, not a necessary rational polynomial, of finite degree? Galois group can indeed permute also the roots of a polynomial, which is not rational. Now one does not however obtain a finite-D extension of rationals.

2. For instance, the cutoff of the product representation of ξ function (https://cutt.ly/5BjcCcv) associated with ζ as a product $\xi(s) = \prod_k (1 - s/s_k)(1 - s/\overline{s}_k)$, assuming that the imaginary parts of the roots are below some upper bound, defines a polynomial P, which is not a rational polynomial and has coefficients, which belong to an extension of rationals, which need not be finite-D or even algebraic. The roots of the polynomial define an extension of this extension. It is implausible that the extension defined by a finite number of roots of ζ can be a finite-D extension of rationals.
This leads to an interesting, possibly testable, conjecture concerning $\xi(s) \equiv \tilde{\xi}(u = s(s-1)) = \sum_k c_k u^k$ and its generalization for the extensions of rationals. Complete p-adic democracy requires that the coefficients have the same meaning irrespective of the number field. This is true if the Taylor coefficients c_k of $\tilde{\xi}(u)$ satisfy $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ defines the hyperbolic analogy for a root of unity and gives rise to a *finite-D* transcendental extension of p-adic numbers. Together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.
3. The hierarchy of L-functions associated with QCC is restricted by the finite-dimensionality of the Galois representation. Although in principle the classical space-time surface contains an infinite amount of potentially representable algebraic information, only a small part of it is represented in terms of quantum states.

18.2.3 About the p-adic variants of L-functions in the TGD framework

In the TGD framework, the existence of p-adic variants of L-functions and modular forms would be highly desirable. The conjecture that the roots of L-functions are algebraic numbers raises the hope that one could define these functions for p-adic integers s satisfying $s = O(p)$.

A stronger hypothesis is that L-functions are analogous to rational polynomials. The strongest meaning of this statement is that their values for rationals are rational. In particular the values of $\zeta(n)$ and $\xi(n)$ should be rational numbers. They are not. A weaker statement would be that the roots of L-functions are algebraic numbers.

The Hadamard product for ξ could make sense p-adically if the sums over the monomials defined by the products of the terms $(s_k\overline{s}_k)^{-1}) = 1/(1/4 + y_k^2)$, define algebraic numbers in the extension of rationals.

Kubota-Leopoldt variant of Dirichlet L-function

There exists proposals for the definitions of p-adic L-functions L_p (https://cutt.ly/wBgafkz). Both their domain and target are p-adic. The Kubota-Leopoldt variant $L_p(s,\chi)$ of Dirichlet L-function $L_p(s,\chi)$ serves as an example.

One starts from Dirichet L-function

$$L(s,\chi_m) = \sum_n \frac{\chi_m(n)}{n^{-s}} = \prod_p \frac{1}{1-\chi_m(p)p^{-s}} \ , \tag{18.2.1}$$

where one has product over primes. $\chi_m(n)$ is Dirichlet character mod integer m (https://cutt.ly/WBKCpZZ), which satisfies $\chi_m(ab) = \chi_m(a)\chi_m(b)$ and vanishes if n is divisible by m. One restricts the consideration to negative integers $s = 1 - n$. The factor $p^{-s} = p^{n-1}$ approaches zero in the p-adic sense for $n \rightarrow \infty$. Unexpectedly, just this Euler factor must be dropped from ζ.

One can express Dirichlet L-function in terms of generalized Bernoulli numbers (https://cutt.ly/DBgajxq) as

$$L(1-n,\chi_m) = -\frac{B_{n,\chi_m}}{n} \ , \tag{18.2.2}$$

where $B_{n,\chi}$ is a generalized Bernoulli number defined by

$$\sum_{n=0}^{\infty} B_{n,\chi_m} \frac{t^n}{n!} = \sum_{a=1}^{f} \frac{\chi_m(a) t e^{at}}{e^{ft}-1} \tag{18.2.3}$$

for χ_m a Dirichlet character (`https://cutt.ly/gBgnhoh`) with conductor f defined as the smallest power of prime for which χ_m is periodic.

The idea of the continuation is that Bernoulli numbers $B_n = -n\zeta(1-n)$, as also generalized Bernoulli numbers, are rational numbers and therefore make sense p-adically.

The Kubota-Leopoldt p-adic L-function $L_p(s,\chi)$ interpolates the Dirichlet L-function with the Euler factor associated with p removed. For positive integers n divisible by $p-1$, one has

$$L_p(1-n,\chi_m) = (1-\chi_m(p)p^{n-1})L(1-n,\chi_m) \ . \tag{18.2.4}$$

When n is not divisible by $p-1$, this does not usually hold but one has

$$L_p(1-n,\chi_m) = (1-\chi_m\omega_p(-n)p^{n-1})L(1-n,\chi_m(n)\omega_p(n)) \ . \tag{18.2.5}$$

Here ω is so called Teichmüller character $\omega(n)$, which is the p:th root of p-adic numbers n (`https://cutt.ly/WBgabxC`).

To my layman understanding, this definition depends on the interpretation of $1-n$ as an ordinary integer. For a p-adic integer, the sign does not have a real meaning so that this definition should make sense also for positive real integers interpreted as p-adic integers so that one can write $1-n = (1-(p-1)/(1-p)n = (1+(p-1)\sum_{k=0}^{\infty} p^k)n = (p+\sum_{k>0}^{\infty} p^k)$. Note that $1-n$ is p-adically of order $O(p)$, which suggests that quite generally this must be the case for the argument of ζ_p.

What could the p-adic variant of a function $f(x)$ mean?

It is not obvious what p-adicization of function $f(x)$ could mean. One can start from a Taylor expansion $f(x) = \sum f_n x^n$. A natural condition is that both the real and p-adic variant converge with an appropriate conditions on the norm of the argument used.

1. The naive approach requires that the coefficients f_n are identical. If algebraic numbers appear in coefficients f_n, an extension of rationals inducing that of p-adic numbers is needed.
   One could replace x with pinary expansion $x = \sum_n x_n p^n$, say identical rational numbers. For instance, for exponent function this would mean that the p-adic variant of $exp(x)$ exists only for $x_p < 1$. Typically, the p-adic expansion in powers p gives an infinite result in the real sense. One could argue that the correspondence must be more physical.
2. A physical correspondence is achieved in p-adic mass calculations [K50] by canonical identification, whose simplest variant is

$$I : x = \sum x_n p^n \rightarrow I(x) = \sum x_n p^{-n} \tag{18.2.6}$$

   mapping p-adic numbers to real numbers. I is continuous and 2-1 for rationals since rationals in real sense have to equivalent expansions as real numbers since one has $1 = (p-1)/p)(1 + 1/p + 1/p^2 + ...)$ implying that the in inverse of I is 2-valued: $1_R \rightarrow 1$ and $1_R = (p-1)/p)(1 + 1/p+1/p^2+...) \rightarrow (p-1)p(1+p+p^2+...))$ (for decimal expansions one has $1.000... = 0.99999...$).
3. For rational coefficients f_n, the simplest correspondence means reinterpretation as a p-adic number r_n/s_n. This would mean that small real values proportional to $1/p^{-n}$ are mapped to values with a large p-adic norm. A way avoid this is canonical identification. One can separate from rational valued f_n power p^k of p and map it to p^{-k} and treate the remaining factor as a p-adic number.
4. One can hope that this generalizes to the case when the coefficients f_n are in an extension of rationals defining extension of p-adic numbers and even in a possibly existing infinite-D extension fo rationals associated with f.

p-Adic Riemann zeta from Hadamard product

p-Adic Riemann zeta function could be obtained from Hadamard product if the roots of zeta are algebraic numbers.

1. The Hadamard product representation of $\zeta(s)$ (see `https://cutt.ly/ABgaQwE` and `https://cutt.ly/BBgaTf6`) is given by

$$\zeta(s) = \frac{e^{[(ln(2\pi)-1-\gamma/2)s]}}{2(s-1)\Gamma(1+s/2)}\prod_\rho (1-\frac{s}{\rho})e^{s/\rho} \ . \quad (18.2.7)$$

   Here γ is the Euler-Mascheroni constant and $\Gamma(s)$ is the Gamma function.

2. The roots $s=-2m$, $m>0$ represent the first problem. The roots with $m=O(p)$ can have an arbitrarily small p-adic norm so that the product of the factors $1-s/\rho$ from the negative real axis does not converge. Therefore one must drop these roots. This corresponds to the dropping of the Euler factor $1/(1-p^{-s})$ from the product form of ζ necessary in the definition of p-adic zeta by Kubota and Leopoldt. Note that this problem is not involved with the ξ function for which the expression of ξ reduces to $\xi(s) = (1-s/\rho_k)(1-s/\overline{\rho}_k)$.

3. Suppose that $s=O(p)$ holds true and the roots ρ of the ζ function are algebraic numbers. RH implies that they have modulus 1. Therefore one can expand $e^{s/\rho}$ in Taylor series and the factors $(1-\frac{s}{\rho})e^{s/\rho})$ as ratios of the Taylor series to the first Taylor polynomial are of form $1+O(p^2)$ so that the product converges.
   The factors $1/(\Gamma(1+s/2)$ and $(1/(s-1)$ can be expanded around $s=1$ to a convergent Taylor series.

4. The problematic term is the factor $e^{[(ln(2\pi)-1-\gamma/2]s]}$. If the coefficient $ln(2\pi)-1-\gamma/2$ is an algebraic number in the extension defined by the roots of zeta then also this exponent converges for p-adic integers $s=O(p)$, which belong to the extension of p-adic numbers conjectured to induced by the extension of rational defined by ζ. The existence of the Kubota-Leopoldt variant of the p-adic zeta indeed suggests that this is the case. If this is not the case, only $\xi(s)$ remains under consideration unless one allows transcendental extensions.

p-Adic ξ function from Hadamard product

ξ function (https://cutt.ly/5BjcCcv) is closely related to ζ and is much simpler. In particular, it lacks the trivial zeros forcing to drop from ζ the Euler factor to get ζ_p. ξ has a very simple representation completely analogous to that for polynomials (`https://cutt.ly/BBgaTf6`):

$$\xi(s) = \frac{1}{2}\prod_k (1-\frac{s}{s_k})(1-\frac{s}{\overline{s}_k}) \ . \quad (18.2.8)$$

Only the non-trivial zeros appear in the product.

1. For $s=O(p)$, this product is finite but need not converge to a well-defined p-adic number in the infinite extension of p-adic numbers. Also the values of $\xi(s)$ at integer points are known to be transcendental so that the interpretation as a generalization of a rational polynomial fails. Note that the presence of an infinite number terms in the product can cause transcendentality of the coefficients of $\xi(s)$. Algebraic numbers are required. $\xi(2n)$ is proportional to π^2 and $\xi(2n+1)$ to $\zeta(2n+1)/\pi^2$. The presence of an infinite number of terms in the expansion of $\xi(s)$ can however cause this.

2. The Hadamard product can be written in the form

$$\xi(s) = \frac{1}{2}\prod_k(1+\frac{s(s-1)}{X_k}) \ , \quad X_k = s_k\overline{s}_k \ , \quad (18.2.9)$$

   in which the $s \leftrightarrow 1-s$ symmetry is manifest. The power series of $\xi(s) = \tilde{\xi}(u) = \sum a_n u^n$, $u=s(s-1)$, should converge for all primes p.
   If regards $s(s-1)$ as p-adic number and apply the inverse of I $s(s-1)$ to get real number.
   If the coefficients a_n of the powers series $\sum_n a_n u^n$ are numbers in an extension of rationals (not necessarily algebraic), the power series in s converges for $s=O(p)$ under rather mild conditioons. For instance, the coefficient of the zeroth order term is $1/2$. The coefficient of the first order term in u is $-(1/2)\sum_k 1(s_k\overline{s}_k^{-1} = -2\sum_k(1+4y_k^2)^{-1}$.

   One can deduce formal expressions for the Taylor coefficiens of $\xi(s)$.

1. Taking $u = s(s-1)$ to be the variable, the coefficients of u^n in $\xi(s) = \tilde{\xi}(u)$ are given by

$$\begin{array}{l} \sum_{U_n} \prod_{k \in U_n} \frac{1}{X_k} \quad , \\ X_k = s_k \overline{s}_k \quad . \end{array} \tag{18.2.10}$$

2. The calculation of the coefficients c_n is simple. In particular, c_1 and c_2 can written as

$$\begin{array}{l} c_1 = \frac{1}{2} \sum_i \frac{1}{X_i} \quad , \\ c_2 = \frac{1}{2} \sum_{i \neq j} \frac{1}{X_i X_j} \\ = \frac{1}{2} \sum_{i,j} \frac{1}{X_i} \frac{1}{X_j} - \frac{1}{2} \sum_i \frac{1}{X_i^2} \\ = \frac{1}{2} c_1^2 - \frac{1}{2} \sum_i \frac{1}{X_i^2} \quad . \end{array} \tag{18.2.11}$$

The calculation reduces to the calculation of sums $\sum_1 /X_i k$, $k = 1, 2$.

3. Also the higher coefficients c_n can be calculated in a similar way recursively by subtracting from the sum $\sum_{i_1 \ldots i_n} \prod_{i_k} X_{i_1}^{-1} = c_1^n$ without the constraint $p_i \neq p_j \neq \ldots$ the sums for which $2, 3, \ldots, n$ primes are identical. One obtains a sum over all partitions of U_n. A given partition $\{i_1, \ldots, i_k\}$ contributes to the sum the term

$$d_{i_1,\ldots,i_k} \prod_{l=1}^{k} c_{i_l} \quad , \quad \sum_{i=1}^{k} n_i = n \quad . \tag{18.2.12}$$

The coefficient $d_{i_1,\ldots,i_k}$ tells the number of different partitions with same numbers $i_1, \ldots, i_k$ of elements, such that the n_i elements of the subset correspond to the same prime so that this subset gives c_{n_i}. Note that the same value of i can appear several times in $\{i_1, \ldots, i_k\}$.
The outcome is that the expressions of c_n reduce to the calculation of the numbers $A_k = \sum_i 1/X_i^k$.

Could one deduce conditions on the coefficients of ξ from number theoretical democracy?

Can one pose additional conditions in the case of ζ or ξ? I have difficulties in avoiding a tendency to bring in some number theoretic mysticism in hope say something interesting of the values of the coefficient X_n in the power series $\xi = c_n u_n$, $u = s(s-1)$, which can calculated from the Hadamard product representation. Number theoretical democracy between p-adic number fields defines one form of mysticism.

There is however also a real problem involved. There is a highly non-trivial problem involved. One can estimate the real coefficients X_k only as a rational approximation since infinite sums of powers of $1/X_k$ are involved. The p-adic norm of the approximation is very sensitive to the approximation.

Therefore it seems that one *must* pose additional conditions and the conditions should be such that the coefficients are mapped to numbers in extension of p-adic numbers by the inverse of I as such so that they should be algebraic numbers or even transcendentals in a finite-D transcendental extension of rationals, if such exists.

1. One could argue that the coefficients c_n must obey a number theoretical democracy, which would mean that they can distinguish p-adically only between the set of primes p_k appearing as divisors of n and the remaining primes. One could require that c_n is a number in a finite-D extension of rationals involving only rational primes dividing n.
2. One could pose an even stronger condition: the coefficients c_n must belong to an n-D algebraic extension of rationals and thus be determined by a polynomial of degree n. Polynomials P of rational coefficients p_n bring in failure of the number theoretic democracy unless one has $p_n \in \{0, \pm 1\}$. For $p = 2$ one does not obtain algebraic numbers. For $p = 3$ this would bring in $\sqrt{5}$.
3. These conditions would guarantee that for a given prime p the coefficients of the expansion would be unaffected by the canonical identification I and at the limit $p \to \infty$ the Taylor coefficients of p-adic ξ_p would be identical with those of ξ.
4. One could allow finite-D transcendental extensions of p-adic numbers. These exist. Since e^p is an ordinary p-adic number, there is an infinite number of extensions with a basis given by the powers roots $e^{k/n}$, $k = 1, \ldots, np - 1$ define a finite-D transcendental extension of p-adics for every prime p.

The strongest hypothesis is that the coefficients c_k are expressible solely as polynomials of this kind of extensions with coefficients, which are algebraic numbers of integers in an extension of rationals by a k:th order polynomial P_k, whose coefficients belong to $\{0, \pm 1\}$.

This picture suggests a connection with the hyperbolic geometry H^2 of the upper half-plane, which is associated with ζ and ξ via Langlands correspondence.

1. The simplest option is that the roots of P_k correspond to the k:th roots x_i of unity satisfying $x_i^k = 1$ so that $cos(n2\pi/k)$ and $sin(n2\pi/k)$ would appear as coefficients in the expression of c_k. The numbers $e^{k/n}$ would be hyperbolic counterparts for the roots of unity.
2. The coefficients c_k would be of form

$$c_k = \sum_{i,j} c_{k,ij} e^{i/k} exp(\sqrt{-1}2\pi(j/n)) \quad , \quad c_{k,rs} \in \{0, \pm 1\} \quad . \tag{18.2.13}$$

The coefficients could be seen as Mellin-Fourier transforms of functions defined in a discretized hyperbolic space H^2 defined by 2-D mass shell such with coordinates $(cosh(\eta), sinh(\eta)cos(phi), sinh(\eta)sin(\phi))$, $\eta) = i/k$, $\phi = 2\pi j/n$. η is the hyperbolic angle defining the Lorentz boost to get the momentum from rest momentum and ϕ defines the direction of space-like part of the momentum. Upper complex plane defines another representation of H^2. The values of functions are in the set $\{0, \pm 1\}$.

3. The points of H^2 associated with a particular c_k would correspond to the orbit of a discrete subgroup of $SO(1,1) \times SO(2) \subset SO(1,2) \subset SL(2,R)$ ($SL(2,R)$ is the covering of $SO(1,2)$). A good guess is that this discretization could be regarded as a tessellation of H^2 and whether other tessellations (there exists an infinite number of them corresponding to discrete subgroups of $SL(2,R)$ could be associated with other L-functions. Mellin transform relates Jacobi theta function (`https://cutt.ly/1B96SSE`), which is a modular form, to $2\xi/s(s-1)$. Therefore $SL(2,C)$, having $SL(2,R)$ as subgroup acting as isometries of H^2, is the appropriate group. Note that the modular forms associated with the representations of algebraic subgroups of $SL(2,C)$ defined by finite algebraic extensions of rationals correspond to L-functions analogous to ζ. Now one would have a hyperbolic extension of rationals inducing a finite-D extension of p-adic numbers.

Just for curiosity and to see how the proposal could fail, one can look at what happens for the first coefficient c_1 in $\xi(s) = \tilde{\xi}(s(s-1)) = \sum c_n s^n$.

1. c_1 would be exceptional since it cannot depend on any prime. c_2 could involve only $p = 2$, and so on.
2. The only way out of the problem is to allow finite-D transcendental extensions of p-adic numbers. These exist. Since e^p is an ordinary p-adic number, there is an infinite number of extensions with a basis given by the powers roots $e^{k/n}$, $k = 1, ..., np-1$ define a finite-D transcendental extension of p-adics for every prime p. For ξ the extension by roots of unity could be infinite-dimensional.
The roots $e^{k/n}$, $k \in 1, ...n$ belong to this extension for all primes p and are in this sense universal. One can construct from the powers of $e^{k/n}$ expressions for c_1 as $c_1 = \sum_k a_k e^{-k/n}$, $a_k \in \{\pm = 0, \pm 1\}$.
3. This would allow to get estimates for n using $x_1 = d\xi/ds(0) \simeq .011547854 = 2c_1$ as an input:

$$c_1 = \sum a_k e^{-k/n} = \frac{x_1}{2} \quad .$$

For instance, the approximation $c_n = e - e^{(n-1)/n}$ would give a rough starting point approximation $n \sim 117$. It is of course far from clear whether a reasonably finite value of n can reproduce the approximate value of c_1.

18.2.4 What about the p-adic variants of modular forms?

What about modular forms as analogs of partition functions? Also they should exist for the same value range for integer conformal weights.

1. Very roughly, L-function is obtained from the Fourier expansion of modular forms

$$Z(s) = \sum c_n q^n \quad , \quad q(s) = e^{i2\pi ns} \tag{18.2.14}$$

by the replacement

$$q^n \rightarrow n^{-s} \quad . \tag{18.2.15}$$

2. A natural condition is that the p-adic variants of $Z(s)$ and $L(s)$ converge for the same range of values of s. The appearance of $i2\pi$ in the exponential is problematic from the point of view of p-adicization.
3. In the p-adic thermodynamics modular form corresponds to a partition function and the natural identification of q is as

$$q = p^{n/T_p} \quad , \tag{18.2.16}$$

where n is conformal weight as eigenvalue $h = n$ of the scaling generator L_0 representing mass squared value and $T_p = 1/k$ is the p-adic temperature. n is interpreted as a p-adic integer so that the partition function converges extremely rapidly in p-adic mass calculations for which p is very large for elementary particles ($= M^{127} = 2^{127} - 1$ for electron).
Note that ordinary Boltzmann weights $exp(-n/T_p)$ would make sense if $1/T_p = O(p)$ holds true. The sum over Boltzmann weights would not however converge since $exp(-n/T_p)$ would have p-adic norm equal to 1. Therefore one must replace e by p: in the real context this would mean only a redefinition of temperature.
4. Naively, the correspondence between modular forms and L-functions should be $q = p^{n/T_p}" = "e^{nln(p)/T} \rightarrow n^s$, $s = O(p)$, by using the definition $\zeta = \sum n^{-s}$. This would suggest the correspondence $1/T_p = k \rightarrow s$. This would conform with the interpretation as p-adic integers but why should one have $k = O(p)$ as required by the definition based on the Hadamard formula? Should one simply assume that $T_p = k \rightarrow s/p$?

Can one make sense of the summand n^{-s}?

1. If n is of form $n = 1 + O(p)$, p-adic logarithm $log_p(n) = log(1 + O(p))$ exists as Taylor series and is of order $O(p)$ and the exponent $exp(log(n)s)$ exists even for $s = O(1)$.
2. p-Adic logarithm can be defined for $p \geq k \geq 0$ by using the finite field property of p-adic integers $0 < x < p$. In this case $log(n)$ contains also an $O(1)$ term so that n^{-s} would make sense only for $s = O(p)$. Therefore there would be a consistency between two definitions for integers n not divisible by p. For $n \propto p^n$ one must have an extension allowing $log(p)$. Should the extension of rationals possibly assignable to zeta contain also logarithms of primes, which are not algebraic numbers?
3. An alternative way is to drop integers n proportional to powers of p from from the definition of ζ. This corresponds to the dropping of the Euler factor $1/(2 - p^{-s})$ associated with p in the product form of zeta used to define zeta for negative integers.
4. One could also restrict the consideration to ξ and use the Hadamard product.

18.2.5 p-Adic thermodynamics and thermal zeta function

The Dirichlet series defines an L-function. The definition of Dirichlet series is following. Consider entities a with integral weight $w(a)$, say quantum states characterized by conformal weight n. Suppose that there are $g(n)$ states with conformal weight n. The sum $\sum w(a)^{-s} = \sum g(n)n^{-s}$ defines the Dirichlet series with nice properties.

This kind of system also has a description in terms of a partition function, which assigns to the partition function an analog of modular form. In the assignment of an L-function to a modular form, the $\sum g(n)exp(-n/T)$ is replaced with $\sum g(n)n^{-s}$ in the real case.

In the p-adic case $\sum g(n)p^{n/T_p)}$ is replaced with a similar sum. The p-adic temperature T_p is quantized to $T_p = 1/n$ for the p-adic partition function. In the p-adic case, the number theoretical existence allows only integer values of $1/T_p$ as a counterpart of s. One can also consider finite-D extensions of rationals for which p-adic extension allows some p-adic roots of integers.

If the p-adic partition function Z for the scaling generator L_0 appearing in the p-adic mass calculations [K50, K21], allows an analog of the zeta function and if it satisfies RH hypothesis, one obtains conformal confinement in weak and strong form and if the roots of the L-function are algebraic numbers, also Galois confinement. This could define a 4-D space-time surface as a classical correlate of the thermal state or its complex square root.

18.2.6 Could elementary particle vacuum functionals define analogs of L-functions?

Elementary particle vacuum functionals (EPVFs) [K21] are defined in the space of conformal equivalences of partonic 2-surfaces and therefore correspond to wave functions in WCW. A partonic 2-surface with a given topology allows a complex structure and moduli space for them. The induced metric defines the conformal equivalence class. Teichmueller space parameterizes this moduli space and is part of WCW. Explanation of the family replication phenomenon is based on hyper-ellipticity.

EPVFs are identified as modular invariant modular forms and are constructed from Jacobi theta functions, which for a given genus g depends on $D = 3g-3$ Teichmueller parameters forming a complex symmetric matrix with positive imaginary part for $g \geq 2$ and on $D = 0$ *resp.* $D = 1$ parameters for g_0 *resp.* $g = 1$. This space can be regarded as a generalization of the upper half of the complex plane (hyperbolic space H^2). For $g = 1$ EPVFs depend on a single theta parameter and the corresponding L-function would satisfy RH.

One can assign to these modular forms L-functions by developing them to Fourier series as $\sum_n c_n q^n$, $q = exp(i2\pi s)$. To this series one can assign an L-function by the replacement $q^n \rightarrow s^{-n}$. I am not quite sure how closely this corresponds to Mellin transform (`https://cutt.ly/NBgnluF` and `https://cutt.ly/dBgncAR`).

The general philosophy described above suggests that it should be possible to assign to EPVF an L-function of a single variable, whose roots would define a space-time surface providing classical representation of the quantum state considered. One should define a multivariable L-function as an analog of poly-zeta and assign to it an L-function of a single variable.

1. One can define multivariable analogs of L-functions. One can imagine a straight forward generalization of the definition of L-function by starting from a multiple Fourier series of Riemann theta function with respect to its arguments, which are Teichmüller parameters Ω_{ij} parameterizing conformal equivalence classes of partonic 2-surfaces. One has $\Omega_{ij} = \Omega_{ji}$, $Im(\Omega_{ij}) > 0$ (one has a higher-D analog of the upper half-plane). The variables s_k are in 1-1 correspondence with the variables Ω_{ij}, $j \geq i$.
   The analogs of L-functions depending on several complex variables $s_1, ..., s_n$ cannot be as such used as a generalization of polynomials. One should identify an L-function of a single variable. One should get rid of the variables $s_2, ..., s_n$.
2. Could one mimic the construction of twistor amplitudes? Could one solve first a residue of a pole of generalized L-function with respect to s_n as a function of $s_1, s_2, ..., s_{n-1}$, after that the residue of the pole with respect s_{n-1} and so on .... At the final step one would get a polynomial of a single variable s_1. Could it be analogous to an L-function of a single variable and have zeros with half-integer valued real part?
   The interpretation would be as a residue integral over variables $s_2, ..., s_n$: similar integrals appear in the construction of twistor amplitudes. There is evidence that his idea might work for ξ functions (`https://cutt.ly/jBgnmOJ`). On the theory of normalized Shintani L-function and its application to Hecke L-function see (`https://cutt.ly/SBgnTUN`).

The following argument provides support for this idea in the case of multiple zeta functions (polyzetas) (see `https://cutt.ly/oBgnO54`, `https://cutt.ly/cBgnXDn` and `https://cutt.ly/ZBgnV6Y`).

1. Poly-zetas have $\{s_1, s_2, ..., s_n\}$ as arguments. One has $\zeta(s_1, ..., s_n) = \sum_{n_1 > n_2 > ... > n_k > 0} \prod_{i=1}^{k} s_i^{-n_i}$. Otherwise one would have a product of ordinary zeta functions.
2. In the Wikipedia article, a variant of polyzeta denoted by $S(s_1, ..., s_n)$ is introduced as $S(s_1, ..., s_n) = \sum_{n_1 \geq n_2 \geq ... \geq n_k > 0} \prod_{i=1}^{k} s_i^{-n_i}$: ">" is replaced with "$\geq$" in the summation.
   By separating from the sum various cases in which 2 or more integers n_i are identical, one can decompose $S(s_1, ..., s_n)$ to a sum over products of the ordinary zeta functions with arguments, which are sums $s_i + s_{i+1} + s_{i+r}$ of subsequent arguments associated with partitions of $\{s_1, ..., s_n\}$ to l subsets $\{s_1, s_2, ..., s_{k_1-1}\}$, $\{s_{k_1}, ..., s_{k_2-1}\}$, ..., $\{s_{k_l}, ..., s_n\}$ respecting the ordering. One can think that the arguments s_i are along a line, and divide the line in all possible ways to segments.
3. One can also form a symmetrized sum $\sum_{\Pi} \zeta(s_{\Pi(1)}, ..., s_{\Pi(k)})$ of $\zeta(s_1, ..., s_k)$ over permutations of $\{s_1, s_2, ..., s_k\}$ to l subsets. The theorem of Hoffmann, mentioned in the Wikipedia article, states that the symmetrized polyzeta reduces to a sum over products of ordinary zetas assigned

over all partitions such that the argument associated with a given subset of partition is the sum $s_{i_1} + ... + s_{i_r}$.

4. If the multiple L-function corresponds to the symmetrized variant of $\zeta(s_1, ..., s_n)$, its $k-1$-fold residue integral decomposes to a sum of residue integrals, which give a vanishing result except in the case of $\zeta(s_1) \times ... \times \zeta(s_k)$ of k zetas assignable to the maximal partition.
If one assumes $s_1 + s_2 + ... + s_k = s$, for the multiple residue integration contour, the integral is proportional $\zeta(s-k)$. The non-trivial zeros are at the critical line $Re(s) = k + 1/2$ and the trivial zeros are at the points $s = k - 2m$, $m \geq 0$. The permutation symmetry of the multiple residue integral suggests that the symmetrization can be performed by using symmetry of the integration measure so that also in this case the outcome is proportional to $\zeta(s-k)$.

18.2.7 Could the tessellations of H^3 be obtained from those of H^2 by holography?

A rather attractive idea is that the 2-D modular forms in 2-D hyperbolic manifolds of H^2 allow a holographic continuation to 3-D modular forms in 3-D hyperbolic manifolds of the mass shell H^3.

1. Compactification of a modular curve is determined by the infinite subgroup Γ of $SL(2,R)$ in the hyperbolic plane H^2 is obtained by adding cusp points located at real axis. The hyperbolic unit cell as hyperbolic 2-manifold has cusps as sharp tips.
2. Does the 2-D hyperbolic manifold extend to a hyperbolic manifold in H^3 having $SL(2,C)$ as a covering group of isometry group $SO(1,3)$? Modular function in H^3. The modular curve as 2-D hyperbolic manifold would be extended to a hyperbolic 3-manifold (`https://cutt.ly/NBgbAYC`) and would have a 2-D hyperbolic manifold as its boundary just like H^2 has real line as a boundary.
Hyperbolic 3-manifold could be identified as a 3-surface at H^3 defining the unit cell of tessellation. Compactication would add points to the counterparts of cusps as singular points, which would naturally correspond to the boundary of the coset space forming a 2-D hyperbolic manifold.
The continuation from H^2 to H^3 would correspond to the extension of γ as a subgroup of $SL(2,R)$ to its complexification as a subgroup of $SL(2,C)$. The extension would be analogous to the continuation of real analytic function to complex analytic function as a form of holography.
3. The physical analogy with the boundary of Fermi torus [L118] is rather obvious. This would conform with the strong form of holography stating that the boundary of 3-surface determines the 3-surface proposed to apply at the light-like boundary of CD. The holography would be however restricted to the mass shells H^3 determined as root of a polynomial and possibly even L-function. An interesting question is whether X^2 fixes also its 3-D light-like orbit by holography. Quantum criticality suggests a failure of a strict determinism.

18.2.8 About the identification of L-group

How could one understand in the TGD framework, the L-group, or ^LG, as a Langlands dual? The standard approach is described in `https://cutt.ly/iBgnMIF`. Langlands dual ^{L}G and L-group are more or less the same. L-group is a semidirect product of $^LG^0$ and Galois group such that the Galois group has natural action in the matrix representation of the algebraic group G with matrix elements. This is the case if G is defined over a field containing the extension of rationals to which the Galois group is associated. Algebraic groups over global fields (extensions of rationals) can be regarded as analogs of Lie groups and the Dynkin diagrams assignable to Lie algebras appear in their classification.

The guess based on TGD vision was following. One assumes global field that is a finite extension of rationals. Lorentz group, $SL(2,C)$, etc. are discretized.

1. In TGD picture, Galois group permutes mass shells. The isotropy group acts on momentum components but keeps them on mass shell. Lorentz group mixes momentum components. Can one form a larger group from these groups by forming the products of group elements.

2. A free group from from G_1 and G_2 with amalgamation is obtained by adding some relations by using a third group U inbedded to both groups by homomorphism (https://cutt.ly/DBgn9Q2). G_1 and G_2 are glued together along U.
   In the recent case, G_1 could corresponds to Galois group and G_2 to Lorentz group $SO(1,3)$ or its covering for a global field extension. U corresponds to a subgroup of Galois group and of Lorentz group. G_2 can correspond to the non-compact groups defined by the truncated Virasoro algebra or symplectic algebra of $\delta M^4 \times CP_2$. U must be a subgroup of Galois group leaving the root fo P defining mass squared invariant.
3. What about Galois singletness in this case? The group obtained in this way permutes mass shells. The automorphic forms in the extended group be invariant under Galois group or its amalgamated product with a discrete infinite subgroup of $SL(2,C)$.
4. The free product and amalgamated free product construction is extremely general. It could work even for an extension of finite field or extension of corresponding p-adic number field and $SL(2,C)$. Here unramified and ramified primes pop up. The induced Galois group looks more natural here.

What about quaternionic automorphisms, which is analog of Galois group? The amalgamated free product of (discrete subgroups of) quaternionic automorphisms with Galois group could be important. Free product with amalgamation would naturally apply to Galois group, quaternionic automorphisms, $SL(2,C)$ and subgroups of conformal transformations.

The identification of candidates for the reductive groups

Extension of irreducible representations of Galois group to representations of reductive groups extended by Galois group, so called L-group, are suggested by the Langlands program and in the TGD framework they would be very natural. These extensions could define WCW spinor fields. What candidates does TGD offer for the reductive groups in question?

1. In TGD, the infinite-D (super-)symplectic group assignable to $\delta M^4_+ \times CP_2$ defines a candidate for the isometries of WCW. The Lie algebra A of this group corresponds to Hamiltonians as functions defined in $\delta M^4_+ \times CP_2$. The basis of Hamiltonians can be assumed to be products of functions defined in δM^4_+ and CP_2. For δM^4_+ one has irreps of $SO(3)$ acting in δM^4_+ and proportional to a power of r^n of the light-like radial coordinates, where n is conformal weight. For CP_2 one has functions defining irreps of $SU(3)$.
2. The Lie-algebra A allows infinite fractal hierarchies formed by sub-algebras A_n with radial conformal weights coming as n-multiples of the conformal weights of the full algebra. The gauge conditions state that A_n and the commutator $[A_n, A]$ annihilate the physical states. These conditions generalize to other symmetry groups assignable to the light-like 3-surfaces defining partonic orbits and to the extended conformal transformations of the metrically 2-D light-cone δM^4_+.
   The first naive guess is that the gauge conditions effectively reduce the symplectic group to finite-D symplectic group $Sp(2m)$ or its reductive subgroup acting linearly. In this case one might have infinite-D representations
3. One can also consider the possibility that the gauge conditions for the radial conformal transformations are weakened to similar conditions as in the case of A. Similar conditions could apply to the algebras associated with the light-like 3-surfaces.

18.2.9 A comment on $M^8 - H$ duality in fermion degrees of freedom in relation to Langlands duality

Gary Ehlenberg sent an URL of a very interesting Quanta Magazine article, which discusses a work related to Langlands program and provides some rather concrete insights how $M^8 - H$ duality [L82, L83], relating the number theoretic and geometric views of TGD, could relate to the Langlands duality.

Langlands duality relates number theory and geometry. At the number theory side one has representations of Galois groups. On the geometry side one has automorphic forms associated with the representations of Lie groups. For instance, in coset spaces of hyperbolic 3-space H^3 in the case of the Lorentz group.

The work could be highly interesting from the TGD perspective. In TGD, the M^8-H duality generalizes momentum-position duality so that it applies to particles represented as 3-surfaces instead of points. M^8-H duality also relates physics as number theory and physics as geometry. Much like Langlands duality. The problem is to understand M^8-H duality as an analog of Langlands duality.

1. $H = M^4 \times CP_2$ is the counterpart of position space and particle corresponds to 3-surface in H. Physics as (differential) geometry applies at this side.
   The orbit of 3-surface is a 4-D space-time surface in H and holography, forced by 4-D general coordinate invariance, implies that space-time surfaces are minimal surfaces irrespective of the action (general coordinate invariant and determined by induced geometry) . They would obey 4-D generalization of holomorphy and this would imply universality.
   These minimal surfaces are also solutions of a nonlinear geometrized version of massless field equations. Field-particle duality has a geometrized variant: minimal surface represents in its interior massless field propagation and as an orbit of 3-D particles the generalization of a light-like geodesic. Hence a connection with electromagnetism mentioned in the popular article, actually metric and all gauge fields of the standard model are geometrized by induction procedure for geometry.
2. M^8, or rather its complexification M^8_c (complexification is only with respect to Minkowski time) corresponds to momentum space and here the orbit of point-like particle in momentum space is replaced with a 4-surface in $\mathrm{M}^8, oractuallyitscomplexification \mathrm{M}^8_c$.
   The 3-D initial data for a given extension of rationals could correspond to a union of hyperbolic 3-manifolds as a union of fundamental regions for a tessellation of H^3 consistent with the extensions, a kind of hyperbolic crystal. These spaces relate closely to automorphic functions and L-functions.
   At the M^8 side polynomials with rational coefficients determine partially the 3-D data associated with number theoretical holography at $\mathrm{M}^8-side. Thenumbertheoreticaldynamicalprinciplestatesthatthe$
   $timesurfaceintheoctonionic \mathrm{M}^8_c$ is associative and initial data correspond to 3-surfaces at mass shells $H^3_c \subset M^4_c \subset M^8_c$ determined by the roots of the polynomial.
3. M^8-H duality maps the 4-surfaces in M^8_c to space-time surfaces in H. At the M^8 side one has polynomials. At the geometric H-side one has naturally the generalizations of periodic functions since Fourier analysis or its generalization is natural for massless fields which space-time surfaces geometrize. L-functions represent a typical example of generalized periodic functions. Are the space-time surfaces at H-side expressible in terms of modular function in H^3?

Here one must stop and take a breath. There are reasons to be very cautious! The proposed general exact solution of space-time surfaces as preferred extremals realizing almost exact holography as analogs of Bohr orbits of 3-D surfaces representing particles relies on a generalization of 2-D holomorphy to its 4-D analog. The 4-D generalization of holomorphic functions [L131] assignable to 4-surfaces in H do not correspond to modular forms in 3-D hyperbolic manifolds assignable to the fundamental regions of tessellations of hyperbolic 3-space H^3 (analogs of lattice cells in E^3).

Fermionic holography reduces the description of fermion states as wave functions at the mass shells of H^3 and their images in H under M^8-H duality, which are also hyperbolic 3-spaces.

1. This brings the modular forms of H^3 naturally into the picture. Single fermion states correspond to wave functions in H^3 instead of E^3 as in the standard framework replacing infinite-D representations of the Poincare group with those of $SL(2,C)$. The modular forms defining the wave functions inside the fundamental region of tessellation of H^3 are analogs of wave functions of a particle in a box satisfying periodic boundary conditions making the box effectively a torus. Now it is replaced with a hyperbolic 3-manifold. The periodicity conditions code invariance under a discrete subgroup $\Gamma \subset SL(2,C)$ and mean that $H^3 = SL(2,C)/U(2)$ is replaced with the double coset space $\Gamma \backslash SL(2,C)/U(2)$.

   Number theoretical vision makes this picture more precise and suggests ideas about the implications of the TGD counterpart of the Langlands duality.

2. Number theoretical approach restricts complex numbers to an extension of rationals. The complex numbers defining the elements $SL(2,C)$ and $U(2,C)$ matrices are replaced with matrices in discrete subgroups $SL(2,F)$ and $U(2,F)$, where F is the extension of rationals associ-

ated with the polynomial P defining the number theoretical holography in M^8 inducing holography in H by M^8-H duality. The group $\Gamma definingtheperiodicboundaryconditionsmustconsistofmatricesin$SL(2,F

3. The modular forms in H^3 as wave functions are labelled by parameters analogous to momenta in the case of E^3: in the case of E^3 they characterize infinite-D irreducible representations of $SL(2,C)$ as covering group of $SO(1,3)$ with partial waves labelled by angular momentum quantum numbers and spin and by the analog of angular momentum associated with the hyperbolic angle (known as rapidity in particle physics): infinitesimal Lorentz boost in the direction of spin axis.

   The irreps are characterized by the values of a complex valued Casimir element of $SL(2,C)$ quadratic in 3 generators of $SL(2,C)$ or equivalently by two real Casimir elements of $SO(1,3)$. Physical intuition encourages the shy question whether the second Casimir operator could correspond to the complex mass squared value defining the mass shell in M^8. It belongs to the extension of rationals considered as a root of P.

   The construction of the unitary irreps of $SL(2,C)$ is discussed in Wikipedia article. The representations are characterized by pairs of half-integer $j_0 = n/2$ and imaginary number $j_1 = i\nu$. Since the representations in question are H^3 analogs of the irreducible representations of Poincare group in M^4 with E^3 replacing H^3 the natural interpretation of j_0 would be as spin. The states of the representation would represent partial waves with definite value of j. In TGD, $j_0 = 1/2$ would be in a special role.

   The values of j_0 and j_1 must be restricted to the extension of rationals associated with the polynomial P defining the number theoretic holography.

4. The Galois group of the extension acts on these quantum numbers. Angular momentum quantum numbers are quantized already without number theory and are integers but the action on the hyperbolic momentum is of special interest. The spectrum of hyperbolic angular momentum must consist of a union of orbits of the Galois group and one obtains Galois multiplets. The Galois group generates from an irrep with a given value of j_1 a multiplet of irreps.

   A good guess is that the Galois action is central for $M^8 - H$ duality as a TGD analog of Langlands correspondence. The Galois group would act on the parameter space of modular forms in $\Gamma(2,F)/U(2,F)$, F and extension of complex rationals and give rise to multiplets formed by the irreps of $SL(2,F)$.

To sum up, $M^8 - H$ duality [L82, L83] is a rather precisely defined notion (I am of course using the standards of physicist).

1. At the M^8 side one has polynomials and roots and at the H-side one has automorphic functions in H^3 and "periods" are interpreted as quantum numbers. What came first in my mind was that understanding of M^8 duality boils down to the question about how the 4-surfaces given by number theoretical holography as associativity of normal space relate to those given by holography (that is generalized holomorphy) in H.
2. However, it seems that the problem should be posed in the fermionic sector. Indeed, above I have interpreted the problem as a challenge to understand what constraints the Galois symmetry on M^8 side poses on the quantum numbers of fermionic wave functions in hyperbolic manifolds associated with H^3 and defined b the extension of rationals in question. I do not know how closely this problem relates to the problem that Ben-Zvi, Sakellaridis and Venkatesh, whose work is discussed in the popular article mentioned in the beginning, have been working with.

18.3 Appendix

In the following some notions of algebraic geometry, group theory, and number theory are briefly explained.

18.3.1 Some notions of algebraic geometry and group theory

Notions related to modular forms and automorphic forms

Fuschian and modular groups are discrete subgroups of $SL(2,R)$ acting as invariance groups of modular functions.

1. Fuschian groups (`https://cutt.ly/hBn0YJU`) is a discrete subgroup of $PSL(2,R)$. The group $PSL(2,R)$ can be regarded equivalently as a group of isometries of the hyperbolic plane, or conformal transformations of the unit disc, or conformal transformations of the upper half plane, so a Fuchsian group can be regarded as a group acting on any of these spaces. There are some variations of the definition: sometimes the Fuchsian group is assumed to be finitely generated, sometimes it is allowed to be a subgroup of $PGL(2,R)$ (so that it contains orientation-reversing elements), and sometimes it is allowed to be a Kleinian group (a discrete subgroup of $PSL(2,C)$), which is conjugate to a subgroup of $PSL(2,R)$.

   Fuchsian groups are used to create Fuchsian models of Riemann surfaces. In this case, the group may be called the Fuchsian group of the surface. In some sense, Fuchsian groups do for non-Euclidean geometry what crystallographic groups do for Euclidean geometry. Some Escher graphics are based on them (for the disc model of hyperbolic geometry).

2. Modular group (`https://cutt.ly/hBgbH9S`) is the projective special linear group $PSL(2,Z)$ of 2×2 matrices with integer coefficients and determinant 1. The matrices A and A are identified. The modular group acts on the upper-half of the complex plane by fractional linear transformations, and the name "modular group" comes from the relation to moduli spaces, such as the moduli space of conformal structures of torus.

   Second presentation is transformations of the complex plane as Möbius transformations $z\rightarrow (az+b)/(cz+d)$ mapping upper plane and real axis to itself. $SL(2,R)/SL(2,Z)$ gives rise to a hyperbolic geometry identifiable as a fundamental domain of the tessellation of H^2 analogous to the lattice cell of the Euclidean planar lattice.

   Modular group is generated by relations generators $z\rightarrow -1/z$ and $T: z\rightarrow z+1$. Modular group has a presentation $S^2=I$, $ST^3=I$. By posing the additional relation $T^n=1$ one obtains a congruence subgroup denoted by $D(2,3,n)$.

   These groups have generalization to discrete groups of $SL(n,C)$ and $Sl(n,R)$.

Modular forms and theta functions are closely related entities as also L-functions and generalize zeta functions.

1. A modular form (`https://cutt.ly/3BgbLsr`) is a (complex) analytic function on the upper half-plane satisfying a certain kind of functional equation with respect to the group action of the modular group, and also satisfying a growth condition. The theory of modular forms therefore belongs to complex analysis but the main importance of the theory has traditionally been in its connections with number theory. Modular forms appear in other areas, such as algebraic topology, sphere packing, and string theory.

   A modular function is a function that is invariant with respect to the modular group, but without the condition that $f(z)$ be holomorphic in the upper half-plane (among other requirements). Instead, modular functions are meromorphic (that is, they are holomorphic on the complement of a set of isolated points, which are poles of the function).

   Modular form theory is a special case of the more general theory of automorphic forms which are functions defined on Lie groups which transform nicely with respect to the action of certain discrete subgroups, generalizing the example of the modular group $\mathrm{SL}_2(\mathbb{Z})\subset \mathrm{SL}_2(\mathbb{R})$.

   For instance, modular forms can be defined in a generalized upper half plane, which consists of symmetric $Gl(n,C)$ matrices such that the imaginary parts of the matrix elements are positive. For certain. values of n these spaces serve as moduli spaces for the conformal equivalence classes of Riemann surfaces and in the TGD framework elementary particle vacuum functionals as "wave functions" in WCW are identified as modular invariant modular forms in Teichmüller spaces [K21].

2. Theta functions (`https://cutt.ly/bBEFAe5`) are special functions of several complex variables. They are involved with Abelian varieties, moduli spaces, quadratic forms, and solitons. As Grassmann algebras, they appear in quantum field theory.

   For instance, the formula for Jacobi's theta function $\theta_1(z,q)$ reads as

$$\theta_1(z,q) = 2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin((2n+1)z) = \sum_{n=-\infty}^{\infty}(-1)^{n-\frac{1}{2}}q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)iz} \quad . \tag{18.3.1}$$

The most common form of theta function is that occurring in the theory of elliptic functions. With respect to one of the complex variables (conventionally called z), a theta function has a property expressing its behavior with respect to the addition of a period of the associated elliptic functions, making it a quasiperiodic function. In the abstract theory this quasiperiodicity comes from the cohomology class of a line bundle on a complex torus, a condition of descent.

One interpretation of theta functions when dealing with the heat equation is that "a theta function is a special function that describes the evolution of temperature on a segment domain subject to certain boundary conditions".

3. Dirichlet series correspond to L-functions and zeta functions. A Dirichlet series `https://cutt.ly/rBgbNKZ` is any series of the form $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$, where s is complex, and a_n is a complex sequence. It is a special case of the general Dirichlet series.

   Dirichlet series play a variety of important roles in analytic number theory. The most usually seen definition of the Riemann zeta function is a Dirichlet series, as are the Dirichlet L-functions.

Modular forms and L-functions correspond to each other.

1. Mapping of modular forms to L-functions assigns to the Fourier sum $\sum a_n q^n$, $q = exp(i2\pi z)$ of a modular form, also known as theta function (`https://cutt.ly/QBEYRfW`), an L-function defined as $\sum a_n n^{-s}$.

   Jacobi theta function $\theta(z) = \sum_{n=1}^{\infty} q^{n^2}$,$q = exp(i\pi z)$ has $\zeta(2s)$ as associated L-function.

2. Mellin transform of function f is defined as $M(f)(s) = \int_0^{\infty} dx x^{s-1} f(x)$ (`https://cutt.ly/gBEbWW4`). $\zeta(s)$ can be written as $(1/\Gamma(s))M(f(x))$, $f(x) = 1/(e^{-x}/(1-e^{-x}))$ identifiable as a partition function of harmonic oscillator with a energy spectrum consisting of positive integers.

Some group theoretic notions

Group theoretical notions.

1. *Reductive groups*

According to the Wikipedia article (`https://cutt.ly/9Bgbv9o`), a reductive group is a linear algebraic group over a field. One definition is that a connected linear algebraic group G over a perfect field (`https://cutt.ly/IBxHw9S`) is reductive if it has a representation with a finite kernel, which is a direct sum of irreducible representations.

Note that for any polynomial over a perfect field K all roots are in K, whereas for algebraically closed field they always have a root in K, as a matter of fact the number of roots equals to the degree of the polynomial in this case.

This does not say much to a layman. The fact that the every finite normal subgroup of a reductive group is central, is more informative. For instance, the Galois groups for extensions of extensions fail to satisfy this condition in general so that only simple Galois groups of Galois groups for which normal subgroups are central, are reductive.

Reductive groups include general linear group $GL(n)$ of invertible matrices, special linear group $SL(n)$ (in particular $SL(2,k)$), the special orthogonal group $SO(n)$, and the symplectic group $Sp(2n)$. Simple algebraic groups (in particular $SU(n)$) and (more generally) semisimple algebraic groups are reductive.

Claude Chevalley showed that the classification of reductive groups is the same over any algebraically closed field. In particular, the simple algebraic groups are classified by Dynkin diagrams, as in the theory of compact Lie groups or complex semisimple Lie algebras. Reductive groups over an arbitrary field are harder to classify, but for many fields such as the real numbers R or a number field, the classification is well understood. The classification of finite simple groups says that most finite simple groups arise as the group $G(k)$ of k-rational points of a simple algebraic group G over a finite field k, or as minor variants of that construction.

2. Borel subgroups, parabolic subgroups and parabolic induction

1. In the theory of algebraic groups, a Borel subgroup (`https://cutt.ly/jBgbmRX`) of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup. In Zariski topology the closed sets are algebraic surfaces, whereas in ordinary topology the set of closed sets is much larger. Zariski topology is therefore rougher than standard topology.

   For example, in the general linear group GL_n, the subgroup of invertible upper triangular matrices is a Borel subgroup. For groups realized over algebraically closed fields, all Borel subgroups are conjugate to this group.

2. Subgroups between a Borel subgroup B and the ambient group G are called parabolic subgroups. Parabolic subgroups P are characterized by the condition that G/P is a complete projective variety defined as by a vanishing conditions for a set homogeneous polynomials so that the solutions possess scale invariance. For algebraically closed fields, the Borel subgroups turn out to be the minimal parabolic subgroups in this sense. Thus B is a Borel subgroup when the homogeneous space G/B is a complete variety, which is "as large as possible".

3. According to the Wikipedia article (`https://cutt.ly/SBxTqTU`), parabolic induction is a method of constructing representations of a reductive group from representations of its parabolic subgroups.

   If G is a reductive algebraic group and $P = MAN$ is the Langlands decomposition of a parabolic subgroup $P \subset G$, then parabolic induction consists of taking a representation of MA, extending it to P by letting N act trivially, and inducing the result from P to G. Induction means extension of the represention of P to G. For instance, the representations of Poincare group can be induced from the representations of $SO(3) \times T^4$. That G/P is a complete projective variety must play an important role in this process.

3. Definition of L-group

According to Wikipedia, in representation theory the Langlands dual ^LG (`https://cutt.ly/cBgbTGs`) of a reductive algebraic group G (also called the L-group of G) is a group that controls the representation theory of G. If G is defined over a field k, then ^LG is an extension of the absolute Galois group of k by a complex Lie group. There is also a variation called the Weil form of the L-group, where the Galois group is replaced by a Weil group. The letter "L" in the name also indicates the connection with the theory of L-functions, particularly the automorphic L-functions. The Langlands dual was introduced by Langlands in a letter to A. Weil.

According to this definition ^LG would be a Lie group and contain the semidirect product of Galois group and of algebraic group over the extension of rationals. Note that amalgamated free product involves a third group U having embeddings to both Gal and $G(k)$ and $G(k)$ and Gal are "glued" along U.

Automorphic representations and automorphic functions

I am not a number theory professional, and in the following I can only try to demonstrate that I have at least done my best in trying to understand the essentials of the description of [A126] for the route from automorphic adelic representations of $GL_e(2, R)$ to automorphic functions defined in upper half-plane. A brief summary of the automorphic representations in Wikipedia involves the following key points.

1. One has an adelic analogy of group algebra, that is the space of functions in the adelic group G satisfying some additional conditions. Representation functions are left invariant with

respect to the algebraic diagonal subgroup G_{diag}. Central character is interpreted as a map ω: $Z(K) \setminus Z(A)^{\times} \to C$.

2. Representation functions are finite sums of the left translates of function f by elements of adelic G. G acts from right on these functions. One speaks of a space of cusp forms with a central character ω.

3. A decomposition of the cuspidal representation into a direct sum of Hilbert spaces with finite multiplicities takes place.

The following describes the construction for $GL(2,Q)$, which is very relevant for TGD since $SL(2,C)$ acts as a covering of the Lorentz group.

1. Characterization of the representation

The representations of $GL_e(2,Q)$ are constructed in the space of smooth bounded functions $GL_e(2,Q)\backslash GL_e(2,A) \to C$ or equivalently in the space of $GL_e(2,Q)$ left-invariant functions in $GL_e(2,A)$. A denotes adeles and $GL_e(2,A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of a compact subgroup K of $GL_e(2,A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in the general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.
To my best non-professional understanding, $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL_e(2,Z_p)$. For ramified primes K_p consists of $SL_e(2,Z_p)$ matrices with $c \in p^{n_p}Z_p$. Here p^{n_p} is the divisor of the conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of the Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ , \tag{18.3.2}$$

where one has

$$X_0\left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right) ,\left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array}\right) \ . \tag{18.3.3}$$

3. The center $A^{\times}$ of $GL_e(2,A)$ consists of $A^{\times}$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^{\times} \to C$ is a character providing a multiplicative representation of $A^{\times}$.
item The so-called cuspidality condition is associated with the cusps. Planar cusp (`https://cutt.ly/sBxd9sH`) corresponds geometrically to a sharp tip. Derivatives of $x(t)$ and $y(t)$ with respect to parameter t become zero at cusp. The direction of the curve changes at the cusp. $x \geq 0$. Cusp catastrophe $x^3 - y^2 = 0$ provides a simple example. The tip of the cusp is added in the compactification of the hyperbolic 2-manifold defined by the space $\Gamma\backslash H^2$.
The cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array}\right) g)du = 0 \tag{18.3.4}$$

is satisfied [A126]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL_e(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL_e(2,A_F)\times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL_e(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL_e(2,Q)\backslash GL_e(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is so called conductor, which is an integer measuring the ramification of the extension [A126] (`https://cutt.ly/DBcgOA2`). This means enormous simplification since one gets rid of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL_e(2,Z)$ consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ c \ mod \ N = 0. \tag{18.3.5}$$

   $_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL_e(2,Z)$ so that also $SL_e(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \to \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL_e(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL_e(2,R)$ *to upper half-plane* $H_u = SL_e(2,R)/SO(2)$

The representations of $(gl(2,C),O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A126]. For the discrete series representation π giving square integrable representation in $SL_e(2,R)$ one has $\rho = k(k-1)/4$, where $k>1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ \ , \ \ X_+ v_\infty = 0 \ \ . \tag{18.3.6}$$

The latter module is in turn generated by the lowest weight vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\infty \ \ . \tag{18.3.7}$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL_e(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL_e(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL_e(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau) \tag{18.3.8}$$

under the action of $\Gamma_0(N)$ The highest weight condition X_+v_∞ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n \ . \tag{18.3.9}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q=0$, which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Wikipedia provides a brief description of Hecke operators (`https://cutt.ly/hBxd5Yb`).

1. Spherical Hecke algebra (, which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL_e(2,Z_p)$ bi-invariant functions on $GL_e(2,Q_p)$ with respect to convolution product. Sub-algebra of group algebra is in question.
2. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators.
3. The normalizations can be chosen so that the second eigenvalue equals unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A126].
4. The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators also have a representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From the TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmüller parameters of the algebraic Riemann surface are affected by the absolute Galois group. This induces $Sl(2g,Z)$ transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g=1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroups of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

An interesting possibility is that these representations can be continued from the hyperbolic 2-manifolds to hyperbolic 3-manifolds assignable to the mass shells H^3 defined by tessellations. The discrete subgroup Γ of $SL(2,R)$ would be continued to a discrete complex subgroup of $SL(2,C)$. There would be left invariance with respect to diagonal $SL(2,C)$. Finite sums over right translates by discrete elements of adelic $SL(2,C)$. Central character associated with Z_2. One could have a holography in the sense that the modular forms associated with the hyperbolic 2-manifold as boundary of hyperbolic manifold would be continued to their counterparts if 3-D hyperbolic manifold.

18.3.2 Some number theoretic notions

Frobenius automorphism

Frobenius automorphism `https://cutt.ly/NBkIudF` maps the element of a finite field $F(p,n)$, or more generally, of a commutative ring with characteristic p, to its p:th power and can there be regarded as an element of Galois group for an extension of finite field. F maps products to products and sums to sums.

For a finite field one has $x^p = x$ by Fermat's little theorem. The elements of F_p determined the roots of the equation $X^p = X$. There are no more roots in any extension. Therefore, if L is an algebraic extension of F_p, F_p is the fixed field of the Frobenius automorphism of L. The Galois group of an extension of a finite field is generated by the iterates of Frobenius automorphism.

The notion of discriminant

The discriminant of the polynomial is the most concrete definition (`https://cutt.ly/GBxfyIm`).

1. For a polynomial $P(x) = a_n x^n + ...$ the discriminant can be defined by the formula

$$\mathrm{Disc}_x(A) = a_n^{2n-2}\prod_{i<j}(r_i - r_j)^2 = (-1)^{n(n-1)/2} a_n^{2n-2}\prod_{i\neq j}(r_i - r_j) \ , \qquad (18.3.10)$$

   This notion applies to extensions of rationals defined by polynomials. For a second order polynomial $ax^2 + bx + x$, one has the familiar formula $\mathrm{Disc} = b^2 - 4ac$.

2. In the recent case the coefficients are rational. D vanishes when the polynomial has two or more identical roots which occurs for suitable values of parameters. The geometric interpretation is that two sheets (roots) of the graph of a root as a many-valued function of parameters a_i co-incide so that the tangent space of the graph is parallel to x. Cusp catastrophe associated with a polynomial of order 3 is the simplests non-trivial example.

3. For a rational polynomials D is a rational number and for the ramified primes dividing D, it vanishes for the finite field variants of the polynomial with coefficients taken modulo p so that there are multiple roots for ramified primes. One can say that p-adically a catastrophe occurs in order $O(p) = 0$. This defines a p-adic variant of quantum criticality and gives an idea about the special physical role of the ramified primes in TGD.

A more abstract definition of the discriminant, which does not depend on the polynomial (`https://cutt.ly/6BxfoQo`). One distinguishes between the absolute discriminant of a number field and the relative discriminant of an extension of a number field. In the TGD framework, both situations are the same since number fields are extensions of rationals or induced by them.

1. One starts directly from the extension of rationals and imbeds the roots as complex numbers to plane. There is a large number of different embeddings. This corresponds to the fact that many polynomials P define the same extension. The counterpart for this non-uniqueness is that any basis elements for the basis for the ring of integers of the extension can define the unit to which the real axis is assigned.

2. There are n choices corresponding to n basic vectors of the integer basis consisting of algebraic integers, which are roots of a monic polynomial. One can choose the monic polynomial so that it is of degree n and the powers of a root define integer basis. Each choice s_i defines a map of the basis vectors e_j to the complex plane. The image vectors $s_i(e_j)$ define a matrix, whose determinant defines the discriminant D of the extension, which is the same as given by the less abstract definition based on the roots of a polynomial.

The notions of valuation and ramification

The notions of valuation and ramification (`https://cutt.ly/bBgb47p`) are easiest to understand in terms of a concrete polynomial representation of extension.

The extension with a given Galois group is obtained in very many ways. For instance, all irreducible polynomials of degree 2 have the same Galois group. Further information comes from the concrete polynomial representation. Ramified primes appear in the discrimant D of P as factors. For ramified primes, the splitting to a product of powers $\mathfrak{p}_i^{e_i}$ of prime ideals $\mathfrak{p}_i$ of extension is such that at least $e_i > 1$ appears. The discriminant is product for the squares of the differences of roots and depends on polynomial. This provides a more precice characterization of the situation than mere Galois group.

Ramified primes are special in the sense that for them the extension of p-adic number field induced by the extension of rationals is has lower dimension than for unramified primes. This is intuitively understandable since the discrimant vanishes in order $O(p)$ at least for the ramified prime. The prime ideals of K can split in to prime ideals of L. Also powers of primes of extension can appear in the splitting and this correspond to ramification. Ramified primes appear as factors in the discriminant.

The extension defined by a polynomials define a basis of algebraic integers and one can define norm by the determinant of the linear transformation defined by multiplication with an integer of the extension. This norm depends on the polynomial P and defines p-adic norm. The logarithm of the norm defines the valuation. When ramification occurs the dimension of p-adic extension l/k restricted to the finite field parts of p-adic numbers is lower than the dimension of extension L/K of rationals. The dimension of the corresponding finite field is lower than that for rationals.

In the abstract approach one does not mention polynomials at all and considers only valuations as norms assigned to an abstract extension of rationals. The equivalence class of valuations replaces the equivalence class of polynomials with the same Galois group and same discriminant if valuation is determined by the powers of ramified primes appearing in the discrimant.

Intuitively, the valuation should correspond to a prime ideal $\mathfrak{p}$ of L and to a norm. For extensions of rationals these prime ideals correspond to the primes defining extensions of p-adic number fields and these primes are special. Ramified primes are those appearing in the discriminant. The catastrophe theoretic picture based on the discriminant of the polynomial defining the catastrophe gives an idea of what is involved. This intuitive helps to make sense of the rather abstract statements below.

1. If there are several prime ideals, there are several valuations, which need not be equivalent (transform to each other by the action of Galois group). This would suggest that G_w transforms to each other prime ideals $\mathfrak{p}$ defining the same evaluation. Valuation ring R_w corresponds to the ring, whose elements have a non-negative norm or equivalenty, a given element x of O or its inverse belongs R_w. Is the valuation ring same as the ring formed by non-negative powers of this prime ideal? Valuation ring has maximal ideal m_w. The maximal ideal m_w of R_w representing the equivalence class of valuation inside the evaluation ring R_w is a key concept.

2. The ramification is characterized using decomposition group G_w and the hierarchy of ramification subgroups, which are normal subgroups of G_w. The decomposition group G_w of a valuation, which is determined by element w, is the subgroup of Galois group acting as the stabilizer group leaving the evaluation invariant.

   G_w must leave invariant the determinant defining the norm. How does G_w relate to the isotropy group of a given root of P? If G_w and the isotropy group are identical and the isotropy group depends on the root, a given polynomial P could allow several evaluations. If the maximal (prime) ideal p of $O(L)$ defines the extension, G_w would transform it to a prime defining an equivalent norm. By Hensel's lemma, the ring of $O(L)$ of L-integsds can be written as $O(L) = O_K(\alpha)$ for some α in $O(L)$.

3. The inertia group I_w of w consists of the elements of Galois group, which leave the elements of R_w invariant modulo m_w. These elements are analogous to p-adic integers numbers smaller than p and the intuitive picture is that ramification means that the generating element of the ring R_w is power of w which is larger than 1.

   Also the functional decomposition of polynomial P defines a hierarchey of normal subgroups as Galois subgroups and factor groups. Hierarchy of ramification groups must correspond to polynomials in a composition of P to polynomials.

   The inertia group of a given equivalence class of valuations is a subgroup of G_w and the stabilizer group of the valuation. It could correspond to the Galois group of the extension E_n associated with $P = P_n \circ ... \circ P_1$ regarded as an extension of the extension E_{n-1} associated with $P_{n-1} \circ ...P_1$.

4. There are also higher normal subgroups in a series associated with Gal. They give additional information about the valuation.

Also the notion of the conductor is involved. The conductor of an extension is an integer serving as measure for the ramification. Qualitatively, the extension is unramified if, and only if, the conductor is zero, and it is tamely ramified if, and only if, the conductor is 1. More precisely, the conductor computes the non-triviality of higher ramification groups. The description of conductor given in the Wikipedia article (`https://cutt.ly/DBcgOA2`) is extremely general and therefore too technical to be understood by a non-specialist.

Artin L-function

Given representation ρ of the Galois group G of the finite extension L/K on a finite-dimensional complex vector space V, the Artin L-function: $L(\rho, s)$ is defined by an Euler product. For each prime ideal $\mathfrak{p}$ in K's ring of integers, there is an Euler factor, which is easiest to define in the case where $\mathfrak{p}$ is unramified in L (true for almost all $\mathfrak{p}$).

In that case, the Frobenius element $\mathbf{Frob}(\mathfrak{p})$ mapping elements of the ring of integers of the extension L/K to its p:th power is identified as a conjugacy class in G. Therefore, the characteristic polynomial of $\rho(\mathbf{Frob}(\mathfrak{p}))$ is well-defined. The Euler factor for $\mathfrak{p}$ is a slight modification of the characteristic polynomial, equally well-defined,

$$\mathrm{charpoly}(\rho(\mathbf{Frob}(\mathfrak{p})))^{-1} = \det\left[I - t\rho(\mathbf{Frob}(\mathfrak{p}))\right]^{-1} \quad , \tag{18.3.11}$$

as rational function in t, evaluated at

$$t = N(\mathfrak{p})^{-s} \quad , \tag{18.3.12}$$

with s a complex variable in the usual Riemann zeta function notation. (Here N is the field norm of an ideal.)

When $\mathfrak{p}$ is ramified, and I is the inertia group which is a subgroup of G, a similar construction is applied, but to the subspace of V fixed (pointwise) by I.

Chapter 19

Finite Fields and TGD

19.1 Introduction

This article represents some material related to two articles discussing number theoretical vision of TGD. The first article [L127] was about the fusion of geometric and number theoretic views of TGD to single coherent theory.

Second article [L125] was about my attempts to understand Langlands correspondence, which postulates a deep correspondence between number theory and geometry, and its relation to the geometric and number theoretic views of TGD. Both articles led to two unexpected new ideas and because of the potential importance of these ideas, I decided to write a separate article raising these ideas to table, as one might say.

19.1.1 Brief summary of the basic mathematical notions behind TGD

The theoretical framework behind TGD involves several different strands and the goal is to unify them to a single coherent whole. This challenge was discussed in [L127].

TGD involves number theoretic and geometric visions about physics and M^8-H duality, analogous to Langlands duality, is proposed to unify them. Also quantum classical correspondence (QCC) is a central aspect of TGD. One should understand both the M^8-H duality and QCC at the level of detail.

The following mathematical notions are expected to be of relevance for this goal.

1. Von Neumann algebras, call them M, in particular hyperfinite factors of type II_1 (HFFs), are in a central role. Both the geometric and number theoretic side, QCC could mathematically correspond to the relationship between M and its commutant M'.

   For instance, symplectic transformations leave induced Kähler form invariant and various fluxes of Kähler form are symplectic invariants and correspond to classical physics commuting with quantum physics coded by the super symplectic algebra (SSA). On the number theoretic side, the Galois invariants assignable to the polynomials determining space-time surfaces are analogous classical invariants.

2. The generalization of ordinary arithmetics to quantum arithmetics obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ allows us to replace the notions of finite and p-adic number fields with their quantum variants. The same applies to various algebras.

3. Number theoretic vision leads to adelic physics involving a fusion of various p-adic physics and real physics and to hierarchies of extensions of rationals involving hierarchies of Galois groups involving inclusions of normal subgroups. The notion of adele can be generalized by replacing various p-adic number fields with the p-adic representations of various algebras.

4. The physical interpretation of the notion of infinite prime has remained elusive although a formal interpretation in terms of a repeated quantization of a supersymmetric arithmetic QFT is highly suggestive. One can also generalize infinite primes to their quantum variants. The proposal is that the hierarchy of infinite primes generalizes the notion of adele.

Second proposal, discussed already in [L127] and to be discussed separately in this article, was that the polynomial Q defining infinite prime at the first level of the hierarchy are identical to the polynomial P defining 4-surface in M^8 and by M^8-H correspondence space-time surface in $H=M^4\times CP_2$. This would realize quantum classical correspondence at very deep level.

The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves f 3 kinds of algebras A; supersymplectic isometries SSA acting on $\delta M^4_+\times CP_2$, affine algebras Aff acting on light-like partonic orbits, and isometries I of light-cone boundary δM^4_+, allowing hierarchies A_n.

The braided Galois group algebras at the number theory side and algebras $\{A_n\}$ at the geometric side define excellent candidates for inclusion hierarchies of HFFs. M^8-H duality suggests that n corresponds to the degree $n$of the polynomial P defining space-time surface and that the n roots of P correspond to n braid strands at H side. Braided Galois group would act in A_n and hierarchies of Galois groups would induce hierarchies of inclusions of HFFs. The ramified primes of P would correspond to physically preferred p-adic primes in the adelic structure formed by p-adic variants of A_n with $+$ and $\times$ replaced with $\oplus$ and $\otimes$.

19.1.2 Langlands correspondence and TGD

In the article [L125], the TGD counterpart of Langlands program was discussed and this led as a side product to a realization how finite fields could serve as basic building blocks of the number theoretic vision of TGD.

1. Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC), which is formulated as a correspondence between the quantum states in the "world of classical worlds" (WCW) characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via Galois group representation partition function. QCC would define a kind of closed loop giving rise to a hierarchy.
2. If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.
3. One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework. Elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters. Multiple residue integral is proposed as a manner to obtain L-functions defining space-time surfaces.
4. One challenge is to construct Riemann zeta and the associated ξ function and the Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k=\sum_{i,j}c_{k,ij}e^{i/k}e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij}\in\{0,\pm1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

   This construction led to the question whether also finite fields could play a fundamental role in the number theoretic vision. Prime polynomial with prime order $n=p$ and integer coefficients smaller than $n=p$ can be regarded as a polynomial in a finite field. If it satisfies the condition that the integer coefficients have no common prime factors, it defines an infinite prime. The proposal is that all physically allowed polynomials are constructible as functional composites of these.

One can end up to the idea that prime polynomials and finite fields could be fundamental in TGD also by a different route.

1. A highly interesting feedback to the number theoretic vision emerges. The rational polynomials P defining space-time surfaces are characterized by ramified primes. Without further conditions, they do not correlate at all with the degree n of P as the physical intuition suggests.

2. In [L127] it was proposed that P can be identified as the polynomial Q defining an infinite prime [K84]: this implies that the coefficients of the integer polynomial P (to which any rational polynomial can be scaled) do not have common prime factors.
3. An additional condition could be that the coefficients of P are smaller than the degree n of P. For $n = p$, P could as such be regarded as a polynomial in a finite field. This proposal is too strong to be true generally but could hold true for so-called prime polynomials of prime order having no functional decomposition to polynomials of lower degree [A103, A160]. The proposal is that all physically allowed polynomials are constructible as functional composites of irreducible prime polynomials. Also finite fields would become fundamental in the TGD framework.

One of the long standing mysteries of TGD is why preferred p-adic primes, characterizing elementary particles and even more general systems, satisfy the p-adic length scale hypothesis. The proposal is that p-adic primes correspond to ramified primes as factors of discriminant D of polynomial $P(x)$. $D = P$ condition reducing discriminant to a single prime is an attractive hypothesis for preferred ramified primes.

$M^8 - H$ duality suggests that the exponent $exp(K)$ of Kähler function corresponds to a negative power D^{-k}. Spin glass character of WCW suggests that the preferred ramified primes for, say prime polynomials of a given degree, and satisfying $D = P$, have an especially large degeneracy for certain ramified primes P, which are therefore of a special physical importance.

Because of the potential importance of this idea, which emerged while writing article about my attempts to understand Langlands correpondence and its relation to TGD, I decided to write a separate article about the role of finite fields in the TGD based world order.

19.2 Infinite primes as a basic mathematical building block

Infinite primes [K84, K43, K52] are one of the key ideas of TGD. Their precise physical interpretation and the role in the mathematical structure of TGD has however remained unclear.

3 new ideas are be discussed. Infinite primes could define a generalization of the notion of adele; quantum arithmetics could replace $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary primes with p-adic representations of say HFFs; the polynomial Q defining an infinite prime could be identified with the polynomial P defining the space-time surface: $P = Q$.

19.2.1 Construction of infinite primes

Consider first the construction of infinite primes [K84].

1. At the lowest level of hierachy, infinite primes (in real sense, p-adically they have unit norm) can be defined by polynomials of the product X of all primes as an analog of Dirac vacuum.

   The decomposition of the simplest infinite primes at the lowest level are of form $aX+b$, where the terms have no common prime divisors. More concretely $a = m_1/n_F$ $b = m_0 n_F$, where n_F is square free integer analogous and the integer m_1 and n_F have no common prime divisors divisors. The divisors of m_2 are divisors of n_F and m_i has interpretation as n-boson state. Power p^k corresponds to k-boson state with momenta p. $n_F = \prod p_i$ has interpretation as many-fermion state satisfying Fermi-Dirac statistics.

   The decomposition of lowest level infinite primes to infinite and finite part has a physical analogy as kicking of fermions from Dirac sea to form the finite part of infinite prime. These states have interpretation as analogs of free states of supersymmetric arithmetic quantum field theory (QFT). There is a temptation to interpret the sum $X/n_F + n_F$ as an analog of quantum superposition. Fermion number is well-defined if one assigns the number of factors of n_F to both n_F and X/n_F.

   These infinite primes define polynomials of ordinary variable x with rational root $m_0 n_F^2/m_1$. This gives all rational roots proportional to square free integers n_F but also the roots $m_0 n_F/m_1$ correspond to infinite primes and run over all possible rational roots. This would require modification of the definition. Fermions corresponding to prime factors of n_F are kicked out of Fermi sea but some of them can be annihilated by dropping some factors of n_F. This definition looks number-theoretically more natural.

2. More general infinite primes correspond to polynomials $Q(X) = \sum_n q_n X^n$ required to define infinite integers, which are not divisible by finite primes or by powers of monomials defined by the infinite primes linear in X so that one has an irreducible polynomial having no rational roots.

   Each summand $q_n X^n$ must be an infinite integer. Note that the signs of q_n can be also negative. This requires that q_n for $n > 0$, is given by $q_n = m_{B,n}/\prod_{i=1}^{n} n_{F,i|n}$ of square free integers $n_{F,i}$ having no common divisors. Let q_0 be the finite part of infinite prime having prime divisors p_i. For given p_i, at least one of the summands $q_n X^n$ must be indivisible by p_i to guarantee the indivisibility of infinite prime by any finite prime. Therefore, for some value $n = n_0$, $\prod_{i=1}^{n} n_{F,i|n}$ must have p_i as a divisor.

   The coefficient $m_{B,n}$ representing bosonic state have no common primes with $\prod n_{F,i|n}$ and there exists no prime p dividing all coefficients $m_{B,n}$, $n > 0$ and q_0: that is there is no boson with momentum p present in all states in the sum.

   These states could have a formal interpretation as bound states of arithmetic supersymmetric QFT. The degree k of Q determines the number of particles in the bound states.

   The products of infinite primes at a given level are infinite primes with respect to the primes at the lower levels but infinite integers at their own level. Sums of infinite primes are not in general infinite primes.

   Notice that since the roots of a polynomial P are not affected by a scaling of P, irreducibility as a criterion for infinite prime property allows the scaling of the infinite prime so that one obtains an irreducible polynomial of X with integer coefficients.

3. At the next step one can form the product of all finite primes and infinite primes constructed in this manner and repeat the process as an analog to second quantization. This procedure can be repeated indefinitely. This repeated quantization a hierarchy of infinite primes, which could correspond to the hierarchy of space-time sheets.

   At the n:th hierarchy level the polynomials are polynomials of n variables X_i. A possible interpretation would be that one has families of infinite primes at the first level labelled by n_1 parameters. If the polynomials $P(x)$ at the first level define space-time surfaces, the interpretation at the level of WCW could be that one has an $n-1$-D surface in WCW parametrized by $n-1$ parameters with rational values and defining a kind of sub-WCW. The WCW spinor fields would be restricted to this surface of WCW.

The Dirac vacuum X brings in mind adele, which is roughly a product of p-adic number fields. The primes of infinite prime could be interpreted as labels for p-adic number fields. Even more generally, they could serve as labels for p-adic representations of various algebras and one could even consider replacing the arithmetic operations with $\oplus$ and $\otimes$ to get the quantum variants of various number fields and of adeles.

The quantum counterparts of infinite primes at the lowest and also at the higher levels of hierarchy could be seen as a generalization of adeles to quantum adeles.

19.2.2 Questions about infinite primes

One can ask several questions about infinite primes.

1. Could $\oplus$ and $\otimes$ replace $+$ and $-$ also for infinite primes. This would allow us to interpret the primes p as labels for algebras realized p-adically. This would give rise to quantal counterparts of infinite primes.

2. What could $+ \to \oplus$ for infinite primes mean physically? Could it make sense in adelic context? Infinite part has finite p-adic norms. The interpretation as direct sum conforms with the fermionic interpretation if the product of all finite primes is interpreted as Dirac sea. In this case, the finite and infinite parts of infinite prime would have the same fermion number.

3. Could adelization relate to the notion of infinite primes? Could one generalize quantum adeles based on $\oplus$ and $\otimes$ so that they would have parts with various degrees of infinity?

19.2.3 $P = Q$ hypothesis

One cannot avoid the idea that that polynomial, call it $Q(X)$, defining an infinite prime at the first level of the hierarchy, is nothing but the polynomial P defining a 4-surface in M^4 and therefore also a space-time surface. $P = Q$ would be a condition analogous to the variational principle defining preferred extremals (PEs) at the level of H.

There is however an objection.

1. $P = Q$ gives very powerful constraints on Q since it must define an infinite integer. The prime polynomials P are expected to be highly non-unique and an entire class of polynomials of fixed degree characterized by the Galois group as an invariant is in question. The same applies to polynomials Q as is easy to see: the only condition is that powers of $a_k X^k$ defining infinite integers have no common prime factors.
2. It seems that a composite polynomial $P_n \circ ... \circ P_1$ satisfying $P_i = Q_i$ cannot define an infinite prime or even infinite integer. Even infinite integer property requires very special conditions.
3. There is however no need to assume $P_i = Q_i$ conditions. It is enough to require that there exists a composite $P_n \circ ... \circ P_1$ of prime polynomials satisfying $P_n \circ ... \circ P_1 = Q$ defining an infinite prime.

   The physical interpretation would be that the interaction spoils the infinite prime property of the composites and they become analogs of off-mass-shell particles. Exactly this occurs for bound many-particle states of particles represented by P_i represented composite polynomials $P_1 \circ ... P_n$. The roots of the composite polynomials are indeed affected for the composite. Note that also products of Q_i are infinite primes and the interpretation is as a free many-particle state formed by bound states Q_i.

There is also a second objection against $P = Q$ property.

1. The proposed physical interpretation is that the ramified primes associated with $P = Q$ correspond to the p-adic primes characterizing particles. This would mean that the ramimied primes appearing in the infinite primes at the first level of the hierarchy should be physically special.
2. The first naive guess is that for the simplest infinite primes $Q(X) = (m_1/n_F)X + m_2 n_F$ at the first level, the finite part $m_2 n_F$ has an identification as the discriminant D of the polynomial $P(X)$ defining the space-time surface. This guess has no obvious generalization to higher degree polynomials $Q(X)$ and the following argument shows that it does not make sense.

   Since Q is a rational polynomial of degree 1 there is only a single rational root and discriminant defined by the differences of distinct roots is ill-defined that $Q = P$ condition would not allow the simplest infinite primes.

   Therefore one must give either of these conjectures and since $P = Q$ conjecture dictates the algebraic structure of the quantum theory for a given space-time surface, it is much more attractive.

The following argument gives $P = Q$. One can assign to polynomial P invariants as symmetric functions of the roots. They are invariants under permutation group S_n of roots containing Galois group and therefore also Galois invariants (for polynomials of second order correspond to sum and product of roots appearing as coefficients of the polynomial in the representation $x^2 + bx + cx$). The polynomial Q having as coefficients these invariants is the original polynomial. This interpretation gives $P = Q$.

19.3 How also finite fields could define fundamental number fields in Quantum TGD?

One can represent two objections against the number theoretic vision.

1. The first problem is related to the physical interpretation of the number theoretic vision. The ramified primes p_{ram} dividing the discriminant of the rational polynomial P have a physical interpretation as p-adic primes defining p-adic length- and mass scales.

The problem is that without further assumptions they do not correlate at all with the degree n of P. However, physical intuition suggests that they should depend on the degree of P so that a small degree n implying a low algebraic complexity should correspond to small ramified primes. This is achieved if the coefficients of P are smaller than n and thus involve only prime factors $p < n$.

2. All number fields except finite fields, that is rationals and their extension, p-adic numbers and their extensions, reals, complex numbers, quaternions, and octonsions appear at the fundamental level in TGD. Could there be a manner to make also finite fields a natural part of TGD?

These problems raise the question of whether one could pose additional conditions to the polynomials P of degree n defining 4-surfaces in M^8 with roots defining mass shells in $M^4 \subset M^8$ (complexification assumed) mapped by $M^8 - H$ duality to space-time surfaces in H.

19.3.1 $P = Q$ condition

One such condition was proposed in [L127]. The proposal is that infinite primes forming a hierarchy are central for quantum TGD. It is proposed that the notion of infinite prime generalizes to that of the notion of adele.

1. Infinite primes at the lowest level of the hierarchy correspond to polynomials of single variable x replaced with the product $X = \prod_p p$ of all finite primes. The coefficients of the polynomial do not have common prime divisors. At higher levels, one has polynomials of several variables satisfying analogous conditions.
2. The notion of infinite prime generalizes and one can replace the argument x with Hilbert space,group representation, or algebra and sum and product of ordinary arithmetics with direct sum $\oplus$ and tensor product $\otimes$.
3. The proposal is $P = Q$: at the lowest level of the hierarchy, the polynomial $P(x)$ defining a space-time surface corresponds to an infinite prime determined by a polynomial $Q(X)$. This would be one realization of quantum classical correspondence. This gives strong constraints to the space-time surface and one might speak of the analog of preferred extremal (PE) at the level of M^8 but does not yet give any special role for the finite fields.
4. The infinite primes at the higher level of the hierarchies correspond to polynomials $Q(x_1, x_2, ..., x_k)$ of several variables. How to assign a polynomial of a single argument and thus a 4-surface to Q? One possibility is that one does as in the case of multiple poly-zeta and performs a multiple residue integral around the pole at infinity and obtains a finite result. The remaining polynomial would define the space-time surface.

19.3.2 Proposal

The speculations related to the p-adicization of the ξ function associated with the Riemann zeta discussed in [L125] inspired the following proposal.

1. The integer coefficients of $P = Q$ are smaller than n. For the most general option for infinite primes, one would have irreducible polynomials equivalent by scaling with polynomials with integer coefficients smaller than n. One could say that the corresponding space-time sheet effectively lives in the ring Z_n instead of integers. For prime value $n = p$ space-time sheet would effectively "live" the finite field F_p and finite fields would gain a fundamental status in the structure of TGD.

   One could allow both signs for the coefficients as the interpretation as rationals would suggest? In this case, finite field interpretation would mean the replacement of -1 with $p - 1$.
2. The construction of the proposed polynomials is very simple. Only integers $a_n < n$, having as their factors primes $p < n$, are possible as coefficients p_n of P and p_n and the condition is that the polynomials are irreducible and therefore do not have rational roots.

   The number of polynomial coefficients is $n + 1$ for an n:th order polynomial, and the number of possible values of a_k is n. This would give $(n + 1)^n$ different polynomials and irreducibility poses additional restrictions. Note that the number of primes smaller than n behaves as $n/log(n)$.

The proposal would solve the two problems mentioned in the beginning.

1. For $n = p$, P would make sense in a finite field F_p if the second condition is true. Finite fields, which have been missing from the hierarchy of numbers fields, would find a natural place in TGD if this condition holds true!
2. Also an upper for ramified primes in terms of order of P emerges and for prime polynomials of order p is given by p^p. This will be discussed in more detail in the sequel.

How does the proposal relate to prime polynomials and polynomials having finite field interpretation?

One can invent an objection against the proposal that the reducible polynomials have coefficients smaller than the order of the polynomial. One of the basic conjectures of the number theoretic vision has been that functional composition of polynomials $P = P_2 \circ P_1$ of degrees m and n giving more complex polynomials is possible. This would give rise to evolutionary hierarchies and could also correspond to the inclusion hierarchies for hyperfinite factors of type II_1 (the additional assumption has been that the polynomials vanish at $x = 0$ that $P_0 = 0$ but this condition could be reconsidered).

Could the proposed conditions hold true for so-called prime polynomials, which are analogous to infinite primes? Prime polynomials are discussed in [L127].

1. Polynomials can be factorized into composites of prime polynomials [A103, A160] (`https://cutt.ly/HXAKDzT` and `https://cutt.ly/5XAKCe2`). A polynomial, which does not have a functional composition to lower degree polynomials, is called a prime polynomial. It is not possible to assign to prime polynomials prime degrees except in special cases. Simple Galois groups with no normal subgroups must correspond to prime polynomials.
2. For a non-prime polynomial, the number N of the factors P_i, their degrees n_i are fixed and only their order can vary so that n_i and $n = \prod n_i$ is an invariant of a prime polynomial and of simple Galois group [A103, A160]. Note that this composition need not exist for monic polynomials even if the Galois group is not simple so that polynomial primes in the monic sense need not correspond to simple Galois groups.

Prime polynomials indeed satisfy the conditions of the proposal.

1. The degree of a composite of polynomials with orders m and n is mn. Therefore a polynomial with a prime degree p does not allow an expression as a composite of polynomials of lower orders so that any polynomial with prime order is a prime polynomial. Any irreducible polynomial with prime order is also a prime polynomial and corresponds to an infinite prime.
2. Polynomials of order m can in principle be functional composites of prime polynomials with orders, which are prime factors of m. All irreducible prime polynomials would satisfy the proposal.
3. The natural conjecture is that the functional composites of irreducible prime polynomials are irreducible. If this is the case, irreducible prime polynomials as counterparts of special infinite primes could be used to construct more general polynomials in correspondence with infinite primes.

These observations suggest the tightening of the proposal. There are two alternative additional conditions.

All physically allowed polynomials P are functional composites of the irreducible prime polynomials P of order $n = p$ or $n = p - 1$ with coefficients smaller than n. For $n = p$ one would have prime polynomials. For $n = p - 1$ the polynomials would have interpretation as polynomials in finite field.

1. The degree $n = p - 1$ required by finite field interpretation is not the same as the degree $n = p$ implied by prime polynomial interpretation. Could both interpretations make sense! Indeed, if one has $P_p = xP_{p-1}$ so that P is reducible, one has both interpretations. $D(P)$ has a general expression as a product of root differences. For $P_p = xP_{p-1}$, $D(P)$ reduces to a product of two terms: the product of roots of P_{p-1} and $D(P_{p-1})$.

   Note that it is not clear whether $P_p = xP_{p-1}$ can be a prime polynomial.

2. The functional composite $P \circ R$ of a polynomial $P = xQ$ with a polynomial R has the property that the roots of R are also the roots of P: $P \circ R$ inherits the roots of R. I have proposed that this inheritance of information could be more than analogous to genetic inheritance [L123]. One would have composition hierarchies of this kind of polynomials? Could they correspond to prime polynomials?

Therefore one can consider also a third alternative:

All physically allowed polynomials P are functional composites of the reducible prime polynomials $P = xQ$ of order $n = p$ such that Q is irreducible polynmial of order $p-1$. In a rather precise sense, finite fields would serve as basic building blocks of the Universe.

19.4 Do elementary particles correspond to polynomials possessing single ramified prime?

The physical motivation for the calculation comes from p-adic mass calculations [K50] and number theoretic vision justifying them.

1. The notion of p-adic prime is central in the p-adic mass calculations. p-Adic primes define the p-adic length scales assignable to elementary particles, actually to any system. p-Adic length/mass scale defines the mass scale of the particle [K50]. p-Adic length scale hypothesis states that these primes are near powers of 2 or possibly also other small primes such as 3 (there is some evidence for this [K62]). One should find a convincing mathematical justification for the p-adic length scale hypothesis.
2. Number theoretical vision suggests the interpretation of p-adic prime as a ramified prime of an extension defined by a rational (or equivalently integer) polynomial $P = Q$ defining the space-time surface by $M^8 - H$ duality. I have proposed the interpretation of ramified primes as
3. There is a long standing interpretational problem related to ramified primes. How are elementary particles distinguished from composite particles and many-particle states?

   Could elementary particles be characterized by only a single ramified prime? Or more generally: could the ramified primes associated with the many-particle state correspond to p-adic mass scales of the particles possibly present in the many-particle state?

   If this were the case, theory would be very predictive: one could identify the polynomials that could give rise to the space-time surfaces associated with the elementary particles!

   This condition becomes even stronger if one assumes prime polynomials of degree $n = p$ or polynomials with finite field interpretation and with degree $n = p - 1$.

19.4.1 Calculation of ramified primes

Consider now the calculational problem.

1. One considers polynomials $P(x) = a_0 + a_1x + a_2x^2 + ...x_nx^n$ (they define space-time surfaces in TGD by $M^8 - H$ duality). P is characterized by the vector $[a_0, a_1, ..., a_n]$. The coefficients a_i are positive or negative integers and satisfy the condition $a_i < n$. This condition is physically very relevant since it implies a correlation between the degree of P and the maximal size for its ramified primes.
2. Especially interesting values of n are primes $p = 2, 3, 5, 7....$ These correspond to prime polynomials having no functional decomposition to polynomials of lower degree.

   Also the values $n = p - 1$ are highly interesting since in this case the polynomial defines a polynomial in finite field F_p.
3. Polynomials are irreducible. This guarantees that P defines what I call infinite prime at the first level of the hierarchy.
4. Example 1: $n = p = 2$. Polynomials of degree 2. $[a_0, a_1, a_2]$. Coefficients are equal to ± 1 or 0.

   Example 2: $n = p = 3$: $[a_0, a_1, a_2, a_3]$. Coefficients are equal ± 2 , ± 1 or 0.

One must calculate the ramified primes of P. They are the primes dividing the discriminant D of P. The definition of D in terms of $[a_n, ..., a_0]$ can be found from Wikipedia (`https://en.wikipedia.org/wiki/Discriminant`). The definition in terms of root differences requires the calculation of roots and remains always approximate.

1. One considers both the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and its derivative

$$P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

2. The resultant of P and P' is the determinant of the Sylvester matrix S (`https://en.wikipedia.org/wiki/Sylvester_matrix`).
Sylvester matrix is defined as the following $(2n-1) \times (2n-1)$ matrix.

$$S = \begin{pmatrix} a_n & a_{n-1} & ... & a_0 & 0 & 0 & ... & 0 \\ 0 & a_n & a_{n-1} & ... & a_0 & 0 & ... & 0 \\ . & & & & & & & \\ . & & & & & & & \\ .0 & 0 & 0 & .... & a_n & a_{n-1} & ... & a_0 \\ na_n & (n-1)a_{n-1} & ... & a_1 & 0 & 0 & ... & 0 \\ & na_n & (n-1)a_{n-1} & ... & a_1 & 0 & ... & 0 \\ . & & & & & & & \\ . & & & & & & & \\ .0 & 0 & 0 & ..... & na_n & (n-1)a_{n-1} & ... & a_1 \end{pmatrix}$$

3. The resultant of P and P' is defined as the determinant of the Sylvester matrix:

$$Res_x(P, P') = det(S)$$

Discriminant $Disc \equiv D$ is defined as

$$Disc \equiv D == (-1)^{n(n-1)/2} \frac{Res_x(P, P')}{a_n} = (-1)^{n(n-1)/2} \frac{det(S)}{a_n}$$

One should calculate D and find whether it has prime values. What one should do is the following.

1. One should calculate the determinant and ramified primes for polynomials or order n. $n = p$ defines prime polynomials. Order $n = p - 1$ allows finite field interpretation.
2. One could study the density of polynomials in the space of arrays $[a_0, ..., a_p]$ having only a single ramified prime. It might be possible to find rather large primes for reasonably small cutoff for p, say around $p = 13$, since the sizes of the individual terms in D have upper bound of order p^{2p+1} and their number is $(2p+1)!$.

The calculation is very straightforward and anyone having access to programs like Mathematica can do it. Unfortunately, as a science dissident living at the income border, I cannot afford this kind of luxury.

1. Build the matrix S for arbitrary integer n. One could also restrict to the cases $n = p$ and $n = p - 1$. Assume $a_k < n$.
2. Calculate the quantity $D = (-1)^{n(n-1)/2} det(S)/a_n$.
3. Calculate ramified primes as the prime factors of D.
4. For each n, one could perform a multiloop over the values of $a_k < n$. One should print the set of ramified primes or prime decomposition of D for each combination and store it in a list. One can use this program to study how ramified primes depend on $n = p$.

Also $n = p - 1$ case, which would correspond to finite fields should be considered.

If one has $P_p(x) = xP_{p-1}(x)$ one can say that one has both the cases P_{p-1} and P_p. In this case, the roots of $P_{p-1}(x)$ are inherited by P_p. The formula of discriminant as a product of root differences gives the discriminant as product $D(xP(x)) = \prod_k r_k D(P) = a_0 D(P)$. Also the prime factors of the coefficient a_0 appear as ramified primes of $x(Px)$ besides those of $P(x)$. For $a_0 = 1$ the ramified primes are the same. It is enough to consider only polynomials $P_{p-1}(x)$ in this case.

19.4.2 Could $D = P$ correspond to a maximum of D or of maximal ramified prime P_{max} for D?

On basis of $M^8 - H$ duality [L82, L83], one can argue that the vacuum functional in WCW defined as exponent $exp(K)$ of Kähler function has a number theoretic counterpart. The most natural number theoretical invariant is the discriminant D for the polynomial $P(x)$ defining the space-time surface by $M^8 - H$ duality. This quantity makes sense also at the continuum limit based on polynomials with continuous coefficients.

One could have $exp(K) = 1/D$. An alternative identification would be as $exp(K) = 1/P_{max}$, where P_{max} is the maximal ramified prime dividing the discriminant D for $P(x)$. This makes sense only for integer coefficients of $P(x)$..

The most probable 3-surfaces correspond to maxima of $exp(K)$. A natural guess is that $D = P$ corresponds to a local maximum of D for the polynomials considered. A weaker hypothesis is that $D = P_{max}$ corresponds to a local maximum of the maximal ramified primed P_{max}.

1. The exponent of the Kähler function for the most probable space-time surfaces in $H = M^4 \times CP_2$ as analogs of Bohr orbits is a local maximum in the "world of classical worlds" (WCW). The space-time surface is that with the highest probability.
2. This conforms with the notion of cognitive representation as a discretization obtained by replacing space-time surface with sets of points, which have coordinates in the extension of rationals defined by $P(x)$. The discretization of WCW would consist of discretizations of the most probable space-time surfaces.
3. $M^8 - H$ duality and number theoretic vision [K9] suggest that the value of vacuum functional as exponent $exp(K)$ of the Kähler function is equal to the p-adic counterpart of the discriminant D for the ramified prime $D = P$: $exp(K) = 1/D$.

   $D = P$ could correspond to either a maximum of $exp(K) = 1/D$ for $D = P$ or maximum of $exp(K) = 1/P_{max}$ for the maximum of P_{max}. The latter form of the hypothesis is weaker. $D = P$ could indeed correspond to a maximum of P_{max} since all other values are at least by a factor 1/2 smaller in the vicinity of the maximum of P_{max}.
4. If the proposed connection between the Kähler function and D or P_{max} is true, one can ask whether D or P_{max} has the largest possible value for polynomials of a given degree. This is so if there is only a single local maximum. However, spin glass property, suggested to be the basic characteristic of the dynamics, suggests a counterpart energy landscape with valleys within valleys [L109] so that a large number of single ramified primes is expected for a polynomial of a given degree.

   This is not surprising. D is proportional to $det(S)$, which is the sum of $(2p + 1)!$ terms which are products of $2p + 1$ matrix elements. The terms in the sum tend to sum up to zero and the terms in which all matrix elements are near the largest possible value give the dominating contribution. The order of magnitude for this kind of term is p^{2p+1}. For $p = 13$ this gives 1.2×10^{30}. Since there are a large number of terms, it is possible to have considerably larger values of $D = P$ than this. Therefore one expects that physically realistic values of ramified primes,the Mersenne prime $M_{127} = 2^{127} - 1$ characterizing electrons in p-adic mass calculations, are possible to relatively small primes p.

19.4.3 Spin glass analogy for WCW geometry as a guide line

The spin glass analogy suggests a physics inspired interpretation of the set of the most probable 4-surfaces defined by a polynomial or even set of polynomials as a discretization of a part of WCW.

1. Spin glass corresponds to a discretized energy landscape that is a fractal and obeys ultrametric topology just like p-adic number fields. For spin glass the notion of ergodicity fails. Global thermodynamic equilibria are impossible because the system tends to stick into a potential well.

   This has spontaneous magnetization and the Higgs mechanism as a very simple analogues. Thermodynamics would suggest no magnetization since there is no preferred direction for it. The magnetization however occurs since the thermodynamic ensemble with even distribution over all magnetization directions is not physically sensible: localization occurs.

   In the case of spin glass, the situation is much more complex: instead of magnetization direction, there are an infinite number of different configurations which correspond to local minima of free energy. The system is typically caught into some local potential well containing smaller potential wells and is unable to get out of the well so that the thermal equilibrium reduces to a smaller scale. In a process known as quenching the system can be brought by reheating and cooling to an increasingly deep potential well.

2. In TGD the exponent of free energy would correspond to $exp(K)$. Number theoretic constraints suggest that it is equal to a negative power of D or P_{max}. The probabilities of individual surfaces characterized by polynomials $P(x) \leftrightarrow [a_0, ...a_k]$ would be proportional to $1/D^k$ or $1/P_{max}^k$.

   One could assign to them probabilities by normalizing these numbers by analog of partition function $Z = \sum d_D D^{-k}$ or $Z = \sum d_{P_{max}} P_{max} - k$. Here d_D *resp.* $d_{P_{max}}$ gives the degeneracy of D *resp.* P_{max} as number of polynomials with this value of D. Z is analogous to Riemann zeta at the point $s = k$ of the real axis. k is analogous to inverse temperature. This thermodynamics is however different from standard thermodynamics in which Boltzman weights are given by $exp(-E/T)$. Now Boltzmann weights would be analogous to powers E^{-k}. One has scaling invariance. Spin glasses indeed correspond to this kind of thermodynamics [L109] and in TGD framework the p-adic thermodynamics is indeed defined by a scaling generator rather than energy.

One can assign to polynomials of a given degree k or degree k smaller than maximum value k_{max} an analog of Riemann zeta, which might be perhaps called TGD zeta.

1. All these zeta functions have a finite number of terms. Also the "full" TGD zeta obtained at the limit $k_{max} \rightarrow \infty$ could make sense. The degree k or its maximal value k_{max} could define the analog for the inverse temperature. This gives a nice connection with the speculations [L125] inspired by the geometry-number theory duality coded by $M^8 - H$ duality in the TGD framework and by Langlands correspondence in pure mathematics.

2. One has also other interpretations for k. The degree k of polynomial $P(x)$ is much smaller than the largest ramified prime P_{max} associated with it. On the other hand, the p-adic length scale hypothesis states that the p-adic primes are near to powers of small primes p, in particular $p = 2$. This suggests that for these physically preferred p-adic primes P, having a very large degeneracy factor $d(P)$, the relationship $P \simeq 2^k$ holds true.

   The interpretation of k as the counterpart of the running Kähler coupling strength α_K is also natural and the quantization of $1/alpha_K$ to integer values is natural by the number theoretic universality. This conformas with the generic logarithmic depends of the Kähler coupling strength on the p-adic length scale. Therefore the logarithmic p-adic coupling constant evolution for α_K could be equivalent with the p-adic length scale hypothesis!

3. Spin glass is never in a complete thermal equilibrium since ergodic theorem fails for it. One can consider various analogs of spin glass ensembles assuming the existence of temperature as a parameter.

   In the case of TGD, the running Kähler coupling strength $1/\alpha_K$ would serve as a temperature like parameter. At the high temperature limit (short scales), analogies of spin glass ensembles involving several degrees $d(P)$ for polynomials $P(x)$ can be considered. At low temperatures (long scales), single degree becomes possible and one can also consider a localization around a single configuration such as a polynomial with $D = P$. Elementary particles could correspond to maximal localization around $D = P$.

   At the number theoretic side, the integer k is analogous to the argument s of the zeta function, and analogous to inverse temperature. $s = 1$ for ζ corresponds to a high temperature limit at which ζ diverges. $k = 1$ would be analogous to the inverse of maximal temperature known as Hagedorn temperature in string models. Large values of k correspond to low temperatures.

4. How does this picture relate to p-adic thermodynamics? In p-adic thermodynamics, one considers single P so that localization is maximal apart from the degeneracy factor d. The p-adic temperature for fermions corresponds to maximal p-adic temperature $T_p = 1$. On the other hand, the localization around single P would suggest a minimal temperature. One should be however cautious in comparisons since the thermodynamics in question are totally different: one with p-adic variants of Boltzmann weights and the second with their scaling covariant analogs.

5. A longstanding open problem of TGD is what determines the preferred ramified primes suggested by p-adic mass calculations to be near powers of small primes, in particular $p = 2$. What these ramified primes correspond to preferred valleys of the spin glass energy landscape? What comes to mind is that some values of $D = P$ (or P_{max}) do occur with a large degeneracy d_D (or $d_{P_{max}}$). Preferred ramified primes could correspond to especially large values of d. The quenching-like processes (cooling and reheating) defined by the cosmic evolution leading to lower temperatures would tend to localize the elementary particles to the wells corresponding to ramified primes satisfying p-adic length scale hypothesis.

19.4.4 The ultrametric topology of discretized WCW

Can one give a concrete interpretation for the ultrametricity of the spin glass energy landscape in the case of WCW?

Ultrametricity can be formulated as a condition for a distance function $d(A, B)$ defined between two valleys of spin glass energy landscape. The distance along a given path from point A to B is the height of the highest mountain at the path and is minimized for the shortest path (MiniMax principle.

It is easy to see that the ultrametricity condition $d_{AB} \leq Max\{d(A, C), d(C, B)\}$ is satisfied. In the recent case, the value of D for a given $P(x)$ in the discretization of WCW by polynomials should naturally define an integer valued height h of the mountain.

There are several questions to be answered.

1. Ultrametricity means the presence of very many p-adic topologies in WCW discretized in terms of polynomials. Somehow this number theoretic WCW decomposes into subsets with different p-adic topologies.

   It would be very natural to assign p-adic topology to some, or more naturally, to all ramified primes dividing the discriminant of a given polynomial $P(x)$. Here the physical picture generalizing the notion of Feynman diagram comes to rescue. The lines of the Feynman diagram become 4-surfaces representing particles and vertices become 4-surfaces defining interaction regions in which external particles arrive.

   Free particles would correspond to $D = P$ and vertices as space-time regions where interactions between particles take place would correspond to discriminants D having a decomposition to several primes labelling the external particles of the Feynman diagram. This would solve the longstanding problem of how particles characterized by different values of p-adic primes P can interact in the same vertex.

2. The notion of p-adic nearness is very different from its real counterpart. Two points of WCW as polynomials can be very far from each other in the real sense but be close to each other p-adically. It is natural to arrange the points of the sub-WCW WCW_P defined by a subset of polynomials to subsets such that points belonging to the same subset have a common ramified prime P.

   The points of WCW_P would allow p-adic topology characterized by P and consist of both particles characterized by $D = P$ and vertices with D divided by P. The subsets WCW_P would intersect along the 4-surfaces with D divided by several primes P.

3. Between the points of this set one can define the p-adic distance function $d_P(A, B)$ using the above general definition using D as a positive integer defining the mountain height. There are two options for the paths involved.

The paths could be paths in WCW and go also through points of discretized WCW, which do not belong to WCW_P or could be contained in WCW_P. These metrics would be analogous to the distance between two points of the space-time surface defined by the shortest path in $X^4 \subset M^4 \times CP_2$ (metric of H) and by the shortest path along X^4 (induced metric).

4. The height function h for a mountain defined by polynomial $P(x)$ with discriminant D could be obtained from D identified as a P-adic number. If t h is identified as the P-adic norm of D, the height function is very rough. A more refined distance function is obtained by the canonical identification $I : \sum x_n p^n \to \sum x_n p^{-n}$ used in the p-adic mass calculations [?]apping $h_P = D = \sum h_n P^n$ to $h_R = h_n P^{-n}$. I maps p-adic numbers to reals in a continuous manner and takes p-adic numbers P^n to P^{-n}.

In the standard ontology, one can predict scattering rates but particle densities cannot be predicted without further assumptions. In ZEO both can be predicted since there is a complete democracy between particles and particle reactions. Physical event as a superposition of deterministic time evolutions becomes the basic notion and both particles and particle reactions correspond to physical events.

The statistical model represents the probabilities of physical events within the quantization volume defined by CD. Particle characterized by $D = P$ and corresponds to a scattering event with a single incoming and outgoing particle, and the statistical model predicts the densities of various particles as probabilities of $D = P$ events. Genuine particle reaction corresponds to $D = \prod P_i$ and the model gives the probabilities of observing these events within CD.

19.4.5 How to study the hypothesis?

There are several ways to study the hypothesis.

1. One could think of finding the polynomial corresponding to the maximum of $D = P$ by considering the coefficients of $P(x)$ as real variables in some region of the coefficient space and finding the nearest polynomial with integer coefficients.
2. One could consider the maximization of D by keeping the polynomial coefficients as real numbers with magnitude below p, say $p = 13$. At maximum the partial derivative of D vanishes unless the point is at the boundary of the region of allowed values. This boundary for allowed values is a $p + 1$-cube and consists of parts for which some coefficients a_k have the maximal value $\pm p$.
3. One could check what one obtains by putting some values of a_k to $a_k = p$. For $k = p$ this would give for the derivative $P' = [pa_p, ...]$ so $a_k = p$ for k near p is favoured.
4. A very simple test for the hypothesis that $D = P$ holds true for a) the maxima of D or b) for the maximal primes P_{max} of D) would be based on small variations of a polynomial $P(x) \leftrightarrow [a_0, a_1, ..., a_n]$, which corresponds to $D = P$: these should be relatively easy to find.

   One could vary the coefficients a_i in the range $a_i + \{-1, 0, +1\}$. This would give 3^{p+1} trials for a prime polynomial $P_p(x)$: this is a rather reasonable number. Finding only a single P for which this is not the case, would kill the hypothesis. If $D < P_0$ is true for all variations, the hypothesis could be tested for further cases $D = P$.

19.5 Gödel's Undecidability Theorem and TGD

$M^8 - H$ duality [L82, L83] relates number theoretic and geometric views of physics [L125, ?]. Gödel's incompleteness theorem relates to number theory. Could one consider a geometric and physical interpretation of Gödel's incompleteness theorem in the TGD framework?

The following response to Lawrence Crowell in the discussion group "The Road to Unifying Relativistic and Quantum Theories" indeed suggests such an interpretation. The topic of discussion related to Gödel's theorem and its possible connection with consciousness proposed by Penrose [J3].

My own view is that quantum jump as state function reduction (SFR) cannot reduce to a deterministic computation and can be seen as a moment of re-creation or discovery of a new truth not following from an existing axiomatic system summarizing the truths already discovered. Zero energy ontology allows to solve the basic paradox of quantum measurement theory [L72, L108].

My emphasis in the sequel is on how the number theoretic vision of the TGD [L82, L83, L125, L120] proposed to provide a mathematical description of (also mathematical) cognition could allow us to interpret the unprovable Gödel sentence and its negation. There is no need to emphasize that these considerations are highly speculative.

19.5.1 What Gödel's theorem could mean in the TGD Universe?

The basic question concerns the physical and consciousness theoretic interpretation of the Göedel's undecidability theorem in the TGD Universe.

Some TGD background

In the following some necessary conceptual background will be introduced.

1. The polynomials P define space-time surfaces and one possible interpretation is that the ramified primes of P define external particles for a space-time region representing particle scattering. The polynomials P which reduce to single ramified prime would represent forward scattering of a single "elementary" particle.
2. In zero energy ontology (ZEO) [L117], ordinary quantum states are replaced by superpositions of almost deterministic time evolutions so that also "elementary" particle would correspond to a scattering event.

   What exists would be events, and what we call states would reduce to particular events. One could call ZEO as an "eastern" ontology. ZEO would predict not only scattering events but densities of particles as single particle scattering events inside a given causal diamond causal diamond (CD) representing quantization volume [L120].
3. Single space-time surface in $H = M^4 \times CP_2$ is obtained by $M^8 - H$ duality from a 4-surface in M^8 and satisfies in H almost exact holography forced by the general coordinate invariance. At the level of M^8 its preimage obeys number theoretic dynamics forcing the associativity of its normal space [L82, L83]. This 4-surface connects mass shells $H^3_a \subset M^4 \subset M^8$, which correspond to the roots of a polynomial P with integer coefficients.

   Almost holographic space-time surfaces represent a profound deviation from the standard physics view. They can be regarded as analogs of computations or proofs of theorems, counterparts of behaviors in neuroscience, and counterparts of biological functions. Quantum states are their superpositions. Number theoretically realized finite measurement resolution means that the superposition of space-time surfaces having the same theoretic discretization effectively represents a single space-time surface.

   Therefore the idea that the SFRs localizing the state to this kind of surfaces, could represent a physical realization of a mathematical theorem, looks natural. Gödel's theorem could correspond to a space-time surface to which localization by SFR is not possible.
4. The additional hypothesis [L120] motivated by $M^8 - H$ duality is that the values of WCW Kähler function H for its maxima defined by preferred extremals in H and analogous to Bohr orbits have values of vacuun functional $exp(K)$, which is equal to $1/D^k$, where the integer k defines analog of temperature and is inversely proportional the discrete running Kähler coupling strength $1/\alpha_k$. Zero energy states correspond to scattering amplitudes so that this would predict the scattering probabilities in WCW geometric degrees of freedom.

   For elementary particles sfor which D reduces to a single prime $D = P$, $1/\alpha_k$ would roughly behave like logarith of P. This would unify the logarithmic dependence of p-adic coupling constant evolution with the p-adic length scale hypothesis [L120].

Gödel numbering in TGD framework and the first for guess for the undecidable statement

Polynomials with integer coefficients (no common factor coefficients) to which all rational polynomials can be scaled without changing the roots define the space-time surfaces. One can pose additional physically well-motivated conditions to these polynomials. These conditions will be discussed later.

What the assignment of a Gödel number to this kind of polynomial could mean? Most of the classical physical content, if not all of it, can be coded by the coefficients $[a_0, , ...a_N]$ of the polynomial.

The Gödel number G associated with polynomial P would be rather naturally

$$G(P) = p_0^{a_0} p_2^{a_1} ... p_N^{a_N} \ ,$$

where p_i is i:th prime and is an injection. Note that one has $p_0 = 2, p_1 = 3, p_2 = 5, ...$

The discriminant D (`https://en.wikipedia.org/wiki/Discriminant`) is the determinant of an $(2N-1) \times (2N-1)$-matrix defined by P and its derivative dP/dx ($[a_1, 2a_2, ..., Na_N]$) and is an integer decomposing to a product of ramified primes of P.

The first guess for Gödels' undecidable statement would that there exist a polynomial P for which one has $G = D$. The number D coding a sentence, whatever it is, would be its own Gödel number. Why this guess? At least this statement is short. Can this statement be undecidable? What undecidability could mean physically?

1. The equation involves both D as a polynomial of a_i and G involving transcendental functions $p_i^{a_i}$ (essentially exponential functions) so that one goes outside the realm of rationals and algebraic numbers.

2. $D = G$ is an analogue of Diophantine equation for $a_1, ..., a_N$ and both powers and exponential $p_i^{a_i}$ appear. If the coefficients a_i are allowed to be a complex numbers, one can ask whether the complex solutions of $G = D$ could form an N-1-D manifold. One can however assume this since $p_i^{a_i}$ leads outside the realm of algebraic numbers and one does not have a polynomial equation.

3. The existence of an integer solution to $D = G$ would mean that the primes p_i for which a_i are non-vanishing, correspond to ramified primes of P with multiplicity a_i so that the polynomials would be very special if solutions exist.

4. It might be possible to solve the equation for any finite field G_p, that is in modulo P approximation. Here one can use Fermat's little theorem $p_i^p = p_i \ mod \ p$. If integer solutions exist, they exist for every G_p.

About the number theoretical content of $G = D$ sentence

It is interesting to look at the number theoretical content of $G = D$ sentence.

1. Integer D would express the sentence/statement. D codes for the ramified primes. Their number is finite and we know them once we know P. Does the unprovable Gödel sentence say that there exists a polynomial P of some degree N, whose ramified primes are the primes p_i associated with a_i? Or does it say that there exists a polynomial satisfying $G = D$ in the set of polynomials of fixed degree N. Note that a priori one does not pose constraints on the values of coefficients a_i.

2. Is it that we cannot prove the existence of integer solution a_i to $P = G$ using a finite computation. Is this due to the appearance of the functions $p_i^{a_i}$ or allowance of arbitrarily large coefficients a_i? The p-adic solutions associated with finite field solutions have an infinite number of coefficients and can be p-adic transcendentals rather than rationals having periodic pinary extensions.

3. Polynomials of degree N satisfying $D = G$ are very special. The ramified primes are contained in a set of $N + 1$ first primes p_i so that D is rather small unless the coefficients a_i are large. D is a determinant of $2N - 1 \times 2N - 1$ matrix so that its maximum value increases rapidly with N even when one poses the constraint $a_i < N$. Rough estimates and explicit numerical calculations demonstrate that determinants involving very large primes are possible, in particular those involving single ramified prime identified as analogues of elementary particles, D can reduce to single large prime: $D = P$.

   What about the polynomials P in the vicinity of points of the space of polynomials of degree N satisfying $D = 0$: they correspond to $N+1$ ramified primes, which are minimal (note that the number of roots is N). D is a product of the root differences and 2 or more roots coincide for $D = 0$. D is a smooth function of real arguments restricted to the integer coefficients.

The value of D in the neighborhood of $D = 0$ can be however rather large. Note that the proposed Gödel numbering fails for $D = 0$, and therefore makes sense only for polynomials without multiple roots.

4. For $D(P) = 0$ one has a problem with the equation $G = D$. $G(P)$ is well-defined also now. The condition $D(P) = 0 = G(P)$ does not however make sense. The first guess is that for 2 identical roots, P is replaced with dP/dx in the definition of D: $D(P) - -> D(dP/dx)$. D is nonvanishing and the ramified primes p_i do exist for dP/dx. Therefore the condition $D(dP/dx) = G(P)$ makes sense. For N identical roots one must use have $D(d^{n-1}P/dx^{n-1}) = G(P)$.

About the physical interpretation of the undecidability

What about the physical interpretation of the undecidabililty in the TGD Universe? What kind of scattering events would these analogues of Gödel sentences correspond? Representations of new mathematical axioms as scattering events, not provable from existing axioms, perhaps?

Exactly what we cannot prove to be true or not true for the possibly existing very special polynomials satisfying $G = D$? What could the $G = D$ sentece state? What "proving" could mean from the point of physics and TGD view of consciousness? Could it mean a conscious experience of proof as a localization to the corresponding space-time surface in WCW? The almost deterministic space-time surface would represent the almost deterministic sequence of logical steps for the proof?

Could $G = D$ sentence be a space-time surface to which a localization in WCW is not possible for the simple reason that the additional natural physical conditions on the physical states do not allow its existence in superpositions definition zero energy states?

1. In TGD, the hypothesis [L120] that the coefficients of polynomials of degree N are smaller than N, is physically very natural and would make the number of polynomials to be considered finite so that in this case one can check the existence of a $G = D$ sentence in a finite time. It looks rather plausible that for given N, no $G = D$ sentence, which satisfies the conditions $a_i \leq N$, does exist.

2. One can of course criticize the hypothesis $a_i \leq N$ implying a strong correlation between the degree N of P and the maximal size of ramified primes of P identified as p-adic primes characterizing elementary particles. One can argue that in absence of this correlation predictivity is lost. This hypothesis also makes also finite fields basic building bricks of number theoretic vision of TGD [L120].

3. Could this give rise to a realization of undecidability at the level of conscious experience and cognition relying on number theoretic notions? How?

   Quantum states are superpositions of space-time surfaces determined by polynomials P and if the holography of consciousness is true, conscious experience reflects the number theoretic properties of these polynomials if associated to a localization to a given polynomial P in a "small" SFR (SSFR). This would be position measurement in the "world of classical worlds" (WCW)? The proof of the statement $G = D$ would mean that a cognizing system becomes conscious of the $G = D$ space-time surface by a localization to it.

   Suppose that for a given finite N and condition $a_i \leq N$, $G = D$ sentences do not exist. Hence one can say that $G = D$ sentences go outside the axiomatic system realized in terms of the polynomials considered. Even the space of all allowed polynomials identified as a union of spaces with varying value for degree N would not allow this. $G = D$ sentences would be undecidable by the condition $a_i \leq N$.

Chapter 20

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

20.1 Introduction

This article deals with two questions.

1. The ideas related to topological quantum computation [L123] suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of Hilbert space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum. I have considered this kind of idea already earli [K65].

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Could one replace the roots labelling states with group representations? Or could even the coefficient field for the state space be replaced with a ring of representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums of algebraic numbers in quantum-classical correspondence interpreted as a kind of category theoretic morphism, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k,C)$, $k=2,3,4$ [A140, A139] to those of $SL(n,C)$. Is there a deep connection between finite subgroups of $SL(n,C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

20.1.1 Could one generalize arithmetics by replacing sum and product with direct sum and tensor product?

In the model for topological quantum computation (TQC) [B8, B7] quantum states in the representations of groups are replaced with entire representations (anyons). One can argue that this helps to guarantee statibility: this generalization could be regarded as error correction code. In TGD, these representations would correspond to irreps of Galois groups or of discrete subgroups of the covering group for automorphisms of quaternions. Also discrete subgroups of $SL(2,C)$ assignable naturally to the tessellations of H^3 can be considered.

Tensor product $\otimes$ and direct sum $\oplus$ are commutative operations and very much like operations of ordinary arithmetics. One can also speak of positive integer multiples of representation. The algebras of irreps of various algebraic structures generated by $\oplus$ and $\otimes$ are applied quite generally in mathematics and especially so in gauge theories and conformal field theories and are known as fusion algebras (https://cutt.ly/TLU3hvJ) and quivers (https://cutt.ly/xLU3zrM).

Could the replacement of the roots of the EDD of the ADE group with representations of the finite subgroup of $SL(2,C)$ associated with the diagram make sense? The trivial representation would correspond to an additional node and lead to an extended Dynkin diagram (EDD).

Could one regard the irreps as quantum roots of an ordinary monic polynomial so that the ordinary algebraic numbers would have representation as state spaces? Could one obtain the full root diagram by a generalization of the Weyl group operation as reflection of root with respect to root? The first guess is that the isotropy group Gal_I of a root acts as a subgroup of Gal defines the polynomial, which gives the roots replaced by irreps and that Gal itself acts in the same role as the Weyl group.

McKay graph characterizes the rules for the tensor product compositions for the irreps of a finite group G, in particular Galois group. There is an excellent description of McKay graphs on the web (see https://cutt.ly/zLzoAwF). The article describes first the special McKay graphs for finite subgroups of $SL(2,C)$ and their geometric interpretation in terms of the geometry of Platonic solids and their denerate versions as regular polygons and shows that they turn out to correspond to EDDs for ADE type Lie algebras. Also general McKay graphs are considered.

20.1.2 McKay graphs and McKay correspondence

The McKay graphs are a special case of quiver diagrams (https://cutt.ly/xLU3zrM) and code for the tensor product decomposition rules for the irreps of finite groups [A166, A152].

For a general finite group, McKay graphs can be constructed in the following way. Consider any finite group G and its irreducible representations (irreps) ξ_i and assign to ξ_i vertices. Select one irrep V and assign also to it a vertex. For all tensor products $\xi_i \otimes V$ and decompose them to a direct sum of irreps ξ_j. If ξ_j is contained to $V \otimes \xi_i$ a_{ij} times, draw a_{ij} directed arrows connecting vertex i to vertex j. One obtains a weighted, directed graph with incidence matrix a_{ij}. Adjacency matrix plays a central role in graph theory.

McKay correspondence is only one of the mysteries related to MacKay graphs for finite subgroups of $SL(k,C)$, $k = 2,3,4$ and presumably also $k > 4$ [A140, A139]. The MacKay graphs correspond to EDDs for ADE type Lie groups having interpretations as Dynkin diagrams for ADE type affine algebras.

The classification of singularities of complex surfaces represents another example of McKay correspondence.

1. ADE Dynkin diagrams provide a classification of Kleinian singularities of complex surfaces having real dimension 4 and satisfying a polynomial equation $P(z_1, z_2, z_3) = 0$ with $P(0,0,0) = 0$ so that the singularity is at origin [A152] (https://cutt.ly/5LQPyhy). The finite subgroups of $SL(2,C)$ naturally appear as symmetries of the singularities at origin.
2. In the TGD framework, this kind of complex surfaces could correspond to surfaces with an Euclidean signature of induced metric as 4-surfaces in $E^2 \times CP_2 \subset M^4 \times CP_2$. What I call CP_2 type extremals have light-like M^4 projection as deformations of the canonically imbedded CP_2. These surfaces could correspond to deformations of CP_2 type extremals. One can ask whether one could assign ADE type affine algebras as affine algebras with these singularities.

20.2 Could the arithmetics based on direct sum and tensor product for the irreps of the Galois group make sense and have physical meaning?

The idea about the generalization of the mathematical structures based on integer arithmetics with arithmetics replacing $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$ raises a bundle of questions. This idea makes sense also for the finite subgroups of $SU(2)$ defining the covering group of quaternion automorphism having a role similar to that of the Galois group.

What motivates this proposal is that the extensions of rationals and their Galois groups are central in TGD. Polynomials P with integer coefficients are proposed to determine space-time surfaces by M^8-H duality in terms of holography based on the realization of dynamics in M^8 in terms of roots of P having interpretation as mass shells. Holography is realized in terms of the condition that the normal space of the space-time surface going through the mass shells has associative normal space [L82, L83].

20.2.1 Questions

The following questions and considerations are certainly very naive from the point of view of a professional mathematician and the main motivation for the mathematical self ridicule is that there are fascinating physical possibilities involved.

The basic question is whether $\otimes$ and $\oplus$ can give rise to quantum variants of rings of integers and even algebraic integers defined in terms of quantum roots of ordinary polynomial equations and could one even generalize the notion of number field: do quantum variants of extensions of rationals, finite fields, and p-adic number fields make sense?

Recall that also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L43, L42] [K59, K40, K41].

Quantum polynomials

To build extensions of rationals, one must have polynomials. The notion of polynomial playing central role in M^8-H duality [L82, L83], or rather the notion of a root of polynomial, generalizes.

1. Polynomials would look exactly like ordinary monic polynomials, with the real unit replaced with identity representation but their quantum roots would be expressible as direct sums of irreps associated with a given extension of rationals.
2. One would obtain roots as direct sums of the generators of the extension which could correspond to irreps of the isotropy group Gal_I of Galois group Gal. McKay graph would define the multiplication rules for the tensor products appearing in the polynomial whose coefficients would be quantum counterparts of ordinary (positive) integers.
3. Also a generalization of an imaginary unit could make sense for p-adic ring and finite fields as a root of a polynomial. Note that $\sqrt{-1}$ can exist for p-adic number fields. Also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L43, L42] [K59, K40, K41].

Does one obtain additive and multiplicative group structures, rings, and fields?

Could one give to the space spanned by irreps a structure of ring or even field?

1. Could one replace algebraic integers of the ordinary extension of rationals with direct sums of the n_C irreps of Galois group G, where n_C is the number of classes of G? Note that the dimensions n_i of irreps satisfy the formula $\sum n_i^2 = n_C$.

   If $\oplus$ corresponds to $+$ for ordinary integers, only non-negative integers can appear as coefficients so that one would have semigroups with respect to both $\oplus$ and $\otimes$.
2. The inverse with respect to $\oplus$ requires that negative multiples of quantum integers make sense. This is possible in p-adic topology: the number -1 would correspond to the quantum part of the integer $(p-1)\sum_{\oplus} p^{\oplus n}$. The summands in this expression would have p-adic norms p^{-n}. This allows to define also the negatives of other roots playing the role of generator of the quantum extension of rationals.
3. Is even the quantum analog of a number field possible? If one requires multiplicative inverse, only the finite field option remains under consideration since the quantum variant of $1/p^k$ does not make sense since one has $p \equiv= 0$. If one requires group structure for only $\oplus$, quantum p-adics remain under consideration.

Can one map the numbers of quantum extensions of rationals the numbers of ordinary extensions?

Concerning the physical interpretation, it would be important to map the quantum variants of algebraic integers to their real counterparts. Mathematicians might talk of some kind of category theoretical correspondence.

1. Since the same polynomial would have ordinary roots and quantum roots, the natural question is whether the quantum roots can be mapped to the ordinary roots.
2. If the quantum roots correspond to roots of the Dynkin diagram as quantum numbers in quantum extension of rationals, it should be able to map all quantum roots of the ADE type affine algebra to ordinary roots. This requires that sums with respect to $\oplus$ correspond to sums with respect to $+$: additivity of quantum numbers would hold true at both levels and one would have category theoretic correspondence as algebraic isomorphisms.

   Note that Galois confinement means that 4-momenta and other quantum numbers of states are integer valued, when one uses the momentum scale defined by causal diamond (CD). This means that they would correspond to $\oplus$ multiples of trivial representation of the Galois group acting as Weyl group.
3. What about the tensor products of roots appearing in the McKay graph? Can one require that the products with respect to $\otimes$ correspond to products with respect to $\times$. Only $\otimes$ does appear in the generation of the quantum roots of a given KM algebra representation.

   What about quantum variants of quantum states? If the quantum variants of p-adic integers or finite fields appear also as a coefficient field of quantum states, one can always express the coefficients as direct sums of quantum roots and map these sums to sums of ordinary polynomial roots, that is algebraic numbers. Extensions of rationals can appear as coefficient fields for Hilbert spaces.

   If one assumes that only quantum variants of p-adic numbers with a finite number of the pinary digits and their negatives are possible, they can be mapped to numbers in algebraic extension. One could overcome the problems related to the definition of inner product when finite field or p-adic numbers define the coefficient field for Hilbert state.
4. For generalized finite fields, the notions of vector space and matrix algebra, hermiticity and unitarity, and eigenvalue problem could be generalized. For instance, eigenvalues of a Hermitian operator could be just real numbers. Also a relatively straightforward looking generalization of group theory can be imagined, and would be obtained by replacing the elements of the matrix group with the elements of a generalized finite field.

20.2.2 Could the notion of quantum arithmetics be useful in the TGD framework?

These ideas might find an application in TGD.

1. The quantum generalization of the notion of rationals, p-adic number fields, and finite fields could be defended as something more than a mere algebraic game. In particular, in TGD the ramified primes of extension of rationals correspond to physically important p-adic primes, especially the largest ramified prime of the extension. Algebraic prime is a generalization of the notion of ordinary prime. Also its generalization could make sense and give rise to the notion of quantum prime.

   Unfortunately, the extension of finite field F_p induced by a given extension of rationals does not exist for the ramified primes appearing as divisors in the discriminant determined by the product of root differences.

   Could the generalization of the notion of finite field save the situation? Topological quantum computations (TQC) relying on Galois representations as counterparts for anyons would mean an increase of the abstraction level replacing numbers of algebraic extension with representations of Galois group as their cognitive representations.

   One can assign also to the possibly unique monic polynomial P_c defining the n_c-dimensional extension, a discriminant, call it D_c. For the primes dividing the discriminant D of P but not D_c, the quantum counterpart of the finite-field extension could make sense.

2. In TGD, the roots of polynomials define 3-D mass and energy shells in M^8 in turn defining holographic data defining 4-D surface in M^8 mapped to space-time surfaces in H by M^8-H duality. Could one consider quantum variants of the polynomial equations defining space-time surfaces by holography in the generalized extensions of rationals based on representations of Galois groups?

   Could monic polynomials define quantum variants of 4-surfaces or at least of discretizations of hyperbolic spaces H^3 as 3-D sections of 4-surface in M^8 defined as roots of polynomial P and containing holographic data as cognitive representation? Mass shells would be mapped by M^8-H duality to light-cone proper time hyperboloids in H.

   The interiors of 4-surfaces in M^8 would contain very few points of cognitive representation as momentum components in the extension of rationals defined by the polynomial P. Mass shells and their H images would be different and represent a kind of cognitive explosion. The presence of fermions (quarks) at the points of cognitive representation of given mass shells would make them active.

3. Could the transition from the classical to a quantum theory, which also describes cognition, replace discrete classical mass shells as roots of a polynomial in M^8 with roots with direct sums of irreps of the Galois group?

   This idea would conform with category theoretic thinking which leaves the internal structure of the basic object, such as point, open. That points of cognitive representations would be actually irreducible representations of the Galois groups would reveal a kind of cognitive hidden variables and quantum cognition.

These ideas are now completely new. I have earlier considered the possibility that points could have an infinite complex internal structure and that the "world of classical worlds" could be actually M^8 or H with points having this structure [K84]. I have also considered the possibility that Hilbert spaces could have arithmetic structure based on $\otimes$ and $\oplus$ with Hilbert spaces with prime dimension defining the primes [K65].

"Do not quantize" has been my motto for all these years but in this framework, it might be possible to talk about quantization of cognition as a deformation of number theory obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary numbers with representations of Galois group. Perhaps this quantization could apply to cognition.

20.3 What could lurk behind McKay correspondence?

The appearance of EDDs in so many contexts having apparently no connection with affine algebras is an almost religious mystery and one cannot avoid the question of whether there is a deep connection between some finite groups G, in particular finite subgroups of $SL(n,C)$, and affine algebras. In the TGD framework M^8-H duality relates number theoretic and differential geometric views about physics and the natural question whether it could provide some understanding of this mystery.

M^8-H duality also suggests how to understand the Langlands correspondence: during years I have tried to understand Langlands correspondence [A126, A124] from the TGD perspective [K46, K47].

20.3.1 McKay correspondence

There is an excellent article of Khovanov [A166] describing the details of McKay correspondence for the discrete subgroups of $SL(2,C)$ (`https://cutt.ly/1LQDqce`). There is also an article "McKay correspondence" by Nakamura about various aspects of McKay correspondence [A152] (`https://cutt.ly/5LQPyhy`).

1. Consider finite subgroups G of $SL(2,C)$. The McKay graph for the tensor products of what is called canonical (faithful) 2-D representation V of G with irreps ξ_i of G corresponds to an extended Dynkin diagram with one node added to a Dynkin diagram. Note that V need not be always irreducible.

The constraints on the graph come from the conditions for the dimension $d = 2d_j$ of the tensor product $V \otimes \xi_i$ satisfies $2d_i = \sum_j a_{ij} d_j$, where the sum is over all vertices directed away from the vertex i. If arrows in both directions are present, there is no arrow. This implies that the dimensions d_j associated with the vertex have G.C.D equal to 1.

2. Dynkin diagram in turn describes the minimal set of roots from which the roots of Lie algebra can be generated by repeated reflections with respect to roots. EDDs can be assigned to affine algebras and for them the eigenvalues of the adjacency matrix are not larger than 2. The maximum of the eigenvalues measures the complexity of the graph.

3. The Weyl group characterizes the symmetries of the root diagram and is generated by reflections of roots with respect to other roots. The Dynkin diagram contains a minimal number of roots needed to generate all roots by reflections as Weyl orbits of the roots of the Dynkin diagram. The action of the Weyl group leads away from the Dynkin diagram since otherwise this set of roots would not be minimal.

   The number of lines characterizes the angle between the roots i and j. For ADE groups $a_{ij} = 1$ codes for angle of 120 degrees $2\pi/3$, $a_{ij} = 2$ corresponds to 135 degrees, and $a_{ij} = 3$ to 150 degrees. $a_{ij} = 0$ means either angle π or $\pi/2$. In the general case, there are 2-valent and 3-valent nodes depending on the number of oriented lines emerging from the node.

   For instance, in the case of a triangle group with 6 elements with irreps $1, 1_1, 1_2$. The canonical representation to 2-D reducible representation decomposes to $1_1 + 1_2$ so that there are 3 vertices involved corresponding to 1_1 and 1_2 and 1. It is easy to see that the adjacency matrix is symmetric and gives rise to an EDD with 3 vertices. From the corresponding Dynkin diagram, representing 2 neighboring roots of the root diagram one obtains the entire root diagram by repeated reflections having 6 roots characterizing the octet representation of A_2 $(SU(3))$.

4. What kind of McKay graphs are associated with other than canonical 2-D representations in the case of rotation groups? Every representation of G belongs to some minimal tensor power $V^{\otimes k}$ and one can study the MacKay diagrams assignable to $V^{\otimes k}$. It is easy to see that the number of paths connecting vertices i and j in the McKay graph $M^k(V)$ for $V^{\otimes k}$ can be understood in terms of the McKay graph $M(V)$ for V. The paths leading from i to j are all k-edged paths along $M(V)$ leading from i to j.

   The symmetry of the adjacency matrix A implies that forth and back movement along $M(V)$ is possible. The adjacency matrix has the same number of nodes and equals the $k:th$ power A^k of A so that extended ADE type Dynkin diagrams are not in question.

20.3.2 Questions

McKay correspondence raises a series of questions which I have discussed several times from the TGD point of view several times [L33, L76, L75]. In the following these questions are discussed by introducing the possibility of quantum arithmetics and cognitive representations as new elements.

Why would $SL(2,C)$ be so special?

$SL(2,C)$ is in a very special role in McKay correspondence. Of course, also the finite subgroups of other groups could have a special role and it is actually known that $SL(n,C)$ $n < 5$ are in the same role, which suggests that all groups $SL(n,C)$ have this role [A140, A139].

Why? In the TGD framework, a possible reason for the special role of $SL(2,C)$ acts as the double covering group of the isometries of the mass shell $H^3 \subset M^4 \subset M^8$ and its counterpart in $M^4 \times CP_2$ obtained by $M^8 - H$ correspondence. $SL(2,C)$ has also natural action on the spinors of H. The finite subgroups relate naturally to the tessellations of the mass shell H^3 leaving the basic unit of tessellation invariant.

The tessellations could naturally force the emergence of ADE type affine algebras as dynamical symmetries in the TGD framework. In fact, the icosa-tetrahedral tessellation plays a key role in the proposed model of the genetic code based on Hamiltonian cycles at icosahedron and tetrahedron [L103].

Why does the faithful representation have a special role?

The mathematical reason for the special role of the faithful canonical representation V is that its tensor powers contain all irreps of the finite group: the tensor product structure for other choices of V can be deduced from that for canonical representations. It is known that any irrep V, which is faithful irrep of G, generates the fusion algebra.

However, this kind of irrep might fail to exist. If G has a normal subgroup H and the irrep χ has H as kernel then the powers of χ contain only the irreps of G/H. In the article "McKay Connectivity Properties of McKay Quivers" by Hazel Brown [A148] (`https://arxiv.org/pdf/2003.09502.pdf`) it was shown that the number of connected components of the McKay quiver is the number of classes of the G, which are contained in H. For instance, the classes associated with the center of G are such (Z_n for $SL(n,C)$).

For simple groups this does not happen but in the case of Galois groups assignable to composite polynomials one has a hierarchy of normal subgroups and this kind of situation can occur since the number of classes of G contained in normal subgroups can be non-vanishing.

2-D representation is also in a special role physically in the TGD framework, the ground states of affine representation correspond to a 2-D spinor representation since quarks are the fundamental particles.

The irreps of the affine representation are obtained as tensor products of the irrep associated with the affine generators with it. Cognitive representations imply a unique discretization and this forces discrete subgroups of $SL(2,C)$ and implies that the irreps of $SL(2,C)$ decompose to irreps of a discrete subgroup. Therefore the quivers for their tensor products appear naturally.

Electroweak gauge group $U(2)$ corresponds to the holonomy group $U(2)$ for CP_2 and for $SU(2)_w$ the McKay correspondence holds true. Also the isometry group $SU(3)$ of CP_2 is assumed to appear as affine algebra. Discretization due to cognitive representations in M^8 induces discretization in H and CP_2. The replacement of $SU(3)$ with its discrete subgroups would decompose irreps for $SU(3)$ to irreps of $SU(3)$. $SL(3,C)$ allows analog of McKay correspondence [A140] so that also the finite subgroups of $SU(3)$ allow it.

What about McKay graphs for more general finite groups?

The obvious question concerns the generality of McKay correspondence. What finite groups and therefore corresponding Galois groups correspond to representations of affine type algebras.

In the general case, the McKay graphs look very different from Dynkin diagrams. The article "Spectral measures for G_2" of Evans and Pugh [A119] (`https://cutt.ly/hLQO7HE`) is of special interest from the TGD point of view since G_2 is the automorphism group of octonions. G_2 however naturally reduces to $SU(3)$ corresponding to color isometries in H. The article discusses in detail McKay graphs for the finite subgroups of G_2. These finite subgroups correspond to those for $SU(2)\times SU(2)$ and $SU(3)$ plus 7 other groups. The McKay graphs for the latter groups contain loops are very complex and contain loops.

What can one say about finite groups, which allow McKay correspondence.

1. ADE diagrams are known to classify the following three finite simple groups, the derived group F'_{24} of the Fischer F_{24}, the Baby monster B and the Monster M are related with E_6, E_7 and E_8 respectively [A152] (`https://cutt.ly/5LQPyhy`). In the TGD framework, this finding inspires the question whether these groups could appear as Galois groups of some polynomial and give rise to E_6, E_7 and E_8 as dynamical symmetries.

   In the TGD framework, one can ask whether also the above mentioned simple groups could appear as Galois groups. What is fascinating that monster would relate to icosahedron and dodecahedron: icosahedron and tetrahedron play key role in TGD inspired model of genetic code, in particular in the proposal that it relates to tetra-icosahedral tessellation of hyperbolic space H^3 [L103].

2. The article [A196](`https://cutt.ly/jLQPgkQ`) mentioned the conjecture that the tensor product structure for the finite subgroups of $SU(3)$ could relate to the integrable characters for some representations of affine algebra associated with $SU(3)$. This encourages the conjecture that this is true also for $SU(n)$.

In TGD, this inspires the question whether finite Galois groups representable as subgroups of $SU(3)$ could give rise to corresponding affine algebras as dynamical symmetries of TGD.

3. Butin and Perets demonstrated McKay correspondence in the article "Branching law for finite subgroups of $SL(3,C)$ and McKay correspondence" [A140] (https://cutt.ly/CLQPvp2) for finite subgroups of $SL(3,C)$ in the sense that branching law defines a generalized Cartan matrix. In the article "Branching Law for the Finite Subgroups of SL(4,C) and the Related Generalized Poincare Polynomials" [A139] (https://cutt.ly/mLQPQnT) shows that the same result holds true for $SL(4,C)$, which suggests that it is true for all $SL(n,C)$.

   A generalization to finite subgroups of $SL(n,C)$ is a natural guess. Therefore Galois groups with this property could be assigned with affine algebras characterized by the generalized Cartan matrices and could correspond to physically very special kind of extensions of rationals,

20.3.3 TGD view about McKay correspondence

The key idea is that one replaces quantum numbers representable as sums of the roots of Lie algebra with representations of the isotropy group of Galois group which is same as a finite subgroup of say $SL(2,C)$ and that Galois groups acts as Weyl group. The Weyl group codes for the differential geometric notion of symmetry realized by Lie groups and Galois group codes for the number theoretic view of symmetry. This correspondence would represent a facet of the duality between number theory and differential geometry.

Quantum roots as direct sums of irreps

Consider first the correspondence between quantum roots (or more generally weights defined as dual space of roots) and ordinary roots (weights) as quantum numbers.

1. The representations of finite group G (say subgroup of $SL(2,C)$) represented by the isotropy group Gal_I of Galois group for a given root, would appear as labels of states rather than as counterparts of states. Galois group Gal itself would act as Weyl group on the roots.

2. Quantum numbers as labels of quantum states would be replaced with representations of Gal_I. The additivity of quantum numbers would correspond to the additivity of representations with respect to $\oplus$. Tensor product for the representations would be analogous to multiplication of quantum quantum numbers so that they would form an algebra. An abstraction or cognitive representation would be in question. Since the roots of the Dynkin diagram correspond to roots of a monic polynomial, one could map them to ordinary algebraic numbers. Same applies to the root of affine representations.

Could also the quantal version of the coeffient field of the state space make sense?

Could also the coefficient field of state space be replaced with a quantum variant of p-adic numbers or of finite field?

1. Here one encounters a technical problem that is encountered already at the level of ordinary p-adics and finite fields. Inner products are bilinear. If norm squared is defined as a sum for the squares of the coefficients of the state in the basis of n states, the non-well-ordered character of p-adics implies that one can have states for which this sum vanishes in p-adic and finite fields.

   In the p-adic case, allowance of only finite number of non-vanishing binary digits for the coefficients might help and would conform with the idea about finite measurement resolution as a pinary cutoff. One could even allow negatives of integers with finite number of pinary digits if the p-adic quantum integers are mapped to the real counterparts.

2. There is also a problem associated with the normalization factors of the states, which cannot be p-adic integers in general. Overall normalization does not however matter so that this problem might be circumvented.

Physical predictions would require the map of the quantum integers to real ones. The fact that quantum integers are $\oplus$ sums of quantum roots of ordinary monic polynomials, makes this possible. The irreps appearing as coefficients of states would be mapped to ordinary algebraic numbers and the normalization of the states could be carried out at the level of the ordinary algebraic numbers.

What about negative multiples of quantum roots

If the quantum roots of a polynomial correspond to irreps of the Galois group, one encounters a technical problem with negative multiples of quantum roots.

1. The negatives of positive roots correspond to -1 multiples of irreps. This does not make sense in ordinary arithmetics. p-Adically -1 corresponds to $(p-1)(1+p+p^2+...)$ and would correspond to infinite $\oplus$-multiple of root but decompose to p^n multiples to which one can assign norm p^{-k} so that the sum converges: $-\xi_i = (p-1)(Id \oplus pId \oplus p^2Id \oplus ...)\xi_i$.

   One has finite measurement resolution so that the appearance of strictly infinite sums is highly questionable. Should one consider only finite sums of positive roots and their negatives but how should one deal with the negatives?

   Could the creation operators labelled by negative roots correspond to annihilation operators with positive roots as in the case of super-Virasoro and affine algebras. Note that if one restricts to ordinary integers at the level of algebra as one must to for supersymplectic and Yangian algebras, one must consider only half-algebras with generators, which have only non-negative conformal weights. This does not make sense for ordinary affine generators.

2. The most plausible solution of the problem relies on the proposed categorical correspondence between quantum roots and ordinary roots as roots of the same monic polynomial. One could map the quantum roots and their direct summands to sums of ordinary roots and this would make sense also for the negatives of positive roots with a finite number of summands. It would be essential that p-adic integers correspond to finite ordinary integers and to their negatives and are mapped to numbers in an extension of rationals. As found, this map would also allow us to circumvent the objections against the quantum variant of the state space.

3. Could zero energy ontology (ZEO) come to the rescue? In zero energy ontology creation and annihilation operators are assigned with the opposite boundaries of causal diamond (CD). Could one assign the negative conformal weights and roots with the members of state pairs located at the opposite boundary of CD?

   This works for the Virasoro and affine generators but this kind of restriction is unphysical in the case of eigenvalues of L_z with both signs? Why would opposite values of L_z be assigned to opposite boundaries of CD?

Wheels and quantum arithmetics

Gary Ehlenberg gave a link to a Wikipedia article telling of Wheel theory (`https://cutt.ly/RZnUB5y`). Wheel theory could be very relevant to the TGD inspired idea about quantum arithmetics.

I understood that Wheel structure is special in the sense that division by zero is well defined and multiplication by zero gives a non-vanishing result. The wheel of fractions, discussed in the Wikipedia article as an example of wheel structure, brings into mind a generalization of arithmetics and perhaps even of number theory to its quantum counterpart obtained by replacing + and - with direct sum $\oplus$ and tensor product $\otimes$ for irreps of finite groups with trivial representation as multiplicative unit: Galois group is the natural group in TGD framework.

Could wheel structure provide a more rigorous generalization of the notions of the additive and multiplicative inverse of the representation in order to build quantum counterparts of rationals, algebraic numbers and p-adics and their extensions?

1. One way to achieve this is to restrict consideration to the quantum analogs of finite fields $G(p,n)$: + and x would be replaced with $\oplus$ and $\otimes$ obtained as extensions by the irreps of the

Galois group in TGD picture. There would be quantum-classical correspondence between roots of quantum polynomials and ordinary monic polynomials.

2. The notion of rational as a pair of integers (now representations) would provide at least a formal solution of the problem, and one could define non-negative rationals.

   p-Adically one can also define quite concretely the inverse for a representation of form $R = 1 \oplus O(p)$ where the representation $O(p)$ is proportional to p (p-fold direct sum) as a geometric series.

3. Negative integers and rationals pose a problem for ordinary integers and rationals: it is difficult to imagine what direct sum of -n irreps could mean.

   The definition of the negative of representation could work in the case of p-adic integers: $-1 = (p-1) \otimes (1 \oplus p * 1 \oplus p^2 * 1 \oplus ...)$ would be generalized by replacing 1 with trivial representation. Infinite direct sum would be obtained but it would converge rapidly in p-adic topology.

4. Could $1/p^n$ make sense in the Wheel structure so that one would obtain the quantum analog of a p-adic number field? The definition of rationals as pairs might allow this since only non-negative powers of p need to be considered. p would represent zero in the sense of Wheel structure but multiplication by p would give a non-vanishing result and also division with p would be well-defined operation.

Galois group as Weyl group?

The action of the Weyl group as reflections could make sense in the quantum arithmetics for quantum variants of extensions of p-adics and finite fields. The generalized Cartan matrix $C_{ij} = d\delta_{ij} - n_{ij}$, where n_{ij} is the number of lines connecting the nodes i and j and d is the dimension of V, is indeed well-defined for any finite group and has integer valued coefficients so that Weyl reflection makes sense also in quantum case.

Can one identify the Weyl group giving the entire root diagram number theoretically? The natural guess is $Gal = W$: Gal would define the Weyl group giving the entire root diagram from the Dynkin diagram by reflections of the roots of the EDD. One can assign to Gal an extension defined by a monic polynomial P with Galois group Gal.

How the group defining the McKay graph is represented?

How the group G defining the McKay graph is represented? The irreps of G should have natural realization and the quarks at mass shells would provide these representations.

One can consider two options. The first option is based on the isotropy group G_I of $Gal = W$ leaving a given root invariant. Second option is based on the finite subgroup of $SU(2)$ as a covering group of quaternion automorphisms.

1. The subgroup $Gal_I \subset Gal$ acting as an isotropy group of a given root of Gal would naturally define the EDD since the action of $Gal = W$ would not leave its nodes as irreps of Gal_I invariant.

   The root diagram should be the orbit of the EDD under $Gal = W$. The irreps of the EDD would correspond to the roots of a monic polynomial P_I associated with Gal_I and having $n_c + 1$ quantum roots. The quantum roots would be in the quantum extension defined by a monic polynomial P for Gal so that the action of Gal on EDD would be well-defined and non-trivial.

2. In the TGD framework, the mass mass squared values assignable to the monic polynomial representing the EDD correspond to different mass squared values. There is no deep reason for why the irreps of Gal_I could not correspond to different mass squared values and in the TGD framework the symmetry breaking $Gal \to Gal_I$ is the analog for the symmetry breaking in the Higgs mechanism.

   In the recent case this symmetry breaking would be associated with $Gal_I \to Gal_{I,I}$ and imply that quantum roots correspond to different mass squared values. At the level of affine algebra this could mean symmetry breaking since the different roots would not have different mass squared values.

If Gal acts as a Weyl group, the McKay graph associated with Gal_I corresponds to the EDD. Gal_I is a subgroup of Gal so that the action of $Gal = Weyl$ on the quantum roots of the monic polynomial P_I would be non-trivial and natural. Could Gal_I be a normal subgroup in which case Gal/Gal_I would be a group and one would have a composite polynomial $P = Q \circ P_1$? This cannot be true generally: for instance for A_p, p prime and E_6 the W is simple. For E^7 and E^8 W is a semidirect product.

3. There is an additional restriction coming from the fact that Gal_I does not affect the rational parts of the 4-momenta. Is it possible to have construct irreps for a finite subgroup of $SL(2,C)$ or even $SL(n,C)$ using many quark states at a given mass shell? The non-rational part of 4-momentum corresponds to the "genuinely" virtual part of virtual momentum and for Galois confined states only the rational parts contribute to the total 4-momentum. Could one say that these representations are possible but only for the virtual states which do not appear as physical states: cognition remains physically hidden.

The very cautious, and perhaps over-optimistic conclusion, would be that only Galois groups, which act as Weyl groups, can give rise to affine algebras as dynamical symmetries. For this option, one would obtain cognitive representations for the isotropy groups of all Galois groups. For Galois groups acting as Weyl groups, EDDs could define cognitive representations of affine algebras. Also cognitive representations for finite subgroups of $SL(n,C)$ and groups like Monster would be obtained.

For the second option in which the subgroup G of quaternionic automorphisms affecting the real parts of 4-momenta is involved. This representation would be possible only for the subgroups of $SL(2,C)$. In this case one would have 3 different groups $Gal = W$, Gal_I and G rather than $Gal = W$ and Gal_I.

1. Quaternionic automorphisms are analogous to the Galois group and one can ask whether the finite subgroups G of quaternionic automorphisms could be directly involved with cognitive representations. This would give McKay correspondence for $SL(2,C)$ only. The quaternionic automorphism would affect the rational part of the 4-momentum in an extension of rationals unlike the Galois group which leaves it invariant. The irrep of G would be realized as many-quark states at a fixed mass shell. Different irreps would correspond to different masses having interpretation in terms of symmetry breaking.

2. Also now one would consider the extension defined by the roots of a monic polynomial P having Galois group $Gal = W$ associated with the corresponding EDD. P_I would give quantum roots defining the Dynkin diagram and define the mass squared values assignable to irreps of G.

3. The situation would differ from the previous one in that the action of G_I on irreps would be replaced by the action of G. Indeed, since G_I leaves the rational part of the 4-momentum invariant, G_I cannot represent G as a genuine subgroup of rotations.

4. The roots would correspond to irreps of a subgroup G of quaternionic automorphisms, which would affect the 4-momenta with a given mass shell and define an irrep of G. Different roots of P would define the mass shells and irreps of G associated with EDD as a McKay graph.

Information about Weyl groups of ADE groups

The Wikipedia article about Coxeter groups (`https://en.wikipedia.org/wiki/Coxeter_group#Properties`), which include Weyl groups, lists some properties of finite irreducible Coxter groups and contains information about Weyl groups. This information might be of interest in the proposed realization as a Galois group.

- $W(A_n) = S_{n+1}$, which is the maximal Galois group associated with a polynomial of degree $n+1$.

- $W(D_n) = Z_2^{n-1} \rtimes S_n$.
- $W(E_6)$ is a unique simple group of order 25920.
- $W(E^7)$ is a direct product of a unique simple group of order 2903040 with Z_2.
- $W(E_8)$ acts as an orthogonal group for F_2 linear automorphisms preserving a norm in Ω/Z_2, where Ω is E_8 lattice (https://mathoverflow.net/questions/230120/the-weyl-group-of-e8-versus-o-230130#230130)
- $W(B_n) = W(C_n) = Z_2^n \rtimes S_n$.
- $W(F_4)$ is a solvable group of order 1152 , and is isomorphic to the orthogonal group $O_4(F_3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the field F_3.
- $W(G_2) = D_6 = Z_2 \rtimes Z_6$.

Candidates for symmetry algebras of WCW, inclusions of hyperfinite factors, and Galois groups acting as Weyl groups

TGD allows several candidates for the symmetry algebras acting in WCW. The intuitive guess is that the isometries and possibly also symplectic transformations of the light-cone boundary $\delta M^4_+ \times CP_2$ define isometries of WCW whereas holonomies of H induce holonomies of WCW.

1. In TGD, supersymplectic algebra SSA could replace affine algebras of string models.
2. By the metric 2-dimensionality of the light-cone boundary δM^4_+, one can assign to it an infinite-dimensional conformal group of sphere S^2 in well-defined sense local with respect to the complex coordinate z of S^2. These transformations can be made local with respect to the light-like coordinate r of δM^4_+. Also a S^2-local radial scaling making these transformations isometries is possible. This is possible only for M^4 and makes it unique.

   Whether SSA or this algebra or both act as isometries of WCW is not clear: see the more detailed discussion in the Appendix of [L119].
3. One can assign this kind of hierarchy also to affine algebras assignable to the holomies of H and Virasoro algebras and their super counterparts. The geometric interpretation of these algebras would be as analogs of holonomy algebras, which serve at the level of H as the counterparts of broken gauge symmetries: isometries would correspond to non-broken gauge symmetries.

All these algebras, refer to them collectively by A, define inclusion hierarchies of sub-algebras A_n with the radial conformal weights given by n-ples of the weights of A.

1. I have proposed that the hierarchy of inclusions of hyperfinite factors of type II_1 to which one could perhaps assign ADE hierarchy could correspond to the hierarchies of subalgebras assignable to SSA and labelled by integer n: the radial conformal weights would be multiples of n. Only non-negative values of n would be allowed.
2. For a given hierarchy A_n, one has $n_1 \mid n_2 \mid .....$, where $\mid$ means "divides". At the n:th level of the hierarchy physical states are annihilated by A_n and $[A_n, A]$. For isometries, the corresponding Noether charges vanish both classically and quantally.
3. The algebra A_n effectively reduces to a finite-D algebra and A_n would be analogous to normal subgroup, which suggests that this hierarchy relates to a hierarchy of Galois groups associated with composite polynomials and having a decomposition to a product of normal subgroups.
4. These hierarchies could naturally relate to the hierarchies of inclusions of hyperfinite factors of type II$_1$ and also to hierarchies of Galois groups for extensions of rationals defined by composites $P_n \circ P_{n-1} \circ ...P_1$ of polynomials.

The Galois correspondence raises questions.

1. Could the Dynkin diagrams for A_n be assigned to the McKay graphs of Galois groups acting as Weyl groups?

2. The Galois groups acting as Weyl group could be assigned to finite subgroups of $SU(2)$ acting as the covering group of quaternion automorphisms and of $SL(2,C)$ as covering group of H^3 isometries acting on tessellations of H^3. Also the finite subgroups of $SL(n,C)$ can be considered.

The proposed interpretation for the hierarchies of inclusions of HFFs is that they correspond to hierarchies for the inclusions of Galois groups defined by hierarchies of composite polynomials $P_n \circ ... \circ P_1$ interpreted as number theoretical evolutionary hierarchies.

If the relative Galois groups act as Weyl groups, they would be associated with the inclusions of HFFs naturally and the corresponding affine algebra (perhaps its finite field or p-adic variant) would characterize the inclusion. The proposed interpretation of the inclusion is in terms of measurement resolution defined by the included algebra. This suggests that a finite field version of the affine algebra could be in question.

This picture would suggest that hierarchies of polynomials for which the relative Galois groups act as Weyl groups are very special and could be selected in the number theoretical fight for survival.

One could argue that since number theoretic degrees of freedom relate to cognition, the quantum arithmetics for the irreps of Galois groups could make possible cognitive representations of the ordinary quantum states: roots would be represented by irreps. Irreps as quantum roots would correspond to ordinary roots as roots of the same monic polynomial and the direct sums of irreps would correspond to ordinary algebraic numbers.

About the interpretation of EDDs

An innocent layman can wonder whether the tensor products for 2-D spinor ground states for the discrete subgroups of the covering group of quaternionic automorphisms or of $SL(2,C)$ as covering group of H^3 isometries could give rise to representations contained by ADE type affine algebras characterized by the same EDD. These representations would be only a small part of the representations and perhaps define representation from which all states can be generated.

1. The reflections for the roots represented as irreps of Gal_I by Weyl group represented as Gal should assign to the irreps of G new copies so that the nodes of the entire root diagram would correspond to a set of representations obtained from the ground state. Infinite number of states labelled by conformal weight n is obtained.

2. Adjacency matrix A should characterize the angles between the roots represented as irreps? If the irreps of Gal_I and their Weyl images correspond to roots of a monic polynomial, they can be mapped to roots of an ordinary algebraic extension of rationals and the angles could correspond to angles between the points of extension regarded as vectors.

How the EDD characterizing the tensor products of the irreps of finite subgroups G with 2-D canonical representation V could define an ADE type affine algebra?

1. Roots are replaced with representations of G, which are in the general case direct sums of irreps. The identity representation should correspond to the scaling generator L_0, whose eigenvalues define integer value conformal weights.

   The inner products between the roots appearing in the Cartan matrix would correspond to the symmetric matrices defined by the structure constant n_{2ij} characterizing the tensor product. One might say that the inner products are matrix elements of the operator $\langle \xi_j | V \otimes \xi_i \rangle$ defined by the tensor product action of V. The diagonal elements of the Cartan matrix have value $+2$ and non-diagonal elements are negative integers or vanish.

2. Weyl reflections of roots with respect to roots involve negatives of the non-diagonal elements of Cartan matrix, which are negative so that the coefficient of the added root is positive represented as a direct sum. The negatives of the positive roots would correspond to negative integers and make sense only p-adically or for finite fields.

The expression for the generalized Cartan matrix for McKay graph is known (`https://cuttly/QLRqrGt`) for the tensor products of representation with dimension d and multiplicities n_{ij}^d and is given by

$$C_{ij}^d = d\delta_{ij} - n_{ij}^d \quad .$$

For Dynkin diagrams the Cartan matrix satisfies additional conditions.

Weyl reflection (`https://cutt.ly/kLRuXBP`) of the root v with respect to root α in the space of roots is defined as

$$s_\alpha v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \quad .$$

where $(.,.)$ is the inner product in V, which now corresponds to extension of rationals associated with Gal.

The Weyl chamber is identified as the set of points of V for which the inner products (α, v) are positive. The Weyl group permutes the Weyl chambers.

3. The root system would be obtained from the roots of the quantum Dynkin diagram by Weyl reflections (Galois group as Weyl group) with respect to other roots. The number N of these roots is $n = d_C + 1$,where d_C is the dimension of Cartan algebra of the Dynkin diagram. The number N_I of irreps is the same: $N = N_I$. The Cartan matrix defines metric in the roots so that the reflections are well-defined also in the generalized picture.

4. It would seem that one must introduce an infinite number of copies of the Lie algebra realized in the usual manner (in terms of oscillator operators) with copies labelled by the conformal weight n. The commutators of these copies would be like for an ordinary affine algebra. Only the roots as labels of generators and possibly also the coefficient field would be replaced with their quantum variants.

5. What about the realization of the scaling generator L_0, whose Sugawara representation involves bilinears of the generators and their Hermitian conjugates with negative conformal weight? In the case of finite fields there are no obvious problems. Also the analog of Virasoro algebra can be realized in the case of finite fields. If one restricts consideration to finite quantum integers and their negatives as conformal weights, the map of the roots to algebraic numbers in extension of rationals is well defined.

20.3.4 Could the inclusion hierarchies of extensions of rationals correspond to inclusion hierarchies of hyperfinite factors?

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (`https://cutt.ly/lXp6MDB`) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving

up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

It is. The number of conjugacy classes $g_i \sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD comes to rescue.

1. In the TGD framework, I am primarily interested in Galois groups, which are finite groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

One would have a deep connection between number theory and HFFs.

20.4 Appendix: Isometries and holonomies of WCW as counterparts of exact and broken gauge symmetries

The detailed interpretation of various candidates for the symmetries of WCW [L69] has remained somewhat obscure. At the level of H, isometries are exact symmetries and analogous to unbroken gauge symmetries assignable to color interactions. Holonomies do not give rise to Noether charges and are analogous to broken gauge symmetries assignable to electroweak interactions. This observation can serve as a principle in attempts to understand WCW symmetries.

The division to isometries and holonomies is expected to take place at the level of WCW and this decomposition would naturally correspond to exact and broken gauge symmetries.

20.4.1 Isometries of WCW

The identification of the isometries of WCW is still on shaky ground.

1. In the H picture, the conjecture has been that symplectic transformations of δM^4_+ act as isometries. The hierarchies of dynamically emerging symmetries could relate to the hierarchies of sub-algebras (SSA_n) of super symplectic algebra SSA [L69] acting as isometries of the "world of classical worlds" (WCW) [K76] [L110].

Each level in the hierarchy of subalgebras SSA_n of SSA corresponds to a transformation in which SSA_n acts as a gauge symmetry and its complement acts as genuine isometries of WCW: gauge symmetry breaking in the complement generates a genuine symmetry, which could correspond to Kac-Moody symmetry. By Noether's theorem, the isometries of WCW would give rise to local integrals of motion: also super-charges are involved. These charges are well-defined but they need not be conserved so that the interpretation as dynamically emerging symmetries must be considered.

The symmetries would naturally correspond to a long range order. The hierarchies of SSA$_n$:s, of relative Galois groups and of inclusions of hyperfinite factors [K99, K33] could relate to each other as $M^8 - H$ duality suggests [L122].

What can one say about the algebras SSA_n and the corresponding affine analogs KM_n (for affine algebras the generalized Cartan matrix is a product of a diagonal matrix with integer entries with a symmetric matrix). If n is prime, one can regard these algebras as local algebras in a finite field $G(p)$. Also extensions $G(p,n)$ of $G(p)$ induced by extensions of rationals can be considered. KM algebras in finite fields define what are called the incomplete Kac-Moody groups. Some of their aspects are discussed in the article "Abstract simplicity of complete Kac-Moody groups over finite fields" [A112]. It is shown that for $p > 3$, affine groups are abstractly simple, that is, have no proper non-trivial closed subgroups. Complete KM groups are obtained as completions of incomplete KM groups and are totally disconnected: this suggests that they define p-adic analogs of Kac-Moody groups. Complete KM groups are known to be simple.

2. There are also different kinds of isometries. Consider first the light-cone boundary $\delta M^4_+ \times CP_2$ as an example of a light-like 3-surface. The isometries of CP_2 are symmetries. ΔM^4_+ is metrically equivalent with sphere S^2. Conformal transformations of S^2, which are made local with light-like coordinate r of δM^4_+, induce a conformal scaling of the metric of S^2 depending on r. It is possible to compensate for this scaling by a local radial scaling of r depending on S^2 coordinates such that the transformation acts as an isometry of δM^4_+.

   These isometries of ΔM^4_+ form an infinite-D group. The transformations of this group differ from those of the symplectic group in that the symplectic group of δM^4_+ is replaced with the isometries of δM^4_+ consisting of r-local conformal transformations of S^2 involving S^2-local radial scaling. There are no localizat of CP_2 isometries. This yields an analog of KM algebra.

   This group induces local spinor rotations defining a realization of KM algebra. Also super-KM algebra defined in terms of conserved super-charges associated with the modified Dirac action is possible. These isometries would be Noether symmetries just like those defined by SSA.

3. What about light-like partonic orbits analogous to $\delta M^4_+ \times CP_2$. Can one assign with them Kac-Moody type algebras acting as isometries?

   The infinite-D group of isometries of the light-cone boundary could generalize. If they leave the partonic 2-surfaces at the ends of the orbit X^3_L, they could be seen as 3-D general coordinate transformations acting as internal isometries of the partonic 3-surface, which cannot be regarded as isometries of a fixed subspace of H. These isometries do not affect the partonic 3-surface as a whole and cannot induce isometries of WCW.

   However, if X^3_L is connected by string world sheets to other partonic orbits, these transformations affect the string world sheets and there is a real physical effect, and one has genuine isometries. Same is true if these transformations do not leave the partonic 2-surfaces at the ends of X^3_L invariant.

20.4.2 Holonomies of WCW

What about holonomies at the level of WCW? The holonomies of H acting on spinors induces a holonomy at the level of WCW: WCW spinors identified as Fock states created by oscillator operators of the second quantized H spinors. This would give a generalized KM-type algebra decomposing to sub-algebras corresponding to spin and electroweak quantum numbers. This algebra would have 3 tensor-factors. p-Adic mass calculations imply that the optimal number of tensor factors in conformal algebra is 5 [K50]. 2 tensor factors are needed.

1. SSA would give 2 tensor factors corresponding to δM^4_+ (effectively S^2) and CP_2. This gives 5 tensor factors which is the optimal number of tensor factors in p-adic mass calculations [K50]. SSA Noether charges are well-defined but not conserved. Could SSA only define a hierarchy of dynamical symmetries. Note however that for isometries of H conservation holds true.

2. Also the isometries of δM^4 and of light-like orbits of partonic 2-surfaces give the needed 2 tensor factors. Also this alternative would give inclusion hierarchies of KM sub-algebras with conformal weights coming as multiples of the full algebra. The corresponding Noether charges are well-defined but can one speak of conservation only in the partonic case? One can even argue that the isometries of $\delta M^4_+ \times CP_2$ define a more plausible candidate for inducing WCW isometries than the symplectic transformations. p-Adic mass calculations conform with this option.

To sum up, WCW symmetries would have a nice geometric interpretation as isometries and holonomies. The details of the interpretation are however still unclear and one must leave the status of SSA open.

Chapter 21

Quantum Arithmetics and the Relationship between Real and p-Adic Physics

21.1 Introduction

The construction of quantum counterparts for various mathematical structures of theoretical physics have been a fashion for decades. Quantum counterparts for groups, Lie algebras, coset spaces, etc... have been proposed often on purely formal grounds. In TGD framework quantum group like structures emerge via the hyper-finite factors of type II_1 (HFFs) about which WCW spinors represent a canonical example [K99]. The inclusions of HFFs provide a very attractive ways to realize mathematically the notion of finite measurement resolution.

In the following a proposal for what might be called quantum integers and quantum matrix groups is discussed. One can imagine two basic definitions of quantum integers n_q: option I and II. For option I the map $n \to n_q$ respects prime decomposition so that one obtains quantum variant of primeness. For option II ordinary primeness in the ordinary sense of word is lost as it is lost also for p-adic numbers (only p is prime for Q_p).

Also quantum rationals belonging to algebraic extension of rationals can be defined as well as their algebraic extensions. Quantum arithmetics differs from the usual one in that quantum sum is defined in such a way that the map $n \to n_q$ commutes also with sum besides the product: $m_q +_q n_q = (m+n)_q$. Quantum matrix groups differ from their standard counterparts in that the matrix elements are not non-commutative. The matrix multiplication involving summation over products is however replaced with quantum summation.

The hope is that these new mathematical structures could allow a better understanding of the relationship between real and p-adic physics for various values of p-adic prime p, to be called l in the sequel because of its preferred physical nature resembling that of l-adic prime in l-adic cohomology. The correspondence with the ordinary quantum groups (see `http://tinyurl.com/3tors5`) [A69] is also considered and suggested to correspond to a discretization following as a correlate of finite measurement resolution.

21.1.1 Overall View About Variants Of Quantum Integers

The starting point of quantum arithmetics is the map $n \to n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \rightarrow p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

Clearly, many variants of quantum integers can be found and it is difficult to decide which of them - if any - has interesting from TGD point of view.

1. If one wants to really model something using quantum integers, the second options is perhaps the realistic one: the reason is that the decomposition into prime factors requires a lot of computation time.

2. A second fictive criterion would be whether the definition is maximally general. Does the definition makes sense for infinite primes? The simplest infinite primes at the first level of hierarchy have physical interpretation as many-particle states consisting of bosons and fermions, whose momentum values correspond to finite primes. The interpretation generalizes to higher levels of the hierarchy. A simple argument show that the option keeping information about prime factorization of the p-adic number allowing also infinite primes as factors makes sense only if prime factors are not expanded in series with respect to the prime p and if p does not correspond to a fermionic mode. The quantum map using prime root of unity therefore makes sense for all but fermionic primes. The presence of exceptional primes in number theory is basic phenomenon: typically they correspond to primes for which factorization is not unique in algebraic extension.

21.1.2 Motivations For Quantum Arithmetics

Quantum arithmetics has several motivations in TGD framework.

Model for Shnoll effect

The model for Shnoll effect [K5] suggests that this effect could be understand in terms of a deformation of probability distribution $f(n)$ (n non-negative integer) for random fluctuations. The deformation would replace the rational parameters characterizing the distribution with new ones obtained by mapping the parameters to new ones by using the analog of canonical identification respecting symmetries.

The idea of the model of Shnoll effect was to modify the quantum map $n \rightarrow n_q$ in such a way that it is consistent with the prime decomposition of ordinary integers. This deformation would involve two parameters: quantum phase $q = exp(i\pi/m)$ and preferred prime l, which need not be independent however: $m = l$ is a highly suggestive restriction.

What could be the deeper mathematics behind dualities?

Dualities certainly represent one of the great ideas of theoretical physics of the last century. On could say that electric-magnetic duality due to Montonen and Olive [B4] is the mother of all dualities. Later a proliferation, one might say even inflation, of dualities has taken place. AdS/CFT correspondence (see `http://tinyurl.com/2zuek8`) [B38] is one example relating to each other perturbative QFT working in short scales and string theory working in long scales.

Also in TGD framework several dualities suggests itself. All of them seem to relate to dichotomies such as weak–strong, perturbative–non-perturbative, point like particle–string. Also number theory seems to be involved in an essential way.

1. If $M^8 - -M^4 \times CP_2$ duality is true it is possible to regard space-times as surfaces in M^8 or $M^4 \times CP_2$ [K86]. The proper treatment of Minkowskian signature requires complexified version M^8_c of M^8 allowing identification as complexified octonions. One way to interpret the duality would as the analog of q-p duality in wave mechanics. Surfaces in M^8 (or M^8_c) would be analogous to momentum space representation of the physical states: space-time surfaces in M^8 would represent in some sense the points for the tangent space of the "world of classical worlds" (WCW) just like tangent for a curve gives the first approximation for the curve near a given point.

   The argument supporting $M^8 - -M^4 \times CP_2$ duality involves the basic facts about classical number fields - in particular octonions and their complexification - and one can understand $M^4 \times CP_2$ in terms of number theory. The analog of the color group in M^8 picture would be the isometry group $SO(4)$ of E^4 which happens to be the symmetry group of the old fashioned hadron physics. Does this mean that $M^4 \times CP_2$ corresponds to short length scales and perturbative QCD whereas M^8 would correspond to long length scales and non-perturbative approach?

2. Second duality would relate partonic 2-surfaces and string world sheets playing a key role in the recent view about preferred extremals of Kähler action [L12]. Partonic 2-surfaces are magnetic monopoles and TGD counterparts of elementary particles, which in QFT approach are regarded as point like objects. The description in terms of partonic 2-surfaces forgetting that they are parts of bigger magnetically neutral structures would correspond to perturbative QFT. The description in terms of string like objects with vanishing magnetic charge is needed in longer length scales. Electroweak symmetry breaking and color confinement would be the natural applications. The essential point is that stringy description corresponds to long length scales (strong coupling) and partonic description to short length scales (weak coupling).

   Number theory seems to be involved also now: string world sheets could be seen as commutative (hyper-complex) 2-surfaces of space-time surface with hyper-quaternionic tangent space structure and partonic 2-surfaces as co-commutative (co-hyper-complex) 2-surfaces. To avoid inflation of clumsy "hyper-"s, the terms "associative"/"co-associative" and "commutative"/"co-commutative" will be used in the sequel.

   The localization of the modes of induced spinor fields to string world sheets and partonic 2-surfaces could be seen as a physical realization this and is implied by the requirement that spinor modes are eigenstates of em charge operator [K100].

3. Space-time surface itself would decompose to associative and co-associative regions and a duality also at this level is suggestive [L10], [K14]. The most natural candidates for dual space-time regions are regions with Minkowskian and Euclidian signatures of the induced metric with latter representing the generalized Feynman graphs. Minkowskian regions would correspond to non-perturbative long length scale description and Euclidian regions to perturbative short length scale description. This duality should relate closely to quantum measurement theory and realize the assumption that the outcomes of quantum measurements are always macroscopic long length scale effects. Again number theory is in a key role.

Real and p-adic physics and their unification to a coherent whole represent the basic pieces of physics as generalized number theory program.

1. p-Adic physics in minimal sense would mean a discretization of real physics relying on effective p-adic topology. p-Adic physics could also mean genuine p-adic physics at p-adic space-time sheets identified as space-time correlates of cognition.

   Real continuity and smoothness is a powerful constraint on short distance physics. p-Adic continuity and smoothness pose similar constraints in short scales an therefore on real physics in long length scales if one accepts that real and space-time surfaces (partonic 2-surfaces for minimal option) intersect along rational points and possible common algebraics in preferred coordinates. p-Adic fractality implying short range chaos and long range correlations is the outcome. Therefore p-adic physics could allow to avoid the landscape problem of M-theory due to the fact that the IR limit is unpredictable although UV behavior is highly unique.

2. The recent argument [L12] suggesting that the areas for partonic 2-surfaces and string world sheets could characterize Kähler action leads to the proposal that the large N_c expansion (see `http://tinyurl.com/ya4xo926`) [B1] in terms of the number of colors defining non-perturbative stringy approach to strong coupling phase of gauge theories could have interpretation in terms of the expansion in powers of $1/\sqrt{p}$, p the p-adic prime. This expansion would converge extremely rapidly since N_c would be of the order of the ratio of the secondary and primary p-adic length scales and therefore of the order of $\sqrt{p}$: for electron one has $p = M_{127} = 2^{127} - 1$.

3. Could there exist a duality between genuinely p-adic physics and real physics? Could the mathematics used in p-adic mass calculations - in particular canonical identification $\sum_n x_n p^n \to \sum x_n p^{-n}$ - be extended to apply to quantum TGD itself and allow to understand the non-perturbative long length scale effects in terms of short distance physics dictated by continuity and smoothness but in different number field? Could a proper generalization of the canonical identification map allow to realize concretely the real–p-adic duality?

Could quantum arithmetics allow a variant of canonical identification respecting both symmetries and continuity?

One could argue that a generalization of the canonical identification [K60] and its variants is needed in order to solve the tension between algebra (symmetries) and topology: the correspondence via common rationals respects algebra and symmetries but is discontinuous. Canonical identification is continuous but does not respect algebra.

Concerning the correspondence between p-adics and reals the notion of p-adic manifolds seems to represent a real step of progress. The notion of p-adic manifold [K101] is based on simple idea. The chart maps of p-adic manifolds (now space-time surfaces) are to real manifolds (space-time surfaces) rather than p-adic counterpart of Euclidian space and realized in terms of some variant of canonical identification restricted to a discrete subset of rational points of manifold- now space-time surface- and preferred extremal property allows to find a space-time surface which contains these points. In accordance with finite measurement resolution, the correspondence is not unique.

The real image is interpreted as realization of intention represented as p-adic space-time surface. The reverse maps providing p-adic charges about real space-time surface are interpreted as cognitive representations. Building of cognitive representation and realization of intention as action could be time reversals of each other in the sense that quantum jump could lead from p-adic sector to real and vice versa: this requires zero energy ontology (ZEO) in order to make sense.

All forms of canonical identification break to some extent symmetries and continuity (this forces the restriction to a discrete subset of space-time points). One could accept this or ask whether a generalization of canonical identification resolving the tension between symmetries and continuity could exist.

It seems that this is not the case. The tension seems to be unresolvable and have interpretation in terms of finite measurement resolution. At best a given continuous symmetry group would be replaced by some of its discrete subgroups. Of course, both real and p-adic variants of symmetries are realized but the problem is that they are very different and canonical identification in its basic form does not give close connection between them.

This chapter was written before the emergence of the notion of p-adic manifold and in the hope that the symmetry respecting generalization of canonical identification might exist. In the new situation quantum variant of canonical identification provides a new variant of the map taking discretization of the p-adic space-time surfaces to its real counterpart.

Quantum integers and preferred extremals of Kähler action

One might hope that quantum integers have some deep function. Somehow the fact that the images of primes $1 < p_i < p$ are algebraic numbers might relate to this. Maybe their function might relate to the notion of p-adic manifold [K101]. The basic challenge is to continue the discrete canonical image of the p-adic space-time points to continuous and differentiable preferred extremal of Kähler action. O_c-real analytic functions (O_c denotes complexified octonions) [K98] defining four-surfaces in M^8_c mappable to space-time surface in H by $M^8 - H$ correspondence might allow to code preferred extremals by real-valued analytic functions. A hierarchy of polynomials with rational or even algebraic arguments suggests itself.

Quantum integers might define discretization of real space-time surface by mapping p-adic integers (continuum) representing preferred embedding space coordinates to a set of quantum integers n_q, $0 \leq n < p$.

The notion of deformation has played central role in attempts to generalize physics and one can see quantum physics as a deformation of classical physics. Suppose that p-adic preferred extremal is characterized by functions which are polynomials/ rational functions. Suppose that one can interpret these functions as functions in the ring of quantum integers. Since differentiability makes sense for the quantum ring one could hope that these functions could define preferred extremal in the ring of quantum integers and perhaps also in real embedding space.

21.1.3 Correspondence Along Common Rationals And Canonical Identification: Two way To Relate Real And P-Adic Physics

The relationship between real and p-adic physics deserves a separate discussion.

Identification along common rationals

The first correspondence between reals and p-adics is based on the idea that rationals are common to all number fields implying that rational points are common to both real and p-adic worlds. This requires preferred coordinates. It also leads to a fusion of different number fields along rationals and common algebraics to a larger structure having a book like structure [K85, K60].

1. Quite generally, preferred space-time coordinates would correspond to a subset of preferred embedding space coordinates, and the isometries of the embedding space give rise to this kind of coordinates which are however not completely unique. This would give rise to a moduli space corresponding to different symmetry related coordinates interpreted in terms of different choices of causal diamonds (CDs: recall that CD is the intersection of future and past directed light-cones.

2. Cognitive representation in the rational (partly algebraic) intersection of real and p-adic worlds would necessarily select certain preferred coordinates and this would affect the physics in a delicate way. The selection of quantization axis would be basic example of this symmetry breaking. Finite measurement resolution would in turn reduce continuous symmetries to discrete ones. It deserves to be mentioned that for color color symmetries $SU(3)$ the space for the choices of quantization axes is flag-manifold $SU(3)/U(1)\times U(1)$ having interpretation as twistor space of CP_2: CP_2 is the only compact 4-manifold allowing twistor space with complex structure. M^4 twistors are assigned with light-like vectors defining plane $M^2 \subset M^4$ in turn defining quantization axis for spin.

3. Typically real and p-adic variants of given partonic 2-surface would have discrete and possibly finite set of rational points plus possible common algebraic points. The intersection of real and p-adic worlds would consist of discrete points. At more abstract level rational functions with rational coefficients used to define partonic 2-surfaces would correspond to common 2-surfaces in the intersection of real and p-adic WCW:s. As a matter fact, the quantum arithmetics would make most points algebraic numbers.

4. The correspondence along common rationals respects symmetries but not continuity: the graph for the p-adic norm of rational point is totally discontinuous. Most non-algebraic reals and p-adics do not correspond to each other. In particular, transcendental at both sides belong to different worlds with some exceptions like e^p which exists p-adically.

Canonical identification and its variants

There is however a totally different view about real–p-adic correspondence.

The predictions of p-adic mass calculations are mapped to real numbers via thecanonical identification applied to the p-adic value of mass squared [K60, K59]. One can imagine several forms of canonical identification but this affects very little the predictions since the convergence in powers of p for the mass squared thermal expectation is extremely fast.

As a matter fact, I proposed for more that 15 years ago that canonical identification could be essential element of cognition mapping external world to p-adic cognitive representations [K62] realized in short length scales and vice versa.

If so, then real–p-adic duality would be a cornerstone of cognition [K62]. Common rational points would relate to the intentionality which is second aspect of the p-adic real correspondence: the transformation of real to p-adic surfaces in quantum jump would be the correlate for the transformation of intention to action. The realization of intention would correspond to the correspondence along rationals and common algebraics (the more common points real and p-adic surface have, the more faithful the realization of intentional action) and the generation of cognitive representations to the canonical identification.

The already mentioned, notion of p-adic manifolds [K101] relies on this notion and provides a very promising approach to the description of space-time correlates of cognition. Various forms of canonical identification would define cognitive representations and their reverses.

Canonical identification is continuous but does does not respect symmetries: the action of the p-adic symmetry followed by a canonical identification to reals is not equal to the canonical identification map followed by the real symmetry.

Can one fuse the two views about real-p-adic correspondence

Could the two views about real-p-adic correspondence be fused if appropriately generalized canonical identification is interpreted as a concrete duality mapping short length scale physics and long length scale physics to each other? There are however hard technical problems involved.

1. Canonical identification is not consistent with general coordinate invariance unless one can identify some physicallyt preferred coordinate system. For embedding spaces the isometries guarantee the existence of rather limited space of this kind of coordinate systems: linear coordinates for M^4 and complex coordinate systems relatd by color isometries for CP_2. This suggests taht canonical identification should be realized at the level of embedding space.

2. Canonical identification would be locally continuous in both directions. Note that for the points with finite pinary expansion (ordinary integers) the map is two-valued. Note also that rationals can be expanded in infinite powers series with respect to p and one can ask whether one should do this or map $q = m/n$ to $I(m)/I(n)$ (the representation of rational is unique if m and n have no common factors). Symmetries represented by matrix groups with rational matrix elements require the latter option.

   One can map rationals by $m/n \rightarrow I(m)/I(n)$. One can also express m and n as power series of p^k as $x = \sum x_n p^{nk}$ and perform the map as $x \rightarrow \sum x_n p^{-nk}$. This allows to preserve symmetries in arbitrary good measurement resolution characterized by the power p^{-k} on real side. The reason would be that rationals m/n with $m < pk$ and $n < p^k$ would be mapped to themselves: algebra wins. If m or n or both are larger than p^k the behavior associated with canonical identification sets in: topology wins.

3. This compromize between algebra and topology looks nice but an additional problem emerges when one brings in more TGD. If one wants to map differentiable p-adic space-time surfaces (preferred extremals of Kähler action) to differentiable real surfaces (preferred extremals of Kähler action), canonical identification cannot work since it is not differentiable. Second

pinary cutoff above which one simply throws out the pinary digits, is needed. p-Adic space-time sheets are discretized and mapped to a discrete subsets of the real space-time sheet. Completion to a preferred extremal is needed and assigning a preferred extremal to a discrete point set becomes the challenge. The p-adic manifold concept relies essentially on this idea about p-adic-real correspondence.

This chapter was originally written few years before the idea of p-adic manifold. The question was whether one could circumvent the tension between symmetries and continuity without approximations? After few years the answer is definitely "No!".

Despite this I have decided to keep this chapter since the quantum variant of canonical identification could also be involved with the definition of p-adic manifold. In particular, the fact that it maps p-adic numbers to algebraic numbers in the algebraic extension defined by p:th root of unity might have some deep meaning and relate to the connection between Galois group of maximal Abelian extension of rationals and adeles consisting of the Cartesian product of real and various p-adic number fields.

Could the canonical identification based on quantum integers provide a generalization of the notion of symmetry itself in order to circumvent ugly constructions? This is the question to be addressed in this chapter.

21.1.4 Brief Summary Of The General Vision

Some of the basic questions of the p-adicization program are following.

1. Is there a duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

   Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant. The physical interpretation could be in terms of inclusions of hyper-finite factors of type II_1 parametrized by quantum phases and allowing to interpret the action of the included algebra as having no effects on the state in the measurement resolution used [K99]. When quantum phase approaches unity one would obtained ordinary canonical identification.

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this [K50])?

   This chapter studies some ideas but does not provide a clearcut answer to these questions.

Two options for quantum integers

In the sequel two options for defininining quantum arithmetics are discussed: Options I and II. These are not the only one imaginable but represent kind of diametrical opposites. The two options are defined in the following way.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to $l \ modp$ (to guarantee positivity of the quantum integer) decomposed into primes $l < p$ and these in turn to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of the product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n) p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

4. Non-negativity of quantum primes is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [K5]. One could also replace quantum prime by its square without losing the map of products to products.

5. At the limit when the quantum phase approaches to unit, ordinary quantum integers with p-adic norm 1 approach to ordinary integers in real sense and ordinary arithmetics results. Ordinary integers in real sense are obtained for option II when the coefficients of the pinary expansion of n are much smaller than p and p approaches infinity. Same is true for option I if the prime factors of the integer are much smaller than p.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

Quantum counterparts of classical groups

Quantum arithmetics inspires the notion of quantum matrix group as a counterpart of quantum group for which matrix elements are non-commuting numbers. Now the elements would be ordinary numbers. Quantum classical correspondence and the notion of finite measurement resolution realized at classical level in terms of discretization suggest that these two views about quantum groups are closely related. The preferred prime p defining the quantum matrix group is identified as p-adic prime or its power and the inversion $p \to 1/p$ is group homomorphism so that symmetries are respected.

Option I gives p-adic counterparts of classical groups. p-Adic numbers are replaced with the ring generated by the quantum images of p-adic numbers, which each correspond to some power of p: this extension gives powers series in p. By requiring the group conditions for a subgroup of special linear group to be satisfied in order $O(p) = 0$ one obtains classical groups for finite fields $G(p, 1)$ by simply requiring that group conditions are satisfied in order $O(p) = 0$. One can also have also classical groups associated with finite fields $G(p, n)$ having p^n elements.

Option II is more interesting and quantum counterparts could be seen as counterparts of classical groups obtained by replacing group elements with the elements of ring defined by Kac-Moody type algebra. The difference to Option I and its variants is that one does not map p-adic integer to $G(p, 1)$ by $n \to n \ mod \ p$ before quantum map but applies it to the entire p-adic integer.

1. The quantum counterparts of special linear groups $SL(n, F)$ exists always. For the covering group $SL(2, C)$ of $SO(3, 1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if quantum arithmetics is characterized by a prime rather than general integer and when the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.

2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k + 7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \to 2^{2r} n$. This means scaling by 2 for the integers appearing in the square sum representation.

3. $r_3(n)$ is proportional to the so called class number function $h(-n)$ telling how many non-equivalent decompositions algebraic integers have in the quadratic algebraic extension generated by $\sqrt{-n}$.

The findings about quantum $SO(3)$ encourages to consider a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.

2. An alternative idea discussed in [K65] is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

21.2 Various options for Quantum Arithmetics

In this section the notion of quantum arithmetics as a deformation of p-adic number field to a ring is discussed. One can imagine several options for quantum arithmetics. Both for Option I and II p-adic integers are mapped to a subset of a ring of quantum integers and the sum operation for the ring has nothing to with that for p-adic numbers. In both cases the elements of ring makes sense as real numbers.

21.2.1 Comparing options I and II

The two options for defining quantum arithmetics are represented in the introduction so that it is no point writing the formulas again. It is interesting to compare these options.

Consider first what is common to these options.

1. For option I all integers are decomposed into products of primes mapped to their quantum counterparts by $p_1 \to p_1 \ mod \ p \to \prod p_i p_i^{k_i}$ followed by the mapping of p_i to its quantum counterpart. The modding operator for p_1 guarantees positivity of the outcome. Hence the information about prime decomposition is not lost. Also the information about p-adic norm is preserved if p is mapped to itself or $1/p$ (this depending on whether one speaks about p-adic or real variant of quantum integer). Quantum image of product is not however product of quantum images. The information about sum is lost.

   For option II the information about prime decomposition is lost.

   For both options it is also possible to decompose the coefficients of powers of p to prime factors. The information about pinary expansion is not lost. This option in turn respects continuity.

2. For both options the quantum image belongs to a ring larger than the image since for neither options the sum of two quantum integers need not be image of p-adic number. This makes possible to assign classical groups to this ring.

3. p-Adic–real duality can be identified as the analog of canonical identification induced by the map $p \rightarrow 1/p$ in the pinary expansion of quantum rational. This maps p-adic and real physics to each other and real long distances to short ones and vice versa. This map is especially interesting as a map for defining cognitive representations. The map $p^n \rightarrow p^{-n}$ is generalization of this map an maps p-adic integers $k < p^n$ to itself. Note that subgroups of $Gl(m, R)$ consisting of matrices with integer valued elements p^n are especially interesting p-adically since one avoid p-adic rationals for which canonical identification map allows several variants.

4. Quantum map $n \rightarrow n_q$ precedes canonical identification so that it could be interpreted as a modification for the chart map defined by canonical identification in the proposed definition for p-adic manifold already mentioned [K101]. My recent view is that this option is not promising. Canonical identification makes sense at the level of probability distributions and Lorentz invariants but not at space-time level since pinary expansion is not general coordinate invariant notion.

The differences between options I and II relate to how one treats integers $n > p$.

1. For option I one decomposes given integer to a product of primes and all primes are mapped to their quantum counterparts so that products go to products. Sums are not however mapped to sums. Quantum primes can be also negative. For $q = exp(i\pi/p)$ integers vanishing modulo p go to zero if one defines n_q by using the general formula for quantum integer. Also the extension of the map to rationals m/n meets with difficulties if n_q can vanish. It seems that p must be mapped to $1/p$ to avoid these problems and this is done in the proposal developed in the model for Shnoll effect [K5]. With this modification the image of integer is always product of quantum primes by some power of p and one does not obtain series in powers of p typical for p-adic numbers and canonical identification.

   If quantum map would respect both product and sum, the quantum counterparts of subgroups of classical matrix groups with elements elements smaller than p^n would exist. This condition cannot be satisfied. It is not clear whether subgroups of matrix groups exist for which their quantum counterparts defined by matrices with matrix elements smaller than p^n are groups too.

   This suggests that one must extend the image of p-adic integers (and its extension to that of p-adic rationals) to a ring defined by quantum sums and assign matrix group acting as symmetries to this ring. Matrix groups for which symmetries preserve volume the determinant of the matrix equals to unity so that the inverse exists always even when number field is replaced wit ring so that the existence of generalized matrix groups does not seem to be a problem.

2. For Option II one expands integer in powers p^k and maps the coefficients $n_k < p$ by quantum map just as for the first option. The quantum counterparts of p-adic integers generate a larger ring via products and sums.

   One obtains the analog of Kac-Moody algebra with coefficients for a given power of p defining an algebra analogy to polynomial algebra. One can define also rationals and obtains a structure analogous to a function field. This field allows projection to p-adic numbers but is much larger than p-adic numbers. The construction works also for more general quantum phases q than those defined by primes and $q = exp(i\pi/p^n)$ is an especially interesting case. For this option the symmetries of quantum p-adics would be preserved in the canonical identification.

21.2.2 About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - \overline{q}^n}{q - \overline{q}} \quad . \tag{21.2.1}$$

Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$p_q = \frac{q^p - \overline{q}^p}{q - \overline{q}}$$
$$n_q = \prod_p p_q^{n_p} \text{ for } n = \prod_p p^{n_p} . \quad (21.2.2)$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$q = exp(\eta)exp(i\pi/m) . \quad (21.2.3)$$

The quantum map is 1-1 for a non-vanishing value of η and the limit $m \to \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteeing a unique definition of quantum sum. In the p-adic context with $m = p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem a^{p-1} mod $p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p-1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers. .

4. $m = p$ the quantum counterparts of pinary digits are non-negative. The model of Sholl effect suggests that the most natural choice. One can however consider also expansions in powers of p^k and now $m = p^k$ is the most natural choice. For general value m it is natural to consider expansions in powers of m but now one loses number field property.

5. For p-adic rationals the quantum map reads as $m/n \to m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [A190] [K32] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one (if any). Certainly Option I for which the quantum map is only part of canonical identification is the simpler one- perhaps quite too simple. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure imbeddable to reals and therefore much more general than that proposed in the model of Shnoll effect [K5] but gives identical predictions when the parameters characterizing the probability distribution $P(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $P(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

21.2.3 Canonical identification for quantum rationals and symmetries

The fate of symmetries in canonical identification map is different for options I and II. Before continuing, one can of course ask why canonical identification should map p-adic symmetries to real symmetries. There is no obvious answer to the question.

1. For option II the prime p in the expansion $\sum x_n l^n$ is interpreted as a symbolic coordinate variable and the product of two quantum integers is analogous to the product of polynomials reducing to a convolution of the coefficient using quantum sum. The coefficient of a given power of p in the product would be just the convolution of the coefficients for factors using quantum sum. In the sum coefficients would be just the quantum sums of coefficients of summands.

2. Option I maps p-adic integers to their quantum counterparts by mapping the prime factors to their quantum counterparts defined by $q = exp(i\pi/p)$. The sums of the resulting quantum integers define a linear space consisting of sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0\leq n<p} q^n$ vanishes. Given p-adic integer is mapped to single phase q^n. The map of all p-adic integers to p quantum phases means loss of information and generation or ring creates information not related to the p-adic numbers themselves.

   (a) One can also define quantum rationals by writing a given rational in unique manner as $r = p^k m/n$, expanding m and n as finite power series in p, and by replacing the coefficients with their quantum counterparts. The mapping of quantum rationals to their real counterparts would be by canonical identification $p \rightarrow 1/p$ in m_q/n_q. Also the completion of quantum rationals obtained by allowing infinite powers series for m and n makes sense and defines by canonical identification what might be called quantum reals.

   (b) Quantum arithmetics defined in this manner does not reflect faithfully the ordinary p-adic arithmetics and also leads to a problem with symmetries. In the product of ordinary p-adic integers the convolution for given power of p can lead to overflow and this leads to the emergence of modulo arithmetics. As a consequence, the canonical identification $\sum x_n l^n \rightarrow \sum x_n l^{-n}$ does not respect product and sum in general (simple example: $I((xl)^2) = x^2 l^{-2} \neq (I(xl))^2 = (x^2 modl)l^{-2} + (x^2 - x^2 modl)l^{-3}$ for $x > l/2$). Therefore canonical identification induced by $l \rightarrow 1/l$ does not respect symmetries represented affinely (as linear transformations and translations) although it is continuous.

   (c) For quantum rationals defined as ratios m_q/n_q of quantum integers and mapped to $I(m_q)/I(n_q)$ the situation improves dramatically but is not cured completely. The breaking of symmetries could have a natural interpretation in terms of finite measurement resolution. For instance, one could argue that p-adic space-time sheets are extrema of Kähler action in algebraic sense and their real counterparts obtained by canonical identification are kind of smoothed out quantum average space-time surfaces, which do not satisfy real field equations and are not even differentiable. In this framework p-adicization would defined quantum average space-time as a p-adically smooth object which nice geometric properties.

   Consider next Option II for quantum p-adics.

1. The original motivation for quantum rationals was to obtain correspondence between p-adics and reals respecting symmetries. For option II this dream can be achieved if the symmetries are defined for quantum rationals rather than p-adic numbers. Whether this means that quantum rationals are somehow deeper notion that p-adic number field is an interesting question. Since quantum rationals are obtained from quantum integers defining a

Kac-Moody type algebra in powers of p^n symmetry conditions for quantum rational matrices reduce to conditions in terms of quantum integers and hold separately for each power of p. Therefore the value of p does not actually matter, and the replacement $p \to 1/p$ respects the symmetries.

For instance, for the quantum counterpart of group $SL(2,Z)$ assuming that p^N is the largest power in the matrix elements the condition $det(A) = 1$ gives $2N+1$ conditions for $4(N+1)$ parameters leaving $2N+3$ parameters. The matrix elements are integers so that actual conditions are more stringent.

2. Quantum integers generate a space in which the space of coefficients of p^n is the module generated by the sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0\leq n<p} q^n$ vanishes. The huge extension of the original space is an obvious problem.

3. For this option non-uniqueness is a potential problem. One can have several quantum integers projecting to the same finite integer in powers of p. The number would be actually infinite when the coefficients of powers of p can occur with both signs. Does the non-uniqueness mean that quantum p-adics are more fundamental than p-adics?

4. The non-uniqueness inspires questions about the relationship between quantum field theory and number theory. Could the sum over different quantum representatives for p-adic integers define the analog of the functional integral in the ideal measurement resolution? Could loop corrections correspond number theoretically to the sum over all the alternatives allowed in a given measurement resolution defined by maximal number of powers of p in expansions of m and n in $r = m/n$? This would extend the vision about physics as generalized number theory considerably.

Note that quantum p-adic numbers are algebraic numbers so that quantum integers are algebraic numbers with prime p remaining ordinary integer.

21.2.4 More about the non-uniqueness of the correspondence between p-adic integers and their quantum counterparts

For both options the projection from quantum integers to p-adic numbers is many-to-one.

For option I p-adic integer is mapped to an integer proportional to a quantum integers proportional to power of p expressing its p-adic norm. Since the primes p_i in the decomposition of n are effectively replaced with $p_i \mod p$, a large number of integers with same p-adic norm is mapped to same quantum integer. A lot of information is lost.

For Option II p-adic number is mapped to a series in powers of p so that information is not lost. It is interesting to have some idea about how many quantum counterparts given p-adic integer has in this case and what might be their physical interpretation. If -1 is mapped to -1 rather than $(p-1)_q(1+p+p^2+...)$ in quantum map and therefore also in canonical identification quantum p-adics form an analog of a function field. The number of quantum p-adics projected to same integer is infinite.

The number of quantum p-adics for which the coefficients of the polynomials of quantum primes $p_1 < p$ regarded as variables are positive is finite. These kind of quantum integers could be called strictly positive. It is easy to count the number of different strictly positive quantum counterparts of p-adic integer $n = n_0 + n_1p + n_2p^2 + ... + n_kp^k$ - that is elements of the ring of quantum integers projected to a given p-adic integer n.

1. For both options the number of quantum integers projected to a given integer n is simply the number of all partitions of to a sum of integers, whose number can vary from 1 to n and thus expressible as the sum $D(n) = \sum_{k=1}^{n} d(n,k)$ of numbers of partitions to k integers. Interestingly, the number of states with total conformal weight n constructible using at most k Virasoro generators equals to $d(n,k)$ and the total number of states with conformal weight n is just $D(n)$. This result follows if one does not assumes that different quantum representatives are really different. One cannot exclude the possibility that the condition $\sum_{n=0}^{p-1} q^n = 0$ for quantum phases implies this kind of dependencies.

Similar situation occurs in the construction of tensor powers of group representations for any additive quantum number for which the basic unit is fixed. Could quantum classical correspondence be realized as a mapping of different states of a tensor product as different quantum p-adic space-time sheets?

2. The partition of n in all possible ways resembles combinatorially the insertion of loop corrections in all possible ways to a Feynman diagram containing corresponds up to p^{k-1}. Maybe the sum over quantum corrections could be reduced to the summation of amplitudes in which p-adic integer is mapped to its quantum counterpart in all possible ways. In zero energy ontology quantum corrections to generalized Feynman diagrams in a new p-adic length scaled defined by p^k indeed more or less reduces to the addition of zero energy states as a new tensor factor in all possible ways so that structurally the process would be like adding tensor factor.

To number of geometric objects to which one can assign quantum counterparts is rather limited. For the points of embedding space with rational coordinates the number of quantum rational counterparts would be finite. If either of the integers appearing in the p-adic rational become infinite as a real integer, the number of quantum rationals becomes infinite and one obtains continuum in p-adic sense since p-adic integers form a continuum.

An infinite number of points of a $D > 0$-dimensional quantum counterpart of p-adic surface project to the same p-adic point. The restriction to a finite number of pinary digits makes sense only at the ends of braid strands at partonic 2-surfaces. This provides additional support for the effective 2-dimensionality and the braid representation for the finite measurement resolution. The selection of braid ends is strongly constrained by the condition that the number of pinary digits for the embedding space coordinates is finite.

The interesting question is whether the summation over the infinite number of quantum copies of the p-adic partonic 2-surface could correspond to the functional integral over partonic 2-surfaces with braid ends fixed and thus having only one term in their pinary expansion. This kind of functional integral is indeed encountered in quantum TGD.

1. The summations in which the quantum positions of braid ends form a finite set would correspond to finite pinary cutoff. Second question is what the quantum summation for partonic 2-surfaces means: certainly there must be correlations between very nearby points if the summation is to make sense. The notion of finite measurement resolution suggests that summation reduces to that over the quantum positions of the braid ends.

2. Indeed, the reduction of the functional integral to a summation over quantum copies makes sense only if it can be carried out as a limit of a discrete sum analogous to Riemann sum and giving as a result what might be called quantum p-adic integral. This limit would mean inclusion of an increasing number of points of the partonic 2-surface to the quantum sum defined by the increasing pinary cutoff. One would also sum over the number of braid strands. This approach could make sense physically if the collection of p-adic partonic 2-surfaces together with their tangent space data corresponds to a maximum of Kähler function. Quantum summation would correspond to a functional integral over small deformations with weight coming from the p-adic counterpart of vacuum functional mapped to its quantum counterpart. Canonical identification would give the real or complex counterpart of the integral.

21.2.5 The three basic options for Quantum Arithmetics

I have proposed two alternative definitions for quantum integers. In [K65] a third option is discussed.

1. For option I quantum counterparts of p-adic integers are identified as products of quantum counterparts for the primes dividing them. Powers of p are mapped to their inverses (straightforward quantum map would take them to zero). The quantum integers can be extended to ring (and algebra) by allowing sum operation. Field property is in general lost.

2. The approach adopted in the sequel is based on Option II based on the identification of quantum p-adics as an analog of Kac-Moody algebra with powers p^n in the same role as the powers z^n for Kac-Moody algebra. The two algebras have identical rules for sum and multiplication, and one does not require the arithmetics to be induced from the p-adic arithmetics (as assumed originally) since this would lead to a loss of associativity in the case of sum. Therefore the quantum counterparts of primes $l \neq p$ generate the algebra. One can also make the limitation $l < p^N$ to the generators. The counterparts of fixed integers in the map of integers to quantum integers are $0, 1, -1$ are $, 0, 1, -1$ as is easy to see. The number of quantum integers projecting to same p-adic integer is infinite.

3. One can consider also quantum m-adic option with expansion $l = \sum l_k m^k$ in powers of integer m with coefficients decomposable to products of primes $l < m$. This option is consistent with p-adic topology for primes p divisible by m and is suggested by the inclusion of hyper-finite factors [K32] characterized by quantum phases $q = exp(i\pi/m)$. Giving up the assumption that coefficients are smaller than m gives what could be called quantum covering of m-adic numbers. For this option all quantum primes l_q are non-vanishing. Phases $q = exp(i\pi/m)$ characterize Jones inclusions of hyper-finite factors of type II_1 assumed to characterize finite measurement resolution.

4. The definition of quantum p-adics discussed in [K65] replaces integers with Hilbert spaces of same dimension and $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions. Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

21.3 Could Lie groups possess quantum counterparts with commutative elements exist?

To begin with, it must be made clear that by commutativity it is meant that the matrix elements of the matrices representing the group elements are commutative numbers, not the matrices themselves.

The proposed definition of quantum rationals involves exceptional prime p expected to define what might be called p-adic prime. In p-adic mass calculations canonical identification is based on the map $p \rightarrow 1/p$ and has several variants but quite generally these variants fail to respect symmetries. Canonical identification for space-time coordinates fails also to be general coordinate invariant unless one has preferred coordinates. A possible interpretation could be that cognition affects physics: the choice of coordinate system to describe physics affects the physics.

The natural question is whether the proposed definition of quantum integers as series of powers of p-adic prime p with coefficients, which are arbitrary quantum rationals not divisible by p with product defined in terms of convolution for the coefficients of the series in powers of p using quantum sum for the summands in the convolution could change (should one say "save"?) the situation.

To see whether this is the case one must find whether the quantum analogues of classical matrix groups exist. To avoid confusion it should be emphasized that these quantum counterparts are distinct from the usual quantum groups having non-commutative matrix elements. Later a possible connection between these notions is discussed. In the recent case matrix elements commute but sum is replaced with quantum sum and the matrix element is interpreted as a powers series or polynomial in symbolic variable $x = p$ or $x = 1/p$, p prime such that coefficients are rationals not divisible by p.

The crucial points are the following ones.

1. All classical groups (see `http://tinyurl.com/y86oror3`) [A15] are subgroups of the special linear groups (see `http://tinyurl.com/3vpk8o8`) [A77] $SL_n(F)$, $F = R, C$, consisting of matrices with unit determinant. One can also replace F with the integers of the field F to get groups like $SL(2,Z)$. Classical groups are obtained by posing additional conditions on $SL_n(F)$ such as the orthonormality of the rows with respect to real, complex or quaternionic inner product. Determinant defines a homomorphism mapping the product of matrices to the product of determinants in the field F.

   Could one generalize rational special linear group (matrices with determinant 1) and its algebraic extensions by replacing the group elements by ratios of polynomials of a formal variable x, which has as its value the preferred prime p such that the coefficients of the polynomials are quantum integers not divisible by p? For Option I the situation one has just ratios of p-adic integers finite as real integers and for Option II the integers are polynomials $x = \sum x_n p^n$, where one has

$$x_n = \sum_{\{n_i\}} N(\{n_i\}) \prod_i x_i^{n_i} \quad , \quad x_i = p_{i,q}, \quad p_i < p \quad , \quad q = exp(i\pi/p) \quad .$$

   Here $N(\{n_i\})$ is integer. Could one perform this generalization in such a way that the canonical identification $p \rightarrow 1/p$ maps this group to an isomorphic group? If quantum p-adic counterpart of the group is non-trivial, this seems to be the case since p plays the role of an argument of a polynomial with a specific values.

2. The identity $det(AB) = det(A)det(B)$ and the fact that the condition $det(A) = 1$ involves at the right hand side only the unit element common to all quantum integers suggests that this generalization could exist. If one has found a set of elements satisfying the condition $det_q(A) = 1$ all quantum products satisfy the same condition and subgroup of rational special linear group is generated.

21.3.1 Quantum counterparts of special linear groups

Special linear groups (see `http://tinyurl.com/3vpk8o8`) [A77] defined by matrices with determinant equal to 1 contain classical groups as subgroups and the conditions for their quantum counterparts are therefore the weakest possible. Special linear group makes sense also when one restricts the matrix elements to be integers of the field so that one has for instance $SL_n(Z)$. Opiton I reduces to that for ordinary p-adics. For Option II each power of p can be treated independently so that the situation is easier. The treatment of conditions in two cases differs only in that overflows in p are possible for Option I. The numbers of conditions are same.

Let us consider $SL_n(Z)$ first.

1. To see that the generalization exists in the case of special linear groups one just just writes the matrix elements a_{ij} in series in powers of p

$$a_{ij} \quad = \quad \sum_n a_{ij}(n) p^n \quad . \qquad (21.3.1)$$

   This expansion is very much analogous to that for the Kac-Moody algebra element and also the product and sum obey similar algebraic structure. p is treated as a symbolic variable in the conditions stating $det_q(A) = 1$. It is essential that $det_q(A) = 1$ holds true when p is treated as a formal symbol so that each power of p gives rise to separate conditions.

2. For SL_n the definition of determinant involves sum over products of n elements. Quantum sums of these elements are in question.

3. Consider now the number of conditions involved. The number of matrix elements is in real case $N^2(k+1)$, where k is the highest power of p involved. $det(A)=1$ condition involves powers of p up to l^{Nk} and the total number of conditions is $kN+1$ - one for each power. For higher powers of p the conditions state the vanishing of the coefficients of p^m. This is achieved elegantly in the sense of modulo arithmetics if the quantum sum involved is proportional to l_q.

   The number of free parameters is

$$\# \ = (k+1)N^2 - kN - 1 = kN(N-1) + N^2 - 1 \ . \tag{21.3.2}$$

   For $N=2, k=0$ one obtains $\#=3$ as expected for SL(2, R). For $N=2, k=1$ one obtains $\#=5$. This can be verified by a direct calculation. Writing $a_{ij}=b_{ij}+c_{ij}p$ one obtains three conditions

$$det_q(B)=1 \ , \ Tr_q(BC)=0 \ , \ det_q(C)=0 \ . \tag{21.3.3}$$

   for the 8 parameters leaving 5 integer parameters.

   Integer values of the parameters are indeed possible. Using the notation

$$b_{ij} = \left(\begin{array}{cc} a_0 & b_0 \\ c_0 & d_0 \end{array} \right) \ , \ c_{ij} = \left(\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) \tag{21.3.4}$$

   one can write the solutions as

$$\begin{array}{l} (a_1,b_1) = k(c_1,d_1) \ , \quad (c_1,d_1) = l(a_0-kc_0, b_0-kd_0) \ , \\ a_0d_0 - b_0c_0 = 1 \ . \end{array} \tag{21.3.5}$$

   Therefore 6 integers characterize the solution.

4. Complex case can be treated in similar manner. In this case the number of three parameters is $2(k+1)N^2$, the number of conditions is $2(kN+1)$ and the number of parameters is

$$\# = 2(k+1)N^2 - 2(kN+1) \ . \tag{21.3.6}$$

5. Since the conditions hold separately for each power of p, the formula $det_q(AB)=det_q(A)det_q(B)$ implies that the matrices satisfying the conditions generate a subgroup of SL_n.

One can generalize the argument to rational values of matrix elements in a simple manner. The matrix elements can be written in the form $A_{ij}=Z_{ij}/K$ and the only modification of the equations is that the zeroth order term in p gives $det(Z)=K^n$ for SL_n. One can expand K^n in powers of p and it gives inhomogenous term to in each power of p. For instance, if K is zeroth order in p, solutions to the conditions certainly exist.

The result means that rational subgroups of special linear groups $SL_n(R)$ and $SL(n,C)$ and also the real and complex counterparts of $SL(n,Z)$ quantum matrix groups characterized by prime p exist in both real and p-adic context and can be related by the map $p\rightarrow 1/p$ mapping short and length scales to each other.

It is remarkable that only the Lorentz groups $SO(2,1)$ and $SO(3,1)$ have covering groups are isomorphic to $SL(2,R)$ and $SL(2,C)$ allow these subgroups. All classical Lie groups involve additional conditions besides the condition that the determinant of the matrix equals to one and all these groups except symplectic groups fail to allow the generalization of this kind for arbitrary values of k. Therefore four-dimensional Minkowski space is in completely exceptional position.

21.3.2 Do classical lie groups allow quantum counterparts?

In the case of classical groups one has additional conditions stating orthonormality of the rows of the matrix in real, complex, or quaternionic number field. It is quite possible that the conditions might not be satisfied always and it turns out that for G_2 and probably also for other exceptional groups this is the case.

1. Non-exceptional classical groups

It is easy to see that all non-exceptional classical groups quantum counterparts in the proposed sense for sufficiently small values of k and in the case of symplectic groups quite generally. In this case one must assume rational values of group elements and one can transform the conditions to those involving integers by writing $A_{ij} = Z_{ij}/K$. The expansion of K gives for orthogonal groups the condition that the lengths of the integer rows defining Z_{ij} have length K^2 plus orthogonality conditions. $det(A) = 1$ condition holds true also now since a subgroup of special linear group is in question.

1. Consider first orthogonal groups $SO(N)$.

   (a) For $q = 1$ there are N^2 parameters. There are N conditions stating that the rows are unit vectors and $N(N-1)/2$ conditions stating that they are orthogonal. The total number of free parameters is $\# = N(N-1)/2$.

   (b) If the highest power of p is k there are $(k+1)N^2$ parameters and $(2k+1)[N+N(N-1)/2] = (2k+1)(N+1)/2$ conditions. The number of parameters is

   $$\# = N^2(k+1) - \frac{N(N+1)(2k+1)}{2} = \frac{N(N-2k+1)}{2} \ . \tag{21.3.7}$$

   This is negative for $k > (N+1)/2$. It is quite not clear how to interpret this result. Does it mean that when one forms products of group elements satisfying the conditions the powers higher than $k_{max} = [(N+1)/2]$ vanish by quantum modulo arithmetics. Or do the conditions separate to separate conditions for factors in AB: this indeed occurs in the unitarity conditions as is easy to verify. For $SO(3)$ and $SO(2,1)$ this would give $k_{max} = 2$. For $SO(3,1)$ one would have $k_{max} = 2$ too. Note that for the covering groups $SL(2,R)$ and $SL(2,C)$ there is no restrictions of this kind.

   (c) The normalization conditions for the coefficients of the highest power of a given row imply that the vector in question has vanishing length squared in quantum inner product. For $q = 1$ this implies that the coefficients vanish. The repeated application of this condition one would obtain that $k = 0$ is the only possible solution. For $q \neq 1$ the conditions can be satisfied if the quantum length squared is proportional to $l_q = 0$. It seems that this condition is absolutely essential and serves as a refined manner to realize p-adic cutoff and quantum group structure and p-adicity are extremely closely related to each other. This conclusion applies also in the case of unitary groups and symplectic groups.

   (d) Complex forms of rotation groups can be treated similarly. Both the number of parameters and the number of conditions is doubled so that one obtains $\# = N^2(k+1) - N(N+1)(2k+1) = N(N-2k+1)$ which is negative for $k > (N+1)/2$.

2. Consider next the unitary groups $U(N)$. Similar argument leads to the expression

   $$\# = 2N^2(k+1) - (2k+1)N^2 = N^2 \tag{21.3.8}$$

   so that the number of three parameters would be N^2- same as for $U(N)$. The determinant has modulus one and the additional conditions requires that this phase is trivial. This is expected to give $k+1$ conditions since the fixed phase has l-adic expansion with $k+1$ powers. Hence the number of parameters for $SU(N)$ is

$$\# = N^2 - k + 1 \quad (21.3.9)$$

giving the condition $k_{max} < N^2 - 1$ which is the dimension of $SU(N)$.

3. Symplectic group can be regarded as a quaternionic unitary group. The number of parameters is $4N^2(k+1)$ and the number of conditions is $(2k+1)(N+2N(N-1)) = N(2N-1)(2k+1)$ so that the number of three parameters is $\# = 4N^2(k+1)-(2k+1)N(N-1) = (2k+3)N^2 + N(2k+1)$. Fixing single quaternionic phase gives 3(k+1) conditions so that the number of parameters reduces to

$$\# = (2k+3)N^2 + (2k+1)N - 3(k+1) = (k+1)(2N^2 + 2N - 3) + N(N-1) \quad (21.3.10)$$

which is positive for all values of N and k so that also symplectic groups are in preferred position. This is rather interesting, since the infinite-dimensional variant of symplectic group associated with the $\delta M^4 \times CP_2$ is in the key role in quantum TGD and one expects that in finite measurement resolution its finite-dimensional counterparts should appear naturally.

2. Exceptional groups are exceptional

Also exceptional groups (see `http://tinyurl.com/y779ldyt`) [A27] [A27] related closely to octonions allow an analogous treatment once the nature of the conditions on matrix elements is known explicitly. The number of conditions can be deduced from the dimension of the ordinary variant of exceptional group in the defining matrix representation to deduce the number of conditions. The following argument allows to expect that exceptional groups are indeed exceptional in the sense that they do not allow non-trivial quantum counterparts.

The general reason for this is that exceptional groups are very low dimensional subgroups of matrix groups so that for the quantum counterparts of these groups the number N_{cond} of group conditions is too large since the number of parameters is $(k+1)N^2$ in the defining matrix representation (if such exists) and the number of conditions is at least $(2k+1)N_{class}$, where N_{class} is the number of condition for the classical counterpart of the exceptional group. Note that r-linear conditions the number of conditions is proportional to $rk+1$.

One can study the automorphism group G_2 (see `http://tinyurl.com/y9rrs7un`) [A31] of octonions as an example to demonstrate that the truth of the conjecture is plausible.

1. G_2 is a subgroup of $SO(7)$. One can consider 7-D real spinor representation so that a representation consists of real 7×7matrices so that one has $7^2 = 49$ parameters. One has $N(N+1)/2$ orthonormality conditions giving for $N = 7$ orthonormality conditions 28 conditions. This leaves 21 parameters. Besides this one has conditions stating that the 7-dimensional analogs of the 3-dimensional scalar-3-products $A \cdot (B \times C)$ for the rows are equal 1, -1, or 0. The number of these conditions is $N(N-1)(N-2)/3!$. For $N = 7$ this gives 35 conditions meaning that these conditions cannot be independent of orthonormalization conditions The number of parameters is $\# = 49 - 35 = 14$ - the dimension of G_2 - so that these conditions must imply orthonormality conditions.

2. Consider now the quantum counterpart of G_2. There are $(k+1)N^2 = 49(k+1)$ parameters altogether. The number of cross product conditions is $(3k+1) \times 35$ since the highest power of p in the scalar-3-product is l^{3k}. This would give

$$\# = -56k + 14 \ . \quad (21.3.11)$$

This number is negative for $k > 0$. Hence G_2 would not allow quantum variant. Could this be interpreted by saying that the breaking of G_2 to $SU(3)$ must take place and indeed occurs in quantum TGD as a consequence of associativity conditions for space-time surfaces.

3. The conjecture is that the situation is same for all exceptional groups.

The general results suggest that both the covering group of the Lorenz group of 4-D Minkowski space and the hierarchy symplectic groups have very special mathematical role and that the notions of finite measurement resolution and p-adic physics have tight connections to classical number fields, in particular to the non-associativity of octonions.

21.3.3 Questions

In the following some questions are introduced and discussed.

How to realize p-adic-real duality at the space-time level?

The concrete realization of p-adic–real duality would require a map from p-adic realm to real realm and vice-versa. The naïve expectation is that it is induced by the map $p \to 1/p$ leading from p-adic number field to real number field or vice versa.

If possible, the realization of p-adic real duality at the space-time level should not pose additional conditions on the preferred extremals themselves. Together with effective 2-dimensionality this suggests that the map from p-adic realm to real realm maps partonic 2-surfaces to partonic 2-surfaces defining at least partially the boundary data for holography.

It turned out that the situation is not so simple. Or putting it correctly - so complex. The point is that the direct mapping of real space-time sheets to real ones requires discretization and length scale cutoff bringing in a lot of arbitrariness and the continuity of the map is in conflict with the preservation of symmetries.

A more realistic view is based on the idea that p-adic space-time sheets indeed define a theory about real space-time sheets. The interaction between real and p-adic number fields would mean that p-adic space-time surfaces define cognitive representations of real space-time surfaces (preferred extremals). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface a cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. String world sheets would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime.

How commutative quantum groups could relate to the ordinary quantum groups?

The interesting question is whether and how the commutative quantum groups relate to ordinary quantum groups.

This kind of question is also encountered when considers what finite measurement resolution means for second quantized induced spinor fields [K100]. Finite measurement resolution implies a cutoff on the number of the modes of the induced spinor fields on partonic 2-surfaces. As a consequence, the induced spinor fields at different points cannot ant-commute anymore. One can however require anti-commutativity at a discrete set of points with the number of points "more or less equal" to the number of modes. Discretization would follow naturally from finite measurement resolution in its quantum formulation.

The same line of thinking might apply to quantum groups. The matrix elements of quantum group might be seen as quantum fields in the field of real or complex numbers or possibly p-adic number field or of its extension. Finite measurement resolution means a cutoff in the number of modes and commutativity of the matrix elements in a discrete set of points of the number field rather than for all points. Finite measurement resolution would apply already at the level of symmetry groups themselves. The condition that the commutative set of points defines a group would lead to the notion of commutative quantum group and imply p-adicity as an additional and completely universal outcome and select quantum phases $exp(i\pi/p)$ in a preferred position. Also the generalization of canonical identification so central for quantum TGD would emerge naturally.

One must of course remember that the above considerations probably generalize so that one should not take the details of the discussion too seriously.

How to define quantum counterparts of coset spaces?

The notion of commutative quantum group implies also a generalization of the notion of coset space G/H of two groups G and $H \subset G$. This allows to define the quantum counterparts of the proper time constant hyperboloid and $CP_2 = SU(3)/U(2)$ as discrete spaces consisting of quantum points identifiable as representatives of cosets of the coset space of discrete quantum groups. This approach is very similar but more precise than the earlier approach in which the points in discretization had angle coordinates corresponding to roots of unity and radial coordinates with discretization defined by p-adic prime.

The infinite-dimensional "world of classical worlds" (WCW) can be seen as a union of infinite-dimensional symmetric spaces (coset spaces) [K24] and the definition as a quantum coset group could make sense also now in finite measurement resolution. This kind of approach has been already suggested and might be made rigorous by constructing quantum counterparts for the coset spaces associated with the infinite-dimensional symplectic group associated with the boundary of causal diamond. The problem is that matrix group is not in question. There are however good hopes that the symplectic group could reduces to a finite-dimensional matrix group in finite measurement resolution. Maybe it is enough to achieve this reduction for matrix representations of the symplectic group.

21.3.4 Quantum P-Adic Deformations Of Space-Time Surfaces As A Representation Of Finite Measurement Resolution?

A mathematically fascinating question is whether one could use quantum arithmetics as a tool to build quantum deformations of partonic 2-surfaces or even of space-time surfaces and how could one achieve this. These quantum space-times would be commutative and therefore not like non-commutative geometries assigned with quantum groups. Perhaps one could see them as commutative semiclassical counterparts of non-commutative quantum geometries just as the commutative quantum groups discussed in [K67] could be seen commutative counterparts of quantum groups.

As one tries to develop a new mathematical notion and interpret it, one tends to forget the motivations for the notion. It is however extremely important to remember why the new notion is needed.

1. In the case of quantum arithmetics Shnoll effect is one excellent experimental motivation. The understanding of canonical identification and realization of number theoretical universality are also good motivations coming already from p-adic mass calculations. A further motivation comes from a need to solve a mathematical problem: canonical identification for ordinary p-adic numbers does not commute with symmetries.

2. There are also good motivations for p-adic numbers. p-Adic numbers and quantum phases can be assigned to finite measurement resolution in length measurement and in angle measurement. This with a good reason since finite measurement resolution means the loss of ordering of points of real axis in short scales and this is certainly one outcome of a finite measurement resolution. This is also assumed to relate to the fact that cognition organizes

the world to objects defined by clumps of matter and with the lumps ordering of points does not matter.

3. Why quantum deformations of partonic 2-surfaces (or more ambitiously: space-time surfaces) would be needed? Could they represent convenient representatives for partonic 2-surfaces (space-time surfaces) within finite measurement resolution?

   (a) If this is accepted, there is no compelling need to assume that this kind of space-time surfaces are preferred extremals of Kähler action.
   (b) The notion of quantum arithmetics and the interpretation of p-adic topology in terms of finite measurement resolution however suggest that they might obey field equations in preferred coordinates but not in the real differentiable structure but in what might be called quantum p-adic differentiable structure associated with prime p.
   (c) Canonical identification would map these quantum p-adic partonic (space-time surfaces) to their real counterparts in a unique continuous manner and the image would be real space-time surface in finite measurement resolution. It would be continuous but not differentiable and would not of course satisfy field equations for Kähler action anymore. What is nice is that the inverse of the canonical identification which is two-valued for finite number of pinary digits would not be needed in the correspondence.
   (d) This description might be relevant also to quantum field theories (QFTs). One usually assumes that minima obey partial differential equations although the local interactions in QFTs are highly singular so that the quantum average field configuration might not even possess differentiable structure in the ordinary sense! Therefore quantum p-adicity might be more appropriate for the minima of effective action.

The cautious conclusion would be that commutative quantum deformations of space-time surfaces could have a useful function in TGD Universe.

Consider now in more detail the identification of the quantum deformations of space-time surfaces.

1. Rationals are in the intersection of real and p-adic number fields and the representation of numbers as rationals $r = m/n$ is the essence of quantum arithmetics. This means that m and n are expanded to series in powers of p and coefficients of the powers of p which are smaller than p are replaced by the quantum counterparts. They are quantum quantum counterparts of integers smaller than p. This restriction is essential for the uniqueness of the map assigning to a give rational quantum rationals.

2. One must get also quantum p-adics and the idea is simple: if the pinary expansions of m and n in positive powers of p are allowed o become infinite, one obtains a continuum very much analogous to that of ordinary p-adic integers with exactly the same arithmetics. This continuum can be mapped to reals by canonical identification. The possibility to work with numbers which are formally rationals is utmost importance for achieving the correct map to reals. It is possible to use the counterparts of ordinary pinary expansions in p-adic arithmetics.

3. One can defined quantum p-adic derivatives and the rules are familiar to anyone. Quantum p-adic variants of field equations for Kähler action make sense.

   (a) One can take a solution of p-adic field equations and by the commutativity of the map $r = m/n \rightarrow r_q = m_q/n_q$ and of arithmetic operations replace p-adic rationals with their quantum counterparts in the expressions of quantum p-adic embedding space coordinates h^k in terms of space-time coordinates x^α.
   (b) After this one can map the quantum p-adic surface to a continuous real surface by using the replacement $p \rightarrow 1/p$ for every quantum rational. This space-time surface does not anymore satisfy the field equations since canonical identification is not even differentiable. This surface - or rather its quantum p-adic pre-image - would represent a space-time surface within measurement resolution. One can however map the induced

metric and induced gauge fields to their real counterparts using canonical identification to get something which is continuous but non-differentiable.

4. This construction works nicely if in the preferred coordinates for embedding space and partonic (space-time) surface itself the embedding space coordinates are rational functions of space-time coordinates with rational coefficients of polynomials (also Taylor and Laurent series with rational coefficients could be considered as limits). This kind of assumption is very restrictive but in accordance with the fact that the measurement resolution is finite and that the representative for the space-time surface in finite measurement resolution is to some extent a convention. The use of rational coefficients for the polynomials involved implies that for polynomials of finite degree WCW reduces to a discrete set so that finite measurement resolution has been indeed realized quite concretely!

Consider now how the notion of finite measurement resolution allows to circumvent the objections against the construction.

1. Manifest GCI is lost because the expression for space-time coordinates as quantum rationals is not general coordinate invariant notion unless one restricts the consideration to rational maps and because the real counterpart of the quantum p-adic space-time surface depends on the choice of coordinates. The condition that the space-time surface is represented in terms of rational functions is a strong constraint but not enough to fix the choice of coordinates. Rational maps of both embedding space and space-time produce new coordinates similar to these provided the coefficients are rational.

2. Different choices for embedding space and space-time surface lead to different quantum p-adic space-time surface and its real counterpart. This is an outcome of finite measurement resolution. Since one cannot order the space-time points below the measurement resolution, one cannot fix uniquely the space-time surface nor uniquely fix the coordinates used. This implies the loss of manifest general coordinate invariance and also the non-uniqueness of quantum real space-time surface. The choice of coordinates is analogous to gauge choice and quantum real space-time surface preserves the information about the gauge.

21.4 Could one understand p-adic length scale hypothesis number theoretically?

p-Adic length scale hypothesis states that primes near powers of two are physically interesting. In particular, both real and Gaussian Mersenne primes seem to be fundamental and can be tentatively assigned to charged leptons and living matter in the length scales between cell membrane thickness and size of the cell nucleus. They can be also assigned to various scaled up variants of hadron physics and with lepto-hadron physics suggested by TGD.

21.4.1 Number theoretical evolution as a selector of the fittest p-adic primes?

How could one understand p-adic length scale hypothesis? The general explanation would be in terms of number theoretic evolution by quantum jumps selecting the primes that are the fittest. The vision discussed in [K65] d leads to the proposal that the fittest p-adic primes are those which do not split in the physically preferred algebraic extensions of rationals. Algebraic extensions are naturally characterized by infinite primes characterizing also stable bound states of particles. Therefore these stable infinite primes or equivalently stable bound states would characterize also the p-adic primes which are fit. This explanation looks rather attractive.

p-Adic evolution would mean also a selection of preferred scales for CDs, instead of integer multiples of CP_2 scale only prime multiples or possibly prime power multiples would be favored and primes near powers of two were especially fit. A possible “biological” explanation is that for the preferred primes the number of quantum states is especially large making possible to build complex sensory and cognitive representations about external world.

The proposed vision about commutative quantum groups encourages to consider a number theoretic explanation for the p-adic length scale hypothesis consistent with the evolutionary explanation is that the quantum counterpart of symmetry groups are especially large for preferred primes. Large symmetries indeed imply large numbers of states related by symmetry transformations and high representational capacity provided by the p-adic–real duality. It is easy to make a rough test of the proposal for $G = SO(3)$, $SU(2)$ or $SU(3)$ associated with p-adic integers modulo p reducing to the counterpart of G for finite field might be especially large for physically preferred primes. Mersenne primes do not however seem to be special in this sense so that the following considerations can be taken as an exercise in the use of number theoretic functions and the reader can quite well skip the section.

21.4.2 Preferred p-adic primes as ramified primes?

As I wrote the first version of this chapter, I had not developed the vision about adelic physics. Adelic physics corresponds to a hierarchy of extensions of rationals inducing extensions of p-adic number fields and the proposal is that ramified primes of extension correspond to preferred p-adic primes.

1. Prime p of number field K can split in the extension L of K to primes P_i of L. Prime p is Galois invariant, which poses strong conditions on the decomposition. p need not split at all, or it splits to maximal number n of primes of extension, which is invariant under Galois group. In some exceptional cases the number of primes can be smaller than the dimension of extension and un this case there is product of primes of extension containing less than the maximal number of P_i. In this case speaks of ramification.

2. Adelic physics suggests that prime p and quite generally, all preferred p-adic primes, could correspond to ramified primes for the extension of rationals defining the adele. Ramified prime divides discriminant $D(P)$ of the irreducible polynomial P (monic polynomial with rational coefficients) defining the extension (see `http://tinyurl.com/oyumsnk`).

   Discriminant $D(P)$ of polynomial whose, roots give rise to extension of rationals, is essentially the resultant $Res(P, P')$ for P and its derivative P' defined as the determinant of so called Sylvester polynomial (see `http://tinyurl.com/p67rdgb`). $D(P)$ is proportional to the product of differences $r_i - r_j$, $i \neq j$ the roots of p and vanishes if there are two identical roots.

   Remark: For second order polynomials $P(x) = x^2 + bx + c$ one has $D = b^2 - 4c$.

3. Ramified primes divide D. Since the matrix defining $Res(P, P')$ is a polynomial of coefficients of p of order $2n - 1$, the size of ramified primes is bounded and their number is finite. The larger coefficients $P(x)$ has, the larger the value of ramified prime can be. Small discriminant means small ramified primes so that polynomials having nearly degenerate roots have also small ramifying primes. Galois ramification is of special interest: for them all primes of extension in the decomposition of p appear as same power. For instance, the polynomial $P(x) = x^2 + p$ has discriminant $D = -4p$ so that primes 2 and p are ramified primes. For Galois extensions one has $e_i = e$ and $f_i = f$ and n equals to the order of Galois group: in this case one has $p = (\prod_{i=1}^{g} P_i)^e$.

   Remark: All polynomials having pair of complex conjugate roots have $p = 2$ as ramified prime.

4. I try to formulate my poor man's understanding about the situation. The expression of the ramified prime p can be written as $p = \prod_{i=1}^{g} P_i^{e_i}$. $e_i > 1$ for some i and $\sum_{i=1}^{g} e_i < n$. The interpretation is that the action of Galois group on each power $P_i^{e_i}$ is non-trivial and its orbit contains f_i points so that one has $\sum_{i=1}^{g} e_i f_i = n$. Although the numbers P^{e_i} are not invariant under Galois group, their product is. f_i can be identified as $f_i = [L/P_i^{e_i} : K/p]$. This says that $P/P_i^{e_i}$ consists of products of Galois images of P^{e_i} with integers $n < p$. The because the integers $n < p$ cannot decompose to a product of form $n = \prod_{i=1}^{g} P_i^{k_i}$ since they would divide p, which is impossible.

Since higher powers of P_i appear in the expression of ramified prime, one has $p \ mod \ P_i=0$ for $e_i > 1$. Why this can take place only for primes dividing D? Galois invariance of p must be involved. D is expressible as a product primes P_i of L and contains only higher powers P_i^k, $k > 1$. D is proportional to $\prod P_i^2$, where P_i are the primes dividing it. Why? Why the orbit consisting of f different integers of L contracts to single integer (this is just the criticality)?

5. What does ramification mean algebraically? The ring $\mathcal{O}(K)/(p)$ of integers of the extension K modulo $p = \pi_i^{e_i}$ can be written as product $\prod_i \mathcal{O}/\pi_i^{e_i}$ (see `http://tinyurl.com/y6yskkas`). If p is ramified, one has $e_i > 1$ for at least one i. Therefore there is at least one nilpotent element in $\mathcal{O}(K)/(p)$.

Could one interpret nilpotency quantum physically?

1. For Galois extensions one has $e_i = e > 1$ for ramified primes. e divides the dimension of extension. For the quadratic extensions ramified primes have $e = 2$. Quadratic extensions are fundamental extensions - kind of conserved genes -, whose further extensions give rise to physically relevant extensions.

   On the other hand, fermionic oscillator operators and Grassmann number used to describe fermions "classically" are nilpotent. Could they correspond to nilpotent elements of order $e_i = e = 2$ in $\mathcal{O}(K)/(p)$? Fermions are building bricks of all elementary particles in TGD. Could this number theoretic analogy for the fermionic statistics have a deeper meaning?

2. What about ramified primes with $e_i = e > 2$? Could they correspond to para-statistics (see `http://tinyurl.com/y4mq6j22`) or braid statistics (see `http://tinyurl.com/psuq45j`)?

   Both parabosonic and parafermionic fields of order n have the representation $\Psi = \sum_{i=1}^n \Psi_i$. For parafermion field one has $\{\Psi_i(x), \Psi_i(y)\} = 0$ and $[\Psi_i(x), \Psi_j(y)] = 0$, $i \neq j$, when x and y have space-like separation. For parabosons the roles of commutator and anti-commutator are changed.

   The states containing N identical parafermions are described by a representation of symmetric group S_N with N rows with at most n columns (anti-symmetrization). For states containing N identical parabosons one has N columns and at most n rows. For parafermions the wave function is symmetric in horizontal direction but the modes are different so that Bose-Einstein condensation is not possible.

   For parafermion of order n operator $\sum_{i=1}^n \Psi_i$ one has $(\sum_{i=1}^n \Psi_i)^n = \prod \Psi_1\Psi_2...\Psi_n$ and higher powers vanish so that one would have $n+1$-nilpotency. Therefore the interpretation for the nilpotent elements of order e in $\mathcal{O}(K)/(p)$ in terms of parafermion of order $n = e-1$ might make sense.

   It seems impossible to build a nilpotent operator from parabosonic field $\Psi = \sum_i \Psi_i$: the reason is that the powers Ψ_i^n are non-vanishing for arbitrarily high values of n.

3. Braid statistics differs from para-statistics and is assigned with quantum groups. It would naturally correspond to quantum phase $exp(i\pi/p)$ assignable to the exchange of particles by braid operation regarded as a homotopy permuting braid strands. Could ramified prime p would correspond to braid statistics and the index $e_i = e$ characterizing it to parafermion statistics of order $e-1$? This possibility cannot be excluded since this exotic physics would be associated in TGD framework to dark matter assigned to algebraic extensions of rationals whose dimension n equals to h_{eff}/h_0.

Why the primes, which do not split maximally in given extension would be physically special?

1. Do ramified primes possess exceptional evolutionary fitness and are ramified primes present for lower-dimensional extensions present also for higher-dimensional extensions? If higher extensions are formed as extensions of already existing extensions, this is the case. Hierarchy of polynomials of polynomials would to this kind of hierarchy with ramified primes of starting point polynomials analogous to conserved genes.

2. Quadratic extensions are the simplest ones and could serve as starting point extensions. Polynomials of form $x^2 - c$ are the simplest among them. Discriminant is now $D = -4c$.

3. Why $c = M_n = 2^n - 1$ allowing $p = 2$ and Mersenne prime $p = M_n$ as ramified primes would be favored? Extension of rationals defined by $x = 2^n$ is non-trivial for odd n and is equivalent with extension containing $\sqrt{2}$. $c = M_n = 2^n - 1$ as a small deformation of $c = 2^n$ gives an extension having both 2 as M_n as ramified primes.

   For $c = M_n$ the number of ramified primes is smallest possible and equal to 2: why minimal number of ramified primes would give rise to a fittest extension? Why smallest number of fermionic p-adic mass scales assignable to the ramified primes would be the fittest option?

   The p-adic length scale corresponding ro M_n would be maximal and mass scale minimal. Could one think that other quadratic extension are unstable against transforming to Mersenne extensions with smallest p-adic mass scale?

21.4.3 Could group theory select the preferred primes?

My recent view about the following considerations is that they are out-of-date. The notion of ramified prime so convincing that group theoretical considerations based on quantum-commutative generalization of Lie groups (matrix elements in commutative ring) look too tricky. I have not however had heart of throwing aways this piece of text yet.

In the following I consider *only the Option I*, which reduces to ordinary p-adic numbers effectively since quantum map induced by $p_i \rightarrow p_{iq}$ for $p_i < p$ can be combined with canonical identification. The arguments developed say nothing about option II. For option I the group transformations for which the conditions hold true modulo p make sense if matrix elements are integers satisfying $a_{ij} < p$. This makes sense for large values of p associated with elementary particles. This implies a reduction to finite field $G(p, 1)$. The original argument was more general and used same condition but involved an error.

1. For $SL(2, C)$ - the covering group of Lorentz group - one obtains no constraints and all quantum phases $exp(i\pi/n)$ are allowed: this would mean that all CDs are in the same position. The rational $SL(2, C)$ matrices whose determinant is zero modulo p form a group assignable to finite field andit might be that for some values of p this group is exceptionally large. $SL(2, C)$ defines also the covering group of conformal symmetries of sphere.

2. For orthogonal, unitary, and symplectic groups only $n = p$, p prime allows $k > 0$ and genuine p-adicity. Since $SO(3, 1)$, $SO(3)$, $SU(2)$ and $SU(3)$ should alow p-adicization this selects CDs with size scale characterized by prime p.

3. For orthogonal, unitary, and symplectic groups one obtains non-trivial solutions to the unitarity conditions only if the highest power of p corresponds quantum image of a vector with zero norm modulo p as follows from the basic properties of quantum arithmetics.

   (a) In the case of $SO(3)$ one has the condition

   $$\sum_{i=1}^{3} x_i^2 = 1 + k \times p \qquad (21.4.1)$$

   Note that this condition can degenerate to a condition stating that a sum of two squares is multiple of prime. As noticed the prime must be large and $x_i^2 < p$ holds true.

   (b) For the covering group $SU(2)$ of $SO(3)$ one has the condition

   $$\sum_{i=1}^{4} x_i^2 = 1 + k \times p \qquad (21.4.2)$$

   since two complex numbers for the row of SU(2) matrix correspond to four real numbers.

(c) For $SU(3)$ one has the condition

$$\sum_{i=1}^{6} x_i^2 = 1 + k \times p \tag{21.4.3}$$

corresponding to 3 complex numbers defining the row of SU(3) matrix.

What can one say about these conditions? The first thing to look is whether the conditions can be satisfied at all. Second thing to look is the number of solutions to the conditions.

21.4.4 Orthogonality conditions for $SO(3)$

The conditions for $SO(3)$ are certainly the strongest ones so that it is reasonable to study this case first.

1. One must remember that there are also integers -in particular primes- allowing representation as a sum of two squares. For instance, Fermat primes whose number is very small, allow representation $F_n = 2^+1$. More generally, Fermat's theorem on sums of two squares states that and odd prime is expressible as sum of two squares only if it satisfies $p \ mod \ 4 = 1$. The second possibility is $p \ mod \ 4 = 3$ so that roughly one half of primes satisfy the $p \ mod \ 4 = 1$ condition: Mersenne primes do not satisfy it.

   The more general condition giving sum proportional to prime is satisfied for all $n = k^2 l$, $k = 1, 2, ...$

2. For the sums of three non-vanishing squares one can use the well-known classical theorem stating that integer n can be represented as a sum of three squares (see `http://tinyurl.com/y6vkccv7`) [A47] only if it is it not of the form

$$n = 2^{2r}(8k+7) \tag{21.4.4}$$

   For instance, squares of odd integers are of form $8k+1$ and multiplied by any power of two satisfy the condition of being expressible as a sum of three squares.

   If n satisfies (does not satisfy) this condition then nm^2 satisfies (doe not satisfy) it for any m this since m^2 gives some power of 2 multiplied by a $8k+1$ type factor so that one can say that square free odd integers for which the condition $n \neq 7 \pmod 8$ generate this set of integers. Note that the integers representable as sums of three non-vanishing squares do not allow a representation using two squares. The product of odd primes $p_1 = 8m_1 + k_1$ and $p_1 = 8m_2 + k_2$ fails to satisfy the condition only if one has $k_1 = 3$ and $k_2 = 5$. The product of n primes $p_i = 8m_i + k_i$ must satisfy the condition $\prod k_i \neq 7 \pmod 8$ in order to serve as a generating square free prime.

   In the recent case one must have $n \ mod \ p = 1$. For Mersenne primes $m = 1 + kM_n$ allows representation as a sum of three squares for most values of k. In particular, for $k = 1$ one obtains $m = 2^n$ allowing at least the representation $m = 2^{n-1} + 2^{n-1}$. One must also remember that all that is needed is that sufficiently small multiples of Mersenne primes correspond to large value of $r_3(n)$ if the proposed idea has any sense.

21.4.5 Number theoretic functions $r_k(n)$ for $k = 2, 4, 6$

The number theoretical functions $r_k(n)$ telling the number of vectors with length squared equal to a given integer n are well-known for $k = 2, 3, 4, 6$ and can be used to gain information about the constraints posed by the existence of quantum groups $SO(2)$, $SO(3)$, $SU(2)$ and $SU(3)$. In the following the easy cases corresponding to $k = 2, 4, 6$ are treated first and after than the more difficult case $k = 3$ is discussed. For the auxiliary function the reader can consult to the Appendix.

The behavior of $r_2(n)$

$r_2(n)$ gives information not only about quantum $SO(2)$ but also about $SO(3)$ since 2-D vectors define 3-D vectors in an obvious manner. The expression for $r_2(n)$ is given by

$$r_2(n) = \sum_{d|n}\chi(d) \ , \ \chi(d) = \left(\frac{-4}{d}\right) \ . \qquad (21.4.5)$$

$\chi(d)$ is so called principal character defined in appendix. For $n = 1 + M_k = 2^k$ only powers of 2 and 1 divide n and for even numbers principal character vanishes so that one obtains $r_2(1+M_k) = \chi(1) = 1$. This corresponds to the representation $2^k = 2^{k-1} + 2^{k-1}$.

The behavior of $r_4(n)$

The expression for $r_4(n)$ reads as

$$r_4(n) = \left\{ \begin{array}{ll} 8\sigma(n) & \text{if } n \text{ is odd } , \\ 24\sigma(m) & \text{if } n = 2^{\nu}m, \ m \text{ odd } . \end{array} \right. \ . \qquad (21.4.6)$$

For $n = M_k + 1 = 2^k$ one has $r_4(n) = 24\sigma(1) = 24$.

The asymptotic behavior of σ function is known so that it is relatively easy to estimate the behavior of $r_4(n)$. The behavior involves random looking local fluctuation which can be understood as reflective the multiplicative character implying correlation between the values associated with multiples of a given prime.

The behavior of $r_6(n)$

The analytic expression for $r_6(n)$ is given by

$$\begin{aligned} r_6(n) &= \sum_{d|n}\left[16\chi(\frac{n}{d}) - 4\chi(d)\right] d^2 \ , \\ \chi(n) &= \left(\frac{-4}{n}\right) = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{array} \right. \end{aligned} \qquad (21.4.7)$$

For $n = M_k + 1 = 2^k$ this gives $r_6(n) = 12 \times 2^{2k} - 4$ so that the number of representation is very large for large Mersenne primes.

21.4.6 What can one say about the behavior of r_3?

The proportionality of $r_3(D)$ to the order of $h(-D)$ (see `http://tinyurl.com/23sp45v`) [A7] of the ideal class group (see `http://tinyurl.com/cbxkhge`) [A45] for quadratic extensions of rationals [A7] inspires some conjectures.

1. The conjecture that preferred primes p correspond to large commutative quantum groups translates to a conjecture that the order of ideal class group is large for the algebraic extension generated by $\sqrt{-p-1}$ or more generally $\sqrt{-kp-1}$ - at least for some values of k. Could suitable integer multiples primes near power of 2 - in particular Mersenne primes - be such primes? Note that only integer multiple is required by the basic argument.

2. Also some kind of approximate fractal behavior $r_k(sp) \simeq r_k(p)f_k(s)$ for some values of s analogous to that encountered for $r_4(D)$ for all values of s might hold true since $k = 3$ is a critical transition dimension between $k = 2$ and $k = 3$. In particular, an approximate periodicity in octaves of primes might hold true: $r_k(2^sp) \simeq r_k(p)$: this would support p-adic length scale hypothesis and make the commutative quantum group large.

Expression of r_3 in terms of class number function

To proceed one must have an explicit expression for the class number function $h(D)$ and the expression of r_3 in terms of $h(D)$.

1. The expression for $h(D)$ discussed in the Appendix reads as gives

$$h(-D) = -\frac{1}{D}\sum_1^D r \times \left(\frac{-D}{r}\right) . \quad (21.4.8)$$

   The symbols$(\left(\frac{-D}{r}\right)$ are Dirichlet and Kronecker symbols defined in the Appendix. Note that for $D = M_k + 1 = 2^k$ the algebraic extension in question reduces to that generated by $\sqrt{-2}$ so that the algebraic extension is definitely special.

2. One can express $r_3(|D|)$ in terms of $h(D)$ as

$$r_3(|D|) = 12(1 - (\frac{D}{2}))h(D) . \quad (21.4.9)$$

   Note that $(\frac{p}{2})$ refers to Kronecker symbol.

3. From Wolfram (see `http://tinyurl.com/ybl4hnp`) one finds the following expressions of $r_3(n)$ for square free integers.

$$\begin{array}{ll} r_3(n) = 24h(-n) & n = 3 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 12h(-4n) & n = 1, 2, 5, 6 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 0 & n = 7 \ (\mathrm{mod}\ 8) \ . \end{array} \quad (21.4.10)$$

4. The generating function for r_3 (see `http://tinyurl.com/ybl4hnp`) [A80] is third power of theta function θ_3.

$$\sum_{n\geq 0} r_3(n)x^n = \theta_3^3(n) = 1 + 6x + 12x^2 + 8x^3 + 6x^4 + 24x^5 + 24x^6 + 12x^8 + 30x^9 + ... \quad (21.4.11)$$

   This representation follows trivially from the definition of θ function as sum $\sum_{n=-\infty}^{\infty} x^{n^2}$.

The behavior of $h(-D)$ for large arguments is not easy to deduce without numerical calculations which probably get too heavy for primes of order M_{127}. The definition involves sum of p terms labeled by $r = 1, ..., p$, and each term is a product is product of terms expressible as a product over the prime factors of of r with over all term being a sign factor. "Interference" effects between terms of different sign are obviously possible in this kind of situation and one might hope that for large primes these effects imply wild fluctuations of $r_3(p)$.

Simplified formula for $r_3(D)$

Recall that the proportionality of $r_3(|D|)$ to the ideal class number $h(D)$ is for $D < -4$ given by

$$r_3(|D|) = 12[1 - \left(\frac{D}{2}\right)]h(D) . \qquad (21.4.12)$$

The expression for the Kronecker symbol appears in the formula as well as formulas to be discussed below and reads as

$$\left(\frac{D}{2}\right) = \begin{cases} 0 & \text{if } D \text{ is even} , \\ 1 & \text{if } D = -1 \pmod 8 , \\ -1 & \text{if } D = \pm 3 \pmod 8 . \end{cases} \qquad (21.4.13)$$

The proportionality factor vanishes for $D = 2^{2r}(8m+7)$ and equals to 12 for even values of D and to 24 for $D = \pm 3 \pmod 8$.

To get more detailed information about r_3 one can begin from class number formula (see `http://tinyurl.com/yaopszpl`) [A14] for $D < -4$ reading as

$$h(D) = \frac{1}{|D|}\sum_{r=1}^{|D|} r\left(\frac{D}{r}\right) . \qquad (21.4.14)$$

Each Jacobi symbol $\left(\frac{D}{r}\right)$ decomposes to a product of Legendre and Kronecker symbols $\left(\frac{D}{p_i}\right)$ in the decomposition of odd integer r to a product of primes p_i.

For $\left(\frac{D}{p_i}\right) = 1$ p_i splits into a product of primes in quadratic extension generated by $\sqrt{D}$. If it vanishes p_i is square of prime in the quadratic extension. In the recent case neither of these options are possible for the primes involved as is easy to see by using the definition of algebraic integers. Hence one has $\left(\frac{D}{p_i}\right) = -1$ for all odd primes to transform the formula for $D < -4$ to the form

$$\begin{aligned} h(D) &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r)-\nu_2(r)} \\ &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[-\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r))} . \end{aligned}$$
$$. \qquad (21.4.15)$$

Here $\nu_2(r)$ characterizes the power of 2 appearing in r and $\Omega(r)$ is the number of prime divisors of r with same divisor counted so many times as it appears. Hence the sign factor is same for all integers r which are obtained from the same square free integer by multiplying it by a product of even powers of primes.

Consider next various special cases.

1. For even values $D < -4$ (say $D = -1 - M_n$) only odd integers r contribute to the sum since the Kronecker symbols vanish for even values of r.

$$h(D = 2d) = \frac{1}{|D|}\sum_{1 \le r < |D| \quad odd} r(-1)^{\Omega(r)}$$
$$. \qquad (21.4.16)$$

2. For $D = \pm 1 \pmod 8$, the factors $\left(\frac{D}{2}\right) = -1$ implies that one can forget the factors of 2 altogether in this case (note that for $D = -1 \pmod 8$ $r_3(|D|)$ vanishes unlike $h(D)$).

$$h(D = \pm 1 (mod\ \ 8)) \quad = \quad \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r))} \\ . \tag{21.4.17}$$

3. For $D = \pm 3 \pmod 8$, the factors $\left(\frac{D}{2}\right) = 1$ implies that one has

$$h(D = \pm 3 (mod\ \ 8)) \quad = \quad \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r) - \nu_2(r)} \\ . \tag{21.4.18}$$

The magnitudes of the terms in the sum increase linearly but the sign factor fluctuates wildly so that the value of $h(-D)$ varies chaotically but must be divisible by p and negative since $r_3(p)$ must be a positive integer.

Could thermodynamical analogy help?

For $D < -4$ $h(D)$ is expressible in terms of sign factors determined by the number of prime factors or odd prime factors modulo two for integers or odd integers $r < D$. This raises hopes that $h(D)$ could be calculated for even large values of D.

1. Consider first the case $D = \pm 1 \pmod 8)$. The function $\lambda(r) = (-1)^{\Omega(r)}$ is known as Liouville function (see `http://tinyurl.com/y883uk5d`) [A52]. From the product expansion of zeta function in terms of "prime factors" it is easy to see that the generating function for $\lambda(r)$

$$\begin{aligned} \sum_n \lambda(n) n^{-s} &= \frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\zeta_F(s)} \ , \\ \zeta_(s) &= \prod_p (1 - p^{-s})^{-1} \ , \ \zeta_F(s) = \prod_p (1 + p^{-s}) \ . \end{aligned} \tag{21.4.19}$$

Recall that $\zeta(s)$ *resp.* $\zeta_F(s)$ has a formal interpretation as partition functions for the thermodynamics of bosonic *resp.* fermionic system. This representation applies to $h(D = \pm 1 (mod 8))$.

2. For $D = 2d$ the representation is obtained just by dropping away the contribution of all even integers from Liouville function and this means division of $(1 + 2^{-s})$ from the fermionic partition function $\zeta_F(s)$. The generating function is therefore

$$\sum_{n\ \ odd} \lambda(n) n^{-s} \quad = \quad \prod_{p\ \ odd} (1 + p^{-s})^{-1} = (1 + 2^{-s}) \frac{1}{\zeta_F(s)} \ . \tag{21.4.20}$$

3. For $h(D = \pm 3(\ \ mod\ \ 8))$. One most modify the Liouville function by replacing $\Omega(r)$ by the number of odd prime factors but allow also even integers r. The generating function is now

$$\sum_n \lambda(n)(-1)^{\nu_2(n)} n^{-s} \quad = \quad \frac{1}{1 - 2^{-s}} \prod_{p\ \ odd} (1 + p^{-s})^{-1} = \frac{1}{1 - 2^{-s}} \frac{1}{\zeta_F(s)} \ . \tag{21.4.21}$$

The generating functions raise the hope that it might be possible to estimate the values of the $h(D)$ numerically for large values of D using a thermodynamical analogy.

1. $h(D)$ is obtained as a kind of thermodynamical average $\langle r(-1)^{\Omega(r)}\rangle$ for particle number r weighted by a sign factor telling the number of divisors interpreted as particle number. s plays the role of the inverse of the temperature and infinite temperature limit $s = 0$ is considered. One can also interpret this number as difference of average particle number for states restricted to contain even *resp.* odd particle number identified as the number of prime divisors with 2 and even particle numbers possibly excluded.

2. The average is obtained at temperature corresponding to $s = 0$ so that $n^{-s} = 1$ holds true identically. The upper bound $r < D$ means cutoff in the partition sum and has interpretation as an upper bound on the energy $log(r)$ of many particle states defined by the prime decomposition. This means that one must replace Riemann zeta and its analogs with their cutoffs with $n \leq |D|$. Physically this is natural.

3. One must consider bosonic system all the cases considered. To get the required sign factor one must associated to the bosonic partition functions assigned with individual primes in $\zeta(s)$ the analog of chemical potential term $exp(-\mu/T)$ as the sign factor $exp(i\pi) = -1$ transforming ζ to $1/\zeta_F$ in the simplest case.

One might hope that one could calculate the partition function without explicitly constructing all the needed prime factorizations since only the number of prime factors modulo two is needed for $r \leq |D|$.

Expression of r_3 in terms of Dirichlet L-function

It is known [A61] that the function $r_3(D)$ is proportional to Dirichlet L-function (see `http://tinyurl.com/yatdk384`) $L(1, \chi(D))$ [A21]:

$$
\begin{aligned}
r_3(|D|) &= \frac{12\sqrt{D}}{\pi}L(1,\chi(D))) \ , \\
L(s,\chi) &= \sum_{n>0}\frac{\chi(n,D)}{n^s} \ ,
\end{aligned}
\tag{21.4.22}
$$

$\chi(n, D)$ is Dirichlet character (see `http://tinyurl.com/2fuudea`) [A20] which is periodic and multiplicative function - essentially a phase factor- satisfying the conditions

$$
\begin{array}{ll}
\chi(n,D) \neq 0 & \text{if n and D have no common divisors} > 1 \ , \\
\chi(n,D) = 0 & \text{if n and D have a common divisor} > 1 \ , \\
\chi(mn,D) = \chi(m,D)\chi(n,D) \ , & \chi(m+D,D) = \chi(m,D) \ , \\
\chi(1,D) = 1 \ . &
\end{array}
\tag{21.4.23}
$$

1. $L(1, \chi(D))$ varies in average sense slowly but fluctuates wildly between certain bounds (see `http://tinyurl.com/yc879v6e`). One can say that there is local chaos.

   The following estimates for the bounds are given in [A118]:

$$
c_1(D) \ \equiv \ k_1log(log(D) < L_1(1,\chi(D)) < c_2(D) \equiv k_2log(log(D)) \ . \tag{21.4.24}
$$

   Also other bounds are represented in the article.

Could preferred integers correspond to the maxima of Dirichlet L-function?

The maxima of Dirichlet L-function are excellent candidates for the local maxima of $r_3(D)$ since $\sqrt{D}$ is slowly varying function.

1. As already found, integers $n = 1 + M_k = 2^k$ cannot represent pronounced maxima of $r_3(n)$ since there are no representation as a sum of three squares and the proportionality constant vanishes. Note that in this case the representation reduces to a representation in terms of two integers. In this special case it does not matter whether L-function has a maximum or not.

   (a) Could just the fact that the representation for $n = 1 + M_k = 2^k$ in terms of three primes is not possible, select Mersenne primes $M_n > 3$ as preferred ones? For $SU(2)$, which is covering group of $SO(3)$ the representation as a sum of four squares is possible. Could it be that the spin 1/2 character of the fermionic building blocks of elementary particles means that a representation as sum of four squares is what matters. But why the non-existence of representation of n as a sum of three squares might make Mersenne primes so special?

2. Could also primes near power of 2 define maxima? Unfortunately, the calculations of [A118] involve averaging, minimum, and maximum over 10^6 integers in the ranges $n \times 10^6 < D < (n+1) \times 10^6$, so that they give very slowly varying maximum and minimum.

3. Could Dirichlet function have some kind of fractal structure such that for any prime one would have approximate factorization? The naïvest guesses would be $L(1, \chi_{kl}) \simeq f_1(k)L(1, \chi_l)$ with $k = 2^s$. This would mean that the primes for which $D(1, \chi_p)$ is maximum would be of special importance.

4. p-Adic fractality and effective p-adic topology inspire the question whether L-function is p-adic fractal in the regions above certain primes defining effective p-adic topology $D(1, \chi_{p^k}) \simeq f_1(k)DK(1, \chi_p)$ for preferred primes.

Interference as a helpful physical analogy?

Could one use physical analog such as interference for the terms of varying sign appearing in L-function to gain some intuition about the situation?

1. One could interpret L-function as a number theoretic Fourier transform with D interpreted as a wave vector and one has an interference of infinite number of terms in position space whose points are labelled by positive integers defining a half -lattice with unit lattice length. The magnitude of n: th summand $1/n$ and its phase is periodic with period $D = kp$. The value of the Fourier component is finite except for $D = 0$ which corresponds to Riemann Zeta at $s = 1$. Could this means that the Fourier component behaves roughly like $1/D$ apart from an oscillating multiplicative factor.

2. The number theoretic counterparts of plane waves are special in that besides D-periodicity they are multiplicative making thema lso analogs of logarithmic waves. For ordinary Fourier components one additivity in the sense that $\Psi(k_1 + k_2) = \Psi(k_1)\Psi(k_2)$. Now one has $\Psi(k_1k_2) = \Psi(k_1)\Psi(k_2)$ so that $log(D)$ corresponds to ordinary wave vector. p-Adic fractality is an analog for periodicity in the sense of logarithmic waves so that powers rather than integer multiples of the basic scale define periodicity. Could the multiplicative nature of Dirichlet characters imply p-adic - or at least 2-adic - fractality, which also means logarithmic periodicity?

3. Could one say that for these special primes a constructive interference takes place in the sum defining the L-function. Certainly each prime represents the analog of fundamental wavelength whose multiples characterize the summands. In frequency space this would mean fundamental frequency and its sub-harmonics.

Period doubling as physical analogy?

1. For $k = 4$ all scales are present because of the multiplicative nature of σ function. Now only the Dirichlet characters are multiplicative which suggests that only few integers define preferred scales? Prime power multiples of the basic scale are certainly good candidates for preferred scales but amongst them must be some very special prime powers. $p = 2$ is the only even prime so that it is the first guess.

2. Could the system be chaotic or nearly chaotic in the sense of period doubling so that octaves of preferred primes interfere constructively? Why constructively? Could complete chaos -interpreted as randomness- correspond to a destructive interference and minimum of the L-function?

3. What about scalings by squares of a given prime? It seems that these scalings cannot be excluded by any simple argument. The point is that $r_3(n)$ contains also the factor $\sqrt{n}$ which must transform by integer in the scaling $n \rightarrow kn$. Therefore k must be power of square.

This leaves two extreme options. Both options are certainly testable by simple numerical calculations for small primes. For instance one can use generating function $\theta_3^3(x) = \sum r_3(n)x^n$ to kill the conjectures.

1. The first option corresponds to scalings by all integers that are squares. This option is also consistent with the condition $n \neq 2^k(8m+7)$ since both the scaling by a square of odd prime and by a square of 2 preserve this condition since one has $n^2 = 1 \pmod 8$ for odd integers. This is also consistent with the finding that $r_3(n) = 1$ holds true only for a finite number of integers. A simple numerical calculation for the sums of 3 squares of 16 first integers demonstrates that the conjecture is wrong.

2. The second option corresponds only to the scaling by even powers of two and is clearly the minimal option. This period quadrupling for n corresponds to period doubling for the components of 3-vector. A calculation of the sums of squares of the 16 first integers demonstrates that for $n = 3, 6, 9, 11, ..$ the conjecture the value of $r_3(n)$ is same so that the conjecture might hold true! If it holds true then Dirichlet L-function should suffer scaling by 2^{-r} in the scaling $n \rightarrow 2^{2r}n$. The integer solutions for n scaled by 2^r are certainly solutions for $2^{2r}n$. Quite generally, one has $r_3(m^2n) \geq r_3(n)$ for any integer m. The non-trivial question is whether some new solutions are possible when the scaling is by 2^{2r}.

   A simple argument demonstrates that there cannot be any other solutions to $\sum_{n_i=1}^{3} m_i^2 = 2^{2r}n$ than the scaled up solutions $m_i = 2n_i$ obtained from $\sum_{n_i=1}^{3} n_i^2 = n$. This is seen by noticing that non-scaled up solutions must contain $1, 2, or 3$ integers m_i, which are odd. For this kind of integers one has $m^2 = 1 \pmod 4$ so that the sum ($\sum_i m_i^2$)= 1, 2, or 3 (mod 4) whereas the right hand side vanishes mod 4.

3. If D is interpreted as wave vector, period quadrupling could be interpreted as a presence of logarithmic wave in wave-vector space with period $2log(2)$.

Does 2-adic quantum arithmetics prefer CD scales coming as powers of two?

For $p = 2$ quantum arithmetics looks singular at the first glance. This is actually not the case since odd quantum integers are equal to their ordinary counterparts in this case. This applies also to powers of two interpreted as 2-adic integers. The real counterparts of these are mapped to their inverses in canonical identification.

Clearly, odd 2-adic quantum quantum rationals are very special mathematically since they correspond to ordinary rationals. It is fair to call them "classical" rationals. This special role might relate to the fact that primes near powers of 2 are physically preferred. CDs with $n = 2^k$ would be in a unique position number theoretically. This would conform with the original - and as such wrong - hypothesis that only these time scales are possible for CDs. The preferred role of powers of two supports also p-adic length scale hypothesis.

The discussion of the role of quantum arithmetics in the construction of generalized Feynman diagrams in [K35] allows to understand how for a quantum arithmetics based on particular prime p

particle mass squared - equal to conformal weight in suitable mass units- divisible by p appears as an effective propagator pole for large values of p. In p-adic mass calculations real mass squared is obtained by canonical identification from the p-adic one. The construction of generalized Feynman diagrams allows to understand this strange sounding rule as a direct implication of the number theoretical universality realized in terms of quantum arithmetics.

21.5 How Quantum Arithmetics could affect basic TGD and TGD inspired view about life and consciousness?

The vision about real and p-adic physics as completions of rational physics or physics associated with extensions of rational numbers is central element of number theoretical universality. The physics in the extensions of rationals are assigned with the interaction of real and p-adic worlds.

1. At the level of the world of classical worlds (WCW) the points in the intersection of real and p-adic worlds are 2-surfaces defined by equations making sense both in real and p-adic sense. Rational functions with polynomials having rational (or algebraic coefficients in some extension of rationals) would define the partonic 2-surface. One can of course consider more stringent formulations obtained by replacing 2-surface with certain 3-surfaces or even by 4-surfaces.

2. At the space-time level the intersection of real and p-adic worlds corresponds to rational points common to real partonic 2-surface obeying same equations (the simplest assumption). This conforms with the vision that finite measurement resolution implies discretization at the level of partonic 2-surfaces and replaces light-like 3-surfaces and space-like 3-surfaces at the ends of causal diamonds with braids so that almost topological QFT is the outcome.

How does the replacement of rationals with quantum rationals modify quantum TGD and the TGD inspired vision about quantum biology and consciousness?

21.5.1 What happens to p-adic mass calculations and Quantum TGD?

The basic assumption behind the p-adic mass calculations and all applications is that one can assign to a given partonic 2-surface (or even light-like 3-surface) a preferred p-adic prime (or possibly several primes).

The replacement of rationals with quantum rationals in p-adic mass calculations implies effects, which are extremely small since the difference between rationals and quantum rationals is extremely small due to the fact that the primes assignable to elementary particles are so large ($M_{127} = 2^{127} - 1$ for electron). The predictions of p-adic mass calculations remains almost as such in excellent accuracy. The bonus is the uniqueness of the canonical identification making the theory unique.

The problem of the original p-adic mass calculations is that the number of common rationals (plus possible algebraics in some extension of rationals) is same for all primes p. What is the additional criterion selecting the preferred prime assigned to the elementary particle?

Could the preferred prime correspond to the maximization of number theoretic negentropy for a quantum state involved and therefore for the partonic 2-surface by quantum classical correspondence? The solution ansatz for the Kähler-Dirac equation indeed allows this assignment [K100]: could this provide the first principle selecting the preferred p-adic prime? Here the replacement of rationals with quantum rationals improves the situation dramatically.

1. Quantum rationals are characterized by a quantum phase $q = exp(i\pi/p)$ and thus by prime p (in the most general but not so plausible case by an integer n). The set of points shared by real and p-adic partonic 2-surfaces would be discrete also now but consist of points in the algebraic extension defined by the quantum phase $q = exp(i\pi/p)$.

2. What is of crucial importance is that the number of common quantum rational points of partonic 2-surface and its p-adic counterpart would depend on the p-adic prime p. For some primes p would be large and in accordance with the original intuition this suggests that the

interaction between p-adic and real partonic 2-surface is stronger. This kind of prime is the natural candidate for the p-adic prime defining effective p-adic topology assignable to the partonic 2-surface and elementary particle. Quantum rationals would thus bring in the preferred prime and perhaps at the deepest possible level that one can imagine.

21.5.2 What happens to TGD inspired theory of consciousness and quantum biology?

The vision about rationals as common to reals and p-adics is central for TGD inspired theory of consciousness and the applications of TGD in biology.

1. One can say that life resides in the intersection of real and p-adic worlds. The basic motivation comes from the observation that number theoretical entanglement entropy can have negative values and has minimum for a unique prime [K53]. Negative entanglement entropy has a natural interpretation as a genuine information and this leads to a modification of Negentropy Maximization Principle (NMP) allowing quantum jumps generating negentropic entanglement. This tendency is something completely new: NMP for ordinary entanglement entropy would force always a state function reduction leading to unentangled states and the increase of ensemble entropy.

   What happens at the level of ensemble in TGD Universe is an interesting question. The pessimistic view (see `http://tinyurl.com/ybm6rxz3`) [K53], [L11] is that the generation of negentropic entanglement (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book) is accompanied by entropic entanglement somewhere else guaranteeing that second law still holds true. Living matter would be bound to pollute its environment if the pessimistic view is correct. I cannot decide whether this is so: this seems like deciding whether Riemann hypothesis is true or not or perhaps unprovable.

2. Replacing rationals with quantum rationals however modifies somewhat the overall vision about what life is. It would be quantum rationals which would be common to real and p-adic variants of the partonic 2-surface. Also now an algebraic extension of rationals would be in question so that the proposal would be only more specific. The notion of number theoretic entropy still makes sense so that the basic vision about quantum biology survives the modification.

3. The large number of common points for some prime would mean that the quantum jump transforming p-adic partonic 2-surface to its real counterpart would take place with a large probability. Using the language of TGD inspired theory of consciousness one would say that the intentional powers are strong for the conscious entity involved. This applies also to the reverse transition generating a cognitive representation if p-adic-real duality induced by the canonical identification is true. This conclusion seems to apply even in the case of elementary particles. Could even elementary particles cognize and intend in some primitive sense? Intriguingly, the secondary p-adic time scale associated with electron defining the size of corresponding CD is.1 seconds defining the fundamental 10 Hz bio-rhythm. Just an accident or something very deep: a direct connection between elementary particle level and biology perhaps?

21.6 Appendix: Some Number Theoretical Functions

Explicit formulas for the number $r_k(n)$ of the solutions to the conditions $\sum_1^k x_k^2 = n$ are known and define standard number theoretical functions closely related to the quadratic algebraic extensions of rationals. The formulas for $r_k(n)$ require some knowledge about the basic number theoretical functions to be discussed first. Wikipedia contains a good overall summary about basic arithmetic functions (see `http://tinyurl.com/23sp45v`) [A7] including the most important multiplicative and additive arithmetic functions.

Included are character functions which are periodic and multiplicative: examples are symbols (m/n) assigned with the names of Legendre, Jacobi, and Kronecker as well as Dirichlet character.

21.6.1 Characters And Symbols

Principal character

Principal character (see `http://tinyurl.com/23sp45v`) [A7] $\chi(n)$ distinguishes between three situations: n is even, $n = 1 \pmod 4$, and $n = 3 \pmod 4$ and is defined as

$$\chi(n) = \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if n=0 (mod 2)} \\ +1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases} \tag{21.6.1}$$

Principal character is multiplicative and periodic with period $k = 4$.

Legendre and Kronecker symbols

Legendre symbol $\left(\frac{n}{p}\right)$ characterizes what happens to ordinary primes in the quadratic extensions of rationals. Legendre symbol is defined for odd integers n and odd primes p as

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } n = 0 \text{ (mod p)} \ , \\ +1 & \text{if } n \neq 0 \text{ (mod p) and } n = x^2 \text{ (mod p)} \ , \\ -1 & \text{if there is no such } x \ . \end{cases} \tag{21.6.2}$$

When D is so called fundamental discriminant- that is discriminant $D = b^2 - 4c$ for the equation $x^2 - bx + c = 0$ with integer coefficients b, c, Legendre symbols tells what happens to ordinary primes in the extension:

1. $\left(\frac{D}{p}\right) = 0$ tells that the prime in question divides D and that p is expressible as a square in the quadratic extension of rationals defined by $\sqrt{D}$.

2. $\left(\frac{D}{p}\right) = 1$ tells that p splits into a product of two different primes in the quadratic extension.

3. For $\left(\frac{D}{p}\right) = -1$ the splitting of p does not occur.

This explains why Legendre symbols appear in the ideal class number $h(D)$ characterizing the number of different splittings of primes in quadratic extension.

Legendre symbol can be generalized to Kronecker symbol well-defined for also for even integers D. The multiplicative nature requires only the definition of $\left(\frac{n}{2}\right)$ for arbitrary n:

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \ , \\ (-1)^{\frac{n^2-1}{8}} & \text{if n is odd} \ . \end{cases} \tag{21.6.3}$$

Kronecker symbol for $p = 2$ tells whether the integer is even, and if odd whether $n = \pm 1$ (mod 8) or $a = \pm 3$ (mod 8) holds true. Note that principal character $\chi(n)$ can be regarded as Dirichlet character $\left(\frac{-4}{n}\right)$.

For $D = p$ quadratic resiprocity (see `http://tinyurl.com/yz2okpf`) [A67] allows to transform the formula

$$\chi_p(n) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\left(\frac{p}{n}\right) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\prod_{p_i|n}\left(\frac{p}{p_i}\right) \ . \tag{21.6.4}$$

Dirichlet character

Dirichlet character (see http://tinyurl.com/2fuudea) [A20] $\left(\frac{a}{n}\right)$ is also a multiplicative function. Dirichlet character is defined for all values of a and odd values of n and is fixed completely by the conditions

$$\begin{aligned} &\chi_D(k) = \chi_D(k+D) \ , \qquad \chi_D(kl) = \chi_D(k)\chi_D(l) \ , \\ &\text{If } D|n \text{ then } \chi_D(n) = 0 \ , \quad \text{otherwise } \chi_D(n) \neq 0 \ . \end{aligned} \tag{21.6.5}$$

Dirichlet character associated with quadratic residues is real and can be expressed as

$$\chi_D(n) \ = \ \left(\frac{n}{D}\right) = \prod_{p_i|D} \left(\frac{n}{p_i}\right) \ . \tag{21.6.6}$$

Here $\left(\frac{n}{p_i}\right)$ is Legendre symbol described above. Note that the primes p_i are odd. $\left(\frac{n}{1}\right) = 1$ holds true by definition.

For prime values of D Dirichet character reduces to Legendre symbol. For odd integers Dirichlet character reduces to Jacobi symbol defined as a product of the Legendre symbols associated with the prime factors. For $n = p^k$ Dirichlet character reduces to $(\left(\frac{p}{n}\right))^k$ and is non-vanishing only for odd integers not divisible by p and containing only odd prime factors larger than p besides power of 2 factor.

21.6.2 Divisor Functions

Divisor functions (see http://tinyurl.com/2qyngq) [A22] $\sigma_k(n)$ are defined in terms of the divisors d of integer n with $d = 1$ and $d = n$ included and are also multiplicative functions. $\sigma_k(n)$ is defined as

$$\sigma_k(n) \ = \ \sum_{d|n} d^k \ , \tag{21.6.7}$$

and can be expressed in terms of prime factors of n as

$$\sigma_k(n) \ = \ \sum_i (p_i^k + p_i^{2k} + ... + p_i^{a_i k}) \ . \tag{21.6.8}$$

$\sigma_1 \equiv \sigma$ appears in the formula for $r_4(n)$.

The figures in Wikipedia (see http://tinyurl.com/y8vrrhx9) [A35] give an idea about the locally chaotic behavior of the sigma function.

21.6.3 Class Number Function And Dirichlet L-Function

In the most interesting $k = 3$ case the situation is more complicated and more refined number theoretic notions are needed. The function $r_3(D)$ is expressible in terms of so called class number function $h(n)$ characterizing the order of the ideal class group for a quadratic extension of rationals associated with D, which can be negative. In the recent case $D = -p$ is of special interest as also $D = -kp$, especially so for $k = 2^r$. $h(n)$ in turn is expressible in terms of Dirichlet L-function so that both functions are needed.

1. Dirichlet L-function (see http://tinyurl.com/yatdk384) [A21] can be regarded as a generalization of Riemann zeta and is also conjectured to satisfy Riemann hypothesis. Dirichlet L-function can be assigned to any Dirichlet character χ_D appearing in it as a function valued parameter and is defined as

$$L(s,\chi_D) = \sum_n \frac{\chi_D(n)}{n^s} \ . \qquad (21.6.9)$$

For $\chi_1 = 1$ one obtains Riemann Zeta. Also L-function has expression as product of terms associated with primes converging for $Re(s) > 1$, and must be analytically continued to get an analytic function in the entire complex plane. The value of L-function at $s = 1$ is needed and for Riemann zeta this corresponds to pole. For Dirichlet zeta the value is finite and $L(1,\chi_{-n})$ indeed appears in the formula for $r_3(n)$.

2. Consider next what class number function h means.

   (a) Class number function (see `http://tinyurl.com/yaopszpl`) [A14] characterizes quadratic extensions defined by $\sqrt{D}$ for both positive and negative values of D. For these algebraic extensions the prime factorization in the ring of algebraic integers need not be unique. Algebraic integers are complex algebraic numbers which are not solutions of a polynomial with coefficients in Z and with leading term with unit coefficient. What is important is that they are closed under addition and multiplication. One can also defined algebraic primes. For instance, for the quadratic extension generated by $\sqrt{\pm 5}$ algebraic integers are of form $m + n\sqrt{\pm 5}$ since $\sqrt{\pm 5}$ satisfies the polynomial equation $x^2 = \pm 5$.

   Given algebraic integer n can have several prime decompositions: $n = p_1p_2 = p_3p_4$, where p_i algebraic primes. In a more advance treatment primes correspond to ideals of the algebra involved: obviously algebra of algebraic integers multiplied by a prime is closed with respect to multiplication with any algebraic integer.

   A good example about non-unique prime decomposition is $6 = 2\times 3 = (1+\sqrt{-5})(\sqrt{1-\sqrt{-5}}$ in the quadratic extension generated by $\sqrt{-5}$.

   (b) Non-uniqueness means that one has what might be called fractional ideals: two ideals I and J are equivalent if one can write $(a)J = (b)I$ where (n) is the integer ideal consisting of algebraic integers divisible by algebraic integer n. This is the counterpart for the non-uniqueness of prime decomposition. These ideals form an Abelian group known as ideal class group (see `http://tinyurl.com/cbxkhge`) [A45]. For algebraic fields the ideal class group is always finite.

   (c) The order of elements of the ideal class group for the quadratic extension determined by integer D can be written as

$$h(D) = \frac{1}{D}\sum_1^{|D|} r\times\left(\frac{D}{r}\right) \ , \quad D < -4 \ . \qquad (21.6.10)$$

   Here $\left(\frac{D}{r}\right)$ denotes the value of Dirichlet character. In the recent case D is negative.

3. It is perhaps not completely surprising that one can express $r_3(|D|)$ characterizing quadratic form in terms of $h(D)$ charactering quadratic algebraic extensions as

$$r_3(|D|) = 12(1-\left(\frac{D}{2}\right))h(D) \ , \quad D < -4 \ . \qquad (21.6.11)$$

Here $\left(\frac{D}{2}\right)$ denotes Kronecker symbol.

Chapter 22

Quantum Adeles

22.1 Introduction

Quantum arithmetics [K67] is a notion which emerged as a possible resolution of long-lasting challenge of finding mathematical justification for the canonical identification mapping p-adics to reals.

22.1.1 What quantum p-adics could be?

The basic idea is that p-adic numbers could have quantum counterparts. This idea has developed through several twists and turns and involved moments of almost despair.

The "*less interesting*" but realistic option

There following proposal seems to be the realistic one and was indeed proposed first. I called it *less interesting*.

1. The earlier work with quantum arithmetics [K67, K5] suggests a modification of p-adic integers numbers by replacing the coefficients a_n p-adic pinary expansions with their quantum counterparts $(a_n)_q$, $q_m = exp(i\pi/m)$. The non-negativity of quantum p-adics is achieved for $m = p$. What is nice is that quantum groups and p-adicity would be very closely related.

2. This definition does not respect the decomposition of integer to prime numbers. One can achieve this by mapping the primes in prime decomposition to their quantum counterparts in the same manner. Products are mapped to products but product of images is not image of product. Sum does not go to sum. Even the pinary coefficients can be decomposed to primes and mapped to their quantum counterparts.

   There are many ways to define quantum integers, and each of them could be seen as carrying information about the number theoretic anatomy of the integer.

3. One can also map rational numbers to their quantum counterparts by mapping numerator and demoninators with no common factors in this manner to quantum integers.

4. One can generalize this description to m-adics expressible in powers of general m defining q_m and obtain as special case $m = p^k$ case. The field property is now lost.

The replacement of numbers with sequences of arithmetic operations and integers with Hilbert spaces

The first attempt to solve the problems related to the definition of $+_q$ and $\times_q$ was inspired by zero energy ontology and led to a replacement of numbers with sequences of arithmetic operations describable by analogs of Feyman diagrams. The comparison with generalized Feynman diagrams allowed to realize how "less-interesting" option could become "interesting": numbers could be replaced with Hilbert spaces and all the conditions would be trivially satisfied! Of course, this can be argued to be mere formal mathematical game but one can also ask whether this might have something to do with physics.

1. The notion of generalized Feynman diagram suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted.

   This kind of reduction to tree diagrams is an old proposal that I gave up as too "romantic" [K12] but which re-emerged from zero energy ontology where the assumption that also internal lines (wormhole throats) are massless and on shell although the sign of energy can be negative, poses extremely powerful kinematical constraints reducing the number of Feynman diagrams. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands.

2. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to $\otimes$ and $\oplus$ obeying also associativity and distributivity and could quantum arithmetics for Hilbert spaces apply to quantum TGD? If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces and quantum states - and in the vertices the incoming states fuse to a direct sum $\oplus$ or tensor product $\otimes$!

3. One could assign to integer n a multiple covering defined by the state basis of n-dimensional Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests cyclic group Z_n. The non-trivial quantum Galois group would thus emerge also for the "less-interesting" but non-risky option so that the conservative approach might work after all!

4. The Hilbert spaces in question could represent physical states - in p-adic context one could speak about cognitive representations. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of embedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the embedding space. Hilbert spaces can be identified as function spaces associated with the discrete point sets of the covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

5. Quantum arithmetics would be arithmetics of Hilbert spaces and of states assigned to them. This arithmetics allows also extension to rationals and algebraic numbers, and even the Hilbert space variants of algebraic complex numbers, quaternions and octonions can be considered. Also quantum adeles can be defined in terms of Hilbert spaces. These generalization are expected to be crucial for the understanding of generalized Feynman diagrams.

22.1.2 Quantum TGD And Hilbert Adeles

Irrespective of whether the isomorphism holds true quantum adeles - if they exist - could provide a very powerful tool also for the formulation of quantum TGD and realize the old intuition that AGG is a symmetry group of quantum TGD [K46].

1. The innocent TGD inspired question posed already earlier is whether the fusion of real and various p-adic physics together could be realized in terms of adeles providing - if not anything else - an ingenious book keeping device allowing to do real physics and all p-adic

physics simultaneously by replacing the whole stuff by single letter A! Now however replaced with A_q.

2. The function spaces associated with quantum adeles decompose to tensor products of function spaces associated with the completions of rationals and one can speak about rational entanglement between different number fields. Rational entanglement can be generalized to algebraic entanglement when one replaces rationals with their algebraic extension and primes with corresponding primes. Could it be that this rational/algebraic entanglement is the rational/algebraic suggested to characterize living matter and to which one can assign negative entanglement entropy having interpretation as a measure for genuine information?

3. The basic vision of TGD inspired quantum bio-physics is that life resides in the intersection of real and p-adic worlds in which rational/algebraic entanglement is natural. One can argue that rational and algebraic entanglement are unstable and that it cannot be realized in any system - living or not. The objection is that Negentropy Maximization Principle (NMP [K53]) favors the generation of negentropic entanglement and once formed between two material systems described by real numbers is stable. Could it be that the mechanism producing this kind of entanglement is the necessary rational/algebraic entanglement between different number fields - between matter and mind one might say - and that quantum jumps transforming p-adic space-time sheets to real ones generates rational/algebraic entanglement between systems consisting of matter. Intention transforming to action would be the interpretation for this process.

4. The construction of generalized Feynman diagrams leads to a picture in which propagator lines give rise to expressions in various p-adic number fields and vertices naturally to multi-p-adic expressions involving p-adic primes of incoming lines. This picture has also natural generalization to quantum variants of p-adic numbers and the expressions are eventually mapped to real numbers by canonical identification induced by $p \rightarrow 1/p$ for quantum rationals appearing in various lines and in vertices of the generalized Feynman diagram. This construct would naturally to a tensor product of state spaces assignable to different p-adic primes and also reals so that M-matrix elements would be naturally in this tensor product. Note that the function space associated with (quantum) adeles is naturally tensor product of functions spaces associated with Cartesian factors of the adele ring with rationals defining the entanglement coefficients. All this of course generalizes by replacing rationals by their algebraic extensions.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

22.2 Earlier Attempts To Construct Quantum Arithmetics

Quantum arithmetics [K67] provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations [K59].

In [K67] two basic options for quantum arithmetics were discussed. For option I products of integers are mapped to products of quantum integers achieved by mapping primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$. For option II this is not the case.

In this chapter a third and much more general option is discussed. In order to give the needed context, the options discussed in [K67] are however briefly discussed first.

22.2.1 Overall View About Variants Of Quantum Integers

The starting point of quantum arithmetics is the map $n \rightarrow n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \rightarrow p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

Clearly, many variants of quantum integers can be found and it is difficult to decide which of them - if any - has interesting from TGD point of view.

1. If one wants to really model something using quantum integers, the second options is perhaps the realistic one: the reason is that the decomposition into prime factors requires a lot of computation time.

2. A second fictive criterion would be whether the definition is maximally general. Does the definition makes sense for infinite primes? The simplest infinite primes at the first level of hierarchy have physical interpretation as many-particle states consisting of bosons and fermions, whose momentum values correspond to finite primes. The interpretation generalizes to higher levels of the hierarchy. A simple argument show that the option keeping information about prime factorization of the p-adic number allowing also infinite primes as factors makes sense only if prime factors are not expanded in series with respect to the prime p and if p does not correspond to a fermionic mode. The quantum map using prime root of unity therefore makes sense for all but fermionic primes. The presence of exceptional primes in number theory is basic phenomenon: typically they correspond to primes for which factorization is not unique in algebraic extension.

Two options for quantum integers

Two options for definining quantum arithmetics are discussed on [K67]: Options I and II. These are not the only one imaginable but represent kind of diametrical opposites. The two options are defined in the following manner.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to $l\ modp$ (to guarantee positivity of the quantum integer) decomposed into primes $l < p$ and these in turn to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of the product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a+b=\sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n+b_n)p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n=\sum_k n_k p^{kr}$, $n_k < p^r$, $r=1,2....$ p^k would serve as cutoff.

4. Non-negativity of quantum primes is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [K5]. One could also replace quantum prime by its square without losing the map of products to products.

5. At the limit when the quantum phase approaches to unit, ordinary quantum integers with p-adic norm 1 approach to ordinary integers in real sense and ordinary arithmetics results. Ordinary integers in real sense are obtained for option II when the coefficients of the pinary expansion of n are much smaller than p and p approaches infinity. Same is true for option I if the prime factors of the integer are much smaller than p.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - \overline{q}^n}{q-\overline{q}} \quad . \tag{22.2.1}$$

   Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$\begin{aligned} p_q &= \frac{q^p - \overline{q}^p}{q-\overline{q}} \\ n_q &= \prod_p p_q^{n_p} \quad \text{for} \quad n=\prod_p p^{n_p} \quad . \end{aligned} \tag{22.2.2}$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$q = exp(\eta)exp(i\pi/m) \quad . \tag{22.2.3}$$

   The quantum map is 1-1 for a non-vanishing value of η and the limit $m \to \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteeing a unique definition of quantum sum. In the p-adic context with $m=p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem $a^{p-1}\ mod\ p = 1$ poses constraints since $p-1$: th root of unity exists as ordinary p-adic number. Hence $m = p-1$: th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers.

4. If q reduces to quantum phase, the $n = 0, 1, -1$ are fixed points of $n \rightarrow n_q$ for ordinary integers so that one could say that all these numbers are common to integers and quantum integers for all values of $q = exp(i\pi/m)$. For p-adic integers $-1 = (p-1)(1 + p + p^2 + ..$ is problematic. Should one use direct formula mapping it to -1 or should one map the expansion to $(p-1)_q(1 + p + p^2 + ...)$? This option looks more plausible.

   (a) For the first option the images under canonical can have both signs and can form a field. For the latter option would obtain only non-negative quantum p-adics for ordinary p-adic numbers. They do not form a field. For algebraic extensions of p-adics by roots of unity one can obtain more general complex numbers as quantum images. For the latter option also the quantum p-adic numbers projecting to a given prime l regarded as p-adic integer form a finite set and correspond to all expansions $l = \sum l_k p^k$ where l_k is product of powers of primes $p_i < p$ but one can have also $l_k > p$.

   (b) Quantum integers containing only the $O(p^0)$ term in the binary expansion for a sub-ring. Corresponding quantum rationals could form a field defining a kind of covering for finite field $G(p, 1)$.

   (c) The image $I(m/n)$ of the pinary expansion of p-adic rational is different from $I(m)/I(n)$. The formula $m/n \rightarrow I(m)/I(n)$ is the correct manner to define canonical identification map. In this case the real counterparts of p-adic quantum integers do not form the analog of function fields since the numbers in question are always non-negative.

5. For p-adic rationals the quantum map reads as $m/n \rightarrow m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [A190] [K32] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one. Certainly Option I for which the quantum map is only part of canonical identification is the simpler one but for this option canonical identification respects discrete symmetries only approximately. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure and therefore much more general than that proposed in the model of Shnoll effect [K5] but gives identical predictions when the parameters characterizing the probability distribution $f(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $f(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

22.2.2 The Third Option For Quantum P-Adics

The definition of quantum p-adics discussed in this chapter replaces integers with Hilbert spaces of same dimension and + and × with direct sum ⊕ and tensor product ⊗. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions.

Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

22.3 The relation between U-Matrix and M-matrices

S-matrix is the key notion in quantum field theories. In Zero Energy Ontology (ZEO) this notion must be replaced with the triplet U-matrix, M-matrix, and S-matrix. U-matrix realizes unitary time evolution in the space for zero energy states realized geometrically as dispersion in the moduli space of causal diamonds (CDs) leaving second boundary (passive boundary) of CD and states at it fixed.

This process can be seen as the TGD counterpart of repeated state function reductions leaving the states at passive boundary unaffected and affecting only the member of state pair at active boundary (Zeno effect) [K53]. In TGD inspired theory of consciousness self corresponds to the sequence of these state function reductions [K96, K7, K78]. M-matrix describes the entanglement between positive and negative energy parts of zero energy states and is expressible as a hermitian square root H of density matrix multiplied by a unitary matrix S, which corresponds to ordinary S-matrix, which is universal and depends only the size scale n of CD through the formula $S(n) = S^n$. M-matrices and H-matrices form an orthonormal basis at given CD and H-matrices would naturally correspond to the generators of super-symplectic algebra.

The first state function reduction to the opposite boundary corresponds to what happens in quantum physics experiments. The relationship between U- and S-matrices has remained poorly understood.

The original view about the relationship was a purely formal guess: M-matrices would define the orthonormal rows of U-matrix. This guess is not correct physically and one must consider in detail what U-matrix really means.

1. First about the geometry of CD [K58]. The boundaries of CD will be called passive and active: passive boundary correspond to the boundary at which repeated state function reductions take place and give rise to a sequence of unitary time evolutions U followed by localization in the moduli of CD each. Active boundary corresponds to the boundary for which U induces delocalization and modifies the states at it.

   The moduli space for the CDs consists of a discrete subgroup of scalings for the size of CD characterized by the proper time distance between the tips and the sub-group of Lorentz boosts leaving passive boundary and its tip invariant and acting on the active boundary only. This group is assumed to be represented unitarily by matrices Λ forming the same group for all values of n.

   The proper time distance between the tips of CDs is quantized as integer multiples of the minimal distance defined by CP_2 time: $T = nT_0$. Also in quantum jump in which the size scale n of CD increases the increase corresponds to integer multiple of T_0. Using the logarithm of proper time, one can interpret this in terms of a scaling parametrized by an integer. The possibility to interpret proper time translation as a scaling is essential for having a manifest Lorentz invariance: the ordinary definition of S-matrix introduces preferred rest system.

2. The physical interpretation would be roughly as follows. M-matrix for a given CD codes for the physics as we usually understand it. M-matrix is product of square root of density matrix and S-matrix depending on the size scale of CD and is the analog of thermal S-matrix. State function at the opposite boundary of CD corresponds to what happens in the state function reduction in particle physics experiments. The repeated state function reductions at same boundary of CD correspond to TGD version of Zeno effect crucial for understanding consciousness. Unitary U-matrix describes the time evolution zero energy states due to the increase of the size scale of CD (at least in statistical sense). This process is dispersion in the moduli space of CDs: all possible scalings are allowed and localization in the space of moduli of CD localizes the active boundary of CD after each unitary evolution.

In the following I will proceed by making questions. One ends up to formulas allowing to understand the architecture of U-matrix and to reduce its construction to that for S-matrix having interpretation as exponential of the generator L_1 of the Virasoro algebra associated with the super-symplectic algebra.

22.3.1 What can one say about M-matrices?

1. The first thing to be kept in mind is that M-matrices act in the space of zero energy states rather than in the space of positive or negative energy states. For a given CD M-matrices are products of hermitian square roots of hermitian density matrices acting in the space of zero energy states and universal unitary S-matrix $S(CD)$ acting on states at the active end of CD (this is also very important to notice) depending on the scale of CD:

$$M^i = H^i \circ S(CD) \ .$$

   Here "$\circ$" emphasizes the fact that S acts on zero energy states at active boundary only. H^i is hermitian square root of density matrix and the matrices H^i must be orthogonal for given CD from the orthonormality of zero energy states associated with the same CD. The zero energy states associated with different CDs are not orthogonal and this makes the unitary time evolution operator U non-trivial.

2. Could quantum measurement be seen as a measurement of the observables defined by the Hermitian generators H^i? This is not quite clear since their action is on zero energy states. One might actually argue that the action of this kind of observables on zero energy states does not affect their vanishing net quantum numbers. This suggests that H^i carry no net quantum numbers and belong to the Cartan algebra. The action of S is restricted at the active boundary of CD and therefore it does not commute with H^i unless the action is in a separate tensor factor. Therefore the idea that S would be an exponential of generators H^i and thus commute with them so that H^i would correspond to sub-spaces remaining invariant under S acting unitarily inside them does not make sense.

3. In TGD framework symplectic algebraas isometries of WCW is analogous to a Kac-Moody algebra with finite-dimensional Lie-algebra replaced with the infinite-dimensional symplectic algebra with elements characterized by conformal weights [K24, K23]. There is a temptation to think that the H^i could be seen as a representation for this algebra or its sub-algebra. This algebra allows an infinite fractal hierarchy of sub-algebras of the super-symplectic algebra isomorphic to the full algebra and with conformal weights coming as n-ples of those for the full algebra. In the proposed realization of quantum criticality the elements of the sub-algebra characterized by n act as a gauge algebra. An interesting question is whether this sub-algebra is involved with the realization of M-matrices for CD with size scale n. The natural expectation is that n defines a cutoff for conformal weights relating to finite measurement resolution.

22.3.2 How does the size scale of CD affect M-matrices?

1. In standard quantum field theory (QFT) S-matrix represents time translation. The obvious generalization is that now scaling characterized by integer n is represented by a unitary S-matrix that is as n:th power of some unitary matrix S assignable to a CD with minimal size:

$S(CD) = S^n$. $S(CD)$ is a discrete analog of the ordinary unitary time evolution operator with n replacing the continuous time parameter.

2. One can see M-matrices also as a generalization of Kac-Moody type algebra. Also this suggests $S(CD) = S^n$, where S is the S-matrix associated with the minimal CD. S becomes representative of phase $exp(i\phi)$. The inner product between CDs of different size scales can n_1 and n_2 can be defined as

$$\langle M^i(m), M^j(n)\rangle = Tr(S^{-m} \circ H^i H^j \circ S^n) \times \theta(n-m) \ ,$$
$$\theta(n) = 1 \ \text{for} \ n \geq 0 \ , \ \theta(n) = 0 \ \text{for} \ n < 0 \ . \tag{22.3.1}$$

Here I have denoted the action of S-matrix at the active end of CD by "$\circ$" in order to distinguish it from the action of matrices on zero energy states which could be seen as belonging to the tensor product of states at active and passive boundary.

It turns out that unitarity conditions for U-matrix are invariant under the translations of n if one assumes that the transitions obey strict arrow of time expressed by $n_j - n_i \geq 0$. This simplifies dramatically unitarity conditions. This gives orthonormality for M-matrices associated with identical CDs. This inner product could be used to identify U-matrix.

3. How do the discrete Lorentz boosts affecting the moduli for CD with a fixed passive boundary affect the M-matrices? The natural assumption is that the discrete Lorentz group is represented by unitary matrices λ: the matrices M^i are transformed to $M^i \circ \lambda$ for a given Lorentz boost acting on states at active boundary only.

One cannot completely exclude the possibility that S acts unitarily at both ends of zero energy states. In this case the scaling would be interpreted as acting on zero energy states rather than those at active boundary only. The zero energy state basis defined by M_i would depend on the size scale of CD in more complex manner. This would not affect the above formulas except by dropping away the "$\circ$".

Unitary U must characterize the transitions in which the moduli of the active boundary of causal diamond (CD) change and also states at the active boundary (paired with unchanging states at the passive boundary) change. The arrow of the experienced flow of time emerges during the period as state function reductions take place to the fixed ("passive") boundary of CD and do not affect the states at it. Note that these states form correlated pairs with the changing states at the active boundary. The physically motivated question is whether the arrow of time emerges statistically from the fact that the size of CD tends to increase in average sense in repeated state function reductions or whether the arrow of geometric time is strict. It turns out that unitarity conditions simplify dramatically if the arrow of time is strict.

22.3.3 What Can One Say About U-Matrix?

1. Just from the basic definitions the elements of a unitary matrix, the elements of U are between zero energy states (M-matrices) between two CDs with possibly different moduli of the active boundary. Given matrix element of U should be proportional to an inner product of two M-matrices associated with these CDs. The obvious guess is as the inner product between M-matrices

$$\begin{aligned} U^{ij}_{m,n} &= \langle M^i(m,\lambda_1), M^j(n,\lambda_2)\rangle \\ &= Tr(\lambda_1^{\dagger} S^{-m} \circ H^i H^j \circ S^n \lambda_2) \\ &= Tr(S^{-m} \circ H^i H^j \circ S^n \lambda_2 \lambda_1^{-1})\theta(n-m) \ . \end{aligned} \tag{22.3.2}$$

Here the usual properties of the trace are assumed. The justification is that the operators acting at the active boundary of CD are special case of operators acting non-trivially at both boundaries.

2. Unitarity conditions must be satisfied. These conditions relate S and the hermitian generators H^i serving as square roots of density matrices. Unitarity conditions $UU^\dagger = U^\dagger U = 1$ is defined in the space of zero energy states and read as

$$\sum_{j_1 n_1} U^{ij_1}_{mn_1} (U^\dagger)^{j_1 j}_{n_1 n} = \delta^{i,j}\delta_{m,n}\delta_{\lambda_1,\lambda_2} \tag{22.3.3}$$

To simplify the situation let us make the plausible hypothesis contribution of Lorentz boosts in unitary conditions is trivial by the unitarity of the representation of discrete boosts and the independence on n.

3. In the remaining degrees of freedom one would have

$$\sum_{j_1, k \geq Max(0,n-m)} Tr(S^k \circ H^i H^{j_1}) Tr(H^{j_1} H^j \circ S^{n-m-k}) = \delta^{i,j}\delta_{m,n} \quad . \tag{22.3.4}$$

The condition $k \geq Max(0, n-m)$ reflects the assumption about a strict arrow of time and implies that unitarity conditions are invariant under the proper time translation $(n, m) \rightarrow (n+r, m+r)$. Without this condition n back-wards translations (or rather scalings) to the direction of geometric past would be possible for CDs of size scale n and this would break the translational invariance and it would be very difficult to see how unitarity could be achieved. Stating it in a general manner: time translations act as semigroup rather than group.

4. Irreversibility reduces dramatically the number of the conditions. Despite this their number is infinite and correlates the Hermitian basis and the unitary matrix S. There is an obvious analogy with a Kac-Moody algebra at circle with S replacing the phase factor $exp(in\phi)$ and H^i replacing the finite-dimensional Lie-algebra. The conditions could be seen as analogs for the orthogonality conditions for the inner product. The unitarity condition for the analog situation would involve phases $exp(ik\phi_1) \leftrightarrow S^k$ and $exp(i(n-m-k)\phi_2) \leftrightarrow S^{n-m-k}$ and trace would correspond to integration $\int d\phi_1$ over ϕ_1 in accordance with the basic idea of non-commutative geometry that trace corresponds to integral. The integration of ϕ_i would give $\delta_{k,0}$ and $\delta_{m,n}$. Hence there are hopes that the conditions might be satisfied. There is however a clear distinction to the Kac-Moody case since S^n does not in general act in the orthogonal complement of the space spanned by H^i.

5. The idea about reduction of the action of S to a phase multiplication is highly attractive and one could consider the possibility that the basis of H^i can be chosen in such a way that H^i are eigenstates of of S. This would reduce the unitarity constraint to a form in which the summation over k can be separated from the summation over j_1.

$$\sum_{k \geq Max(0,n-m)} exp(iks_i - (n-m-k)s_j) \sum_{j_1} Tr(H^i H^{j_1}) Tr(H^{j_1} H^j) = \delta^{i,j}\delta_{m,n} \quad . \tag{22.3.5}$$

The summation over k should gives a factor proportional to δ_{s_i,s_j}. If the correspondence between H^i and eigenvalues s_i is one-to-one, one obtains something proportional to $\delta(i,j)$ apart from a normalization factor. Using the orthonormality $Tr(H^i H^j) = \delta^{i,j}$ one obtains for the left hand side of the unitarity condition

$$exp(is_i(n-m))\sum_{j_1} Tr(H^i H^{j_1})Tr(H^{j_1}H^j) = exp(is_i(n-m))\delta_{i,j} \ . \tag{22.3.6}$$

Clearly, the phase factor $exp(is_i(n-m))$ is the problem. One should have Kronecker delta $\delta_{m,n}$ instead. One should obtain behavior resembling Kac-Moody generators. H^i should be analogs of Kac-Moody generators and include the analog of a phase factor coming visible by the action of S.

22.3.4 How to obtain unitarity correctly?

It seems that the simple picture is not quite correct yet. One should obtain somehow an integration over angle in order to obtain Kronecker delta.

1. A generalization based on replacement of real numbers with function field on circle suggests itself. The idea is to the identify eigenvalues of generalized Hermitian/unitary operators as Hermitian/unitary operators with a spectrum of eigenvalues, which can be continuous. In the recent case S would have as eigenvalues functions $\lambda_i(\phi) = exp(is_i\phi)$. For a discretized version ϕ would have has discrete spectrum $\phi(n) = 2\pi k/n$. The spectrum of λ_i would have n as cutoff. Trace operation would include integration over ϕ and one would have analogs of Kac-Moody generators on circle.

2. One possible interpretation for ϕ is as an angle parameter associated with a fermionic string connecting partonic 2-surface. For the super-symplectic generators suitable normalized radial light-like coordinate r_M of the light-cone boundary (containing boundary of CD) would be the counterpart of angle variable if periodic boundary conditions are assumed.

   The eigenvalues could have interpretation as analogs of conformal weights. Usually conformal weights are real and integer valued and in this case it is necessary to have generalization of the notion of eigenvalues since otherwise the exponentials $exp(is_i)$ would be trivial. In the case of super-symplectic algebra I have proposed that the generating elements of the algebra have conformal weights given by the zeros of Riemann zeta. The spectrum of conformal weights for the generators would consist of linear combinations of the zeros of zeta with integer coefficients. The imaginary parts of the conformal weights could appear as eigenvalues of S.

3. It is best to return to the definition of the U-matrix element to check whether the trace operation appearing in it can already contain the angle integration. If one includes to the trace operation appearing the integration over ϕ it gives $\delta_{m,n}$ factor and U-matrix has elements only between states assignable to the same causal diamond. Hence one must interpret U-matrix elements as functions of ϕ realized factors $exp(i(s_n - s_m)\phi)$. This brings strongly in mind operators defined as distributions of operators on line encountered in the theory of representations of non-compact groups such as Lorentz group. In fact, the unitary representations of discrete Lorentz groups are involved now.

4. The unitarity condition contains besides the trace also the integrations over the two angle parameters ϕ_i associated with the two U-matrix elements involved. The left hand side of the unitarity condition reads as

$$\begin{aligned}
\sum_{k\geq Max(0,n-m)} & I(ks_i)I((n-m-k)s_j)\times\sum_{j_1} Tr(H^iH^{j_1})Tr(H^{j_1}H^j) \\
= & \ \delta^{i,j}\delta_{m,n} \ , \\
I(s) = \frac{1}{2\pi}\times\int d\phi exp(is\phi) = \delta_{s,0} \ . &
\end{aligned} \tag{22.3.7}$$

Integrations give the factor $\delta_{k,0}$ eliminating the infinite sum obtained otherwise plus the factor $\delta_{n,m}$. Traces give Kronecker deltas since the projectors are orthonormal. The left hand side equals to the right hand side and one achieves unitarity. It seems that the proposed ansatz works and the U-matrix can be reduced by a general ansatz to S-matrix.

5. It should be made clear that the use of eigenstates of S is only a technical trick, the physical states need not be eigenstates. If the active parts of zero energy states where eigenstates of S, U-matrix would not have matrix elements between different H^i and projection operator could not change during time evolution.

22.3.5 What about the identification of S?

1. S should be exponential of time the scaling operator whose action reduces to a time translation operator along the time axis connecting the tips of CD and realized as scaling. In other words, the shift $t/T_0 = m \rightarrow m+n$ corresponds to a scaling $t/T_0 = m \rightarrow km$ giving $m+n = km$ in turn giving $k = 1+n/m$. At the limit of large shifts one obtains $k \simeq n/m \rightarrow \infty$, which corresponds to QFT limit. nS corresponds to $(nT_0)\times(S/T_0) = TH$ and one can ask whether QFT Hamiltonian could corresponds to $H = S/T_0$.

2. It is natural to assume that the operators H^i are eigenstates of radial scaling generator $L_0 = ir_M d/dr_M$ at both boundaries of CD and have thus well-defined conformal weights. As noticed the spectrum for super-symplectic algebra could also be given in terms of zeros of Riemann zeta.

3. The boundaries of CD are given by the equations $r_M = m^0$ and $r_M = T - m_0$, m_0 is Minkowski time coordinate along the line between the tips of CD and T is the distance between the tips. From the relationship between r_M and m_0 the action of the infinitesimal translation $H \equiv i\partial/\partial_{m^0}$ can be expressed as conformal generator $L_{-1} = i\partial/\partial_{rM} = r_M^{-1}L_0$. Hence the action is non-diagonal in the eigenbasis of L_0 and multiplies with the conformal weights and reduces the conformal weight by one unit. Hence the action of U can change the projection operator. For large values of conformal weight the action is classically near to that of L_0: multiplication by L_0 plus small relative change of conformal weight.

4. Could the spectrum of H be identified as energy spectrum expressible in terms of zeros of zeta defining a good candidate for the super-symplectic radial conformal weights. This certainly means maximal complexity since the number of generators of the conformal algebra would be infinite. This identification might make sense in chaotic or critical systems. The functions $(r_M/r_0)^{1/2+iy}$ and $(r_M/r_0)^{-2n}$, $n>0$, are eigenmodes of r_M/dr_M with eigenvalues $(1/2+iy)$ and $-2n$ corresponding to non-trivial and trivial zeros of zeta.

   There are two options to consider. Either L_0 or iL_0 could be realized as a hermitian operator. These options would correspond to the identification of mass squared operator as L_0 and approximation identification of Hamiltonian as iL_1 as iL_0 making sense for large conformal weights.

   (a) Suppose that $L_0 = r_M d/dr_M$ realized as a hermitian operator would give harmonic oscillator spectrum for conformal confinement. In p-adic mass calculations the string model mass formula implies that L_0 acts essentially as mass squared operator with integer spectrum. I have proposed conformal confinent for the physical states net conformal weight is real and integer valued and corresponds to the sum over negative integer valued conformal weights corresponding to the trivial zeros and sum over real parts of non-trivial zeros with conformal weight equal to 1/2. Imaginary parts of zeta would sum up to zero.

   (b) The counterpart of Hamiltonian as a time translation is represented by $H = iL_0 = ir_M d/dr_M$. Conformal confinement is now realized as the vanishing of the sum for the real parts of the zeros of zeta: this can be achieved. As a matter fact the integration measure dr_M/r_M brings implies that the net conformal weight must be 1/2. This is achieved if the number of non-trivial zeros is odd with a judicious choice of trivial zeros. The eigenvalues of Hamiltonian acting as time translation operator could correspond to the linear combination of imaginary part of zeros of zeta with integer coefficients. This is an attractive hypothesis in critical systems and TGD Universe is indeed quantum critical.

22.3.6 What about Quantum Classical Correspondence?

Quantum classical correspondence realized as one-to-one map between quantum states and zero modes has not been discussed yet.

1. M-matrices would act in the tensor product of quantum fluctuating degrees of freedom and zero modes. The assumption that zero energy states form an orthogonal basis implies that the hermitian square roots of the density matrices form an orthonormal basis. This condition generalizes the usual orthonormality condition.

2. The dependence on zero modes at given boundary of CD would be trivial and induced by 1-1 correspondence $|m\rangle \to z(m)$ between states and zero modes assignable to the state basis $|m_{\pm}$ at the boundaries of CD, and would mean the presence of factors $\delta_{z_+,f(m_+)} \times \delta_{z_-,f(n_-)}$ multiplying M-matrix $M^i_{m,n}$.

To sum up, it seems that the architecture of the U-matrix and its relationship to the S-matrix is now understood and in accordance with the intuitive expectations the construction of U-matrix reduces to that for S-matrix and one can see S-matrix as discretized counterpart of ordinary unitary time evolution operator with time translation represented as scaling: this allows to circumvent problems with loss of manifest Poincare symmetry encountered in quantum field theories and allows Lorentz invariance although CD has finite size. What came as surprise was the connection with stringy picture: strings are necessary in order to satisfy the unitary conditions for U-matrix. Second outcome was that the connection with super-symplectic algebra suggests itself strongly. The identification of hermitian square roots of density matrices with Hermitian symmetry algebra is very elegant aspect discovered already earlier. A further unexpected result was that U-matrix is unitary only for strict arrow of time (which changes in the state function reduction to opposite boundary of CD).

22.4 Hilbert P-Adics, Hilbert Adeles, And TGD

One can imagine also a third generalization of the number concept. One can replace integer n with n-dimensional Hilbert space and sum and product with direct sum and tensor product and introduced their co-operations, the definition of which is non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. It would also generalize the notion of algebraic holography.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the embedding space. Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. The definition of co-operations would define quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers.

22.4.1 Could The Notion Of Hilbert Mathematics Make Sense?

After having worked one month with the iea I found myself in a garden of branching paths and realized that something must be wrong. Is the idea about quantum p-adics a disgusting fix idee or is it something deeper?

The successful manner to make progress in this this kind of situation has been the combination of existing firmly established ideas with the newcomer. Could the attempt to relate quantum p-adics to generalized Feynman graphs, infinite primes, and hierarchy of Planck constants help?

Second good strategy is maximal simplification. In the recent case this encourages sticking to the most conservative option for which quantum p-adics are obtained from ordinary p-adics by mapping the coefficients of powers of p to quantum integers. This option has also a variant for which one has expansion in powers of p^N defining pinary cutoff. At the level of p-adic numbers different values of N make no difference but at the level of finite measurement resolution situation is different. Also quantum m-adicity would have natural interpretation in terms of measurement resolution rather than fundamental algebra.

Replacing integers with Hilbert spaces

Consider now the argument leading to the interpretation of p-adic integers as Hilbert space dimensions and the formulation of quantum p-adics as p-adic Hilbert spaces whose state basis defines a multiple covering of integer defining the dimension of the Hilbert space.

1. The notion of generalized Feynman diagram and zero energy ontology suggest suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. This approach indeed led to the discovery that integers could be replaced by Hilbert spaces.

2. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted.

3. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands for generalized Feynman diagrams.

4. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to tensor product and direct sum obeying also associativity and distributivity?! If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces - and in vertices the incoming states fuse to direct sum of tensor product!

5. What this would mean is that one could assign to each p-adic integer a multiple covering defined by the state basis of the corresponding Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests just cyclic group. The non-trivial quantum Galois group would emerge also for the "less-interesting" but non-risky option so that the conservative approach might work!

6. The Hilbert spaces in question could represent physical states - maybe cognitively in the p-adic context. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of embedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the embedding space and Hilbert spaces can be identified as function spaces associated with the discrete point sets of covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

This approach works for the ordinary p-adic integers. There is no need to allow coefficients $a_n > p$ ("interesting" option) in the expansion $\sum a_n p^n$ of p-adic numbers but still consisting of primes $l < p$. "Interesting" option would emerge as one takes finite measurement resolution into account by mapping the Hilbert spaces defining coefficients of Hilbert space pinary expansion with their quantum counterparts. More precisely.

1. At Hilbert space level pinary expansion of p-adic Hilbert space becomes direct sum $\oplus_n a_n \otimes p^n$. $a_n = \otimes_i p_i$, $p_i < p$, denotes tensor product of prime Hilbert spaces for which I use the same label as for p-adic numbers. p^n denotes Hilbert space with dimension p^n. In real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. Quantum p-adic numbers would be obtained by mapping the Hilbert space valued coefficients a_n of the to their quantum counterparts $(a_n)_q$, which are conjectured to allow precise definition in terms of inclusions of hyper-finite factors with Jones inclusions associated with the quantum counterpart of 2-D Hilbert space. The quantum map would reduce to the mapping of the tensor factors p_1 of a_n to $(p_1)_q$. Same would apply to quantum states. The map would be defined as $\oplus a_n \otimes p^n \to \oplus (a_n)_q \otimes p^n$, $(a_n)_q = \otimes_{p_1} (p_1)_q$. The map $p_1 \to (p_1)_q$ would take into account finite measurement resolution.

   "Interesting" option would be obtained as follows. It is possible to express given p-adic number in many ways if one only requires that the coefficients a_n in the direct sum are tensor products of prime Hilbert spaces with dimension $p_1 < p$ but does not assume $a_n < p$. For instance, for $p = 3$ and $n = 8$ one has $8 = 2 \oplus 2 \otimes$ or $8 = 2 \otimes 2 \otimes 2$. These representations are p-adically equivalent. Quantum map however spoils this equivalence. $2 \oplus 2 \otimes 3 \to 2_q \oplus 2_q \otimes 3$ and $8 = 2 \otimes 2 \otimes 2 \to 2_q \otimes 2_q \otimes 2_q$ are not same quantum Hilbert spaces. The "interesting" option would thus emerge as one takes into account the finite measurement resolution.

4. One could say that the quantum Hilbert spaces associated with a given p-adic Hilbert space form a covering space like structure. Quantum Galois group identified as a subgroup of permutations of these quantum Hilbert spaces need not make sense however.

After this lengthy motivating introduction I want to describe some details of the arithmetics of p-adic Hilbert spaces. This arithmetics is formally identical with the ordinary integer arithmetics. What is however interesting is that one can generalize it so that one obtains something that one could call Hilbert spaces of dimension which is negative, rational, algebraic, or even complex, and even quaternionic or octonionic. It might be necessary to have these generalizations if one wants full generality.

1. Consider first what might be called p-adic Hilbert spaces. For brevity I will denote Hilbert spaces in the same manner as p-adic numbers: reader can replace "n" with "H_n" if this looks more appropriate. p-Adic Hilbert spaces have direct sum expansions of form

$$n = \oplus_k a_k \otimes p^k \quad .$$

   All integers appearing in the formula can be also interpreted as Hilbert space dimensions. In the real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. How to define Hilbert spaces with negative dimension? In p-Adic context this is not a problem. Hilbert space with dimension -1 is given by Hilbert spaces with dimension $(p-1)/(1-p) = (p-1)(1+p+p^2+...)$ converging p-adically and given by

$$-1 = \oplus_k (p-1) \otimes p^k \quad .$$

   In real context one must consider pairs of Hilbert spaces (m,n) and define equivalence $(m,n) = (m+k, n+k)$. In canonical representation Hilbert space with positive dimension m corresponds to $(m,0)$ and Hilbert spaces with negative dimension $-m$ to $(0,m)$. This procedure is familiar from the theory of vector bundles where one subtracts vector bundles and defines their negatives.

3. In p-adic context one can also define p-adic Hilbert spaces with rational dimension if the p-adic norm of the rational (m/n) is smaller than 1. This is achieved simply by the expansion

$$\frac{m}{n} = \oplus_k a_k \otimes p^k \quad .$$

In real context tone can define Hilbert spaces with rational valued dimension just as one defines rational numbers - that is as pairs of Hilbert spaces (m, n) with equivalence $(m, n) \equiv (km, kn)$.

4. One can even define Hilbert spaces with dimensions in algebraic extensions of rationals.

   (a) Consider first the real case and the extension defined by Gaussian integers for which integers are of form $m + in \equiv (m, n)$. What is needed is just the product rule: $(m, n) \otimes (r, s) = (m \otimes r - \oplus(-n \otimes s), m \otimes s \oplus r \otimes n)$. This expression is completely well-defined in the p-adic context and also in real context if one accepts the proposed defined of integer Hilbert spaces as pairs of ordinary Hilbert spaces. For $Q(\sqrt{5})$ one would have $(m, n) \times (r, s) = (m \otimes r \oplus 5 \otimes n \otimes s, m \otimes s \oplus r \otimes n)$. In n-dimensional case one just replaces Hilbert spaces with n-multiple of ordinary Hilbert spaces and uses the multiplication rules.

   (b) In p-adic context similar approach works when the algebraic extension requires also extension of p-adic numbers. In p-adic context however many algebraic numbers can exist as ordinary p-adic numbers. For instance, for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as well as its Hilbert space counterpart. For quadratic extensions of $p > 2$-adic numbers the 4-D extension involving the addition of two square roots all square roots except that of p exist -adically.

Quantum Hilbert spaces and generalization to extensions of rationals

The map of p-adic integers to their quantum counterparts generalizes so that it applies to Hilbert spaces. This means that prime Hilbert spaces are mapped to the quantum counterparts. What this means is not quite obvious. Quantum groups appearing in the context of Jones inclusions lead to the emergence of quantum spinors that is quantum counterparts of 2-D Hilbert spaces. This suggest that more general inclusions lead to prime-dimensional quantum Hilbert spaces. The idea is simple: quantum matrix algebra M/N with quantum dimension $(2_q)^2$ is defined as a coset space of hyper-finite factor M and included factor $N \subset M$. This quantum matrix algebra acts in quantum spinor space of dimension 2_q. The generalization would introduce p_q-dimensional quantum Hilbert spaces.

A good test for the proposal is whether it generalizes naturally to algebraic extensions of rationals.

1. For algebraic extensions some ordinary primes split into products of primes associated with the extension. The problem is that for these algebraic primes the factors $exp(i\pi/P)$ fail to be algebraic numbers and finite roots of unity and its is not at all clear whether the naïve generalization of the notion of quantum p-adic makes sense. This suggests that only the ordinary primes which do not split into products of primes of extension remain and one can define quantum p-adics only for these whereas the other primes correspond to ordinary algebraic extension of p-adic numbers. This would make algebraic extension of rationals the coefficient group of adele consisting of p-adic numbers fields associated with non-split primes only. Note that rationals or their extension would naturally appear as tensor factor of adeles meaning that their action can be thought to affect any of the factors of the adele.

2. For split primes the p-adic Hilbert spaces must be defined for their algebraic prime factors. The proposed procedure of defining Hilbert space counterparts for algebraic extensions of rationals provides a recipe for how to achieve this. These Hilbert spaces the quantum map would be trivial.

3. Hilbert space counterpart for the algebraic extension of rationals and of p-adics makes also sense. The Hilbert space assigned with integer which splits into primes of extension splits

also to a tensor product of prime Hilbert spaces assignable with the extension. The splitting of integers and primes is highly analogous to the decomposition of hadron to quarks and gluons. This decomposition is not seen at the level of rationals representing observed.

What about Hilbert spaces with real number valued dimension?

What Hilbert space variant of a real number could mean? What Hilbert space with dimension equal to arbitrary real number could mean? One can imagine two approaches.

1. The first approach is based on the map of Hilbert p-adics to real p-adics by a map used to map p-adic numbers to reals. The formula would be $\oplus_n a_n \otimes p^n \rightarrow \oplus (a_n)_q \otimes p^{-n}$. $(a_n)_q = \otimes_l l_q^{e_l}$, were l_q is quantum Hilbert space of prime dimension. Also the Hilbert space p^{-n} would be well-defined as a Hilbert rational defined as a pair of Hilbert spaces.

   For hyper-finite factors of type II_1 Hilbert spaces with continuous dimension emerge naturally. The reason is that the dimension of the Hilbert space is defined as quantum trace of identity operator characterized by quantum phase this dimension is finite and continuous. This allows a spectrum of sub-Hilbert spaces with continuously varying real dimension. The appearance of quantum Hilbert spaces in the canonical identification map conforms with this and even for dimension $0 < n < p$ gives rise to quantum Hilbert space with algebraic quantum dimension given as $n = \prod l_q^{e^l}$ for $n = \prod_l l^{e_l}$.

2. Second approach relies on the mimicry of the completion of ordinary rationals to real numbers. One can define Hilbert space analogs of rationals and algebraics by defining positive and negative rationals as pairs of Hilbert spaces with equivalence relation $(m, n) \equiv (m \oplus r, n \oplus r)$. Taking pairs of these pairs with equivalence relation $(M, N) \equiv (K \otimes M, K \otimes N)$ one obtains Hilbert spaces corresponding to rational numbers. Algebraic extensions are obtained similarly. By taking limits just in the same manner as for real numbers one would obtain Hilbert reals with transcendental dimensions. For instance, e could be defined as the limit of tensor power $(1 \oplus 1/n)^n$, $n \rightarrow \infty$.

Again one must remember that the co-vertices define the hard part of the problem and their definition means postulate of quantum dynamics. This would be the genuinely new element and transform mathematics to quantum physics. Every sequences of algebraic operations having a realization as Feynman diagram involving arithmetic operations as positive energy part of Feynman diagrams and co-operations as the negative energy part of diagram connected by single line.

It should not go un-noticed that the direct sum and tensor product decompositions of possibly infinite-dimensional Hilbert spaces are very essential for the interpretation. For infinite-dimensional Hilbert spaces these decompositions would be regarded as equivalent for an abstract definition of Hilbert space. In physical applications tensor product and direct sum representations have also very concrete physical content.

Hilbert calculus?

What this approach suggests is a generalization of calculus in both real and and p-adic context. The first thing to do is to define Hilbert functions as Hilbert space valued functions as $x \rightarrow f(x)$. This could be done formally by assigning to Hilbert space associated with point x Hilbert space associated with the point $f(x)$ for all values of x. Function could have representation as Taylor series or Laurent series with sum replaced with direct sum and products with tensor products. The correspondence $x \rightarrow f(x)$ would have as a counterpart the analog of Feynman diagram describing the Taylor series with final line defining the value $f(x)$. Also derivatives and integrals would be at least formally defined. This would requite separate diagram for every point x. One can consider also the possibility of more abstract definition of $f(x)$. For instance the set of coefficients $\{f_n\}$ in the Taylor series of f would defined a collection of Hilbert spaces.

One should be able to define also co-functions in terms of co-vertices. The value of co-function at point y would give all the values of x for which one has $f(x) = y$. Co-function would correspond to a quantum superposition of values of inverse function and to time reversed zero energy states. The breaking of time reversal would be inherent in the very definition of function as an arrow from one Hilbert set to another Hilbert set and typically the functions involved would be many-valued form beginning. Perhaps it would be better to speak from the beginning about relations between two sets rather than functions. The physical realization of Hilbert calculus would be obtained by assigning to incoming arguments represented as Hilbert space quantum states.

Quantum mathematics?

Could one transform entire mathematics to quantum mathematics - or would it be better to say Hilbert mathematics? Reader can decide. Consider first Hilbert set theory. The idea wold be to replace numbers with Hilbert spaces. This would give Hilbert structure. By replacing Hilbert spaces with their quantum counterparts characterized by quantum dimensions n_q one would obtain which might be called quantum Hilbert structure.

1. At the level of set theory this would mean replacement of sets with Hilbert sets. A set with n elements would correspond intuitively to n-dimensional Hilbert space. Therefore Hilbert sets would provide much more specific realization of set theory than abstract set theory in which the elements of set can be anything. For n-dimensional Hilbert space however the ordering of the elements of basis induces automatically the ordering of the elements of the set. Does the process of counting the elements of set corresponds to this ordering. Direct sum would be the counterpart of set theoretic union. One could construct natural numbers inductively as direct sums $(n+1) = n \oplus 1$. To be subset would correspond to sub-Hilbert space property. Intersection of two Hilbert sets would correspond to the direct sum of common direct summands. Also set difference and symmetric difference could be defined.

2. The set theoretic realization of Boolean logic would have Hilbert variant. This would mean that logical statements could be formulated using Hilbert variants of basic logical functions.

3. Cartesian product of sets would correspond to a tensor product of Hilbert spaces. This would bring in the notion of prime since Hilbert integers would have decomposition into tensor products of Hilbert primes. Note that here one can consider the symmetrization of tensor product modulo phase factor and this could give rise to bosonic and fermionic statistics and perhaps also to anyonic statistics when the situation is 2-dimensional as it indeed is for partonic 2-surfaces.

4. What about sets of sets?

   (a) The elements of n-dimensional Hilbert space consist of numbers in some number field. By replacing these numbers with corresponding Hilbert spaces one would obtain Hilbert space of Hilbert spaces as a counterpart for sets of sets. One would have Hilbert space whose points are Hilbert spaces: Hilbert-Hilbert space!. This process could be continued indefinitely and would give rise to a hierarchy formed by Hilbert^n-spaces. This would be obviously something new and mean self-referential property. For Hilbert^n-spaces one would the points at n: th level of hierarchy with points of the number field involved and obtain a concrete realization. The construction of infinite primes involves formations of sets of rationals and sets of these sets, etc.... and would have also interpretation as formation of a hierarchy of Hilbert sets of sets of.....

   (b) Power set as set of subsets of set would be obtained from direct sum of Hilbert spaces, by replacing the points of each Hilbert space with corresponding Hilbert spaces.

   (c) One could define the analog of set theoretic intersection also for tensor products as the set of common prime Hilbert factors for two Hilbert sets. For ordinary integers defined as sets the intersection in this sense would correspond to the common prime factors. In Cartesian product the intersection would correspond to common Cartesian factors.

5. The completely new and non-trivial element bringing in the quantum dynamics is brought in by co-operations for union and intersection. The solution to the equation $f(x) = y$ could be represented as a number theoretic Feynman diagram in zero energy ontology. Positive energy part would correspond to y and diagram beginning from y would represent co-function of $f(x)$ identifiable as its inverse. Negative energy state would represent a quantum superposition of the values of x representing the solutions.

6. One can ask whether a Feynman diagrammatic representation for the statements like $\exists x \in A$ such that $f(x) = g(x)$ and $\forall x \in A f(x) = g(x)$ exists. One should be able to construct quantum state which is superposition of solutions to the condition $f(x) = g(x)$. If this state is non-vanishing the solution exists.

   This kind of statements are statements of first order logic involving existential quantifiers whereas the statements of predicate logic would correspond simply to a calculation of a value of function at given point. The hierarchy of Hilbert^n spaces brings in mind strongly the hierarchy of infinite primes assigned already earlier to a hierarchy of logics. Could the statements of n: th order logic require the use of Hilbert^n- spaces. The replacement of numbers with Hilbert spaces could correspond to formation of statements of first order logic. The individual quantum states satisfying the statement would represent the statements of predicate logic.

   The construction of infinite primes can be regarded as repeated second quantization in which the many particle states of the previous level become single particle states of the new level. Maybe also the hierarchy of Hilbert^n-spaces could be seen in terms of a hierarchy of second quantizations.

   Infinite primes lead to the notion of algebraic holography meaning that real point has infinitely rich number theoretical anatomy due to the existence of real units expressible as ratios of infinite integers reducing to real unit in real topology. The possibility to replace the points of space-time with Hilbert spaces and to continue this process indefinitely would realize the same idea.

Number theoretic Feynman diagrams

Could one imagine a number theoretical quantum dynamics in which integers are replaced with sequences of arithmetic operations? If numbers are replaced with Hilbert spaces and if one can assigns to each number a state of the Hilbert space accompanying it, this seems to be possible.

1. All algebraic functions would be replaced with their algebraic expressions, which would be interpreted as analogs of zero energy states in which incoming arguments would represent positive energy part and the result of operation outgoing state. This would also unify algebra and co-algebra thinking and the information about the values of the arguments of the function would not be forgotten in the operations.

2. The natural constraints on the dynamics would be trivial. In $+_q$ vertex a direct sum of incoming states and in $\times_q$ gives rise to tensor product. This also at the level of Hilbert spaces involved. The associativity and commutativity of direct sum and tensor product guarantee automatically the these properties for the vertices. The associativity and commutativity conditions are analogous to associativity conditions for 3-point functions of conformal field theories. Distributivity condition is something new. Co-product and co-sum obey completely analogous constraints as product and sum.

3. For product the total numbers of prime factors is conserved for each prime appearing in the product meaning that the total momenta $n_i log(p_i)$ are conserved separately for each prime in the process involving only products. This kind of conservation law is natural also for infinite primes and one can indeed map the simplest infinite primes at the lowest level analogous to free Fock states of bosons and fermions to ordinary rationals so that the addition of Galois degrees of freedom tentatively identified as cyclic permutations of the state basis for Hilbert space associated with given prime would give for a particle labelled by prime p additional internal degrees of freedom. In fact, one can illustrate infinite prime as in terms of two braids corresponding to the numerator and denominator of corresponding rational and the primes appearing in rationals take the role of braid strands. For $\times_q$ the conservation of quantum numbers would correspond to conservation of representations. This guarantees commutativity and associativity of product. One can also allow co-product and co-sum and they obey completely analogous constraints as product and sum and they have counterparts at the level of Hilbert spaces two studied in the theory of quantum groups.

One can represent algebraic operations using the analogs of Feynman diagrams and there is an intriguing analogy with generalized Feynman diagrams which forces to ask whether the generalized Feynman diagrams of quantum TGD could be interpreted in terms of quantum counterparts algebraic equations transformed if one extends the number field to quaternions and their possibly existing p-adic counterparts.

1. Multiplicative and additive inverses - in the case that they exist - can be seen as kind of conjugation operations analogous to C and P which commute with each other. Their product $n \rightarrow -1/n$ could be seen as the analog of T if $CPT = 1$ is taken as identity. Co-product and co-sum would would be obtained from product and sum by CP or T.

2. One can represent the integer $X = X(\{n_k\})$ resulting from a sequence of algebraic operations $+_q$ and $\times_q$ performed for integers n_k appearing as inputs of a Feynman diagram having the value of X as outgoing line. $n_{+,k}$ represent incoming external lines and intermediate products of algebraic operations appear as internal "off-mass-shell" lines. $+_q$ and $\times_q$ represent the basic vertices. This gives only tree diagrams with single outgoing line representing the (quantum value) of X.

   Associativity and commutativity for $+_q$ resp. $\times_q$ would mean that the lines of diagram with 3 incoming particles and two vertices can be modified by permuting the incoming lines in all possible ways. Distributivity $a \times_q (b +_q c) = a \times_q b +_q a \times_q c$ does not correspond anything familiar from conformal field theories since the line representing a appears twice on the right hand side of the identity and there are 3 vertices whereas left hand side involves 2 vertices. In TGD framework the interpretation of the analogs of stringy decay vertices in terms of propagation along two different paths allows however to interpret these vertices as counterparts of $+_q$ whereas the TGD counterparts of vertices of Feynman diagrams would correspond to $\times_q$. $_+q$ would correspond at state space level to direct sum and $\times_q$ to tensor product.

3. The lines of Feynman diagrams are naturally replaced with braids - just as in quantum TGD. The decomposition of the incoming quantum rational $q = m/n$ to primes defines a braid with two colors of braid strands corresponding to the primes appearing in m and n so that a close connection with braid diagrams emerges. This of course raises the question whether one could allow non-trivial braiding operation for two braid strands represented by primes. Non-triviality would mean that $p_1p_2 = p_2p_1$ would not hold true only in projective sense so that the exchange would induce a phase factor. This would suggest that the commutativity of the basic operations - or at least multiplication - might hold true only apart from quantum phase factor. This would not be too surprising since quantum phases are the essence of what it is to be quantum integer.

4. The diagrammatical counterparts of co-operations are obtained by time reversal transforming incoming to outgoing lines and vice versa. If one adds co-products and sums to the algebraic operations producing X one obtains diagrams with loops. If ordinary algebraic rules generalizes the diagrams with loops must be transformable to diagrams without them by algebraic "moves". The simplification of arithmetic formulas that we learn in elementary school would correspond to a sequence of "moves" leading to a tree diagram with single internal line at the middle and representing $X = Y$. One can form also diagrams of form $X = Y = Z = ...$ just as in zero energy ontology.

5. In zero energy ontology a convenient manner to represent a identity $X = Y$ - call it a "quantum correlate for mathematical thought" - involving only sums and products and therefore no loops is as a tree diagram involving only two kinds of 3-vertices, namely $+_q$ and $\times_q$ and their co-algebra vertices representing time reversed processes. In zero energy ontology this kind of representation would correspond to either the condition $X/Y = 1$ or as $X - Y = 0$. In both cases one can say that the total quantum numbers would be conserved - that is net quantum numbers assignable to prime factors of X vanish for zero energy state. The diagram involves always single integral line representing the identical values of X and Y. Line representing X would be preceded by a tree diagram involving only product and sum vertices and Y would involve only co-product and co-sum. For ordinary arithmetics every algebraic

operation is representable in this kind of diagram, which suggests that infinite number of different diagrams involving loops are equivalent to this diagram with single internal line.

6. The resulting braid Feynman diagrammatics would obey extremely powerful rules due to the possibility of the "moves". All possible independent equations $X = Y$ would define the basis of zero energy states. In quantum TGD the breaking of time reversal invariance is unavoidable and means that only the positive or negative energy parts of the diagram can have well defined quantum numbers. The direct translation would be that the zero energy states correspond to sums over all diagrams for which either positive/negative energy part corresponds to given rationals and the negative/positive energy part of the state is superposition of states consisting of rationals. This would mean non-trivial U-matrix dictated by the coefficients of the superpositions and genuine arithmetic quantum dynamics. The general architecture of U-matrix is discussed in [K58].

22.4.2 Hilbert p-adics, hierarchy of Planck constants, and finite measurement resolution

The hierarchy of Planck constants assigns to the N-fold coverings of the embedding space points N-dimensional Hilbert spaces. The natural identification of these Hilbert spaces would be as Hilbert spaces assignable to space-time points or with points of partonic 2-surfaces. There is however an objection against this identification.

1. The dimension of the local covering of embedding space for the hierarchy of Planck constants is constant for a given region of the space-time surface. The dimensions of the Hilbert space assignable to the coordinate values of a given point of the embedding space are defined by the points themselves. The values of the 8 coordinates define the algebraic Hilbert space dimensions for the factors of an 8-fold Cartesian product, which can be integer, rational, algebraic numbers or even transcendentals and therefore they vary as one moves along space-time surface.

2. This dimension can correspond to the locally constant dimension for the hierarchy of Planck constants only if one brings in finite measurement resolution as a pinary cutoff to the pinary expansion of the coordinate so that one obtains ordinary integer-dimensional Hilbert space. Space-time surface decomposes into regions for which the points have same pinary digits up to p^N in the p-adic case and down to p^{-N} in the real context. The points for which the cutoff is equal to the point itself would naturally define the ends of braid strands at partonic 2-surfaces at the boundaries of CD: s.

3. At the level of quantum states pinary cutoff means that quantum states have vanishing projections to the direct summands of the Hilbert spaces assigned with pinary digits p^n, $n > N$. For this interpretation the hierarchy of Planck constants would realize physically pinary digit representations for number with pinary cutoff and would relate to the physics of cognition.

One of the basic challenges of quantum TGD is to find an elegant realization for the notion of finite measurement resolution. The notion of resolution involves observer in an essential manner and this suggests that cognition is involved. If p-adic physics is indeed physics of cognition, the natural guess is that p-adic physics should provide the primary realization of this notion.

The simplest realization of finite measurement resolution would be just what one would expect it to be except that this realization is most natural in the p-adic context. One can however define this notion also in real context by using canonical identification to map p-adic geometric objets to real ones.

Does discretization define an analog of homology theory?

Discretization in dimension D in terms of pinary cutoff means division of the manifold to cube-like objects. What suggests itself is homology theory defined by the measurement resolution and by the fluxes assigned to the induced Kähler form.

1. One can introduce the decomposition of n-D sub-manifold of the embedding space to n-cubes by $n-1$-planes for which one of the coordinates equals to its pinary cutoff. The construction works in both real and p-adic context. The hyperplanes in turn can be decomposed to $n-1$-cubes by $n-2$-planes assuming that an additional coordinate equals to its pinary cutoff. One can continue this decomposition until one obtains only points as those points for which all coordinates are their own pinary cutoffs. In the case of partonic 2-surfaces these points define in a natural manner the ends of braid strands. Braid strands themselves could correspond to the curves for which two coordinates of a light-like 3-surface are their own pinary cutoffs.

2. The analogy of homology theory defined by the decomposition of the space-time surface to cells of various dimensions is suggestive. In the p-adic context the identification of the boundaries of the regions corresponding to given pinary digits is not possible in purely topological sense since p-adic numbers do not allow well-ordering. One could however identify the boundaries sub-manifolds for which some number of coordinates are equal to their pinary cutoffs or as inverse images of real boundaries. This might allow to formulate homology theory to the p-adic context.

3. The construction is especially interesting for the partonic 2-surfaces. There is hierarchy in the sense that a square like region with given first values of pinary digits decompose to p square like regions labelled by the value $0, ..., p-1$ of the next pinary digit. The lines defining the boundaries of the 2-D square like regions with fixed pinary digits in a given resolution correspond to the situation in which either coordinate equals to its pinary cutoff. These lines define naturally edges of a graph having as its nodes the points for which pinary cutoff for both coordinates equals to the actual point.

4. I have proposed earlier [K18] what I have called symplectic QFT involving a triangulation of the partonic 2-surface. The fluxes of the induced Kähler form over the triangles of the triangulation and the areas of these triangles define symplectic invariants, which are zero modes in the sense that they do not contribute to the line element of WCW although the WCW metric depends on these zero modes as parameters. The physical interpretation is as non-quantum fluctuating classical variables. The triangulation generalizes in an obvious manner to quadrangulation defined by the pinary digits. This quadrangulation is fixed once internal coordinates and measurement accuracy are fixed. If one can identify physically preferred coordinates - say by requiring that coordinates transform in simple manner under isometries - the quadrangulation is highly unique.

5. For 3-surfaces one obtains a decomposition to cube like regions bounded by regions consisting of square like regions and Kähler magnetic fluxes over the squares define symplectic invariants. Also Kähler Chern-Simons invariant for the 3-cube defines an interesting almost symplectic invariant. 4-surface decomposes in a similar manner to 4-cube like regions and now instanton density for the 4-cube reducing to Chern-Simons term at the boundaries of the 4-cube defines symplectic invariant. For 4-surfaces symplectic invariants reduce to Chern-Simons terms over 3-cubes so that in this sense one would have holography. The resulting structure brings in mind lattice gauge theory and effective 2-dimensionality suggests that partonic 2-surfaces are enough.

The simplest realization of this homology theory in p-adic context could be induced by canonical identification from real homology. The homology of p-adic object would the homology of its canonical image.

1. Ordering of the points is essential in homology theory. In p-adic context canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ map to reals induces this ordering and also boundary operation for p-adic homology can be induced. The points of p-adic space would be represented by n-tuples of sequences of pinary digits for n coordinates. p-Adic numbers decompose to disconnected sets characterized by the norm p^{-n} of points in given set. Canonical identification allows to glue these sets together by inducing real topology. The points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ having p-adic norms p^{-n} and p^{-n-1} are mapped to the same real point p^{-n} under canonical identification and therefore the points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ can be said to define the endpoints of a continuous interval

in the induced topology although they have different p-adic norms. Canonical identification induces real homology to the p-adic realm. This suggests that one should include canonical identification to the boundary operation so that boundary operation would be map from p-adicity to reality.

2. Interior points of p-adic simplices would be p-adic points not equal to their pinary cutoffs defined by the dropping of the pinary digits corresponding p^n, $n > N$. At the boundaries of simplices at least one coordinate would have vanishing pinary digits for p^n, $n > N$. The analogs of $n-1$ simplices would be the p-adic points sets for which one of the coordinates would have vanishing pinary digits for p^n, $n > N$. $n-k$-simplices would correspond to points sets for which k coordinates satisfy this condition. The formal sums and differences of these sets are assumed to make sense and there is natural grading.

3. Could one identify the end points of braid strands in some natural manner in this cohomology? Points with $n \leq N$ pinary digits are closed elements of the cohomology and homologically equivalent with each other if the canonical image of the p-adic geometric object is connected so that there is no manner to identify the ends of braid strands as some special points unless the zeroth homology is non-trivial. In [K6] it was proposed that strand ends correspond to singular points for a covering of sphere or more general Riemann surface. At the singular point the branches of the covering would co-incide.

   The obvious guess is that the singular points are associated with the covering characterized by the value of Planck constant. As a matter fact, the original assumption was that *all* points of the partonic 2-surface are singular in this sense. It would be however enough to make this assumption for the ends of braid strands only. The orbits of braid strands and string world sheet having braid strands as its boundaries would be the singular loci of the covering.

Does the notion of manifold in finite measurement resolution make sense?

A modification of the notion of manifold taking into account finite measurement resolution might be useful for the purposes of TGD.

1. The chart pages of the manifold would be characterized by a finite measurement resolution and effectively reduce to discrete point sets. Discretization using a finite pinary cutoff would be the basic notion. Notions like topology, differential structure, complex structure, and metric should be defined only modulo finite measurement resolution. The precise realization of this notion is not quite obvious.

2. Should one assume metric and introduce geodesic coordinates as preferred local coordinates in order to achieve general coordinate invariance? Pinary cutoff would be posed for the geodesic coordinates. Or could one use a subset of geodesic coordinates for $\delta CD \times CP_2$ as preferred coordinates for partonic 2-surfaces? Should one require that isometries leave distances invariant only in the resolution used?

3. A rather natural approach to the notion of manifold is suggested by the p-adic variants of symplectic spaces based on the discretization of angle variables by phases in an algebraic extension of p-adic numbers containing n^{th} root of unity and its powers. One can also assign p-adic continuum to each root of unity [K100]. This approach is natural for compact symmetric Kähler manifolds such as S^2 and CP_2. For instance, CP_2 allows a coordinatization in terms of two pairs (P^k, Q^k) of Darboux coordinates or using two pairs $(\xi^k, \overline{\xi}^k)$, $k = 1, 2$, of complex coordinates. The magnitudes of complex coordinates would be treated in the manner already described and their phases would be described as roots of unity. In the natural quadrangulation defined by the pinary cutoff for $|\xi^k|$ and by roots of unity assigned with their phases, Kähler fluxes would be well-defined within measurement resolution. For light-cone boundary metrically equivalent with S^2 similar coordinatization using complex coordinates $(z, \overline{z})$ is possible. Light-like radial coordinate r would appear only as a parameter in the induced metric and pinary cutoff would apply to it.

Hierachy of finite measurement resolutions and hierarchy of p-adic normal Lie groups

The formulation of quantum TGD is almost completely in terms of various symmetry group and it would be highly desirable to formulate the notion of finite measurement resolution in terms of symmetries.

1. In p-adic context any Lie-algebra gG with p-adic integers as coefficients has a natural grading based on the p-adic norm of the coefficient just like p-adic numbers have grading in terms of their norm. The sub-algebra g_N with the norm of coefficients not larger than p^{-N} is an ideal of the algebra since one has $[g_M, g_N] \subset g_{M+N}$: this has of course direct counterpart at the level of p-adic integers. g_N is a normal sub-algebra in the sense that one has $[g, g_N] \subset g_N$. The standard expansion of the adjoint action $g g_N g^{-1}$ in terms of exponentials and commutators gives that the p-adic Lie group $G_N = exp(tpg_N)$, where t is p-adic integer, is a normal subgroup of $G = exp(tpg)$. If indeed so then also G/G_N is group, and could perhaps be interpreted as a Lie group of symmetries in finite measurement resolution. G_N in turn would represent the degrees of freedom not visible in the measurement resolution used and would have the role of a gauge group.

2. The notion of finite measurement resolution would have rather elegant and universal representation in terms of various symmetries such as isometries of embedding space, Kac-Moody symmetries assignable to light-like wormhole throats, symplectic symmetries of $\delta CD \times CP_2$, the non-local Yangian symmetry, and also general coordinate transformations. This representation would have a counterpart in real context via canonical identification I in the sense that $A \rightarrow B$ for p-adic geometric objects would correspond to $I(A) \rightarrow I(B)$ for their images under canonical identification. It is rather remarkable that in purely real context this kind of hierarchy of symmetries modulo finite measurement resolution does not exist. The interpretation would be that finite measurement resolution relates to cognition and therefore to p-adic physics.

3. Matrix group G contains only elements of form $g = 1 + O(p^m)$, $m \geq 1$ and does not therefore involve matrices with elements expressible in terms roots of unity. These can be included by writing the elements of the p-adic Lie-group as products of elements of above mentioned G with the elements of a discrete group for which the elements are expressible in terms of roots of unity in an algebraic extension of p-adic numbers. For p-adic prime p p: th roots of unity are natural and suggested strongly by quantum arithmetics [K67].

22.4.3 Quantum Adeles

Before saying anything about Hilbert space adeles it is better to consider ordinary adeles.

1. Fusing reals and quantum p-adic integers for various values of prime p to Cartesian product $A_Z = R \times (\prod_p Z_p)$ gives the ring of integer adeles. The tensor product $Q \otimes_Z A_Z$ gives rise to rational adeles. $_Z$ means the equivalence $(nq, a) \equiv (q, na)$. This definition generalization to any number field including algebraic extensions of rationals. It is not quite clear to me how essential the presence of R as Cartesian factor is. One can define ideles as invertible adeles by inverting individual p-adic numbers and real number in the product. If the component in the Cartesian product vanishes, the component of inverse also vanishes.

2. The definition of a norm of adele is not quite straightforward.

   (a) The norm of quantum adeles defined as product of real and p-adic norms is motivated by the formula for the norm of rational numbers as the product of its p-adic norms. This definition of norm however looks non-physical and non-mathematical. For instance, it requires that all p-adic components of quantum adele are non-vanishing and most of them have norm equal to one and are therefore p-adic integers of norm one. This condition would also break general coordinate invariance at the level of quantum adelic embedding space very strongly. Also for adelic spinors and adelic Hilbert space this condition is definitely non-sensical.

(b) The physically acceptable norm for adeles should reflect the basic properties of p-adic norm for a given p-adic field in the product but should also have the characteristic property of Hilbert space norm that the norm squared is sum of the norms squared for the factors of the adele. The solution to these demands seems to be simple: map the p-adic number to its quantum counterpart in each factor and map this number to real number by canonical identification. After this form the real Hilbert space norm of the resulting element of infinite-dimensional real Hilbert space. This norm generalizes in a natural manner to linear spaces possessing adeles as components. Most importantly, for this norm the elements of adele having finite number of components have a non-vanishing norm and field property is possible.

Consider now what happens when one replaces p-adic integers with p-adic Hilbert spaces and p-adic numbers as components of the vectors of the Hilbert space.

1. As far as arithmetics is considered, the definition of Hilbert space adeles for p-adic number fields is formally the same as that of ordinary adeles. It of course takes time to get accustomed to think that rationals correspond to a pair of Hilbert spaces and their product is formulated for this pair.

2. p-Adic Hilbert spaces would be linear spaces with p-adic coefficients that is vectors with p-adic valued components. Inner product and norm would be defined by mapping the components of vectors to real/complex numbers by mapping them first to quantum p-adics and them to reals by canonical identification. Note that the attempts to define p-adic Hilbert spaces using p-adic norm or formal p-adic valued norm mapped to real number by canonical identification lead to difficulties since already in 2-D case the equation $x^2 + y^2 = 0$ has solution $y = \sqrt{-1}x$ for $p \ mod \ 4 = 1$ since in this case $\sqrt{-1}$ exists p-adically.

3. A possible problem relates to the fact that all p-adic numbers are mapped to non-negative real numbers under canonical identification if the coefficients a_n in the expansion $\sum_n a_n p^n$ consists of primes $l < p$ for which quantum counterpart is non-negative. For ordinary p-adic numbers orthogonal vectors in a given basis would be simply vectors with no common non-vanishing components. Does this mean the existence of a preferred basis with elements $(0, .., 0, 1, 0...)$ so that any other unitarily related basis would be impossible. Or should one introduce cyclic algebraic extension of p-adic numbers with n-elements $exp(i2\pi k/n)$ for which one obtains linear superposition and can form new unitarily related basis taking into account the restrictions posed by p-adicity. This option is suggested also by the identification of the Hilbert space as wave functions in the local singular covering of embedding space. The phases form also in a natural manner cyclic group Z_n identifiable as quantum Galois group assignable to integer n and decomposing to a product of cyclic groups Z_{p_i}, $p_i|n$.

Also real numbers form a Cartesian factor of adeles. The question what Hilbert spaces with dimension equal to arbitrary real number could mean has been already discussed and there are two approaches to the problem: one based on canonical identification and quantum counterparts of p-adic numbers and one to a completion of Hilbert rationals.

22.5 Generalized Feynman Diagrams As Quantum Arithmetic Feynman Diagrams?

The idea that the generalized Feynman diagrams of TGD could have interpretation in terms of arithmetic QFT is not new but the quantum arithmetic Feynman diagrams give much more precise content to this idea.

1. The possibility to eliminate all loops is by "moves" is an old idea (briefly discussed in [K12]), which I introduced as a generalization of the old fashioned s-t duality of string models. One motivation was of course the resulting cancellation of diverges. I however gave up this idea as too romantic [K12]. The properties of the counterparts of twistor diagrams in zero energy ontology re-inspires this idea.

2. The basic question concerns the possible physical interpretation of the two kinds of 3-vertices and their co-vertices, which are also included and mean decomposition of incoming particle characterized by integer m to quantum superposition of two particle states characterized by integers n, p satisfying $m = n + p$ for the co-sum and $m = n \times p$ for co-product. The amplitudes of different state pairs n, p in fact determine the quantum dynamics and typically the irreversible dynamics leading from state with well-defined quantum number characterized by integers would be due to the presence of co-vertices meaning de-localization.

3. If quantum p-adic integers correspond to Hilbert spaces then the identifications $+_q = \oplus$ and $\times_q = \otimes$ become possible. The challenge is to fix uniquely their co-vertices and this procedure fixes completely number theoretic Feynman amplitudes. Quantum dynamics would reduce to co-arithmetics. Or should one say that mathematics could reduce to quantum dynamics?

4. $\times_q$ and $+_q$ alone look very quantal and the generalization of string model duality means that besides cyclic permutations arbitrary permutations of incoming *resp.* outgoing lines act as symmetries. The natural question is whether this symmetry generalizes to permutations of all lines. This of course if commutativity in strict sense holds true also for quantum arithmetics: it could be that it holds true only in projective sense. Distributivity has however no obvious interpretation in terms of standard quantum field theory. The arithmetics for integers would naturally reflect the arithmetics of Hilbert spaces dimensions induced by direct sum and tensor product

22.5.1 Quantum TGD Predicts Counterparts For $\times_Q$ And $+_Q$ Vertices

Also quantum TGD allows two kinds of vertices identifiable in terms of the arithmetic vertices and this gives strong physical constraints on $+_q$ vertices.

1. First kind of vertices are the direct topological analogs of vertices of ordinary Feynman diagrams and there are good arguments suggesting that only 3-vertices are possible and would mean joining of 3 light-like 3-surfaces representing lines of generalized Feyman diagram along their 2-dimensional ends. At the these vertices space-time fails to be a manifold but 3-surface and partonic 2-surface are manifolds. These vertices correspond naturally to $\times_q$ or equivalently $\otimes$.

2. The vertices of second kind correspond to the stringy vertices, in particular the analog of stringy trouser vertices. The TGD based interpretation - different from stringy interpretation- is that no decay takes place for a particle: rather the same particles travels along different routes. These vertices correspond to four-surfaces, which are manifolds but 3-surfaces and partonic 2-surface fail to be manifolds at the vertex. There is a strong temptation to interpret $+_q$ - or equivalently $\oplus$ - as the counterpart of stringy vertices so that the two lines entering to $+_q$ would represent same incoming particle and should have in some sense same quantum numbers in the situation when the particle is an eigenstate of the quantum numbers in question? This would allow to understand the strange looking quantum distributivity and also to deduce what can happen in $+_q$ vertex.

3. What does the conservation of quantum numbers mean for quantum Galois quantum numbers identified in the proposed manner as quantum number associated with the cyclic groups assignable to the integers appearing in the vertex? For $\times_q$ vertex the answer is simple since tensor product is formed. This means that the number theoretic momentum is conserved. For direct sum one obtains direct sum of the incoming states and one cannot speak about conservation of quantum numbers since the final state does not possess well-defined quantum numbers.

22.5.2 How Could Quantum Numbers Of Physical States Relate To The Number Theoretic Quantum Numbers?

Quite generally, the above proposal would allow to represent all n-plets of rationals as zero energy states with either positive or negative arrow of time and one could assign to these states M-matrices as entanglement coefficients and define quantum jump as a sequence of two state function reductions occurring to states with opposite arrow of time. This kind of strong structural similarities with quantum TGD are hardly not a accident when one takes into account the connection with infinite primes and one could hope that zero energy states and generalized Feynman graphs could represent the arithmetics of Hilbert adeles with very dramatic consequences due to the arithmetic moves allowing to eliminate loops and permuted incoming lines without affecting the diagram except by a phase factor. The hierarchy of infinite primes suggests strongly the generalization of this picture since the resulting states would correspond only to the infinite integers at the lowest level of the hierarchy and identifiable in terms of free Fock states of super-symmetric arithmetic QFT.

The possible reduction of generalized Feynman diagrams to Hilbert adelic arithmetics raises several questions and one can try to proceed by requiring consistency with the earlier speculations.

1. How the quantum numbers like momentum, spin and various internal quantum numbers relate to the number theoretic quantum numbers $k = n2\pi/p$ defined only modulo p? The natural idea is that they find a representation in the number theoretical anatomy of the state so that these quantum numbers corresponds to waves with these momenta at the orbits of quantum Galois group. Momentum UV cutoff would have interpretation in terms of finite measurement resolution completely analogous to that encountered in condensed matter physics for lattice like systems. This would realize self-reference in the sense that cognitive part of the quantum state would represent quantum numbers characterizing the real part of the quantum state.

2. What about the quantum p-adics themselves characterizing incoming and out-going states in number theoretic vertices? There would be a conservation of number theoretical "momentum" characterized by logarithm of a rational in $\times_q$ vertex. Does this momentum have any concrete physical counterpart? Perhaps not since it would be associated with quantum p-adic degrees of freedom serving as correlates for cognition. In fact, the following argument suggest interpretation in terms of a finite dimension (finite by finite measurement resolution) of a Hilbert space associated with the orbit of a partonic 2-surface.

   (a) The prime factors of integer characterizing the orbit of a partonic 2-surface correspond naturally to braid strands for generalized Feynman diagrams. This suggests that the primes in question can be assigned with braid strands and would be indeed something new. The product of the primes associated with the particles entering $\times_q$ vertex would be same as the product of primes leaving this vertex. In the case of $+_q$ vertex the integer associated with each line would be same. One cannot identify these primes as p-adic primes since entire orbit of partonic 2-surface and therefore all braid strands are characterized by single common p-adic prime p.

   (b) Hilbert spaces with prime dimension are in a well-defined sense primes for tensor product, and any finite-dimensional Hilbert space decomposes into a product of prime Hilbert spaces. Hence the integer n associated with the line of a generalized Feynman diagram could characterize the dimension of the finite-dimensional Hilbert space (by finite measurement resolution) associated with it. The decomposition of n to prime factors would correspond to a decomposition of this Hilbert space to a tensor product of prime factors assignable to braid strands. This would define a direct Hilbert space counterpart for the decomposition of braid into braid strands and would be very natural physically and actually define the notion of elementarity. The basic selection rule for $\times_q$ vertex would be that the prime factors of incoming Hilbert spaces recombining to form Hilbert spaces of outgoing particles. For the $+_q$ incoming Hilbert spaces of dimensions n_1 and n_b would fuse to $n_1 + n_2$ dimensional direct sum. $a(b + c) = ab + ac$ would state that the tensor product with direct sum is sum of tensor products with direct summands. Therefore the two kind of vertices as well as corresponding vertices of quantum TGD would correspond to basic algebraic operations for finite-dimensional Hilbert spaces very natural for finite measurement resolution.

   (c) Could the different quantum versions of p-adic prime $l > p$ correspond to different direct sum decompositions of a Hilbert space with prime dimension to Hilbert spaces with prime dimensions appearing in the quantum pinary expansion in powers of p?

The coefficients of powers of p defined as products of quantum primes $l < p$ would be quantum dimensions and reflect effects caused by finite measurement resolution whereas the powers of p would correspond to ordinary dimensions. This decomposition would correspond to a natural decomposition to a direct sum by some natural criterion related to finite measurement resolution. For instance, power p^n could correspond to n-ary p-adic length scale. The decomposition would take place for every strand of braid.

The objection is that for algebraic extensions of rationals the primes of the extension can be algebraic number so that the corresponding Hilbert space dimension would be complex algebraic number. It seems that only the primes $l > p$ which do not split for the algebraic extension used (and thus label quantum p-adic number fields in the adele) can be considered as prime dimensions for the Hilbert spaces associated with braid strands. The latter option is more natural and would mean that the number theoretic evolution generating increasingly higher-dimensional algebraic extensions implies selection of both preferred p-adic primes and preferred prime dimensions for state spaces. One implication would be that the quantum Galois group assignable to given p-adic integer would in general be smaller for an algebraic extension of rationals than for rationals since only the non-splittable primes in its factorization would contribute to the quantum Galois group.

(d) As already discussed, the most plausible interpretation is that the pair of co-prime integers defining the quantum rational defines a pair of Hilbert space dimensions possibly assignable to fermions and bosons respectively. Interestingly, for the simplest infinite primes representing Fock states and mappable to rationals m/n the integers m and n could be formally associated with many-boson and many-fermion states.

(e) Because of multiplicative conservation law in $\times_q$ vertex quantum p-adic numbers does not have a natural interpretation as ordinary quantum numbers - say momentum components. The problem is that the momentum defined as logarithm of multiplicatively conserved number theoretic momentum would not be p-adic number without the introduction of an infinite-dimensional transcendental extension to guarantee the existence of logarithms of primes.

(f) If this vision is correct, the representation of ordinary quantum numbers as quantum Galois quantum numbers would be a representation in a state space formed by (quantum) state spaces of various quantum dimensions and thus rather abstract but quite possible in TGD framework. This is of course a huge generalization from the simple wave mechanical picture based on single Hilbert space but in spirit with abstract category-theoretical thinking about what integers are category-theoretically. The integers appearing as integers in the Cartesian factors of adeles would represent Hilbert space dimensions in the case of generalized Feynman diagrams. The arithmetic Feynman rules would be only a part of story: as such very abstract but made concrete by braid representation.

3. Note that the interpretation of $+$ and $\times$ vertices in terms of Hilbert space dimensions makes sense also in the real context whereas the further decomposition into direct sum in powers of p^n does not make sense anymore.

22.5.3 Number Theoretical Quantum Numbers And Hierarchy Of Planck Constants

What could be the TGD inspired physical interpretation of these mysterious looking Hilbert spaces possessing prime dimensions and having no obvious identification in standard physics context?

How the Hilbert space dimension relates to the value of Planck constant?

The first question is how the Hilbert space dimension assigned to a given line of a generalized Feynman diagram relates to the value of Planck constant.

1. As already noticed, the decomposition of integer to primes would naturally correspond to its decomposition to braid strands to which one can assign Hilbert spaces of prime-valued

dimension $D = l$ appearing as factors of integer n. This suggests a Hilbert space is defined by wave functions in a set B_n with n points, . This Hilbert space naturally decomposes into a tensor product of Hilbert spaces with Hilbert spaces associated with point sets B_l containing l of points with $l|n$.

2. The only space of this kind that comes in mind relates to the proposed hierarchy of (effective) Planck constants coming as integer multiples of ordinary Planck constant. For the simplest option Planck constant $\hbar_n = n\hbar_0$ would correspond to a local (singular) covering of the embedding space due to the n-valuedness of the time derivatives of the embedding space coordinates as function of canonical momentum densities which is due to the huge vacuum degeneracy of Kähler action.

3. The discrete group Z_n would act as a natural symmetry of the covering and would decomposes a $Z_n = \prod_{l|n} Z_l^{e_l}$ and the orbits of Z_l in the covering would define naturally the sets B_l. Given prime l in the decomposition would correspond to an l-fold covering of a braid strand and to a wave function in this space.

4. The proposal for the hierarchy of Planck constants assumes that different sheets of this singular covering degenerate to single sheet at partonic 2-surfaces at the ends of CD. Furthermore, the integers n would decompose to products $n = n_1n_2$ corresponding to directions of time-like braids along wormhole throat and along the space-like 3-surface at the end of CD defining by effective 2-dimensionality (strong form of holography) two space-time coordinates playing the role of time coordinate in the field equations for preferred extremals. Note that the information about the presence of covering would be carried at partonic 2-surfaces by the tangent space data characterized by the n_i-valued normal derivatives.

5. The simplest option is that Hilbert space dimension corresponds to Planck constant for a given line of generalized Feynman diagram. This would predict that in the multiplicative vertex also the values of Planck constants characterizing the numbers of sheets for many-sheeted coverings would satisfy the condition $n_3 = n_1n_2$. The assumption that the multiplicative vertex corresponds to the gluing of incoming lines of generalized Feynman diagram together along their ends seems however to require $n_1 = n_2 = n_3$. Furthermore, the identification of Hilbert space dimension as Planck constant is also inconsistent with the vision about book like structure of the embedding space explaining the darkness as relative darkness due to the fact that only particles with the same value of Planck constant can appear in the same vertex [K32].

   The way out of the difficulty is to assume that the value of Planck constant $\hbar = n\hbar_0$ corresponds to $n = n_3 = n_1n_2$ or has n_3 as a factor. For $n = n_3$ the states with Hilbert space dimensions n_1 and n_2 are invariant under cyclic groups Z_{n_2} and Z_{n_1} respectively. For n containing n_3 as a genuine divisor analogous conditions would hold true.

6. p-Adic prime p would make itself manifest in the further decomposition of the l-dimensional Hilbert spaces to a direct sum of sub-Hilbert spaces with dimensions given by the terms $l_{n,q}p^n$ in the expression of l as quantum integer. The fact that the only prime ideal for p-adic integers is pQ_p should relate to this. It is quite possible that this decomposition occurs only for the p-adic sectors of the Hilbert adelic embedding space.

   What suggests itself is symmetry breaking implying the decomposition of the covering A_n of braid strand to subsets $A_{n,m}$ with numbers of elements given by $\#_{n,m} = l_mp^m$ with l_m divisible only by primes $p_1 < p$. Wave functions would be localized to the sets $A_{n,m}$, and inside $A_{n,m}$ one would have tensor product of wave functions localized into the sets A_l with $l < p$ and $l|l_m$.

   Hilbert space dimensions would be now quantum dimensions associated with the quantum phase $exp(i\pi/l)$: this should be due to the finite measurement resolution and relate to the fact that one has divided away the hyper-finite factor N from the factor $M \supset N$.

   The index characterizing Jones inclusion [A190] [K32] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M:N]=l_q^2$, $l<p$ prime and $q=exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q=n$-adicity. Note that $l_q<l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q\simeq l$.

If the above identification is correct, the conservation laws in $\times_q$ and $+_q$ vertices would give rather precise information about what can happen for the values of Planck constants in thes vertices. In $\times_q$ co-vertices Hilbert space-timensions would combine multiplicatively to give the common value of Planck constant and in $\oplus_q$ co-vertices additively. The phase transitions changing Planck constant, for instance for photons, are central for quantum TGD and the selection rules would not allow them only if they correspond to a formation of a Bose-Einstein condensate like state or its decay by $\times_q$- or $+_q$-vertex.

Could one identify the Hilbert space dimension as value of Planck constant?

It has been already seen that the identification of Hilbert space dimension with Planci constant it is not consistent with the idea that product vertex means that the lines of generalized Feynman graph are glued along their 2-D ends together. I did not however realize this when I wrote the first version of this section and I decided to keep the earlier discussion about the option for which Planck constants correspond to Hilbert space dimensions so that $n_3=n_1n_2$ holds true for Planck constants. The question was whether it could be consistent with the idea of dark matter as matter with non-standard value of Planck constant. By replacing "Planck constant" with "Hilbert space dimension" below one obtains a discussion giving information about the selection rules for Hilbert space dimensions.

1. In $\times_q$-vertex the Planck constants for the outgoing particles would be smaller and factors of incoming Planck constant. In $\times_q$ co-vertex Planck constant would increase. I have considered analogous selection rules already earlier. $\times_q$ vertex does not allow the fusion of photons with the ordinary value of Planck constant to fuse to photons with larger value of Planck constant.

   By conservation of energy the frequency of a photon like state resulting in the fusion is given by $f=\sum n_kf_k/N_{out}\prod_k n_k$ for $\hbar_k=n_k\hbar_0$, where N_{in} and N_{out} are the numbers of quanta in the initial and final state. For a common incoming frequency $f_k=f_0$ this gives $f/f_0=\sum_k n_k/(N_{out}\prod_k n_k)$. If one assumes that spin unit for photon increases to $\prod_k n_k\hbar_0$ and spins are parallel one obtains from angular momentum conservation $N_{out}\prod_k n_k=N_{in}\sum n_k$ giving $N_{out}=\prod_k n_kN_{in}/\sum n_k=n^{N_{in}}/N_{in}n$, which in turn gives $f/f_0=1/N_{in}$. This looks rather natural.

   In the presence of a feed of $r=\hbar/\hbar_0$¿1 particles $\times_q$ vertex could lead to a phase transition generating particles with large values of Planck constant. Large values of Planck constant are in a key role in TGD based model of living matter since Compton lengths and other quantum scales are proportional to $\hbar$ so that large values of $\hbar$ make possible macroscopic quantum phases. The phase transition leading to living matter could be this kind of phase transition in presence of feed of $r>1$ particles.

2. For $+_q$ co-vertex $r=\hbar/\hbar_0$ could be additive and for incoming photons with same frequency and Planck constants $\hbar_k$ the outgoing state with Planck constant $\sum_k\hbar_k$ energy conservation is guaranteed if the frequency stays same. This vertex would allow the transformation of ordinary photons to photons with large Planck constant, and one could say that effectively the photons fuse to form single photon. This is consistent with the quantization of spin since the unit of spin increases. For this option the presence of particles with ordinary value of Planck constant would be enough to generate particles with $r>1$ and this in turn could lead to a the phase transition generation living matter.

3. One can of course ask whether it should be $r-1=\hbar/\hbar_0+1$, which corresponds to the integer n. For this option the third particle of $+_q$ vertex with two incoming particles with ordinary Planck constant would have ordinary Planck constant. For $\times_q$ vertex containing two incoming particles with $r=n$, $n=1$ ($n=2$), also the third particle would have $n=1$ ($n=2$). $\times_q$ and $+_q$ vertices could not generate $n>1$ particles from particles with ordinary Planck constant. The phase transition leading from inanimate to living matter would require

$n > 1$ states as a seed (one has $2+2 \rightarrow 3$ for $_+q$ vertex). A quantum jump generating a CD containing this kind of particles could lead to this kind of situation.

4. These selection rules would mean a deviation of the earlier proposal that only particles with same values of Planck constant can appear in a given vertex [K32]. This assumption explains nicely why dark matter identified as phases with non-standard value of Planck constant decouples from ordinary matter at vertices. Now this explanation would be modified. If $\times_q$ vertex contains two particles with $r = n+1$ for $r = n$ option ($r = 1$ or 2 for $r = n+1$ option), also the third particle has ordinary value of Planck constant so that ordinary matter effectively decouples from dark matter. For $+_q$ vertex the decoupling of the ordinary from dark matter occurs for $r = n+1$ option but not for $r = n$ option. Hence $r = n+1$ could explain the virtual decoupling of dark and ordinary matter from each other. The assumption that Planck constant is same for all incoming lines and corresponds to $n_3 = n_1 n_2$ defines however much more plausible option.

What happens in phase transitions changing the value of Planck constant?

The phase transitions changing the value of Planck constant are in a central role in TGD inspired quantum biology. The typical phase transition of this kind would change the Planck constant of photon. This phase transition would formally correspond to a 2-vertex changing the value of Planck constant. Can one pose selection rules to the change of Planck constant? By the above assumptions both the incoming and outgoing line correspond to Hilbert space dimension which is a factor of the integer defining Planck constant. If the value of the Hilbert space dimension is not changed in the process, the incoming and outgoing Planck constants must have this dimension as a common factor.

22.5.4 What is the relation to infinite primes?

Already quantum p-adics would mean a dramatic generalization of number concept by assigning to rationals ane even algebraic numbers Hilbert spaces and their states. Quantum adeles would mean a further generalization of number concept by gluing together reals and Hilbert space variants of p-adic number fields.

TGD leads also to another generalization of number concept based on the hierarchy of infinite primes [K84]. This generalization also leads to a generalization of real number in the sense that one can construct infinite number of real units as infinite rationals which reduce to units in real sense. This would mean that space-time point has infinitely complex number theoretic anatomy not visible at the level of real physics [K86].

The possibly existing relationship between these speculative generalizations is of course interesting. Galois groups for extensions of rationals would be central symmetry of quantum TGD and would permute the sheets space-time surfaces regarded as covering spaces. Infinite primes can be mapped to polynomial primes and this means that one can assign to them algebraic extensions of rationals and corresponding Galois groups and in [K52]. I discussed a conjecture that the elements of these Galois groups could be represented as symplectic flows assignable to braids which emerge naturally as counterparts of partonic 2-surfaces in finite measurement resolution. This would suggest a possible relationship.

The construction of infinite primes relies on the product $X = \prod_p p$ of finite primes interpreted physical as analog of Dirac vacuum with all negative energy states filled. Simplest infinite primes are constructed by kicking away fermions from this vacuum and by adding also bosons labeled by primes. One obtains also the analogs of bound states as infinite primes which can be mapped to irreducible polynomials. The roots of the polynomial code for the infinite prime and the algebraic extension. The infinite primes corresponding to nth order polynomials decompose to products of n simplest infinite primes of algebraic extension so that the corresponding Galois group emerges naturally.

The construction can be repeated endlessly by taking the infinite primes of the existing highest level and forming the product X of them and repeating the process. What these means that the many-particle states of the previous level define single particle states of the new level. One can map these infinite primes to polynomial primes for polynomials of several variables. Also this hierarchy might allow generalization obtained by assigning to infinite primes the orbits of their Galois groups. The earlier considerations [K56] suggest strongly a reduction of the description to the lowest level and involving only algebraic numbers.

A possible physical interpretation for this repeated second quantization would be as a construction of polynomials of single variable at first level, polynomials of two variables as polynomials of second variable having the polynomials of first variable as as coefficients at second level, and so on. These polynomials could be realized as spinor harmonics of world of classical worlds (WCW).

What do we understand about infinite primes?

Let us first try to summarize what we understand about infinite primes. What seems very natural is the postulate that arithmetic QFT associated with infinite primes conserves multiplicative number theoretic momenta defined by ordinary primes with separate conservation law for each prime. This law would hold for $\times_q$ vertices very naturally whereas for $+_q$ vertices it would be broken. Recall that these two vertices correspond to the TGD counterparts of 3-vertices for Feynman diagrams and stringy diagrams respectively and also to tensor product and direct sum.

1. What seems clear is that infinite prime characterizes an algebraic extension of rationals (or of its extension) in the case that infinite primes is defined in terms of finite primes of extension. Infinite prime dictates also the p-adic primes which are possible and appear in the quantum adele assignable to infinite prime.

2. The integer exponents of ordinary primes appearing in the infinite and finite part of the simplest lowest level infinite prime could define infinite number of conserved number theoretic momenta, one for each prime p and having $log(p)$, p prime, as a unit. Separate conservation follows from the algebraic independence. These number theoretic momenta do not make sense p-adically, which means that in p-adic context the multiplicative form of the conservation law is the appropriate one. Therefore it is appropriate to speak of multiplicative momenta. Therefore the relationship with ordinary additively conserved momenta does not lok plausible.

   Arithmetic QFT interpretation allows also to interpret the numbers n_p in p^{n_p} as particle numbers assignable to bosonic quanta and fermionic quanta in the case of the simplest infinite primes with "small part" representing fermions kicked out from the Dirac sea possibly accompanied by bosonic quanta. The conservation law at $\times_q$ vertices would mean conservation of total particle numbers assignable to primes p.

3. For the simplest primes at the lowest level identifiable as linear polynomials with integer coefficients there are two separate integers defining number theoretic momenta. The first integer corresponds to the finite part of infinite prime and the second one to the finite part of the infinite prime to which one assigns number theoretic fermions. These two parts are separately conserved. Since the integers have no common prime factors, one can also speak about rational valued multiplicative number theoretic momentum. The physical interpretation for the absence of common factors would be that given mode cannot simultaneously containing and not contain fermionic excitation. For higher irreducible polynomials of order n interpreted in terms of bound states there are $n+1$ integers defining a collection of number theoretic momenta. For the representation as a monic polynomial one has a collection of n rational valued number theoretic momenta.

4. The notion of multiplicative number theoretic momentum generalizes.

   (a) At the second level of the hierarchy ordinary primes are replaced with prime polynomials $P_n(x)$ of single variable. At the n^{th} level they are prime polynomials $P_n(x_1, ..., x_{n-1})$ of $n-1$ variables. The value of the number theoretic momentum at n^{th} s level can be said to be a polynomial $P_n(x_1, ..., x_{n-1})$ rather than integer.

   (b) This looks very abstract but can be concretized. For instance, each coefficient of $P_n(x, y)$ at second level as polynomial of y defines a polynomial $P_k(x)$ at the first level and $P_k(x)$ is characterized by a collection of number theoretic momenta defined by its integer coefficients in the representation as a polynomial with integer coefficients. Therefore $P_k(x)$ can be identified as the collection of $k+1$ integer coefficients or k rational coefficients in the monic representation identified as number theoretic momenta for a k-particle state. $P_n(x,y)$ in turn corresponds to a collection of n many-particles states with i^{th} one containing k_i particles, $i = 1, ...n$. The interpretation in terms of n-braid with braid strands decomposing to k_i braid strands is natural and conforms with the fractality of TGD Universe.

(c) This example allows to deduce the number theoretic interpretation of the polynomial at the n^{th} level and one can continue this abstraction hierarchy ad infinitum. Eventually each prime at a given level of hierarchy reduces to a collection of number theoretic momenta defined by ordinary integers grouped in a way characterized by the infinite prime. Physically this would characterize how these number theoretic elementary particles group to particles at the first level, these to particles at second level, and so on.

(d) The possibility to express the irreducible polynomial as a product of first order polynomials with zeros which algebraic numbers gives for the bound state a representation as free many-particle state but with number theoretic momenta which are algebraic rationals in algebraic extension of rationals. These number theoretic momenta can be also complex and therefore do not allow interpretation as Hilbert space dimensions. This decomposition is analogous to a decomposition of hadron to quarks. The rational coefficients expressible in terms of the roots of the polynomial code for Galois invariants analogous to the observables assignable to hadrons and accessible to the experimenter.

5. The basic conservation law of arithmetic QFT and of TGD would be that the multiplicative number theoretic momenta labelled by finite primes are separately conserved in $\times_q$ vertices but not in $+_q$ vertices. The conservation number theoretic quantum numbers allows the interpretation of Hilbert space dimensions in terms of the hierarchy of Planck constants, and this leads to a proposal that infinite primes code the pairs of finite integers with no common factors assignable to the pairs of time-like and space-like braid strands.

   If one takes seriously the notion of number theoretic fermion, one could assign to space-like braid strands only bosonic excitations and to time-like braid strands fermion and possibly also bosonic excitations. The interpretation could be in terms of the super-conformal algebras containing both fermionic and bosonic generators. The hierarchy of infinite primes would correspond to a hierarchy of braids containing lower level braids as their strands as suggested already earlier (see `http://tinyurl.com/yc2pu5wd`) [K52]. What would be new would be a concrete assignment of primes to braid strands and detailed identification in terms of time-like and space-like braids.

   This kind of assignment would mean a rather dramatic step of progress in the understanding of the complexities of generalized Feynman diagrams. One not completely settled old question is what selects the p-adic prime assignable to given partonic 2-surface.

This is the stable looking part of the vision about infinite primes, and any attempt to relate it to quantum p-adics and quantum adeles should respect this picture.

Hyper-octonionic primes correspond to p-adic primes in extension of rationals

The earlier interpretation hyper-complex and appropriately defined quaternionic and octonionic generalizations is in terms of standard model quantum numbers [K100]. It seems that also this identification survives under the selective pressures by new ideas but that one cannot replace hyper-complex primes with their infinite counterparts. Rather, hyper-complex prime generalizes p-adic prime as a preferred prime by replacing ordinary integers with hyper-complex integers. The definition of infinite primes in quaternionic and octonionic context is plagued by the problems caused by non-commutativity and associativity so that the conclusion is well-come.

1. The solutions of Kähler-Dirac equation (see `http://tinyurl.com/ycb247qp` and `http://tinyurl.com/ycc9qe95`) suggest the interpretation of the M^2 projections of four-momenta as "hyper-complex" primes or perhaps more realistically, their integer multiples. These momenta are conserved additively rather than multiplicatively at vertices to which $\times_q$ is assigned and only their exponents - naturally phase factors - would be conserved multiplicatively.

2. Could this identification generalize from hyper-octonionic primes to hyper-octonionic infinite primes? This does not seem to be the case. The multiplicative conservation in $\times_q$ vertices for number theoretic momenta is in conflict with additive conservation for ordinary quantum numbers. Additive conservation is also in conflict with interpretation in terms Hilbert space dimensions allowing concretization in terms of the hierarchy of Planck constants. Of course, hyper-complex Hilbert space dimension does not make sense either.

3. One must remember that there are many kinds of primes involved and a little list helps to see what the correct interpretation for hyper-complex primes could be.

   (a) There are the primes l appearing in the decomposition of infinite primes and having interpretation in terms of Hilbert space dimensions. The conservation of multiplicative number theoretical momenta is natural at $\times_q$ vertices.

   (b) There are the p-adic primes p, and on basis of p-adic mass calculations it is this prime to which it is natural to assign additively conserved momenta. p characterizes the "active" sector of adeles and therefore also the various quantum variants of the prime l in which quantum primes $p_1 < p$ appear as factors. p characterizes partonic 2-surface.

   (c) The Abelizanization of the quantum Galois group assignable to prime l decomposes into prime factors Z_{p_2} and the phases $exp(i\pi/p_1)$ might provide cognitive representations in finite measurement resolution for various standard model quantum numbers.

4. The only reasonable interpretation seems to be that the hyper-complex momenta and possible other quantum numbers assignable to them correspond to p-adic prime p for rationals or for an algebraic extension of rationals to the ring hyper-complex rationals. The failure of field property implies that the inverse of hyper-complex number fails to exist when it defines a light-like vector of M^2. This has however a concrete physical interpretation and light-like hyper-complex momentum for a massless state is massless only when the momentum of the state transverse to M^2 vanishes so that also propagator defined by M^2 momentum diverges.

What the identification of M^2 momenta as hyper-complex integers really means, deserves some comments.

1. Suppose that particle's p-adic mass squared is of form $m^2 = np$ as predicted by p-adic mass calculations. Assume that m^2 corresponds to M^2 momentum squared with preferred M^2 characterizing given causal diamond CD. Assume also that total M^4 mass squared vanishes in accordance with the idea that all states - even those representing virtual particles - carried by wormhole throats are massless. In accordance with the adelic vision, assume that the prime p does not split in the algebraic extension of rationals used (simplest extension would be $Q[\sqrt{-1}]$). This requires $p \ mod \ 4 = 3$ in accordance with Mersenne prime hypothesis. The idea is that p does not split for ordinary algebraic extension but splits in the ring of hyper-complex numbers.

2. The preferred plane $M^2 \subset M^4$ corresponds to a preferred hyper-complex plane of complexified (by commuting imaginary unit i) hyper-octonionic space M^8. M^2-momentum has therefore purely number theoretic interpretation being due to the slitting of $M^2 = np$ to a product of hypercomplex integer $N = N_0 + eN_z$ and its conjugate $N_0 - eN_z$). The hyper-complex imaginary unit $e = iI$ satisfying $e^2 = 1$ and $I^2 = -i^2 = -1$ would correspond to z-axes of M^2. Here is I is the preferred octonionic imaginary unit and i an imaginary unit commuting with it. One could say that 2-D particle momentum emerges via the emergence of hyper-complex extension of rationals of their extension. This would also generalize to quaternions and one could say that M^4 momentum emerges via extension of rationals to hyper-quaternions.

3. M^2 momentum squared would satisfy $P_0^2 - P_z^2 = (P_0 - eP_z)(P_0 + eP_z) = np$. The prime p does not split in the algebraic extension of rationals used but splits in the ring of hyper-complex numbers. Assume first $n = 1$. In this case the splitting of $p \ mod \ = 3$ ($p \ mod \ = 1$) to $p = (p_0 + ep_z)(p_0 - ep_z)$ implies p_0 is even (odd) and p_z is odd (even). For $n > 1$ one must have $(n_0 - en_1)(n_0 + en_1) = n$ and similar conditions apply to n so that one would have for M^2 momentum $P_0 + eP_z = (n_0 \pm en_z)(p_0 \pm ep_z)$.

4. Momentum components are hyper-complex integer multiples of hyper-complex prime so that the allowed momenta would form an ideal of hyper-complex numbers. This is mathematically very nice but might be quite too strong a condition physically although it is typically encountered in systems in which particle is enclosed in box. Now the box would correspond to CD with periodic boundary conditions at the ends of CD for the Kähler-Dirac equation. One could consider also a weaker condition for with the integer n is replaced with a rational (m/n) such that neither m nor n contains p as a prime factor.

5. The peculiar looking prediction would be that M^2 momentum cannot be purely time-like. In other words, the particle cannot be at rest M^2. Observer for which CD defines the rest system could not perform a state function reduction leading to a situation in which the particle is at rest with respect to the observer! In fact, this kind of situation is encountered also for particle in box since boundary conditions do not allow constant mode. If one recalls that all particles would be massless in M^4 sense, this condition does not look so strange.

Infinite primes and Hilbert space dimensions

Arithmetic QFT picture would strongly suggests that the number theoretic momenta at the lowest level are conserved in $\times_q$ vertices at least. For $+_q$ vertices the conservation cannot hold true. The conservation could mean that the total number of powers of given prime in state is same for positive and negative energy states.

Of course, much richer spectrum of conservation laws can be imagined since one could require similar conservation laws also at the higher levels of hierarchy, where various number theoretic momenta correspond to numbers prime polynomials at lower level present in the state. The physical interpretation would be that the numbers of bound states particles are conserved meaning that these particles can be regarded as stable. On physical grounds this kind of conservation laws can be only approximate.

1. Infinite primes mean that infinite numbers have detailed number theoretical anatomy. Could infinite primes label infinite-dimensional prime Hilbert spaces as finite primes do? Could the interpretation for the object $X = \prod_p p$ be in terms of a tensor product of all prime-dimensional Hilbert spaces. Infinite primes with positive finite part would have interpretation as direct sums of this space and finite integer-dimensional Hilbert space. When the finite part of the infinite prime is negative the interpretation would not be so straightforward, and this option does not look attractive.

2. A much more plausible option is that infinite prime at the first level defines an algebraic extension of rationals (or of its extension) and that this gives rise to a collection of norm for algebraic extension induced by complex norm. As a matter fact, these points at which this norm vanishes might have interpretation as complex coordinates for a corresponding braid strand in n-strand bound state braid in preferred complex coordinates for the partonic 2-surface. A possible geometric interpretation for these points inspired by the notion of dessins d'enfant is that the partonic 2-surface as an abstract Riemann surface representable as a covering of sphere becomes singular at these points as several sheets of covering co-incide.

3. The infinite primes of the lowest level of the hierarchy formally representing Fock states of free bosons and fermions can be mapped to rationals. These rationals could define pairs of Hilbert space dimensions assignable to bosonic and fermionic parts of the state and could this allow identification as quantum p-adic integer in each sector of the adele and the identification in terms of integer dimension in the real sector of quantum adeles. The fact that the two integers have no common factors would only mean that given mode cannot both contain and not contain fermionic excitation.

   One could even consider the possibility of concrete assignment of the first dimension in terms of fermionic braid strands with bosonic excitations and second dimension in terms of purely bosonic braid strands. This interpretation is very natural since the super-conformal algebras creating states have both purely bosonic and purely fermionic generators. These braids could correspond to space-like and time-like (actually light-like) braids having their ends at partonic 2-surfaces.

The Galois groups associated with primes appearing as factors of the primes would correspond naturally to additional internal degrees of freedom. This identification makes sense also for the infinite primes represented by irreducible polynomials since the coefficients of the polynomial representable in terms of the roots of polynomials define rationals having interpretation as number theoretic momenta. Therefore the interpretation in terms of Hilbert space dimensions makes sense when rationals are interpreted as pairs of dimensions for Hilbert spaces.

4. What about the infinite primes representing bound states and mappable to irreducible polynomials with rational coefficients and defining polynomial primes characterized by a collection of roots (see `http://tinyurl.com/yag8tvpx`) [K56]. These roots define an algebraic extension of rationals and this suggests that the quantum adele associated with the infinite prime in question is defined accordingly. The infinite primes mappable to nth order monic polynomials would have interpretation as many particle states consisting of single particle states which correspond to algebraic number rather than rational. The rational coefficients of the monic polynomial would define the rationals defining pairs of Hilbert space dimensions.

5. The natural identification for the Hilbert spaces in question would be in terms of the singular local coverings of embedding space associated with the hierarchy of Planck constants suggested to emerge from the vacuum degeneracy of Kähler action. The integer n decomposing to primes would correspond to sub-braids labeled by prime factors l of n and consisting of l strands in the l-fold sub-covering.

The consistency with the quantum adeles would force the following highly speculative picture. Main justification comes from its internal consistency and consistency with generalize Feynman graphs.

1. Infinite prime (integer, rational) defines the algebraic extension used and the allowed quantum p-adic number fields contributing as factors to the corresponding quantum adele. p-Adic primes, which can be also algebraic primes if one starts from extension of rationals, by definition do not split in the algebraic extension. Infinite primes assignable to particle states obey the conservation of multiplicative number theoretic momenta and define naturally collections of pairs if Hilbert space dimensions assignable to the particles and decomposing to primes l assignable to braid strands. The integers characterizing the rational defining number theoretic momentum correspond to time-like and space-like braid strands and only the time- or space-like strand carries fermionic quantum numbers.

2. These Hilbert spaces have a natural interpretation in terms of the hierarchy of Planck constants realizable in terms of local singular coverings of the embedding space forced by the enormous vacuum degeneracy of Kähler action.

3. Hyper-complex primes are identifiable as generalizations of p-adic primes and have nothing to do with infinite primes. They could code for standard model quantum number.

4. The quantum Galois quantum numbers assignable to primes l for given p-adic prime p and appearing in the infinite prime characterizing the state would provide a cognitive representation of the standard model quantum numbers.

5. Mersenne primes and primes near powers of 2 and $p = 2$ also should be selected as a p-adic prime in this manner.

6. The basic uncertain aspect of the scenario is whether the notion of quantum p-adic with coefficients in quantum pinary expansion satisfying only the condition $x_n < p^N$ for $N > 1$, with N dictated by the pinary cutoff, makes sense. Physically $N > 1$ is very natural generalization. Most of the preceding considerations remain intact even if $N = 1$ is the only internally consistent option. What is lost is the representation of quantum numbers using quantum Galois group and the crazy proposal that quantum Galois group could be isomorphic to AGG.

This is only the simplest possibility that I can imagine now and reader is encouraged to imagine something better!

The relationship between the infinite primes of TGD and of algebraic number theory

While preparing this chapter I experienced quite a surprise as I learned that something called infinite primes emerges in algebraic number theory (see http://tinyurl.com/pw7cr5c) [A5]. Infinite primes in this sense looked first to me like a heuristic concept characterizing norms for algebraic extensions of rationals induced by the complex norm for the embeddings of the extension to complex plane. The nomenclature is motivated by the analogy with p-adic norms defined by algebraic primes. It however turns out that there is a close connection with infinite primes at the first level of the hierarchy.

1. The embeddings (ring homomorphisms) of Galois extension to complex plane induce a collection of norms induced by the complex norm. The analogy with p-adic norms labelled by primes serves as a partial motivation for calling these norms infinite primes. The embeddings are induced by the embeddings of the roots of an irreducible monic polynomials $P_n(x) = x^n + ...$ with rational coefficients, which defines a polynomial prime so that infinite primes in the sense of algebraic number theory correspond to a polynomial primes.

2. The embeddings (ring homomorphisms) of the extension of K in C an be defined to those reducing to embeddings in R and those not. The embeddings to R correspond in one-one manner to real roots and complex embeddings come in pairs corresponding to complex root and its conjugate. The norm is defined as $|z - z_k|$, where z_k is the root. The number of embeddings and therefore of norms is $r = r_1 + 2r_2$, where r is the degree of the extension K/Q and also the degree of its Galois group for Galois extensions (defined by polynomials with rational coefficients).

3. Also in TGD framework the infinite primes at the lowest level of hierarchy can be mapped to irreducible monic polynomials of single variable: at n^{th} level polynomials of n variables is required. Now however also polynomials $P_1(x)$, whose roots are rationals and have interpretation in terms of free Fock states, are included. Note that the replacement of the variable z with $z - m/n$ shifts the roots of a monic polynomial by m/n so that the corresponding algebraic extension is not modified. For the simplest infinite primes the norm would correspond to $|z - m/n|$. Therefore infinite prime indeed characterizes the algebraic extension and its embeddings and the "real" factor of quantum adeles is identifiable with this algebraic extension endowed with any of these norms.

22.5.5 What selects preferred primes in number theoretical evolution?

Preferred p-adic length scales seem to correspond to primes near powers of two, in particular Mersenne primes. The proposed explanation is that number theoretic evolution as emergence of higher-dimensional extensions of rationals and also of p-adics somehow selects Mersenne primes as fittest. But what fitness could mean? Could it mean stability in some sense or perhaps criticality - living systems are indeed critical and in TGD inspired quantum biology quantum criticality is central aspect.

The primes p of the field K split into primes of extension L of K as $p = \prod_{i=1}^{g} P_i^{e_i}$. One has $\sum_{i=1}^{g} e_i f_i = n$, where n is the dimension of L as extension of K and f_i is so called inertia degree of P_i over p equal to $[\mathcal{O}_L/P_j : \mathcal{O}_K/p]$, $\mathcal{O}_K$ denotes algebraic integers of K.

For maximally splitting primes one has $e_i = 1$, $g = 1$, $f = n$. For $e_i > 1$ for some i one has ramification. For $g = 1, e_1 = 1, f = n$ one has inert prime. Inert primes are stable and one can ask whether they could be special. As will be found below, for quadratic extensions the number inert primes is infinite so that stability does not seem to be an attractive criterion for fitness.

For ramified primes there is an analogy with the multiple roots of a polynomial assignable to criticality. Since TGD Universe is quantum critical, one can ask whether the fittest primes could be ramified. One could of course argue that its maximally splitting primes that are the most stable ones. However, the fact that the number of ramified primes is finite suggests that preferred p-adic primes correspond to the ramified ones.

Ramified prime divides discriminant $D(P)$ of the irreducible polynomial P (monic polynomial with rational coefficients) defining the extension (see http://tinyurl.com/oyumsnk).

1. Discriminant $D(P)$ of polynomial whose, roots give rise to extension of rationals, is essentially the resultant $Res(P, P')$ for P and its derivative P' defined as the determinant of so called Sylvester polynomial (see `http://tinyurl.com/p67rdgb`). $D(P)$ is proportional to the product of differences $r_i - r_j$, $i \neq j$ the roots of p and vanishes if there are two identical roots. Therefore ramified primes divides the differences $r_i - r_j$ for the roots of the polynomial. In particular, all polynomials having pair of complex conjugate roots have $p = 2$ as ramified prime.

   Remark: For second order polynomials $P(x) = x^2 + bx + c$ one has $D = b^2 - 4c$.

2. Ramified primes divide D. Since the matrix defining $Res(P, P')$ is a polynomial of coefficients of p of order $2n - 1$, the size of ramified primes is bounded and their number is finite. The larger coefficients $P(x)$ has, the larger the value of ramified prime can be. Small discriminant means small ramified primes so that polynomials having nearly degenerate roots have also small ramifying primes. Galois ramification is of special interest: for them all primes of extension in the decomposition of p appear as same power. For instance, the polynomial $P(x) = x^2 + p$ has discriminant $D = -4p$ so that primes 2 and p are ramified primes.

For Galois extensions one has $e_i = e$, which is factor of n. Maximimal ramification corresponds to $p = P^e$, $e = n$. If the dimension of extension is prime $n = p_1$, p_1 prime, one has maximal ramification $e = p_1$ for Galois extensions. This makes extensions with prime dimension interesting. Cyclic extensions with prime dimension generated by $exp^{i2\pi/p_1}$ are an example of this kind of extensions. Cyclic extensions with prime dimension equal to Mersenne prime M_n could be of special importance physically $p = 2$ divides the discriminant and $P = 2^{1/M_n}$ would define prime of extension.

Could the following statement catch something about reality? The ramified primes are not stable but the criticality of the ramified primes is stable in the process of generation of algebraic extensions. The ramified primes would be conserved for extensions of extensions constructible as polynomials of polynomials and would be analogous to conserved genes. I have checked the conservation for quadratic extensions of quadratic extensions representable as quadratic polynomials $P_2(y)$, where $y = Q_2(x)$ is also quadratic polynomial. If ramified primes like Mersennes are present for the simplest extensions, which are quadratic, they characterize also the extensions of these.

Why should Mersenne primes be ramified? Why should quadratic polynomials have Mersenne primes as ramified primes? $P_2(x) = x^2 - M_n$ with discriminant $D = 4M_n$ is example of this kind of polynomial? Are these polynomials especially stable against transformation to different second order polynomials physically? If Mersennes are special, also the infinite primes defining these special algebraic extensions via corresponding polynomials are in a special physical role. A possible physical interpretation for these infinite primes would be in terms of bound states. Could the criticality of Mersenne primes translate to the criticality of the bound state represented by corresponding infinite prime.

The splitting to primes need not be unique (if it is one speaks of principal ideal domain). For instance, in $Q[\sqrt{-5}]$ for which factorization to algebraic primes is not unique (but is unique to prime ideals): $6 = 2\times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. In this kind of situation it is better to speak about prime ideals since this makes the splitting unique for what is known as Dedekind domains. The ideal class group characterizes the non-uniqueness of splitting to primes and consists of equivalence classes of fractional ideals (essentially integers defined by some fixed integer) under equivalence defined by multiplication by a rational of extension. The non-uniqueness of the factorization is characterized by so called ideal class group (see `http://tinyurl.com/cbxkhge`) [A45].

Quadratic fields (see `http://tinyurl.com/35w4jkv`) $Q[\sqrt{d}]$ are the simplest algebraic extensions of rationals since they correspond to second order prime polynomials and are also relatively well-studied so that one can look them at first. For $Q[\sqrt{d}]$ there are general results about the splitting of primes.

1. Quite generally, given prime p can be inert, split to a product of two distinct prime ideals, or can be ramified. The so called discriminant D characterizes the situation: for $d \ mod \ 4 = 1$ equals to $D = d$ and otherwise to $D = 4d$.

2. If p - say M_k - is an odd prime not dividing d, p splits only if one has

$$D \ mod \ p = x^2$$

In this case one has $(D/p) = 1$, where (D/p) is Legendre symbol (see `http://tinyurl.com/ykfudjq`) having values in the set $\{0, 1, -1\}$. $(D/p) = -1$ means stability of p against slitting.

Legendre symbol is a multiplicative function in the set of integers D meaning that if p splits under D_1 and D_2 it splits also under D_1D_2, and if p does not split under D_1 nor under D_2 it splits under D_1D_2. The multiplicative property implies $(4p_1/p) = (2/p)^2 \times (p_1/p) = (p_1/p)$. It is obviously enough to check whether the splitting occurs for primes p_1. Non-splitting prime p_1 gives rise to a set of non-splitting integers obtained by multiplying p_1 with any splitting prime. Also odd powers of non-splitting p_1 define this kind of sets.

3. Also the following properties of Legendre symbol are useful. One has $(D/p) = (p/D)$ if either $D \ mod \ 4 = 1$ or $p \ mod \ 4 = 1$ holds true. $D \ mod \ 4 = 3$ and $p \ mod \ 4 = 3$ one has $(D/p) = -(p/D)$. One has also $(-1/p) = (-1)^{(p-1)/2}$ and $(2/p) = (-1)^{(p^2-1)/8}$.

4. If the p-adic number fields, which do not allow $\sqrt{-1}$ as ordinary p-adic number are in special role then there might be hopes about the understanding of the special role of Mersenne primes. Mersenne primes are also stable for Gaussian integers and quadratic extensions $Q[\sqrt{\pm d}]$ of rationals defined by positive integers d, which are products $d = d_1d_2$ of two integers. d_1 factorizes to a product of primes $p_1 \ mod \ 4 = 1$ splitting M_k, and d_2 is a product of an odd number of primes $p_1 \ mod \ 4 = 1$ not splitting M_k.

5. One must also distinguish between the algebraic extensions of rationals and finite dimensional extensions of p-adic numbers (also powers e^k, $k < p$ define finite-dimensional extension). For instance, one can consider a quadratic extension $Q[\sqrt{-1}]$ for rationals defining similar extension for the allowed p-adic primes $p \ mod \ 4 = 3$ and fuse it with a quadratic extension $Q[\sqrt{d}]$ for which $d \ mod \ 4 = 1$ holds true. For adeles the extension of rationals and the extensions of p-adic numbers can be said to separate.

Some special examples are in order to make the situation more concrete.

1. A good example about physically very relevant quadratic extension is provided by Gaussian integers, which correspond to Galois extension (see `http://tinyurl.com/h9528pl`) $Q[\sqrt{-1}]$ [A32]. $p = 2$ splits as $2 = (1+i)(1-i) = -i(1+i)^2 = i(1-i)^2$ and the splitting to primes is non-unique. The splitting to prime ideals is however unique so that $p = 2$ is not ramified.

   The primes $p \ mod \ 4 = 1$ split also as stated by Fermat's theorem of two squares. Mersenne primes satisfy $p \ mod \ 4 = 3$ but some additional criterion is needed to select them. Primes $p \ mod \ 4 = 3$ do not and cannot define p-adic primes appearing in quantum adele for Gaussian rationals. Note that for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as p-adic number, which might cause problems in the p-adic formulation of quantum mechanics. These observations suggest that p-adic primes $p \ mod 4 = 1$ suffer extinction when $\sqrt{-1}$ emerges in the number theoretic evolution and only the primes $p \ mod 4 = 3$ remain. One could also start from the extension $Q[\sqrt{-1}]$ rather than rationals as the role of $\sqrt{-1}$ in quantum theory suggests so that the primes $p \ mod 4 = 3$ would be the only allowed quantum p-adic primes.

2. for $Q[\sqrt{2}]$ for which 2-adicity would not be possible. What happens for Mersenne primes? One can write $M_3 = 7 = (\sqrt{2}+3)(-\sqrt{2}+3)$ where $3 \pm \sqrt{2}$ is an algebraic integer as a root of a monic polynomial $P(x) = x^2 - 6x + 7$ so that the splitting of M_3 occurs in $Q[\sqrt{2}]$. Therefore it seems that the absence of $\sqrt{2}$ and allowance of 2-adicity is necessary for Mersenne-adicity. This conforms with the naïve physical picture that the p-adic scales defined by Mersennes are in excellent approximation n-ary 2-adic length scales.

   One should check whether the extension defined by $\sqrt{2}$ is somehow special as compared to the extensions defined by odd primes. Certainly the fact that this prime is the only even prime makes it rather special. It allows extension with $\sqrt{-1}$ and p-adic extension allowing all square roots except those of 2 is spanned by four square roots unlike similar extensions for other p-adic numbers fields which require only two square roots.

3. Suppose $D = p_1$ with $p_1 \ mod \ 4 = 1$. For $p = M_k$ quadratic resiprocity (see http://tinyurl.com/yz2okpf) implies that the condition is equivalent with $M_k \ mod \ p_1 = x^2$. Neither the extensions $Q[\sqrt{p_1}]$ nor $Q[\sqrt{-p_1}]$ induce splitting of M_k for $p_1 \ mod \ 4 = 1$. For $M_3 = 7$ and $p_1 \in \{5, 13, 17\}$ no splitting of M_3 takes place but for $p_1 = 29$ splitting occurs. This suggests that there is no general rule guaranteeing the stability of Mersenne primes in this case.

4. Suppose $D = p_1 \ mod \ 4 = 3$. One has $(4p_1, M_k) = (p_1, M_k)$ by the multiplicative character of the Legendre symbol. Quadratic resiprocity gives now $(p_1, M_k) = -(M_k, p_1)$ so that splitting occurs for M_k only if it does not occur for p_1. If splitting occurs for p_1 it does not occur for $-p_1$ and vice versa. $p_1 = 7$ and $M_2 = 3$ serve as a testing sample. One has $(3, 7) = 1$ so that the splitting of $M_2 = 3$ takes place for $Q[\sqrt{7}]$ but not for $Q(\sqrt{-7})$ and the splitting of $M_3 = 7$ takes place for $Q[\sqrt{-3}]$ but not for $Q(\sqrt{3})$.No obvious general rule seems to hold.

22.5.6 Generalized Feynman Diagrams And Adeles

The notion of Hilbert adeles seems to fit nicely with the recent view about generalized Feynman diagrams. The basic heuristic idea is the idea about fusion of physics in various number fields. p-Adic mass calculations lead to the conclusion that elementary particles are characterize by p-adic primes and inside hadron quarks obeying different effective or real p-adic topologies are present. One can speak about real and p-adic space-time sheets and real and p-adic spinors and also WCW has real and p-adic sectors. There is a hierarchy of algebraic extensions of rationals and presumably of also p-adic numbers. Even more general finite-dimensional extensions containing for instance Neper number e and its roots are also possible and involve extensions of p-adic numbers.

At the level of Feynman graphs this means that different lines correspond to different p-adic topologies and I have already proposed how this could give rise p-adic length scale hypothesis when the Feynman amplitudes in the tensor product of quantum variants p-adic number fields are mapped to reals by canonical identification [K35]. Rational or even more general entanglement between different number fields would be essential.

The vertices of generalized Feynman diagrams for different incoming p-adic number fields could be multi-p p-adic objects in quantum sense involving powers expansions in powers of integer n decomposed to product of powers of quantum primes associated with its factors with coefficients not divisible by the factors. An alternative option is that vertices are rational numbers common to all number fields serving as entanglement coefficients. A third option is that they are real numbers in corresponding tensor factor. One should also formulate symmetries in p-adic sectors and the simplest option is that symmetries represented as affine transformations simply reduce to products of the symmetries in various p-adic sectors of the embedding space.

The challenge is to formulate all this in a concise and elegant manner. It seems that adeles generalized to Hilbert adeles might indeed provide this formulation. The naïve basic recipe would be extremely simple: whenever you have a real number, replace it with Hilbert adele. You can even replace the points of Hilbert spaces involved with corresponding Hilbert spaces! One could replace embedding space, space-time surfaces, and WCW as well as embedding space spinors and spinor fields and WCW spinors and spinor fields with the hierarchy of their Hilbert adelic counterparts obtaining in this manner what might be interpreted as cognitive representations.

22.6 Quantum Mathematics And Quantum Mechanics

Quantum Mathematics (QM) suggests that the basic structures of Quantum Mechanics (QM) might reduce to fundamental mathematical and metamathematical structures, and that one even consider the possibility that Quantum Mechanics reduces to Quantum Mathematics with mathematician included or expressing it in a concise manner: QM=QM!

The notes below were stimulated by an observation raising a question about a possible connection between multiverse interpretation of quantum mechanics and quantum mathematics. The heuristic idea of multiverse interpretation is that quantum state repeatedly branches to quantum states which in turn branch again. The possible outcomes of the state function reduction would correspond to different branches of the multiverse so that one could save keep quantum mechanics deterministic if one can give a well-defined mathematical meaning to the branching. Could quantum mathematics allow to somehow realize the idea about repeated branching of the quantum universe? Or at least to identify some analog for it? The second question concerns the identification of the preferred state basis in which the branching occurs.

Quantum Mathematics replaces numbers with Hilbert spaces and arithmetic operations $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$.

1. The original motivation comes from quantum TGD where direct sum and tensor product are naturally assigned with the two basic vertices analogous to stringy 3-vertex and 3-vertex of Feynman graph. This suggests that generalized Feynman graphs could be analogous to sequences of arithmetic operations allowing also co-operations of $\oplus$ and $\otimes$.

2. One can assign to natural numbers, integers, rationals, algebraic numbers, transcendentals and their p-adic counterparts for various prime p Hilbert spaces with formal dimension given by the number in question. Typically the dimension of these Hilbert spaces in the ordinary sense is infinite. Von Neuman algebras known as hyper-finite factors of type II_1 assume as a convention that the dimension of basic Hilbert space is one although it is infinite in the standard sense of the word. Therefore this Hilbert space has sub-spaces with dimension which can be any number in the unit interval. Now however also negative and even complex, quaternionic and octonionic values of Hilbert space dimension become possible.

3. The decomposition to a direct sum matters unlike for abstract Hilbert space as it does also in the case of physical systems where the decomposition to a direct sum of representations of symmetries is standard procedure with deep physical significance. Therefore abstract Hilbert space is replaced with a more structured objects. For instance, the expansion $\sum_n x_n p^n$ of a p-adic number in powers of p defines decomposition of infinite-dimensional Hilbert space to a direct sum $\oplus_n x_n \otimes p^n$ of the tensor products $x_n \otimes p^n$. It seems that one must modify the notion of General Coordinate Invariance since number theoretic anatomy distinguishes between the representations of space-time point in various coordinates. The interpretation would be in terms of cognition. For instance, the representation of Neper number requires infinite number of pinary digits whereas finite integer requires onlya finite number of them so that at the level of cognitive representations general coordinate invariance is broken.

   Note that the number of elements of the state basis in p^n factor is p^n and $m \in \{0, ..., p-1\}$ in the factor x_n. Therefore the Hilbert space with dimension $p^n > x_n$ is analogous to the Hilbert space of a large effectively classical system entangled with the microscopic system characterized by x_n. p-Adicity of this Hilbert space in this example is for the purpose of simplicity but raises the question whether the state function reduction is directly related to cognition.

4. On can generalize the concept of real numbers, the notions of manifold, matrix group, etc... by replacing points with Hilbert spaces. For instance, the point $(x_1, .., x_n)$ of E^n is replaced with Cartesian product of corresponding Hilbert spaces. What is of utmost importance for the idea about possible connection with the multiverse idea is that also this process can be also repeated indefinitely. This process is analogous to a repeated second quantization since intuitively the replacement means replacing Hilbert space with Hilbert space of wave functions in Hilbert space. The finite dimension and its continuity as function of space-time point must mean that there are strong constraints on these wave functions. What does this decomposition to a direct sum mean at the level of states? Does one have super-selection rules stating that quantum interference is possible only inside the direct summands?

5. Could one find a number theoretical counterpart for state function reduction and preparation and unitary time evolution? Could zero energy ontology have a formulation at the level of the number theory as earlier experience with infinite primes suggest? The proposal was that zero energy states correspond to ratios of infinite integers which as real numbers reduce to real unit. Could zero energy states correspond to states in the tensor product of Hilbert spaces for which formal dimensions are inverses of each other so that the total space has dimension 1?

22.6.1 The basic idea of Quantum Mathematics

The minimal view (see `http://tinyurl.com/yblbzk6x`) about unitary process and state function reduction is provided by ZEO [K7, K58].

1. Zero energy states correspond to a superposition of pairs of positive and negative energy states. The M-matrix defining the entanglement coefficients is product of Hermitian square root of density matrix and unitary S-matrix, and various M-matrices are orthonormal as products of orthonormal Hermitian square roots of density matrices and universal S-matrix $S(CD) = S^n$, where integer n characterizes the size scale of CD. Quantum theory is square root of thermodynamics. This is true even at single particle level. The square root of the density matrix could be also interpreted in terms of finite measurement resolution.

2. It is natural to assume that zero energy states have well-defined single particle quantum numbers at the either end of CD as in particle physics experiment. This means that state preparation has taken place and the prepared end represents the initial state of a physical event. Since either end of CD can be in question, both arrows of geometric time identifiable as the Minkowski time defined by the tips of CD are possible.

3. The simplest identification of the U-matrix is as the unitary U-matrix relating to each other the state basis for which M-matrices correspond to prepared states at two opposite ends of CD. Let us assume that the preparation has taken place at the "lower" end, the initial state. State function reduction for the final state means that one measures the single particle observables for the "upper" end of CD. This necessarily induces the loss of this property at the "lower" end. Next preparation in turn induces localization in the "lower" end. One has a kind of time flip-flop and the breaking of time reversal invariance would be absolutely essential for the non-triviality of the process.

The basic idea of Quantum Mathematics is that M-matrix is characterized by Feynman diagrams representing sequences of arithmetic operations and their co-arithmetic counterparts. The latter ones give rise to a superposition of pairs of direct summands (factors of tensor product) giving rise to same direct sum (tensor product). This vision would reduce quantum physics to generalized number theory. Universe would be calculating and the consciousness of the mathematician would be in the quantum jumps performing the state function reductions to which preparations reduce.

Note that direct sum, tensor product, and the counterpart of second quantization for Hilbert spaces in the proposed sense would be quantum mathematics counterpart for set theoretic operations, Cartesian product and formation of the power set in set theory.

22.6.2 ZEO, state function reduction, unitary process, and Quantum Mathematics

State function reduction acts in a tensor product of Hilbert spaces. In the p-adic context to be discussed n the following $x_n \otimes p^n$ is the natural candidate for this tensor product. One can assign a density matrix to a given entangled state of this system and calculate the Shannon entropy. One can also assign to it a number theoretical entropy if entanglement coefficients are rationals or even algebraic numbers, and this entropy can be negative. One can apply Negentropy Maximization Principle to identify the preferred states basis as eigenstates of the density matrix. For negentropic entanglement the quantum jump does not destroy the entanglement.

Could the state function reduction take place separately for each subspace $x_n \otimes p^n$ in the direct sum $\oplus_n x_n \otimes p^n$ so that one would have quantum parallel state function reductions? This is an old proposal motivated by the many-sheeted space-time. The direct summands in this case would correspond to the contributions to the states localizable at various space-time sheets assigned to different powers of p defining a scale hierarchy. The powers p^n would be associated with zero modes by the previous argument so that the assumption about independent reduction would reflect the super-selection rule for zero modes. Also different values of p-adic prime are present and tensor product between them is possible if the entanglement coefficients are rationals or even algebraics. In the formulation using adeles the needed generalization could be formulated in a straightforward manner.

How can one select the entangled states in the summands $x_n \otimes p^n$? Is there some unique choice? How do unitary process and state function reduction relate to this choice? Could the dynamics of Quantum Mathematics be a structural analog for a sequence of state function reductions taking place at the opposite ends of CD with unitary matrix U relating the state basis for which single particle states have well defined quantum numbers either at the upper or lower end of CD? Could the unitary process and state function reduction be identified solely from the requirement that zero energy states correspond to tensor products Hilbert spaces, which correspond to inverses of each other as numbers? Could the extension of arithmetics to include co-arithmetics make the dynamics in question unique?

22.6.3 What multiverse branching could mean?

Could QM allow to identify a mathematical counterpart for the branching of quantum states to quantum states corresponding to preferred basis? Could one can imagine that a superposition of states $\sum c_n \Psi_n$ in a direct summand $x_n \otimes p^n$ is replaced by a state for which Ψ_n belong to different direct summands and that branching to non-interfering sub-universes is induced by the proposed super-selection rule or perhaps even induces state function reduction? These two options seem to be equivalent experimentally. Could this de-coherence process perhaps correspond to the replacement of the original Hilbert space characterized by number x with a new Hilbert space corresponding to number y inducing the splitting of $x_n \otimes p^n$? Could the interpretation of finite integers x_n and p^n as p-adic numbers $p_1 \neq p$ induce the de-coherence?

This kind of situation is encountered also in symmetry breaking. The irreducible representation of a symmetry group reduces to a direct sum of representations of a sub-group and one has in practice super-selection rule: one does not talk about superpositions of photon and Z^0. In quantum measurement the classical external fields indeed induce symmetry breaking by giving different energies for the components of the state. In the case of the factor $x_n \otimes p^n$ the entanglement coefficients define the density matrix characterizing the preferred state basis. It would seem that the process of branching decomposes this state space to a direct sum 1-D state spaces associated with the eigenstates of the density matrix. In symmetry breaking superposition principle holds true and instead of quantum superposition for different orientations of "Higgs field" or magnetic field a localization selecting single orientation of the "Higgs field" takes place. Could state function reduction be analogous process? Could non-quantum fluctuating zero modes of WCW metric appear as analogs of "Higgs fields". In this picture quantum superposition of states with different values of zero modes would not be possible, and state function reduction might take place only for entanglement between zero modes and non-zero modes.

22.6.4 The replacement of a point of hilbert space with Hilbert space as a second quantization

The fractal character of the Quantum Mathematics is what could make it a good candidate for understanding the self-referentiality of consciousness. The replacement of the Hilbert space with the direct sum of Hilbert spaces defined by its points would be the basic step and could be repeated endlessly corresponding to a hierarchy of statements about statements or hierarchy of n^{th} order logics. The construction of infinite primes leads to a similar structure.

What about the step leading to a deeper level in hierarchy and involving the replacement of each point of Hilbert space with Hilbert space characterizing it number theoretically? What could it correspond at the level of states?

1. Suppose that state function reduction selects one point for each Hilbert space $x_n \otimes p^n$. The key step is to replace this direct sum of points of these Hilbert spaces with direct sum of Hilbert spaces defined by the points of these Hilbert spaces. After this one would select point from this very big Hilbert space. Could this point be in some sense the image of the Hilbert space state at previous level? Should one imbed Hilbert space $x_n \otimes p^n$ isometrically to the Hilbert space defined by the preferred state $x_n \otimes p^n$ so that one would have a realization of holography: part would represent the whole at the new level. It seems that there is a canonical manner to achieve this. The interpretation as the analog of second quantization

suggest the identification of the embedding map as the identification of the many particle states of previous level as single particle states of the new level.

2. Could topological condensation be the counterpart of this process in many-sheeted space-time of TGD? The states of previous level would be assigned to the space-time sheets topologically condensed to a larger space-time sheet representing the new level and the many-particle states of previous level would be the elementary particles of the new level.

3. If this vision is correct, second quantization performed by theoreticians would not be a mere theoretical operation but a fundamental physical process necessary for cognition! The above proposed unitary embedding would imbed the states of the previous level as single particle states to the new level. It would seem that the process of second quantization, which is indeed very much like self-reference, is completely independent from state function reduction and unitary process. This picture would conform with the fact that in TGD Universe the theory about the Universe is the Universe and mathematician is in the quantum jumps between different solutions of this theory.

Returning to the motivating question: it seems that the endless branching of the states in multiverse interpretation cannot correspond to a repeated second quantization but could have interpretation as a de-coherence identifiable as de-localization in zero modes. If state function is allowed, it corresponds to a localization in zero modes analogous to Higgs mechanism. The Quantum Mathematics realization for a repeated second quantization would represent a genuinely new kind of process which does not reduce to anything already known.

22.7 Speculations related to Hilbert adelization

This section contains further speculations related to realization of number theoretical universality in terms of Hilbert adeles and to the notion of number theoretic emergence. One can construct infinite hierarchy of Hilbert adeles by replacing the points of Hilbert spaces with Hilbert spaces repeatedly: this generalizes the repeated second quantization used to construct infinite primes and realizes also algebraic holography since the points of space have infinitely complex structure. There are strong restrictions on the values of coordinates of Hilbert space for the p-adic sectors of the adele and the number of state basis satisfying orthonormality conditions is very restricted: a good guess is that unitary transformations reduce to a permutation group and that its cyclic subgroup defines quantum Galois group. Also the Hilbert counterpart of real factor of adeles is present and in this case there are no such restrictions.

A logical use of terms is achieved if one refers by term “quantum Hilbert adele” to the adele obtained by replacing the Hilbert space coefficients $a_n < p$ of pinary expansions with their quantum Hilbert spaces. On the other hand the hierarchy of Hilbert adeles is very quantal since it is analogous to a hierarchy of second quantizations so that Hilbert adeles could be also called quantum adeles. Reader can decide.

22.7.1 Hilbert adelization as a way to realize number theoretical universality

Hilbert adelization is highly suggestive realization of the number theoretical universality. The very construction of adeles and their Hilbert counterparts is consistent with the idea that rational numbers are common to all completions of rationals. This suggests a generalization of the formalism of physics allowing to realize number theoretical universality in terms of adeles and their Hilbert counterparts. What this would mean the replacement of real numbers everywhere by adeles containing real numbers as one Cartesian factor. Field equations make sense for the adeles separately in each Cartesian factor.

If one can define differential calculus for the Hilbert reals and p-adics as seems to be the case, this abstraction might make sense. There seems to be no obvious objection for field property and the entire hierarchy of n-Hilbert spaces could be seen as a cognitive self-referential representation of the mathematical structure allowing perhaps also physical realization if the structure is consistent with the general axioms.

Field equations would thus make sense also for an infinite hierarchy formed by Hilbert^n adeles. The fascinating conjecture is that quantum physics reduces to quantum mathematics and one might hope that TGD provides a realization for this physics because of its very strong ties with number theory.

Hilbert adelication at embedding space level

The Hilbert adelization at the level of embedding space makes senses if adelization works so that one can consider only adelization.

1. Could embedding space coordinates regarded as adeles? In the p-adic sectors general coordinate invariance would require some preferred coordinate choices maybe unique enough by symmetry considerations. One can also consider a spontaneous breaking of GCI by cognitive representations. Adelization would code field equations in various p-adic number fields to single field equation for adeles and would not bring anything new.

2. What could field equations mean for Hilbert adeles? One could imagine that ordinary field equations as local algebraic statements are expressed separately at each point of space-time surface giving infinite number of equations of form $F^k(x) = 0$, where k labels embedding space coordinates. Moving to the first level of hierarchy would mean that one replaces the points of Hilbert spaces involved with Hilbert spaces. The connection with the first order logic would suggest that the points of the Hilbert spaces representing points of embedding space and space-time - in general infinite-dimensional for real and p-adic numbers - represent points of embedding space and of space-time. This second quantization would transform infinite number of statements of predicate logic to a statement of first order logic.

   This certainly sounds hopelessly abstract and no-one would seriously consider solving field equations in this manner. But maybe mathematical thinking relying on quantum physics could indeed do it like this? At the next level of hierarchy one might dream of combining field equations for entire families of solutions of field equations to single equation and so on. Maybe these families could correspond to supports of WCW spinor fields in WCW . At the next level statements would be about families of WCW spinors fields and so on - ad infinitum. In fact, WCW spinors can be seen as quantum superpositions of logical statements in fermionic Fock space and WCW spinor fields would assign to WCW a direct sum of this kind of statements, one to each point of WCW . This sounds infinitely infinite but one must remember that the sub- WCW consisting of surfaces expressible in terms of rational functions is discrete.

3. The conjecture that field equations reduce to octonion real-analyticity requires that octonions and quaternions make sense also p-adically. The problem is that the p-adic variants of octonions and quaternions do not form a field: the reason is that even the equation $x^2 + y^2 = 0$ can have solutions in p-adic number fields so that the inverses of quaternions and octonions, and even p-adic complex numbers need not make sense. The p-adic counterparts of quaternions and octonions however exist as a ring so that one could speak about polynomials and Taylor series whereas the definition of rationals and therefore rational functions would involve problems. Octonion real-analyticity and quaternion real-analyticity and therefore also space-time surfaces defined by polynomials or even by infinite Taylor series could make sense also for the p-adic variants of octonions and quaternions.

Could embedding space spinors be regarded as adelic and even Hilbert adelic spinors? Again the problems reduce to the adelic level.

1. Adelization could be perhaps seen as a convenient book keeping device allowing to encapsulate the infinite number of physics in various quantum p-adic number fields to single physics. Hilbert adelic structures could however provide much deeper realization of physics as generalized number theory. One can indeed ask whether the action of the p-adic quantum counterparts of various symmetries could representable in the quantum quantum Galois groups for Hilbert adeles: these groups might reduce to cyclic groups and might relate to cyclic coverings of embedding space at the level of physics.

The minimal interpretation would be as a cognitive representation of quantum numbers of physical states at the first "material" level of hierarchy using the number theoretic Hilbert space anatomy of the point to achieve the representation. The representative capacity would be infinite for transcendental numbers with infinite number of pinary digits and finite for rational numbers. For real unit if would be miminal and zero could not represent anything. Quantum entanglement would be possible for tensor product coefficients and quantum superposition would be possible due to direct sum of pinary digits.

2. Embedding space spinor fields could be regarded as Cartesian products (direct sums) of spinor fields in real and various p-adic embedding spaces having values in the same number field. Also the induced metric and spinor connection would correspond to Cartesian product rather than tensor product. The isometries of the embedding space would have matrix representation in terms of adeles on the adelic components of spinors and embedding space coordinates.

Hilbert adelication at the level of WCW

What about quantum TGD at the level of WCW ? Could Hilbert adelication apply also at this level? Could one use the same general recipes to adelize? The step from adele to the hierarchy of Hilbert adeles does not seem to be a conceptual problem and the basic problem is to understand what adele means.

1. Could WCW described in terms of generalized number theory? Could adelic WCW be defined as the Cartesian product of real WCW and p-adic WCW s? The observations about dessins d'enfant [A19] [K6] suggest that the description of WCW could be reduced to the description in terms of orbits of algebraic 2-surfaces identified as partonic 2-surfaces at the boundaries of CDs (also the 4-D tangent space data at them codes for physics).

2. For a Cartesian product of finite-dimensional spaces spinors are formed as tensor products associated with with the Cartesian factors. Adelic WCW is Cartesian sum of real and p-adic variants. Could Hilbert adelic WCW spinors be identified as a tensor product of WCW spinors defined in the Hilbert adelic variant of WCW . This would conform with the physical vision that real and p-adic physics (matter and cognition) correspond to tensor factors of a larger state space. Furthermore, spinos generalizes scalar functions and the function space for adele valued functions with adelic argument forms in a natural manner tensor product of function spaces for various completions of reals. Note that one can speak about rational quantum entanglement since rational numbers are common to all the Cartesian factors.

3. Could also the moduli space of conformal equivalence classes of partonic 2-surfaces be regarded as adele in the sense that Teichmueller parameters from adele. This requires that the Teichmueller space of conformal equivalence classes of Riemann surfaces corresponds to the p-adic version of real Teichmueller space: this has been actually assumed in p-adic mass calculations [K21, K50].

   One could start from the observation that algebraic Riemann surfaces are dense in the space of all Riemann surfaces. This means that the algebraic variant of Teichmueller space is able to characterize the conformal equivalence classes. What happens when one adds the Riemann surfaces for which the coefficients of the Belyi function and rational functions defining are allowed to be in real or p-adic completion of rationals. A natural guess is that completion of the algebraic variant of Teichmueller space results in this manner. If this is argument makes sense then adelic moduli space makes sense too.

   There are however technical delicacies involved. Teichmueller parameters are defined as values of 1-forms for the homology generators of Riemann surface. What does one mean with the values of these forms when one has a surface containing only algebraic points and ordinary integral is not well-defined? Also in the p-adic context the definition of the integral is problematic and I have devoted a lot of time and energy to this problem (see for instance [K52]). Could the holomorphy of these forms help to define them in terms of residue calculus? This option looks the most plausible one.

What about the partial well-ordering of p-adic numbers induced by the map $n \to n_q$ combined with canonical identification: could this allow an elegant notion of integration by using the partial well-ordering. Note that one cannot say which of the numbers 1 and $-(p-1)\sum n = 1^{\infty} p^n$ is bigger in this ordering, and this induces similar problem for all p-adic integers which have finite number of pinary digits.

Problems to solutions and new questions

Usually one becomes fully conscious of a problem only after one has found the solution of the problem. The vision about Hilbert adeles - as a matter fact, already adeles- solves several nasty nuisances of this kind and I have worked hardly to prevent these problems from running off under the rug.

1. What one means with integer -1 is not a problem for p-adic mathematics. It becomes a problem for physical interpretation when one must relate real and p-adic physics to each other since canonical identification maps p-adic numbers to non-negative reals. This leads to problems with Hilbert space inner product but algebraic extensions of p-adic numbers by roots of unity allow to define p-adic Hilbert spaces but it seems that the allowed state basis are very restricted since the number of unitary isometries of Hilbert space is restricted dramatically by number theoretical existence requirement. The optimistic interpretation would that full quantum superposition is highly restricted in cognitive sectors by the condition of number theoretic existence.

2. What one means with complex p-adics is second problem. $\sqrt{-1}$ exists p-adically for $p \ \ mod \ \ 4 = 1$ so that one cannot introduce it via algebraic extension of p-adics in this case. This is a problem of p-adic quantum mechanics. Allowance of only p-adic primes p which do not split for the extension containing imaginary unit seems to be a general solution of the problem.

3. p-Adic counterparts of quaternions, and octonions do not exist for the simple reason that the p-adic norm can be vanishing even for p-adic complex number for p-adic fields allowing $\sqrt{-1}$. This problem can be circumvented by giving up the requirement that one has number field.

4. The norm for adeles exist as a product of real and norm and p-adic norms but is not physical. Also the assignment of Hilbert space structure to adeles is problematic. Canonical identification combined with $n \to n_q$ allows the mapping p-adic components of adele to real numbers and this allows to define natural inner product and norm analogous to Hilbert space norm for adeles and their Hilbert counterparts.

5. p-Adic numbers are not well ordered. This implies that difficulties with the definition of integral since definite integral relies heavily on well-orderedness of reals. Canonical identification suggests that quantum p-adics are well ordered: $a < b$ holds true if it holds true for the images under canonical identification. This gives hopes about defining also definite integral. For integrable functions the natural definition of quantum p-adic valued integral would be by using substitution for integral function. One - and rather ugly - option is to define the integral as ordinary real integral for the canonical image of the quantum p-adic valued function. This because this image is not expected to be smooth in real sense even if p-adic function is smooth.

6. p-Adic integration is plagued also by the problem that already for rational integrals one obtains numbers like $log(n)$ and π and is forced to introduce infinite-dimensional extension of p-adic numbers. For $log(n)$ one could restrict the consideration to p-adic primes p satisfying $n \ \ mod \ \ p = 1$ but this looks like a trick. Could this difficulty be circumvented somehow for p-adic numbers? The only possibility that one can imagine would be canonical identification map combined with $n \to n_q$ and the interpretation of integral as a real number.

   This could provide also the trick to interpret the integrals involving powers of π possible emerging from Feynman diagrams in sensible manner. All integrals can be reduced with the use of Laurent series to integrals of powers of x so that integral calculus would exist in analytic sense for analytic functions of quantum p-adic numbers.

7. What does one mean with the p-adic counterpart of CP_2 or more generally, with the p-adic counterpart of any non-linear manifold? What does one mean with the complex structure of p-adic CP_2 for $p \ mod \ 4 = 1$? Should one restrict the consideration to $p \ mod \ 4 = 3$? What does one mean with groups and coset spaces? One can inceed have a satisfactory looking definition based on algebraic extensions and effective discretization by introducing roots of unity replacing complex phases as continuous variables [K52].

   One could consider two options.

   (a) Could the p-adic counterpart of real $M^4 \times CP_2$ be M^8? The objection is that algebraic groups are however fundamental for mathematics and typically non-linear manifolds. Therefore there are excellent motivations for their (Hilbert) adelic existence. Projective spaces are in turn central in algebraic geometry and in this spirit one might hope that CP_2 could have non-trivial p-adic counterpart defined as quantum p-adic projective space.

   (b) Another option accepts that adeles contain only those p-adic number fields as Cartesian factors for which the prime does not split. This excludes automatically $p \ mod \ 4 = 1$ if $\sqrt{-1}$ is present from the beginning in the algebraic extension of rationals defining the adeles. What happens if one does not assume this. Does CP_2 degenerate to real projective space RP_2? What happens to M^4 if regarded as a Cartesian product of hyper-complex numbers and complex numbers. Does it reduce to M^2. Could the not completely well understood role of M^2 in quantum TGD relate to this kind of reduction?

The new view raises also questions challenging previous basic assumptions.

1. Could adeles and their octonionic counterpart allow to understand the origin of commutative complexification for quaternions and octonions in number theoretic vision about TGD? How could the commutative imaginary unit emerge number theoretically?

2. One must also reconsider $M^8 - M^4 \times CP_2$ duality. For instance, could M^8 be the natural choice in p-adic sectors and $M^4 \times CP_2$ in the real sector?

3. The preferred extremals of Kähler action are conjectured to be quaternionic in some sense. There are two proposals for what this means. Could it be that the sense in which the space-time surfaces are quaternionic depends on whether the surface is real or quantum p-adic?

4. The idea that rationals are in the intersection of reals and p-adics is central in the applications of TGD. How does this vision change? For $p = 2$ quantum rationals in the sense that pinary coefficients are quantum integer, are ordinary rational numbers. For $p > 2$ the pinary coefficients are in general mapped to algebraic numbers involving l_q, $0 < l < p$. The common points with reals would in general algebraic numbers.

Do basic notions require updating in the Hilbert adelic context?

In the adelic context one must take a fresh look to what one means with phrases like "embedding space" and "space-time surfaces". The phrase "space-time surface as a preferred extremal of Kähler action" might be quite too strong a statement in adelic context and could actually make sense only in the real sector of the quantum adelic embedding space. Also the phrase "p-adic variant of $M^4 \times CP_2$" might involve un-necessarily strong implicit assumptions since for p-adic integers one has automatically the counterparts of compactness even for M^8. The proposed identification of the quantum p-adic numbers as Hilbert p-adic quantum numbers reduces the question to whether p-adic counterparts of various structures exist or are needed as such.

1. We "know" that the real embedding space must be $M^4 \times CP_2$. What about p-adic counterpart of the embedding space? Is it really possible to have a p-adic counterpart of CP_2 or could non-linearity destroy this kind of hopes? Are there any strong reasons for having the counterpart of $M^4 \times CP_2$ in p-adic sectors? Could one have $M^4 \times CP_2$ only in real sector and M^8 in p-adic sectors. Complex structure of CP_2 requires $p \ mod \ 4 = 3$. This is not a problem if one assumes that adeles contain only the p-adic primes which do not split in the extension

of rationals containing imaginary unit. Definition as coset space $CP_2 = SU(3)/U(2)$ is one possible manner to proceed and seems to work also.

One can also wonder whether octonion real-analyticity really makes sense for $M^4 \times CP_2$ and its p-adic variants. The fact that real analyticity makes sense for S^2 suggests that it does. In any case, octonion real-analyticity would make life very easy for p-adic sectors if regarded as octonionic counterpart of M^8 rather than $M^4 \times CP_2$.

2. If the p-adic factors are identified as linear spaces with M^8 regarded as sub-space of the ring of complexified p-adic octonions, octonion real-analyticity for polynomial functions with rational coefficients could replace field equations in the ring formed by Z_p. Note however that octonion real-analyticity requires the Wick rotation mapping to ordinary octonions, the identification of the 4-surface from the vanishing of the imaginary part of the octonion real-analytic function, and map back to Minkowski space by Wick rotation. This is well-defined procedure used routinely in quantum field theories but could be criticized as mathematically somewhat questionable. One could consider also the definition of Minkowski space inner product as real part of $z_1 z_2$ for quaternions and use similar formula for octonions. This would give Minkowski norm squared for $z_1 = z_2$.

   Linear space would also allow to realize the idea that partonic 2-surfaces are in some sense trivial in most sectors reducing to points represented most naturally by the tips of causal diamonds (CDs). For p-adic sectors CP_2 would be replaced with E^4 and for most factors M^8_p the partonic 2-surfaces would reduce to the point $s = 0$ of E^4 representing the origin of coordinates in which E^4 rotations act linearly.

3. The conjecture is that preferred extremals correspond to loci for the zeros of the imaginary or real part of octonion real-analytic function. Is this identification really necessary? Could it be that in the *real* sector the extremals correspond to quaternionic 4-surfaces in the sense that they have quaternionic tangent spaces? And could the identification as loci for the zeros of the imaginary or real part of octonion real-analytic function be the sensible option in the p-adic sectors of the adelic embedding space: in particular if these sectors correspond to octonionic M^8. If this were the case, $M^8 - M^4 \times CP_2$ duality would have a meaning differing from the original one and would relate the real sector of adelic embedding space to its p-adic sectors in manner analogous to the expression of real rational as a Cartesian product of powers of p-adic primes in various sectors of adele.

My cautious conclusion is that the earlier vision is correct: $M^4 \times CP_2$ makes sense in all sectors.

22.7.2 Could number theoretic emergence make sense?

The observations made in this and previous sections encourage to ask whether some kind of number theoretic emergence could make sense. One would end up step by step from rationals to octonions by performing algebraic extensions and completions. At some step also the attribute "Hilbert" would lead to a further abstraction and relate closely to the evolution of cognition. This would mean something like follows.

Rationals → algebraic extensions → algebraic numbers → completions of rationals to reals and p-adics → completions of algebraic 2-surfaces to real and p-adic ones in algebraic extensions reals and classical number fields→ hierarchy of Hilbert variants of these structures as their cognitive representations.

The Maximal Abelian Galois group (MAGG) for rationals is isomorphic to the multiplicative group of ideles and involves reals and various p-adic number fields. How could one interpret the Hilbert variant of this structure. Could some kind of physical and cognitive evolution lead from rationals to octonions and eventually to Universe according to TGD? Could it be that the gradual emergence of algebraic numbers and AGG (Absolute Galois Group defined as Galois group of algebraic numbers as extension of rationals) brings in various completions of rationals and further extensions to quaternions and octonions and symmetry groups like SU(2) acting as automorphisms of quaternions as extension of reals and $SU(3) \subset G_2$ where G_2 acts as Galois for the extension of octonions as extension of reals?

Objections against emergence

The best manner to develop a new idea is by inventing objections against it. This applies also to the notion of algebraic emergence. The objections actually allow to see the basic conjectures about preferred extremals of Kähler action in new light.

1. Algebraic numbers emerge via extensions of rationals and complex numbers via completion of algebraic numbers. But can higher dimensions really emerge? This is possible but only when they correspond to those of classical number fields: reals, quaternions, and octonions. This is enough in TGD framework. Adelization could lead to the emergence of real space-time and its p-adic variants. Completion of solutions of algebraic equations to p-adic and real number fields is natural. Also the extensions of reals and complex numbers to quaternions and octonions are natural and could be seen as emergence.

2. All algebraic Riemann surfaces are compact but the reverse of this does not hold true. Partonic 2-surfaces are fundamental in TGD framework. Once the induced metric of the compact partonic 2-surface is known, one can regard it as a Riemann surface. Only if it is algebraic surface, the action of Galois group on it is well-defined as an action on the algebraic coefficients appearing in rational functions defining the surface. This is consistent with the basic vision about life as something in the intersection of real and p-adic worlds and therefore having as correlates algebraic partonic 2-surfaces. The non-algebraic partonic 2-surfaces are naturally present and if they emerge they must do so via completion to reals occurring also at adelic level.

   All partonic 2-surfaces allow a representation as projective varieties in CP_3 which forces again the question about possible connection with twistors.

   Representation as algebraic projective varieties in say CP_3 does not imply this kind of representation in $\delta CD \times CP_2$. This kind of representation can make sense for 3-surfaces consisting of light like geodesics emanating from the tip of the CD. If one wants to obtain 2-surfaces one must restrict light-like radial coordinate r to be a real function of complex variables so that the 2-surface cannot be algebraic surface defined as a null locus of holomorphic functions unless r is taken to be a constant equal to algebraic number. Note that the light rays of 3-D light-cone are parametrized by S^2, which corresponds to $CP_1 \subset CP_3$. This kind of partonic 2-surfaces might correspond to maxima for Kähler function.

3. Could one do without the non-algebraic partonic 2-surfaces? This is not the case if one believes on the notion of number theoretic entanglement entropy which can be negative for rational or even algebraic entanglement and presumably also for its quantum variant. Non-algebraic partonic 2-surfaces would naturally correspond to reals as a Cartesian factor of adeles. All partonic 2-surfaces which do not allow a representation as algebraic surfaces would belong to this factor of adelic embedding space. The ordinary real number based physics would prevail in this sector and entanglement in this sector would be in generic case real so that ordinary definition of entropy would work. In quantum p-adic sectors entanglement probabilities would be quantum rational (in the sense of $n \rightarrow n_q$) and the generalization of number theoretic entanglement entropy should make sense. Completion must be taken as would be part of the emergence.

Could embedding space spinors really emerge? The dimension of the space of embedding space spinors is dictated by the dimension of the embedding space. Therefore it is difficult to image how 8+8-complex-dimensional spinors could emerge from spinors in the set of algebraic numbers since these spinors are naturally 2-dimensional for algebraic numbers which are geometrically 2-dimensional. Does this mean that one must introduce algebraic octonions and their complexifications from the very beginning? Not necessarily.

1. The idea that also the embedding space spinors emerge algebraically suggests that embedding space spinors in p-adic sectors are octonionic (p-adic octonions form a ring but this might be enough). In real sector both interpretations might make sense and have been considered [K91, K11]. For octonionic spinors ordinary gamma matrices are replaced with the

analogs of gamma matrices obtained as tensor products of sigma matrices having quaternionic interpretation and of octonionic units. For these gamma matrices SO(1, 7) as vielbein group is replaced with G_2. Physically this corresponds to the presence of a preferred time direction defined by the line connecting the tips of CD. It would seem that $SO(1,7)$ must be assigned with the ordinary embedding space spinors assignable to the reals as a factor of quantum adeles. The relationship between the ordinary and octonionic embedding space spinors is unclear. One can however ask whether the p-adic spinors in various factors of adelic spinors could correspond to the octonionic modification of gamma matrices so that these spinors would be 1-D spinors algebraically extended to octonionic spinors.

2. Also quaternionic spinors make sense and could emerge in a well-defined sense. The basic conjecture is that the preferred extremals of Kähler action are quaternionic surfaces in some sense. This could mean that the octonionic tangent space reduces to quaternionic one at each point of the space-time surface. This condition involves partial derivatives and these make sense for p-adic number fields. The "real" gamma matrices would be ordinary gamma matrices. In p-adic sectors at least octonion real-analyticity would be the natural condition allowing to identify quaternionic 4-surfaces [K86] if one allows only Taylor series expansions.

Emergence of reals and p-adics via quantum adeles?

MAGG (Maximal Abelian Galois Group) brings in reals and various p-adic number fields although one starts from algebraic numbers as maximal abelian extension of rationals. Does this mean emergence?

1. Could one formulate the theory by starting from algebraic numbers? The proposal that octonion real-analytic functions can be used to define what quaternionicity looks sensible for quantum p-adic space-time surfaces. For real space-time surfaces octonion real-analyticity might be an unrealistic condition and quaternionicity as the condition that octonionic gamma matrices generate quaternionic algebra in the tangent space looks more plausible alternative. Quantum p-adic space-time surfaces would be naturally algebraic but in real context also non-algebraic space-time surfaces and partonic 2-surfaces are possibe. In real sector partial differential equations would prevail and in quantum p-adic sectors algebraic equations would dictate the dynamics.

2. The p-adic variants of quaternions and octonions do not exist as fields. The vanishing of the sum of Euclidian norm for quaternions and octonions for p-adic octonions and quaternions makes it impossible to define p-adic quaternion and octonionic fields. There are also problems due to the fact that $\sqrt{-1}$ exists as p-adic number for $p \ mod \ 4 = 1$.

3. The notion of quaternionic space-time surface requires complexified octonions with additional imaginary unit i commuting with octonionic imaginary units I_k. Space-time surfaces are identified as surfaces in the sub-space of complexified octonions of form $o_0 + i\sum o_k I^k$. Could i relate to the algebraic extensions of rationals and could complexified quantum p-adic embedding spaces have complex coordinates $x + iy$?

4. Polynomial equations with real algebraic coefficients make sense even if adeles where not a field and one can assign to the roots of polynomials with quaternionic and octonionic argument Galois group if one restricts to solution which reduce to complex solutions in some complex plane defined by preferred imaginary unit. For quaternions Galois group consist of rotations in SO(3) acting via adjoint action combined with AAG. For octonions Galois group consists of G_2 elements combined with AAG. $SU(3)$ leaves the preferred imaginary unit invariant and $U(2)$ the choice of quaternionic plane. Are there any other solutions of polynomial equations than those reducing to complex plane?

Is it really necessary to introduce p-adic space-time sheets?

The (Hilbert) adelization of embedding space, space-time, and WCW as well as spinors fields of embedding space and WCW would be extremely elegant manner to realize number theoretic universality. One must however keep the skeptic attitude. The definition of p-adic embedding space and space-time surfaces is not free of technical problems. The replacement of $M^4 \times CP_2$ with M^8 in p-adic sectors could help solve these problems. The conservative approach would be based on giving up p-adicization in embedding space degrees of freedom. It is certainly not an imaginative option but must be considered as a manner to gain additional insights.

1. p-Adic mass calculations do not mention anything about the p-adicization of space-time sheets unless one wants to answer the question what is the concrete realizations of various conformal algebras. Only p-adic and adelic interpretation of conformal weights would be needed. Adelic interpretation of conformal weights makes sense. The replacement $n \rightarrow n_q$ (interpreted originally as quantum p-adicization) brings in only $O(p^2)$ corrections which are typically extremely small in elementary particle scales.

2. Is the notion of p-adic or Hilbert p-adic (Hilbert adelic) spinor field in embedding space absolutely necessary? If one has p-adic spinors one must have also p-adic spinor connection. This does not require p-adic embedding space and space-time surface if one restricts the consideration to algebraic points and if the components of connection are algebraic numbers or even rational numbers and allow p-adic interpretation. This assumption is however in conflict with the universality of adelization.

3. What about Hilbert adelic WCW spinor fields. They are needed to give both p-adic and real quantum states. These fields should have adelic values. Their arguments could be algebraic partonic surfaces. There would be no absolute need to perform completions of algebraic partonic 2-surfaces although this would be very natural on basis of number theoretical universality.

4. The vision about life in the intersection of real and p-adic worlds is very attractive. The p-adicization of algebraic surfaces is very natural as completion meaning that one just solves the algebraic equations using series in powers of p. Imaginary unit is key number of quantum theory and the fact that $\sqrt{-1}$ exists for $p \mod 4 = 1$ is potential problem for p-adic quantum mechanics. For these primes also splitting occurs in the ring of Gaussian integers. For quantum adeles this problem disappears if one allows only the p-adic number fields for which p does not slit in algebraic extension (now Gaussian rationals).

22.8 Appendix: some possibly motivating considerations

The path to the idea that quantum adeles could represent algebraic numbers originated from a question having no obvious relation to quantum p-adics or quantum adeles and I will proceed in the following by starting from this question.

Function fields are much simpler objects to handle than rationals and their algebraic extensions. In particular, the objects of function fields have inverses and inverse is well defined also for sum of elements. This is not true in the ring of adeles. This is the reason why geometric Langlands is easier than the number theoretic one. Also the basic idea of Langlands correspondence is that it is possible to translate problems of classical number theory (rationals and their extensions) to those involving function fields. Could it be possible to represent the field of rationals as a function field in some sense? Quantum arithmetics gives a slight hope that this might be possible.

22.8.1 Analogies between number theoretic and function field theoretic ramification

Consider first the analogies between number theoretic and geometric ramification (probably trivialities for professionals but not for a physicist like me!). The relationship between number theoretic and geometric ramification is interesting and mathematician could of course tell a lot about it. My comments are just wonderings of a novice.

1. The number theoretic ramification takes place for the primes of number field when it is extended. If one knows the roots of the polynomials involved with the rational function $f(z)$ defining Belyi function one knows the coefficient field F of polynomial and its algebraic extension K and can deduce the representations of ordinary primes as products of those of

F and of the primes of the coefficient field F as products of those of K. In particular, one can find the ramified primes of ordinary integers and of integers of F.

2. The ramification however occurs also for ordinary integers and means that their decomposition to primes involves higher powers of some primes: $n = \prod_l l^{e_l}$ with $e_l > 1$ for some primes l dividing n. Could one introduce an extension of some ring structure in which ordinary primes would be analogous to the primes in the extension of rationals?

3. Geometric ramification takes place for polynomials decomposing to products of first order monomials $P(z) = z - z_k$ with roots which are in algebraic extension of coefficients. The polynomials can however fail to be irreducible meaning that they have multiple roots. For multiple roots one obtains a ramified zero of a root and for Belyi functions these critical points correspond to zeros which are ramified when the degree is larger than zero. The number theoretic ramification implies that the polynomials involved have several algebraic roots and when they coincide, a geometric ramification takes place. Degeneration of roots of polynomial implies ramification.

4. Ordinary integers clearly correspond to the space of polynomials and the integers, which are not square free are analogous to polynomials with multiple roots. The ramification of prime in the extension of rationals and also the appearance of higher powers of p in non-square free integer is analogous to the degeneration of roots of polynomial.

22.8.2 Could one assign analog of function field to integers and analogs prime polynomials to primes?

Could one assign to integer (prime) a map analogous to (prime) polynomial? Prime polynomial can be labeled by its zero and polynomial by its zeros. What kind of maps could represent ordinary primes and integers. What could be the argument of this kind of maps and do zeros of these map label them? What could be the ring in which the counterparts of polynomials are defined?

Could quantum arithmetics [K67] help to answer these questions?

1. Quantum arithmetics involves the map $f_q : n = \prod_{l|n} l^{e_l} \rightarrow n_q = \prod_{l|n} l_q^{e_l}$, where l are primes in the prime decomposition of n and quantum primes s $l_q = (q^l - q^{-ln})/(q - q^{-1})$ are defined by the phase $q = exp(i\pi/p)$, where p is the preferred prime. Note that one has $p_q = 0$ and $(p+1)_q = -1$. Note also that one has $q = exp(i\pi/p)$ rather than $q = exp(i2\pi/p)$ (as in the earlier version of article). This is necessary to get the denominator correctly also for $p = 2$ and to make quantum primes l_q non-negative for $l < p$. Under $n \rightarrow n_q$ all integers n divisible by p are mapped to zero. This would suggest that the counterparts of prime polynomials are the maps f_q, $q = q_p$ and that the analogs of polynomials are products $\prod_p f_{q_p}$ defined in some sense.

2. The more conventional view about quantum integers defines analogous map as $n \rightarrow n_q = (q^n - q^{-n})/(q - q^{-1})$. Choosing $q = exp(i\pi/p)$ one finds also now that integers divisible by p are mapped to zero. By finding the primes for which n is mapped to zero one finds the prime decomposition of n. Now one does not however have a decomposition to a product of quantum primes as above. Similar statement is of course true also for the above definition of quantum decomposition: the maps $n \rightarrow n_q$ are analogous to polynomials and primes are analogous to the zeros of these polynomials.

3. One can also consider $q = exp(i\pi/m)$ and used decomposition primes which are smaller than m. This would give non-vanishing quantum integers. They would correspond to quantum q-adicity with $q = m$ integer: q-adic numbers do not form a field. q could be even rational. As a special case these numbers give rise to multi-p p-adicity. The Jones inclusions of hyperfinite factors of type II_1 [K32] suggests that also these quantum phases should be considered. The index $[M : N] = 4cos^2(2\pi/n)$ of the inclusion would correspond to quantum matrix dimension 2_q^2, for $q = exp(i\pi/n)$ corresponding to quantum 2-spinors so that quantum dimension p_q could be interpreted as dimension of p-dimensional quantum Hilbert space.

Chapter 23

About Absolute Galois Group

23.1 Introduction

Langlands correspondence represents extremely abstract mathematics - perhaps too abstract for a simple minded physicist with rather mundane thinking habits. It takes years to get just a grasp about the basic motivations and notions, to say nothing about technicalities. Therefore I hope that my own prattlings about Langlands correspondence could be taken with a merciful understanding attitude. I cannot do anything for it: I just want desperately to understand what drives these mathematical physicists and somehow I am convinced that this exotic mathematics could be extremely useful for my attempts to develop the TGD view about Universe and everything. Writing is for me the only way to possibly achieve understanding - or at least a momentary illusion of understanding - and I can only apologize if the reader has feeling of having wasted time by trying to understand these scribblings.

Ed Frenkel (see `http://tinyurl.com/y8sgk672`) lectured again about geometric Langlands correspondence and quantum field theories and this inspired a fresh attempt to understand what the underlying notions could mean in TGD framework. Frenkel has also article about the relationship between geometric Langlands program and conformal field theories [A126]. My own attempt might be regarded as hopeless but to my view it is worth of reporting.

The challenge of all challenges for a number theorist is to understand the Galois group of algebraic numbers regarded as extension of rationals - by its fundamental importance this group deserves to be called Absolute Galois Group (see `http://tinyurl.com/yaffmruw`) (AGG, [A2]). This group is monstrously big since it is in some sense union of all finite-D Galois groups. Another fundamental Galois group is the Maximal Abelian Galois Group (MAGG) associated with maximal Abelian extension of rationals (see `http://tinyurl.com/y8dosjut`) [A56]. This group is isomorphic with a subgroup assignable to the ring of adeles (see `http://tinyurl.com/64pgerm`) [A4].

23.1.1 Could AGG Act As Permutation Group For Infinite Number Of Objects?

My own naive proposal for years ago is that AGG could be identified as infinite-dimensional permutation group S_∞ [?]. What the subscript ∞ means is of course on non-trivial question. The set of all finite permutations for infinite sequence of objects at integer positions (to make this more concrete) or also of permutations which involve infinite number of objects? Do these object reside along integer points of half-line or the entire real line? In the latter case permutations acting as integer shifts along the real line are possible and bring in discrete translation group.

A good example is provided by 2-adic numbers. If only sequences consisting of a finite number of non-vanishing bits are allowed, one obtains ordinary integers - a discrete structure. If sequences having strictly infinite number of non-vanishing bits are allowed, one obtains 2-adic integers forming a continuum in 2-adic topology, and one can speak about differential calculus. Something very similar could take place in the case of AGG and already the example of maximal Abelian Galois group which has been shown to be essentially Cartesian product of real numbers and all p-adic number fields Q_p divided by rationals suggests that Cartesian product of all p-adic continuums is involved.

What made this proposal so interesting from TGD point of view is that the group algebra of S_∞ defined in proper way is hyper-finite factor of II_1 (HFF) [?]. HFFs are fundamental in TGD: WCW spinors form as a fermionic Fock spaces HFF. This would bring in the inclusions of HFFs, which could provide new kind understanding of AGG. Also the connection with physics might become more concrete. The basic problem is to identify how AGG acts on quantum states and the obvious guess is that they act on algebraic surfaces by affecting the algebraic number valued coefficients of the polynomials involved. How to formulate this with general coordinate invariant (GCI) ways is of course a challenge: one should be able to identify preferred coordinates or at least class of them related by linear algebraic transformations if possible. Symmetries make possible to consider candidates for this kind of coordinates but it is far from obvious that p-adic CP_2 makes sense - or is even needed!

In [?] I proposed a realization of AGG or rather- its covering replacing elements of permutation group with flows - in terms of braids. Later I considered the possibility to interpret the mapping of the Galois groups assignable to infinite primes to symplectic flows on braids [K52]. This group is covering group of AGG with permutations being replaced with flows which in TGD framework could be realized as symplectic flows. Again GCI is the challenge. I have discussed the symplectic flow representation of generalized Galois groups assigned with infinite primes (allowing mapping to polynomial primes) in [K52] speculating in the framework provided by the TGD inspired physical picture. Here the notion of finite measurement resolution leading to finite Galois groups played a key role.

23.1.2 Dessins D'Enfant

Any algebraic surface defined as a common zero locus of rational (in special case polynomial) functions with algebraic coefficients defines a geometric representation of AGG. The action on algebraic coefficients is induced the action of AGG on algebraic numbers appearing as coefficients and in the roots of the polynomials involved. One can study many things: the subgroups of AGG leaving given algebraic surface invariant, the orbits of given algebraic surface under AGG, the subgroups leaving the elements at the orbit invariant, etc.... This looks simple but is extremely difficult to realize in practice.

One working geometric approach of this kind to AGG relies on so called dessins d'enfant (see `http://tinyurl.com/y927ebvd`) [A19] to be discussed later. These combinatorial objects provide an amazingly simple diagrammatic approach allowing to understand concretely what the action of AGG means geometrically at the level of algebraic Riemann surfaces. What is remarkable that every algebraic Riemann surface (with polynomials involved having algebraic coefficients) is compact by Belyi's theorem (see `http://tinyurl.com/ydxzerkr`) [A9] and bi-holomorphisms generate non-algebraic ones from these.

In TGD partonic 2-surfaces are the basic objects and necessarily compact. This puts bells ringing and suggests that the old idea about AGG as symmetry group of WCW might make sense in the algebraic intersection of real and p-adic worlds at the level of WCW identifies as the seat of life in TGD inspired quantum biology. Could this mean that AGG acts naturally on partonic 2-surfaces and its representations assign number theoretical quantum numbers to living systems? An intriguing additional result is that all compact Riemann surfaces can be representation as projective varities in CP_3 assigned to twistors. Could there be some connection?

23.1.3 Langlands Program

Another approach to AGG is algebraic and relies on finite-dimensional representations of AGG. If one manages to construct a matrix representation of AGG, one can identify AGG invariants as eigenvalues of the matrices characterizing their AGG conjugacy class. Langlands correspondence (see `http://tinyurl.com/ybmcnqh8`) [A126, A124] is a conjecture stating that the representations of adelic variants of algebraic matrix groups (see `http://tinyurl.com/yde5mras`) [A3].

Adelic representations are obtained by replacing the matrix elements with elements in the ring of rational adeles which is tensor product of rationals with Cartesian product of real numbers and all p-adic number fields with and they provide representations of AGG. Ideles represent elements of abelianization of AGG. Various completions of rationals are simply collected to form single super structure.

Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this way an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

The conjecture is that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the complex analysis.

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L14].

23.2 Langlands Program

Langlands programs starts from the idea that finite-dimensional representations of AGG provide information about AGG. If one manages to construct a matrix representation of AGG, one can identify AGG invariants as eigenvalues of the matrices characterizing their AGG conjugacy class. Langlands correspondence (see `http://tinyurl.com/ybmcnqh8`) [A126, A124] is a conjecture stating that the representations of adelic variants of algebraic matrix groups (see `http://tinyurl.com/yde5mras`) [A3].

Adelic representations are obtained by replacing the matrix elements with elements in the ring of adeles and they provide representations of AGG. Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this manner an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

Langlands conjecture states that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the analysis. I hasten to admit that I have failed to understand intuitively the deeper motivations for this conjecture but there is support for it.

23.2.1 Adeles

This approach leads to adeles [A4].

1. AGG is extremely complex and the natural approach is to try something less ambitious first and construct representations of the Maximal Abelian Galois Group of rationals (MAGG) [A56] assigned to an extension containing all possible roots of unity. One can show that MAGG is isomorphic to the group of invertible adeles divided by rationals. This is something concrete as compared to AGG albeit still something extremely complex.

2. The ring of rational adeles (see http://tinyurl.com/64pgerm) [A4] discovered by Chevalley is formed by the Cartesian product of all p-adic number fields and of reals and its non-vanishing elements have the property that only finite number of p-adic numbers in $(....,a_{p_n},....)\times a$ are not p-adic integers (that is possess norm >1). Algebraic operations are purely local: multiplications in every completion of rationals involved. One can also understand this space as a tensor product of rationals with integer adeles defined by the cartesian product of reals and various p-adic integers. One can say that adeles organize reals and all p-adic number fields to infinite-dimensional Cartesian product and that identified rational numbers as common to all of them so that multiplication by rational acts just as it act in a finite dimensional Cartesian product. The idea that rationals are common to all completions of rationals is fundamental for quantum TGD so that adeles are expected to be important.

3. The ring property of adeles makes possible to talk about polynomials of adele valued argument having rational coefficients and one can extend algebraic geometry to adeles as long as one talks about varieties defined by polynomials. Existence of polynomials makes it possible to talk about matrices with adele valued elements. The notion of determinant is well-defined and one can also define the inverse of adele matrix so that classical algebraic groups have also adele counterpart. This is of utmost significance in Langlands program and means a breathtaking achievement in book keeping: all the p-adic number fields would be caught under single symbol "A" !

4. Ideles are rational adeles with inverse. Ideles form a group but sum of two ideles is not always idele so that ideles do not form a number field and one cannot dream of constructing genuine differential calculus of ideles or talking about rational functions of ideles. Also rational functions fail to make sense. This means quite a strong constraint: if one wants adelic generalization of physics the solutions of field equations must be representable in terms of polynomials or infinite Taylor series.

The conjecture of Langlands is that the algebraic groups with matrix elements replaced with adeles provide finite-dimensional representations of adeles in what can be loosely called group algebra of adelic algebraic group.

The construction of representation uses complex valued functions defined in the ring of adeles. This function algebra decomposes naturally to a tensor product of function algebras associated with reals and various p-adic number fields and one can speak about rational entanglement between these functions. From the TGD point of view this is very interesting since rational entanglement plays a key role in TGD inspired quantum biology.

23.2.2 Construction Of Representations Of Adelic Gl_2

I have explained some details about the construction of the representation of adelic Gl_2 in the Appendix and earlier in [K46].

1. The basic idea is to start from the tensor product of representations in various completions of rationals using the corresponding group algebras. It is natural to require that the functions are invariant under the *left* multiplication by $Gl_2(Q)$ and eigenstates of $Gl_2(R)$ Casimir operator C under the *right* multiplication. The functions are smooth in the sense that they are smooth in $Gl_2(R)$ and locally constant in $Gl_2(Q_p)$.

2. The diagonal subgroup $Z(A)$ consists of products of diagonal matrices in $Gl_2(A)$. Characters (see http://tinyurl.com/ybeheayk) are defined in $Z(A)$ as group homomorphisms to complex numbers. The maximal compact subgroup $K\subset Gl_2(A)$ is the Cartesian product of $Gl_2(Z_p)$ and $O_2(R)$ and finite-dimensionality under the action of these groups is also a natural condition.

3. The representations functions satisfy various constraints described in detail in the appendix and in the article of Frenkel (see http://tinyurl.com/y7fhl75f) [A126]. I just try to explain what I see as the basic ideas.

(a) Functions f form a finite-dimensional vector space under the action of elements of the maximal compact subgroup K. Multiplication from left by diagonal elements reduces to a multiplication with character. The functions are eigenstates of the Casimir operator of $Gl_2(R)$ acting from left with a discrete spectrum of eigen values. they are bounded in $Gl_2(A)$. These conditions are rather obvious.

(b) Besides this the functions satisfy also the so called cuspidality conditions, the content of which is not obvious for a novice like me. These conditions imply that the functions are invariant under the action for $Gl_2(Z_p)$ apart from finite number of primes called ramified. For these primes invariance holds true only under subgroup $\Gamma_0(p^{n_k})$ of $Sl_2(Z_p)$ consisting of 2×2-matrices for which the elements $a_{21}\equiv c$ vanish modulo p^n.

(c) What is non-trivial and looks like a miracle to a physicist is that one can reduce everything to the study of so called automorphic functions (see `http://tinyurl.com/ybwzg73x`) [A8] defined in $\Gamma_0(N)/Sl(2,R)$, $N=\prod p^{n_k}$. Intuitively one might try to understand this from the idea that adeles for which elements in Z_p are powers of p represent rational numbers. That various p-adic physics somehow factorize the real physics would be the misty idea which in TGD inspired theory of consciousness translates to the idea that various p-adic physics make possible cognitive representations of real physics. Somehow the whole adele effectively reduces to a real number. Automorphic functions have a number theoretic interpretation and this is certainly one of the key motivations between Langlands program.

4. Automorphic functions reduce to complex analytic functions in the upper half plane $H=SL_2(R)/O(2)$ transforming in a simple manner under $\Gamma_0(N)$ (modular form of weight k). What one is left with are modular forms of weight k and level N in upper half plane.

   (a) The overall important cuspidality conditions characterized by integer N imply that the automorphic functions vanish at the cusp points of the algebraic Riemann surface defined as $H/\Gamma_0(N)$. The modular form can be expanded in Fourier series $f=\sum a_n q^n$ in powers of $q=exp(i2\pi\tau)$, where τ parameterizes upper half plane.

   (b) The Fourier coefficients a_n satisfy the condition $a_{mn}=a_m a_n$ and one ends up with the conclusion that for each elliptic curve (see `http://tinyurl.com/ybsdt65r`) [L81] $y^2=x^3+ax+b$ (a and b are rational numbers satisfying $4a^3+27b^2\neq 0$ and reduce to integer is the recent case) there should exist a modular form with the property that a_p codes for the numbers of points of this elliptic curve in finite field F_p for all but finite number of primes! This is really amazing and mysterious looking result.

   (c) τ can be interpreted as a complex coordinate parametrizing the conformal moduli of tori. Is this a pure accident or could this relate to the fact that the coefficients turn out to give numbers of roots for algebraic elliptic surfaces, which are indeed tori? Could cuspidality conditions have interpretation as vanishing of the modular forms for tori with moduli corresponding to cusps: could these be are somehow singular as elliptic surfaces? The objection is that the elliptic surfaces as sub-manifolds of C^2 have a unique induced metric and therefore correspond to a unique conformal modulus τ. But what about other Kähler metrics than the standard metric for C^2 and embeddings to other complex spaces as algebraic surfaces? Could adelic Gl_2 representations generalize to adelic representations of Gl_{2g} acting on Teichmueller parameters of Riemann surface with genus g?

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

23.3 Compactness Is Guaranteed By Algebraicity: Dessins D'Enfant

This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focussed. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.

This piece of text was written by Grothendieck. He described here the profound impact of the notion of dessins d'enfant (see `http://tinyurl.com/y927ebvd`) [A19] on him. The translation of the notion to english is "child's drawings". These drawings are graphical representations of Riemann surfaces (see `http://tinyurl.com/cgl2pj`) understood as pairs formed by an algebraic Riemann surface and its universal covering space from which Riemann surface is obtained as a projection which can be many-to-one one map. This diagram allows to construct the Riemann surface modulo bi-holomorphism. Algebraic Riemann surface means that the equations defining it involve only rational functions with coefficients which are algebraic numbers. This implies that the action of AGG on the algebraic Riemann surface is well defined as action on the coefficients. One can assign to the dessin d'enfant combinatorial invariants for the action of AGG.

23.3.1 Dessins D'Enfant

1. Dessin d'enfant is a bipartite graph (see `http://tinyurl.com/3x2cjf`) [A192] meaning that it is possible to label the nodes of the graphs by black and white points in such a way that the black and white points alternate along edge paths. One can identify black and white nodes as sets U and V and every edge of the graph connects points of U and V. For instance, bipartite graph does not posses any odd edge cycles. Every tree is bipartite and every planar graphs with even number of edges is bipartite. The vertices of the bipartite graph are topologically characterized by the number of lines emerging to the vertex and also 2-vertices are possible. The surface and the embedding can be described combinatorially using rotation system assigned with each vertex of the graph and telling the order in which the edges would be crossed by a path that travels clockwise on the surface around the vertex.

2. The notions of dessin d'enfant and counterpart for Belyi function [A9] defining the projection from the covering of sphere to sphere dates back to the work of Felix Klein. A very deep and very surprising theorem by Belyi (`http://tinyurl.com/ydxzerkr`) states that all algebraic curves represent compact Riemann surfaces. These surfaces are ramified coverings of the Riemann sphere ramified at three points only which in suitable complex coordinates can be taken to be the rational points 0, 1, ∞ of real axis. Ramification means that the rational function f with algebraic number coefficients - known as Belyi's function - projecting the Riemann surface as covering of sphere to sphere has critical points which are pre-images of these three points. In the neighborhood of the critical points the projection map known as Belyi's function is characterized by degree telling how many points are mapped to single point of sphere. At the critical point itself these points coincide. A simplified example of criticality is z^n at origin.

   The Riemann surface in question can be taken to be H/Γ compactified by finite number of cusp points. Here H is upper half plane Γ a subgroup of modular group having finite index

3. Dessin d'enfant allows to code combinatorially the data about the Belyi function so that one can construct both the surface and its Belyi function from this data apart from bi-holomorhism. The interpretation as projection from covering allows to get grasp about the geometric meaning of dessin d'enfant. Physicist reader is probably familiar with the graphical representation of cusp catastrophe. The projection of the critical points and curves of cusp catastrophe as function of the two control parameters to the control parameter plane replaced in the recent case by complex plane is highly analogous to dessin d'enfant. The boundary of cusp catastrophe in which cusp projection is three-to-one has V-shape and at the sides of V the covering of plane is 2-to-1 and and at the vertex and outside cusp region 1-to-1. The

edges of V correspond to the edges of the dessin d'enfant and the vertex of V to a node of dessin d'enfant.

The number of edges entering given critical point tells the degree of the Belyi function at that critical point. Dessin d'enfant is imbedded on an oriented surface - plane in the simplest situation but also sphere and half plane can be considered. The lines of the graph correspond to curves at which two branches of the covering coincide.

The Wikipedia article (see `http://tinyurl.com/y927ebvd`) [A19] about dessin d'enfant discusses a nice example about the construction of dessin d'enfant and is recommended for the reader.

4. The Belyi function could be any holomorphic function from X to Riemann sphere having only 0, 1, and ∞ as critical values and the function f is determined only up to bi-holomorphism. If X is algebraic surface, f is rational function with algebraic coefficients.

5. What makes the dessin d'enfant so remarkable is that AGG has natural action on the algebraic coefficients of the rational functions defining algebraic Riemann surfaces and therefore on dessin d'enfant. For instance, the sequence of integers form by the degrees of the projection map at the critical points is geometric Galois invariant. One can identify the stabilize of dessin as the sub-group of AGG leaving dessin d'enfant invariant. One can identify the orbit of dessin d'enfant under AGG and the subgroup of AGG leaving the points of orbit invariant.

23.3.2 Could One Combine Quantum Adelic Representations With Dessin D'Enfant Representations?

As already noticed, dessin d'enfant representation of AGG allows to have representations of AGG at the orbits of dessins d'enfant. If the orbit consists of a finite number n of points, one obtains representations of AGG in the finite-dimensional discrete Hilbert space spanned by the points, and representation matrices are $n \times n$ matrices.

Suppose that the Galois group of quantum adeles is indeed isomorphic with the commutator group of AGG. If this is the case then quantum adele valued amplitudes defined in the discrete space formed by the orbits of dessins d'enfant would provide a representation of AGG with commutator group acting on the fiber analogous to spin degrees of freedom and AGG on the base space having role analogous to that of Minkowski space.

One can imagine an approach mimicking the construction of induced representations (see `http://tinyurl.com/y9nfp438`) [A46] of Mackey inspired by the representations of Poincare group. In this approach one identifies orbit of group G as a space carrying the fields with spin. The subgroup H of G leaving a given point of representation space invariant is same at all points of orbit apart from conjugation. The field would have values in H or group algebra of H or in space in which H acts linearly. In the recent case H could adelic Galois group of quantum adeles identified as AGG or the subgroup G_I of AGG leaving the dessins d'enfant invariant.

What can one say about G_I. How large it is? Can one identify it or its abelization A_{G_I} and assign it to the points of orbits to construct analogs of induced representations?

1. If the orbit of dessin d'enfant is finite as the fact that the number of its points is invariant under the action of AGG suggests, G_I must be infinite. This would suggests that also A_{G_I} is infinite. Does A_{G_I} possess adele representation? Is this adele representation identifiable as a sub-adele of A_{AGG} in some sense? Could it be obtained by dropping some quantum variants of Z_p: from the decomposition of adele? What the interpretation of these lacking primes could be? Could these primes correspond to the primes which split in the extensions. If this is the case one could consider the representations in which A_{G_I} forms the fiber space at each point of dessin d'enfant.

2. One can consider also weaker option for which only so called ramified primes are dropped from the adele for rationals to obtain the adele for algebraic extension. In adele construction there are problematic primes p. For rational primes (or corresponding ideals) the representation of p is as a product of primes of extension as $p = \prod P_i^{e_i}$ e_i are called degrees of ramification. For some $e_i > 1$ one has ramification analogous to the dependence of form $(z-z_0)^n$, $n > 1$

of holomorphic function around critical point have interpretation as ramified primes and corresponding factors Z_p are dropped from the adele. To eliminate the problems cause by number theoretic ramification one can drop ramified primes from the adele in the extensions of algebraic numbers associated with the roots of the polynomials appearing in the Belyi map. Could the resulting adele be the counterpart for the reduced MGGA?

23.3.3 Dessins D'Enfant And TGD

What might be the relevance of Belyi's theorem and dessins d'enfant for TGD?

1. In TGD framework effective 2-dimensionality implies that basic objects are partonic 2-surfaces together with their data related to the 4-D tangent space a them. I have already earlier proposed that Absolute Galois group could have a natural action in the world of the classical worlds (WCW). The horrible looking problem is how to achieve General Coordinate Invariance (GCI) for this action.

   Partonic 2-surfaces are compact so that they allow a representation as algebraic surfaces. The notion of dessin d'enfant suggests that partonic 2-surfaces could be described as simple combinatorial objects defined by dessin d'enfant as far as the action of Galois group is considered. This representation would be manifestly general coordinate invariant and would allow to construct representations as Galois group in terms of discrete wave functions at the orbits of dessin d'enfant. One can also expect that the representation reduces to those of finite Galois groups.

2. Second central problem is the notion of braid which is proposed to provide a realization for the notion of finite measurement resolution.The recent view is that time-like braids on light like surfaces and space-like braids at the 3-surfaces defining the ends of space-time surfaces contain braid strands as Legendrian knots for which the projection of Kähler gauge potential has vanishing inner product with the tangent vector of the braid strand. For light-like 3-surfaces this does not imply that the tangent vector of strand is orthogonal to the strand: if the tangent vector is light-like the condition is automatically satisfied and light-like braid strands define a good but - as it seems - not a unique guess for what the braid strands are. Note however that the condition that braid strands correspond to boundaries of string world sheets gives additional conditions. At space-like 3-surfaces orthogonality to induced Kähler gauge potential fixes the direction of the tangent vector field only partially.

   Suppose one manages to fix completely the equations for braid strands - say by the identification as light-like strands. What about the end points of strands? How uniquely their positions are determined? Number theoretical universality suggests that the end points are rational or algebraic points as points of embedding space but again GCI poses a problem. Symmetry arguments suggest that one could use group theoretically preferred coordinates for M^4 and CP_2 and identify also the coordinates of partonic 2-surface as embedding space coordinates for their projections to geodesic spheres of $\delta M^4_\pm$ and geodesic sphere of CP_2.

   A possible resolution of this problem comes from the fact that partonic 2-surface allows an interpretation as algebraic surface. Braid ends could correspond to the critical points of the Belyi function defining the projection from the covering so that they would be algebraic points in the complex coordinates of partonic 2-surfaces fixed apart from algebraic bi-holomorphism. One would a concrete topological interpretation for why the braid ends are so special. I have already earlier proposed that braid ends could correspond to singularities associated with coordinate patches.

3. Is it possible to have compact Riemann which cannot be represented as algebraic surfaces? Belyi's theorem does not deny this. For instance rational functions with real coefficients for polynomials are possible and must give rise to compact surfaces. Inherently non-algebraic partonic 2-surfaces are possible and for them one cannot define representations of AGG at the orbits of dessin d'denfant since the action of AGG on f is not well defined now.

   This relates in an interesting manner to the conjecture [K53] that life resides in the in the intersection of real and p-adic worlds. At WCW level this would mean that the equations for the partonic 2-surfaces makes sense in any completion of rationals. For algebraic partonic

2-surfaces this is indeed the case if arbitrary high-dimensional algebraic extensions of p-adic numbers are allowed. Taking this seriously one can ask whether the existence of the representations of Galois group at the level of WCW is an essential aspect of what it is to be living. Could one assign Galois quantum numbers to the quantum states of living system? These would be realized in the discrete space provided by different quantum counterparts of a given integer and one would have discrete wave functions in these discrete spaces.

4. One also learns from Wikipedia (see `http://tinyurl.com/cgl2pj`) that any compact Riemann surface is a projective variety and thus representable using polynomial equations in projective space. It also allows an embedding as as a surface n 3-dimensional complex projective space CP_3. Wikipedia states that if compactness condition is added the Riemann the surface is necessarily algebraic: here however algebraic means rational functions with arbitrary real or complex coefficients. Above it means algebraic coefficients. Whether this CP_3 could have anything to do with the twistor space appearing in Witten's twistor string model [B21] and also in the speculated twistorial formulation of TGD [K91] remains an open question.

5. Modular invariance plays central role in TGD [K21], and a natural additional condition on the representations of AGG would be that the quantum states in WCW are modular invariant. The action of AGG induces a well-defined action on the conformal moduli of the partonic 2-surfaces and therefore on Teichmueller parameters. This discrete action need not be simple - say linear- but it would be action in n-dimensional space. Modular invariance requires that the action of AGG transformation induces a conformal scaling of the induced metric and changes the conformal moduli by an action of modular group $Sl(2g, Z)$. For torus topology this group is Sl(2, Z) appearing in modular invariant functions assigned to the representations of AGG in the group algebra of adelic algebraic groups.

6. Could the combination of dessins d'enfant as a geometric representation and adelic matrix representations for the abelianizer of the isotropy group G_I of dessin d'enfant provide additional insights in to Langlands conjecture? The problem is that AGG elements do not leave MGGA invariant.

7. Bi-partite graphs (see `http://tinyurl.com/3x2cjf`) appear also in the construction of inclusions of hyper-finite factors of type II_1 (HFF). The TGD inspired proposal that AGG allows identification as S_∞ and the group algebra of permutation group S_∞ is HFF. In optimistic mood one might see dessins d'enfant as a piece of evidence for this identification of AGG and adele formed from the Galois group of quantum p-adic integers as its commutator group.

23.4 Appendix: Basic Concepts And Ideas Related To The Number Theoretic Langlands Program

The following representation of the basic ideas of Langlands program reflects my very limited understanding of the extremely refined conceptual framework involved. This pieces of text can be found almost as such also in [K46] and Ed Frenkel provides more detailed discussion in his article [A126, A124].

23.4.1 Langlands Correspondence And AGG

The representations of group carry information about the group and the natural question is how to represent the AGG and deduce invariants of AGG in this manner. Eigenvalues for the representation matrices are invariants characterizing conjugacy classes of the group. The generators of MAGG abelled by primes define so called Frobenius elements and the eigenvalues and traces for their representation matrics defined invariants of this kind. The big question is how to construct representations of the AGG. Langlands program is an attempt to answer this question.

1. 1-D representations of AGG corresponds those of maximal Abelian Galois group which is the factor group of AGG by its commutator group. The natural intuitive guess is that the

n-dimensional representations of AGG in the group algebra of adelic algebraic group $Gl(n)$ could provide higher-dimensional representations of AGG. $Gl(n)$ would give rise to a kind of AGG spin. The action of AGG commutator group would be mapped to$GL_n(A)$ action. Does this mean that AGG is mapped homomorphically to adelic matrices in $Gl_n(A)$ as one might first think? I am not able to answer the question. From Wikipedia one learns that so called Langlands dual (see `http://tinyurl.com/yclcloaj`) [A50] extends AGG by the algebraic Lie group G_L so that one obtains semi-direct product of complex G_L with the AGG which acts on the algebraic root data of G_L. The adelic representations of G_L are said to control those of G. In this form the correspondence gives information about group representations rather than number theory.

Remark: One naïve guess would be that one could realize the representations of AGG by adjoint action $x \rightarrow gxg^{-1}$ in the commutator subgroup of AGG, which is maximal normal subgroup and closed with respect to this action. Also the adjoint action of the factor group defined my maximal Abelian group in this group could define representation? The guess of the outsider is that the practical problem is that the commutator group is not known.

2. Number theoretic Langlands program is however more than study of the relationships between representations of $Gl(F)$ and its adelic variant $Gl(A_F)$. The basic conjecture is the existence of duality between number theory and harmonic analysis. On number theoretical side one typically studies algebraic curves. Typical question concerns the number of rational points in modulo p approximation to the equations determining the algebraic curve. The conjecture about number theoretic Langlands correspondence was inspired by the observation that Fourier series expansions of automorphic forms code via their coefficients this kind of data and the proof of Fermat's theorem can be seen as application of this correspondence.

   There is support for the conjecture that adelic representations carry purely number theoretic information in the case of $Gl(n)$. The number theoretical invariants defined by the trace for the representation matrix for the Frobenius element generating the Abelian Galois group would corresponds to the trace of so called Hecke operator at the side of the harmonic analysis.

3. Intuitive motivations for the Langlands duality come from the fact the notion of algebraic surface defined by a polynomials with integer coefficients is number theoretically universal: the argument can belong to finite field, rational numbers or their extension, real numbers, or any p-adic number field and can represent even element of function field. Function fields defined algebraic functions at algebraic curves in finite fields are somehow between classical number fields and function fields associated with Riemann surfaces to which one can apply the tools of harmonic analysis.

23.4.2 Abelian Class Field Theory And TGD

The context leading to the discovery of adeles (`http://tinyurl.com/64pgerm`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \ mod \ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (see `http://tinyurl.com/y8aefmg2`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

Adeles and ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://tinyurl.com/y8aefmg2`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [K46, A126, A124]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where embedding space is replaced with Cartesian product of real embedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://tinyurl.com/64pgerm`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod_p' Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://tinyurl.com/yc3yrcxx`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

23.4.3 Langlands Correspondence And Modular Invariance

A strong motivation for Langlands correspondence is modular invariance - or rather its restricted form - which emerges in both number theory and in the automorphic representations of Gl_2 and relates directly to the ramification of primes for Galois extensions- now maximal Abelian extension. In TGD framework the restricted modular invariance could have interpretation in terms of concrete representations of AGG involving the action of AGG on the adelic variants of Teichmueller parameters characterizing the algebraic surfaces its variants in various number fields.

It is not necessary to know the explicit action of AGG to modular parameters. What is however needed is modular invariance in some sense. The first - and hard-to-realize - option is that allowed subgroup of AGG leaves the conformal equivalence class of Riemann surface invariant. Second option is that the action of both AGG and modular group $Sl(2g, Z)$ or its subgroup leave the states of representation invariant. This is the case if AGG induces Gl_{2g} transformations in each Cartesian factor of the adele and the states defined in the group algebra of Gl_{2g} are invariant. For ramified primes however modular invariance can break down to subgroup of Sl_{2g}. These conditions lead to automorphic modular forms.

These arguments are very heuristic and following arguments due to Frenkel give better view about the situation.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL_e(2, A)$ and more generally $GL_e(n, A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A58], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL_e(n, A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL_e(n, A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n, R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1, .., n$ acting in group algebra of $Gl(n, R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weyl group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weyl group to the Langlands dual $G_e^L(F)$ of $G(F)$.

23.4.4 Correspondence Between N-Dimensional Representations Of $Gal(\overline{F}/F)$ And Representations Of $Gl_E(N, A_F)$ In The Space Of Functions In $Gl_E(N, F)\backslash Gl_E(N, A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^{\times}$, where $Z_p^{\times}$ consists of invertible p-adic integers [A126].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^{\times}\backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^{\times}\backslash A_F^{\times}$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of* $Gal(\overline{F}/F)$ *correspond to representations of* $GL_e(1, A_F)$ *in the space of functions defined in* $GL_e(1, F)\backslash GL_e(1, A_F)$.

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of* $Gal(\overline{F}/F)$ *correspond to representations of* $GL_e(n, A_F)$ *in the space of functions defined in* $GL_e(n, F)\backslash GL_e(n, A_F)$.

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal embedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the embedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [A73]. In the case of elliptic curves [A126] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL_e(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL_e(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL_e(n, E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative ways to state the same thing.

Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphism Fr_v is by definition the generator of the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen.

The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A126]. The representations can be also l-adic being defined in $GL_e(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A126] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p, 1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^{\times}$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^{\times}|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A126] for the route from automorphic adelic representations of $GL_e(2, R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL_e(2, Q)$ are constructed in the space of smooth bounded functions $GL_e(2, Q)\backslash GL_e(2, A) \rightarrow C$ or equivalently in the space of $GL_e(2, Q)$ left-invariant functions in $GL_e(2, A)$. A denotes adeles and $GL_e(2, A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL_e(2, A)$. The motivating idea is the central role of double coset decompositions $G = K_1 A K_2$, where K_i are compact subgroups and A denotes the space of double cosets $K_1 g K_2$ in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL_e(2, Z_p)$. For ramified primes K_p consists of $SL_e(2, Z_p)$ matrices with $c \in p^{n_p} Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2, R)$ with eigenvalue ρ so that irreducible representations of $gl(2, R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$

where one has

$$X_0 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix} \ .$$

3. The center $A^\times$ of $GL_e(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \rightarrow C$ is a character providing a multiplicative representation of $A^\times$.

4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)du = 0$$

is satisfied [A126]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL_e(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL_e(2,A_F)\times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL_e(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL_e(2,Q)\backslash GL_e(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [A126]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL_e(2,Z)$ consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ c \ mod \ N = 0.$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL_e(2,Z)$ so that also $SL_e(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL_e(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL_e(2,R)$ to upper half-plane $H_u = SL_e(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A126]. For the discrete series representation π giving square integrable representation in $SL_e(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ , \ X_+ v_\infty = 0 \ .$$

The latter module is in turn generated by the lowest weight vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\infty \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL_e(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL_e(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL_e(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \ = \ \sum_{n=0}^{\infty} a_n q^n \ . \tag{23.4.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL_e(2,Z_p)$ bi-invariant functions on $GL_e(2,Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A126].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g, Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

Chapter i

Appendix

A-1 Introduction

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of embedding space and related spaces are discussed and the relationship of CP_2 to the standard model is summarized. The basic vision is simple: the geometry of the embedding space $H = M^4 \times CP_2$ geometrizes standard model symmetries and quantum numbers. The assumption that space-time surfaces are basic objects, brings in dynamics as dynamics of 3-D surfaces based on the induced geometry. Second quantization of free spinor fields of H induces quantization at the level of H, which means a dramatic simplification.

The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Also the relationship to string models is discussed briefly.

Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure analogous to adele [L43, L42]. In the recent view of quantum TGD [L127], both notions reduce to physics as number theory vision, which relies on $M^8 - H$ duality [L82, L83] and is complementary to the physics as geometry vision.

Zero energy ontology (ZEO) [L72] [K103] has become a central part of quantum TGD and leads to a TGD inspired theory of consciousness as a generalization of quantum measurement theory having quantum biology as an application. Also these aspects of TGD are briefly discussed.

A-2 Embedding space $M^4 \times CP_2$

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that embedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Embedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) [L72, L108] [K103] causal diamond (CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure [A150] so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2.1 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrate that it codes for the symmetries of standard models via its isometries and holonomies.

CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A134] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.2}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.3}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://tgdtheory.fi/appfigures/cp2.jpg`

Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i\bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-2.5}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.6}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r,\phi) = (tan(\theta/2),\phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-2.7}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-2.8}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-2.9}$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , \quad e^1 = \frac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} \ , \quad e^3 = \frac{r\sigma_3}{F} \ . \end{aligned} \tag{A-2.10}$$

The explicit representations of vierbein vectors are given by

$$\begin{array}{lllllll} e^0 & = & \frac{dr}{F} \ , & & e^1 & = & \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 & = & \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} & , & e^3 & = & \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{array} \quad \text{(A-2.11)}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \quad \text{(A-2.12)}$$

From this expression one finds that at coordinate infinity $r = \infty$ line element reduces to $\frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2)$ of S^2 meaning that 3-sphere degenerates metrically to 2-sphere and one can say that CP_2 is obtained by adding to R^4 a 2-sphere at infinity.

The vierbein connection satisfying the defining relation

$$de^A = -V^A_B \wedge e^B \ , \quad \text{(A-2.13)}$$

is given by

$$\begin{array}{llllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ . \end{array} \quad \text{(A-2.14)}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{llllll} R_{01} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , & R_{23} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} & = & e^0 \wedge e^2 - e^3 \wedge e^1 \ , & R_{31} & = & -e^0 \wedge e^2 + e^3 \wedge e^1 \ , \\ R_{03} & = & 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , & R_{12} & = & 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \quad \text{(A-2.15)}$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J = -is_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \quad \text{(A-2.16)}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k_{\ r} J^{rl} = -s^{kl} \ . \quad \text{(A-2.17)}$$

The condition states that J and g give representations of real unit and imaginary units related by the formula $i^2 = -1$.

Kähler form is expressible locally in terms of Kähler gauge potential

$$J = dB \ , \quad \text{(A-2.18)}$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

$dJ = ddB = 0$ gives the topological half of Maxwell equations (vanishing of magnetic charges and Faraday's induction law) and self-duality $*J = J$ reduces the remaining equations to $dJ = 0$. Hence the Kähler form can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling).

The magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{aligned} B &= 2re^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2} dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F} sin\Theta d\Theta \wedge d\Phi \ . \end{aligned} \quad \text{(A-2.19)}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1, 1).

Useful coordinates for CP_2 are the so called canonical (or symplectic or Darboux) coordinates in which the Kähler potential and Kähler form have very simple expressions

$$\begin{aligned} B &= \sum_{k=1,2} P_k dQ_k \ , \\ J &= \sum_{k=1,2} dP_k \wedge dQ_k \ . \end{aligned} \quad \text{(A-2.20)}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned} P_1 &= -\frac{1}{1+r^2} \ , \\ P_2 &= -\frac{r^2 cos\Theta}{2(1+r^2)} \ , \\ Q_1 &= \Psi \ , \\ Q_2 &= \Phi \ . \end{aligned} \quad \text{(A-2.21)}$$

Spinors In CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A114]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0, 1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1-factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential

$exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4\times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the embedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A185] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]]\in t \ \text{ for } \ X,Y,Z\in t \ . \tag{A-2.22}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1=\bar{\xi}^2 \ \text{ or equivalently } \ (\Theta=\pi/2,\Psi=0) \ ,$$

$$S^2_{II} \ : \ \xi^1=\xi^2 \ \text{ or equivalently } \ (\Theta=\pi/2,\Phi=0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2.2 CP_2 geometry and Standard Model symmetries

Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B40] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e=\pm 1$, CP_2-chirality l,r and M^4-chirality L,R are defined by the condition

$$\begin{aligned}\Gamma\Psi &= e\Psi \ ,\\ e &= \pm 1 \ ,\end{aligned} \tag{A-2.23}$$

where Γ denotes the matrix $\Gamma_9=\gamma_5\otimes\gamma_5$, $1\otimes\gamma_5$ and $\gamma_5\otimes 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4)$ having as its covering group $SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \qquad \text{(A-2.24)}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ , \end{array} \qquad \text{(A-2.25)}$$

and

$$B = 2re^3 \ , \qquad \text{(A-2.26)}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23} I_L^1 + 2V_{13} I_L^2 \ , \qquad \text{(A-2.27)}$$

where one have defined

$$\begin{array}{lll} I_L^1 & = & \frac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\ I_L^2 & = & \frac{(\Sigma_{02} - \Sigma_{13})}{2} \ . \end{array} \qquad \text{(A-2.28)}$$

A_{ch} is clearly left handed so that one can perform the identification of the gauge potential as

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} \ , \qquad \text{(A-2.29)}$$

where $W^\pm$ denotes the charged intermediate vector boson.

The covariantly constant curvature tensor is given by

$$\begin{array}{l} R_{01} = -R_{23} = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} = -R_{31} = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \\ R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , \\ R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \qquad \text{(A-2.30)}$$

The charged part of the curvature tensor is left handed.

This is to be compared with the Weyl tensor, which defines a representation of quaternionic imaginary units.

$$\begin{array}{l} W_{03} = W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ W_{01} = W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ W_{02} = W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \end{array} \tag{A-2.31}$$

The charged part of the Weyl tensor is right-handed and that the relative sign of the two terms in the curvature tensor and Weyl tensor are opposite.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$\begin{aligned} X &= re^3 \ , \\ Y &= \frac{e^3}{r} \ , \end{aligned} \tag{A-2.32}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\begin{aligned} \bar{\gamma} &= aX + bY \ , \\ \bar{Z}^0 &= cX + dY \ , \end{aligned} \tag{A-2.33}$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned} A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\ &+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \end{aligned} \tag{A-2.34}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5\Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \ . \tag{A-2.35}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.36}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$\begin{aligned} Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ , \\ I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \end{aligned} \tag{A-2.37}$$

The fields γ and Z^0 are defined via the relations

$$\begin{aligned} \gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ , \\ Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \end{aligned} \tag{A-2.38}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} , \tag{A-2.39}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of the Weinberg angle is a dynamical problem. The original approach was based on the assumption that it makes sense to talk about electroweak action defined at fundamental level and introduce a symmetry breaking by adding an additional term proportional to Kähler action. The recent view is that Kähler action plus volume term defines the fundamental action.

The Weinberg angle is completely fixed if one requires that the electroweak action contains no cross term of type γZ^0. This leads to a definite value for the Weinberg angle.

One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle. As a matter fact, color gauge action identifying color gauge field as proportional to $H^A J_{\alpha\beta}$ is proportional to Kähler action. A possible interpretation would be as a sum of electroweak and color gauge interactions.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) , \tag{A-2.40}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) , \end{aligned} \tag{A-2.41}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) . \tag{A-2.42}$$

Evaluating the expressions above, one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} , \\ Z^0 &= 2R_{03} . \end{aligned} \tag{A-2.43}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) . \tag{A-2.44}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) , \end{aligned} \tag{A-2.45}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$
\begin{aligned}
X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\
K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \qquad \text{(A-2.46)}
\end{aligned}
$$

This parameter can be calculated by substituting the values of quark and lepton charges and weak isospins.

In the general case the value of the coefficient K is given by

$$
K = \sum_i \left[-\frac{(18+2n_i^2)sin^2\theta_W}{9}\right] \ , \qquad \text{(A-2.47)}
$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18+n_i^2))} \ . \qquad \text{(A-2.48)}
$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \qquad \text{(A-2.49)}
$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is not far from the typical value 9/24 of GUTs at high energies [B11]. The experimental value at the scale length scale of the electron can be deduced from the ratio of W and Z boson masses as $sin^2\theta_W = 1-(m_W/m_Z)^2 \simeq .22290$. This ratio and also the weak boson masses depend on the length scale.

If one interprets the additional term proportional to J as color action, one could perhaps interpret the value of Weinberg angle as expressing a connection between strong and weak coupling constant evolution. The limit $f \rightarrow 0$ should correspond to an infinite value of color coupling strength and at this limit one would have $sin^2\theta_W = \frac{9}{28}$ for $f/g^2 \rightarrow 0$. This does not make sense since the Weinberg angle is in the standard model much smaller in QCD scale Λ corresponding roughly to pion mass scale. The Weinberg angle is in principle predicted by the p-adic coupling constant evolution fixed by the number theoretical vision of TGD.

One could however have a sum of electroweak action, correction terms changing the value of Weinberg angle, and color action and coupling constant evolution could be understood in terms of the coupling parameters involved.

Electroweak symmetry breaking

One of the hardest challenges in the development of the TGD based view of weak symmetry breaking was the fact that classical field equations allow space-time surfaces with finite but arbitrarily large size. For a fixed space-time surface, the induced gauge fields, including classical weak fields, are long ranged. On the other hand, the large mass for weak bosons would require a short correlation length. How can one understand this together with the fact that a photon has a long correlation length?

In zero energy ontology quantum states are superpositions of space-time surfaces as analogs of almost unique Bohr orbits of particles identified as 3-D surfaces. For some reason the superposition should be such that the quantum averages of weak gauge boson fields vanish below the weak scale whereas the quantum average of electromagnetic fields is non-vanishing.

This is indeed the case.

1. The supersymplectic symmetries form isometries of the world of classical worlds (WCW) and they act in CP_2 degrees of freedom as symplectic transformations leaving the CP_2 symplectic form J invariant and therefore also its contribution to the electromagnetic field since this part is the same for all space-time surfaces in the superposition of space-time surfaces as a representation of supersymplectic isometry group (as a special case a representation of color group).

2. In TGD, color and electroweak symmetries acting as holonomies are not independent and for the $SU(2)_L$ part of induced spinor connection the symplectic transformations induces $SU(2)_L \times U(1)_R$ gauge transformation. This suggests that the quantum expectations of the induced weak fields over the space-time surfaces vanish above the quantum coherence scale. The averages of W and of the left handed part of Z^0 should therefore vanish.

3. $\langle Z^0 \rangle$ should vanish. For $U(1)_R$ part of Z^0, the action of gauge transformation is trivial in gauge theory. Now however the space-time surface changes under symplectic transformations and this could make the average of the right-handed part of Z^0 vanishing. The vanishing of the average of the axial part of the Z^0 is suggested by the partially conserved axial current hypothesis.

One can formulate this picture quantitatively.

1. The electromagnetic field [L136] contains, besides the induced Kähler form, also the induced curvature form R_{12}, which couples vectorially. Conserved vector current hypothesis suggests that the average of R_{12} is non-vanishing. One can express the neutral part of the induced gauge field in terms of induced spinor curvature and Kähler form J as

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) = J + 2e^0 \wedge e^3 \quad , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \quad , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) = 3J - 2e^0 \wedge e^3 \quad , \end{aligned} \tag{A-2.50}$$

2. The induced fields γ and Z^0 (photon and Z- boson) can be expressed as

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \quad , \\ Z^0 &= 2R_{03} = 2(J + 2e^0 \wedge e^3) \quad \text{(A-2.51)} \\ per. & \quad \text{(A-2.52)} \end{aligned}$$

The condition $\langle Z^0 \rangle = 0$ gives $2\langle e^0 \wedge e^3 \rangle = -2J$ and this in turn gives $\langle R_{12} \rangle = 4J$. The average over γ would be

$$\langle \gamma \rangle = (3 - 4sin^2\theta_W)J \quad .$$

For $sin^2\theta_W = 3/4\ langle \gamma \rangle$ would vanish.

The quantum averages of classical weak fields quite generally vanish. What about correlation functions?

1. One expects that the correlators of classical weak fields as color invariants, and perhaps even symplectic invariants, are non-vanishing below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as hadron.

2. The intuitive picture is that in longer length scales one has disjoint 3-surfaces with a size scale of Compton length. If the states associated with two disjoint 3-surfaces are separately color invariant there are no correlations in color degrees of freedom and correlators reduce to the products of expectations of classical weak fields and vanish. This could also hold when the 3-surfaces are connected by flux tube bonds.

   Below the Compton length weak bosons would thus behave as correlated massless fields. The Compton lengths of weak bosons are proportional to the value of effective Planck constant h_{eff} and in living systems the Compton lengths are proposed to be even of the order of cell size. This would explain the mysterious chiral selection in living systems requiring large parity violation.

3. What about the averages and correlators of color gauge fields? Classical color gauge fields are proportional to the products of Hamiltonians of color isometries induced Kähler form and the expectations of color Hamiltonians give vanishing average above Compton length and therefore vanishing average. Correlators are non-vanishing below the hadron scale. Gluons do not propagate in long scales for the same reason as weak bosons. This is implied by color confinement, which has also classical description in the sense that 3-surfaces have necessarily a finite size.

   A large value of h_{eff} allows colored states even in biological scales below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as dark hadron.

Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

1. Symmetries must be realized as purely geometric transformations.

2. Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B14] .

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.53}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.54}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.55}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

A-3.1 Induction procedure for gauge fields and spinor connection

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has embedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as a base space the imbedded manifold, whose points have as fiber the fiber if embedding space at their image points. In the recent case the embedding of space-time surface to embedding space defines the induction procedure. The induced gauge potentials and gauge fields are projections of the spinor connection of the embedding space to the space-time surface (see `http://tgdtheory.fi/appfigures/induct.jpg`).

Induction procedure makes sense also for the spinor fields of embedding space and one obtains geometrization of both electroweak gauge potentials and of spinors. The new element is induction of gamma matrices which gives their projections at space-time surface.

As a matter fact, the induced gamma matrices cannot appear in the counterpart of massless Dirac equation. To achieve super-symmetry, Dirac action must be replaced with Kähler-Dirac action for which gamma matrices are contractions of the canonical momentum currents of Kähler action with embedding space gamma matrices. Induced gamma matrices in Dirac action would correspond to 4-volume as action.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://tgdtheory.fi/appfigures/induct.jpg`.

A-3.2 Induced gauge fields for space-times for which CP_2 projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP_2 projection, only vacuum extremals and space-time surfaces for which CP_2 projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP_2 projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP_2 projection color rotations and weak symmetries commute.

A-3.3 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four embedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. `http://tgdtheory.fi/appfigures/manysheeted.jpg`

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. `http://tgdtheory.fi/appfigures/wormholecontact.jpg`

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg`

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of embedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)") as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-3.4 Embedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

1. Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

2. Spinor harmonics of embedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of embedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

1. Although the embedding space spinor connection carries W gauge potentials one can say that the embedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

2. The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the

spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

3. Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

4. One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

5. This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-3.5 About induced gauge fields

In the following the induced gauge fields are studied for general space-time surface without assuming the preferred extremal property (Bohr orbit property). Therefore the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta)d\Phi) + \frac{r^2}{2F} sin(\Theta)d\Theta \wedge d\Phi \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-3.1}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta)d\Phi) + (3+p)\frac{r^2}{2F} sin(\Theta)d\Theta \wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.2}$$

where Θ_W denotes Weinberg angle.

1. The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2 F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p)sin(\Theta) = 0 \ , \end{aligned} \tag{A-3.3}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C = k + cos(\Theta_0) \ , \ D = \frac{r_0^2}{1+r_0^2} \ , \ \epsilon = \frac{3+p}{3+2p} \ , \end{aligned} \quad \text{(A-3.4)}$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1 + r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k+1 \ ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \quad \text{(A-3.5)}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

2. The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

3. The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\ \gamma &= -\frac{p}{2} Z^0 \ . \end{aligned} \quad \text{(A-3.6)}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \quad \text{(A-3.7)}$$

and is useful in the construction of vacuum embedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$\begin{aligned} \Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\ \Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ . \end{aligned} \quad \text{(A-3.8)}$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global embedding for, say a constant magnetic field. Although global embedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \quad \text{(A-3.9)}$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-4 The relationship of TGD to QFT and string models

The recent view of the relationship of TGD to QFT and string models has developed slowly during years and it seems that in a certain sense TGD means a return to roots: instead of QFT like description involving path integral one would have wave mechanics for 3-surfaces.

A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces

The first vision of TGD was as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

The later work has revealed that TGD could be seen as a generalization of the wave mechanism based on the replacement of a point-like particle with 3-D surface. This is due to holography implied by general coordinate invariance. The definition of the metric of the "world of classical worlds" (WCW) must assign a unique or at least almost unique space-time surface to a given 3-surface. This 4-surface is analogous to Bohr orbit so that also Bohr orbitology becomes an exact part of quantum physics. The failure of strict determinism forces to replace 3-surfaces with 4-surfaces and this leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K42, K24, K76] [L110, L127].

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://tgdtheory.fi/appfigures/particletgd.jpg`

A-4.2 Extension of superconformal invariance

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in a completely unique position as far as symmetries are considered.

In fact, this leads to a generalization of the Kac-Moody type symmetries of string models. $\delta M^4_+ \times CP_2$ allows huge supersymplectic symmetries for which the radial light-like coordinate of δM^4_+ plays the role of complex string coordinate in string models. These symmetries are assumed to act as isometries of WCW.

A-4.3 String-like objects and strings

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of the TGD Universe and the inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string-like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situations in Minkowskian regions of space-time surface. `http://tgdtheory.fi/appfigures/fermistring.jpg`

A-4.4 TGD view of elementary particles

The TGD based view about elementary particles has two key aspects.

1. The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidean signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

2. At the level of H Fermion number is carried by the modes of the induced spinor field. In space-time regions with Minkowski signature the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle orbit corresponds to a 4-D generalization of a world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidean signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacts carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompanied by fermionic strings connecting the throats at the same sheet: the strings do not extend inside the wormhole contacts. `http://tgdtheory.fi/appfigures/elparticletgd.jpg`

Particle interactions involve both stringy and QFT aspects.

1. The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having a hadronic string as a physical counterpart. Long strings should be distinguished from wormhole contacts which due to their superconformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

2. Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats behave like virtual bosons so that one has BFF type vertices typically.

3. In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particles along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. `http://tgdtheory.fi/appfigures/tgdgraphs.jpg`

A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K42, K76].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

A-5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [K79] [L110, L115, L116] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and

$T(CP_2)$ of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A150] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

1. Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

2. Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

3. The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

   For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

1. The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

2. Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

   For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

A-5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

1. If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

2. WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L127]. There are also other important algebras but these will not be discussed now.

1. The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

   The super symplectic algebra A has an infinite hierarchy of sub-algebras [L127] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

   This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

2. The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

   The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on M^8-H duality [L127] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

1. The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by M^8-H duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

2. This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L127] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P)=n(braid)=n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

3. The actions related to different values of $n(P)=n(braid)=n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

   WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

4. Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

   A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L120] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i=2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r=1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

   A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

5. A subset of p-adic primes allowed by the p-adic length scale hypothesis $p\simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K54, K55]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

1. Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

   P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L120], and the forward scattering amplitude corresponds to the propagator.

   This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

2. According to [L120], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

   A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L120].

3. p-Adic length scale hypothesis [L128] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

   For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

   Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

1. The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

   In [L127] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

2. The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K42, K24]. As isometries they would naturally permute the maxima with each other.

A-6 Number theoretic vision of TGD

Physics as number theory vision is complementary to the physics as geometry vision and has developed gradually since 1993. Langlands program is the counterpart of this vision in mathematics [L125].

The notion of p-adic number fields emerged with the motivation coming from the observation that elementary particle mass scales and mass ratios could be understood in terms of the so-called p-adic length scale hypothesis [K59, K50, K21]. The fusion of the various p-adic physics leads to what I call adelic physics [L43, L42]. Later the hypothesis about hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter emerged [K27, K28, K29, K29].

Eventually this led to that the values of effective Planck constant could be identified as the dimension of an algebraic extension of rationals assignable to polynomials with rational coefficients. This led to the number theoretic vision in which so-called M^8-H duality [L82, L83] plays a key role. M^8 (actually a complexification of real M^8) is analogous to momentum space so that the duality generalizes momentum position duality for point-like particles. M^8 has an interpretation as complexified octonions.

The dynamics of 4-surfaces in M^8 is coded by polynomials with rational coefficients, whose roots define mass shells H^3 of $M^4 \subset M^8$. It has turned out that the polynomials satisfy stringent additional conditions and one can speak of number theoretic holography [L120, L125]. Also the ordinary 3→ 4 holography is needed to assign 4-surfaces with these 3-D mass shells. The number theoretic dynamics is based on the condition that the normal space of the 4-surface in M^8 is associative (quaternionic) and contains a commutative complex sub-space. This makes it possible to assign to this surface space-time surface in $H = M^4 \times CP_2$.

At the level of H the space-time surfaces are by holography preferred extremals and are assumed to be determined by the twistor lift of TGD [K79] giving rise to an action which is sum of the Kähler action and volume term. The preferred extremals would be minimal surfaces analogous to soap films spanned by frames. Outside frames they would be simultaneous extremals of the Kähler action, which requires a generalization of the holomorphy characterizing string world sheets.

In the following only p-adic numbers and hierarchy of Planck constants will be discussed.

A-6.1 p-Adic numbers and TGD

p-Adic number fields

p-Adic numbers (p is prime: 2, 3, 5, ...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A110]. p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k,\ x(k) = 0,....,p-1 \ . \tag{A-6.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} . \tag{A-6.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) , \tag{A-6.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} . \tag{A-6.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D . \tag{A-6.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

2. Distances of points x and y inside single class are smaller than distances between different classes.

3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B35]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

1. Basic form of the canonical identification

There exists a natural continuous map $I : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} , \\ y_k &\in \{0, 1, .., p-1\} . \end{aligned} \tag{A-6.6}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{A-6.7}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{A-6.8}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

2. The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{A-6.9}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{A-6.10}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.11}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3. Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.12}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4. Generalization of number concept and notion of embedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic embedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real embedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \ mod \ p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real embedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles".

Fig. 16. The basic idea between p-adic manifold. `http://tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

1. Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

2. Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

3. Canonical identification violates general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic embedding space with chart maps to real embedding space and assuming preferred coordinates made possible by isometries of embedding space: one however obtains several inequivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-6.2 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the embedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of embedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of embedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-6.3 $M^8 - H$ duality as it is towards the end of 2021

The view of $M^8 - H$ duality (see Appendix **??**) has changed considerably towards the end 2021 [L110] after the realization that this duality is the TGD counterpart of momentum position duality of wave mechanics, which is lost in QFTs. Therefore M^8 and also space-time surface is analogous to momentum space. This forced us to give up the original simple identification of the points $M^4 \subset M^4 \times E^4 = M^8$ and of $M^4 \times CP_2$ so that it respects Uncertainty Principle (UP).

The first improved guess for the duality map was the replacement with the inversion $p^k \rightarrow m^k = \hbar_{eff} p^k/p^2$ conforming in spirit with UP but turned out to be too naive.

The improved form [L110] of the $M^8 - H$ duality map takes mass shells $p^2 = m^2$ of $M^4 \subset M^8$ to cds with size $L(m) = \hbar_{eff}/m$ with a common center. The slicing by mass shells is mapped to a Russian doll like slicing by cds. Therefore would be no CDs in M^8 contrary to what I believed first.

Quantum classical correspondence (QCC) inspires the proposal that the point $p^k \in M^8$ is mapped to a geodesic line corresponding to momentum p^k starting from the common center of cds. Its intersection with the opposite boundary of cd with size $L(m)$ defines the image point. This is not yet quite enough to satisfy UP but the additional details [L110] are not needed in the sequel.

The 6-D brane-like special solutions in M^8 are of special interest in the TGD inspired theory of consciousness. They have an M^4 projection which is $E = E_n$ 3-ball. Here E_n is a root of the real polynomial P defining $X^4 \subset M^8_c$ (M^8 is complexified to M^8_c) as a "root" of its octonionic continuation [L82, L83]. E_n has an interpretation as energy, which can be complex. The original interpretation was as moment of time. For this interpretation, $M^8 - H$ duality would be a linear identification and these hyper planes would be mapped to hyperplanes in $M^4 \subset H$.

This motivated the term "very special moment in the life of self" for the image of the $E = E_n$ section of $X^4 \subset M^8$ [L63]. This notion does not make sense at the level M^8 anymore.

The modified $M^8 - H$ duality forces us to modify the original interpretation [L110]. The point $(E_n, p = 0)$ is mapped $(t_n = \hbar_{eff}/E_n, 0)$. The momenta (E_n, p) in $E = E_n$ plane are mapped to the boundary of cd and correspond to a continuous time interval at the boundary of CD: "very special moment" becomes a "very special time interval".

The quantum state however corresponds to a set of points corresponding to quark momenta, which belong to a cognitive representation and are therefore algebraic integers in the extension determined by the polynomial. These active points in E_n are mapped to a discrete set at the boundary of cd(m). A "very special moment" is replaced with a sequence of "very special moments".

So called Galois confinement [L101] forces the total momenta for bound states of quarks and antiquarks to be rational integers invariant under Galois group of extension of rationals determined by the polynomial P [L110]. These states correspond to states at boundaries of sub-CDs so that one obtains a hierarchy. Galois confinement provides a universal number theoretic mechanism for the formation of bound states.

A-7 Zero energy ontology (ZEO)

ZEO is implied by the holography forced in the TGD framework by general coordinate invariance.

A-7.1 Basic motivations and ideas of ZEO

The following gives a brief summary of ZEO [L72] [K103].

1. In ZEO quantum states are not 3-dimensional but superpositions of 4-dimensional deterministic time evolutions connecting ordinary initial 3-dimensional states. By holography they are equivalent to pairs of ordinary 3-D states identified as initial and final states of time evolution. One can say that in the TGD framework general coordinate invariance implies holography and the slight failure of its determinism in turn forces ZEO.

   Quantum jumps replace this state with a new one: a superposition of deterministic time evolutions is replaced with a new superposition. Classical determinism of individual time evolution is not violated and this solves the basic paradox of quantum measurement theory. There are two kinds of quantum jumps: ordinary ("big") state function reductions (BSFRs) changing the arrow of time and "small" state function reductions (SSFRs) (weak measurements) preserving it and giving rise to the analog of Zeno effect [L72].

2. To avoid getting totally confused it is good to emphasize some aspects of ZEO.

   (a) ZEO does not mean that physical states in the usual 3-D sense as snapshots of time evolution would have zero energy state pairs defining zero energy states as initial and final states have same conserved quantities such as energy. Conservation implies that one can adopt the conventions that the values of conserved quantities are opposite for these states so that their sum vanishes: one can think that incoming and outgoing particles come from geometric past and future is the picture used in quantum field theories.

   (b) ZEO means two times: subjective time as sequence of quantum jumps and geometric time as space-time coordinate. These times are identifiable but are strongly correlated.

3. In BSFRs the arrow of time is changed and the time evolution in the final state occurs backwards with respect to the time of the external observer. BSFRs can occur in all scales since TGD predicts a hierarchy of effective Planck constants with arbitrarily large values. There is empirical support for BSFRs.

   (a) The findings of Minev et al [L59] in atomic scale can be explained by the same mechanism [L59]. In BSFR a final zero energy state as a superposition of classical deterministic time evolutions emerges and for an observer with a standard arrow of time looks like a superposition of deterministic smooth time evolutions leading to the final state. Interestingly, once this evolution has started, it cannot be stopped unless one changes the stimulus signal inducing the evolution in which case the process does not lead to anywhere: the interpretation would be that BSFR back to the initial state occurs!

(b) Libets' experiments about active aspects of consciousness [J1] can be understood. Subject person raises his finger and neural activity starts before the conscious decision to do so. In the physicalistic framework it is thought to lead to raising of the finger. The problem with the explanation is that the activity beginning .5 seconds earlier seems to be dissipation with a reversed arrow of time: from chaotic and disordered to ordered at around .15 seconds. ZEO explanation is that macroscopic quantum jump occurred and generated a signal proceeding backwards in time and generated neural activity and dissipated to randomness.

(c) Earthquakes involve a strange anomaly: they are preceded by ELF radiation. One would expect that they generate ELF radiation. The identification as BSFR would explain the anomaly [L62]. In biology the reversal of the arrow of time would occur routinely and be a central element of biological self-organization, in particular self-organized quantum criticality (see [L68, L140].

A-7.2 Some implications of ZEO

ZEO has profound implications for understanding self-organization and self-organized quantum criticality in terms of dissipation with non-standard arrow of time looking like generation of structures [L68, L140]. ZEO could also allow understanding of what planned actions - like realizing the experiment under consideration - could be.

1. Second law in the standard sense does not favor - perhaps even not allow - realization of planned actions. ZEO forces a generalization of thermodynamics: dissipation with a non-standard arrow of time for a subsystem would look like self-organization and planned action and its realization.

   Could most if not all planned action be like this - induced by BSFR in the geometric future and only apparently planned? There would be however the experience of planning and realizing induced by the signals from geometric future by a higher level in the hierarchy of conscious entities predicted by TGD! In long time scales we would be realizing our fates or wishes of higher level conscious entities rather than agents with completely free will.

2. The notion of magnetic body (MB) serving as a boss of ordinary matter would be central. MB carries dark matter as $h_{eff} = nh_0$ phases of ordinary matter with n serving as a measure for algebraic complexity of extension of rationals as its dimension and defining a kind of universal IQ. There is a hierarchy of these phases and MBs labelled by extension of rationals and the value of n.

   MBs would form a hierarchy of bosses - a realization for master slave hierarchy. Ordinary matter would be at the bottom and its coherent behavior would be induced from quantum coherence at higher levels. BSFR for higher level MB would give rise to what looks like planned actions and experienced as planned action at the lower levels of hierarchy. One could speak of planned actions inducing a cascade of planned actions in shorter time scales and eventually proceeding to atomic level.

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred embedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes. There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. `http://tgdtheory.fi/appfigures/padictoreal.jpg`

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. `http://tgdtheory.fi/appfigures/sharing.jpg`

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see **Fig.** `http://tgdtheory.fi/appfigures/timemirror.jpg` or **Fig.** 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. `http://tgdtheory.fi/appfigures/timemirror.jpg`

Acknowledgements: I am grateful for Dainis Zeps for enlightnening discussions concerning CP_2 geometry.

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