TOWARDS M-MATRIX: PART II

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0.1 PREFACE

This book belongs to a series of online books summarizing the recent state Topological Geometrodynamics (TGD) and its applications. TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students at seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 37 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of imbedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, main stream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to same multiplet of the gauge group implying instability of proton.

There have been also longstanding problems.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. Furthermore, the imbeddings of Robertson-Walker cosmologies turned out to be vacuum extremals with respect to the inertial energy. About 25 years was needed to realize that the sign of the inertial energy can be also negative and in cosmological scales the density of inertial energy vanishes: physically acceptable universes are creatable from vacuum. Eventually this led to the notion of zero energy ontology (ZEO) which deviates dramatically from the standard ontology being however consistent with the crossing symmetry of quantum field theories. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of so called causal diamonds defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle as it is expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields.

  It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to dark matter. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem, and 32 years was needed to discover that what I call weak form of electric-magnetic duality gives a satisfactory solution of the problem and provides also surprisingly powerful insights to the mathematical structure of quantum TGD.

  The latest development was the realization that the well- definedness of electromagnetic charge as quantum number for the modes of the induced spinors field requires that the CP_2 projection of the region in which they are non-vanishing carries vanishing W boson field and is 2-D. This implies in the generic case their localization to 2-D surfaces: string world sheets and possibly also partonic 2-surfaces. This localization applies to all modes except covariantly constant right handed neutrino generating supersymmetry and mplies that string model in 4-D space-time is part of TGD. Localization is possible only for Kähler-Dirac assigned with Kähler action defining the dynamics of space-time surfaces. One must however leave open the question whether W field might vanish for the space-time of GRT if related to many-sheeted space-time in the proposed manner even when they do not vanish for space-time sheets.

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but the first discovery made already during first year of TGD was that these formalisms might be useless due to the extreme non-linearity and enormous vacuum degeneracy of the theory. This turned out to be the case.

- It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as 3-dimensional surfaces. Still few years had to pass before I understood that general coordinate invariance leads to a more or less unique solution of the problem and in positive energyontology implies that space-time surfaces are analogous to Bohr orbits. This in positive energy ontology in which space-like 3-surface is basic object. It is not clear whether Bohr orbitology is necessary also in ZEO in which space-time surfaces connect space-like 3-surfaces at the light-like boundaries of causal diamond CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). The reason is that the pair of 3-surfaces replaces the boundary conditions at single 3-surface involving also time derivatives. If one assumes Bohr orbitology then strong correlations between the 3-surfaces at the ends of CD follow. Still a couple of years and I discovered that quantum states of the Universe can be identified as classical spinor fields in WCW. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

- TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Pronciple with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

- One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

  During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

  Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

  The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. It has indeed turned out that the dream about explicit formula is unrealistic before one has understood what happens in quantum jump. Symmetries and general physical principles have turned out to be the proper guide line here. To give some impressions about what is required some highlights are in order.

- With the emergence of ZEO the notion of S-matrix was replaced with M-matrix defined between positive and negative energy parts of zero energy states. M-matrix can be interpreted as a complex square root of density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally. M-matrices in turn bombine to form the rows of unitary U-matrix defined between zero energy states.

- A decisive step was the strengthening of the General Coordinate Invariance to the requirement that the formulations of the theory in terms of light-like 3-surfaces identified as 3-surfaces at which the induced metric of space-time surfaces changes its signature and in terms of space-like 3-surfaces are equivalent. This means effective 2-dimensionality in the sense that partonic 2-surfaces defined as intersections of these two kinds of surfaces plus 4-D tangent space data at partonic 2-surfaces code for the physics. Quantum classical correspondence requires the coding of the quantum numbers characterizing quantum states assigned to the partonic 2-surfaces to the geometry of space-time surface. This is achieved by adding to the modified Dirac action a measurement interaction term assigned with light-like 3-surfaces.

- The replacement of strings with light-like 3-surfaces equivalent to space-like 3-surfaces means enormous generalization of the super conformal symmetries of string models. A further generalization of these symmetries to non-local Yangian symmetries generalizing the recently discovered Yangian symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is highly suggestive. Here the replacement of point like particles with partonic 2-surfaces means the replacement of conformal symmetry of Minkowski space with infinite-dimensional super-conformal algebras. Yangian symmetry provides also a further refinement to the notion of conserved quantum numbers allowing to define them for bound states using non-local energy conserved currents.

- A further attractive idea is that quantum TGD reduces to almost topological quantum field theory. This is possible if the Kähler action for the preferred extremals defining WCW Kähler function reduces to a 3-D boundary term. This takes place if the conserved currents are so called Beltrami fields with the defining property that the coordinates associated with flow lines extend to single global coordinate variable. This ansatz together with the weak form of electric-magnetic duality reduces the Kähler action to Chern-Simons term with the condition that the 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric magnetic duality. It is the latter constraint which prevents the trivialization of the theory to a topological quantum field theory. Also the identification of the Kähler function of WCW as Dirac determinant finds support as well as the description of the scattering amplitudes in terms of braids with interpretation in terms of finite measurement resolution coded to the basic structure of the solutions of field equations.

- In standard QFT Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

  In TGD framework this generalization of Feynman diagrams indeed emerges unavoidably. Light-like 3-surfaces replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. The approximate localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In TGD framework fermionic variant of twistor Grassmann formalism leads to a stringy variant of twistor diagrammatics in which basic fermions can be said to be on mass-shell but carry non-physical helicities in the internal lines. This suggests the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras.

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 32 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks.

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Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family and Kalevi and Ritva Tikkanen and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 45 lonely years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During the last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss my work. Pertti Kärkkäinen is my old physicist friend and has provided continued economic support for a long time. I have also had stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Tommi Ullgren has provided both economic support and encouragement during years. Pekka Rapinoja has offered his help in this respect and I am especially grateful to him for my Python skills.

During the last five years I have had inspiring discussions with many people in Finland interested in TGD. We have had video discussions with Sini Kunnas and had podcast discussions with Marko Manninen related to the TGD based view of physics and consciousness. Marko has also helped in the practical issues related to computers and quite recently he has done a lot of testing of chatGPT helping me to get an overall view of what it is. The discussions in a Zoom group involving Marko Manninen, Tuomas Sorakivi and Rode Majakka have given me the valuable opportunity to clarify my thoughts.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation in CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. I am grateful to Mark McWilliams, Paul Kirsch, Gary Ehlenberg, and Ulla Matfolk and many others for providing links to possibly interesting websites and articles. We have collaborated with Peter Gariaev and Reza Rastmanesh. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the public through the iron wall of academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as an individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without direct support from power holders- even in archives like arXiv.org.

Situation changed as Andrew Adamatsky proposed the writing of a book about TGD when I had already gotten used to the thought that my work would not be published during my lifetime. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loophole. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy.

And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his 73th birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During the period when the right wing held political power this trend was steadily strengthening and the situation is the same as I am writing this. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

Karkkila, August 30, 2023, Finland

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Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K3].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A54]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

   M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of

electromagnetic fields are nonvanishing. The correlations functions for weak fields are non-vanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question: super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A18] [B36, B26, B27]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B25]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4 \times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n \times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A43, A53, A31, A49].

The identification of the space-time as a sub-manifold [A44, A75] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. 9.4** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

[1] There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.1.5 Construction of scattering amplitudes

Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A65, A80, A89]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K61]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, M^8-H duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and M^8-H duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and M^8-H duality would map the algebraic physics in M^8 to the ordinary physics in $M^4\times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their M^8-H duals in M^8_c are defined for both reals and various p-adic number fields. This is true if they are defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p=3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired “Universe as Computer” vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.
2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since $M^8 - H$ duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K57].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to “mind stuff”, the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{(Re(E)^2 - Im(E)^2)}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{(Re(E)^2 - Im(E)^2)}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} . \quad (1.1.1)$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to M^8-H duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.
2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.
3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as M^8-H duality requires and deduce an explicit formula for the point. This means a realization of the dream.
4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the M^8-H image of Y^4 satisfies the generalized holomorphy.
5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same.

The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of M^8-H duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4\times CP_2$?

1. The selection of $SU(3)\subset G_2$ for ordinary M^8-H duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3\subset M^8_c$ and its image $H^3\subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?
2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3)\subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.
3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of M^8-H duality as something more fundamental than the ordinary M^8-H duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3\times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1)\times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1\oplus 3\oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3\oplus\overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8\oplus 3\oplus\overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3)\times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x)\times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X^3_i.

1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that

Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K85].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K76, K77, K74]) support the view that dark matter might be a key player in living matter.

Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [L10]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A54]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   $M^8 - H$ duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L57].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L42]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see http://tinyurl.com/yyhwvbqb) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see http://tinyurl.com/yyvkx7as) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t-m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

1.2 Bird's Eye of View about the Topics of "Towards M-matrix"

This book is devoted to a detailed representation of quantum TGD in its recent form. Quantum TGD relies on two different views about physics: physics as an infinite-dimensional spinor geometry and physics as a generalized number theory.

Number theoretic vision leads to the notion of adelic physics fusing real physics with p-adic physics as physics of cognition. It also leads to M^8-H duality raising classical number fields in central role and reducing the dynamics of space-time surfaces in $M^4\times CP_2$ determined by action principle and subject to infinite number of analogs of gauge conditions to purely algebraic dynamics in M^8. Twistor lift of TGD is a further central notion.

The most important guiding principle is quantum classical correspondence, whose most profound implications follow almost trivially from the basic structure of the classical theory forming an exact part of quantum theory. A further mathematical guideline is the mathematics associated with hyper-finite factors of type II_1 about which the spinors of the world of classical worlds represent a canonical example.

1.2.1 Zero energy ontology

1. The new view about energy and time finding a justification in the framework of zero energy ontology (ZEO) means that the sign of the inertial energy depends on the time orientation of the space-time sheet and that negative energy space-time sheets serve as correlates for communications to the geometric future. This alone leads to profoundly new views about metabolism, long term memory, and realization of intentional action. ZEO has led to a new view about quantum measurement theory extending it to a theory of consciousness solving the basic paradox of quantum measurement theory in its standard form.

2. Classical theory is in a well-defined sense exact part of quantum TGD. Action principle should assign to a given 3-surface unique space-time surface analogous to Bohr orbit. In zero energy ontology (ZEO) 3-surface is identified as a disjoint pair of 3-surfaces with members located at the opposite boundaries of causal diamond (CD) being analogous to initial and final states

of a unique classical time evolution represented by preferred extremals. What the action principle is and what *preferred* does mean? During years I have considered several answers to these questions.

For a long time action was identified as 4-D Kähler action but the emergence of the twistor lift of TGD changed this view. 4-D space-time surface is replaced with the analog of its 6-D twistor-space represented as 6-D surfac having the structure of S^2 bundle with base space identifiable as 4-D space-time surface. Twistor structure of this 6-surface is induced from the 12-D Cartesian product of 6-D twistor spaces $T(M^4)$ and $T(CP_2)$ having Kähler structure only for M^4 and CP_2. This allows to define 6-D Kähler action whose dimensionally reduced extremals induce of twistor structure to the 6-D surface. Quantum criticality suggests that all preferred extremals are minimal surfaces apart from 2-D singular surfaces identifiable as string world sheets and partonic 2-surfaces. The reason is that the dynamics in this case is independent of coupling parameters (Kähler coupling strength).

The dimensionally reduced action is sum of Kähler action and volume term having interpretation in terms of cosmological constant. Minimal surfaces are extremals of both volume term and Kähler action separately. Therefore all extremals of Kähler action with non-vanishing Kähler form are also minimal surfaces so that no changes emerge. Therefore I have kept the old chapters studying extremals of Kähler action as such.

3. The differences between the Kähler action with volume term and mere Kähler action emerge only in the vacuum sector. For non-vanishing value of cosmological constant the vacuum extremals with vanishing induced Kähler form are not possible but one can consider the possibility that the dynamically determined cosmological constant [L57] can vanish at the limiting situation when the space-time surfaces have infinite size. The emerging huge vacuum degeneracy and the failure of the classical determinism in the conventional sense, would have strong implications.

   One would have near vacuum extremals of Kähler action a strongly interacting theory defined by volume action with a small cosmological constant with large quantum fluctuations characterizing quantum criticality playing a key role. Vacuum degeneracy implies spin glass degeneracy in 4-D sense. Whether this nearly vacuum degeneracy is a fundamental characteristic of TGD Universe in long length scales, remains an open question.

1.2.2 Quantum classical correspondence

Quantum classical correspondence has turned out to be the most important guiding principle concerning the interpretation of the theory.

1. Quantum classical correspondence and the properties of the simplest extremals of Kähler action have served as the basic guideline in the attempts to understand the new physics predicted by TGD. The most dramatic predictions follow without even considering field equations in detail by using quantum classical correspondence and form the backbone of TGD and TGD inspired theory of living matter in particular.

   The notions of many-sheeted space-time, topological field quantization and the notion of field/magnetic body, follow from simple topological considerations. The observation that space-time sheets can have arbitrarily large sizes and their interpretation as quantum coherence regions forces to conclude that in TGD Universe macroscopic and macro-temporal quantum coherence are possible in arbitrarily long scales.

2. Also long ranged classical color and electro-weak fields are an unavoidable prediction It however took a considerable time to make the obvious conclusion: TGD Universe is fractal containing fractal copies of standard model physics at various space-time sheets and labeled by the collection of p-adic primes assignable to elementary particles and by the level of dark matter hierarchy characterized partially by the value of Planck constant labeling the pages of the book like structure formed by singular covering spaces of the imbedding space $M^4 \times CP_2$ glued together along a four-dimensional back. Particles at different pages are dark relative to each other since purely local interactions defined in terms of the vertices of Feynman diagram involve only particles at the same page.

3. The detailed study of the simplest extremals of Kähler action interpreted as correlates for asymptotic self organization patterns provides additional insights. CP_2 type extremals representing elementary particles, cosmic strings, vacuum extremals, topological light rays ("massless extremal", ME), flux quanta of magnetic and electric fields represent the basic extremals. Pairs of wormhole throats identifiable as parton pairs define a completely new kind of particle carrying only color quantum numbers in ideal case and I have proposed their interpretation as quantum correlates for Boolean cognition. MEs and flux quanta of magnetic and electric fields are of special importance in living matter.

   Topological light rays have interpretation as space-time correlates of "laser beams" of ordinary or dark photons or their electro-weak and gluonic counterparts. Neutral MEs carrying em and Z^0 fields are ideal for communication purposes and charged W MEs ideal for quantum control. Magnetic flux quanta containing dark matter are identified as intentional agents quantum controlling the behavior of the corresponding biological body parts utilizing negative energy W MEs. Bio-system in turn is populated by electrets identifiable as electric flux quanta.

1.2.3 Physics as infinite-dimensional geometry in the "world of classical worlds"

Physics as infinite-dimensional Kähler geometry of the "world of classical worlds" with classical spinor fields representing the quantum states of the universe and gamma matrix algebra geometrizing fermionic statistics is the first vision.

The mere existence of infinite-dimensional non-flat Kähler geometry has impressive implications. WCW must decompose to a union of infinite-dimensional symmetric spaces labelled by zero modes having interpretation as classical dynamical degrees of freedom assumed in quantum measurement theory. Infinite-dimensional symmetric space has maximal isometry group identifiable as a generalization of Kac Moody group obtained by replacing finite-dimensional group with the group of canonical transformations of $\delta M^4_+ \times CP_2$, where δM^4_+ is the boundary of 4-dimensional future light-cone. The infinite-dimensional Clifford algebra of configuration space gamma matrices in turn can be expressed as direct sum of von Neumann algebras known as hyper-finite factors of type II_1 having very close connections with conformal field theories, quantum and braid groups, and topological quantum field theories.

1.2.4 Physics as a generalized number theory

Second vision is physics as a generalized number theory. This vision forces to fuse real physics and various p-adic physics to a single coherent whole having rational physics as their intersection and poses extremely strong conditions on real physics. This led eventually to what I call adelic physics [L42, L43]. One of the outcomes was a proposal for a number theoretical interpretation for the hierarchy of Planck constants: the integer defining effective Planck constant $h_{eff} = n \times h_0$ would correspond to the dimension of the extension of rationals defining the adele.

A further aspect of this vision is the reduction of the classical dynamics of space-time sheets to number theory with space-time sheets identified as what I christened quaternionic sub-manifolds of complexified octonionic imbedding space M^8_c.

$M^8 - H$ duality leads to a concrete proposal stating that space-time surfaces in 16-D M^8_c consist of regions for which either real or imaginary part of a complexified-octonion valued polynomial (additional imaginary unit i commutes with octonion units) vanishes. Imaginary and real part refer now to complexified quaternions $o_c = q_{1,c} + J_4 q_{2,c}$ so that 2×4 conditions give 8-D complexified space-time surface. 4-D space-time surfaces in M^8 could correspond to projections of these with respect to M^8, that is time coordinate would be real and remaining 7 coordinates imaginary.

The development of ideas involved a rather strange quirk, which I noticed while doing the updating in 2019.

1. The original idea that I forgot too soon was that the notion of calibration (see `http://tinyurl.com/y3lyead3`) generalizes and could be relevant for TGD. A calibration in Riemann manifold M means the existence of a k-form ϕ in M such that for any orientable k-D sub-

manifold the integral of ϕ over M equals to its k-volume in the induced metric. One can say that metric k-volume reduces to homological k-volume.

Calibrated k-manifolds are minimal surfaces in their homology class. Kähler calibration is induced by the k^{th} power of Kähler form and defines calibrated sub-manifold of real dimension $2k$. Calibrated sub-manifolds are in this case precisely the complex sub-manifolds. In the case of CP_2 they would be complex curves (2-surfaces) as has become clear.

2. By the Minkowskian signature of M^4 metric, the generalization of calibrated sub-manifold so that it would apply in $M^4 \times CP_2$ is non-trivial. Twistor lift of TGD however forces to introduce the generalization of Kähler form in M^4 (responsible for CP breaking and matter antimatter asymmetry) and calibrated manifolds in this case would be naturally analogs of string world sheets and partonic 2-surfaces as minimal surfaces. Cosmic strings are Cartesian products of string world sheets and complex curves of CP_2. Calibrated manifolds, which do not reduce to Cartesian products of string world sheets and complex surfaces of CP_2 should also exist and one expects that they are minimal surfaces.

   One can also have 2-D calibrated surfaces and they could correspond to string world sheets and partonic 2-surfaces which also play key role in TGD. Even discrete points assignable to partonic 2-surfaces play key role and would trivially correspond to calibrated surfaces.

3. Much later I ended up with the identification of preferred extremals as minimal surfaces by totally different route without realizing the possible connection with the generalized calibrations. Twistor lift and the notion of quantum criticality led to the proposal that preferred extremals for the twistor lift of Kähler action containing also volume term are minimal surfaces. Preferred extremals would be separately minimal surfaces and extrema of Kähler action and generalization of complex structure to what I called Hamilton-Jacobi structure would be an essential element. Quantum criticality would be realized as decoupling of the two parts of action. Could all preferred extremals be regarded as calibrated in some generalized sense.

   If so, the dynamics of preferred extremals would define a homology theory in the sense that each homology class would contain single preferred extremal. TGD would define a generalized topological quantum field theory with conserved Noether charges (in particular rest energy) serving as generalized topological invariants having extremum in the set of topologically equivalent 3-surfaces.

Infinite primes, integers, and rationals define the third aspect of this vision. The construction of infinite primes is structurally similar to a repeated second quantization of an arithmetic quantum field theory and involves also bound states. Infinite rationals can be also represented as space-time surfaces somewhat like finite numbers can be represented as space-time points.

1.2.5 Towards M-matrix or towards S-matrix?

S-matrix codes the predictions of quantum field theory and the challenge is to construct the analogy or generalization of S-matrix.

1. In ZEO one is forced to challenge the usual notion of S-matrix. Ordinary S-matrix is between ordinary quantum states associated with time=constant snapshot of time evolution S-matrix. Now these states are replaced by zero energy states formed by these pairs with members at boundaries of CD.

   The first proposal was that S-matrix is replaced with M-matrix between zero energy states and identifiable as time-like entanglement coefficients between positive and negative energy parts of zero energy states assignable to the past and future boundaries of 4-surfaces inside causal diamond defined as intersection of future and past directed light-cones.

   M-matrix would be a product of diagonal density matrix and unitary S-matrix and there are reasons to believe that S-matrix is universal. Generalized Feynman rules based on the generalization of Feynman diagrams obtained by replacing lines with light-like 3-surfaces and vertices with 2-D surfaces at which the lines meet.

   In M-matrix approach without any constraints the state would be superposition of pairs of states with S-matrix defining entanglement coefficients. This zero energy state with sum over states associated with all CDs. The square root of density matrix could take care of the normalization: without it the state has infinite norm. For hyper-finite factors this state could be normalized to unity and one could also require that the normal unitary conditions hold true when one fixes the boundaries of CD and looks for the scattering rates for fixed states at the passive boundary of CD. This should give S-matrix components from given initial state at passive boundary of CD to states and the active boundary of CD.

It is however far from clear what unitary time evolution following preparation of initial state could mean in this picture. It seems that the standard view about quantum measurement requires that the second boundary of CD - the passive bound - and states at it must be regarded as fixed and that unitary evolution affects only the active boundary and states at it.

Remark: After the emergence of ZEO the name of this chapter has fluctuated between "T"owards S-matrix and "T"owards M-matrix. This reflects my fluctuating views about what the counterpart of S-matrix could be in ZEO.

2. Later it turned out that the generalization of quantum measurement theory to a theory of consciousness indeed requires a more conservative view. Observer, conscious entity, or self corresponds to a sequence of unitary time evolutions followed by state function reductions for which the active boundary of CD shifts farther away from the passive boundary, which remains unchanged.

   The states at active boundary are changed by unitary time evolution implying also time delocalization of the active boundary in the moduli space of CDs with fixed passive boundary. The state function reduction induces localization in this moduli space and is analogous to weak measurement. The localization means also time localization since the temporal distance between the tips of CD is fixed. Eventually all observables are measured in the sense that there are no state function reductions not affective the states at passive boundary. The roles of passive and active boundary are changed. One can say that self dies and reincarnates as self living in opposite direction of time since its is the former passive boundary which shifts farther away from former active boundary. The distance between the tips can also increase in statistical sense only.

   S-matrix would be associated with the unitary evolution assignable to the active boundary of CD and involving shift of this boundary farther away from the passive boundary.

1.2.6 Organization of "T"owards M-matrix: Part II

The book consists of 3 parts.

1. The 5 chapters in the 1st part of the book are devoted to twistor lift of TGD. The motivation for twistor lift came from the twistor Grassmannian approach. The basic idea was that light-likeness in 8-D sense natural in M^8 and $H = M^4 \times CP_2$ pictures allows to overcome the basic problem of 4-D twistor approach since particles massless in 8-D sense can be massive in 4-D sense. In M^8 picture the 8-momenta would be quaternionic and light-like.

   Second key idea was that M^4 (and E^4) and CP_2 are completely unique as the only 4-D spaces allowing twistor space with Kähler structure. Here twistor space is identified as its geometric variant which is $M^4 \times S^2$ for M^4. The existence of 6-D Kähler action for the 6-D surface representing the counterpart of twistor bundle of space-time surface would completely fix TGD.

2. The five chapters of the 2nd part of the book are devoted to various ideas inspired by category theory. I hope that the reader forgives the fact that in these chapters I am moving at the outer boundaries of my mathematical skill profile.

3. The 3rd part with title "M"iscellaneous topics contains two chapters about the question whether space-time supersymmetry has TGD counterpart or not. The question remains still unsettled. The third chapter is taken as an example about side track and is about possibility of assigning coupling constant evolution to the zeros of Riemann zeta. Zeros of zeta appear in more convincing manner in the recent view about coupling constant evolution reducing to that for cosmological constant.

1.3 Sources

The eight online books about TGD [K101, K96, K79, K67, K20, K62, K42, K88] and nine online books about TGD inspired theory of consciousness and quantum biology [K93, K15, K73, K14, K40, K51, K53, K87, K92] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://tinyurl.com/ybv8dt4n`) contains a lot of material about TGD. In particular, a TGD glossary at `http://tinyurl.com/yd6jf3o7`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://tinyurl.com/ycyrxj4o` founded by Lian Sidorov and in *Prespacetime Journal* (`http://tinyurl.com/ycvktjhn`), *Journal of Consciousness Research and Exploration* (`http://tinyurl.com/yba4f672`), and *DNA Decipher Journal* (`http://tinyurl.com/y9z52khg`), all of them founded by Huping Hu. One can find the list about the articles published at `http://tinyurl.com/ybv8dt4n`. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.4 The contents of the book

1.4.1 PART I: TWISTORS AND TGD

TGD Variant of Twistor Story

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D embedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the embedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD and classical TGD defined by the extremals of Kähler action. In the following I summarize the background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams.

There is also a very closely analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. The landscape is replaced with twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach. Furthermore, one ends up to a formulation of the scattering amplitudes in terms of Yangian of the super-symplectic algebra relying on the idea that scattering amplitudes are sequences consisting of algebraic operations (product and co-product) having interpretation as vertices in the Yangian extension of super-symplectic algebra. These sequences connect given initial and final states and having minimal length. One can say that Universe performs calculations.

From Principles to Diagrams

The recent somewhat updated view about the road from general principles to diagrams is discussed. A more explicit realization of twistorialization as lifting of the preferred extremal X^4 of Kähler action to corresponding 6-D twistor space X^6 identified as surface in the 12-D product of twistor spaces of M^4 and CP_2 allowing Kähler structure suggests itself. Contrary to the original expectations, the twistorial approach is not mere reformulation but leads to a first principle identification of cosmological constant and perhaps also of gravitational constant and to a modification of the dynamics of Kähler action however preserving the known extremals and basic properties of Kähler action and allowing to interpret induced Kähler form in terms of preferred imaginary unit defining twistor structure.

Second new element is the fusion of twistorial approach with the vision that diagrams are representations for computations. This as also quantum criticality demands that the diagrams should allow huge symmetries allowing to transform them to braided generalizations of tree-diagrams. Several guiding principles are involved and what is new is the observation that they indeed seem to form a coherent whole.

About Twistor Lift of TGD

The twistor lift of classical TGD is attractive physically but it is still unclear whether it satisfies all constraints. The basic implication of twistor lift would be the understanding of gravitational and cosmological constants. Cosmological constant removes the infinite vacuum degeneracy of Kähler action but because of the extreme smallness of cosmological constant Λ playing the role of inverse of gauge coupling strength, the situation for nearly vacuum extremals of Kähler action in the recent cosmology is non-perturbative. Cosmological constant and thus twistor lift make sense only in zero energy ontology (ZEO) involving causal diamonds (CDs) in an essential manner.

One motivation for introducing the hierarchy of Planck constants was that the phase transition increasing Planck constant makes possible perturbation theory in strongly interacting system. Nature itself would take care about the converge of the perturbation theory by scaling Kähler coupling strength α_K to α_K/n, $n = h_{eff}/h$. This hierarchy might allow to construct gravitational perturbation theory as has been proposed already earlier. This would for gravitation to be quantum coherent in astrophysical and even cosmological scales.

In this chapter twistor lift is studied in detail.

1. The first working hypothesis is that the values of $\alpha_K(M^4)$ and $\alpha_K(CP_2)$ are widely different with $\alpha_K(M^4)$ being extremely large so that M^4 part of the 6-D Kähler action gives in dimensional reduction extremely small cosmological term. The first interesting finding is that allowing Kähler coupling strength $\alpha_K(CP_2)$ to correspond to zeros of zeta implies that for complex zeros the preferred extremals for $\alpha_K(M^4)$ having different phase are mimimal surface extremals of Kähler action so that the values of coupling constants do not matter and extremals depend on couplings only through the boundary conditions stating the vanishing of certain super-symplectic conserved charges.
2. The other working hypothesis is $\alpha_K(M^4) = \alpha_K(CP_2)$. The small effective value of cosmological constant is obtained if the Kähler action and volume term tend to cancel each other. In this case minimal surface extremals of Kähler action correspond naturally to asymptotic dynamics near the boundaries of CDs. This option looks more natural.

Both options lead to a generalization of Chladni mechanism to a "dynamics of avoidance" meaning that at least asymptotically the two dynamics decouple. This leads to an interpretation with profound implications for the views about what happens in particle physics experiment and in quantum measurement, for consciousness theory and for quantum biology.

A related observation is that a fundamental length scale of biology - size scale of neuron and axon - would correspond to the p-adic length scale assignable to vacuum energy density assignable to cosmological constant and be therefore a fundamental physics length scale.

Some Questions Related to the Twistor Lift of TGD

In this chapter I consider questions related to both classical and quantum aspects of twistorialization.

1. The first group of questions relates to the twistor lift of classical TGD. What does the induction of the twistor structure really mean? Can the analog of Kähler form assignable to M^4 suggested by the symmetry between M^4 and CP_2 and by number theoretical vision appear in the theory. What would be the physical implications? How does gravitational coupling

emerge at fundamental level? Could one regard the localization of spinor modes to string world sheets as a localization to Lagrangian sub-manifolds of space-time surface with vanishing induced Kähler form. Lagrangian sub-manifolds would be commutative in the sense of Poisson bracket. How this relates to the idea that string world sheets correspond complex (commutative) surfaces of quaternionic space-time surface in octonionic embedding space?

During the re-processing of the details related to twistor lift, it became clear that the earlier variant for the twistor lift can be criticized and allows an alternative. This option led to a much simpler view about twistor lift, to the conclusion that minimal surface extremals of Kähler action represent only asymptotic situation (external particles in scattering), and also to a re-interpretation for the p-adic evolution of the cosmological constant: cosmological term would correspond to the *entire* 4-D action and the cancellation of Kähler action and cosmological term would lead to the small value of the effective cosmological constant.

2. Second group of questions relates to the construction of scattering amplitudes. The idea is to generalize the usual construction for massless states. In TGD all single particle states are massless in 8-D sense and this gives excellent hopes about the applicability of 8-D twistor approach. M^8-H duality turns out to be the key to the construction. Also the holomorphy of twistor amplitudes in helicity spinors λ_i and independence on $\tilde{\lambda}_i$ is crucial. The basic vertex corresponds to 4-fermion vertex for which the simplest expression can be written immediately. $n>4$-fermion scattering amplitudes can be also written immediately.

   If scattering diagrams correspond to computations as number theoretic vision suggests, the diagrams should be reducible to tree diagrams by moves generalizing the old-fashioned hadronic duality. This condition reduces to the vanishing of loops which in terms of BCFW recursion formula states that the twistor diagrams correspond to closed objects in what might be called WCFW homology.

The Recent View about Twistorialization in TGD Framework

The recent view about the twistorialization in TGD framework is discussed.

1. A proposal made already earlier is that scattering diagrams as analogs of twistor diagrams are constructible as tree diagrams for CDs connected by free particle lines. Loop contributions are not even well-defined in zero energy ontology (ZEO) and are in conflict with number theoretic vision. The coupling constant evolution would be discrete and associated with the scale of CDs (p-adic coupling constant evolution) and with the hierarchy of extensions of rationals defining the hierarchy of adelic physics.
2. Logarithms appear in the coupling constant evolution in QFTs. The identification of their number theoretic versions as rational number valued functions required by number-theoretical universality for both the integer characterizing the size scale of CD and for the hierarchy of Galois groups leads to an answer to a long-standing question what makes small primes and primes near powers of them physically special. The primes $p\in\{2,3,5\}$ indeed turn out to be special from the point of view of number theoretic logarithm.
3. The reduction of the scattering amplitudes to tree diagrams is in conflict with unitarity in 4-D situation. The imaginary part of the scattering amplitude would have discontinuity proportional to the scattering rate only for many-particle states with light-like total momenta. Scattering rates would vanish identically for the physical momenta for many-particle states.

   In TGD framework the states would be however massless in 8-D sense. Massless pole corresponds now to a continuum for M^4 mass squared and one would obtain the unitary cuts from a pole at $P^2=0$! Scattering rates would be non-vanishing only for many-particle states having light-like 8-momentum, which would pose a powerful condition on the construction of many-particle states.

   This idea does not make sense for incoming/outgoing particles, which light-like momenta unless they are parallel: their total momentum cannot be light-like in the general case. Rather, $P^2=0$ applies to the states formed inside CDs from groups of incoming and outgoing particles. BCFW deformation $p_i\rightarrow p_i+zr_i$ describes what happens for the single-particle momenta: they cease to be light-like but the total momenta for subgroups of particles in factorization channels are complex and light-like. This strong form of conformal symmetry has highly non-trivial implications concerning color confinement.

4. The key idea is number theoretical discretization in terms of "cognitive representations" as space-time time points with M^8-coordinates in an extension of rationals and therefore shared by both real and various p-adic sectors of the adele. Discretization realizes measurement resolution, which becomes an inherent aspect of physics rather than something forced by observed as outsider. This fixes the space-time surface completely as a zero locus of real or imaginary part of octonionic polynomial.

   This must imply the reduction of "world of classical worlds" (WCW) corresponding to a fixed number of points in the extension of rationals to a finite-dimensional discretized space with maximal symmetries and Kähler structure.

   The simplest identification for the reduced WCW would be as complex Grassmannian - a more general identification would be as a flag manifold. More complex options can of course be considered. The Yangian symmetries of the twistor Grassmann approach known to act as diffeomorphisms respecting the positivity of Grassmannian and emerging also in its TGD variant would have an interpretation as general coordinate invariance for the reduced WCW. This would give a completely unexpected connection with supersymmetric gauge theories and TGD.

5. M^8 picture implies the analog of SUSY realized in terms of polynomials of super-octonions whereas H picture suggests that supersymmetry is broken in the sense that many-fermion states as analogs of components of super-field at partonic 2-surfaces are not local. This requires breaking of SUSY. At M^8 level the breaking could be due to the reduction of Galois group to its subgroup G/H, where H is normal subgroup leaving the point of cognitive representation defining space-time surface invariant. As a consequence, local many-fermion composite in M^8 would be mapped to a non-local one in H by $M^8 - H$ correspondence.

TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors

In this chapter 4 topics are discussed. McKay correspondence, SUSY, and twistors are discussed from TGD point of view, and new aspects of $M^8 - H$ duality are considered.

1. McKay correspondence in TGD framework

There are two mysterious looking correspondences involving ADE groups. McKay correspondence between McKay graphs characterizing tensor products for finite subgroups of $SU(2)$ and Dynkin diagrams of affine ADE groups is the first one. The correspondence between principal diagrams characterizing inclusions of hyper-finite factors of type II_1 (HFFs) with Dynkin diagrams for a subset of ADE groups and Dynkin diagrams for affine ADE groups is the second one.

These correspondences are discussed from number theoretic point of view suggested by TGD and based on the interpretation of discrete subgroups of $SU(2)$ as subgroups of the covering group of quaternionic automorphisms $SO(3)$ (analog of Galois group) and generalization of these groups to semi-direct products $Gal(K) \triangleleft SU(2)_K$ of Galois group for extension K of rationals with the discrete subgroup $SU(2)_K$ of $SU(2)$ with representation matrix elements in K. The identification of the inclusion hierarchy of HFFs with the hierarchy of extensions of rationals and their Galois groups is proposed.

A further mystery whether $Gal(K) \triangleleft SU(2)_K$ could give rise to quantum groups or affine algebras. In TGD framework the infinite-D group of isometries of "world of classical worlds" (WCW) is identified as an infinite-D symplectic group for which the discrete subgroups characterized by K have infinite-D representations so that hyper-finite factors are natural for their representations. Symplectic algebra SSA allows hierarchy of isomorphic sub-algebras SSA_n. The gauge conditions for SSA_n and $[SSA_n, SSA]$ would define measurement resolution giving rise to hierarchies of inclusions and ADE type Kac-Moody type algebras or quantum algebras representing symmetries modulo measurement resolution.

A concrete realization of ADE type Kac-Moody algebras is proposed. It relies on the group algebra of $Gal(K) \triangleleft SU(2)_K$ and free field representation of ADE type Kac-Moody algebra identifying the free scalar fields in Kac-Moody Cartan algebra as group algebra elements defined by the traces of representation matrices (characters).

2. New aspects of $M^8 - H$ duality

M^8-H duality is now a central part of TGD and leads to new findings. M^8-H duality can be formulated both at the level of space-time surfaces and light-like 8-momenta. Since the choice of M^4 in the decomposition of momentum space $M^8 = M^4\times E^4$ is rather free, it is always possible to find a choice for which light-like 8-momentum reduces to light-like 4-momentum in M^4 - the notion of 4-D mass is relative. This leads to what might be called $SO(4)-SU(3)$ duality corresponding to the hadronic and partonic views about hadron physics. Particles, which are eigenstates of mass squared are massless in $M^4\times CP_2$ picture and massive in M^8 picture. The massivation in this picture is a universal mechanism having nothing to do with dynamics and results in zero energy ontology automatically if the zero energy states are superpositions of states with different masses. p-Adic thermodynamics describes this massivation. Also a proposal for the realization of ADE hierarchy emerges.

4-D space-time surfaces correspond to roots of octonionic polynomials $P(o)$ with real coefficients corresponding to the vanishing of the real or imaginary part of $P(o)$. These polynomials however allow universal roots, which are not 4-D but analogs of 6-D branes and having topology of S^6. Their M^4 projections are time =constant snapshots $t=r_n, r_M\leq r_n$ 3-balls of M^4 light-cone (r_n is root of $P(x)$). At each point the ball there is a sphere S^3 shrinking to a point about boundaries of the 3-ball. These special values of M^4 time lead to a deeper understanding of ZEO based quantum measurement theory and consciousness theory.

3. Is the identification of twistor space of M^4 really correct?

The critical questions concerning the identification of twistor space of M^4 as $M^4\times S^2$ led to consider a more conservative identification as CP_3 with hyperbolic signature (3,-3) and replacement of H with $H=cd_{conf}\times CP_2$, where cd_{conf} is CP_2 with hyperbolic signature (1,-3). This approach reproduces the nice results of the earlier picture but means that the hierarchy of CDs in M^8 is mapped to a hierarchy of spaces cd_{conf} with sizes of CDs. This conforms with Poincare symmetry from which everything started since Poincare group acts in the moduli space of octonionic structures of M^8. Note that also the original form of M^8-H duality continues to make sense and results from the modification by projection from $CP_{3,h}$ to M^4 rather than $CP_{2,h}$.

The outcome of octo-twistor approach applied at level of M^8 together with modified M^8-H duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor (super-)Grassmannian approach. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. This part of article is not a mere side track since by M^8-H duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

This article deals with two questions.

1. The ideas related to topological quantum computation suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of state space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum.

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Or could one replace the roots labelling states with representations? Or could even the coefficient field for state space be replaced with the representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums in quantum-classical correspondence, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ to those of $SL(n, C)$ at least. Is there a deep connection between finite subgroups of $SL(n, C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

In the TGD framework $M^8 - H$ duality relates number theoretic and differential geometric views about physics: could it provide some understanding of this mystery? The proposal is that for cognitive representations associated with extended Dynkin diagrams (EEDs), Galois group Gal acts as Weyl group on McKay diagrams defined by irreps of the isotropy group Gal_I of given root of a polynomial which is monic polynomial but with roots replaced with direct sums of irreps of Gal_I. This could work for p-adic number fields and finite fields. One also ends up with a more detailed view about the connection between the hierarchies of inclusion of Galois groups associated with functional composites of polynomials and hierarchies of inclusions of hyperfinite factors of type II_1 assignable to the representation of super-symplectic algebra.

TGD as it is towards end of 2021

This article tries to give a rough overall view about Topological Geometrodynamics (TGD) as it is towards the end of 2021. The two views about TGD and their relationship are discussed at the general level.

1. The first view generalizes Einstein's program for the geometrization of physics. Entire quantum physics is geometrized in terms of the notion of "world of classical worlds" (WCW), which by its infinite dimension has unique Kähler geometry.
2. Second vision reduces physics to number theory. Classical number fields (reals, complex numbers, quaternions, and octonions) are central as also p-adic number fields and extensions of rationals. The physics is classically coded by algebraic 4-surfaces in complexified M^8 having octonionic structure and "roots" of octonionic polynomials obtained as algebraic continuations of real polynomials with rational coefficients. M^8_c has an interpretation as an analog of momentum space.

The preparation of this summary led to considerable progress in several aspects of TGD.

1. The mutual entanglement of fermions (bosons) as elementary particles is always maximal so that only fermionic and bosonic degrees can entangle in QFTs. The replacement of point-like particles with 3-surfaces forces us to reconsider the notion of identical particles from the category theoretical point of view. The number theoretic definition of particle identity seems to be the most natural and implies that the new degrees of freedom make possible geometric entanglement.

   Also the notion particle generalizes: also many-particle states can be regarded as particles with the constraint that the operators creating and annihilating them satisfy commutation/anticommutation relations. This leads to a close analogy with the notion of infinite prime.
2. The understanding of the details of the $M^8 - H$ duality forces us to modify the earlier view. The notion of causal diamond (CD) central to zero energy ontology (ZEO) emerges as a prediction at the level of H. The pre-image of CD at the level of M^8 is a region bounded by two mass shells rather than CD. $M^8 - H$ duality maps the points of cognitive representations as momenta of quarks with fixed mass in M^8 to either boundary of CD in H. Mass shell (its positive and negative energy parts) is mapped to a light-like boundary of CD with size $T = h_{eff}/m$, m the mass associated with momentum.
3. Galois confinement at the level of M^8 is understood at the level of momentum space and is found to be necessary. Galois confinement implies that quark momenta in suitable units are algebraic integers but integers for Galois singlet just as in ordinary quantization for a particle in a box replaced by CD. Galois confinement could provide a universal mechanism for the formation of all bound states.

4. There is considerable progress in the understanding of the quantum measurement theory based on ZEO. From the point of view of cognition BSFRs would be like heureka moments and the sequence of SSFRs would correspond to an analysis having as a correlate the decay of 3-surface to smaller 3-surfaces.

Article includes also a section about neutrinos and TGD. The motivation is that the recent results related to neutrino mixing led to a dramatic progress in the understanding of the role of right-handed neutrino solving long-standing problems of quantum TGD.

About TGD counterparts of twistor amplitudes

The twistor lift of TGD, in which $H = M^4 \times CP_2$ is replaced with the product of twistor spaces $T(M^4)$ and $T(CP_2)$, and space-time surface $X^4 \subset H$ with its 6-D twistor space as 6-surface $X^6 \subset T(M^4) \times T(CP_2)$, is now a rather well-established notion and $M^8 - H$ duality predicts it at the level of M^8.

Number theoretical vision involves $M^8 - H$ duality. At the level of H the quark mass spectrum is determined by the Dirac equation in H. In M^8 mass squared spectrum is determined by the roots of the polynomial P determining space-time surface and are in general complex. By Galois confinement the momenta are integer valued when p-adic mass is used as a unit and mass squared spectrum is also integer valued. This raises hope about a generalization of the twistorial construction of scattering amplitudes to TGD context.

It is always best to start from a problem and the basic problem of the twistor approach is that physical particles are not massless.

1. The intuitive TGD based proposal has been that since quark spinors are massless in H, the masslessness in the 8-D sense could somehow solve the problems caused by the massivation in the construction of twistor scattering amplitudes. However, no obvious mechanism has been identified. One step in this direction was the realization that in H quarks propagate with well-defined chiralities and only the square of Dirac equation is satisfied. For a quark of given helicity the spinor can be identified as helicity spinor.
2. M^8 quark momenta are in general complex as algebraic integers. They are the counterparts of the area momenta x_i of momentum twistor space whereas H momenta are identified as ordinary momenta. Total momenta of Galois confined states have as components ordinary integers.
3. The M^8 counterpart of the 8-D massless condition in H is the restriction of momenta to mass shells $m^2 = r_n$ determined as roots of P. The M^8 counterpart of Dirac equation in H is octonionic Dirac equation, which is algebraic as everything in M^8 and analogous to massless Dirac equation. The solution is a helicity spinor $\tilde{\lambda}$ associated with the massive momentum.

The outcome is an extremely simple proposal for the scattering amplitudes.

1. Vertices correspond to trilinears of Galois confined many-quark states as states of super symplectic algebra acting as isometries of the "world of classical worlds" (WCW). Quarks are on-shell with H momentum p and M^8 momenta x_i, x_{i+1}, $p_i = x_{i+1} - x_i$. Dirac operator $x_i^k \gamma_k$ restricted to fixed helicity L, R appears as a vertex factor and has an interpretation as a residue of a pole from an on-mass-shell propagator D so that a correspondence with twistorial construction becomes obvious. D is expressible in terms of the helicity spinors of given chirality and gives two independent holomorphic factors as in case of massless theories.
2. The scattering amplitudes would be rational functions in accordance with the number theoretic vision. The absence of logarithmic radiative corrections is not a problem: the coupling constant evolution would be discrete and defined by the hierarchy of extensions of rationals.
3. The scattering amplitudes for a single 4-surface X^4 characterixzing interaction region are determined by a polynomial P. External particles are Galois singlets consisting of off-mass shell quarks with mass squared values coming as roots of the polynomial P characterizing the interaction region. External particles are characterized by polynomials P_i satisfying $P_i(0) = 0$. P is identified as the functional composite of P_i since it inherits the masses of incoming particles as their roots. This allows only cyclic permutations of P_i. The scattering event is essentially a re-combination of incoming Galois singlets to new Galois singlets and quarks

propagate freely: hence OZI rule generalizes. Also a connection with the dual resonance models emerges.

4. The integration over WCW is replaced with a summation of polynomials characterized by rational coefficients. Monic polynomials are highly suggestive. A connection with p-adicization emerges via the identification of the p-adic prime as one of the ramified primes of P. Only (monic) polynomials having a common p-adic prime are allowed in the sum. The counterpart of the vacuum functional $exp(-K)$ is naturally identified as the discriminant D of the extension associated with P and p-adic coupling constant evolution emerges from the identification of $exp(-K)$ with D.

Unitary, locality, and the failure to find the twistorial counterparts of non-planar Feynman diagrams are the basic problems of the twistor Grassmannian approach. Also the non-existence of twistor spaces for most Riemannian manifolds is a problem in GRT framework but in TGD the existence of twistor spaces for M^4 and CP_2 solves this problem. In the TGD framework, the replacement of point-like particles with 3-surfaces leads to the loss of locality at the fundamental level. The analogs of non-planar diagrams are eliminated since only cyclic permutations of P_i are allowed.

This leaves only the problem with unitarity. The TGD counterpart of unitarity realized in terms of Kähler geometry of fermionic state space is very natural in the geometrization of quantum physics. Scattering probabilities are identified as products of covariant and contravariant matrix elements of the metric, and unitary conditions are replaced by the definition of the contravariant metric. Probabilities are complex but real and imaginary parts are separately conserved. The interpretation in terms of Fisher information is possible. Due to the infinite-D character of the state space, the Kähler geometry exists only if it has a maximal group of isometries and is a unique constant curvature geometry. Also the interpretation of this approach in zero energy ontology is discussed.

There are physical motivations for considering the number theoretic generalizations of the amplitudes. For an iterate of fixed P (say large number of gravitons), the roots of the iterate of P defined virtual mass squared values, approach to the Julia set of P. The construction of scattering amplitudes thus leads to chaos theory at the limit of an infinite number of identical particles.

The construction generalizes also to the surfaces defined by real analytic functions and the fermionic variant of Riemann zeta and elliptic functions are discussed as examples.

1.4.2 PART II: CATEGORY THEORY AND TGD

Category Theory and Quantum TGD

Possible applications of category theory to quantum TGD are discussed. The so called 2-plectic structure generalizing the ordinary symplectic structure by replacing symplectic 2-form with 3-form and Hamiltonians with Hamiltonian 1-forms has a natural place in TGD since the dynamics of the light-like 3-surfaces is characterized by Chern-Simons type action. The notion of planar operad was developed for the classification of hyper-finite factors of type II_1 and its mild generalization allows to understand the combinatorics of the generalized Feynman diagrams obtained by gluing 3-D light-like surfaces representing the lines of Feynman diagrams along their 2-D ends representing the vertices.

The fusion rules for the symplectic variant of conformal field theory, whose existence is strongly suggested by quantum TGD, allow rather precise description using the basic notions of category theory and one can identify a series of finite-dimensional nilpotent algebras as discretized versions of field algebras defined by the fusion rules. These primitive fusion algebras can be used to construct more complex algebras by replacing any algebra element by a primitive fusion algebra. Trees with arbitrary numbers of branches in any node characterize the resulting collection of fusion algebras forming an operad. One can say that an exact solution of symplectic scalar field theory is obtained.

Conformal fields and symplectic scalar field can be combined to form symplecto-formal fields. The combination of symplectic operad and Feynman graph operad leads to a construction of Feynman diagrams in terms of n-point functions of conformal field theory. M-matrix elements with a finite measurement resolution are expressed in terms of a hierarchy of symplecto-conformal n-point functions such that the improvement of measurement resolution corresponds to an algebra homomorphism mapping conformal fields in given resolution to composite conformal fields in improved resolution. This expresses the idea that composites behave as independent conformal fields. Also other applications are briefly discussed.

Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals of Kähler action would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity). The duality would allow to construct new preferred extremals of Kähler action.

Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD?

TGD Universe is extremely simple locally but the presence of various hierarchies make it to look extremely complex globally. Category theory and quantum groups, in particular Yangian or its TGD generalization are most promising tools to handle this complexity. The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.

2. TGD view about generalized Feynman diagrams relies on two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to tree diagrams with braiding. This picture fits very nicely with the picture suggested by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction but vertices expressed in terms of twistors depend on momenta of fermions.

3. Braiding transformations take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time regions with Minkowskian and Euclidian signature of induced metric respectively permuting two braid strands. R-matrix satisfying Yang-Baxter equation characterizes this operation algebraically.

4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. Reconnection induces an exchange of braid strands defined by the boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries of string world sheets. A generalization of quantum algebras to include also algebraic representation for reconnection is needed. Also reconnection might reduce to a braiding type operation.

Yangians look especially natural quantum algebras from TGD point of view. They are bi-algebras with co-product Δ. This makes the algebra multi-local raising hopes about the understanding of bound states. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Kac-Moody algebras (SKMAs) involved and even with super-conformal algebra (SSA), which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some subalgebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question.

Are higher structures needed in the categorification of TGD?

The notion of higher structures promoted by John Baez looks very promising notion in the attempts to understand various structures like quantum algebras and Yangians in TGD framework. The stimulus for this article came from the nice explanations of the notion of higher structure by Urs Screiber. The basic idea is simple: replace "=" as a blackbox with an operational definition with a proof for $A = B$. This proof is called homotopy generalizing homotopy in topological sense. n-structure emerges when one realizes that also the homotopy is defined only up to homotopy in turn defined only up...

In TGD framework the notion of measurement resolution defines in a natural manner various kinds of "="s and this gives rise to resolution hierarchies. Hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adeles in adelic TGD and corresponding hierarchy of Galois groups represented geometrically, hierarchy of infinite primes, self hierarchy, etc...

In this article the idea of n-structure is studied in more detail. A rather radical idea is a formulation of quantum TGD using only cognitive representations consisting of points of space-time surface with embedding space coordinates in extension of rationals defining the level of adelic hierarchy. One would use only these discrete points sets and Galois groups. Everything would reduce to number theoretic discretization at space-time level perhaps reducing to that at partonic 2-surfaces with points of cognitive representation carrying fermion quantum numbers.

Even the "world of classical worlds " (WCW) would discretize: cognitive representation would define the coordinates of WCW point. One would obtain cognitive representations of scattering amplitudes using a fusion category assignable to the representations of Galois groups: something diametrically opposite to the immense complexity of the WCW but perhaps consistent with it. Also a generalization of McKay's correspondence suggests itself: only those irreps of the Lie group associated with Kac-Moody algebra that remain irreps when reduced to a subgroup defined by a Galois group of Lie type are allowed as ground states. Also the relation to number theoretic Langlands correspondence is very interesting.

Is Non-associative Physics and Language Possible Only in Many-Sheeted Space-Time?

Language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D embedding space $M^4 \times CP_2$ is one of the key conjectures of TGD. But what about many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes- that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA. The mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity.

1.4.3 PART III: MISCELLANEOUS TOPICS

Does the QFT Limit of TGD Have Space-Time Super-Symmetry?

Contrary to the original expectations, TGD seems to allow a generalization of the space-time super-symmetry. This became clear with the increased understanding of the Kähler-Dirac action. The introduction of a measurement interaction term to the action allows to understand how stringy propagator results and provides profound insights about physics predicted by TGD.

The appearance of the momentum (and possibly also color quantum numbers) in the measurement interaction couples space-time degrees of freedom to quantum numbers and allows also to define SUSY algebra at fundamental level as anti-commutation relations of fermionic oscillator operators. Depending on the situation a finite-dimensional SUSY algebra or the fermionic part of super-conformal algebra with an infinite number of oscillator operators results. The addition of a fermion in particular mode would define particular super-symmetry. Zero energy ontology implies that fermions as wormhole throats correspond to chiral super-fields assignable to positive or negative energy SUSY algebra whereas bosons as wormhole contacts with two throats correspond to the direct sum of positive and negative energy algebra and fields which are chiral or antichiral with respect to both positive and negative energy theta parameters. This super-symmetry is badly broken due to the dynamics of the Kähler-Dirac operator which also mixes M^4 chiralities inducing massivation. Since righthanded neutrino has no electro-weak couplings the breaking of the corresponding super-symmetry should be weakest.

The question is whether this SUSY has a realization as a SUSY algebra at space-time level and whether the QFT limit of TGD could be formulated as a generalization of SUSY QFT. There are several problems involved.

1. In TGD framework super-symmetry means addition of fermion to the state and since the number of spinor modes is larger states with large spin and fermion numbers are obtained. This picture does not fit to the standard view about super-symmetry. In particular, the identification of theta parameters as Majorana spinors and super-charges as Hermitian operators is not possible.
2. The belief that Majorana spinors are somehow an intrinsic aspect of super-symmetry is however only a belief. Weyl spinors meaning complex theta parameters are also possible. Theta parameters can also carry fermion number meaning only the supercharges carry fermion number and are non-hermitian. The general classification of super-symmetric theories indeed demonstrates that for $D = 8$ Weyl spinors and complex and non-hermitian super-charges are possible. The original motivation for Majorana spinors might come from MSSM assuming that right handed neutrino does not exist. This belief might have also led to string theories in D=10 and D=11 as the only possible candidates for TOE after it turned out that chiral anomalies cancel.
3. The massivation of particles is basic problem of both SUSYs and twistor approach. The fact that particles which are massive in M^4 sense can be interpreted as massless particles in $M^4 \times CP_2$ suggests a manner to understand super-symmetry breaking and massivation in TGD framework. The octonionic realization of twistors is one possibility in this framework and quaternionicity condition guaranteing associativity leads to twistors which are almost equivalent with ordinary 4-D twistors.
4. The first approach is based on an approximation assuming only the super-multiplets generated by right-handed neutrino or both right-handed neutrino and its antineutrino. The assumption that right-handed neutrino has fermion number opposite to that of the fermion associated with the wormhole throat implies that bosons correspond to $\mathcal{N} = (1,1)$ SUSY and fermions to $\mathcal{N} = 1$ SUSY identifiable also as a short representation of $\mathcal{N} = (1,1)$ SUSY algebra trivial with respect to positive or negative energy algebra. This means a deviation from the standard view but the standard SUSY gauge theory formalism seems to apply in this case.
5. A more ambitious approach would put the modes of induced spinor fields up to some cutoff into super-multiplets. At the level next to the one described above the lowest modes of the induced spinor fields would be included. The very large value of $\mathcal{N}$ means that $\mathcal{N} \leq \ni\in$ SUSY cannot define the QFT limit of TGD for higher cutoffs. One must generalize SUSYs gauge theories to arbitrary value of $\mathcal{N}$ but there are reasons to expect that the formalism becomes

rather complex. More ambitious approach working at TGD however suggest a more general manner to avoid this problem.

(a) One of the key predictions of TGD is that gauge bosons and Higgs can be regarded as bound states of fermion and antifermion located at opposite throats of a wormhole contact. This implies bosonic emergence meaning that it QFT limit can be defined in terms of Dirac action. The resulting theory was discussed in detail in [?] and it was shown that bosonic propagators and vertices can be constructed as fermionic loops so that all coupling constant follow as predictions. One must however pose cutoffs in mass squared and hyperbolic angle assignable to the momenta of fermions appearing in the loops in order to obtain finite theory and to avoid massivation of bosons. The resulting coupling constant evolution is consistent with low energy phenomenology if the cutoffs in hyperbolic angle as a function of p-adic length scale is chosen suitably.

(b) The generalization of bosonic emergence that the TGD counterpart of SUSY is obtained by the replacement of Dirac action with action for chiral super-field coupled to vector field as the action defining the theory so that the propagators of bosons and all their super-counterparts would emerge as fermionic loops.

(c) The huge super-symmetries give excellent hopes about the cancelation of infinities so that this approach would work even without the cutoffs in mass squared and hyperbolic angle assignable to the momenta of fermions appearing in the loops. Cutoffs have a physical motivation in zero energy ontology but it could be an excellent approximation to take them to infinity. Alternatively, super-symmetric dynamics provides cutoffs dynamically.

6. The condition that $\mathcal{N} = \infty$ variants for chiral and vector superfields exist fixes completely the identification of these fields in zero energy ontology.

(a) In this framework chiral fields are generalizations of induced spinor fields and vector fields those of gauge potentials obtained by replacing them with their super-space counterparts. Chiral condition reduces to analyticity in theta parameters thanks to the different definition of hermitian conjugation in zero energy ontology (θ is mapped to a derivative with respect to theta rather than to $\overline{\theta}$) and conjugated super-field acts on the product of all theta parameters.

(b) Chiral action is a straightforward generalization of the Dirac action coupled to gauge potentials. The counterpart of YM action can emerge only radiatively as an effective action so that the notion emergence is now unavoidable and indeed basic prediction of TGD.

(c) The propagators associated with the monomials of n theta parameters behave as $1/p^n$ so that only $J = 0, 1/2, 1$ states propagate in normal manner and correspond to normal particles. The presence of monomials with number of thetas higher than 2 is necessary for the propagation of bosons since by the standard argument fermion and scalar loops cancel each other by super-symmetry. This picture conforms with the identification of graviton as a bound state of wormhole throats at opposite ends of string like object.

(d) This formulation allows also to use Kähler-Dirac gamma matrices in the measurement interaction defining the counterpart of super variant of Dirac operator. Poincare invariance is not lost since momenta and color charges act on the tip of CD rather than the coordinates of the space-time sheet. Hence what is usually regarded as a quantum theory in the background defined by classical fields follows as exact theory. This feeds all data about space-time sheet associated with the maximum of Kähler function. In this approach WCW as a Kähler manifold is replaced by a cartesian power of CP_2, which is indeed quaternionic Kähler manifold. The replacement of light-like 3-surfaces with number theoretic braids when finite measurement resolution is introduced, leads to a similar replacement.

(e) Quantum TGD as a "complex square root" of thermodynamics approach suggests that one should take a superposition of the amplitudes defined by the points of a coherence region (identified in terms of the slicing associated with a given wormhole throat) by weighting the points with the Kähler action density. The situation would be highly analogous to a spin glass system since the Kähler-Dirac gamma matrices defining the propagators would be analogous to the parameters of spin glass Hamiltonian allowed to have a spatial dependence. This would predict the proportionality of the coupling strengths to Kähler coupling strength and bring in the dependence on the size of CD coming as a power of 2 and give rise to p-adic coupling constant evolution. Since TGD Universe is analogous to 4-D spin glass, also a sum over different preferred extremals assignable to a given coherence regions and weighted by $exp(K)$ is probably needed.

(f) In TGD Universe graviton is necessarily a bi-local object and the emission and absorption of graviton are bi-local processes involving two wormhole contacts: a pair of particles rather than single particle emits graviton. This is definitely something new and defies a description in terms of QFT limit using point like particles. Graviton like states would be entangled states of vector bosons at both ends of stringy curve so that gravitation could be regarded as a square of YM interactions in rather concrete sense. The notion of emergence would suggest that graviton propagator is defined by a bosonic loop. Since bosonic loop is dimensionless, IR cutoff defined by the largest CD present must be actively involved. At QFT limit one can hope a description as a bi-local process using a bi-local generalization of the QFT limit. It turns out that surprisingly simple candidate for the bi-local action exists.

This statement has become somewhat misleading. It has turned out that all elementary particle in TGD framework are bi-local objects: one can assign to them both closed magnetic flux tubes behaving like strings and closed strings carrying fermion number. For other elementary particles than graviton second wormhole contact carries only neutrino pair neutralizing electroweak-isospin so that above weak scale they correspond to single em charged wormhole contact.

Could $\mathcal{N}=2$ Super-conformal Theories Be Relevant For TGD?

TGD has as is symmetries super-conformal symmetry (SCS), which is a huge extension of the ordinary SCS. For instance, the infinite-dimensional symplectic group plays the role of finite-dimension Lie-group as Kac-Moody group and the conformal weights for the generators of algebra corresponds to the zeros of fermionic zeta and their number of generators is therefore infinite.

The relationship of TGD SCS to super-conformal field theories (SCFTs) known as minimal models has remained without definite answer. The most general super-conformal algebra (SCA) assignable to string world sheets by strong form of holography has $\mathcal{N}$ equal to the number of spin states of leptonic and quark type fundamental spinors but the space-time SUSY is badly broken for it. Covariant constancy of the generating spinor modes is replaced with holomorphy - kind of "half covariant constancy". Right-handed neutrino and antineutrino are excellent candidates for generating $\mathcal{N}=2$ SCS with a minimal breaking of the corresponding space-time SUSY.

$\mathcal{N}=2$ SCS has also some inherent problems. The critical space-time dimension is $D=4$ but the existence of complex structure seems to require the space-time has metric signature different from Minkowskian: here TGD suggests a solution. $\mathcal{N}=2$ SCFTs are claimed also to reduce to topological QFTs under some conditions: this need not be a problem since TGD can be characterized as almost topological QFT. What looks like a further problem is that p-adic mass calculations require half-integer valued negative conformal weight for the ground state (and vanishing weight for massless states). One can however shift the scaling generator L_0 to get rid of problem: the shift has physical interpretation in TGD framework and must be half integer valued which poses the constraint $h=K/2$, $K=0,1,2..$ on the representations of SCA.

$\mathcal{N}=2$ SCA allows a spectral flow taking Ramond representations to Neveu-Scwartz variant of algebra. The physical interpretation is as super-symmetry mapping fermionic states to bosonic states. The representations of $\mathcal{N}=2$ SCA allowing degenerate states with positive central charge c and non-vanishing ground state conformal weight h give rise to minimal models allowing ADE classification, construction of partition functions, and even of n-point functions. This could make S-matrix of TGD exactly solvable in the fermionic sector. The ADE hierarchy suggests a direct interpretation in terms of orbifold hierarchy assignable to the hierarchy of Planck constants associated with the super-symplectic algebra: primary fields would correspond to orbifolds identified as coset spaces of ADE groups. Also an interpretation in terms of inclusions of hyper-finite factors is highly suggestive.

Does Riemann Zeta Code for Generic Coupling Constant Evolution?

A general model for the coupling constant evolution is proposed. The analogy of Riemann zeta and fermionic zeta $\zeta_F(s)/\zeta_F(2s)$ with complex square root of a partition function natural in Zero Energy Ontology suggests that the the poles of $\zeta_F(ks)$, $k = 1/2$, correspond to complexified critical temperatures identifiable as inverse of Kähler coupling strength itself having interpretation as inverse of critical temperature. One can actually replace the argument s of ζ_F with Möbius transformed argument $w = (as+b)/(cs+d)$ with a, b, c, d real numbers, rationals, or even integers. For α_K $w = (s+b)/2$ is proper choices and gives zeros of $\zeta(s)$ and $s = 2-b$ as poles. The identification $\alpha_K = \alpha_{U(1)}$ leads to a prediction for α_{em}, which deviates by .7 per cent from the experimental value at low energies (atomic scale) if the experimental value of the Weinberg angle is used. The conjecture generalizes also to weak, color and gravitational interactions when general Möbius transformation leaving upper half-plane invariant is allowed. One ends up with a general model predicting successfully the entire electroweak coupling constant evolution successfully from the values of fine structure constant at atomic or electron scale and in weak scale.

Part I

TWISTORS AND TGD

Chapter 2

TGD variant of Twistor Story

2.1 Introduction

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D embedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the embedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD [K96, K67, K79] and classical TGD [K20] defined by the extremals of Kähler action.

In the following I summarize first the basic results and problems of the twistor approach. After that I describe some of the mathematical background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams having as lines space-time surfaces with Euclidian signature of induced metric and having wormhole contacts as basic building bricks.

There is also a very close analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds [A1, A86] and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry [B16]emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. Twistor space has space-time as base-space rather than forming with it Cartesian factors of a 10-D space-time. The Calabi-Yau landscape is replaced with the space of twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings [B29]. The space of twistor spaces is the lift of the "world of classical worlds" (WCW) by adding the CP_1 fiber to the space-time surfaces so that the analog of landscape has beautiful geometrization.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach.

1. The notion of quaternion analyticity extending the notion of ordinary analyticity to 4-D context is highly attractive but has remained one of the long-standing ideas difficult to take quite seriously but equally difficult to throw to paper basked. Four-manifolds possess almost quaternion structure. In twistor space context the formulation of quaternion analyticity becomes possible and relies on an old notion of tri-holomorphy about which I had not been aware earlier. The natural formulation for the preferred extremal property is as a condition stating that various charges associated with generalized conformal algebras vanish for preferred ex-

tremals. This leads to ask whether Euclidian space-time regions could be quaternion-Kähler manifolds for which twistor spaces are so called Fano spaces. In Minkowskian regions so called Hamilton-Jacobi property would apply.

2. The generalization of Witten's twistor theory to TGD framework is a natural challenge and the 2-surfaces studied defining scattering amplitudes in Witten's theory could correspond to partonic 2-surfaces identified as algebraic surfaces characterized by degree and genus. Besides this also string world sheets are needed. String worlds have 1-D lines at the light-like orbits of partonic 2-surfaces as their boundaries serving as carriers of fermions. This leads to a rather detailed generalization of Witten's approach using the generalization of twistors to 8-D context.

3. The generalization of the twistor Grassmannian approach to 8-D context is second fascinating challenge. If one requires that the basic formulas relating twistors and four-momentum generalize one must consider the situation in tangent space M^8 of embedding space (M^8-H duality) and replace the usual sigma matrices having interpretation in terms of complexified quaternions with octonionic sigma matrices.

   The condition that octonionic spinors are are equivalent with ordinary spinors has strong consequences. Induced spinors must be localized to 2-D string world sheets, which are (co-)commutative sub-manifolds of (co-)quaternionic space-time surface. Also the gauge fields should vanish since they induce a breaking of associativity even for quaternionic and complex surface so that CP_2 projection of string world sheet must be 1-D. If one requires also the vanishing of gauge potentials, the projection is geodesic circle of CP_2 so that string world sheets are restricted to Minkowskian space-time regions. Although the theory would be free in fermionic degrees of freedom, the scattering amplitudes are non-trivial since vertices correspond to partonic 2-surfaces at which partonic orbits are glued together along common ends. The classical light-like 8-momentum associated with the boundaries of string world sheets defines the gravitational dual for 4-D momentum and color quantum numbers associated with imbedding space spinor harmonics. This leads to a more detailed formulation of Equivalence Principle which would reduce to M^8-H duality basically.

   Number theoretic interpretation of the positivity of Grassmannians is highly suggestive since the canonical identification maps p-adic numbers to non-negative real numbers. A possible generalization is obtained by replacing positive real axis with upper half plane defining hyperbolic space having key role in the theory of Riemann surfaces. The interpretation of scattering amplitudes as representations of permutations generalizes to interpretation as braidings at surfaces formed by the generalized Feynman diagrams having as lines the light-like orbits of partonic surfaces. This because 2-fermion vertex is the only interaction vertex and induced by the non-continuity of the induced Dirac operator at partonic 2-surfaces. OZI rule generalizes and implies an interpretation in terms of braiding consistent with the TGD as almost topological QFT vision. This suggests that non-planar twistor amplitudes are constructible as analogs of knot and braid invariants by a recursive procedure giving as an outcome planar amplitudes.

4. Yangian symmetry is associated with twistor amplitudes and emerges in TGD from completely different idea interpreting scattering amplitudes as representations of algebraic manipulation sequences of minimal length (preferred extremal instead of path integral over space-time surfaces) connecting given initial and final states at boundaries of causal diamond. The algebraic manipulations are carried out in Yangian using product and co-product defining the basic 3-vertices analogous to gauge boson absorption and emission. 3-surface representing elementary particle splits into two or vice versa such that second copy carries quantum numbers of gauge boson or its super counterpart. This would fix the scattering amplitude for given 3-surface and leave only the functional integral over 3-surfaces.

2.2 Background And Motivations

In the following some background plus basic facts and definitions related to twistor spaces are summarized. Also reasons for why twistor are so relevant for TGD is considered at general level.

2.2.1 Basic Results And Problems Of Twistor Approach

The author describes both the basic ideas and results of twistor approach as well as the problems.

Basic results

There are three deep results of twistor approach besides the impressive results which have emerged after the twistor resolution.

1. Massless fields of arbitrary helicity in 4-D Minkowski space are in 1-1 correspondence with elements of Dolbeault cohomology in the twistor space CP_3. This was already the discovery of Penrose..The connection comes from Penrose transform. The light-like geodesics of M^4 correspond to points of 5-D sub-manifold of CP_3 analogous to light-cone boundary. The points of M^4 correspond to complex lines (Riemann spheres) of the twistor space CP_3: one can imagine that the point of M^4 corresponds to all light-like geodesics emanating from it and thus to a 2-D surface (sphere) of CP_3. Twistor transform represents the value of a massless field at point of M^4 as a weighted average of its values at sphere of CP_3. This correspondence is formulated between open sets of M^4 and of CP_3. This fits very nicely with the needs of TGD since causal diamonds which can be regarded as open sets of M^4 are the basic objects in zero energy ontology (ZEO).
2. Self-dual instantons of non-Abelian gauge theories for SU(n) gauge group are in one-one correspondence with holomorphic rank-N vector bundles in twistor space satisfying some additional conditions. This generalizes the correspondence of Penrose to the non-Abelian case. Instantons are also usually formulated using classical field theory at four-sphere S^4 having Euclidian signature.
3. Non-linear gravitons having self-dual geometry are in one-one correspondence with spaces obtained as complex deformations of twistor space satisfying certain additional conditions. This is a generalization of Penrose's discovery to the gravitational sector.

Complexification of M^4 emerges unavoidably in twistorial approach and Minkowski space identified as a particular real slice of complexified M^4 corresponds to the 5-D subspace of twistor space in which the quadratic form defined by the SU(2,2) invariant metric of the 8-dimensional space giving twistor space as its projectivization vanishes. The quadratic form has also positive and negative values with its sign defining a projective invariant, and this correspond to complex continuations of M^4 in which positive/negative energy parts of fields approach to zero for large values of imaginary part of M^4 time coordinate.

Interestgingly, this complexification of M^4 is also unavoidable in the number theoretic approach to TGD: what one must do is to replace 4-D Minkowski space with a 4-D slice of 8-D complexified quaternions. What is interesting is that real M^4 appears as a projective invariant consisting of light-like projective vectors of C^4 with metric signature (4,4). Equivalently, the points of M^4 represented as linear combinations of sigma matrices define hermitian matrices.

Basic problems of twistor approach

The best manner to learn something essential about a new idea is to learn about its problems. Difficulties are often put under the rug but the thesis is however an exception in this respect. It starts directly from the problems of twistor approach. There are two basic challenges.

1. Twistor approach works as such only in the case of Minkowski space. The basic condition for its applicability is that the Weyl tensor is self-dual. For Minkowskian signature this leaves only Minkowski space under consideration. For Euclidian signature the conditions are not quite so restrictive. This looks a fatal restriction if one wants to generalize the result of Penrose to a general space-time geometry. This difficulty is known as "googly" problem.

   According to the thesis MHV construction of tree amplitudes of $\mathcal{N} = 4$ SYM based on topological twistor strings in CP_3 meant a breakthrough and one can indeed understand also have analogs of non-self-dual amplitudes. The problem is however that the gravitational theory assignable to topological twistor strings is conformal gravity, which is generally regarded as non-physical. There have been several attempts to construct the on-shell scattering amplitudes of Einstein's gravity theory as subset of amplitudes of conformal gravity and also thesis considers this problem.

2. The construction of quantum theory based on twistor approach represents second challenge. In this respect the development of twistor approach to $\mathcal{N} = 4$ SYM meant a revolution and one can indeed construct twistorial scattering amplitudes in M^4.

2.2.2 Results About Twistors Relevant For TGD

First some background.

1. The twistors originally introduced by Penrose (1967) have made breakthrough during last decade. First came the twistor string theory of Edward Witten [B29] proposed twistor string theory and the work of Nima-Arkani Hamed and collaborators [B34] led to a revolution in the understanding of the scattering amplitudes of scattering amplitudes of gauge theories [B22, B20, B36]. Twistors do not only provide an extremely effective calculational method giving even hopes about explicit formulas for the scattering amplitudes of $\mathcal{N} = 4$ supersymmetric gauge theories but also lead to an identification of a new symmetry: Yangian symmetry [A18], [B26, B27], which can be seen as multilocal generalization of local symmetries.

   This approach, if suitably generalized, is tailor-made also for the needs of TGD. This is why I got seriously interested on whether and how the twistor approach in empty Minkowski space M^4 could generalize to the case of $H = M^4 \times CP_2$. The twistor space associated with H should be just the cartesian product of those associated with its Cartesian factors. Can one assign a twistor space with CP_2?

2. First a general result [A54] deserves to be mentioned: any oriented manifold X with Riemann metric allows 6-dimensional twistor space Z as an almost complex space. If this structure is integrable, Z becomes a complex manifold, whose geometry describes the conformal geometry of X. In general relativity framework the problem is that field equations do not imply conformal geometry and twistor Grassmann approach certainly requires conformal structure.

3. One can consider also a stronger condition: what if the twistor space allows also Kähler structure? The twistor space of empty Minkowski space M^4 (and its Euclidian counterpart S^4 is the Minkowskian variant of $P_3 = SU(2,2)/SU(2,1) \times U(1)$ of 3-D complex projective space $CP_3 = SU(4)/SU(3) \times U(1)$ and indeed allows Kähler structure.

   The points of the Euclidian twistor space $CP_3 = SU(4)/SU(3) \times U(1)$ can be represented by any column of the 4×4 matrix representing element of $SU(4)$ with columns differing by phase multiplication being identified. One has four coordinate charts corresponding to four different choices of the column. The points of its Minkowskian variant $CP_{2,1} = SU(2,2)/SU(2,1) \times U(1)$ can be represented in similar manner as U(1) gauge equivalence classes for given column of SU(3,1) matrices, whose rows and columns satisfy orthonormality conditions with respect to the hermitian inner product defined by Minkowskian metric $\epsilon = (1, 1, -1, -1)$.

   Rather remarkably, there are *no other space-times* with Minkowski signature allowing twistor space with Kähler structure [A54]. Does this mean that the empty Minkowski space of special relativity is much more than a limit at which space-time is empty?

   This also means a problem for GRT. Twistor space with Kähler structure seems to be needed but general relativity does not allow it. Besides twistor problem GRT also has energy problem: matter makes space-time curved and the conservation laws and even the definition of energy and momentum are lost since the underlying symmetries giving rise to the conservation laws through Noether's theorem are lost. GRT has therefore two bad mathematical problems which might explain why the quantization of GRT fails. This would not be surprising since quantum theory is to high extent representation theory for symmetries and symmetries are lost. Twistors would extend these symmetries to Yangian symmetry but GRT does not allow them.

4. What about twistor structure in CP_2? CP_2 allows complex structure (Weyl tensor is self-dual), Kähler structure plus accompanying symplectic structure, and also quaternion structure. One of the really big personal surprises of the last years has been that CP_2 twistor space

indeed allows Kähler structure meaning the existence of antisymmetric tensor representing imaginary unit whose tensor square is the negative of metric in turn representing real unit.

The article by Nigel Hitchin, a famous mathematical physicist, describes a detailed argument identifying S^4 and CP_2 as the only compact Riemann manifolds allowing Kählerian twistor space [A54]. Hitchin sent his discovery for publication 1979. An amusing co-incidence is that I discovered CP_2 just this year after having worked with S^2 and found that it does not really allow to understand standard model quantum numbers and gauge fields. It is difficult to avoid thinking that maybe synchrony indeed a real phenomenon as TGD inspired theory of consciousness predicts to be possible but its creator cannot quite believe. Brains at different side of globe discover simultaneously something closely related to what some conscious self at the higher level of hierarchy using us as instruments of thinking just as we use nerve cells is intensely pondering.

Although 4-sphere S^4 allows twistor space with Kähler structure, it does not allow Kähler structure and cannot serve as candidate for S in $H = M^4 \times S$. As a matter of fact, S^4 can be seen as a Wick rotation of M^4 and indeed its twistor space is CP_3.

In TGD framework a slightly different interpretation suggests itself. The Cartesian products of the intersections of future and past light-cones - causal diamonds (CDs) - with CP_2 - play a key role in ZEO (ZEO) [K7]. Sectors of "world of classical worlds" (WCW) [K45, K24] correspond to 4-surfaces inside $CD \times CP_2$ defining a the region about which conscious observer can gain conscious information: state function reductions - quantum measurements - take place at its light-like boundaries in accordance with holography. To be more precise, wave functions in the moduli space of CDs are involved and in state function reductions come as sequences taking place at a given fixed boundary. This kind of sequence is identifiable as self and give rise to the experience about flow of time. When one replaces Minkowski metric with Euclidian metric, the light-like boundaries of CD are contracted to a point and one obtains topology of 4-sphere S^4.

5. Another really big personal surprise was that there are *no other* compact 4-manifolds with Euclidian signature of metric allowing twistor space with Kähler structure! The embedding space $H = M^4 \times CP_2$ is not only physically unique since it predicts the quantum number spectrum and classical gauge potentials consistent with standard model but also mathematically unique!

   After this I dared to predict that TGD will be the theory next to GRT since TGD generalizes string model by bringing in 4-D space-time. The reasons are many-fold: TGD is the only known solution to the two big problems of GRT: energy problem and twistor problem. TGD is consistent with standard model physics and leads to a revolution concerning the identification of space-time at microscopic level: at macroscopic level it leads to GRT but explains some of its anomalies for which there is empirical evidence (for instance, the observation that neutrinos arrived from SN1987A at two different speeds different from light velocity [?] has natural explanation in terms of many-sheeted space-time). TGD avoids the landscape problem of M-theory and anthropic non-sense. I could continue the list but I think that this is enough.

6. The twistor space of CP_2 is 3-complex dimensional flag manifold $F_3 = SU(3)/U(1) \times U(1)$ having interpretation as the space for the choices of quantization axes for the color hypercharge and isospin. This choice is made in quantum measurement of these quantum numbers and a means localization to single point in F_3. The localization in F_3 could be higher level measurement leading to the choice of quantizations for the measurement of color quantum numbers.

   F_3 is symmetric space meaning that besides being a coset space with $SU(3)$ invariant metric it also has involutions acting as a reflection at geodesics through a point remaining fixed under the involution. As a symmetric space with Fubini-Study metric F_3 is positive constant curvature space having thus positive constant sectional curvatures. This implies Einstein space property. This also conforms with the fact that F_3 is CP_1 bundle over CP_2 as base space (for more details see `http://tinyurl.com/ychdeqjz`).

   The points of flag manifold $SU(3)/U(1) \times U(1)$ can be represented locally by identifying $SU(3)$ matrices for which columns differ by multiplication from left with exponential $e^{i(aY+bI_3)}$, a and b arbitrary real numbers. This transformation allows what might be called a "gauge choice". For instance, first two elements of the first row can be made real in this manner. These coordinates are not global.

7. Analogous interpretation could make sense for M^4 twistors represented as points of P_3. Twistor corresponds to a light-like line going through some point of M^4 being labelled by 4 position coordinates and 2 direction angles: what higher level quantum measurement could involve a choice of light-like line going through a point of M^4? Could the associated spatial direction specify spin quantization axes? Could the associated time direction specify preferred rest frame? Does the choice of position mean localization in the measurement of position? Do momentum twistors relate to the localization in momentum space? These questions remain fascinating open questions and I hope that they will lead to a considerable progress in the understanding of quantum TGD.

8. It must be added that the twistor space of CP_2 popped up much earlier in a rather unexpected context [K39]: I did not of course realize that it was twistor space. Topologist Barbara Shipman [A25] has proposed a model for the honeybee dance leading to the emergence of F_3. The model led her to propose that quarks and gluons might have something to do with biology. Because of her position and specialization the proposal was forgiven and forgotten by community. TGD however suggests both dark matter hierarchies and p-adic hierarchies of physics [K35, ?]. For dark hierarchies the masses of particles would be the standard ones but the Compton scales would be scaled up by $h_{eff}/h = n$ [?]. Below the Compton scale one would have effectively massless gauge boson: this could mean free quarks and massless gluons even in cell length scales. For p-adic hierarchy mass scales would be scaled up or down from their standard values depending on the value of the p-adic prime.

2.2.3 Basic Definitions Related To Twistor Spaces

One can find from web several articles explaining the basic notions related to twistor spaces and Calabi-Yau manifolds. At the first look the notions of twistor as it appears in the writings of physicists and mathematicians don't seem to have much common with each other and it requires effort to build the bridge between these views. The bridge comes from the association of points of Minkowski space with the spheres of twistor space: this clearly corresponds to a bundle projection from the fiber to the base space, now Minkowski space. The connection of the mathematician's formulation with spinors remains still somewhat unclear to me although one can understand CP_1 as projective space associated with spinors with 2 complex components. Minkowski signature poses additional challenges. In the following I try my best to summarize the mathematician's view, which is very natural in classical TGD.

There are many variants of the notion of twistor depending on whether how powerful assumptions one is willing to make. The weakest definition of twistor space is as CP_1 bundle of almost complex structures in the tangent spaces of an orientable 4-manifold. Complex structure at given point means selection of antisymmetric form J whose natural action on vector rotates a vector in the plane defined by it by $\pi/2$ and thus represents the action of imaginary unit. One must perform this kind of choice also in normal plane and the direct sum of the two choices defines the full J. If one chooses J to be self-dual or anti-self-dual (eigenstate of Hodge star operation), one can fix J uniquely. Orientability makes possible the Hodge star operation involving 4-dimensional permutation tensor.

The condition $i^1 = -1$ is translated to the condition that the tensor square of J equals to $J^2 = -g$. The possible choices of J span sphere S^2 defining the fiber of the twistor spaces. This is not quite the complex sphere CP_1, which can be thought of as a projective space of spinors with two complex components. Complexification must be performed in both the tangent space of X^4 and of S^2. Note that in the standard approach to twistors the entire 6-D space is projective space P_3 associated with the C^8 having interpretation in terms of spinors with 4 complex components.

One can introduce almost complex structure also to the twistor space itself by extending the almost complex structure in the 6-D tangent space obtained by a preferred choice of J by identifying it as a point of S^2 and acting in other points of S^2 identified as antisymmetric tensors. If these points are interpreted as imaginary quaternion units, the action is commutator action divided by 2. The existence of quaternion structure of space-time surfaces in the sense as I have proposed in TGD framework might be closely related to the twistor structure.

Twistor structure as bundle of almost complex structures having itself almost complex structure is characterized by a hermitian Kähler form ω defining the almost complex structure of the twistor space. Three basic objects are involved: the hermitian form h, metric g and Kähler form ω satisfying $h = g + i\omega$, $g(X,Y) = \omega(X, JY)$.

In the base space the metric of twistor space is the metric of the base space and in the tangent space of fibre the natural metric in the space of antisymmetric tensors induced by the metric of the base space. Hence the properties of the twistor structure depend on the metric of the base space.

The relationship to the spinors requires clarification. For 2-spinors one has natural Lorentz invariant antisymmetric bilinear form and this seems to be the counterpart for J?

One can consider various additional conditions on the definition of twistor space.

1. Kähler form ω is not closed in general. If it is, it defines symplectic structure and Kähler structure. S^4 and CP_2 are the only compact spaces allowing twistor space with Kähler structure [A54].

2. Almost complex structure is not integrable in general. In the general case integrability requires that each point of space belongs to an open set in which vector fields of type (1, 0) or (0, 1) having basis ∂/∂_{z^k} and $\partial/\partial_{\overline{z}^k}$ expressible as linear combinations of real vector fields with complex coefficients commute to vector fields of same type. This is non-trivial conditions since the leading names for the vector field for the partial derivatives does not yet guarantee these conditions.

   This necessary condition is also enough for integrability as Newlander and Nirenberg have demonstrated. An explicit formulation for the integrability is as the vanishing of Nijenhuis tensor associated with the antisymmetric form J (see (`http://tinyurl.com/ybp9vsa5` and `http://tinyurl.com/y8j36p4m`). Nijenhuis tensor characterizes Nijenhuis bracket generalizing ordinary Lie bracket of vector fields (for detailed formula see `http://tinyurl.com/y83mbnso`).

3. In the case of twistor spaces there is an alternative formulation for the integrability. Curvature tensor maps in a natural manner 2-forms to 2-forms and one can decompose the Weyl tensor W identified as the traceless part of the curvature tensor to self-dual and anti-self-dual parts W^+ and W^-, whose actions are restricted to self-dual resp. antiself-dual forms (self-dual and anti-self-dual parts correspond to eigenvalue + 1 and -1 under the action of Hodge * operation: for more details see `http://tinyurl.com/ybkhj4np`). If W^+ or W^- vanishes - in other worlds W is self-dual or anti-self-dual - the assumption that J is self-dual or anti-self-dual guarantees integrability. One says that the metric is anti-self-dual (ASD). Note that the vanishing of Weyl tensor implies local conformal flatness (M^4 and sphere are obviously conformally flat). One might think that ASD condition guarantees that the parallel translation leaves J invariant.

   ASD property has a nice implication: the metric is balanced. In other words one has $d(\omega\wedge\omega) = 2\omega \wedge d\omega = 0$.

4. If the existence of complex structure is taken as a part of definition of twistor structure, one encounters difficulties in general relativity. The failure of spin structure to exist is similar difficulty: for CP_2 one must indeed generalize the spin structure by coupling Kähler gauge potential to the spinors suitably so that one obtains gauge group of electroweak interactions.

5. One could also give up the global existence of complex structure and require symplectic structure globally: this would give $d\omega = 0$. A general result is that hyperbolic 4-manifolds allow symplectic structure and ASD manifolds allow complex structure and hence balanced metric.

2.2.4 Why Twistor Spaces With Kähler Structure?

I have not yet even tried to answer an obvious question. Why the fact that M^4 and CP_2 have twistor spaces with Kähler structure could be so important that it could fix the entire physics? Let us consider a less general question. Why they would be so important for the classical TGD - exact part of quantum TGD - defined by the extremals of Kähler action [K12] ?

1. Properly generalized conformal symmetries are crucial for the mathematical structure of TGD [K24, K106, K23, L10]. Twistor spaces have almost complex structure and in these two special cases also complex, Kähler, and symplectic structures (note that the integrability of the almost complex structure to complex structure requires the self-duality of the Weyl tensor of the 4-D manifold).

   For years ago I considered the possibility that complex 3-manifolds of $CP_3 \times CP_3$ could have the structure of S^2 fiber space and have space-time surfaces as base space. I did not realize that these spaces could be twistor spaces nor did I realize that CP_2 allows twistor space with Kähler structure so that $CP_3 \times F_3$ looks a more plausible choice.

   The expectation was that the Cartesian product $CP_3 \times F_3$ of the two twistor spaces with Kähler structure is fundamental for TGD. The obvious wishful thought is that this space makes possible the construction of the extremals of Kähler action in terms of holomorphic surfaces defining 6-D twistor sub-spaces of $CP_3 \times F_3$ allowing to circumvent the technical problems due to the signature of M^4 encountered at the level of $M^4 \times CP_2$. It would also make the magnificent machinery of the algebraic geometry so powerful in string theories a tool of TGD. Here CP_3 could be replaced with its non-compact form and the problem is that one can have only compactification of M^4 for which metric is defined only modulo conformal scaling. There is however a problem: the compactified Minkowski space or its complexification has a metric defined only modulo conformal factor. This is not a problem in conformally invariant theories but becomes a problem if one wants to speak of induced metric.

   The next realization was that M^4 allows twistor bundle also in purely geometric sense and this bundle is just $T(M^4) = M^4 \times CP_2$. The two variants of twistor space would naturally apply at the level of momentum space and embedding space.

2. Every 4-D orientable Riemann manifold allows a twistor space as 6-D bundle with CP_1 as fiber and possessing almost complex structure. Metric and various gauge potentials are obtained by inducing the corresponding bundle structures. Hence the natural guess is that the twistor structure of space-time surface defined by the induced metric is obtained by induction from that for $T(M^4) \times F_3$ by restricting its twistor structure to a 6-D (in real sense) surface of $T(M^4) \times F_3$ with a structure of twistor space having at least almost complex structure with CP_1 as a fiber. For the embedding of the twistor space of space-time this requires the identification of S^2 fibers of $T(M^4)$ and F_3. If so then one can indeed identify the base space as 4-D space-time surface in $M^4 \times CP_2$ using bundle projections in the factors $T(M^4)$ and F_3.

3. There might be also a connection to the number theoretic vision about the extremals of Kähler action. At space-time level however complexified quaternions and octonions could allow alternative formulation. I have indeed proposed that space-time surfaces have associative of co-associative meaning that the tangent space or normal space at a given point belongs to quaternionic subspace of complexified octonions.

2.3 The Identification Of 6-D Twistor Spaces As Sub-Manifolds Of 12-D Twistor Space

How to identify the 6-D sub-manifolds with the structure of twistor space? Is this property all that is needed? Can one find a simple solution to this condition? What is the relationship of twistor spaces to the Calabi-Yau manifolds of super string models? In the following intuitive considerations of a simple minded physicist. Mathematician could probably make much more interesting comments.

2.3.1 Conditions For Twistor Spaces As Sub-Manifolds

Consider the conditions that must be satisfied using local trivializations of the twistor spaces. It will be assumed that the twistor space $T(M^4)$ is CP_3 or its Minkowskian variant. It has turned out that a more reasonable option $T(M^4) = M^4 \times CP_1$ is possible. The following consideration is however for CP_3 option. Before continuing let us introduce complex coordinates $z_i = x_i + iy_i$ resp. $w_i = u_i + iv_i$ for CP_3 resp. F_3.

1. 6 conditions are required and they must give rise by bundle projection to 4 conditions relating the coordinates in the Cartesian product of the base spaces of the two bundles involved and thus defining 4-D surface in the Cartesian product of compactified M^4 and CP_2.

2. One has Cartesian product of two fiber spaces with fiber CP_1 giving fiber space with fiber $CP_1^1 \times CP_1^2$. For the 6-D surface the fiber must be CP_1. It seems that one must identify the two spheres CP_1^i. Since holomorphy is essential, holomorphic identification $w_1 = f(z_1)$ or $z_1 = f(w_1)$ is the first guess. A stronger condition is that the function f is meromorphic having thus only finite numbers of poles and zeros of finite order so that a given point of CP_1^i is covered by CP_1^{i+1}. Even stronger and very natural condition is that the identification is bijection so that only Möbius transformations parametrized by SL(2, C) are possible.

3. Could the Möbius transformation $f : CP_1^1 \rightarrow CP_1^2$ depend parametrically on the coordinates z_2, z_3 so that one would have $w_1 = f_1(z_1, z_2, z_3)$, where the complex parameters a, b, c, d $(ad - bc = 1)$ of Möbius transformation depend on z_2 and z_3 holomorphically? Does this mean the analog of local SL(2, C) gauge invariance posing additional conditions? Does this mean that the twistor space as surface is determined up to SL(2, C) gauge transformation?

   What conditions can one pose on the dependence of the parameters a, b, c, d of the Möbius transformation on (z_2, z_3)? The spheres CP_1 defined by the conditions $w_1 = f(z_1, z_2, z_3)$ and $z_1 = g(w_1, w_2, w_3)$ must be identical. Inverting the first condition one obtains $z_1 = f^{-1}(w_1, z_2, z_3)$. If one requires that his allows an expression as $z_1 = g(w_1, w_2, w_3)$, one must assume that z_2 and z_3 can be expressed as holomorphic functions of (w_2, w_3): $z_i = f_i(w_k)$, $i = 2, 3$, $k = 2, 3$. Of course, non-holomorphic correspondence cannot be excluded.

4. Further conditions are obtained by demanding that the known extremals - at least non-vacuum extremals - are allowed. The known extremals [K12] can be classified into CP_2 type vacuum extremals with 1-D light-like curve as M^4 projection, to vacuum extremals with CP_2 projection, which is Lagrangian sub-manifold and thus at most 2-dimensional, to massless extremals with 2-D CP_2 projection such that CP_2 coordinates depend on arbitrary manner on light-like coordinate defining local propagation direction and space-like coordinate defining a local polarization direction, and to string like objects with string world sheet as M^4 projection (minimal surface) and 2-D complex sub-manifold of CP_2 as CP_2 projection, . There are certainly also other extremals such as magnetic flux tubes resulting as deformations of string like objects. Number theoretic vision relying on classical number fields suggest a very general construction based on the notion of associativity of tangent space or co-tangent space.

5. The conditions coming from these extremals reduce to 4 conditions expressible in the holomorphic case in terms of the base space coordinates (z_2, z_3) and (w_2, w_3) and in the more general case in terms of the corresponding real coordinates. It seems that holomorphic ansatz is not consistent with the existence of vacuum extremals, which however give vanishing contribution to transition amplitudes since WCW ("world of classical worlds") metric is completely degenerate for them.

   The mere condition that one has CP_1 fiber bundle structure does not force field equations since it leaves the dependence between real coordinates of the base spaces free. Of course, CP_1 bundle structure alone does not imply twistor space structure. One can ask whether non-vacuum extremals could correspond to holomorphic constraints between (z_2, z_3) and (w_2, w_3).

6. The metric of twistor space is not Kähler in the general case. However, if it allows complex structure there is a Hermitian form ω, which defines what is called balanced Kähler form [A84] satisfying $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$: ordinary Kähler form satisfying $d\omega = 0$ is special case about this. The natural metric of compact 6-dimensional twistor space is therefore balanced. Clearly, mere CP_1 bundle structure is not enough for the twistor structure. If the Kähler and symplectic forms are induced from those of $CP_3 \times Y_3$, highly non-trivial conditions are obtained for the embedding of the twistor space, and one might hope that they are equivalent with those implied by Kähler action at the level of base space.

7. Pessimist could argue that field equations are additional conditions completely independent of the conditions realizing the bundle structure! One cannot exclude this possibility. Mathematician could easily answer the question about whether the proposed CP_1 bundle structure with some added conditions is enough to produce twistor space or not and whether field equations could be the additional condition and realized using the holomorphic ansatz.

2.3.2 Twistor Spaces By Adding CP_1 Fiber To Space-Time Surfaces

The physical picture behind TGD is the safest starting point in an attempt to gain some idea about what the twistor spaces look like.

1. Canonical embeddings of M^4 and CP_2 and their disjoint unions are certainly the natural starting point and correspond to canonical embeddings of CP_3 and F_3 to $CP_3 \times F_3$.
2. Deformations of M^4 correspond to space-time sheets with Minkowskian signature of the induced metric and those of CP_2 to the lines of generalized Feynman diagrams. The simplest deformations of M^4 are vacuum extremals with CP_2 projection which is Lagrangian manifold.

   Massless extremals represent non-vacuum deformations with 2-D CP_2 projection. CP_2 coordinates depend on local light-like direction defining the analog of wave vector and local polarization direction orthogonal to it.

   The simplest deformations of CP_2 are CP_2 type extremals with light-like curve as M^4 projection and have same Kähler form and metric as CP_2. These space-time regions have Euclidian signature of metric and light-like 3-surfaces separating Euclidian and Minkowskian regions define parton orbits.

   String like objects are extremals of type $X^2 \times Y^2$, X^2 minimal surface in M^4 and Y^2 a complex sub-manifold of CP_2. Magnetic flux tubes carrying monopole flux are deformations of these.

   Elementary particles are important piece of picture. They have as building bricks wormhole contacts connecting space-time sheets and the contacts carry monopole flux. This requires at least two wormhole contacts connected by flux tubes with opposite flux at the parallel sheets.
3. Space-time surfaces are constructed using as building bricks space-time sheets, in particular massless exrremals, deformed pieces of CP_2 defining lines of generalized Feynman diagrams as orbits of wormhole contacts, and magnetic flux tubes connecting the lines. Space-time surfaces have in the generic case discrete set of self intersections and it is natural to remove them by connected sum operation. Same applies to twistor spaces as sub-manifolds of $CP_3 \times F_3$ and this leads to a construction analogous to that used to remove singularities of Calabi-Yau spaces [A84].

Physical intuition suggests that it is possible to find twistor spaces associated with the basic building bricks and to lift this engineering procedure to the level of twistor space in the sense that the twistor projections of twistor spaces would give these structure. Lifting would essentially mean assigning CP_1 fiber to the space-time surfaces.

1. Twistor spaces should decompose to regions for which the metric induced from the $CP_3 \times F_3$ metric has different signature. In particular, light-like 5-surfaces should replace the light-like 3-surfaces as causal horizons. The signature of the Hermitian metric of 4-D (in complex sense) twistor space is (1, 1, -1, -1). Minkowskian variant of CP_3 is defined as projective space $SU(2,2)/SU(2,1) \times U(1)$. The causal diamond (CD) (intersection of future and past directed light-cones) is the key geometric object in ZEO (ZEO) and the generalization to the intersection of twistorial light-cones is suggestive.
2. Projective twistor space has regions of positive and negative projective norm, which are 3-D complex manifolds. It has also a 5-dimensional sub-space consisting of null twistors analogous to light-cone and has one null direction in the induced metric. This light-cone has conic singularity analogous to the tip of the light-cone of M^4.

   These conic singularities are important in the mathematical theory of Calabi-You manifolds since topology change of Calabi-Yau manifolds via the elimination of the singularity can be associated with them. The S^2 bundle character implies the structure of S^2 bundle for the base of the singularity (analogous to the base of the ordinary cone).

3. Null twistor space corresponds at the level of M^4 to the light-cone boundary (causal diamond has two light-like boundaries). What about the light-like orbits of partonic 2-surfaces whose light-likeness is due to the presence of CP_2 contribution in the induced metric? For them the determinant of induced 4-metric vanishes so that they are genuine singularities in metric sense. The deformations for the canonical embeddings of this sub-space (F_3 coordinates constant) leaving its metric degenerate should define the lifts of the light-like orbits of partonic 2-surface. The singularity in this case separates regions of different signature of induced metric.

   It would seem that if partonic 2-surface begins at the boundary of CD, conical singularity is not necessary. On the other hand the vertices of generalized Feynman diagrams are 3-surfaces at which 3-lines of generalized Feynman digram are glued together. This singularity is completely analogous to that of ordinary vertex of Feynman diagram. These singularities should correspond to gluing together 3 deformed F_3 along their ends.

4. These considerations suggest that the construction of twistor spaces is a lift of construction space-time surfaces and generalized Feynman diagrammatics should generalize to the level of twistor spaces. What is added is CP_1 fiber so that the correspondence would rather concrete.

5. For instance, elementary particles consisting of pairs of monopole throats connected buy flux tubes at the two space-time sheets involved should allow lifting to the twistor level. This means double connected sum and this double connected sum should appear also for deformations of F_3 associated with the lines of generalized Feynman diagrams. Lifts for the deformations of magnetic flux tubes to which one can assign CP_3 in turn would connect the two F_3s.

6. A natural conjecture inspired by number theoretic vision is that Minkowskian and Euclidian space-time regions correspond to associative and co-associative space-time regions. At the level of twistor space these two kinds of regions would correspond to deformations of CP_3 and F_3. The signature of the twistor norm would be different in this regions just as the signature of induced metric is different in corresponding space-time regions.

   These two regions of space-time surface should correspond to deformations for disjoint unions of CP_3s and F_3s and multiple connected sum form them should project to multiple connected sum (wormhole contacts with Euclidian signature of induced metric) for deformed CP_3s. Wormhole contacts could have deformed pieces of F_3 as counterparts.

There are interesting questions related to the detailed realization of the twistor spaces of space-time surfaces.

1. In the case of CP_2 J would naturally correspond to the Kähler form of CP_2. Could one identify J for the twistor space associated with space-time surface as the projection of J? For deformations of CP_2 type vacuum extremals the normalization of J would allow to satisfy the condition $J^2 = -g$. For general extremals this is not possible. Should one be ready to modify the notion of twistor space by allowing this?

2. Or could the associativity/co-associativity condition realized in terms of quaternionicity of the tangent or normal space of the space-time surface guaranteeing the existence of quaternion units solve the problem and J could be identified as a representation of unit quaternion? In this case J would be replaced with vielbein vector and the decomposition 1+3 of the tangent space implied by the quaternion structure allows to use 3-dimensional permutation symbol to assign antisymmetric tensors to the vielbein vectors. Also the triviality of the tangent bundle of 3-D space allowing global choices of the 3 imaginary units could be essential.

3. Does associativity/co-associativity imply twistor space property or could it provide alternative manner to realize this notion? Or could one see quaternionic structure as an extension of almost complex structure. Instead of single J three orthogonal J: s (3 almost complex structures) are introduced and obey the multiplication table of quaternionic units? Instead of S^2 the fiber of the bundle would be $SO(3) = S^3$. This option is not attractive. A manifold with quaternionic tangent space with metric representing the real unit is known as quaternionic Riemann manifold and CP_2 with holonomy U(2) is example of it. A more restrictive condition is that all quaternion units define closed forms: one has quaternion Kähler manifold, which is Ricci flat and has in 4-D case Sp(1)=SU(2) holonomy. (see `http://tinyurl.com/y9qtoebe`).

4. Anti-self-dual property (ASD) of metric guaranteeing the integrability of almost complex structure of the twistor space implies the condition $\omega \wedge d\omega = 0$ for the twistor space. What does this condition mean physically for the twistor spaces associated with the extremals of Kähler action? For the 4-D base space this property is of course identically true. ASD property need of course not be realized.

2.3.3 Twistor Spaces As Analogs Of Calabi-Yau Spaces Of Super String Models

CP_3 is also a Calabi-Yau manifold in the strong sense that it allows Kähler structure and complex structure. Witten's twistor string theory considers 2-D (in real sense) complex surfaces in twistor space CP_3 or its Minkowskian variant. This choice does not however seem to be natural from the point of view of the induced geometry although it looks natural at the level of momentum space. It is less well-known that M^4 allows also second twistor space $T(M^4) = M^4 \times CP_1$, and this looks very natural concerning twistor lift of TGD replacing space-time surfaces in H with their twistor spaces in $T(H) = T(M^4) \times T(CP_2)$.

The original identification $T(M^4)$ with CP_3 or its Minkowskian variant has been given up bit it inspired some questions discussed in the sequel.

1. Could TGD in twistor space formulation be seen as a generalization of this theory?
2. General twistor space is not Calabi-Yau manifold because it does does not have Kähler structure. Do twistor spaces replace Calabi-Yaus in TGD framework?
3. Could twistor spaces be Calabi-Yau manifolds in some weaker sense so that one would have a closer connection with super string models.

Consider the last question.

1. One can indeed define non-Kähler Calabi-Yau manifolds by keeping the hermitian metric and giving up symplectic structure or by keeping the symplectic structure and giving up hermitian metric (almost complex structure is enough). Construction recipes for non-Kähler Calabi-Yau manifold are discussed in [A84]. It is shown that these two ways to give up Kähler structure correspond to duals under so called mirror symmetry [B16] which maps complex and symplectic structures to each other. This construction applies also to the twistor spaces.
2. For the modification giving up symplectic structure, one starts from a smooth Kähler Calabi-Yau 3-fold Y, such as CP_3. One assumes a discrete set of disjoint rational curves diffeomorphic to CP_1. In TGD framework work they would correspond to special fibers of twistor space.

   One has singularities in which some rational curves are contracted to point - in twistorial case the fiber of twistor space would contract to a point - this produces double point singularity which one can visualize as the vertex at which two cones meet (sundial should give an idea about what is involved). One deforms the singularity to a smooth complex manifold. One could interpret this as throwing away the common point and replacing it with connected sum contact: a tube connecting the holes drilled to the vertices of the two cones. In TGD one would talk about wormhole contact.
3. Suppose the topology looks locally like $S^3 \times S^2 \times R_\pm$ near the singularity, such that two copies analogous to the two halves of a cone (sundial) meet at single point defining double point singularity. In the recent case S^2 would correspond to the fiber of the twistor space. S^3 would correspond to 3-surface and $R_\pm$ would correspond to time coordinate in past/future direction. S^3 could be replaced with something else.

   The copies of $S^3 \times S^2$ contract to a point at the common end of R_+ and R_- so that both the based and fiber contracts to a point. Space-time surface would look like the pair of future and past directed light-cones meeting at their tips.

   For the first modification giving up symplectic structure only the fiber S^2 is contracted to a point and $S^2 \times D$ is therefore replaced with the smooth "bottom" of S^3. Instead of sundial one has two balls touching. Drill small holes two the two S^3s and connect them by connected sum contact (wormhole contact). Locally one obtains $S^3 \times S^3$ with k connected sum contacts.

   For the modification giving up Hermitian structure one contracts only S^3 to a point instead of S^2. In this case one has locally two CP_3: s touching (one can think that CP_n is obtained by replacing the points of C^n at infinity with the sphere CP_1). Again one drills holes and connects them by a connected sum contact to get k-connected sum of CP_3.

For k CP_1s the outcome looks locally like to a k-connected sum of $S^3 \times S^3$ or CP_3 with $k \geq 2$. In the first case one loses symplectic structure and in the second case hermitian structure. The conjecture is that the two manifolds form a mirror pair.

The general conjecture is that all Calabi-Yau manifolds are obtained using these two modifications. One can ask whether this conjecture could apply also the construction of twistor spaces representable as surfaces in $CP_3 \times F_3$ so that it would give mirror pairs of twistor spaces.

4. This smoothing out procedures isa actually unavoidable in TGD because twistor space is sub-manifold. The 6-D twistor spaces in 12-D $T(M^4) \times F_3$ have in the generic case self intersections consisting of discrete points. Since the fibers CP_1 cannot intersect and since the intersection is point, it seems that the fibers must contract to a point. In the similar manner the 4-D base spaces should have local foliation by spheres or some other 3-D objects with contract to a point. One has just the situation described above.

   One can remove these singularities by drilling small holes around the shared point at the two sheets of the twistor space and connected the resulting boundaries by connected sum contact. The preservation of fiber structure might force to perform the process in such a way that local modification of the topology contracts either the 3-D base (S^3 in previous example or fiber CP_1 to a point.

The interpretation of twistor spaces is of course totally different from the interpretation of Calabi-Yaus in superstring models. The landscape problem of superstring models is avoided and the multiverse of string models is replaced with generalized Feynman diagrams! Different twistor spaces correspond to different space-time surfaces and one can interpret them in terms of generalized Feynman diagrams since bundle projection gives the space-time picture. Mirror symmetry means that there are two different Calabi-Yaus giving the same physics. Also now twistor space for a given space-time surface can have several embeddings - perhaps mirror pairs define this kind of embeddings.

To sum up, the construction of space-times as surfaces of H lifted to those of (almost) complex sub-manifolds in $T(M^4) times F_3$ with induced twistor structure shares the spirit of the vision that induction procedure is the key element of classical and quantum TGD. It also gives deep connection with the mathematical methods applied in super string models and these methods should be of direct use in TGD.

2.4 Witten's Twistor String Approach And TGD

The twistor Grassmann approach has led to a phenomenal progress in the understanding of the scattering amplitudes of gauge theories, in particular the $\mathcal{N} = 4$ SUSY.

As a non-specialist I have been frustrated about the lack of concrete picture, which would help to see how twistorial amplitudes might generalize to TGD framework. A pleasant surprise in this respect was the proposal of a particle interpretation for the twistor amplitudes by Nima Arkani Hamed *et al* in the article "Unification of Residues and Grassmannian Dualities" [B37] (see `http://tinyurl.com/y86mad5n`)

In this interpretation incoming particles correspond to spheres CP_1 so that n-particle state corresponds to $(CP_1)^n/Gl(2)$ (the modding by $Gl(2)$ might be seen as a kind of formal generalization of particle identity by replacing permutation group S_2 with $Gl(2)$ of 2×2 matrices). If the number of "wrong" helicities in twistor diagram is k, this space is imbedded to $CP^n_{k-1}/Gl(k)$ as a surface having degree $k-1$ using Veronese map to achieve the embedding. The embedding space can be identified as Grassmannian $G(k,n)$. This surface defines the locus of the multiple residue integral defining the twistorial amplitude.

The particle interpretation brings in mind the extension of single particle configuration space E^3 to its Cartesian power E^{3n}/S_n for n-particle system in wave mechanics. This description could make sense when point-like particle is replaced with 3-surface or partonic 2-surface: one would have Cartesian product of WCWs divided my S_n. The generalization might be an excellent idea as far calculations are considered but is not in spirit with the very idea of string models and TGD that many-particle states correspond to unions of 3-surfaces in H (or light-like boundaries of causal diamond (CD) in Zero Energy Ontology (ZEO).

Witten's twistor string theory [B29] is more in spirit with TGD at fundamental level since it is based on the identification of generalization of vertices as 2-surfaces in twistor space.

1. There are several kinds of twistors involved. For massless external particles in eigenstates of momentum and helicity null twistors code the momentum and helicity and are pairs of 2-spinor and its conjugate. More general momenta correspond to two independent 2-spinors.

   One can perform twistor Fourier transform for the conjugate 2-spinor to obtain twistors coding for the points of compactified Minkowski space. Wave functions in this twistor space characterized by massless momentum and helicity appear in the construction of twistor amplitudes. BCFW recursion relation [B20] allows to construct more complex amplitudes assuming that intermediate states are on mass shells massless states with complex momenta.

   One can perform twistor Fourier transformation (there are some technical problems in Minkowski signature) also for the second 2-spinor to get what are called momentum twistors providing in some aspects simpler description of twistor amplitudes. These code for the four-momenta propagating between vertices at which the incoming particles arrive and the differences if two subsequent momenta are equal to massless external momenta.

2. In Witten's theory the interactions of incoming particles correspond to amplitudes in which the twistors appearing as arguments of the twistor space wave functions characterized by momentum and helicity are localized to complex curves X^2 of twistor space CP_3 or its Minkowskian counterpart. This can be seen as a non-local twistor space variant of local interactions in Minkowski space.

   The surfaces X^2 are characterized by their degree d (of the polynomial of complex coordinates defining the algebraic 2-surface) the genus g of the algebraic surface, by the number k of "wrong" (helicity violating) helicities, and by the number of loops of corresponding diagram of SUSY amplitude: one has $d = k - 1 + l$, $g \leq l$. The interaction vertex in twistor space is not anymore completely local but the n particles are at points of the twistorial surface X^2.

In the following a proposal generalizing Witten's approach to TGD is discussed.

1. The fundamental challenge is the generalization of the notion of twistor associated with massless particle to 8-D context, first for $M^4 = M^4 \times E^4$ and then for $H = M^4 \times CP_2$. The notion of twistor space solves this question at geometric level. As far as construction of the TGD variant of Witten's twistor string is considered, this might be quite enough.

2. $M^8 - H$ duality and quantum-classical correspondence however suggest that M^8 twistors might allow tangent space description of four-momentum, spin, color quantum numbers and electroweak numbers and that this is needed. What comes in mind is the identification of fermion lines as light-like geodesics possessing M^8 valued 8-momentum, which would define the long sought gravitational counterparts of four-momentum and color quantum numbers at classical point-particle level. The M^8 part of this four-momentum would be equal to that associated with embedding space spinor mode characterizing the ground state of superconformal representation for fundamental fermion.

   Hence one might also think of starting from the 4-D condition relating Minkowski coordinates to twistors and looking what it could mean in the case of M^8. The generalization is indeed possible in $M^8 = M^4 \times E^4$ by its flatness if one replaces gamma matrices with octonionic gamma matrices.

   In the case of $M^4 \times CP_2$ situation is different since for octonionic gamma matrices $SO(1,7)$ is replaced with G_2, and the induced gauge fields have different holonomy structure than for ordinary gamma matrices and octonionic sigma matrides appearing as charge matrices bring in also an additional source of non-associativity. Certainly the notion of the twistor Fourier transform fails since CP_2 Dirac operator cannot be algebraized.

   Algebraic twistorialization however works for the light-like fermion lines at which the ordinary and octonionic representations for the induced Dirac operator are equivalent. One can indeed assign 8-D counterpart of twistor to the particle classically as a representation of light-like hyper-octonionic four-momentum having massive M^4 and CP_2 projections and CP_2 part perhaps having interpretation in terms of classical tangent space representation for color and electroweak quantum numbers at fermionic lines.

If all induced electroweak gauge fields - rather than only charged ones as assumed hitherto - vanish at string world sheets, the octonionic representation is equivalent with the ordinary one. The CP_2 projection of string world sheet should be 1-dimensional: inside CP_2 type vacuum extremals this is impossible, and one could even consider the possibility that the projection corresponds to CP_2 geodesic circle. This would be enormous technical simplification. What is important that this would not prevent obtaining non-trivial scattering amplitudes at elementary particle level since vertices would correspond to re-arrangement of fermion lines between the generalized lines of Feynman diagram meeting at the vertices (partonic 2-surfaces).

3. In the fermionic sector one is forced to reconsider the notion of the induced spinor field. The modes of the embedding space spinor field should co-incide in some region of the space-time surface with those of the induced spinor fields. The light-like fermionic lines defined by the boundaries of string world sheets or their ends are the obvious candidates in this respect. String world sheets is perhaps too much to require.

   The only reasonable identification of string world sheet gamma matrices is as induced gamma matrices and super-conformal symmetry requires that the action contains string world sheet area as an additional term just as in string models. String tension would correspond to gravitational constant and its value - that is ratio to the CP_2 radius squared, would be fixed by quantum criticality.

4. The generalization of the Witten's geometric construction of scattering amplitudes relying on the induction of the twistor structure of the embedding space to that associated with space-time surface looks surprisingly straight-forward and would provide more precise formulation of the notion of generalized Feynman diagrams forcing to correct some wrong details. One of the nice outcomes is that the genus appearing in Witten's formulation naturally corresponds to family replication in TGD framework.

2.4.1 Basic Ideas About Twistorialization Of TGD

The recent advances in understanding of TGD motive the attempt to look again for how twistor amplitudes could be realized in TGD framework. There have been several highly non-trivial steps of progress leading to a new more profound understanding of basic TGD.

1. $M^4 \times CP_2$ is twistorially unique [L10] in the sense that its factors are the only 4-D geometries allowing twistor space with Kähler structure (M^4 corresponds to S^4 in Euclidian signature) [A54]. The twistor spaces in question are CP_3 for S^4 and its Minkowskian variant for M^4 (I will use P^3 as short hand for both twistor spaces) and the flag manifold $F = SU(3)/U(1) \times U(1)$ parametrizing the choices of quantization axes for color group $SU(3)$ in the case of CP_2. Recall that twistor spaces are S^2 bundles over the base space and that all orientable four-manifolds have twistor space in this sense. Note that space-time surfaces allow always almost quaternionic structure and that preferred extremals are suggested to be quaternionic [L10].

2. The light-likeness condition for twistors in M^4 is fundamental in the ordinary twistor approach. In 8-D context light-likeness holds in generalized sense for the spinor harmonics of H: the square of the Dirac operator annihilates spinor modes. In the case M^8 one can indeed define twistors by generalizing the standard approach by replacing ordinary gamma matrices with octonionic ones [?] For H octonionic and ordinary gamma matrices are equivalent at the fermionic lines defined by the light-like boundaries of string world sheets and at string world sheets if they carry vanishing induced electro-weak gauge fields that is have 1-D CP_2 projection.

3. Twistor spaces emerge in TGD framework as lifts of space-time surfaces to corresponding twistor spaces realized as 6-surfaces in the lift of $M^4 \times CP_2$ to $T(H) = P^3 \times F$ having as base spaces space-time surfaces. This implies that that generalized Feynman diagrams and also generalized twistor diagrams can be lifted to diagrams in T and that the construction of twistor spaces as surfaces of T has very concrete particle interpretation.

   The modes of the embedding space spinor field defining ground states of the extended conformal algebras for which classical conformal charges vanish at the ends of the space-time surface (this defines gauge conditions realizing strong form of holography [K106]) are lifted to the products of modes of spinor fields in $T(H)$ characterized by four-momentum and helicity in M^4 degrees of freedom and by color quantum numbers and electroweak quantum numbers in F degrees of freedom. Thus twistorialization provides a purely geometric representation of spin and electro-weak spin and it seems that twistorialization allows to a formulation without H-spinors.

What is especially nice, that twistorialization extends to from spin to include also electroweak spin. These two spins correspond correspond to M^4 and CP_2 helicities for the twistor space amplitude, and are non-local properties of these amplitudes. In TGD framework only twistor amplitudes for which helicities correspond to that for massless fermion and antifermion are possible and by fermion number conservation the numbers of positive and negative helicities are identical and equal to the fermion number (or antifermion number). Separate lepton and baryon number conservation realizing 8-D chiral symmetry implies that M^4 and CP_2 helicities are completely correlated.

For massless fermions in M^4 sense helicity is opposite for fermion and antifermion and conserved. The contributions of initial and final states to k are same and equal to $k_i = k_f = 2(n(F) - n(\overline{F})$. This means restriction to amplitudes with $k = 2(n(F) - n(\overline{F})$. If fermions are massless only in M^8 sense, chirality mixing occurs and this rule does not hold anymore. This holds true in quark and lepton sector separately.

4. All generalized Feynman graphs defined in terms of Euclidian regions of space-time surface are lifted to twistor spaces [K23]. Incoming particles correspond quantum mechanically to twistor space amplitudes defined by their momenta and helicities and and classically to the entire twistor space of space-time surface as a surface in the twistor space of H. Of course, also the Minkowskian regions have this lift. The vertices of Feynman diagrams correspond to regions of twistor space in which the incoming twistor spaces meet along their 5-D ends having also S^2 bundle structure over space-like 3-surfaces. These space-like 3-surfaces correspond to ends of Euclidian and Minkowskian space-time regions separated from each other by light-like 3-surfaces at which the signature of the metric changes from Minkowskian to Euclidian. These "partonic orbits" have as their ends 2-D partonic surfaces. By strong form of General Coordinate Invariance implying strong of holography, these 2-D partonic surfaces and their 4-D tangent space data should code for quantum physics. Their lifts to twistor space are 4-D S^2 bundles having partonic 2-surface X^2 as base.

5. The well-definedness of em charge for the spinor modes demands that they are localized at 2-D string world sheets [K106] and also these world sheets are lifted to sub-spaces of twistor space of space-time surface. If one demands that octonionic Dirac operator makes sense at string world sheets, they must carry vanishing induced electro-weak gauge fields and string world sheets could be minimal surfaces in $M^4 \times S^1$, $S^1 \subset CP_2$ a geodesic circle.

   The boundaries of string world sheets at partonic orbits define light-like curves identifiable as carriers of fermion number and they define the analogs of lines of Feynman diagrams in ordinary sense. The only purely fermionic vertices are 2-fermion vertices at the partonic 2-surfaces at which the end of space-time surface meet. As already explained, the string world sheets can be seen as correlates for the correlations between fermion vertices at different wormhole throats giving rise to the counterpart of bosonic propagator in quantum field theories.

   The localization of spinor fields to 2-D string world sheets corresponds to the localization of twistor amplitudes to their 4-D lifts, which are S^2 bundles and the boundaries of string world sheets are lifted to 3-D twistor lifts of fermion lines. Clearly, the localization of spinors to string world sheets would be absolutely essential for the emergence of twistor description.

6. All elementary particles are many particle bound states of massless fundamental fermions: the non-collinearity (and possible complex character) of massless momenta explains massivation. The fundamental fermions are localized at wormhole throats defining the light-like orbits of partonic 2-surfaces. Throats are associated with wormhole contacts connecting two space-time sheets. Stability of the contact is guaranteed by non-vanishing monopole magnetic flux through it and this requires the presence of second wormhole contact so that a closed magnetic flux tube carrying monopole flux and involving the two space-time sheets is formed. The net fermionic quantum numbers of the second throat correspond to particle's quantum numbers and above weak scale the weak isospins of the throats sum up to zero.

7. Fermionic 2-vertex is the only *local* many-fermion vertex [K23] being analogous to a mass insertion. The non-triviality of fundamental 4-fermion vertex is due to classical interactions between fermions at opposite throats of worm-hole. The non-triviality of the theory involves also the analog of OZI mechanism: the fermionic lines inside partonic orbits are redistributed in vertices. Lines can also turn around in time direction which corresponds to creation or annihilation of a pair. 3-particle vertices are obtained only in topological sense as 3 space-time surfaces are glued together at their ends. The interaction between fermions at different wormhole throats is described in terms of string world sheets.
8. The earlier proposal was that the fermions in the internal fermion lines are massless in M^4 sense but have non-physical helicity so that the algebraic M^4 Dirac operator emerging from the residue integration over internal four-momentum does not annihilate the state at the end of the propagator line. Now the algebraic induced Dirac operator defines the propagator at fermion lines. Should one assume generalization of non-physical helicity also now?
9. All this stuff must be lifted to twistorial level and one expects that the lift to S^2 bundle allows an alternative description of fermions and spinor structure so that one can speak of induced twistor structure instead of induced spinor structure. This approach allows also a realization of M^4 conformal symmetries in terms of globally well-defined linear transformations so that it might be that twistorialization is not a mere reformulation but provides a profound unification of bosonic and fermionic degrees of freedom.

2.4.2 The Emergence Of The Fundamental 4-Fermion Vertex And Of Boson Exchanges

The emergence of the fundamental 4-fermion vertex and of boson exchanges deserves a more detailed discussion.

1. I have proposed that the discontinuity of the Dirac operator at partonic two-surface (corner of fermion line) defines both the fermion boson vertex and TGD analog of mass insertion (not scalar but embedding space vector) giving rise to mass parameter having interpretation as Higgs vacuum expectation and various fermionic mixing parameters at QFT limit of TGD obtained by approximating many-sheeted space-time of TGD with the single sheeted region of M^4 such that gravitational field and gauge potentials are obtained as sums of those associated with the sheets.
2. Non-trivial scattering requires also correlations between fermions at different partonic 2-surfaces. Both partonic 2-surfaces and string world sheets are needed to describe these correlations. Therefore the string world sheets and partonic 2-surfaces cannot be dual: both are needed and this means deviation from Witten's theory. Fermion vertex corresponds to a "corner" of a fermion line at partonic 2-surface at which generalized 4-D lines of Feynman diagram meet and light-like fermion line changes to space-like one. String world sheet with its corners at partonic 2-surfaces (wormhole throats) describes the momentum exchange between fermions. The space-like string curve connecting two wormhole throats serves as the analog of the exchanged gauge boson.
3. Two kinds of 4-fermion amplitudes can be considered depending on whether the string connects throats of single wormhole contact (CP_2 scale) or of two wormhole contacts (p-adic length scale - typically of order elementary particle Compton length). If string worlds sheets have 1-D CP_2 projection, only Minkowskian string world sheets are possible. The exchange in Compton scale should be assignable to the TGD counterpart of gauge boson exchange and the fundamental 4-fermion amplitude should correspond to single wormhole contact: string need not to be involved now. Interaction is basically classical interaction assignable to single wormhole contact generalizing the point like vertex.
4. The possible TGD counterparts of BCFW recursion relations [B20] should use the twistorial representations of fundamental 4-fermion scattering amplitude as seeds. Yangian invariance poses very strong conditions on the form of these amplitudes and the exchange of massless bosons is suggestive for the general form of amplitude.

   The 4-fermion amplitude assignable to two wormhole throats defines the analog of gauge boson exchange and is expressible as fusion of two fundamental 4-fermion amplitudes such that the 8-momenta assignable to the fermion and anti-fermion at the opposite throats of exchanged wormhole contact are complex by BCFW shift acting on them to make the exchanged momenta massless but complex. This entity could be called fundamental boson (not elementary particle).

5. Can one assume that the fundamental 4-fermion amplitude allows a purely formal composition to a product of $F\overline{F}B_v$ amplitudes, B_v a purely fictive boson? Two 8-momenta at both $F\overline{F}B_v$ vertices must be complex so that at least two external fermionic momenta must be complex. These external momenta are naturally associated with the throats of the a wormhole contact defining virtual fundamental boson. Rather remarkably, without the assumption about product representation one would have general four-fermion vertex rather than boson exchange. Hence gauge theory structure is not put in by hand but emerges.

2.4.3 What About SUSY In TGD?

Extended super-conformal symmetry is crucial for TGD and extends to quaternion-super-conformal symmetry giving excellent hopes about calculability of the theory. $\mathcal{N}=4$ space-time supersymmetry is in the key role in the approach of Witten and others.

In TGD framework space-time SUSY could be present as an approximate symmetry.

1. The many fermion states at partonic surfaces are created by oscillator operators of fermionic Clifford algebra having also interpretation as a supersymmetric algebra but in principle having $\mathcal{N}=\infty$. This SUSY is broken since the generators of SUSY carry four-momentum.
2. More concrete picture would be that various SUSY multiplets correspond to collinear many-fermion states at the same wormhole throat. By fermionic statistics only the collinear states for which internal quantum numbers are different are realized: other states should have anti-symmetric wave function in spatial degrees of freedom implying wiggling in CP_2 scale so that the mass of the state would be very high. In both quark and lepton sectors one would have $\mathcal{N}=4$ SUSY so that one would have the analog $\mathcal{N}=\forall$ SUSY (color is not spin-like quantum number in TGD).

   At the level of diagrammatics single line would be replaced with "line bundle" representing the fermions making the many-fermion state at the light-like orbit of the partonic 2-surface. The fusion of neighboring collinear multifermion stats in the twistor diagrams could correspond to the process in which partonic 2-surfaces and single and many-fermion states fuse.
3. Right handed neutrino modes, which are not covariantly constant, are also localized at the fermionic lines and defines the least broken $\mathcal{N}=2$ SUSY. The covariantly constant mode seems to be a pure gauge degree of freedom since it carriers no quantum numbers and the SUSY norm associated with it vanishes. The breaking would be smallest for $\mathcal{N}=2$ variant assignable to right-handed neutrino having no weak and color interactions with other particles but whose mixing with left-handed neutrino already induces SUSY breaking.

Why this SUSY has not been observed? One can consider two scenarios [K84].

1. The first scenario relies on the absence of weak and color interactions: one can argue that the bound states of fermions with right-handed neutrino are highly unstable since only gravitational interaction so that sparticle decays very rapidly to particle and right-handed or left-handed neutrino. By Uncertainty Principle this makes sparticle very massive, maybe having mass of order CP_2 mass. Neutrino mixing caused by the mixing of M^4 and CP_2 gamma matrices in induced gamma matrices is the weak point of this argument.
2. The mixing of left and right-handed neutrinos could be characterized by the p-adic mass scale of neutrinos and be long. Sparticles would have same p-adic mass scale as particles and would be dark having non-standard value of Planck constant $h_{eff}=n\times h$: this would scale up the lifetime by factor n and correlate with breaking of conformal symmetry assignable to the mixing [K84].

What implications the approximate SUSY would have for scattering amplitudes?

1. $k=2(n(F)-n(\overline{F})$ condition reduces the number of amplitudes dramatically if the fermions are massless in M^4 sense but the presence of weak iso-spin implies that the number of amplitudes

is 2^n as in non-supersymmetric gauge theories. One however expects broken SUSY with generators consisting of fermionic oscillator operators at partonic 2-surfaces with symmetry breaking taking place only at the level of physical particles identifiable as many particle bound states of massless (in 8-D sense) particles. This motivates the guess that the formal $F\overline{F}B_v$ amplitudes defining fundamental 4-fermion vertex are expressible as those associated with $\mathcal{N} = 4$ SUSY in quark and lepton sectors respectively. This would reduce the number of independent amplitudes to just one.

2. Since SUSY and its breaking emerge automatically in TGD framework, super-space can provide a useful technical tool but is not fundamental.

   Side note: The number of external fermions is always even suggesting that the super-conformal anomalies plaguing the amplitudes with odd n (`http://tinyurl.com/yb85tnvc`) [B58] are absent.

2.4.4 What Does One Really Mean With The Induction Of Embedding Space Spinors?

The induction of spinor structure is a central notion of TGD but its detailed definition has remained somewhat obscure. The attempt to generalize Witten's approach has made it clear that the mere restriction of spinor fields to space-time surfaces is not enough and that one must understand in detail the correspondence between the modes of embedding space spinor fields and those of induced spinor fields.

Even the identification of space-time gamma matrices is far from obvious at string world sheets.

1. The simplest notion of the space-time gamma matrices is as projections of embedding space gamma matrices to the space-time surface - induced gamma matrices. If one assumes that induced spinor fields are defined at the entire space-time surfaces, this notion fails to be consistent with fermionic super-conformal symmetry unless one replaces Kähler action by space-time volume. This option is certainly unphysical.

2. The notion of Kähler-Dirac matrices in the interior of space as gamma matrices defined as contractions of canonical momentum densities of Kähler with embedding space gamma matrices allows to have conformal super-symmetry with fermionic super charges assignable to the modes of the induced spinor field. Also Chern-Simons action could define gamma matrices in the same manner at the light-like 3-surfaces between Minkowskian and Euclidian space-time regions and at space-like 3-surfaces at the ends of space-time surface. Chern-Simons-Dirac matrices would involve only CP_2 gamma matrices.

It is however not quite clear whether the spinor fields in the interior of space-time surface are needed at all in the twistorial approach and they seem to be only an un-necessary complication. For instance, their modes would have well-defined electromagnetic charge only when induced W gauge fields vanish, which implies that CP_2 projection is 2-dimensional. This forces to consider very seriously the possibility that induced spinor fields reside at string world sheets only (their ends are at partonic 2-surfaces). This option supported also by strong form of holography and number theoretic universality.

What about the space-time gamma matrices at string world sheets and their boundaries?

1. The first option would be reduction of Kähler-Dirac gamma matrices by requiring that they are parallel to the string world sheets. This however poses additional conditions besides the vanishing of W fields already restricting the dimension to two in the generic case. The conditions state that the embedding space 1-forms defined by the canonical momentum densities of Kähler action involve only 2 linearly independent ones and that they are proportional to embedding space coordinate gradients: this gives Frobenius conditions. These conditions look first too strong but one can also think that one fixes first string world sheets, partonic 2-surfaces, and perhaps also their light-like orbits, requires that the normal components of canonical momentum currents at string world sheets vanish, and deduces space-time surface from this data. This would be nothing but strong form of holography!

   For this option the string world sheets could emerge in the sense that it would be possible to express Kähler action as an area of string world sheet in the effective metric defined by the anticommutator of K-D gamma matrices appearing also in the expressions for the matrix elements of WCW metric. Gravitational constant would be a prediction of the theory.

2. Second possibility is to use induced gamma matrices automatically parallel to the string world sheet so that no additional conditions would result. This would also conform with the ordinary view about string world sheets and spinors.

   Supersymmetry would require the addition of the area of string world sheet to the action defining Kähler function in Euclidian regions and its counterpart in Minkowskian regions. This would bring in gravitational constant, which otherwise remains a prediction. Quantum criticality could fix the ratio of $\hbar G/R^2$ (R is CP_2 radius). The vanishing of induced weak gauge fields requires that string world sheets have 1-D CP_2 projection and are thus restricted to Minkowskian regions with at most 3-D CP_2 projection. Even stronger condition would be that string world sheets are minimal surfaces in $M^4 \times S^1$, S^1 a geodesic sphere of CP_2.

   There are however grave objections. The presence of a dimensional parameter G as fundamental coupling parameter does not encourage hopes about the renomalizibility of the theory. The idea that strings connecting partonic 2-surfaces gives rise to the formation of gravitationally bound states is suggested by AdS/CFT correspondence. The problem is that the string tension defined by gravitational constant is so large that only Planck length sized bound states are feasible. Even the replacement $\hbar \rightarrow \hbar_{eff}$ fails to allow gravitationally bound states with length scale of order Schwartschild radius. For the K-D option the string tension behaves like $1/\hbar^2$ and there are no problems in this respect.

At this moment my feeling is that the first option - essentially the original view - is the correct one. The short belief that the second option is the correct choice was a sidetrack, which however helped to become convinced that the original vision is indeed correct, and to understand the general mechanism for the formation of bound states in terms of strings connection partonic 2-surfaces (in the earlier picture I talked about magnetic flux tubes carrying monopole flux: the views are equivalent).

Both options have the following nice features.

1. Induced gammas at the light-like string boundaries would be light-like. Massless Dirac equation would assign to spinors at these lines a light-like space-time four-momentum and twistorialize it. This four-momentum would be essentially the tangent vector of the light-like curve and would not have a constant direction. Light-likeness for it means light-likeness in 8-D sense since light-like curves in H correspond to non-like momenta in M^4. Both M^4 mass squared and CP_2 mass would be conserved. Even four-momentum could be conserved if M^4 projection of stringy curve is geodesic line of M^4.

2. A new connection with Equivalence Principle (EP) would emerge. One could interpret the induced four-momentum as gravitational four-momentum which would be massless in 4-D sense and correspond to inertial 8-momentum. EP wold state in the weakest form that only the M^4 masses associated with the two momenta are identical. Stronger condition would be that the Minkowski parts of the two momenta co-incide at the ends of fermion lines at partonic 2-surfaces. Even stronger condition is that the 8-momentum is 8-momentum is conserved along fermion line. This is certainly consistent with the ordinary view about Feynman graphs. This is guaranteed if the light-like curve is light-like geodesic of embedding space.

The induction of spinor fields has also remained somewhat imprecise notion. It how seems that quantum-classical correspondence forces a unique picture.

1. Does the induced spinor field co-incide with embedding space spinor harmonic in some domain? This domain would certainly include the ends of fermionic lines at partonic 2-surfaces associated with the incoming particles and vertices. Could it include also the boundaries of string world sheets and perhaps also the string world sheets? The Kähler-Dirac equation certainly does not allow this for entire space-time surface.

2. Strong form of holography suggest that the light-like momenta for the Dirac equation at the ends of light-like string boundaries has interpretation as 8-D light-like momentum has M^4 projection equal to that of H spinor-harmonic. The mass squared of M^4 momentum would be

same as the CP_2 momentum squared in both senses. Unless the gravitational four-momentum assignable to the induced Dirac operato r is conserved one cannot pose stronger condition.

3. If the induced spinor mode equals to embedding space-spinor mode also at fermion line, the light like momentum is conserved. The fermion line would be also light-like geodesic of the embedding space so that twistor polygons would have very concrete interpretation. This condition would be clearly analogous to the conditions in Witten's twistor string theory. A stronger condition would be that the mode of the embedding space spinor field co-incides with induced spinor field at the string world sheet.
4. p-Adic mass calculations require that the massive excitations of embedding space spinor fields with CP_2 mass scale are involved. The thermodynamics could be for fermion line, wormhole throat carrying possible several fermions, or wormhole contact carrying fermion at both throats. In the case of fermions physical intuition suggests that p-adic thermodynamics must be associated with single fermionic line. The massive excitations would correspond to light-like geodesics of the embedding space.

To minimize confusion I must confess that until recently I have considered a different options for the momenta associated with fermionic lines.

1. The action of the Kähler-Dirac operator on the induced spinor field at the fermionic line equals to that of 4-D Dirac operator $p^k\gamma_k$ for a massless momentum identified as M^4 momentum [K23].
   Now the action reduces to that of 8-D massless algebraic Dirac operator for light-like string boundaries in the case of induced gamma matrices. Explicit calculation shows that in case of K-D gamma matrices and for light-like string boundaries it can happen that the 8-momentum of the mode can be tachyonic. Intriguingly, p-adic mass calculations require a tachyonic ground state?
2. For this option the helicities for virtual fermions were assumed to be non-physical in order to get non-vanishing fermion lines by residue integration: momentum integration for Dirac operator would replace Dirac propagators with Dirac operators. This would be the counterpart for the disappearance of bosonic propagators in residue integration.
3. This option has problems: quantum classical correspondence is not realized satisfactorily and the interpretation of p-adic thermodynamics is problematic.

2.4.5 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [L10] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

The case of $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K91] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?
2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.
4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

   This is not the only possibility that I have by-passingly considered [K23].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?
2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.
3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.
2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.
   (a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

(b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can coincide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

(c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

2.4.6 How To Generalize Witten's Twistor String Theory To TGD Framework?

The challenge is to lift the geometric description of particle like aspects of twistorial amplitudes involving only algebraic curves (2-surfaces) in twistor space to TGD framework.

1. External particles correspond to the lifts of H-spinor harmonics to spinor harmonics in the twistor space and are labeled by four-momentum, helicity, color, and weak helicity (isospin) so that there should be no need to included fermions explicitly. The twistorial wave functions would be superpositions of the eigenstates of helicity operator which would become a non-local property in twistor space. Light-likeness would hold true in 8-D sense for spinor harmonics as well as for the corresponding twistorial harmonics.

2. The surfaces X^2 in Witten's theory would be replaced with the lifts of partonic 2-surfaces X^2 to 4-D surfaces with bundle structure with X^2 as base and S^2 as fiber. S^2 would be non-dynamical. Whether X^2 or its lift to 4-surface allows identification as algebraic surface is not quite clear but it seems that X^2 could be the relevant object identifiable as surface of the base space of $T(X^4)$. If X^2 is the basic object one would have the additional constraint (not present in Witten's theory) that it belongs to the base space X^4. The genus of the lift of X^2 would be that of its base space X^2. One obtains a union of partonic 2-surfaces rather than single surface and lines connecting them as boundaries of string world sheets.

   The n points of given X^2 would correspond to the ends of boundaries of string world sheets at the partonic 2-surface X^2 carrying fermion number so that the helicities of twistorial spinor modes would be highly fixed unless M^4 chiralities mix making fermions massive in M^4 sense. This picture is in accordance with the fact that the lines of fundamental fermions should correspond to QFT limit of TGD.

3. In TGD genus g of the partonic 2-surface labels various fermion families and $g < 3$ holds true for physical fermions. The explanation could be that Z^2 acts as global conformal symmetry (hyper-ellipticity) for $g < 3$ surfaces irrespective of their conformal moduli but for $g > 3$ only in for special moduli. Physically for $g > 2$ the additional handles would make the partonic 2-surface to behave like many-particle state of free particles defined by the handles.

   This assumption suggests that assigns to the partonic surface what I have called modular invariant elementary particle vacuum functional (EVPF) in modular degrees of freedom such that for a particle characterized by genus g one has $l \geq g$ and $l > g$ amplitudes are possible because the EPVF leaks partially to higher genera [K21]. This would also induce a mixing of boundary topologies explaining CKM mixing and its leptonic counterpart. In this framework it would be perhaps more appropriate to define the number of loops as $l_1 = l - g$.

   A more precise picture is as follows. Elementary particles have actually four wormhole throats corresponding to the two wormhole contacts. In the case of fermions the wormhole throat carrying the electroweak quantum numbers would have minimum value g of genus characterized by the fermion family. Furthermore, the universality of the standard model physics requires that the couplings of elementary fermions to gauge bosons do not depend on genus. This is the case if one has quantum superposition of the wormhole contacts carrying the quantum numbers of observed gauge bosons at their opposite throats over the three lowest genera $g = 0, 1, 2$ with identical coefficients. This meas SU(3) singlets for the dynamical SU(3) associated with genus degeneracy. Also their exotic variants - say octets - are in principle possible.

4. This description is not complete although already twistor string description involves integration over the conformal moduli of the partonic 2-surface. One must integrate over the "world of classical worlds" (WCW) -roughly over the generalized Feynman diagrams and their complements consisting of Minkowskian and Euclidian regions. TGD as almost topological QFT reduces this integration to that of the boundaries of space-time regions.

   By quaternion conformal invariance [L10] this functional integral could reduce to ordinary integration over the quaternionic-conformal moduli space of space-time surfaces for which the moduli space of partonic 2-surfaces should be contained (note that strong form of holography suggests that only the modular invariants associated with the tangent space data should enter the description). One might hope that twistor space approach allows an elegant description of the moduli assignable to the tangent space data.

2.4.7 Yangian Symmetry

One of the victories of the twistor Grassmannian approach is the discovery of Yangian symmetry [A18], [B27, B36], [L10], whose variant associated with extended super-conformal symmetries is expected to be in key role in TGD.

1. The very nature of the residue integral implies that the complex surface serving as the locus of the integrand of the twistor amplitude is highly non-unique. Indeed, the Yangian symmetry [L10] acting as multi-local symmetry and implying dual of ordinary conformal invariance acting on momentum twistors, has been found to reduce to diffeomorphisms of $G(k,n)$ respecting positivity property of the decomposition of $G(k,n)$ to polyhedrons. It is quite possible that this symmetry corresponds to quaternion conformal symmetries in TGD framework.

2. Positivity of a given regions means parameterization by non-negative coordinates in TGD framework a possible interpretation is based on the observation that canonical identification mapping reals to p-adic number and vice versa is well-defined only for non-negative real numbers. Number theoretical Universality of spinor amplitudes so that they make sense in all number fields, would therefore be implied.

3. Could the crucial Yangian invariance generalizing the extended conformal invariance of TGD could reduce to the diffeomorphisms of the extended twistor space $T(H)$ respecting positivity. In the case of CP_2 all coordinates could be regarded as angle coordinates and be replaced by phase factors coding for the angles which do not make sense p-adically. The number theoretical existence of phase factors in p-adic case is guaranteed if they belong to some algebraic extension of rationals and thus also p-adics containing these phases as roots of unity. This implies discretization of CP_2.

   ZEO allows to reduce the consideration to causal diamond CD defined as an intersection of future and past directed light-cones and having two light-like boundaries. CD is indeed a natural counterpart for S^4. One could use as coordinates light-cone proper time a invariant under Lorentz transformations of either boundary of CD, hyperbolic angle η and two spherical angles (θ,ϕ). The angle variables allow representation in terms of finite algebraic extension. The coordinate a is naturally non-negative and would correspond to positivity. The diffeomorphisms perhaps realizing Yangian symmetry would respect causality in the sense that they do not lead outside CD.

   Quaternionic conformal symmetries the boundaries of $CD\times CP_2$ continued to the interior by translation of the light-cones serve as a good candidates for the diffeomorphisms in question since they do not change the value of the Minkowski time coordinate associated with the line connecting the tips of CD.

2.4.8 Does BCFW Recursion Have Counterpart In TGD?

Could BCFW recursion for tree diagrams and its generalization to diagrams with loops have a generalization in TGD framework? Could the possible TGD counterpart of BCFW recursion have a representation at the level of the TGD twistor space allowing interpretation in terms of geometry of partonic 2-surfaces and associated string world sheets? Supersymmetry is essential ingredient in obtaining this formula and the proposed SUSY realized at the level of amplitudes and broken at the level of states gives hopes for it. One could however worry about the fact that spinors are Dirac spinors in TGD framework and that Majorana property might be essential element.

How to produce Yangian invariants

Nima Arkani-Hamed *et al* [B36] (`http://tinyurl.com/y97rlzqb`) describe in detail various ways to form Yangian invariants defining the singular parts of the integrands of the amplitudes allowing to construct the full amplitudes. The following is only a rough sketch about what is involved using particle picture and I cannot claim of having understood the details.

1. One can *add* particle ($(k,n) \rightarrow (k+1,n+1)$) to the amplitude by deforming the momentum twistors of two neighboring particles in a way depending on the momentum twistor of the added particle. One inserts the new particle between n-1:th and 1st particle, modifies their momentum twistors without changing their four-momenta, and multiplying the resulting amplitude by a twistor invariant known as $[n-2, n-1, n, 1, 2]$ so that there is dependence on the added n:th momentum twistor.
2. One can *remove* particle ($(k,n) \rightarrow (k-1,n-1)$) by contour integrating over the momentum twistor variable of one particle.
3. One can *fuse* invariants simply by multiplying them.
4. One can *merge* invariants by identifying momentum twistors appearing in the two invariants. The integration over the common twistor leads to an elimination of particle.
5. One can form a *BCFW bridge* between n_1+1-particle invariant and n_2+1-particle invariant to get $n = n_1 + n_2$-particle invariant using the operations listed. One starts with the *fusion* giving the product $I_1(1, ..., n_1, I)I_2(n_1+1, ..n, I)$ of Yangian invariants followed by *addition* of n_1+1 to I_1 between n_1 and I and 1 to I_2 between I and n_1+1 (see the first item for details). After that follows the *merging* of lines labelled by I next to n_1 in I_1 and the precedessor of n_1+1 in I_2 reducing particle number by one unit and followed by residue integration over Z_I reducing particle number further by one unit so that the resulting amplitude is n-particle amplitude.
6. One can perform *entangled removal* of two particles. One could remove them one-by-one by independent contour integrations but one can also perform the contour integrations in such a way that one first integrates over two twistors at the same complex line and then over the lines: this operation adds to n-particle amplitude loop.

BCFW recursion formula

BCFW recursion formula allows to express n-particle amplitudes with l loops in terms of amplitudes with amplitudes having at most $l-1$ loops. The basic philosophy is that singularities serve as data allowing to deduce the full integrands of the amplitudes by generalized unitarity and other kinds of arguments.

Consider first the arguments behind the BCFW formula.

1. BCFW formula is derived by performing the canonical momentum twistor deformation $Z_n \rightarrow z_n + zZ_{n-1}$, multiplying by $1/z$ and performing integration along small curve around origin so that one obtains original amplitude from the residue inside the curve. One obtains also and alternative of the residue integral expression as sum of residues from its complement. The singularities emerge by residue integral from poles of scattering amplitudes and eliminate two lines so that the recursion formula for n-particle amplitude can involve at most $n+2$-particle amplitudes.

   It seems that one must combine all n-particle amplitudes to form a single entity defining the full amplitude. I do not quite understand what how this is done. In ZEO zero energy state involving different particle numbers for the final state and expressible in terms of S-matrix (actually its generalization to what I call M-matrix) might allow to understand this.
2. In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L + n_R = n + 2$, $k_L + k_R = k - 1$, and $l_L + l_R = l$.

3. The singularities are easy to understand in the case of tree amplitudes: they emerge from the poles of the conformally invariant quantities in the denominators of amplitudes. Physically this means that the sum of the momenta for a subset of particles corresponds to a complex pole (BCFW deformation makes two neighboring momenta complex). Hence one obtains sum over products of $j+1$-particle amplitudes BCFW bridged with $n-j$-particle amplitude to give n-particle amplitude by the merging process.

4. This is not all that is needed since the diagrams could be reduced to products of 1 loop 3-particle amplitudes which vanish by the triviality of coupling constant evolution in $\mathcal{N}=4$ SUSY. Loop amplitudes serving as a kind of source in the recursion relation save the situation. There is indeed also a second set of poles coming from loop amplitudes.

   One-loop case is the simplest one. One begins from $n+2$ particle amplitude with $l-1$ loops. At momentum space level the momenta the neighboring particles have opposite light-like momenta: one of the particles is not scattered at all. This is called forward limit. This limit suffers from collinear divergences in a generic gauge theory but in supersymmetric theories the limit is well-defined. This forward limit defines also a Yangian invariant at the level of twistor space. It can be regarded as being obtained by entangled removal of two particles combined with merge operation of two additional particles. This operation leads from $(n+2, l-1)$ amplitude to (n, l) amplitude.

Does BCFW formula make sense in TGD framework?

In TGD framework the four-fermion amplitude but restricted so that two outgoing particles have (in general) complex massless 8-momenta is the basic building brick. This changes the character of BCFW recursion relations although the four-fermion vertex effectively reduces to $F\overline{F}B$ vertex with boson identified as wormhole contact carrying fermion and antifermion at its throats.

The fundamental 4-fermion vertices assignable to wormhole contact could be formally expressed in terms of the product of two $F\overline{F}B_v$ vertices (MHV expression), where B_v is purely formal gauge boson, using the analog of MHV expression and taking into account that the second $F\overline{F}$ pair is associated with wormhole contact analogous to exchanged gauge boson.

If the fermions at fermion lines of the same partonic 2-surface can be assumed to be collinear and thus to form single coherent particle like unit, the description as superspace amplitude seems appropriate. Consequently, the effective $F\overline{F}B_v$ vertices could be assumed to have supersymmetry defined by the fermionic oscillator operator algebra at the partonic 2-surface (Clifford algebra). A good approximation is to restrict this algebra to that generating various spinor components of embedding space spinors so that $\mathcal{N}=4$ SUSY is obtained in leptonic and quark sector. Together these give rise to $\mathcal{N}=8$ SUSY at the level of vertices broken however at the level of states.

Side note: The number of external fermions is always even suggesting that the superconformal anomalies plaguing the SUSY amplitudes with odd n (`http://tinyurl.com/yb85tnvc`) [B58] are absent in TGD: this would be basically due to the decomposition of gauge bosons to fermion pairs.

The leading singularities of scattering amplitudes would naturally correspond to the boundaries of the moduli space for the unions of partonic 2-surfaces and string world sheets.

1. The tree contribution to the gauge boson scattering amplitudes with $k=0,1$ vanish as found by Parke and Taylor who also found the simple twistorial form for the $k=2$ case (`http://tinyurl.com/y7nas26b`). In TGD framework, where lowest amplitude is 4-fermion amplitude, this situation is not encountered. According to Wikipedia article the so called CSW rules inspired by Witten's twistor theory have a problem due to the vanishing of $++-$ vertex which is not MHV form unless one changes the definition of what it is to be "wrong helicity". $++-$ is needed to construct $++++$ amplitude at one loop which does not vanish in YM theory. In SUSY it however vanishes.

   In TGD framework one does not encounter these problems since 4-fermion amplitudes are the basic building bricks. Fermion number conservation and the assumption that helicities do not mix (light-likeness in M^4 rather than only M^8-sense) implies $k=2(n(F)-n(\overline{F})$.

   In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L+n_R=n+2$, $k_L+k_R=k-1$. If the TGD counterpart of the bridge eliminates two antifermions with the same "wrong" helicity -1/2, and one indeed has $k_L+k_R=k-1$

if fermions have well-defined M^4 helicity rather than being in superposition in completely correlated M^4 and CP_2 helicities.

2. In string theory loops correspond to handles of a string world sheet. Now one has partonic 2-surfaces and string world sheets and both can in principle have handles. The condition $l \geq g$ of Witten's theory suggests that $l - g$ defines the handle number for string world sheet so that l is the total number of handles.

   The identification of loop number as the genus of partonic 2-surface is second alternative: one would have $l = g$ and string world sheets would not contain handles. This might be forced by the Minkowskian signature of the induced metric at string world sheet. The signature of the induced metric would be presumably Euclidian in some region of string world sheet since the M^4 projection of either homology generator assignable with the handle would presumably define time loop in M^4 since the derivative of M^4 time coordinate with respect to string world sheet time should vanish at the turning points for M^4 time. Minimal surface property might eliminate Euclidian regions of the string world sheet. In any case, the area of string world sheet would become complex.

3. In the moduli space of partonic 2-surfaces first kind of leading singularities could correspond to pinches formed as n partonic 2-surfaces decomposes to two 2-surfaces having at least single common point so that moduli space factors into a Cartesian product. This kind of singularities could serve as counterparts for the merge singularities appearing in the BCFW bridging of amplitudes. The numbers of loops must be additive and this is consistent with both interpretations for l.

4. What about forward limit? One particle should go through without scattering and is eliminated by entangled removal. In ZEO one can ask whether there is also quantum entanglement between the positive and negative energy parts of this single particle state and state function reduction does not occur. The addition of particle and merging it with another one could correspond to a situation in which two points of partonic 2-surface touch. This means addition of one handle so that loop number l increases.

   It seems that analytically the loop is added by the entangled removal but at the level of partonic surface it is added by the merging. Also now both $l > g$ and $l = g$ options make sense.

2.4.9 Possible Connections Of TGD Approach With The Twistor Grassmannian Approach

For a non-specialist lacking the technical skills, the work related to twistors is a garden of mysteries and there are a lot of questions to be answered: most of them of course trivial for the specialist. The basic questions are following.

How the twistor string approach of Witten and its possible TGD generalization relate to the approach involving residue integration over projective sub-manifolds of Grassmannians $G(k, n)$?

1. In [B37] Nima *et al* argue that one can transform Grassmannian representation to twistor string representation for tree amplitudes. The integration over $G(k, n)$ translates to integration over the moduli space of complex curves of degree $d = k - 1 + l$, $l \geq g$ is the number of loops. The moduli correspond to complex coefficients of the polynomial of degree d and they form naturally a projective space since an overall scaling of coefficients does not change the surfaces. One can expect also in the general case that moduli space of the partonic 2-surfaces can be represented as a projective sub-manifold of some projective space. Loop corrections would correspond to the inclusion of higher degree surfaces.

2. This connection gives hopes for understanding the integration contours in $G(k, n)$ at deeper level in terms of the moduli spaces of partonic 2-surfaces possibly restricted by conformal gauge conditions.

Below I try to understand and relate the work of Nima Arkani Hamed *et al* with twistor Grassmannian approach to TGD.

The notion of positive Grassmannian

The notion of positive Grassmannian is one of the central notions introduced by Nima et al.

1. The claim is that the sub-spaces of the real Grassmannian $G(k,n)$ contributing to the amplitudes for $++--$ signature are such that the determinants of the $k\times k$ minors associated with ordered columns of the $k\times n$ matrix C representing point of $G(k,n)$ are positive. To be precise, the signs of all minors are positive or negative simultaneously: only the ratios of the determinants defining projective invariants are positive.
2. At the boundaries of positive regions some of the determinants vanish. Some k-volumes degenerate to a lower-dimensional volume. Boundaries are responsible for the leading singularities of the scattering amplitudes and the integration measure associated with $G(k,n)$ has a logarithmic singularity at the boundaries. These boundaries would naturally correspond to the boundaries of the moduli space for the partonic 2-surfaces. Here also string world sheets could contribute to singularities.
3. This condition has a partial generalization to the complex case: the determinants whose ratios serve as projectively invariant coordinates are non-vanishing. A possible further manner to generalize this condition would be that the determinants have positive real part so that apart from rotation by $\pi/2$ they would reside in the upper half plane of complex plane. Upper half plane is the hyperbolic space playing key role in complex analysis and in the theory of hyperbolic 2-manifolds for which it serves as universal covering space by a finite discrete subgroup of Lorentz group $SL(2,C)$. The upper half-plane having a deep meaning in the theory of Riemann surfaces might play also a key role in the moduli spaces of partonic 2-surfaces. The projective space would be based - not on projectivization of C^n but that of H^n, H the upper half plane.

   Could positivity have some even deeper meaning?

1. In TGD framework the number theoretical universality of amplitudes suggests this. Canonical identification maps $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ p-adic number to non-negative reals. p-Adicization is possible for angle variables by replacing them by discrete phases, which are roots of unity. For non-angle like variables, which are non-negative one uses some variant of canonical identification involving cutoffs [K107]. The positivity should hold true for all structures involved, the $G(2,n)$ points defined by the twistors characterizing momenta and helicities of particles (actually pairs of orthogonal planes defined by twistors and their conjugates), the moduli space of partonic 2-surfaces, etc...
2. p-Adicization requires discretization of phases replacing angles so that they come as roots of unity associated with the algebraic extension used. The p-adic valued counterpart of Riemann or Lebesque integral does not make sense p-adically. Residue integrals can however allow to define p-adic integrals by analytic continuation of the integral and discretization of the phase factor along the integration contour does not matter (not however the p-adically troublesome factor 2π!).
3. TGD suggests that the generalization of positive real projectively invariant coordinates to complex coordinates of the hyperbolic space representable as upper half plane, or equivalently as unit disk obtained from the upper half plane by exponential mapping $w = exp(iz)$: positive coordinate α would correspond to the radial coordinate for the unit disk (Poincare hyperbolic disk appearing in Escher's paintings). The real measure $d\alpha/\alpha$ would correspond to $dz = dw/w$ restricted to a radial line from origin to the boundary of the unit disk. This integral should correspond to integral over a closed contour in complex case. This is the case if the integrand is discontinuity over a radial cut and equivalent with an integral over curve including also the boundary of the unit disk. This integral would reduce to the sum of the residues of poles inside the unit disk.

The notion of amplituhedron

The notion of amplituhedron is the latest step of progress in the twistor Grassmann approach [B15, B14]. What is so remarkable, is the simplicity of the expressions for all-loop amplitudes and the fact that positivity implies locality and unitarity for $\mathcal{N} = 4$ SUSY.

Consider first tree amplitudes with general value of k.

1. The notion of amplituhedron relies on the mapping of $G(k,n)$ to $G_+(k,k+m)$ $n \geq k+m$. $G_+(k,k+m)$ is positive Grassmannian characterized by the condition that all $k\times k$- minors $k\times(k+m)$ matrix representing the point of $G_+(k,k+m)$ are non-negative and vanish at the boundaries $G_+(k,k+m)$. The value of m is $m=4$ and follows from the conditions that amplitudes come out correctly. The constraint $Y = C\cdot Z$, where Y corresponds to point of $G_+(k,k+4)$ and Z to the point of $G(k,n)$ performs this mapping, which is clearly many-to one. One can decompose $G_+(k,k+4)$ to positive regions intersecting only along their common boundary portions. The decomposition of a convex polygon in plane represent the basic example of this kind of decomposition.
2. Each decomposition defines a sum of contributions to the scattering amplitudes involving integration of a projectively invariant volume form over the positive region in question. The form has a logarithmic singularity at the boundaries of the integration region but spurious singularities cancel so that only the contribution of the genuine boundary of $G_+(k,k+4)$ remains. There are additional delta function constraints fixing the integral completely in real case.
3. In complex case one has residue integral. The proposed generalization to the complex case is by analytic continuation. TGD inspired proposal is that the positivity condition in the real case is generalized to the condition that the positive coordinates are replaced by complex coordinates of hyperbolic space representable as upper half plane or equivalently as the unit disk obtained from upper half plane by exponential mapping $w = exp(iz)$. The measure $d\alpha/\alpha$ would correspond to $dz = dw/w$. If taken over boundary circle labelled by discrete phase factors $exp(i\phi)$ given by roots of unity the integral would be numerically a discrete Riemann sum making no sense p-adically but residue theorem could allow to avoid the discretizaton and to define the p-adic variant of the integral by analytic continuation. These conditions would be completely general conditions on various projectively invariant moduli involved.
4. One must extend the bosonic twistors Z_a of external particles by adding k coordinates. Somewhat surprisingly, these coordinates are anticommutative super-coordinates expressible as linear combinations of fermionic parts of super-twistor using coefficients, which are also Grassmann numbers. Integrating over these one ends up with the standard expression of the amplitude using canonical integration measure for the regions in the decomposition of amplituhedron.

   What looks to me intriguing is that there is only super-integration involved over the additional k degrees of freedom. In Witten's approach $k-1$ corresponds to the minimum degree of the polynomial defining the string world sheet representing tree diagram. In TGD framework $k+1$ (rather than $k-1$) could correspond to the minimum degree of partonic 2-surface. In TGD approximate SUSY would correspond to Grassmann algebra of fermionic oscillator operators defined by the spinor basis for embedding space spinors. The interpretation could be that each fermion whose helicity differs from that allowed by light-likeness in M^4 sense (this requires non-vanishing M^4 mass), contributes $\Delta k = 1$ to the degree of corresponding partonic 2-surface. Since the partonic 2-surface is common for all particles, one must have $d = k+1$ at least. The k-fold super integration would be basically integral over the moduli characterizing the polynomials of degree k realizing quantum classical correspondence in fermionic degrees of freedom.

BFCW recursion formula involves also loop amplitudes for which amplituhedron provides also a very nice representation.

1. The basic operation is the addition of a loop to get (n,k,l) amplitude from $(n+2,k,l-1)$ amplitude. That 2 particles must be removed for each loop is not obvious in $\mathcal{N}=4$ SUSY but follows from the condition that positivity of the integration domain is preserved. This procedure removes from $(n+2,k,l-1)$-amplitude 2 particles with opposite four-momenta so that (n,k,l) amplitude is obtained. In the case of L-loops one extends $G(k,n)$ by adding its "complement" as a Cartesian factor $G(n-k,n)$ and imbeds to $G(n-k,n)$ 2-plane for each loop. Positivity conditions can be generalized so that they apply to $(k+2l)\times(k+2l)$-minors associated with matrices having as rows $0 \leq l \leq L$ ordered D_{i_k}:s and of C. The general

expressions of the loop contributions are of the same form as for tree contributions: only the number of integration variables is $4 \times (k + L)$.

2. As already explained, in TGD framework the addition of loop would correspond to the formation of a handle to the partonic surface by fusing two points of partonic 2-surface and thus creating a surface intermediate between topologies with g and g+1 handles. g would correspond to the genus characterizing fermion family and one would have $L \geq g$. In elementary particle wave functionals loop [K21] contributions would correspond to higher genus contributions $l_1 = l - g > 0$ with basic contribution coming from genus g. For scattering amplitudes loop contributions would involve the change of the genus of the incoming wormhole throat so that they correspond to singular surfaces at the boundaries of their moduli space identifiable as loop corrections. $l_1 = l - g > 0$ would represent the number of pinches of the partonic 2-surface.

What about non-planar amplitudes?

Non-planar Feynman diagrams have remained a challenge for the twistor approach. The problem is simple: there is no canonical ordering of the extrenal particles and the loop integrand involving tricky shifts in integrations to get finite outcome is not unique and well-defined so that twistor Grasmann approach encounters difficulties.

Recently Nima Arkani-Hamed *et al* have considered also non-planar MHV diagrams [B39] (having minimal number of "wrong" helicities) of N=4 SUSY, and shown that they can be reduced to non-planar diagrams for different permutations of vertices of planar diagrams ordered naturally. There are several integration regions identified as positive Grassmannians corresponding to different orderings of the external lines inducing non-planarity. This does not however hold true generally.

At the QFT limit the crossings of lines emerges purely combinatorially since Feynman diagrams are purely combinatorial objects with the ordering of vertices determining the topological properties of the diagram. Non-planar diagrams correspond to diagrams, which do not allow crossing-free embedding to plane but require higher genus surface to get rid of crossings.

1. The number of the vertices of the diagram and identification of lines connecting them determines the diagram as a graph. This defines also in TGD framework Feynman diagram like structure as a graph for the fermion lines and should be behind non-planarity in QFT sense.
2. Could 2-D Feynman graphs exists also at geometric rather than only combinatorial level? Octonionization at embedding space level requires identification of preferred $M^2 \subset M^4$ defining a preferred hyper-complex sub-space. Could the projection of the Fermion lines defined concrete geometric representation of Feynman diagrams?
3. Despite their purely combinatorial character Feynman diagrams are analogous to knots and braids. For years ago [K46] I proposed the generalization of the construction of knot invariants in which one gradually eliminates the crossings of the knot projection to end up with a trivial knot is highly suggestive as a procedure for constructing the amplitudes associated with the non-planar diagrams. The outcome should be a collection of planar diagrams calculable using twistor Grassmannian methods. Scattering amplitudes could be seen as analogs of knot invariants. The reduction of MHV diagrams to planar diagrams could be an example of this procedure.

One can imagine also analogs of non-planarity, which are geometric and topological rather than combinatorial and not visible at the QFT limit of TGD.

1. The fermion lines representing boundaries of string world sheets at the light-like orbits of partonic 2-surfaces can get braided. The same can happen also for the string boundaries at space-like 3-surfaces at the ends of the space-time surface. The projections of these braids to partonic 2-surfaces are analogs of non-planar diagrams. If the fermion lines at single wormhole throat are regarded effectively as a line representing one member of super-multiplet, this kind of braiding remains below the resolution used and cannot correspond to the braiding at QFT limit.
2. 2-knotting and 2-braiding are possible for partonic 2-surfaces and string world sheets as 2-surfaces in 4-D space-time surfaces and have no counterpart at QFT limit.

2.4.10 Permutations, Braidings, And Amplitudes

In [B33] Nima Arkani-Hamed demonstrates that various twistorially represented on-mass-shell amplitudes (allowing light-like complex momenta) constructible by taking products of the 3-particle amplitude and its conjugate can be assigned with unique permutations of the incoming lines. The article describes the graphical representation of the amplitudes and its generalization. For 3-particle amplitudes, which correspond to $++-$ and $+--$ twistor amplitudes, the corresponding permutations are cyclic permutations, which are inverses of each other. One actually introduces double cover for the labels of the particles and speaks of decorated permutations meaning that permutation is always a right shift in the integer and in the range $[1, 2 \times n]$.

Amplitudes as representation of permutations

It is shown that for on mass shell twistor amplitudes the definition using on-mass-shell 3-vertices as building bricks is highly reducible: there are two moves for squares defining 4-particle sub-amplitudes allowing to reduce the graph to a simpler one. The first ove is topologically like the s-t duality of the old-fashioned string models and second one corresponds to the transformation black $\leftrightarrow$ white for a square sub-diagram with lines of same color at the ends of the two diagonals and built from 3-vertices.

One can define the permutation characterizing the general on mass shell amplitude by a simple rule. Start from an external particle a and go through the graph turning in in white (black) vertex to left (right). Eventually this leads to a vertex containing an external particle and identified as the image $P(a)$ of the a in the permutation. If permutations are taken as right shifts, one ends up with double covering of permutation group with $2 \times n!$ elements - decorated permutations. In this manner one can assign to any any line of the diagram two lines. This brings in mind 2-D integrable theories where scattering reduces to braiding and also topological QFTs where braiding defines the unitary S-matrix. In TGD parton lines involve braidings of the fermion lines so that an assignment of permutation also to vertex would be rather nice.

BCFW bridge has an interpretation as a transposition of two neighboring vertices affecting the lines of the permutation defining the diagram. One can construct all permutations as products of transpositions and therefore by building BCFW bridges. BCFW bridge can be constructed also between disjoint diagrams as done in the BCFW recursion formula.

Can one generalize this picture in TGD framework? There are several questions to be answered.

(a) What should one assume about the states at the light-like boundaries of string world sheets? What is the precise meaning of the supersymmetry: is it dynamical or gauge symmetry or both?

(b) What does one mean with particle: partonic 2-surface or boundary line of string world sheet? How the fundamental vertices are identified: 4 incoming boundaries of string world sheets or 3 incoming partonic orbits or are both aspects involved?

(c) How the 8-D generalization of twistors bringing in second helicity and doubling the M^4 helicity states assignable to fermions does affect the situation?

(d) Does the crucial right-left rule relying heavily on the possibility of only 2 3-particle vertices generalize? Does M^4 massivation imply more than 2 3-particle vertices implying many-to-one correspondence between on-mass-shell diagrams and permutations? Or should one generalize the right-left rule in TGD framework?

Fermion lines for fermions massless in 8-D sense

What does one mean with particle line at the level of fermions?

(a) How the addition of CP_2 helicity and complete correlation between M^4 and CP_2 chiralities does affect the rules of $\mathcal{N} = 4$ SUSY? Chiral invariance in 8-D sense guarantees fermion number conservation for quarks and leptons separately and means conservation of the product of M^4 and CP_2 chiralities for 2-fermion vertices. Hence only M^4 chirality need to be considered. M^4 massivation allows more 4-fermion vertices than $\mathcal{N} = 4$ SUSY.

(b) One can assign to a given partonic orbit several lines as boundaries of string world sheets connecting the orbit to other partonic orbits. Supersymmetry could be understoond in two ways.

   i. The fermions generating the state of super-multiplet correspond to boundaries of different string world sheets which need not connect the string world sheet to same partonic orbit. This SUSY is dynamical and broken. The breaking is mildest breaking for line groups connected by string world sheets to same partonic orbit. Right handed neutrinos generated the least broken $\mathcal{N} = 2$ SUSY.

   ii. Also single line carrying several fermions would provide realization of generalized SUSY since the multi-fermion state would be characterized by single 8-momentum and helicity. One would have $\mathcal{N} = 4$ SUSY for quarks and leptons separately and $\mathcal{N} = 8$ if both quarks and leptons are allowed. Conserved total for quark and antiquarks and leptons and antileptons characterize the lines as well.
   What would be the propagator associated with many-fermion line? The first guess is that it is just a tensor power of single fermion propagator applied to the tensor power of single fermion states at the end of the line. This gives power of $1/p^{2n}$ to the denominator, which suggests that residue integral in momentum space gives zero unless one as just single fermion state unless the vertices give compensating powers of p. The reduction of fermion number to 0 or 1 would simplify the diagrammatics enormously and one would have only 0 or 1 fermions per given string boundary line. Multi-fermion lines would represent gauge degrees of freedom and SUSY would be realized as gauge invariance. This view about SUSY clearly gives the simplest picture, which is also consistent with the earlier one, and will be assumed in the sequel

(c) The multiline containing n fermion oscillator operators can transform by chirality mixing in 2^n ways at 4-fermion vertex so that there is quite a large number of options for incoming lines with n_i fermions.

(d) In 4-D Dirac equation light-likeness implies a complete correlation between fermion number and chirality. In 8-D case light-likeness should imply the same: now chirality correspond to fermion number. Does this mean that one must assume just superposition of different M^4 chiralities at the fermion lines as 8-D Dirac equation requires. Or should one assume that virtual fermions at the end of the line have wrong chirality so that massless Dirac operator does not annihilate them?

Fundamental vertices

One can consider two candidates for fundamental vertices depending on whether one identifies the lines of Feynman diagram as fermion lines or as light-like orbits of partonic 2-surfaces. The latter vertices reduces microscopically to the fermionic 4-vertices.

(a) If many-fermion lines are identified as fundamental lines, 4-fermion vertex is the fundamental vertex assignable to single wormhole contact in the topological vertex defined by common partonic 2-surface at the ends of incoming light-like 3-surfaces. The discontinuity is what makes the vertex non-trivial.

(b) In the vertices generalization of OZI rule applies for many-fermion lines since there are no higher vertices at this level and interactions are mediated by classical induced gauge fields and chirality mixing. Classical induced gauge fields vanish if CP_2 projection is 1-dimensional for string world sheets and even gauge potentials vanish if the projection is to geodesic circle. Hence only the chirality mixing due to the mixing of M^4 and CP_2 gamma matrices is possible and changes the fermionic M^4 chiralities. This would dictate what vertices are possible.

(c) The possibility of two helicity states for fermions suggests that the number of amplitudes is considerably larger than in $\mathcal{N} = 4$ SUSY. One would have 5 independent fermion amplitudes and at each 4-fermion vertex one should be able to choose between 3 options if the right-left rule generalizes. Hence the number of amplitudes is larger than the number of permutations possibly obtained using a generalization of right-left rule to right-middle-left rule.

(d) Note however that for massless particles in M^4 sense the reduction of helicity combinations for the fermion and antifermion making virtual gauge boson happens. The fermion and antifermion at the opposite wormhole throats have parallel four-momenta in good approximation. In M^4 they would have opposite chiralities and opposite helicities so that the boson would be M^4 scalar. No vector bosons would be obtained in this manner.

In 8-D context it is possible to have also vector bosons since the M^4 chiralities can be same for fermion and anti-fermion. The bosons are however massive, and even photon is predicted to have small mass given by p-adic thermodynamics [K52]. Massivation brings in also the M^4 helicity 0 state. Only if zero helicity state is absent, the fundamental four-fermion vertex vanishes for $++++$ and $----$ combinations and one extend the right-left rule to right-middle-left rule. There is however no good reason for he reduction in the number of 4-fermion amplitudes to take place.

Partonic surfaces as 3-vertices

At space-time level one could identify vertices as partonic 2-surfaces.

(a) At space-time level the fundamental vertices are 3-particle vertices with particle identified as wormhole contact carrying many-fermion states at both wormhole throats. Each line of BCFW diagram would be doubled. This brings in mind the representation of permutations and leads to ask whether this representation could be re-interpreted in TGD framework. For this option the generalization of the decomposition of diagram to 3-particle vertices is very natural. If the states at throats consist of bound states of fermions as SUSY suggests, one could characterize them by total 8-momentum and helicity in good approximation. Both helicities would be however possible also for fermions by chirality mixing.

(b) A genuine decomposition to 3-vertices and lines connecting them takes place if two of the fermions reside at opposite throats of wormhole contact identified as fundamental gauge boson (physical elementary particles involve two wormhole contacts).

The 3-vertex can be seen as fundamental and 4-fermion vertex becomes its microscopic representation. Since the 3-vertices are at fermion level 4-vertices their number is greater than two and there is no hope about the generalization of right-left rule.

OZI rule implies correspondence between permutations and amplitudes

The realization of the permutation in the same manner as for $\mathcal{N}=4$ amplitudes does not work in TGD. OZI rule following from the absence of 4-fermion vertices however implies much simpler and physically quite a concrete manner to define the permutation for external fermion lines and also generalizes it to include braidings along partonic orbits.

(a) Already $\mathcal{N}=4$ approach assumes decorated permutations meaning that each external fermion has effectively two states corresponding to labels k and $k+n$ (permutations are shifts to the right). For decorated permutations the number of external states is effectively 2^n and the number of decorated permutations is $2\times n!$. The number of different helicity configurations in TGD framework is 2^n for incoming fermions at the vertex defined by the partonic 2-surface. By looking the values of these numbers for lowest integers one finds $2n \geq 2^n$: for $n=2$ the equation is saturated. The inequality $log(n!) > nlog(n)/e)+1$ (see `http://tinyurl.com/2bjk5h`). gives

$$\frac{log(2n!)}{log(2^n)} \geq \frac{log(2)+1+nlog(n/e)}{nlog(2)} = log(n/e)/log(2) + O(1/n)$$

so that the desired inequality holds for all interesting values of n.

(b) If OZI rule holds true, the permutation has very natural physical definition. One just follows the fermion line which must eventually end up to some external fermion since the only fermion vertex is 2-fermion vertex. The helicity flip would map $k \rightarrow k+n$ or vice versa.

(c) The labelling of diagrams by permutations generalizes to the case of diagrams involving partonic surfaces at the boundaries of causal diamond containing the external fermions and the partonic 2-surfaces in the interior of CD identified as vertices. Permutations generalize to braidings since also the braidings along the light-like partonic 2-surfaces are allowed. A quite concrete generalization of the analogs of braid diagrams in integrable 2-D theories emerges.

(d) BCFW bridge would be completely analogous to the fundamental braiding operation permuting two neighboring braid strands. The almost reduction to braid theory - apart from the presence of vertices conforms with the vision about reduction of TGD to almost topological QFT.

To sum up, the simplest option assumes SUSY as both gauge symmetry and broken dynamical symmetry. The gauge symmetry relates string boundaries with different fermion numbers and only fermion number 0 or 1 gives rise to a non-vanishing outcome in the residue integration and one obtains the picture used hitherto. If OZI rule applies, the decorated permutation symmetry generalizes to include braidings at the parton orbits and $k \rightarrow k \pm n$ corresponds to a helicity flip for a fermion going through the 4-vertex. OZI rules follows from the absence of non-linearities in Dirac action and means that 4-fermion vertices in the usual sense making theory non-renormalizable are absent. Theory is essentially free field theory in fermionic degrees of freedom and interactions in the sense of QFT are transformed to non-trivial topology of space-time surfaces.

3. If one can approximate space-time sheets by maps from M^4 to CP_2, one expects General Relativity and QFT description to be good approximations. GRT space-time is obtained by replacing space-time sheets with single sheet - a piece of slightly deformed Minkowski space but without assupmtion about embedding to H. Induced classical gravitational field and gauge fields are sums of those associated with the sheets. The generalized Feynman diagrams with lines at various sheets and going also between sheets are projected to single piece of M^4. Many-sheetedness makes 1-homology non-trivial and implies analog of braiding, which should be however invisible at QFT limit.

A concrete manner to eliminate line crossing in non-planar amplitude to get nearer to non-planar amplitude could proceed roughly as follows. This is of course a pure guess motivated only by topological considerations. Professional might kill it in few seconds.

1. If the lines carry no quantum numbers, reconnection allows to eliminate the crossings. Consider the crossing line pair connecting AB in the initial state to CD in final state. The crossing lines are AD and BC. Reconnection can take place in two ways: AD and BC transform either to AB and CD or to AC and BD: neither line pair has crossing. The final state of the braid would be quantum superposition of the resulting more planar braids.

2. The crossed lines however carry different quantum numbers in the generic situation: for instance, they can be fermionic and bosonic. In this particular case the reconnection does not make sense since a line carrying fermion number would transform to a line carrying boson.

   In TGD framework all lines are fermion lines at fundamental level but the constraint due to different quantum numbers still remains and it is easy to see that mere reconnection is not enough. Fermion number conservation allows only one of the two alternatives to be realized. Conservation of quantum numbers forces to restrict gives an additional constraint which for simplest non-planar diagram with two crossed fermion lines forces the quantum numbers of fermions to be identical.

   It seems also more natural to consider pairs of wormhole contacts defining elementary particles as "lines" in turn consisting of fermion lines. Yangian symmetry allows to develop a more detailed view about what this decomposition could mean.

   Quantum number conservation demands that reconnection is followed by a formation of an additional internal line connecting the non-crossing lines obtained by reconnection. The additional line representing a quantum number exchange between the resulting non-crossing lines would guarantee the conservation of quantum numbers. This would bring in two additional vertices and one additional internal line. This would be enough to reduce planarity. The repeated application of this transformation should produced a sum of non-planar diagrams.

3. What could go wrong with this proposal? In the case of gauge theory the order of diagram increases by g^2 since two new vertices are generated. Should a multiplication by $1/g^2$ accompany this process? Or is this observation enough to kill the hypothesis in gauge theory framework? In TGD framework the situation is not understood well enough to say anything. Certainly the critical value of α_K implies that one cannot regard it as a free parameter and cannot treat the contributions from various orders as independent ones.

2.5 Could The Universe Be Doing Yangian Arithmetics?

One of the old TGD inspired really crazy ideas about scattering amplitudes is that Universe is doing some sort of arithmetics so that scattering amplitude are representations for computational sequences of minimum length. The idea is so crazy that I have even given up its original form, which led to an attempt to assimilate the basic ideas about bi-algebras, quantum groups [K11], Yangians [L10], and related exotic things. The work with twistor Grassmannian approach inspired a reconsideration of the original idea seriously with the idea that super-symplectic Yangian could define the arithmetics. I try to describe the background, motivation, and the ensuing reckless speculations in the following.

2.5.1 Do Scattering Amplitudes Represent Quantal Algebraic Manipulations?

I seems that tensor product $\otimes$ and direct sum $\oplus$ - very much analogous to product and sum but defined between Hilbert spaces rather than numbers - are naturally associated with the basic vertices of TGD. I have written about this a highly speculative chapter - both mathematically and physically [K69]. The chapter [K11] is a remnant of earlier similar speculations.

1. In $\otimes$ vertex 3-surface splits to two 3-surfaces meaning that the 2 "incoming" 4-surfaces meet at single common 3-surface and become the outgoing 3-surface: 3 lines of Feynman diagram meeting at their ends. This has a lower-dimensional shadow realized for partonic 2-surfaces. This topological 3-particle vertex would be higher-D variant of 3-vertex for Feynman diagrams.
2. The second vertex is trouser vertex for strings generalized so that it applies to 3-surfaces. It does not represent particle decay as in string models but the branching of the particle wave function so that particle can be said to propagate along two different paths simultaneously. In double slit experiment this would occur for the photon space-time sheets.
3. The idea is that Universe is doing arithmetics of some kind in the sense that particle 3-vertex in the above topological sense represents either multiplication or its time-reversal co-multiplication.

The product, call it $\circ$, can be something very general, say algebraic operation assignable to some algebraic structure. The algebraic structure could be almost anything: a random list of structures popping into mind consists of group, Lie-algebra, super-conformal algebra quantum algebra, Yangian, etc.... The algebraic operation $\circ$ can be group multiplication, Lie-bracket, its generalization to super-algebra level, etc...). Tensor product and thus linear (Hilbert) spaces are involved always, and in product operation tensor product $\otimes$ is replaced with $\circ$.

1. The product $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is analogous to a particle reaction in which particles A_k and A_l fuse to particle $A_k \otimes A_l \rightarrow C = A_k \circ A_l$. One can say that $\otimes$ between reactants is transformed to $\circ$ in the particle reaction: kind of bound state is formed.
2. There are very many pairs A_k, A_l giving the same product C just as given integer can be divided in many ways to a product of two integers if it is not prime. This of course suggests that elementary particles are primes of the algebra if this notion is defined for it! One can use some basis for the algebra and in this basis one has $C = A_k \circ A_l = f_{klm} A_m$, f_{klm} are the structure constants of the algebra and satisfy constraints. For instance, associativity $A(BC) = (AB)C$ is a constraint making the life of algebraist more tolerable and is almost routinely assumed.

For instance, in the number theoretic approach to TGD associativity is proposed to serve as fundamental law of physics and allows to identify space-time surfaces as 4-surfaces with associative (quaternionic) tangent space or normal space at each point of octonionic embedding space $M^4 \times CP_2$. Lie algebras are not associative but Jacobi-identities following from the associativity of Lie group product replace associativity.

3. Co-product can be said to be time reversal of the algebraic operation $\circ$. Co-product can be defined as $C = A_k \rightarrow \sum_{lm} f_k^{lm} A_l \otimes A_m$, where f_k^{lm} are the structure constants of the algebra. The outcome is quantum superposition of final states, which can fuse to C (the "reaction" $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is possible). One can say that $\circ$ is replaced with $\otimes$: bound state decays to a superposition of all pairs, which can form the bound states by product vertex.

There are motivations for representing scattering amplitudes as sequences of algebraic operations performed for the incoming set of particles leading to an outgoing set of particles with particles identified as algebraic objects acting on vacuum state. The outcome would be analogous to Feynman diagrams but only the diagram with minimal length to which a preferred extremal can be assigned is needed. Larger ones must be equivalent with it.

The question is whether it could be indeed possible to characterize particle reactions as computations involving transformation of tensor products to products in vertices and co-products to tensor products in co-vertices (time reversals of the vertices). A couple of examples gives some idea about what is involved.

1. The simplest operations would preserve particle number and to just permute the particles: the permutation generalizes to a braiding and the scattering matrix would be basically unitary braiding matrix utilized in topological quantum computation.
2. A more complex situation occurs, when the number of particles is preserved but quantum numbers for the final state are not same as for the initial state so that particles must interact. This requires both product and co-product vertices. For instance, $A_k \otimes A_l \rightarrow f_{kl}^m A_m$ followed by $A_m \rightarrow f_m^{rs} A_r \otimes A_s$ giving $A_k \rightarrow f_{kl}^m f_m^{rs} A_r \otimes A_s$ representing 2-particle scattering. State function reduction in the final state can select any pair $A_r \otimes A_s$ in the final state. This reaction is characterized by the ordinary tree diagram in which two lines fuse to single line and defuse back to two lines. Note also that there is a non-deterministic element involved. A given final state can be achieved from a given initial state after large enough number of trials. The analogy with problem solving and mathematical theorem proving is obvious. If the interpretation is correct, Universe would be problem solver and theorem prover!
3. More complex reactions affect also the particle number. 3-vertex and its co-vertex are the simplest examples and generate more complex particle number changing vertices. For instance, on twistor Grassmann approach on can construct all diagrams using two 3-vertices. This encourages the restriction to 3-vertice (recall that fermions have only 2-vertices)
4. Intuitively it is clear that the final collection of algebraic objects can be reached by a large - maybe infinite - number of ways. It seems also clear that there is the shortest manner to end up to the final state from a given initial state. Of course, it can happen that there is no way to achieve it! For instance, if $\circ$ corresponds to group multiplication the co-vertex can lead only to a pair of particles for which the product of final state group elements equals to the initial state group element.
5. Quantum theorists of course worry about unitarity. How can avoid the situation in which the product gives zero if the outcome is element of linear space. Somehow the product should be such that this can be avoided. For instance, if product is Lie-algebra commutator, Cartan algebra would give zero as outcome.

2.5.2 Generalized Feynman Diagram As Shortest Possible Algebraic Manipulation Connecting Initial And Final Algebraic Objects

There is a strong motivation for the interpretation of generalized Feynman diagrams as shortest possible algebraic operations connecting initial and final states. The reason is that in TGD one does not have path integral over all possible space-time surfaces connecting the 3-surfaces at the ends of CD. Rather, one has in the optimal situation a space-time surface unique apart from conformal gauge degeneracy connecting the 3-surfaces at the ends of CD (they can have disjoint components).

Path integral is replaced with integral over 3-surfaces. There is therefore only single minimal generalized Feynman diagram (or twistor diagram, or whatever is the appropriate term). It would be nice if this diagram had interpretation as the shortest possible computation leading from the initial state to the final state specified by 3-surfaces and basically fermionic states at them. This would of course simplify enormously the theory and the connection to the twistor Grassmann approach is very suggestive. A further motivation comes from the observation that the state basis created by the fermionic Clifford algebra has an interpretation in terms of Boolean quantum logic and that in ZEO the fermionic states would have interpretation as analogs of Boolean statements $A \rightarrow B$.

To see whether and how this idea could be realized in TGD framework, let us try to find counterparts for the basic operations $\otimes$ and $\circ$ and identify the algebra involved. Consider first the basic geometric objects.

1. Tensor product could correspond geometrically to two disjoint 3-surfaces representing 3-particles. Partonic 2-surfaces associated with a given 3-surface represent second possibility. The splitting of a partonic 2-surface to two could be the geometric counterpart for co-product.
2. Partonic 2-surfaces are however connected to each other and possibly even to themselves by strings. It seems that partonic 2-surface cannot be the basic unit. Indeed, elementary particles are identified as pairs of wormhole throats (partonic 2-surfaces) with magnetic monopole flux flowing from throat to another at first space-time sheet, then through throat to another sheet, then back along second sheet to the lower throat of the first contact and then back to the thirst throat. This unit seems to be the natural basic object to consider. The flux tubes at both sheets are accompanied by fermionic strings. Whether also wormhole throats contain strings so that one would have single closed string rather than two open ones, is an open question.
3. The connecting strings give rise to the formation of gravitationally bound states and the hierarchy of Planck constants is crucially involved. For elementary particle there are just two wormhole contacts each involving two wormhole throats connected by wormhole contact. Wormhole throats are connected by one or more strings, which define space-like boundaries of corresponding string world sheets at the boundaries of CD. These strings are responsible for the formation of bound states, even macroscopic gravitational bound states.

2.5.3 Does Super-Symplectic Yangian Define The Arithmetics?

Super-symplectic Yangian would be a reasonable guess for the algebra involved.

1. The 2-local generators of Yangian would be of form $T_1^A = f_{BC}^A T^B \otimes T^C$, where f_{BC}^A are the structure constants of the super-symplectic algebra. n-local generators would be obtained by iterating this rule. Note that the generator T_1^A creates an entangled state of T^B and T^C with f_{BC}^A the entanglement coefficients. T_n^A is entangled state of T^B and T_{n-1}^C with the same coefficients. A kind replication of T_{n-1}^A is clearly involved, and the fundamental replication is that of T^A. Note that one can start from any irreducible representation with well defined symplectic quantum numbers and form similar hierarchy by using T^A and the representation as a starting point.

   That the hierarchy T_n^A and hierarchies irreducible representations would define a hierarchy of states associated with the partonic 2-surface is a highly non-trivial and powerful hypothesis about the formation of many-fermion bound states inside partonic 2-surfaces.
2. The charges T^A correspond to fermionic and bosonic super-symplectic generators. The geometric counterpart for the replication at the lowest level could correspond to a fermionic/bosonic string carrying super-symplectic generator splitting to fermionic/bosonic string and a string carrying bosonic symplectic generator T^A. This splitting of string brings in mind the basic gauge boson-gauge boson or gauge boson-fermion vertex.

   The vision about emission of virtual particle suggests that the entire wormhole contact pair replicates. Second wormhole throat would carry the string corresponding to T^A assignable to gauge boson naturally. T^A should involve pairs of fermionic creation and annihilation operators as well as fermionic and anti-fermionic creation operator (and annihilation operators) as in quantum field theory.

3. Bosonic emergence suggests that bosonic generators are constructed from fermion pairs with fermion and anti-fermion at opposite wormhole throats: this would allow to avoid the problems with the singular character of purely local fermion current. Fermionic and anti-fermionic string would reside at opposite space-time sheets and the whole structure would correspond to a closed magnetic tube carrying monopole flux. Fermions would correspond to superpositions of states in which string is located at either half of the closed flux tube.

4. The basic arithmetic operation in co-vertex would be co-multiplication transforming T_n^A to $T_{n+1}^A = f^A_{BC} T_n^B \otimes T^C$. In vertex the transformation of T_{n+1}^A to T_n^A would take place. The interpretations would be as emission/absorption of gauge boson. One must include also emission of fermion and this means replacement of T^A with corresponding fermionic generators F^A, so that the fermion number of the second part of the state is reduced by one unit. Particle reactions would be more than mere braidings and re-grouping of fermions and anti-fermions inside partonic 2-surfaces, which can split.

5. Inside the light-like orbits of the partonic 2-surfaces there is also a braiding affecting the M-matrix. The arithmetics involved would be therefore essentially that of measuring and "co-measuring" symplectic charges.

   Generalized Feynman diagrams (preferred extremals) connecting given 3-surfaces and many-fermion states (bosons are counted as fermion-anti-fermion states) would have a minimum number of vertices and co-vertices. The splitting of string lines implies creation of pairs of fermion lines. Whether regroupings are part of the story is not quite clear. In any case, without the replication of 3-surfaces it would not be possible to understand processes like e-e scattering by photon exchange in the proposed picture.

It is easy to hear the comments of the skeptic listener in the back row.

1. The attribute "minimal" - , which could translate to minimal value of Kähler function - is dangerous. It might be very difficult to determine what the minimal diagram is - consider only travelling salesman problem or the task of finding the shortest proof of theorem. It would be much nicer to have simple calculational rules.

   The original proposal might help here. The generalization of string model duality was in question. It stated that it is possible to move the positions of the vertices of the diagrams just as one does to transform s-channel resonances to t-channel exchange. All loops of generalized diagrams could be be eliminated by transforming the to tadpoles and snipped away so that only tree diagrams would be left. The variants of the diagram were identified as different continuation paths between different paths connecting sectors of WCW corresponding to different 3-topologies. Each step in the continuation procedure would involve product or co-product defining what continuation between two sectors means for WCW spinors. The continuations between two states require some minimal number of steps. If this is true, all computations connecting identical states are also physically equivalent. The value of the vacuum functional be same for all of them. This looks very natural.

   That the Kähler action should be same for all computational sequences connecting the same initial and final states looks strange but might be understood in terms of the vacuum degeneracy of Kähler action.

2. QFT perturbation theory requires that should have superposition of computations/continuations. What could the superposition of QFT diagrams correspond to in TGD framework?

   Could it correspond to a superposition of generators of the Yangian creating the physical state? After all, already quantum computer perform superpositions of computations. The fermionic state would not be the simplest one that one can imagine. Could AdS/CFT analogy allow to identify the vacuum state as a superposition of multi-string states so that single super-symplectic generator would be replaced with a superposition of its Yangian counterparts with same total quantum numbers but with a varying number of strings? The weight of a given superposition would be given by the total effective string world sheet area. The sum of diagrams would emerge from this superposition and would basically correspond to functional integration in WCW using exponent of Kähler action as weight. The stringy functional integral ("functional" if also wormhole contacts contain string portion, otherwise path integral) would give the perturbation theory around given string world sheet. One would have effective reduction of string theory.

2.5.4 How Does This Relate To The Ordinary Perturbation Theory?

One can of course worry about how to understand the basic results of the usual perturbation theory in this picture. How does one obtain a perturbation theory in powers of coupling constant, what does running coupling constant mean, etc...? I have already discussed how the superposition of diagrams could be understood in the new picture.

1. The QFT picture with running coupling constant is expected at QFT limit, when many-sheeted space-time is replaced with a slightly curved region of M^4 and gravitational field and gauge potentials are identified as sums of the deviations of induced metric from M^4 metric and classical induced gauge potentials associated with the sheets of the many-sheeted space-time. The running coupling constant would be due to the dependence of the size scale of CD, and p-adic coupling constant evolution would be behind the continuous one.
2. The notion of running coupling constant is very physical concept and should have a description also at the fundamental level and be due to a finite computational resolution, which indeed has very concrete description in terms of Noether charges of super-symplectic Yangian creating the states at the ends of space-time surface at the boundaries of CD. The space-time surface and the diagram associated with a given pair of 3-surfaces and stringy Noether charges associated with them can be characterized by a complexity measured in terms of the number of vertices (3-surface at which three 3-surfaces meet).

   For instance, 3-particle scattering can be possible only by using the simplest 3-vertex defined by product or co-product for pairs of 3-surfaces. In the generic case one has more complex diagram and what looks first 3-particle vertex has complex substructure rather than being simple product or co-product.
3. Complexity seems to have two separate aspects: the complexities of the positive and negative parts of zero energy state as many-fermion states and the complexity of associated 3-surfaces. The generalization of AdS/CFT however suggests that once the string world sheets and partonic 2-surfaces appearing in the diagram have been fixed, the space-time surface itself is fixed. The principle also suggests that the fixing partonic 2-surface and the strings connecting them at the boundaries of CD fixes the 3-surface apart from the action of sub-algebra of Yangian acting as gauge algebra (vanishing classical Noether charges). If one can determine the minimal sequence of allowed algebraic operation of Yangian connecting initial and final fermion states, one knows the minimum number of vertices and therefore the topological structure of the connecting minimal space-time surface.
4. In QFT spirit one could describe the finite measurement resolution by introducing effective 3-point vertex, which is need not be product/co-produce anymore. 3-point scattering amplitudes in general involve microscopic algebraic structure involving several vertices. One can however give up the nice algebraic interpretation and just talk about effective 3-vertex for practical purposes. Just as the QFT vertex described by running coupling constant decomposes to sum of diagrams, product/co-product in TGD could be replaced with effective product/co-product expressible as a longer computation. This would imply coupling constant evolution.

   Fermion lines could however remain as such since they are massless in 8-D sense and mass renormalization does not make sense.

   Similar practical simplification could be done the initial and final states to get rid of superposition of the Yangian generators with different numbers of strings ("cloud of virtual particles"). This would correspond to wave function renormalization.
5. The number of vertices and wormhole contact orbits serves as a measure for the complexity of the diagram. Since fermion lines are associated with wormhole throats assignable with wormhole contacts identifiable as deformations CP_2 type vacuum extremals, one expects that the exponent of the Kähler function defining vacuum functional is in the first approximation the total CP_2 volume of wormhole contacts giving a measure for the importance of the

contribution in functional integral. If it converges very rapidly only Gaussian approximation around maximum is needed.

6. Convergence depends on how large the fraction of volume of CP_2 is associated with a given wormhole contact. The volume is proportional to the length of the wormhole contact orbit. One expects exponential convergence with the number of fermion lines and their lengths for long lines. For short distances the exponential damping is small so that diagrams with microscopic structure of diagrams are needed and are possible. This looks like adding small scale details to the algebraic manipulations.
7. One must be of course be very cautious in making conclusions. The presence of $1/\alpha_K \propto h_{eff}$ in the exponent of Kähler function would suggest that for large values of h_{eff} only the 3-surfaces with smallest possible number of wormhole contact orbits contribute. On the other hand, the generalization of AdS/CFT duality suggests that Kähler action reducible to area of string world sheet in the effective metric defined by canonical momentum currents of Kähler action behaves as $\alpha_K^2 \propto 1/h_{eff}^2$. What does this mean?

To sum up, the identification of vertex as a product or co-product in Yangian looks highly promising approach. The Nother charges of the super-symplectic Yangian are associated with strings and are either linear or bilinear in the fermion field. The fermion fields associated with the partonic 2-surface defining the vertex are contracted with fermion fields associated with other partonic 2-surfaces using the same rule as in Wick expansion in quantum field theories. The contraction gives fermion propagator for each leg pair associated with two vertices. Vertex factor is proportional to the contraction of spinor modes with the operators defining the Noether charge or super charge and essentially Kähler-Dirac gamma matrix and the representation of the action of the symplectic generator on fermion realizable in terms of sigma matrices. This is very much like the corresponding expression in gauge theories but with gauge algebra replaced with symplectic algebra. The possibility of contractions of creation and annihilation operator for fermion lines associated with opposite wormhole throats at the same partonic 2-surface (for Noether charge bilinear in fermion field) gives bosonic exchanges as lines in which the fermion lines turns in time direction: otherwise only regroupings of fermions would take place.

2.5.5 This Was Not The Whole Story Yet

The proposed amplitude represents only the value of WCW spinor field for single pair of 3-surfaces at the opposite boundaries of given CD. Hence Yangian construction does not tell the whole story.

1. Yangian algebra would give only the vertices of the scattering amplitudes. On basis of previous considerations, one expects that each fermion line carries propagator defined by 8-momentum. The structure would resemble that of super-symmetric YM theory. Fermionic propagators should emerge from summing over intermediate fermion states in various vertices and one would have integrations over virtual momenta which are carried as residue integrations in twistor Grassmann approach. 8-D counterpart of twistorialization would apply.
2. Super-symplectic Yangian would give the scattering amplitudes for single space-time surface and the purely group theoretical form of these amplitudes gives hopes about the independence of the scattering amplitude on the pair of 3-surfaces at the ends of CD near the maximum of Kähler function. This is perhaps too much to hope except approximately but if true, the integration over WCW would give only exponent of Kähler action since metric and poorly defined Gaussian and determinants would cancel by the basic properties of Kähler metric. Exponent would give a non-analytic dependence on α_K.

   The Yangian supercharges are proportional to $1/\alpha_K$ since covariant Kähler-Dirac gamma matrices are proportional to canonical momentum currents of Kähler action and thus to $1/\alpha_K$. Perturbation theory in powers of $\alpha_K = g_K^2/4\pi\hbar_{eff}$ is possible after factorizing out the exponent of vacuum functional at the maximum of Kähler function and the factors $1/\alpha_K$ multiplying super-symplectic charges.

   The additional complication is that the characteristics of preferred extremals contributing significantly to the scattering amplitudes are expected to depend on the value of α_K by quantum interference effects. Kähler action is proportional to $1/\alpha_K$. The analogy of AdS/CFT correspondence states the expressibility of Kähler function in terms of string area in the effective metric defined by the anti-commutators of K-D matrices. Interference effects eliminate string length for which the area action has a value considerably larger than one so that the string length and thus also the minimal size of CD containing it scales as h_{eff}. Quantum interference effects therefore give an additional dependence of Yangian super-charges on h_{eff} leading to a perturbative expansion in powers of α_K although the basic expression for scattering amplitude would not suggest this.

3. Non-planar diagrams of quantum field theories should have natural counterpart and linking and knotting for braids defines it naturally. This suggests that the amplitudes can be interpreted as generalizations of braid diagrams defining braid invariants: braid strands would appear as legs of 3-vertices representing product and co-product. Amplitudes could be constructed as generalized braid invariants transforming recursively braided tree diagram to an un-braided diagram using same operations as for braids. In [L18] I considered a possible breaking of associativity occurring in weak sense for conformal field theories and was led to the vision that there is a fractal hierarchy of braids such that braid strands themselves correspond to braids. This hierarchy would define an operad with subgroups of permutation group in key role. Hence it seems that various approaches to the construction of amplitudes converge.

2.6 Appendix: Some Mathematical Details About Grasmannian Formalism

In the following I try to summarize my amateurish understanding about the mathematical structure behind the Grassmann integral approach. The representation summarizes what I have gathered from the articles of Arkani-Hamed and collaborators [B34, B36]. These articles are rather sketchy and the article of Bullimore provides additional details [B55] related to soft factors. The article of Mason and Skinner provides excellent introduction to super-twistors [B27] and dual super-conformal invariance. I apologize for unavoidable errors.

Before continuing a brief summary about the history leading to the articles of Arkani-Hamed and others is in order. This summary covers only those aspects which I am at least somewhat familiar with and leaves out many topics about existence which I am only half-conscious.

1. It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a\tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle\lambda,\mu\rangle &= \epsilon_{ab}\lambda^a\mu^b \ , \\ \left[\tilde{\lambda},\tilde{\mu}\right] &= \epsilon_{a'b'}\lambda^{a'}\mu^{b'} \ , \\ p\cdot q &= \langle\lambda,\mu\rangle\left[\tilde{\lambda},\tilde{\mu}\right] \ , \ (q_{aa'}=\mu_a\tilde{\mu}_{a'}) \ . \end{aligned} \quad (2.6.1)$$

If the particle has spin one can assign it a positive or negative helicity $h=\pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a\tilde{\lambda}_{a'}}{\langle\mu,\lambda\rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a\tilde{\mu}_{a'}}{\left[\tilde{\mu},\tilde{\lambda}\right]} \ , \ \text{negative helicity} \ . \end{aligned} \quad (2.6.2)$$

In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

2. Tree amplitudes are considered and it is convenient to drop the group theory factor $Tr(T_1T_2\cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{2.6.3}$$

When the sign of the helicities is changed $\langle .. \rangle$ is replaced with [..].

3. The article of Witten [B29] proposed that twistor approach could be formulated as a twistor string theory with string world sheets "living" in 6-dimensional CP_3 possessing Calabi-Yau structure and defining twistor space. In this article Witten introduced what is known as half Fourier transform allowing to transform momentum integrals over light-cone to twistor integrals. This operation makes sense only in space-time signature $(2,2)$. Witten also demonstrated that maximal helicity violating (MHV) twistor amplitudes (two gluons with negative helicity) with n particles with $k+2$ negative helicities and l loops correspond in this approach to holomorphic 2-surfaces defined by polynomials defined by polynomials of degree $D=k-1+l$, where the genus of the surface satisfies $g \leq l$. AdS/CFT duality provides a second stringy approach to $\mathcal{N}=4$ theory allowing to understand the scattering amplitudes in terms of Wilson loops with light-like edges: about this I have nothing to say. In any case, the generalization of twistor string theory to TGD context is highly attractive idea and will be considered later.

4. In the article [B22] Cachazo, Svrcek, and Witten propose the analog of Feynman diagrammatics in which MHV amplitudes can be used as analogs of vertices and ordinary $1/P^2$ propagator as propagator to construct tree diagrams with arbitrary number of negative helicity gluons. This approach is not symmetric with respect to the change of the sign of helicities since the amplitudes with two positive helicities are constructed as tree diagrams. The construction is non-trivial because one must analytically continue the on mass shell tree amplitudes to off mass shell momenta. The problem is how to assign a twistor to these momenta. This is achieved by introducing an arbitrary twistor $\eta^{a'}$ and defining λ_a as $\lambda_a = p_{aa'}\eta^{a'}$. This works for both massless and massive case. It however leads to a loss of the manifest Lorentz invariance. The paper however argues and the later paper [B20, B20] shows rigorously that the loss is only apparent. In this paper also BCFW recursion formula is introduced allowing to construct tree amplitudes recursively by starting from vertices with 2 negative helicity gluons. Also the notion which has become known as BCFW bridge representing the massless exchange in these diagrams is introduced. The tree amplitudes are not tree amplitudes in gauge theory sense where correspond to leading singularities for which 4 or more lines of the loop are massless and therefore collinear. What is important that the very simple MHV amplitudes become the building blocks of more complex amplitudes.

5. The nex step in the progress was the attempt to understand how the loop corrections could be taken into account in the construction BCFW formula. The calculation of loop contributions to the tree amplitudes revealed the existence of dual super-conformal symmetry which was found to be possessed also by BCFW tree amplitudes besides conformal symmetry. Together these symmetries generate infinite-dimensional Yangian symmetry [B27].

6. The basic vision of Arkani-Hamed and collaborators is that the scattering amplitudes of $\mathcal{N}=4$ SYM are constructible in terms of leading order singularities of loop diagrams. These singularities are obtained by putting maximum number of momenta propagating in the lines of the loop on mass shell. The non-leading singularities would be induced by the leading singularities by putting smaller number of momenta on mass shell are dictated by these terms. A related idea serving as a starting point in [B34] is that one can define loop integrals as residue integrals in momentum space. If I have understood correctly, this means that one an imagine the possibility that the loop integral reduces to a lower dimensional integral for on mass shell particles in the loops: this would resemble the approach to loop integrals based on unitarity and analyticity. In twistor approach these momentum integrals defined

as residue integrals transform to residue integrals in twistor space with twistors representing massless particles. The basic discovery is that one can construct leading order singularities for n particle scattering amplitude with $k+2$ negative helicities as Yangian invariants $Y_{n,k}$ for momentum twistors and invariants constructed from them by canonical operations changing n and k. The correspondence $k = l$ does not hold true for the more general amplitudes anymore.

2.6.1 Yangian Algebra And Its Super Counterpart

The article of Witten [B26] gives a nice discussion of the Yangian algebra and its super counterpart. Here only basic formulas can be listed and the formulas relevant to the super-conformal case are given.

Yangian algebra

Yangian algebra $Y(G)$ is associative Hopf algebra. The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [B26] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{2.6.4}$$

Besides this Serre relations are satisfied. These have more complex and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right]+\left[J^{(1)B},\left[J^{(1)C},J^A\right]\right]+\left[J^{(1)C},\left[J^{(1)A},J^B\right]\right]\\
&\qquad = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM}\{J_D, J_E, J_F\} \quad ,\\
&\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^C,J^{(1)D}\right]\right]+\left[\left[J^{(1)C},J^{(1)D}\right],\left[J^A,J^{(1)B}\right]\right]\\
&\qquad = \frac{1}{24}(f^{AGL} f^{BEM} f_K^{CD}\\
&\qquad + f^{CGL} f^{DEM} f_K^{AB}) f^{KFN} f_{LMN}\{J_G, J_E, J_F\} \quad .
\end{aligned} \tag{2.6.5}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A, B, C\}$ denotes the symmetrized product of three generators.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} are given as

$$J^{(1)A} \quad = \quad f_{BC}^A \sum_{i<j} J_i^B J_j^C \quad . \tag{2.6.6}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N}=4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A\otimes 1 + 1\otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A}\otimes 1 + 1\otimes J^{(1)A} + f^A_{BC}J^B\otimes J^C \end{aligned} \tag{2.6.7}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B26].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

a and d representing the bosonic part of the algebra are $n\times n$ matrices and $m\times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n\times m$ and $m\times n$ matrices, whose anti-commutator is the direct sum of $n\times n$ and $n\times n$ matrices. For $n=m$ bosonic generators transform like Lie algebra generators of $SU(n)\times SU(n)$ whereas fermionic generators transform like $n\otimes\overline{n}\oplus\overline{n}\otimes n$ under $SU(n)\times SU(n)$. Supertrace is defined as $Str(x)=Tr(a)-Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n\neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n=m$ is an important delicacy since this case corresponds to $\mathcal{N}=4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R\otimes\overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'}J^{(1)C'} = g_{CC'}f_{AB}^{C'}\sum_{i<j}J_i^AJ_j^B \\ &= g_{CC'}f_{AB}^{C'}g^{AA'}g^{BB'}\sum_{i<j}J_{A'}^iJ_{B'}^j \ . \end{aligned} \quad (2.6.8)$$

Here $g_{AB} = Str(J_AJ_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

Generators of super-conformal Yangian symmetries

The explicit formula for the generators of super-conformal Yangian symmetries in terms of ordinary twistors is given by

$$\begin{aligned} j_B^A &= \sum_{i=1}^{n} Z_i^A\partial_{Z_i^B} \ , \\ j_B^{(1)A} &= \sum_{i<j}(-1)^C\left[Z_i^A\partial_{Z_j^C}Z_j^C\partial_{Z_j^B}\right] \ . \end{aligned} \quad (2.6.9)$$

This formula follows from completely general formulas for the Yangian algebra discussed above and allowing to express the dual generators $j_N^{(1)}$ as quadratic expression of j_N involving structures constants. In this rather sketchy formula twistors are ordinary twistors. Note however that in the recent case the lattice is replaced with its finite cutoff corresponding to the external particles of the scattering amplitude. This probably corresponds to the assumption that for the representations considered only finite number of lattice points correspond to non-trivial quantum numbers or to cyclic symmetry of the representations.

In the expression for the amplitudes the action of transformations is on the delta functions and by partial integration one finds that a total divergence results. This is easy to see for the linear generators but not so for the quadratic generators of the dual super-conformal symmetries. A similar formula but with j_B^A and $j_B^{(1)A}$ interchanged applies in the representation of the amplitudes as Grassmann integrals using ordinary twistors. The verification of the generalization of Serre formula is also straightforward.

2.6.2 Twistors And Momentum Twistors And Super-Symmetrization

In [B27] the basics of twistor geometry are summarized. Despite this it is perhaps good to collect the basic formulas here.

Conformally compactified Minkowski space

Conformally compactified Minkowski space can be described as $SO(2,4)$ invariant (Klein) quadric

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \ . \quad (2.6.10)$$

The coordinates (T,V,W,X,Y,Z) define homogenous coordinates for the real projective space RP^5. One can introduce the projective coordinates $X_{\alpha\beta} = -X_{\beta\alpha}$ through the formulas

$$\begin{aligned} &X_{01} = W - V \ , \quad X_{02} = Y + iX \ , \quad X_{03} = \frac{i}{\sqrt{2}}T - Z \ , \\ &X_{12} = -\frac{i}{\sqrt{2}}(T+Z) \ , \quad X_{13} = Y - iX \ , \quad X_{23} = \frac{1}{2}(V+W) \ . \end{aligned} \quad (2.6.11)$$

The motivation is that the equations for the quadric defining the conformally compactified Minkowski space can be written in a form which is manifestly conformally invariant:

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} = 0 \ per. \tag{2.6.12}$$

The points of the conformally compactified Minkowski space are null separated if and only if the condition

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta} = 0 \tag{2.6.13}$$

holds true.

Correspondence with twistors and infinity twistor

One ends up with the correspondence with twistors by noticing that the condition is equivalent with the possibility to expression $X_{\alpha\beta}$ as

$$X_{\alpha\beta} = A_{[\alpha}B_{\beta]} \ , \tag{2.6.14}$$

where brackets refer to antisymmetrization. The complex vectors A and B define a point in twistor space and are defined only modulo scaling and therefore define a point of twistor space CP_3 defining a covering of 6-D Minkowski space with metric signature $(2,4)$. This corresponds to the fact that the Lie algebras of $SO(2,4)$ and $SU(2,2)$ are identical. Therefore the points of conformally compactified Minkowski space correspond to lines of the twistor space defining spheres CP_1 in CP_3.

One can introduce a preferred scale for the projective coordinates by introducing what is called infinity twistor (actually a pair of twistors is in question) defined by

$$I_{\alpha\beta} = \begin{pmatrix} \epsilon^{A'B'} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{2.6.15}$$

Infinity twistor represents the projective line for which only the coordinate X_{01} is non-vanishing and chosen to have value $X_{01} = 1$.

One can define the contravariant form of the infinite twistor as

$$I^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} \ . \tag{2.6.16}$$

Infinity twistor defines a representative for the conformal equivalence class of metrics at the Klein quadric and one can express Minkowski distance as

$$(x-y)^2 = \frac{X^{\alpha\beta}Y_{\alpha\beta}}{I_{\alpha\beta}X^{\alpha\beta}I_{\mu\nu}Y^{\mu\nu}} \ . \tag{2.6.17}$$

Note that the metric is necessary only in the denominator. In twistor notation the distance can be expressed as

$$(x-y)^2 = \frac{\epsilon(A,B,C,D)}{\langle AB\rangle\langle CD\rangle} \ . \tag{2.6.18}$$

Infinite twistor $I_{\alpha\beta}$ and its contravariant counterpart project the twistor to its primed and unprimed parts usually denoted by $\mu^{A'}$ and λ^A and defined spinors with opposite chiralities.

Relationship between points of M^4 and twistors

In the coordinates obtained by putting $X_{01} = 1$ the relationship between space-time coordinates $x^{AA'}$ and $X^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{A'B'}x^2 & -ix^{A'}_{B} \\ ix^{B'}_{A} & \epsilon_{A,B} \end{pmatrix} \quad , \quad X^{\alpha\beta} = \begin{pmatrix} \epsilon_{A'B}{}'x^2 & -ix_{A'}^{B} \\ ix^{A}_{B'} & -\frac{1}{2}\epsilon^{AB}x^2 \end{pmatrix} \quad , \tag{2.6.19}$$

If the point of Minkowski space represents a line defined by twistors (μ_U, λ_U) and (μ_V, λ_V), one has

$$x^{AC'} = i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)^{AC'}}{\langle UV\rangle} \tag{2.6.20}$$

The twistor μ for a given point of Minkowski space in turn is obtained from λ by the twistor formula by

$$\mu^{A'} \quad = \quad -ix^{AA'}\lambda_A \quad . \tag{2.6.21}$$

Generalization to the super-symmetric case

This formalism has a straightforward generalization to the super-symmetric case. CP_3 is replaced with $CP_{3|4}$ so that Grassmann parameters have four components. At the level of coordinates this means the replacement $[W_I] = [W_\alpha, \chi_\alpha]$. Twistor formula generalizes to

$$\mu^{A'} = -ix^{AA'}\lambda_A \quad , \quad \chi_\alpha = \theta^A_\alpha\lambda_A \quad . \tag{2.6.22}$$

The relationship between the coordinates of chiral super-space and super-twistors generalizes to

$$(x, \theta) \quad = \quad \left(i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)}{\langle UV\rangle}, \frac{(\chi_V\lambda_U - \chi_U\lambda_V)}{\langle UV\rangle}\right) \tag{2.6.23}$$

The above formulas can be applied to super-symmetric variants of momentum twistors to deduce the relationship between region momenta x assigned with edges of polygons and twistors assigned with the ends of the light-like edges. The explicit formulas are represented in [B27]. The geometric picture is following. The twistors at the ends of the edge define the twistor pair representing the region momentum as a line in twistor space and the intersection of the twistor lines assigned with the region momenta define twistor representing the external momenta of the graph in the intersection of the edges.

Basic kinematics for momentum twistors

The super-symmetrization involves replacement of multiplets with super-multiplets

$$\Phi(\lambda, \tilde{\lambda}, \eta) = G^+(\lambda, \tilde{\lambda}) + \eta_i\Gamma^a\lambda, \tilde{\lambda}) + \cdots + \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d G^-(\lambda, \tilde{\lambda}) \quad . \tag{2.6.24}$$

Momentum twistors are dual to ordinary twistors and were introduced by Hodges. The light-like momentum of external particle a is expressed in terms of the vertices of the closed polygon defining the twistor diagram as

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = \lambda_i\tilde{\lambda}_i \quad , \quad \theta_i - \theta_{i+1} = \lambda_i\eta_i \quad . \tag{2.6.25}$$

One can say that massless momenta have a conserved super-part given by $\lambda_i\eta_i$. The dual of the super-conformal group acts on the region momenta exactly as the ordinary conformal group acts on space-time and one can construct twistor space for dua region momenta.

Super-momentum conservation gives the constraints

$$\sum p_i = 0 \ , \quad \sum \lambda_i \eta_i = 0 \ . \tag{2.6.26}$$

The twistor diagrams correspond to polygons with edges with lines carrying region momenta and external massless momenta emitted at the vertices.

This formula is invariant under overall shift of the region momenta x_a^μ. A natural interpretation for x_a^μ is as the momentum entering to the vertex where p_a is emitted. Overall shift would have interpretation as a shift in the loop momentum. x_a^μ in the dual coordinate space is associated with the line $Z_{a-1}Z_a$ in the momentum twistor space. The lines $Z_{a-1}Z_a$ and Z_aZ_{a+1} intersect at Z_a representing a light-like momentum vector p_a^μ.

The brackets $\langle abcd\rangle \equiv \epsilon_{IJKL}Z_a^IZ_b^JZ_c^KZ_d^L$ define fundamental bosonic conformal invariants appearing in the tree amplitudes as basic building blocks. Note that Z_a define points of 4-D complex twistor space to be distinguished from the projective twistor space CP_3. Z_a define projective coordinates for CP_3 and one of the four complex components of Z_a is redundant and one can take $Z_a^0 = 1$ without a loss of generality.

2.6.3 Brief Summary Of The Work Of Arkani-Hamed And Collaborators

The following comments are an attempt to summarize my far from complete understanding about what is involved with the representation as contour integrals. After that I shall describe in more detail my impressions about what has been done.

Limitations of the approach

Consider first the limitations of the approach.

1. The basis idea is that the representation for tree amplitudes generalizes to loop amplitudes. On other words, the amplitude defined as a sum of Yangian invariants expressed in terms of Grassmann integrals represents the sum of loops up to some maximum loop number. The problem is here that shifts of the loop momenta are essential in the UV regularization procedure. Fixing the coordinates $x_1, \cdots, x_n$ having interpretation as momenta associated with lines in the dual coordinate space allows to eliminate the non-uniqueness due to the common shift of these coordinates.
2. It is not however not possible to identify loop momentum as a loop momentum common to different loop integrals unless one restricts to planar loops. Non-planar diagrams are obtained from a planar diagram by permuting the coordinates x_i but this means that the unique coordinate assignment is lost. Therefore the representation of loop integrands as Grassmann integrals makes sense only for planar diagrams. From TGD point of view one could argue that this is one good reason for restricting the loops so that they are for on mass shell particles with non-parallel on mass shell four-momenta and possibly different sign of energies for given wormhole contact representing virtual particle.
3. IR regularization is needed even in $\mathcal{N} = 4$ for SYM given by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory.

What has been done?

The article proposes a generalization of the BCFW recursion relation for tree diagrams of $\mathcal{N} = 4$ for SYM so that it applies to planar diagrams with a summation over an arbitrary number of loops.

1. The basic goal of the article is to generalize the recursion relations of tree amplitudes so that they would apply to loop amplitudes. The key idea is following. One can formally represent loop integrand as a contour integral in complex plane whose coordinate parameterizes the deformations $Z_n \rightarrow Z_n + \epsilon Z_{n-1}$ and re-interpret the integral as a contour integral with oppositely oriented contour surrounding the rest of the complex plane which can be imagined also as being mapped to Riemann sphere. What happens only the poles which correspond to lower number of loops contribute this integral. One obtains a recursion relation with respect

to loop number. This recursion seems to be the counterpart for the recursive construction of the loops corrections in terms of absorptive parts of amplitudes with smaller number of loop using unitarity and analyticity.

2. The basic challenge is to deduce the Grassmann integrands as Yangian invariants. From these one can deduce loop integrals by integration over the four momenta associated with the lines of the polygonal graph identifiable as the dual coordinate variables x_a. The integration over loop momenta can induce infrared divergences breaking Yangian symmetry. The big idea here is that the operations described above allow to construct loop amplitudes from the Yangian invariants defining tree amplitudes for a larger number of particles by removing external particles by fusing them to form propagator lines and by using the BCFW bridge to fuse lower-dimensional invariants. Hence the usual iterative procedure (bottom-up) used to construct scattering amplitudes is replaced with a recursive procedure (top-down). Of course, once lower amplitudes has been constructed they can be used to construct amplitudes with higher particle number.
3. The first guess is that the recursion formula involves the same lower order contributions as in the case of tree amplitudes. These contributions have interpretation as factorization of channels involving single particle intermediate states. This would however allow to reduce loop amplitudes to 3-particle loop amplitudes which vanish in $\mathcal{N}=4$ SYM by the vanishing of coupling constant renormalization. The additional contribution is necessary and corresponds to a source term identifiable as a "forward limit" of lower loop integrand. These terms are obtained by taking an amplitude with two additional particles with opposite four-momenta and forming a state in which these particles are entangled with respect to momentum and other quantum numbers. Entanglement means integral over the massless momenta on one hand. The insertion brings in two momenta x_a and x_b and one can imagine that the loop is represented by a branching of propagator line. The line representing the entanglement of the massless states with massless momentum define the second branch of the loop. One can of course ask whether only massless momentum in the second branch. A possible interpretation is that this state is expressible by unitarity in terms of the integral over light-like momentum.
4. The recursion formula for the loop amplitude $M_{n,k,l}$ involves two terms when one neglects the possibility that particles can also suffer trivial scattering (cluster decomposition). This term basically corresponds to the Yangian invariance of n arguments identified as Yangian invariant of $n-1$ arguments with the same value of k.
   (a) The first term corresponds to single particle exchange between particle groups obtained by splitting the polygon at two vertices and corresponds to the so called BCFW bridge for tree diagrams. There is a summation over different splittings as well as a sum over loop numbers and dimensions k for the Grassmann planes. The helicities in the two groups are opposite.
   (b) Second term is obtained from an amplitude obtained by adding of two massless particles with opposite momenta and corresponds to $n+2, k+1, l-1$. The integration over the light-like momentum together with other operations implies the reduction $n+2 \rightarrow n$. Note that the recursion indeed converges. Certainly the allowance of added zero energy states with a finite number of particles is necessary for the convergence of the procedure.

2.6.4 The General Form Of Grassmannian Integrals

If the recursion formula proposed in [B36] is correct, the calculations reduce to the construction of N^kMHV (super) amplitudes. MHV refers to maximal helicity violating amplitudes with 2 negative helicity gluons. For N^kMHV amplitude the number of negative helicities is by definition $k+2$ [B34]. Note that the total right handed R-charge assignable to 4 super-coordinates η_i of negative helicity gluons can be identified as $R=4k$. BCFW recursion formula [B20, B20] allows to construct from MHV amplitudes with arbitrary number of negative helicities.

The basic object of study are the leading singularities of color-stripped n-particle N^kMHV amplitudes. The discovery is that these singularities are expressible in terms Yangian invariants $Y_{n,k}(Z_1,\cdots,Z_n)$, where Z_i are momentum super-twistors. These invariants are defined by residue integrals over the compact $nk-1$-dimensional complex space $G(n,k)=U(n)/U(k)\times U(n-k)$ of k-planes of complex n-dimensional space. n is the number of external massless particles, k is the number negative helicity gluons in the case of N^kMHV amplitudes, and Z_a, $i = 1, \cdots, n$ denotes the projective 4-coordinate of the super-variant $CP^{3|4}$ of the momentum twistor space CP_3 assigned to the massless external particles is following. $Gl(n)$ acts as linear transformations in the n-fold Cartesian power of twistor space. Yangian invariant $Y_{n,k}$ is a function of twistor variables Z^a having values in super-variant $CP_{3|3}$ of momentum twistor space CP_3 assigned to the massless external particles being simple algebraic functions of the external momenta.

It is also possible to define N^kMHV amplitudes in terms of Yangian invariants $L_{n,k+2}(W_1, \cdots, W_n)$ by using ordinary twistors W_a and identical defining formula. The two invariants are related by the formula $L_{n,k+2}(W_1, \cdots, W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1, \cdots, Z_n)$. Here M^{tree}_{MHV} is the tree contribution to the maximally helicity violating amplitude for the scattering of n particles: recall that these amplitudes contain two negative helicity gluons whereas the amplitudes containing a smaller number of them vanish [B22]. One can speak of a factorization to a product of n-particle amplitudes with $k-2$ and 2 negative helicities as the origin of the duality. The equivalence between the descriptions based on ordinary and momentum twistors states the dual conformal invariance of the amplitudes implying Yangian symmetry. It has been conjectured that Grassmannian integrals generate all Yangian invariants.

The formulas for the Grassmann integrals for twistors and momentum twistors appearing in the expressions of N^kMHV amplitudes are given by following expressions.

1. The integrals $L_{n,k}(W_1, \cdots, W_n)$ associated with $N^{k-2}MHV$ amplitudes in the description based on ordinary twistors correspond to k negative helicities and are given by

$$\begin{aligned} L_{n,k}(W_1, \cdots, W_n) &= \frac{1}{Vol(GL(2))} \int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)} \times \\ &\times \prod_{\alpha=1}^{k} d^{4|4}Y_\alpha \prod_{i=1}^{n} \delta^{4|4}(W_i - C_{\alpha i}Y_\alpha) \ . \end{aligned} \tag{2.6.27}$$

   Here $C_{\alpha a}$ denote the $n \times k$ coordinates used to parametrize the points of $G_{k,n}$.

2. The integrals $Y_{n,k}(W_1, \cdots, W_n)$ associated with N^kMHV amplitudes in the description based on momentum twistors are defined as

$$Y_{n,k}(Z_1, \cdots, Z_n) = \frac{1}{Vol(GL(k))} \times \int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)} \times \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}Z_a) \ . \tag{2.6.28}$$

   The possibility to select $Z^0_a = 1$ implies $\sum_k C_{\alpha k} = 0$ allowing to eliminate $C_{\alpha n}$ so that the actual number of coordinates Grassman coordinates is $nk-1$. As already noticed, $L_{n,k+2}(W_1, \cdots, W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1, \cdots, Z_n)$. Momentum twistors are obviously calculationally easier since the value of k is smaller by two units.

The $4k$ delta functions reduce the number of integration variables of contour integrals from nk to $(n-4)k$ in the bosonic sector (the definition of delta functions involves some delicacies not discussed here). The n quantities $(m, \cdots m+k)$ are $k \times k$-determinants defined by subsequent columns from m to $m+k-1$ of the $k \times n$ matrix defined by the coordinates $C_{\alpha a}$ and correspond geometrically to the k-volumes of the k-dimensional parallel-pipeds defined by these column vectors. The fact that the scalings of twistor space coordinates Z_a can be compensated by scalings of $C_{\alpha a}$ deforming integration contour but leaving the residue integral invariant so that the integral depends on projective twistor coordinates only.

Since the integrand is a rational function, a multi-dimensional residue calculus allows to deduce the values of these integrals as residues associated with the poles of the integrand in a recursive manner. The poles correspond to the zeros of the $k \times k$ determinants appearing in the integrand or equivalently to singular lower-dimensional parallel-pipeds. It can be shown that local residues are determined by $(k-2)(n-k-2)$ conditions on the determinants in both cases. The value of the integral depends on the explicit choice of the integration contour for each variable

$C_{\alpha a}$ left when delta functions are taken into account. The condition that a correct form of tree amplitudes is obtained fixes the choice of the integration contours.

For the ordinary twistors W the residues correspond to projective configurations in CP_{k-1}, or more precisely in the space $CP^n_{k-1}/Gl(k)$, which is $(k-1)n-k^2$-dimensional space defining the support for the residues integral. $Gl(k)$ relates to each other different complex coordinate frames for k-plane and since the choice of frame does not affect the plane itself, one has $Gl(k)$ gauge symmetry as well as the dual $Gl(n-k)$ gauge symmetry.

CP_{k-1} comes from the fact that C_{α_k} are projective coordinates: the amplitudes are indeed invariant under the scalings $W_i \rightarrow t_i W_i$, $C_{\alpha i} \rightarrow t C_{\alpha i}$. The coset space structure comes from the fact that $Gl(k)$ is a symmetry of the integrand acting as $C_{\alpha i} \rightarrow \Lambda_\alpha^{\ \beta} C_{\beta_i}$. This analog of gauge symmetry allows to fix k arbitarily chosen frame vectors $C_{\alpha i}$ to orthogonal unit vectors. For instance, one can have $C_{\alpha i} = \delta_{\alpha_i}$ for $\alpha = i \in 1, \cdots, k$. This choice is discussed in detail in [B34]. The reduction to CP_{k-1} implies the reduction of the support of the integral to line in the case of MHV amplitudes and to plane in the case of NMHV as one sees from the expression $d\mu = \prod_\alpha d^{4|4}Y_\alpha \prod_{i=1}^n \delta^{4|4}(W_i - C_{\alpha i}Y_\alpha)$. For $(i_1, \cdots, i_k) = 0$ the vectors $i_1, ..i_k$ belong to $k-2$-dimensional plane of CP_{k-1}. In the case of $NMHV$ (N^2MHV) amplitudes this translates at the level of twistors to the condition that the corresponding twistors $\{i_1, i_2, i_3\}$ $(\{i_1, i_2, i_3, i_4\})$ are collinear (in the same plane) in twistor space. This can be understood from the fact that the delta functions in $d\mu$ allow to express W_i in terms of $k-1$ Y_α: s in this case.

The action of conformal transformations in twistor space reduces to the linear action of $SU(2,2)$ leaving invariant Hermitian sesquilinear form of signature $(2,2)$. Therefore the conformal invariance of the Grassmannian integral and its dual variant follows from the possibility to perform a compensating coordinate change for $C_{\alpha a}$ and from the fact that residue integral is invariant under small deformations of the integration contour. The above described relationship between representations based on twistors and momentum twistors implies the full Yangian invariance.

2.6.5 Canonical Operations For Yangian Invariants

General l-loop amplitudes can be constructed from the basic Yangian invariants defined by N^kMHV amplitudes by various operations respecting Yangian invariance apart from possible IR anomalies. There are several operations that one can perform for Yangian invariants $Y_{n,k}$ and all these operations appear in the recursion formula for planar all loop amplitudes. These operations are described in [B36] much better than I could do it so that I will not go to any details. It is possible to add and remove particles, to fuse two Yangian invariants, to merge particles, and to construct from two Yangian invariants a higher invariant containing so called BCFW bridge representing single particle exchange using only twistorial methods.

Inverse soft factors

Inverse soft factors add to the diagram a massless collinear particles between particles a and b and by definition one has

$$O_{n+1}(a,c,b,\cdots) = \frac{\langle ab\rangle}{\langle ac\rangle\langle cb\rangle} O_n(a'b') \ . \tag{2.6.29}$$

At the limit when the momentum of the added particle vanishes both sides approach the original amplitude. The right-handed spinors and Grassmann parameters are shifted

$$\begin{aligned} \tilde{\lambda}'_a &= \tilde{\lambda}_a + \tfrac{\langle cb\rangle}{\langle ab\rangle}\tilde{\lambda}_c \ , \quad \tilde{\lambda}'_b = \tilde{\lambda}_b + \tfrac{\langle ca\rangle}{\langle ba\rangle}\tilde{\lambda}_c \ , \\ \eta'_a &= \eta_a + \tfrac{\langle cb\rangle}{\langle ab\rangle}\eta_c \ , \quad \eta'_b = \eta_b + \tfrac{\langle ca\rangle}{\langle ba\rangle}\eta_c \ . \end{aligned} \tag{2.6.30}$$

There are two kinds of inverse soft factors.

1. The addition of particle leaving the value k of negative helicity gluons unchanged means just the re-interpretation

$$Y'_{n,k}(Z_1,\cdots,Z_{n-1},Z_n) = Y_{n-1,k}(Z_1,\cdots,Z_{n-1}) \tag{2.6.31}$$

without actual dependence on Z_n. There is however a dependence on the momentum of the added particle since the relationship between momenta and momentum twistors is modified by the addition obtained by applying the basic rules relating region super momenta and momentum twistors (light-like momentum determines λ_i and twistor equations for x_i and λ_i,η_i determine (μ_i,χ_i)) is expressible assigned to the external particles [B55]. Modifications are needed only for the new vertex and its neighbors.

2. The addition of a particle increasing k with single unit is a more complex operation which can be understood in terms of a residue of $Y_{n,k}$ proportional to $Y_{n-1,k-1}$ and Yangian invariant $[z_1\cdots z_5]$ with five arguments constructed from basic Yangian invariants with four arguments. The relationship between the amplitudes is now

$$Y'_{n,k}(..,Z_{n-1}Z_n,Z_1\cdots) = [n-2\ n-1\ n\ 1\ 2]\times Y_{n-1,k-1}(\cdots\hat{Z}_{n-1},\hat{Z}_1,\cdots)\ . \tag{2.6.32}$$

Here

$$[abcde] = \frac{\delta^{0|4}(\eta_a\langle bcde\rangle + cyclic)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}\ . \tag{2.6.33}$$

denoted also by $R(a,b,c,d,e)$ is the fundamental R-invariant appearing in one loop corrections of MHV amplitudes and will appears also in the recursion formulas. $\langle abcd\rangle$ is the fundamental super-conformal invariant associated with four super twistors defined in terms of the permutation symbol.

$\hat{Z}_{n-1},\hat{Z}_1$ are deformed momentum twistor variables. The deformation is determined from the relationship between external momenta, region momenta and momentum twistor variables. $\hat{Z}^1$ is the intersection $\hat{Z}^1=(n-2\ n-1\ 2)\cap(12)$ of the line (12) with the plane $(n-2\ n-1\ 2)$ and $\hat{Z}^{n-1}$ the intersection $\hat{Z}^1=(12n)\cap(n-2\ n-1)$ of the line $(n-2\ n-1)$ with the plane $(12n)$. The interpretation for the intersections at the level of ordinary Feynman diagrams is in terms of the collinearity of the four-momenta involved with the underlying box diagram with parallel on mass shell particles. These result from unitarity conditions obtained by putting maximal number of loop momenta on mass shell to give the leading singularities.

The explicit expressions for the momenta are

$$\begin{aligned}\hat{Z}^1 &\equiv (n-2\ n-1\ 2)\cap(12)Z_1 = \langle 2\ n-2\ n-1\ n\rangle + Z_2\langle n-2\ n-1\ n\ 1\rangle\ ,\\ \hat{Z}^{n-1} &\equiv (12n)\cap(n-2\ n-1) = Z_{n-2}\langle n-2\ n-1\ n\ 2\rangle + Z_{n-1}\langle n\ 1\ 2\ n-2\rangle\ .\end{aligned} \tag{2.6.34}$$

These intersections also appear in the expressions defining the recursion formula.

Removal of particles and merge operation

Particles can be also removed. The first manner to remove particle is by integrating over the twistor variable characterizing the particle. This reduces k by one unit. Merge operation preserves the number of loops but removes a particle particle by identifying the twistor variables of neighboring particles. This operation corresponds to an integral over on mass shell loop momentum at the level of tree diagrams and by Witten's half Fourier transform can be transformed to twistor integral.

The product

$$Y'(Z_1,\cdots Z_n) = Y_1(Z_1,\cdots Z_m)\times Y_2(Z_{m+1},\cdots Z_n) \tag{2.6.35}$$

of two Yangian invariants is again a Yangian invariant. This is not quite trivial since the dependence of region momenta and momentum twistors on the momenta of external particles makes the operation non-trivial.

Merge operation allows to construct more interesting invariants from the products of Yangian invariants. One begins from a product of Yangian invariants (Yangian invariant trivially) represented cyclically as points of circle and identifies the last twistor argument of given invariant with the first twistor argument of the next invariant and performs integrals over the momentum twistor variables appearing twice. The soft k-increasing and preserving operations can be described also in terms of this operation for Yangian invariants such that the second invariant corresponds to 3-vertex. The cyclic merge operation applied to four MHV amplitudes gives NMHV amplitudes associated with on mass shell momenta in box diagrams. By applying similar operation to NMHV amplitudes and MHV amplitudes one obtains 2-loop amplitudes. In [B36] examples about these operations are described.

BCFW bridge

BCFW bridge allows to build general tree diagrams from MHV tree diagrams [B20, B20] and recursion formula of [B36] generalizes this to arbitrary diagrams. At the level of Feynman diagrams it corresponds to a box diagram containing general diagrams labeled by L and R and MHV and $\overline{MHV}$ 3-vertices ($\overline{MHV}$ 3-vertex allows expression in terms of MHV diagrams) with the lines of the box on mass shell so that the three momenta emanating from the vertices are parallel and give rise to a one-loop leading singularity.

At the level of Feynman diagrams BCFW bridge corresponds to so called "two-mass hard" leading singularities associated with box diagrams with light-like momenta at the four lines of the diagram [B34]. The motivation for the study of these diagrams comes from the hypothesis the leading order singularities obtained by putting as many particles as possible on mass shell contain the data needed to construct scattering amplitudes of $\mathcal{N}=4$ SYM completely. This representation of the leading singularities generalizes to arbitrary loops. The recent article is a continuation of this program to planar amplitudes.

Also BCFW bridge allows an interpretation as a particular kind fusion for Yang invariants and involves all the basic operations. One starts from the amplitudes $Y^L_{n_L,k_L}$ and $Y^R_{n_R,k_R}$ and constructs an amplitude $Y'_{n_L+n_R,k_L+k_R+1}$ representing the amplitude which would correspond to a generalization of the MHV diagrams with the two tree diagrams connected by the MHV propagator (BCFW bridge) replaced with arbitrary loop diagrams. Particle "1" *resp.* "j+1" is added by the soft k-increasing factor to Y_{n_L+1,k_L+1} *resp.* Y_{n_R+1,k_R+1} giving amplitude with $n+2$ particles and with k-charge equal to k_L+k_R+2. The subsequent operations must reduce k-charge by one unit. First repeated "1" and "j+1" are identified with their copies by k conserving merge operation, and after that one performs an integral over the twistor variable Z^I associated with the internal line obtained and reducing k by one unit. The soft k-increasing factors bring in the invariants $[n-1\ n\ 1\ I\ j+2]$ associated with Y_L and $[1\ I\ j+1\ j\ j-1]$ associated with Y_R. The integration contour is chosen so that it selects the pole defined by $\angle n-1\ n\ 1\ I\rangle$ in the denominator of $[n-1\ n\ 1\ I\ j+2]$ and the pole defined by $\langle 1\ I\ j+1\ j\rangle$ in the denominator of $[1\ I\ j+1\ j\ j-1]$.

The explicit expression for the BCFW bridge is very simple:

$$\begin{aligned} (Y_L \otimes_{BCFW} Y_R)(1,\cdots,n) &= [n-1\ n\ 1\ j\ j+1]\times Y_R(1,\cdots,j,I)Y_L(I,j+1,\cdots,n-1,\hat{n})\ , \\ \hat{n} &= (n-1\ n)\cap(j\ j+1\ 1)\ ,\ \ I=(j\ j+1)\cap(n-1\ n\ 1)\ . \end{aligned} \quad (2.6.36)$$

Single cuts and forward limit

Forward limit operation is used to increase the number of loops by one unit. The physical picture is that one starts from say 1-loop amplitude and cuts one line by assigning to the pieces of the line opposite light-like momenta having interpretation as incoming and outgoing particles. The resulting amplitude is called forward limit. The only reasonable interpretation seems to be that the loop integration is expressed by unitarity as forward limit meaning cutting of the line carrying the loop momentum. This operation can be expressed in a manifestly Yangian invariant way as entangled removal of two particles with the merge operation meaning the replacement $Z_n\rightarrow Z_{n-1}$.

Particle $n+1$ is added adjacent to A, B as a k-increasing inverse soft factor and then A and B are removed by entangled integration, and after this merge operation identifies $n+1$ and 1.

Forward limit is crucial for the existence of loops and for Yangian invariants it corresponds to the poles arising from $\langle (AB)_q Z_n(z) Z_1)\rangle$ the integration contour $Z_n + zZ_{n-1}$ around Z_b in the basic formula $M = \oint (dz/z) M_n$ leading to the recursion formula. A and B denote the momentum twistors associated with opposite light-like momenta. In the generalized unitarity conditions the singularity corresponds to the cutting of line between particles n and 1 with momenta q and $-q$, summing over the multiplet of stats running around the loop. Between particles n_2 and 1 one has particles $n-1$, n with momenta $q, -q$. $q = x_1 - x_n = -x_n + x_{n-1}$ giving $x_1 = x_{n-1}$. Light-likeness of q means that the lines $(71) = (76)$ and (15) intersect. At the forward limit giving rise to the pole Z_6 and Z_7 approach to the intersection point $(76) \cap (15)$. In a generic gauge theories the forward limits are ill-defined but in super-symmetric gauge theories situation changes.

The corresponding Yangian operation removes two external particles with opposite four-momenta and involves integration over two twistor variables Z_a and Z_b and gives rise to the following expression

$$\int_{GL(2)} Y(\cdots, Z_n, Z_A, Z_B, Z_1, \cdots) \ . \qquad (2.6.37)$$

The integration over $GL(2)$ corresponds to integration over twistor variables associated Z_A and Z_B. This operation allows addition of a loop to a given amplitude. The line $Z_a Z_b$ represents loop momentum on one hand and the dual x-coordinate identified as momentum propagating along the line on the other hand.

The integration over these variables is equivalent to an integration over loop momentum as the explicit calculation of [B36] (see pages 12-13) demonstrates. If the integration contours are products in the product of twistor spaces associated with a and b the and gives lower order Yangian invariant as answer. It is however also possible to choose the integration contour to be entangled in the sense that it cannot be reduced to a product of integration contours in the Cartesian product of twistor spaces. In this case the integration gives a loop integral. In the removal operation Yangian invariance can be broken by IR singularities associated with the integration contour and the procedure does not produce genuine Yangian invariant always.

What is highly interesting from TGD point of view is that this integral can be expressed as a contour integral over $CP_1 \times CP_1$ combined with integral over loop momentum. If TGD vision about generalized Feynman graps in zero energy ontology is correct, the loop momentum integral is discretized to an an integral over discrete mass shells and perhaps also to a sum over discretized momenta and one can therefore avoid IR singularities.

2.6.6 Explicit Formula For The Recursion Relation

Recall that the recursion formula is obtained by considering super-symmetric momentum-twistor deformation $Z_n \to Z_n + zZ_{n-1}$ and by integrating over z to get the identity

$$M_{n,k,l} \ = \ \oint \frac{dz}{z} \hat{M}_{n,k,l}(z) \ . \qquad (2.6.38)$$

This integral equals to integral with reversed integration contour enclosing the exterior of the contour. The challenge is to deduce the residues contributing to the residue integral and the claim of [B36] is that these residues reduce to simple basic types.

1. The first residue corresponds to a pole at infinity and reduces the particle number by one giving a contribution $M_{n-1,k,l}(1, \cdots, n-1)$ to $M_{n,k,l}(1, \cdots, n-1, n)$. This is not totally trivial since the twistor variables are related to momenta in different manner for the two amplitudes. This gives the first contribution to the right hand side of the formula below.
2. Second pole corresponds to the vanishing of $\langle Z_n(z) Z_1 Z_j Z_{j+1}\rangle$ and corresponds to the factorization of channels. This gives the second BCFW contribution to the right hand side of the formula below. These terms are however not enough since the recursion formula would imply the reduction to expressions involving only loop corrections to 3-loop vertex which vanish in $\mathcal{N} = 4$ SYM.

3. The third kind of pole results when $\langle (AB)_q Z_n(z) Z_1\rangle$ vanishes in momentum twistor space. $(AB)_q$ denotes the line in momentum twistor space associated with q: th loop variable.

The explicit formula for the recursion relation yielding planar all loop amplitudes is obtained by putting all these pieces together and reads as

$$\begin{array}{lcl} M_{n,k,l}(1,\cdots,n) & = & M_{n-1,k,l}(1,\cdots,n-1) \\ & + & \sum_{n_L,k_L,l_L;j} [j\ j+1\ n-1\ n\ 1]\, M^R_{n_R,k_R,l_R}(1,\cdots,j,I_j)\times M^L_{n_L,k_L,l_L}(I_j,j+1,\cdots,\hat{n}_j) \\ & + & \int_{GL(2)} [AB\ n-1\ n\ 1]\, M_{n+2,k+1,n,k-1}(1,\cdots,\hat{n}_{AB},\hat{A},B) \ , \\ n_L & + & n_R = n+2 \ ,\ k_L+k_R = k-1 \ ,\ l_R+l_L = l \ . \end{array} \tag{2.6.39}$$

The momentum super-twistors are given by

$$\begin{array}{ll} \hat{n}_j = (n-1\ n)\cap(j\ j+1\ 1) \ , & I_j = (j\ j+1\ 1)\cap(n-1\ n\ 1) \ , \\ \hat{n}_{AB} = (n-1\ n)\cap(AB\ 1) \ , & \hat{A} = (AB)\cap(n-1\ n\ 1) \ . \end{array} \tag{2.6.40}$$

The index l labels loops in $n+2$-particle amplitude and the expression is fully symmetrized with equal weight for all loop integration variables $(AB)_l$. A and B are removed by entangled integration meaning that $GL(2)$ contour is chosen to encircle points where both points A,B on the line (AB) are located at the intersection of the line (AB) with the plane $(n-1\ n\ 1)$. GL(2) integral can be done purely algebraically in terms of residues.

In [B36] and [B55] explicit calculations for N^kMHV amplitudes are carried out to make the formulas more concrete. For N^1MHV amplitudes second line of the formula vanishes and the integrals are rather simple since the determinants are 1×1 determinants.

Chapter 3

From Principles to Diagrams

3.1 Introduction

The generalization of twistor diagrams to TGD framework has been very inspiring (and also frightening) mission impossible and allowed to gain deep insights about what TGD diagrams could be mathematically. I of course cannot provide explicit formulas but the general structure for the construction of twistorial amplitudes in $\mathcal{N}=4$ SUSY suggests an analogous construction in TGD thanks to huge symmetries of TGD and unique twistorial properties of $M^4 \times CP_2$. The twistor program in TGD framework has been summarized in [L10].

Contrary to the original expectations, the twistorial approach is not a mere reformulation but leads to a first principle identification of cosmological constant and perhaps also of gravitational constant and to a modification of the dynamics of Kähler action however preserving the known extremals and basic properties of Kähler action and allowing to interpret induced Kähler form in terms of preferred imaginary unit defining twistor structure.

There are some new results forcing a profound modification of the recent view about TGD but consistent with the general picture. A more explicit realization of twistorialization as lifting of the preferred extremal X^4 of Kähler action to corresponding 6-D twistor space X^6 identified as surface in the 12-D product of twistor spaces of M^4 and CP_2 allowing Kähler structure suggests itself. The fiber F of Minkowskian twistor space must be identified with sphere S^2 with signature $(-1,-1)$ and would be a variant of the complex space with complex coordinates associated with S^2 and transversal space E^2 in the decomposition $M^4 = M^2 \times E^2$ and one hyper-complex coordinate associated with M^2.

The action principle in 6-D context is also Kähler action, which dimensionally reduces to Kähler action plus cosmological term. This brings in the radii of spheres $S^2(M^4)$ and $S^2(CP_2)$ associated with the twistors space of M^4 and CP_2. For $S(CP_2)$ the radius is of order CP_2 radius R. $R(S^2(M^4))$ could be of the order of Planck length l_P, which would thus become purely classical parameter contrary the expectations. An alternative option is $R(S^2(M^4)) = R$ The radius of S^2 associated with space-time surface is determined by the induced metric and is emergent length scale. The normalization of 6-D Kähler action by a scale factor $1/L^2$ with dimension, which is inverse length squared brings in a further length scale closely related to cosmological constant which is also dynamical and has correct sign to explain accelerated expansion of the Universe. The order of magnitude for L must be radius of the $S^2(X^4)$ and therefore small. This could mean a gigantic cosmological constant. Just as in GRT based cosmology!

This issue can be solved by using the observation that thanks to the decomposition $H = M^4 \times CP_2$ 6-D Kähler action is a sum of two independent terms. The first term corresponds to the 6-D lift of the ordinary Kähler action and for it the contribution from $S^2(CP_2)$ fiber is assumed to be absent: this could be due to the imbedding of $S^2(X^4)$ reducing to identification $S^2(M^4)$ and is not true generally. Second term in action is assumed to come from the $S^2(M^4)$ fiber of twistor space $T(M^4)$. The independency implies that couplings strengths are independent for them.

The analog for Kähler coupling strength (analogous to critical temperature) associated with $S^2(M^4)$ must be extremely large - so large that one has $\alpha_K(M^4) \times R(M^4)^2 \sim L^2$, L size scale of the recent Universe. This makes possible the small value of cosmological constant assignable to the volume term given by this part of the dimensionally reduced action. Both Kähler coupling strengths are assumed to have a spectrum determined by quantum criticality and the spectrum of $\alpha_K(M^4)$ comes essentially as p-adic primes satisfying p-adic length scale hypothesis $p \simeq 2^k$, k prime. In fact, it turns that one can assumed that the entire 6-D Kähler action contributes if one assumes that the winding numbers (w_1, w_2) for the map $S^2(X^4) \to S^2(M^4) \times S^2(CP_2)$ satisfy $(w_1, w_2) = (n, 0)$ in cosmological scales. The identification of w_1 as $h_{eff}/h = n$ is highly suggestive.

The dimensionally reduced dynamics is a highly non-trivial modification of the dynamics of Kähler action however preserving the known extremals and basic properties of Kähler action and allowing to interpret induced Kähler form in terms of preferred imaginary unit defining twistor structure. Strong constraints come also from the condition that induced spinor structure coming from that for twistor space $T(H)$ is essentially that coming from that of H.

Second new element is the fusion of the twistorial approach with the vision that diagrams are representations for computations. This as also quantum criticality demands that the diagrams should allow huge symmetries allowing to transform them to braided generalizations of tree-diagrams. Several guiding principles are involved and what is new is the observation that they indeed seem to form a coherent whole.

In the sequel I will discuss the recent understanding of twistorizalization, which is considerably improved from that in the earlier formulation. I formulate the dimensional reduction of 6-D Kähler action and consider the physical interpretation. There are considerable uncertainties at the level of details I dare believe that basically the situation is understood. After that I proceed to discuss the basic principles behind the recent view about scattering amplitudes as generalized Feynman diagrams.

3.2 Twistor lift of Kähler action

First I will try to clarify the mathematical details related to the twistor spaces and how they emerge in the recent context. I do not regard myself as a mathematician in technical sense and I can only hope that the representation based on physical intuition does not contain serious mistakes.

3.2.1 Embedding space is twistorially unique

It took roughly 36 years to learn that M^4 and CP_2 are twistorially unique. Space-times are surfaces in $H = M^4 \times CP_2$. M^4 and CP_2 are unique 4-manifolds in the sense that both allow twistor space with Kähler structure: Kähler structure is the crucial concept. Strictly speaking, it is E^4 and S^4 allow twistor space with Kähler structure [A54] : in the case of M^4 signature could cause problems. The standard identification for the twistor space of M^4 would be Minkowskian variant $PT = P_3 = SU(2,2)/SU(2,1) \times U(1)$ of 6-D twistor space $PT = CP_3 = SU(4)/SU(3) \times U(1)$ of E^4. The twistor space of CP_2 is 6-D $T(CP_2) = SU(3)/U(1) \times U(1)$, the space for the choices of quantization axes of color hypercharge and isospin.

The case of M^4 is however problematic. It is often stated that the twistor space is $PT = CP_3 = SU(4)/SU(3) \times U(1)$. The metric of twistor space does not appear in the construction of twistor amplitudes. Already the basic structure of PT suggests that this identification cannot be correct.

As if the situation were not complicated enough, there are two notions of twistor space: the twistor space identified as P_3 and as a trivial sphere bundle $M^4 \times CP_1$ having Kähler structure - what Kähler structure actually means in case of M^4 is hower not quite clear.

These considerations lead to a proposal - just a proposal - for the formulation of TGD in which space-time surfaces X^4 in H are lifted to twistor spaces X^6, which are sphere bundles over X^4 and such that they are surfaces in 12-D product space $T(M^4) \times T(CP_2)$ such the twistor structure of X^4 are in some sense induced from that of $T(M^4) \times T(CP_2)$. In the following $T(M^4)$ therefore denotes the trivial sphere bundle $M^4 \times CP_1$ over M^4 and twistorialization of scattering amplitudes would involve the projection from $T(M^4)$ to P_3. What is nice in this formulation is that one could use all the machinery of algebraic geometry so powerful in superstring theory (Calabi-Yau manifolds).

3.2.2 Some basic definitions

What twistor structure in Minkowskian signature does really mean geometrically has remained a confusing question for me. The problems associated with the Minkowskian signature of the metric are encountered also in twistor Grassmann approach to the scattering amplitudes but are circumvented by performing Wick rotation that is using E^4 or S^4 instead of M^4 and applying algebraic continuation. Also complexification of Minkowksi space for momenta is used. These tricks do not apply now.

To make this more concrete, let us sum up the basic definitions.

1. Bi-spinors in representations (1/2,0) and (0,1/2) of Lorentz group are the building bricks of twistors. Bi-spinors v^a and their conjugates $v^{a'}$ have the following inner products:

$$\langle vw\rangle = \epsilon_{ab}v^a w^b \ , \quad [vw] = \epsilon_{a'b'}v^{a'}w^{b'} \ , \\ \epsilon_{ab} = (0,1;-1,0) \ , \quad \epsilon_{a'b'} = (0,1;-1,0) \ . \tag{3.2.1}$$

   Unprimed spinor and its primed variant of the spinor are related by complex conjugation. Index raising is by the inverse ϵ^{ab} of ϵ_{ab}.

2. Twistors are identified as pairs of 2-spinor and its conjugate

$$Z^\alpha = (\lambda_a, \mu^{a'}) \ , \quad \overline{Z}_\alpha = (\overline{\mu}^a, \lambda_{a'}) \tag{3.2.2}$$

   The norm for Z^α is defined as

$$Z^\alpha \overline{Z}^\alpha = \langle \lambda\overline{\mu}\rangle + [\overline{\lambda}\mu] \ . \tag{3.2.3}$$

   One can write the metric explicitly as direct sum of terms of form $dudv$ (metric of M^2) and each of the can be taken to diagonal form (1,-1). Hence the metric can be written as $diag(1,1,1,1,-1,-1,-1,-1)$.

3. This norm allows to decompose PT to 3 parts PT_+,PT_- and PN in a projectively invariant manner depending on whether the sign of the norm is negative, positive, or whether it vanishes. PT_+ and PT_- serve as loci for the twistor lifts of positive and negative energy modes of massless fields. PN corresponds to the 5-D boundary of the lightcone of $M(2,4)$. By projective identification along light-like radial coordinate it reduces to what is known as conformal compactification of M^4, whose metric is defined only apart from a conformal factor. The natural metric of $PT = P_3$ does not seem to play any role in the construction of the amplitudes relying on projective invariants. The signature of M^4 metric however makes itself visible in the structure of PT: for the Euclidian variant of twistor space one would not have this decomposition to three parts.

Another definition of twistor space - to be used in the geometrization of twistor approach to be proposed - is as a trivial S^2 bundle $M^4 \times CP_1$ over M^4. Since the twistor spheres associated with the points of M^4 with light-like separation intersect, these two definitions cannot be equivalent. In fact, the proper definition of twistor space relies on double fibration involving both views about twistor space discussed in [B64] (see `http://tinyurl.com/yb4bt74l`).

1. The twistor bundle denoted as PS is the product $M^4 \times CP_1$ with CP_1 realized as projective space and having coordinates $(x^{aa'}, \lambda_a)$, $\{x^{aa'}\} \leftrightarrow x^\mu \sigma_\mu$, where the spinor λ_a is projective 2-spinor in $(1/2, 0)$ representation.

2. The twistors defined in this manner have a trivial projection q to M^4 and non-trivial projection p to P_3 with local projective coordinates $(\lambda_a, \mu^{a'})$. The projection p is defined by the projectively invariant incidence relation

$$\mu^{a'} = ix^{aa'}\lambda_a$$

If $y^{aa'}$ and $a^{aa'}$ differ by light-like vector there exists spinor λ annihilated by the difference vector and there exists twistor $(\lambda_a, \mu^{a'})$ to which both (x,λ) and (y,λ) are mapped by the incidence relation. Thus the images of twistor spheres associated for points with light-like separation intersect so that one does not have a proper CP_1 bundle structure.

3. The trivial twistor bundle $T(M^4) = M^4 \times CP_1$ would define the twistor space of M^4 in geometric sense. For this space the metric matters and the radius of CP_1 turns out to allow identification in terms of Planck length. Gravitational interaction would bring in Planck length as a basic scale in this manner. PT in turn would define the twistor space in which the twistor lifts of embedding space-spinor fields are defined. For this space the metric, which is degenerate and seems to be only projectively defined should not be relevant as the construction of twistorial amplitudes suggests. Note however that the identification as the Minkowskian variant of P_3 allows also the introduction of metric.

This picture has an important immediate implication for the construction of quantum TGD. Positive and negative energy parts of zero energy states are defined at light-like boundaries of $CD \times CP_2$, where CD is the intersection of future and past directed light-cones. The twistor lifts of the amplitudes from $\delta CD \times CP_2$ must be single valued. The strongest condition guaranteing this is that they do not depend on the radial light-like coordinate at δCD. Super-symplectic symmetry implying the analog of conformal gauge symmetry for the radial light-like coordinate could guarantee this. There is however a hierarchy of conformal gauge symmetry breakings corresponding to the inclusion hierarchy of isomorphic sub-algebras so that this condition is too strong. A weaker condition is that the amplitude $F(m,\lambda)$ in $T(M^4)$ is constant along the light-like ray for the λ associated with the m along this ray. An even stronger condition is that $F(m,\lambda)$ vanishes along the ray. Particle would not propagate along δCD and would avoid remaining at the boundary of CD, a condition which is perfectly sensible physically.

3.2.3 What does twistor structure in Minkowskian signature really mean?

The following considerations relate to $T(M^4)$ identified as trivial bundle $M^4 \times CP_1$ with natural coordinates $(m^{aa'}, \lambda_a)$, where λ_a is projective spinor. The challenge is to generalize the complex structure of twistor space of E^4 to that for M^4. It turns out that the assumption that twistor space has ordinary complex structure fails. The first guess was that the fiber of twistor space is hyperbolic sphere with metric signature $(1,-1)$ having infinite area so that the 6-D Kähler action would be infinite. This makes no sense. The only alternative, which comes in mind is a hypercomplex generalization of the Kähler structure for M^4 lifted to twistor space, which locally means only adding of S^2 fiber with metric signature $(-1,-1)$.

1. To proceed one must make an explicit the definition of twistor space. The 2-D fiber S^2 consists of antisymmetric tensors of X^4 which can be taken to be self-dual or anti-self-dual by taking any antisymmetric form and by adding to its plus/minus its dual. Each tensor of this kind defines a direction - point of S^2. These points can be also regarded as quaternionic imaginary units. One has a natural metric in S^2 defined by the X^4 inner product for antisymmetric tensors: this inner product depends on space-time metric. Kähler action density is example of a norm defined by this inner product in the special case that the antisymmetric tensor is induced Kähler form. Induced Kähler form defines a preferred imaginary unit and is needed to define the imaginary part $\omega(X,Y) = ig(X,-JY)$ of hermitian form $h = h + i\omega$.
2. To define the analog of Kähler structure for M^4, one must start from a decomposition of $M^4 = M^2 \times E^2$ (M^2 is generated by light-like vector and its dual) and E^2 is orthogonal to it. M^2 allows hypercomplex structure, which light-like coordinates $(u = t - z, v = t + z)$ and E^2 complex structure and the metric has form $ds^2 = dudv + dzd\overline{z}$. Hypercomplex numbers can be represented as $h = t + iez$, $i^2 = -1, e^2 = -1$ $i^2 = -1, e^2 = -1$. Hyper-complex numbers do not define number field since for light-like hypercomplex numbers $t + iez$, $t = \pm z$ do not have finite inverse. Hypercomplex numbers allow a generalization of analytic functions used routinely in physics. Kähler form representing hypercomplex imaginary unit would be replaced with eJ. One would consider sub-spaces of complexified quaternions spanned by real unit and units eI_k, $k = 1,2,3$ as representation of the tangent space of space-time surfaces in Minkowskian regions. This is familiar already from M^8 duality [K104].

$M^4 = M^2 \times E^2$ decomposition can depend on point of M^4 (polarization plane and light-like momentum direction depend on point of M^4. The condition that this structure allows global coordinates analogous to $(u, v, z, \overline{z})$ requires that the distributions for M^2 and E^2 are integrable and thus define 2-D surfaces. I have christened this structure Hamilton-Jacobi structure. It emerges naturally in the construction of extremals of Kähler action that I have christened massless extremals (MEs, [K12]) and also in the proposal for the generalization of complex structure to Minkowskian signature.

One can define the analog of Kähler form by taking sum of induced Kähler form J and its dual $*J$ defined in terms of permutation tensor. The normalization condition is that this form integrates to the negative of metric $(J \pm *J)^2 = -g$. This condition is possible to satisfy.

3. How to lift the Hamilton Jacobi structure of M^4 to Kähler structure of its twistor space? The basic definition of twistors assumes that their exists a field of time-like directions, and that one considers projections of 4-D antisymmetric tensors to the 3-space orthogonal to the time-like direction at given point. One can say that the projection yields magnetic part of the antisymmetric tensor (say induced Kähler form J) with positive norm with respect to natural metric induced to the twistor fiber from the inner product between two-forms. This unique time direction would be defined the light-like vector defining M^2 and its dual. Therefore the signature of the metric of S^2 would be $(-1, -1)$. In quaternionic picture this direction corresponds to real quaternionic unit.

4. To sum up, the metric of the Minkowskian twistor space has signature $(-1, -1, 1, -1, -1, -1)$. The Minkowskian variant of the twistor space would give 2 complex coordinates and one hyper-complex coordinate. Cosmological term would be finite and the sign of the cosmological term in the dimensionally reduced action would be positive as required. Also metric determinant would be imaginary as required. At this moment I cannot invent any killer objection against this option.

It must be made clear that the proposed definition of twistor space of M^4 does not seem to be equivalent with the twistor space assignable to conformally compactified M^4. One has trivial S^2 bundle and Hamilton-Jacobi structure, which is hybrid of complex and hyper-complex structure.

3.2.4 What does the induction of the twistor structure to space-time surface really mean?

Consider now what the induction of the twistor structure to space-time surface X^4 could mean.

1. The induction procedure for Kähler structure of 12-D twistor space T requires that the induced metric and Kähler form of the base space X^4 of X^6 obtained from T is the same as that obtained by inducing from $H = M^4 \times CP_2$. Since the Kähler structure and metric of T is lift from H this seems obvious. Projection would compensate the lift.

2. This is not yet enough. The Kähler structure and metric of S^2 projected from T must be same as those lifted from X^4. The connection between metric and ω implies that this condition for Kähler form is enough. The antisymmetric Kähler forms in fiber obtained in these two ways co-incide. Since Kähler form has only one component in 2-D case, one obtains single constraint condition giving a commutative diagram stating that the direct projection to S^2 equals with the projection to the base followed by a lift to fiber. The resulting induced Kähler form is not covariantly constant but in fiber S^2 one has $J^2 = -g$.

   As a matter of fact, this condition might be trivially satisfied as a consequence of the bundle structure of twistor space. The Kähler form from $S^2 \times S^2$ can be projected to S^2 associated with X^4 and by bundle projection to a two-form in X^4. The intuitive guess - which might be of course wrong - is that this 2-form must be same as that obtained by projecting the Kähler form of CP_2 to X^4. If so then the bundle structure would be essential but what does it really mean?

3. Intuitively it seems clear that X^6 must decompose locally to a product $X^4 \times S^2$ in some sense. This is true if the metric and Kähler form reduce to direct sums of contributions from the tangent spaces of X^4 and S^2. This guarantees that 6-D Kähler action decomposes to a sum of 4-D Kähler action and Kähler action for S^2.

This could be however too strong a condition. Dimensional reduction occurs in Kaluza-Klein theories and in this case the metric can have also components between tangent spaces of the fiber and base being interpreted as gauge potentials. This suggests that one should formulate the condition in terms of the matrix $T \leftrightarrow g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}$ defining the norm of the induced Kähler form giving rise to Kähler action. T maps Kähler form $J \leftrightarrow J_{\alpha\beta}$ to a contravariant tensor $J_c \leftrightarrow J^{\alpha\beta}$ and should have the property that $J_c(X^4)$ ($J_c(S^2)$) does not depend on $J(S^2)$ ($J(X^4)$).

One should take into account also the self-duality of the form defining the imaginary unit. In X^4 the form $S = J \pm *J$ is self-dual/anti-self dual and would define twistorial imaginary unit since its square equals to $-g$ representing the negative of the real unit. This would suggest that 4-D Kähler action is effectively replaced with $(J \pm *J) \wedge (J \pm *J) = J^*J \pm J \wedge J$, where $*J$ is the Hodge dual defined in terms of 4-D permutation tensor ϵ. The second term is topological term (Abelian instanton term) and does not contribute to field equations. This in turn would mean that it is the tensor $T \pm \epsilon$ for which one can demand that $S_c(X^4)$ ($S_c(S^2)$) does not depend on $S(S^2)$ ($S(X^4)$).

4. The preferred quaternionic imaginary unit should be represented as a projection of Kähler form of 12-D twistor space $T(H)$. The preferred imaginary unit defining twistor structure as sum of projections of both $T(CP_2)$ and $T(M^4)$ Kähler forms would guarantee that vacuum extremals like canonically imbedded M^4 for which $T(CP_2)$ Kähler form contributes nothing have well-defined twistor structure. $T(M^4)$ or $T(CP_2)$ are treated completely symmetrically but the maps of $S^2(X^4)$ to $S^2(M^4)$ and $S^2(CP_2)$ characterized by winding numbers induce symmetry breaking.

   For Kähler action $M^4 - CP_2$ symmetry does not make sense. 4-D Kähler action to which 6-D Kähler action dimensionally reduces can depend on CP_2 Kähler form only. I have also considered the possibility of covariantly constant self-dual M^4 term in Kähler action but given it up because of problems with Lorentz invariance. One should couple the gauge potential of M^4 Kähler form to induced spinors. This would mean the existence of vacuum gauge fields coupling to sigma matrices of M^4 so that the gauge grop would be non-compact $SO(3,1)$ leading to a breakdown of unitarity.

There is still one difficulty to be solved.

1. The normalization of 6-D Kähler action by a scale factor $1/L^2$ with dimension, which is inverse length squared, brings in a further length scale. The first guess is that $1/L^2$ is closely related to cosmological constant, which is also dynamical and $1/L^2$ has indeed correct sign to explain accelerated expansion of the Universe. Unfortunately, if $1/L^2$ is of order cosmological constant, the value of the ordinary Kähler coupling strength α_K would be enormous. As a matter of fact, the order of magnitude for L^2 must be equal to the area of $S^2(X^4)$ and in good approximation equal to $L^2 = 4\pi R^2(S^2(M^4))$ and therefore in the same range as Planck length l_P and CP_2 radius R. This would imply a gigantic value of cosmological constant. Just as in GRT based cosmology!

2. This issue can be solved by using the observation that thanks to the decomposition $H = M^4 \times CP_2$, 6-D Kähler action is sum of two independent terms. The first term corresponds to the 6-D lift of the ordinary Kähler action. For it the contribution from $S^2(CP_2)$ fiber is absent if the embedding of $S^2(X^4)$ to $S^2(M^4) \times S^2(CP_2)$ reduces to identification with $S^2(M^4)$ so that $S^2(CP_2)$ is effectively absent: this is not true generally. Second term in the action is assumed to come from the $S^2(M^4)$ fiber of twistor space $T(M^4)$, which can indeed contribute without breaking of Lorentz symmetry. In fact, one can assume that also the Kähler form of M^4 contributes as will be found.

3. The independency implies that Kähler couplings strengths are independent for them. If one wants that cosmological constant has a reasonable order of magnitude, $L \sim R(S^2(M^4))$ must hold true and the analog $\alpha_K(S^2(M^4))$ of the ordinary Kähler coupling strength (analogous to critical temperature) must be extremely large - so large that one has

$$\alpha_K(M^4) \times 4\pi R(M^4)^2 \sim L^2 \quad ,$$

   where L is the size scale of the recent Universe.

This makes possible the small value of cosmological constant assignable to the volume term given by this part of dimensionally reduced action. Both Kähler coupling strengths are assumed to have a spectrum determined by quantum criticality and the spectrum of $\alpha_K(M^4)$ would be essentially as p-adic primes satisfying p-adic length scale hypothesis $p \simeq 2^k$, k prime. One can criticize this identification of 6-D Kähler action as artificial but it seems to be the only option that works. Interestingly also the contribution from M^4 Kähler form can be allowed since it is also extremely small. For canonically imbedded M^4 this contribution vanishes by self-duality of M^4 Kähler form and is extremely small for the vacuum extremals of Kähler action.

4. For general winding numbers of the map $S^2(X^4) \rightarrow S^2(M^4) \times S^2(CP_2)$ also $S^2(CP_2)$ Kähler form contributes and cosmological constant is gigantic. It would seem that only the winding numbers $(w_1, w_2) = (n, 0)$ are consistent with the observed value of cosmological constant. Hence it seems that there is no need to pose any additional conditions to the Kähler action if one uses the fact that $T(M^4)$ and $T(CP_2)$ parts are independent!

It is good to list the possible open issues related to the precise definition of the twistor structure and of M^4 Kähler action.

1. The proposed definition of M^4 twistor space a Cartesian product of M^4 and $S^2(M^4)$ parts involving Hamilton-Jacobi structure does not seem to be equivalent with the twistor identification as $SU(2,2)/SU(2,1) \times U(1)$ having conformally compactified M^4 as base space. There exists an entire moduli space of Hamilton-Jacobi structures. If the M^4 part of Kähler form participates in dynamics, one must include the specification of the Hamilton-Jacobi structure to the definition of CD and integrate over Hamilton Jacobi-structures as part of integral over WCW in order to gain Lorentz invariance. Note that Hamilton-Jacobi structure enters to dynamics also through the construction of massless extremals [K12].
2. The presence of M^4 part of Kähler form in action implies breaking of Lorentz invariance for extremals of lifted Kähler action. The same happens at the level of induced spinors if this Kähler form couples to embedding space spinors. If $T(M^4)$ is trivial bundle, one can include only the $T(S^2(M^4))$ part of Kähler form to Kähler action and couple only this to the spinors of $T(H)$. The integration over Hamilton-Jacobi structures becomes un-necessary.
3. If one includes M^4 part of Kähler form to 6-D Kähler action, one has several options. One can have sum of the Kähler actions for $T(M^4)$ and $T(CP_2)$ or Kähler action defined by the sum $J(T(M^4)/g_K$ and $J(T(CP_2)/\alpha_K$ with $\alpha_K(M^4) = g_K^2(M^4)/4\pi\hbar$ and $\alpha_K = g_K^2/4\pi\hbar$ with a proper normalization to guarantee that the squares of induced Kähler forms give sum of Kähler actions as in the first option. In this case one obtains interference term proportional to $Tr(J(M^4)J(CP_2)$. For the proposed value of α_K also the interference term is extremely small as compared to Kähler action in recent cosmology.

3.2.5 Could M^4 Kähler form introduce new gravitational physics?

The introduction of M^4 Kähler form could bring in new gravitational physics.

1. As found, the twistorial formulation of TGD assigns to M^4 a self dual Kähler form whose square gives Minkowski metric. It can (but need not if M^4 twistor space is trivial as bundle) contribute to the 6-D twistor counterpart of Kähler action inducing M^4 term to 4-D Kähler action vanishing for canonically imbedded M^4.
2. Self-dual Kähler form in empty Minkowski space satisfies automatically Maxwell equations and has by Minkowskian signature and self-duality a vanishing action density. Energy momentum tensor is proportional to the metric so that Einstein Maxwell equations are satisfied for a non-vanishing cosmological constant! M^4 indeed allows a large number of self dual Kähler fields (I have christened them as Hamilton-Jacobi structures). These are probably the simplest solutions of Einstein-Maxwell equations that one can imagine!
3. There however exist quite a many Hamilton-Jacobi structures. However, if this structure is to be assigned with a causal diamond (CD) it must satisfy additional conditions, say $SO(3)$ symmetry and invariance under time translations assignable to CD. Alternatively, covariant constancy and $SO(2) \subset SO(3)$ symmetry might be required.

This raises several questions. Could M^4 Kähler form replace CP_2 Kähler form in the picture for how gravitational interaction is mediated at quantal level? Could one speak of flux tubes of the magnetic part of this Kähler form? Or should one consider the Kähler field as a sum of the two Kähler forms weighted by the inverses $1/g_K$ of corresponding Kähler couplings. If so then M^4 contribution would be negligible except for canonically imbedded M^4 in the recent cosmology. Note that α_K and $\alpha_K(M^4)$ have interpretation as analogs of quantum critical temperatures but can depend on the p-adic lengths scale defining the cosmology.

1. The natural expectation is that Kähler form characterizes CD having preferred time direction suggested strongly by number theoretical considerations involving quaternionic structure with preferred direction of time axis assignable to real unit quaternion.

   Self-duality gives rise to Kähler magnetic and electric fields in the same spatial direction identifiable as a local quantization axis for spin assignable to CD assignable to observer. CD indeed serves as a correlate for conscious entity in TGD inspired theory of consciousness. Flux tube would connect mass M to mass m assignable to observer and flux tube direction would define spin quantization axes for the CD of the observer. Spin quantization axis would be naturally in the direction of magnetic field, which is direction of the flux tube.

2. The self-dual Kähler form could be spherically symmetric for CDs and represent self dual magnetic monopole field (dyon) with monopole charge at the line connecting the tips of CD and have non-vanishing components $J^{tr} = \epsilon^{tr\theta\phi}J_{\theta\phi}$, $J_{\theta\phi} = sin(\theta)$. One would have genuine monopole, which is somewhat questionable feature. Only the entire radial flux would be quantized. CD could be associated with the mass M of the central object. The gauge potential associated with J could be chosen to be $A_\mu \leftrightarrow (1/r, 0, 0, cos(\theta)$. I have considered this kind of possibility earlier in context of TGD inspired model of anyons but gave up the idea.

   The moduli space for CDs with second tip fixed would be hyperbolic space $H^3 = SO(3,1)/SO(3)$ or a space obtained by identifying points at the orbits of some discrete subgroup of $SO(3,1)$ as suggested by number theoretic considerations. This induced Kähler field could make the blackholes with center at this line to behave like M^4 magnetic monopoles if the M^4 part of Kähler form is induced into the 6-D lift of Kähler action with extremely small coefficients of order of magnitude of cosmological constant. Cosmological constant and the possibility of CD monopoles would thus relate to each other.

3. The self-dual M^4 Kähler form could be also covariantly constant ($J_{tz} = J_{xy} = 1$) and represent electric and magnetic fluxes in a fixed direction identifiable as a quantization axes for spin and characterizing CD. In this case the CD would be associated with the mass m of observer. The moduli space of CDs would be now $SO(3,1)/SO(1,1) \times SO(2)$ which is completely analogous to the twistor space $SU(3)/U(1) \times U(1)$.

4. Boundary conditions (allowing no boundaries!) demand that the flux tubes have closed cross section - say sphere S^2 - rather than disk: stability is guaranteed if the S^2 cross section is mapped to homologically non-trivial surface of CP_2 or is projection of it. This would give monopole flux also for CP_2 Kähler form so that the original hypothesis would be correct.

5. Radial flux tubes are possible both spherically symmetric and covariantly constant Kähler form possibly mediating gravitational interaction but the flux is not quantized unless preferred extremal property implies this: in any case M^4 flux would be very small unless one has large value of gravitational Planck constant implying n-sheeted covering of M^4 and flux is scale up by n since every sheet gives a contribution. For spherically symmetric M^4 Kähler form the flux tubes would have naturally conical structure spanning a constant solid angle. For covariantly constant Kähler form the flux tubes would be cylindrical.

There are further interpretational problems.

1. The classical coupling of M^4 Kähler gauge potential to induced spinors is not small. Can one really tolerate this kind of coupling equivalent to a coupling to a self dual monopole field carrying electric and magnetic charges? One could of course consider the condition that the string world sheets carrying spinor modes are such that the induced M^4 Kähler form vanishes and gauge potential become pure gauge. M^4 projection would be 2-D Lagrange manifold whereas CP_2 projection would carry vanishing induce W and possibly also Z^0 field in order

that em charge is well defined for the modes. These conditions would fix the string world sheets to a very high degree in terms of maps between this kind of 2-D sub-manifolds of M^4 and CP_2. Spinor dynamics would be determined by the avoidance of interaction!

Recall that one could interpret the localization of spinor modes to 2-surfaces in the sense of strong form of holography: one can continued induced spinor fields to the space-time interior as indeed assumed but the continuation is completely determined by the data at 2-D string world sheets.

It must be emphasized that the embedding space spinor modes characterizing the ground states of super-symplectic representations would not couple to the monopole field so that at this level Poincare invariance is not broken. The coupling would be only at the space-time level and force spinor modes to Lagrangian sub-manifolds.

2. At the static limit of GRT and for $g_{ij} \simeq \delta_{ij}$ implying $SO(3)$ symmetry there is very close analogy with Maxwell's equations and one can speak of gravi-electricity and gravi-magnetism with 4-D vector potential given by the components of $g_{t\alpha}$. The genuine $U(1)$ gauge potential does not however relate to the gravimagnetism in GRT sense. Situation would be analogous to that for CP_2, where one must add to the spinor connection $U(1)$ term to obtain respectable spinor structure. Now the $U(1)$ term would be added to trivial spinor connection of flat M^4: its presence would be justified by twistor space Kähler structure. If the induced M^4 Kähler form is present as a classical physical field it means genuinely new contribution to U(1) electroweak of standard model. If string world sheets carry vanishing M^4 Kähler form, this contribution vanishes classically.

3.2.6 A connection with the hierarchy of Planck constants?

A connection with the hierarchy of Planck constants is highly suggestive. Since also a connection with the p-adic length scale hierarchy suggests itself for the hierarchy of p-adic length scales it seems that both length scale hierarchies might find first principle explanation in terms of twistor lift of Kähler action.

1. Cosmological considerations encourage to think that $R_1 \simeq l_P$ and $R_2 \simeq R$ hold true. One would have in early cosmology $(w_1, w_2) = (1, 0)$ and later $(w_1, w_2) = (0, 1)$ guaranteeing R_D grows from l_P to R during cosmological evolution. These situations would correspond the solutions $(w_1 = n, 0)$ and $(0, w_2 = n)$ one has $A = n4\pi R_1^2$ and $A = n \times 4\pi R_2^2$ and both Kähler coupling strengths are scaled down to α_K/n. For $\hbar_{eff}/h = n$ exactly the same thing happens!

   There are further intriguing similarities. $h_{eff}/h = n$ is assumed to correspond *multi-sheeted* (to be distinguished from *many-sheeted*!) covering space structure for space-time surface. Now one has covering space defined by the lift $S^2(X^4) \to S^2(M4) \times S^2(CP_2)$. These lifts define also lifts of space-time surfaces.

   Could the hierarchy of Planck constants correspond to the twistorial surfaces for which $S^2(M^4)$ is n-fold covering of $S^2(X^4)$? The assumption has been that the n-fold multi-sheeted coverings of space-time surface for $h_{eff}/h = n$ are singular at the ends of space-time surfaces at upper and lower boundaries if causal diamond (CD). Could one consider a more precise definition of twistor space in such a way that CD replaces M^4 and the covering becomes singular at the light-like boundaries of CD - the branches of space-time surface would collapse to single one.

   Does this collapse have a clear geometric meaning? Are the projections of various branches of the S^2 lift automatically identical so that one would have the original picture in which one has n identical copies of the same space-time surface? Or can one require identical projections only at the light-like boundaries of CD?

2. $w_1 = w_2 = w$ is essentially the first proposal for conditions associated with the lifting of twistor space structure. $w_1 = w_2 = n$ gives $ds^2 = (R_1^2 + R_2^2)(d\theta^2 + w^2 d\phi^2)$ and $A = n \times 4\pi(R_1^2 + R_2^2)$. Also now Kähler coupling strength is scaled down to α/n. Again a connection with the hierarchy of Planck constants suggests itself.

3. One can consider also the option $R_1 = R_2$ option giving $ds^2 = R_1^2(2d\theta^2 + (w_1^2 + w_2^2)d\phi^2$. If the integers w_i define Pythagorean square one has $w_1^2 + w_2^2 = n^2$ and one has $R_1 = R_2$ option

that one has $A = n \times 4\pi R^2$. Also now the connection with the hierarchy of Planck constants might make sense.

3.2.7 Twistorial variant for the embedding space spinor structure

The induction of the spinor structure of embedding space is in key role in quantum TGD. The question arises whether one should lift also spinor structure to the level of twistor space. If so one must understand how spinors for $T(M^4)$ and $T(CP_2)$ are defined and how the induced spinor structure is induced.

1. In the case of CP_2 the definition of spinor structure is rather delicate and one must add to the ordinary spinor connection U(1) part, which corresponds physically to the addition of classical U(1) gauge potential and indeed produces correct electroweak couplings to quarks and leptons. It is assumed that the situation does not change in any essential manner: that is the projections of gauge potentials of spinor connection to the space-time surface give those induced from $M^4 \times CP_2$ spinor connection plus possible other parts coming as a projection from the fiber $S^2(M^2) \times S^2(CP_2)$. As a matter of fact, these other parts should vanish if dimensional reduction is what it is meant to be.

2. The key question is whether the complications due to the fact that the geometries of twistor spaces $T(M^4)$ and $T(CP_2)$ are not quite Cartesian products (in the sense that metric could be reduced to a direct sum of metrics for the base and fiber) can be neglected so that one can treat the sphere bundles approximately as Cartesian products $M^4 \times S^2$ and $CP_2 \times S^2$. This will be assumed in the following but should be carefully proven.

3. Locally the spinors of the twistorspace $T(H)$ are tensor products of embedding spinors and those for of $S^2(M^4) \times S^2(CP_2)$ expressible also as tensor products of spinors for $S^2(M^4)$ and $S^2(CP_2)$. Obviously, the number of spinor components increases by factor $2 \times 2 = 4$ unless one poses some additional conditions taking care that one has dimensional reduction without the emergence of any new spin like degrees of freedom for which there is no physical evidence. The only possible manner to achieve this is to pose covariant constancy conditions already at the level of twistor spaces $T(M^4)$ and $T(CP_2)$ leaving only single spin state in these degrees of freedom.

4. In CP_2 covariant constancy is possible for right-handed neutrino so that CP_2 spinor structure can be taken as a model. In the case of CP_2 spinors covariant constancy is possible for right-handed neutrino and is essentially due to the presence of U(1) part in spinor connection forced by the fact that the spinor structure does not exist otherwise. Ordinary S^2 spinor connection defined by vielbein exists always. One can however add a coupling to a suitable multiple of Kähler potential satisfying the quantization of magnetic charge (the magnetic flux defined by $U(1)$ connection is multiple of 2π so that its imaginary exponential is unity).

   S^2 spinor connections must must have besides ordinary vielbein part determined by S^2 metric also $U(1)$ part defined by Kähler form coupled with correct coupling so that the curvature form annihilates the second spin state for both $S^2(M^4)$ and $S^2(CP_2)$. $U(1)$ part of the spinor curvature is proportional to Kähler form $J \propto sin(\theta)d\theta d\phi$ so that this is possible. The vielbein and $U(1)$ parts of the spinor curvature ear proportional Pauli spin matrix $\sigma_z = (1,0;0,-1)/2$ and unit matrix $(1,0;0,1)$ respectively so that the covariant constancy is possible to satisfy and fixes the spin state uniquely.

5. The covariant derivative for the induced spinors is defined by the sum of projections of spinor gauge potentials for $T(M^4)$ and $T(CP_2)$. With above assumptions the contributions gauge potentials from $T(M^4)$ and $T(CP_2)$ separately annihilate single spinor component. As a consequence there are no constraints on the winding numbers w_i, $i = 1,2$ of the maps $S^2(X^4) \to S^2(M^4)$ and $S^2(X^4) \to S^2(CP_2)$. Winding number w_i corresponds to the embedding map $(\Theta_i = \theta, \Phi_i = w_i\phi)$.

6. If the square of the Kähler form in fiber degrees of freedom gives metric to that its square is metric, one obtains just the area of S^2 from the fiber part of action. This is given by the area $A = 4\pi\sqrt{2(w_1^2R_1^2 + w_2^2R_2^2)}$ since the induced metric is given by $ds^2 = (R_1^2 + R_2^2)d\theta^2 + (w_1^2R_1^2 + w_2^2R_2^2)d\phi^2$ for $(\Theta_1 = \theta, \Phi = n_1\phi, \Phi_2 = n_2\phi)$.

3.2.8 Twistor googly problem transforms from a curse to blessing in TGD framework

There was a nice story with title "Michael Atiyah's Imaginative State of Mind" about mathematician Michael Atyiah in Quanta Magazine (see http://tinyurl.com/jta2va8). The works of Atyiah have affected profoundly the development of theoretical physics. What was pleasant to hear that Atyiah belongs to those scientists who do not care what others think. As he tells, he can afford this since he has got all possible prices. This is consoling and encouraging even for those who have not cared what others think and for this reason have not earned any prizes. Nor even a single coin from what they have been busily doing their whole lifetime!

In the beginning of the story "twistor googly problem" was mentioned. I had to refresh my understanding about googly problem. In twistorial description the modes of massless fields (rather than entire massless fields) in space-time are lifted to the modes in its 6-D twistor-space and dynamics reduces to holomorphy. The analog of this takes place also in string models by conformal invariance and in TGD by its extension.

One however encounters what is known as googly problem: one can have twistorial description for circular polarizations with well-defined helicity +1/-1 but not for general polarization states - say linear polarizations, which are superposition of circular polarizations. This reflects itself in the construction of twistorial amplitudes in twistor Grassmann program for gauge fields but rather implicitly: the amplitudes are constructed only for fixed helicity states of scattered particles. For gravitons the situation gets really bad because of non-linearity.

Mathematically the most elegant solution would be to have only +1 or -1 helicity but not their superpositions implying very strong parity breaking and chirality selection. Parity parity breaking occurs in physics but is very small and linear polarizations are certainly possible! The discusion of Penrose with Atyiah has inspired a possible solution to the problem known as "palatial twistor theory" (see http://tinyurl.com/hr7hmh2). Unfortunately, the article is behind paywall too high for me so that I cannot say anything about it.

What happens to the googly problem in TGD framework? There is twistorialization at space-time level and embedding space level.

1. One replaces space-time with 4-surface in $H = M^4 \times CP_2$ and lifts this 4-surface to its 6-D twistor space represented as a 6-surface in 12-D twistor space $T(H) = T(M^4) \times T(CP_2)$. The twistor space has Kähler structure only for M^4 and CP_2 so that TGD is unique. This Kähler structure is needed to lift the dynamics of Kähler action to twistor context and the lift leads to the a dramatic increase in the understanding of TGD: in particular, Planck length and cosmological constant with correct sign emerge automatically as dimensional constants besides CP_2 size.
2. Twistorialization at embedding space level means that spinor modes in H representing ground states of super-symplectic representations are lifted to spinor modes in T(H). M^4 chirality is in TGD framework replaced with H-chirality, and the two chiralities correspond to quarks and leptons. But one cannot superpose quarks and leptons! "Googly problem" is just what the superselection rule preventing superposition of quarks and leptons requires in TGD!

One can look this in more detail.

1. Chiral invariance makes possible for the modes of massless fields to have definite chirality: these modes correspond to holomorphic or antiholomorphic amplitudes in twistor space and holomorphy (antiholomorphy is holomorphy with respect to conjugates of complex coordinates) does not allow their superposition so that massless bosons should have well-defined helicities in conflict with experimental facts. Second basic problem of conformally invariant field theories and of twistor approach relates to the fact that physical particles are massive in 4-D sense. Masslessness in 4-D sense also implies infrared divergences for the scattering amplitudes. Physically natural cutoff is required but would break conformal symmetry.
2. The solution of problems is masslessness in 8-D sense allowing particles to be massive in 4-D sense. Fermions have a well-defined 8-D chirality - they are either quarks or leptons depending on the sign of chirality. 8-D spinors are constructible as superpositions of tensor products of M^4 spinors and of CP_2 spinors with both having well-defined chirality so that tensor product has chiralities (ϵ_1, ϵ_2), $\epsilon_i = \pm 1$, $i = 1, 2$. H-chirality equals to $\epsilon = \epsilon_1 \epsilon_2$. For quarks one

has $\epsilon = 1$ (a convention) and for leptons $\epsilon = -1$. For quark states massless in M^4 sense one has either $(\epsilon_1, \epsilon_2) = (1, 1)$ or $(\epsilon_1, \epsilon_2) = (-1, -1)$ and for massive states superposition of these. For leptons one has either $(\epsilon_1, \epsilon_2) = (1, -1)$ or $(\epsilon_1, \epsilon_2) = (-1, 1)$ in massless case and superposition of these in massive case.

3. The twistor lift to $T(M^4) \times T(CP_2)$ of the ground states of super-symplectic representations represented in terms of tensor products formed from H-spinor modes involves only quark and lepton type spinor modes with well-defined H-chirality. Superpositions of amplitudes in which different M^4 helicities appear but M^4 chirality is always paired with completely correlating CP_2 chirality to give either $\epsilon = 1$ or $\epsilon = -1$. One has never a superposition of of different chiralities in either M^4 or CP_2 tensor factor. I see no reason forbidding this kind of mixing of holomorphicities and this is enough to avoid googly problem. Linear polarizations and massive states represent states with entanglement between M^4 and CP_2 degrees of freedom. For massless and circularly polarized states the entanglement is absent.

4. This has interesting implications for the massivation. Higgs field cannot be scalar in 8-D sense since this would make particles massive in 8-D sense and separate conservation of B and L would be lost. Theory would also contain a dimensional coupling. TGD counterpart of Higgs boson is actually CP_2 vector, and one can say that gauge bosons and Higgs combine to form 8-D vector. This correctly predicts the quantum numbers of Higgs. Ordinary massivation by constant vacuum expectation value of vector Higgs is not an attractive idea since no covariantly constant CP_2 vector field exists so that Higgsy massivation is not promising except at QFT limit of TGD formulated in M^4. p-Adic thermodynamics gives rise to 4-D massivation but keeps particles massless in 8-D sense. It also leads to powerful and correct predictions in terms of p-adic length scale hypothesis.

Anonymous reader gave me a link to the paper of Penrose and this inspired further more detailed considerations of googly problem.

1. After the first reading I must say that I could not understand how the proposed elimination of conjugate twistor by quantization of twistors solves the googly problem, which means that both helicities are present (twistor Z and its conjugate) in linearly polarized classical modes so that holomorphy is broken classically.

2. I am also very skeptic about quantizing of either space-time coordinates or twistor space coordinates. To me quantization is natural only for linear objects like spinors. For bosonic objects one must go to higher abstraction level and replace superpositions in space-time with superpositions in field space. Construction of "World of Classical Worlds" (WCW) in TGD means just this.

3. One could however think that circular polarizations are fundamental and quantal linear combination of the states carrying circularly polarized modes give rise to linear and elliptic polarizations. Linear combination would be possible only at the level of field space (WCW in TGD), not for classical fields in space-time. If so, then the elimination of conjugate of Z by quantization suggested by Penrose would work.

4. Unfortunately, Maxwell's equations allow classically linear polarisations! In order to achieve classical-quantum consistency, one should modify classical Maxwell's equations somehow so that linear polarizations are not possible. Googly problem is still there!

What about TGD?

1. Massless extremals representing massless modes are very "quantal": they cannot be superposed classically unless both momentum and polarisation directions for them (they can depend space-time point) are exactly parallel. Optimist would guess that the classical local classical polarisations are circular. No, they are linear! Superposition of classical linear polarizations at the level of WCW can give rise to local linear but not local circular polarization! Something more is needed.

2. The only sensible conclusion is that only gauge boson quanta (not classical modes) represented as pairs of fundamental fermion and antifermion in TGD framework can have circular polarization! And indeed, massless bosons - in fact, all elementary particles- are constructed from fundamental fermions and they allow only two M^4, CP_2 and $M^4 \times CP_2$ helicities/-chiralities

analogous to circular polarisations. B and L conservation would transform googly problem to a superselection rule as already described.

To sum up, both the extreme non-linearity of Kähler action, the representability of all elementary particles in terms of fundamental fermions and antifermions, and the generalization of conserved M^4 chirality to conservation of H-chirality would be essential for solving the googly problem in TGD framework.

3.3 Surprise: Twistorial Dynamics Does Not Reduce to a Trivial Reformulation of the Dynamics of Kähler Action

I have thought that twistorialization classically means only an alternative formulation of TGD. This is definitely not the case as the explicit study demonstrated. Twistor formulation of TGD is in terms of of 6-D twistor spaces $T(X^4)$ of space-time surfaces $X^4 \subset M^4 \times CP_2$ in 12-dimensional product $T = T(M^4) \times T(CP_2)$ of 6-D twistor spaces of $T(M^4)$ of M^4 and $T(CP_2)$ of CP_2. The induced Kähler form in X^4 defines the quaternionic imaginary unit defining twistor structure: how stupid that I realized it only now! I experienced during single night many other "How stupid I have been" experiences.

Classical dynamics is determined by 6-D variant of Kähler action with coefficient $1/L^2$ having dimensions of inverse length squared. Since twistor space is bundle, a dimensional reduction of 6-D Kähler action to 4-D Kähler action plus a term analogous to cosmological term - space-time volume - takes place so that dynamics reduces to 4-D dynamics also now. Here one must be careful: this happens provided the radius of $S2$ associated with X^4 does not depend on point of X^4. The emergence of cosmological term was however completely unexpected: again "How stupid I have been" experience. The scales of the spheres and the condition that the 6-D action is dimensionless bring in 3 fundamental length scales!

3.3.1 New scales emerge

The twistorial dynamics gives to several new scales with rather obvious interpretation. The new fundamental constants that emerge are the radii of the spheres associated with $T(M^4)$ and $T(CP_2)$. The radius of the sphere associated with X^4 is not a fundamental constant but determined by the induced metric. By above argument the fiber is sphere for both Euclidian signature and Minkowskian signatures.

1. For CP_2 twistor space the radius of $S^2(CP_2)$ must be apart from numerical constant equal to CP_2 radius R. For $S^2(M^4)$ one an consider two options. The first option is that also now the radius for $S^2(M^4)$ equals to $R(M^4) = R$ so that Planck length would not emerge from fundamental theory classically as assumed hitherto. Second imaginable option is that it does and one has $R(M^4) = l_P$.
2. If the signature of $S^2(M^4)$ is $(-1,-1)$ both Minkowskian and Euclidian regions have $S^2(X^4)$ with the same signature $(-1,-1)$. The radius R_D of $S^2(X^4)$ is dynamically determined.

Recall first how the cosmological constant emerges from TGD framework. The key point is that the 6-D Kähler action contains two terms.

1. The first term is essentially the ordinary Kähler action multiplied by the area of $S^2(X^4)$ which is compensated by the length scale, which can be taken to be the area $4\pi R^2(M^4)$ of $S^2(M^4)$. This makes sense for winding numbers $(w_1, w_2) = (1, 0)$ meaning that $S^2(CP_2)$ is effectively absent but $S^2(M^4)$ is present.
2. Second term is the analog of Kähler action assignable assignable to the projection of $S^2(M^4)$ Kähler form. The corresponding Kähler coupling strength $\alpha_K(M^4)$ is huge - so huge that one has $\alpha_K(M^4) 4\pi R^2(M^4) \equiv L^2$, where $1/L^2$ is of the order of cosmological constant and thus of the order of the size of the recent Universe. $\alpha_K(M^4)$ is also analogous to critical temperature and the earlier hypothesis that the values of L correspond to p-adic length scales implies that the values of come as $\alpha_K(M^4) \propto p \simeq 2^k$, p prime, k prime.

The assignment of different value of α_K to M^4 and CP_2 degrees of freedom can be criticized as ad hoc assumption. In [L45] a scenario in which the value of α_K is universal. This option has very nice properties and one can overcome the problem associated with cosmological constant by assuming that it the *entire* 4-D action corresponds to the effective cosmological constant. The cancellation between Kähler action and volume term would give rise to very small cosmological constant and also its p-adic evolution could be understood.

3. One can get an estimate for the relative magnitude of the Kähler action $S(CP_2) = \pi/8\alpha_K$ assignable to CP_2 type vacuum extremal and the corresponding cosmological term. The magnitude of the volume term is of order $1/4\pi\alpha_K(M^4)$ with $\alpha_K(M^4)$ given by $\alpha_K(M^4) = L^2/4\pi R^2(M^4)$. The sequel the magnitude of L is estimated to be $L = (2^{3/2}\pi l_P/R_D) \times R_U$, where R_U is the recent size of the Universe. This estimate follows from the identification of the volume term as cosmological constant term.

   For $R_D = R_M = l_P$ this gives $\alpha_K(M^4) = 2\pi(R_U/l_P)^2 \sim 2 \times 10^{18}$. For $\alpha_K \simeq 1/137$ the ratio of the two terms is of order 10^{-20}. The cosmological terms is completely negligible in elementary particle scales. For vacuum extremals the situation changes and the overall effect is presumably the transformation of 4-D spin glass degeneracy so that the potentials wells in the analog spin glass energy landscape do not correspond to vacuum extremal anymore and perturbation theory around them is in principle possible. The huge value of $\alpha_K(M^4)$ implies that the system corresponds mathematically to an extremely strongly interacting system so that perturbation theory fails to converge. The geometry of "world of classical worlds" (WCW) provides the needed non-perturbative approach and leads to strong form of holography.

4. One could argue that the Kähler form assignable to M^4 cannot contribute to the action since it does not contribute to spinor connection of M^4 - an assumption that can be challenged. For canonically imbedded M^4 self-duality implies that this contribution to action vanishes. For vacuum extremals of ordinary Kähler action the contribution to the action density is proportional to the CP_2 part of induced metric and to $1/\alpha_K(M^4)$, and therefore extremely small.

   The breaking of Lorentz invariance can be seen as a possible problem for the induced spinor fields coupling to the self-dual Kähler potential. This corresponds to coupling to constant magnetic field and constant electric field, which are duals of each other. This would give rise to the analogs of cyclotron energy states in transversal directions and to the analogs of states in constant electric field in longitudinal directions. Could this extremely small effect serve as a seed for the generation of Kähler magnetic flux tubes carrying longitudinal electric fields in various scales? Note also that the value of $\alpha_K(M^4)$ is predicted to decrease as p-adic length scale so that the effect would be larger in early cosmology and in short length scales.

Hence one can consider the possibility that the action is just the sum of full 6-D Kähler actions assignable to $T(M^4)$ and $T(CP_2)$ but with different values of α_K if one has $(w_1, w_2) = (n, 0)$. Also other $w_2 \neq 0$ is possible but corresponds to gigantic cosmological constant.

Given the parameter L^2 as it is defined above, one can deduce an expression for cosmological constant Λ and show that it is positive.

1. 6-D Kähler action has dimensions of length squared and one must scale it by a dimensional constant: call it $1/L^2$. L is a fundamental scale and in dimensional reduction it gives rise to cosmological constant. Cosmological constant Λ is defined in terms of vacuum energy density as $\Lambda = 8\pi G\rho_{vac}$ can have two interpretations. Λ can correspond to a modification of Einstein-Hilbert action or - as now - to an additional term in the action for matter. In the latter case positive Λ means negative pressure explaining the observed accelerating expansion. It is actually easy to deduce the sign of Λ.

   $1/L^2$ multiplies both Kähler action - $F^{ij}F_{ij}$ ($\propto E^2 - B^2$ in Minkowskian signature). The energy density is positive. For Kähler action the sign of the multiplier must be positive so that $1/L^2$ is positive. The volume term is fiber space part of action having same form as Kähler action. It gives a positive contribution to the energy density and negative contribution to the pressure.

   In $\Lambda = 8\pi G\rho_{vac}$ one would have $\rho_{vac} = \pi/L^2R_D^2$ as integral of the $-F^{ij}F_{ij}$ over S^2 given the π/R_D^2 (no guarantee about correctness of numerical constants). This gives $\Lambda = 8\pi^2G/L^2R_D^2$.

Λ is positive and the sign is same as as required by accelerated cosmic expansion. Note that super string models predict wrong sign for Λ. Λ is also dynamical since it depends on R_D, which is dynamical. One has $1/L^2 = k\Lambda$, $k = 8\pi^2 G/R_D^2$ apart from numerical factors.

The value of L of deduced from Euclidian and Minkowskian regions in this formal manner need not be same. Since the GRT limit of TGD describes space-time sheets with Minkowskian signature, the formula seems to be applicable only in Minkowskian regions. Again one can argue that one cannot exclude Euclidian space-time sheets of even macroscopic size and blackholes and even ordinary concept matter would represent this kind of structures.

2. L is not size scale of any fundamental geometric object. This suggests that L is analogous to α_K and has value spectrum dictated by p-adic length scale hypothesis. In fact, one can introduce the ratio of $\epsilon = R^2/L^2$ as a dimensionless parameter analogous to coupling strength what it indeed is in field equations. If so, L could have different values in Minkowskian and Euclidian regions.

3. I have earlier proposed that $R_U \equiv 1/\sqrt{1/\Lambda}$ is essentially the p-adic length scale $L_p \propto \sqrt{p} = 2^{k/2}$, $p \simeq 2^k$, k prime, characterizing the cosmology at given time and satisfies $R_U \propto a$ meaning that vacuum energy density is piecewise constant but on the average decreases as $1/a^2$, a cosmic time defined by light-cone proper time. A more natural hypothesis is that L satisfies this condition and in turn implies similar behavior or R_U. p-Adic length scales would be the critical values of L so that also p-adic length scale hypothesis would emerge from quantum critical dynamics! This conforms with the hypothesis about the value spectrum of α_K labelled in the same manner [L17].

4. At GRT limit the magnetic energy of the flux tubes gives rise to an average contribution to energy momentum tensor, which effectively corresponds to negative pressure for which the expansion of the Universe accelerates. It would seem that both contributions could explain accelerating expansion. If the dynamics for Kähler action and volume term are coupled, one would expect same orders of magnitude for negative pressure and energy density - kind of equipartition of energy.

Consider first the basic scales emerging also from GRT picture. $R_U \sim \sqrt{1/\Lambda} \sim 10^{26}$ m $= 10$ Gly is not far from the recent size of the Universe defined as $c \times t \sim 13.8$ Gly. The derived size scale $L_1 \equiv (R_U \times l_P)^{1/2}$ is of the order of $L_1 = .5 \times 10^{-4}$ meters, the size of neuron. Perhaps this is not an accident. To make life of the reader easier I have collected the basic numbers to the following table.

$$\begin{array}{lll} m(CP_2) \simeq 5.7 \times 10^{14}\ \ GeV\ , & m_P = 2.435 \times 10^{18}\ \ GeV\ , & \frac{R(CP_2)}{l_P} \simeq 4.1 \times 10^3\ , \\ R_U = 10\ \ Gy\ , & t = 13.8\ \ Gy\ , & L_1 = \sqrt{l_P R_U} = .5 \times 10^{-4}\ \ m\ . \end{array} \quad (3.3.1)$$

Let us consider now some quantitative estimates. $R(X^4)$ depends on homotopy equivalence classes of the maps from $S^2(X^4) \to S^2(M^4)$ and $S^2(X^4) \to S^2(CP_2)$ - that is winding numbers w_i, $i = 1, 2$ for these maps. The simplest situations correspond to the winding numbers $(w_1, w_2) = (1, 0)$ and $(w_1, w_2) = (0, 1)$. For $(w_1, w_2) = (1, 0)$ M^4 contribution to the metric of $S^2(X^4)$ dominates and one has $R(X^4) \simeq R(M^4)$. For $R(M^4) = l_P$ so Planck length would define a fundamental length and Planck mass and Newton's constant would be quantal parameters. For $(w_1, w_2) = (0, 1)$ the radius of sphere would satisfy $R_D \simeq R$ (CP_2 size): now also Planck length would be quantal parameter.

Consider next additional scales emerging from TGD picture.

1. One has $L = (2^{3/2}\pi l_P/R_D) \times R_U$. In Minkowskian regions with $R_D = l_P$ this would give $L = 8.9 \times R_U$: there is no obvious interpretation for this number in recent cosmology. For ($R_D = R$) one obtains the estimate $L = 29$ Mly. The size scale of large voids varies from about 36 Mly to 450 Mly (see `http://tinyurl.com/jyqcjhl`).

2. Consider next the derived size scale $L_2 = (L \times l_P)^{1/2} = \sqrt{L/R_U} \times L_1 = \sqrt{2^{3/2}\pi l_P/R_D} \times L_1$. For $R_D = l_P$ one has $L_2 \simeq 3L_1$. For $R_D = R$ making sense in Euclidian regions, this is of

the order of size of neutrino Compton length: 3 μm, the size of cellular nucleus and rather near to the p-adic length scale $L(167) = 2.6$ m, corresponds to the largest miracle Gaussian Mersennes associated with $k = 151, 157, 163, 167$ defining length scales in the range between cell membrane thickness and the size of cellular nucleus. Perhaps these are co-incidences are not accidental. Biology is something so fundamental that fundamental length scale of biology should appear in the fundamental physics.

The formulas and predictions for different options are summarized by the following table.

$$\begin{array}{lll} Option & L = \frac{2^{3/2}\pi l_P}{R_D} \times R_U & L_2 = \sqrt{Ll_P} = \sqrt{\frac{2^{3/2}\pi l_P}{R_D}} \times L_1 \\ R_D = R \ , & 29 \ \ Mly \ , & \simeq 3 \ \ \mu m \ , \\ R_D = l_P \ , & 8.9 R_U \ , & \simeq 3L_1 = 1.5 \times 10^{-4} \ \ m \ , \end{array} \quad (3.3.2)$$

In the case of M^4 the radius of S^2 cannot be fixed it remains unclear whether Planck length scale is fundamental constant or whether it emerges.

3.3.2 Estimate for the cosmic evolution of R_D

One can actually get estimate for the evolution of R_D as function of cosmic time if one accepts Friedman cosmology as an approximation of TGD cosmology.

1. Assume critical mass density so that one has

$$\rho_{cr} = \frac{3H^2}{8\pi G} \ .$$

2. Assume that the contribution of cosmological constant term to the mass mass density dominates. This gives $\rho \simeq \rho_{vac} = \Lambda/8\pi G$. From $\rho_{cr} = \rho_{vac}$ one obtains

$$\Lambda = 3H^2 \ .$$

3. From Friedman equations one has $H^2 = ((da/dt)/a)^2$, where a corresponds to light-cone proper time and t to cosmic time defined as proper time along geodesic lines of space-time surface approximated as Friedmann cosmology. One has

$$\Lambda = \frac{3}{g_{aa}a^2}$$

in Robertson-Walker cosmology with $ds^2 = g_{aa}da^2 - a^2 d\sigma_3^2$.

4. Combining this equations with the TGD based equation

$$\Lambda = \frac{8\pi^2 G}{L^2 R_D^2}$$

one obtains

$$\frac{8\pi^2 G}{L^2 R_D^2} = \frac{3}{g_{aa}a^2} \ . \quad (3.3.3)$$

5. Assume that quantum criticality applies so that L has spectrum given by p-adic length scale hypothesis so that one discrete p-adic length scale evolution for the values of L. There are two options to consider depending on whether p-adic length scales are assigned with light-cone proper time a or with cosmic time t

$$T = a \ \ (Option \ \ I) \ , \quad T = t \ \ (Option \ \ II) \quad (3.3.4)$$

Both options give the same general formula for the p-adic evolution of $L(k)$ but with different interpretation of $T(k)$.

$$\frac{L(k)}{L_{now}} = \frac{T(k)}{T_{now}} \quad , \quad T(k) = L(k) = 2^{(k-151)/2} \times L(151) \quad , \quad L(151) \simeq 10 \ \ nm \ . \qquad (3.3.5)$$

Here $T(k)$ is assumed to correspond to primary p-adic length scale. An alternative - less plausible - option is that $T(k)$ corresponds to secondary p-adic length scale $L_2(k) = 2^{k/2}L(k)$ so that $T(k)$ would correspond to the size scale of causal diamond. In any case one has $L \propto L(k)$. One has a discretized version of smooth evolution

$$L(a) = L_{now} \times \frac{T}{T_{now}} \ . \qquad (3.3.6)$$

6. Feeding into this to Eq. 3.3.3 one obtains an expression for $R_D(a)$

$$\frac{R_D}{l_P} = (\frac{8}{3})^{1/2}\pi \times \frac{a}{L(a)} \times g_{aa}^{1/2} \ . \qquad (3.3.7)$$

Unless the dependences on cosmic time compensate each other, R_D is dynamical and becomes very small at very early times since g_{aa} becomes very small. $R(M^4) = l_P$ however poses a lower boundary since either of the maps $S^2(X^4) \to S^2(M^4)$ and $S^2(X^4) \to S^2(CP_2)$ must be homotopically non-trivial. For $R(M^4) = l_P$ one would obtain $R_D/l_P = 1$ at this limit giving also lower bound for g_{aa}. For $T = t$ option $a/L(a)$ becomes large and g_{aa} small.
As a matter of fact, in very early cosmic string dominated cosmology g_{aa} would be extremely small constant [K86]. In late cosmology $g_{aa} \to 1$ holds true and one obtains at this limit

$$\frac{R_D(now)}{l_P} = (\frac{8}{3})^{1/2}\pi \times \frac{a_{now}}{L_{now}} \times l_P \simeq 4.4\frac{a_{now}}{L_{now}} \ . \qquad (3.3.8)$$

7. For $T = t$ option R_D/l_P remains constant during both matter dominated cosmology, radiation dominated cosmology, and string dominated cosmology since one has $a \propto t^n$ with $n = 1/2$ during radiation dominated era, $n = 2/3$ during matter dominated era, and $n = 1$ during string dominated era [K86]. This gives

$$\frac{R_D}{l_P} = (\frac{8}{3})^{1/2}\pi \times \frac{a}{t}\sqrt{g_{aa}}\frac{t(end)}{L(end)} = (\frac{8}{3})^{1/2}\frac{\pi}{n}\frac{t(end)}{L(end)} \ .$$

Here "end" refers the end of the string or radiation dominated period or to the recent time in the case of matter dominated era. The value of n would have evolved as $R_D/l_P \propto (1/n)(t_{end}/L_{end})$, $n \in \{1, 3/2, 2\}$. During radiation dominated cosmology $R_D \propto a^{1/2}$ holds true. The value of R_D would be very nearly equal to $R(M^4)$ and $R(M^4)$ would be of the same order of magnitude as Planck length. In matter dominated cosmology would would have $R_D \simeq 2.2(t(now)/L(now)) \times l_P$.

8. For $R_D(now) = l_P$ one would have

$$\frac{L_{now}}{a_{now}} = (\frac{8}{3})^{1/2}\pi \simeq 4.4 \ .$$

In matter dominated cosmology $g_{aa} = 1$ gives $t_{now} = (2/3) \times a_{now}$ so that predictions differ only by this factor for options I and II. The winding number for the map $S^2(X^4) \to S^2(CP_2)$ must clearly vanish since otherwise the radius would be of order R.

9. For $R_D(now) = R$ one would obtain

$$\frac{a_{now}}{L_{now}} = (\frac{8}{3})^{1/2} \times \frac{R}{l_P} \simeq 2.1 \times 10^4 \ .$$

One has $L_{now} = 10^6$ ly: this is roughly the average distance scale between galaxies. The size of Milky Way is in the range $1 - 1.8 \times 10^5$ ly and of an order of magnitude smaller.

10. An interesting possibility is that $R_D(a)$ evolves from $R_D \sim R(M^4) \sim l_P$ to $R_D \sim R$. This could happen if the winding number pair $(w_1, w_2) = (1, 0)$ transforms to $(w_1, w_2) = (0, 1)$ during transition to from radiation (string) dominance to matter (radiation) dominance. R_D/l_P radiation dominated cosmology would be related by a factor

$$\frac{R_D(rad)}{R_D(mat)} = (3/4)\frac{t(rad,end)}{L(rad,end)} \times \frac{L(now)}{t(now)}$$

to that in matter dominated cosmology. Similar factor would relate the values of R_D/l_P in string dominated and radiation dominated cosmologies. The condition $R_D(rad)/R_D(mat) = l_P/R$ expressing the transformation of winding numbers would give

$$\frac{L(now)}{L(rad,end)} = \frac{4}{3}\frac{l_P}{R}\frac{t(now)}{t(rad,end)} \quad .$$

One has $t(now)/t(rad,end) \simeq .5 \times 10^6$ and $l_P/R = 2.5 \times 10^{-4}$ giving $L(now)/L(rad,end) \simeq 125$, which happens to be near fine structure constant.

11. For the twistor lifts of space-time surfaces for which cosmological constant has a reasonable value , the winding numbers are equal to $(w_1, w_2) = (n, 0)$ so that $R_D = \sqrt{n}R(S^2(M^4))$ holds true in good approximation. This conforms with the observed constancy of R_D during various cosmological eras, and would suggest that the ratio $\frac{t(end)}{L(end)}$ characterizing these periods is same for all periods. This determines the evolution for the values of $\alpha_K(M^4)$.

$R(M^4) \sim l_P$ seems rather plausible option so that Planck length would be fundamental classical length scale emerging naturally in twistor approach. Cosmological constant would be coupling constant like parameter with a spectrum of critical values given by p-adic length scales.

3.3.3 What about the extremals of the dimensionally reduced 6-D Kähler action?

It seems that the basic wisdom about extremals of Kähler action remains unaffected and the motivations for WCW are not lost in the case that M^4 Kähler form does not contribute to 6-D Kähler action (the case to be considered below): otherwise the predicted effects are extremely small in the recent Universe. What is new is that the removal of vacuum degeneracy is forced by twistorial action.

1. All extremals, which are minimal surfaces remain extremals. In fact, all the known extremals except vacuum extremals. For minimal surfaces the dynamics of the volume term and 4-D Kähler action separate and field equations for them are separately satisfied. The vacuum degeneracy motivating the introduction of WCW is preserved. The induced Kähler form vanishes for vacuum extremals and the imaginary unit of twistor space is ill-defined. Hence vacuum extremals cannot belong to WCW. This correspond to the vanishing of WCW metric for vacuum extremals.

2. For non-minimal surfaces Kähler coupling strength does not disappear from the field equations and appears as a genuine coupling very much like in classical field theories. Minimal surface equations are a generalization of wave equation and Kähler action would define analogs of source terms. Field equations would state that the total isometry currents are conserved. It is not clear whether other than minimal surfaces are possible, I have even conjectured that all preferred extremals are always minimal surfaces having the property that being holomorphic they are almost universal extremals for general coordinate invariant actions.

3. Thermodynamical analogy might help in the attempts to interpret. Quantum TGD in zero energy ontology (ZEO) corresponds formally to a complex square root of thermodynamics. Kähler action can be identified as a complexified analog of free energy. Complexification follows both from the fact that $\sqrt{g}$ is real/imaginary in Euclidian/Minkowskian space-time regions. Complex values are also implied by the proposed identification of the values of Kähler coupling strength in terms of zeros and pole of Riemann zeta in turn identifiable as poles of the so called fermionic zeta defining number theoretic partition function for fermions [K104]

[L17, L19]. The thermodynamical for Kähler action with volume term is Gibbs free energy $G = F - TS = E - TS + PV$ playing key role in chemistry.

4. The boundary conditions at the ends of space-time surfaces at boundaries of CD generalize appropriately and symmetries of WCW remain as such. At light-like boundaries between Minkowskian and Euclidian regions boundary conditions must be generalized. In Minkowkian regions volume can be very large but only the Euclidian regions contribute to Kähler function so that vacuum functional can be non-vanishing for arbitrarily large space-time surfaces since exponent of Minkowskian Kähler action is a phase factor.

5. One can worry about almost topological QFT property. Although Kähler action from Minkowskian regions at least would reduce to Chern-Simons terms with rather general assumptions about preferred extremals, the extremely small cosmological term does not. Could one say that cosmological constant term is responsible for "almost"?

   It is interesting that the volume of manifold serves in algebraic geometry as topological invariant for hyperbolic manifolds, which look locally like hyperbolic spaces $H_n = SO(n,1)/SO(n)$ [A21] [K56]. See also the article "Volumes of hyperbolic manifolds and mixed Tate motives" (see `http://tinyurl.com/yargy3uw`). Now one would have $n = 4$. It is probably too much to hope that space-time surfaces would be hyperbolic manifolds. In any case, by the extreme uniqueness of the preferred extremal property expressed by strong form of holography the volume of space-time surface could also now serve as topological invariant in some sense as I have earlier proposed. What is intriguing is that AdS_n appearing in AdS/CFT correspondence is Lorentzian analogue H_n.

6. $\alpha(M^4)$ is extremely large so that there is no hope of quantum perturbation theory around canonically imbedded M^4 although the propagator for CP_2 coordinate exists. In the new framework WCW can be seen as a solution to how to construct non-perturbative quantum TGD.

To sum up, I have the feeling that the final formulation of TGD has now emerged and it is clear that TGD is indeed a quantum theory of gravitation allowing to understand standard model symmetries. The existence of twistorial formulation is all that is needed to fix the theory completely. It makes possible gravitation and predicts standard model symmetries. This cannot be said about any competitor of TGD.

3.4 Basic Principles Behind Construction of Amplitudes

Basic principles of the construction summarized in this section could be seen as axioms trying to abstract the essentials. The explicit construction of amplitudes is too heavy challenge at this stage and at least for me.

3.4.1 Embedding space is twistorially unique

It took roughly 36 years to learn that M^4 and CP_2 are twistorially unique.

1. As already explained, M^4 and CP_2 are unique 4-manifolds in the sense that both allow twistor space with Kähler structure: Kähler structure is the crucial concept as one might guess from the fact that the projection of Kähler form naturally defines the preferred quaternionic imaginary unit defining the twistor structure for space-time surface. Both M^4 and its Euclidian variant E^4 allow twistor space. The first guess is that the twistor space of M^4 is Minkowskian variant $T(M^4) = SU(2,2)/SU(2,1) \times U(1)$ of 6-D twistor space $CP_3 = SU(4)/SU(3) \times U(1)$ of E^4. This is sensible assumption at the level of momentum space but the second candidate, which is simply $T(M^4) = M^4 \times CP_1$, is the only sensible option at space-time level. The twistor space of CP_2 is 6-D $T(CP_2) = SU(3)/U(1) \times U(1)$, the space for the choices of quantization axes of color hypercharge and isospin.

2. This leads to a proposal for the formulation of TGD in which space-time surfaces X^4 in H are lifted to twistor spaces X^6, which are sphere bundles over X^4 and such that they are surfaces in 12-D product space $T(M^4) \times T(CP_2)$ such the twistor structure of X^4 are in some sense induced from that of $T(M^4) \times T(CP_2)$.

What is nice in this formulation is that one might be able to use all the machinery of algebraic geometry so powerful in superstring theory (Calabi-Yau manifolds) provided one can generalize the notion of Kähler structure from Euclidian to Minkowskian signature. It has been already described how this approach leads to a profound understanding of the relationship between TGD and GRT. Planck length emerges whereas fundamental constant as also cosmological constant emerges dynamically from the length scale parameter appearing in 6-D Kähler action. One can say, that twistor extension is absolutely essential for really understanding the gravitational interactions although the modification of Kähler action is extremely small due to the huge value of length scale defined by cosmological constant.

3. Masslessness (masslessness in complex sense for virtual particles in twistorialization) is essential condition for twistorialization. In TGD massless is masslessness in 8-D sense for the representations of superconformal algebras. This suggests that 8-D variant of twistors makes sense. 8-dimensionality indeed allows octonionic structure in the tangent space of embedding space. One can also define octonionic gamma matrices and this allows a possible generalization of 4-D twistors to 8-D ones using generalization of sigma matrices representing quaternionic units to octonionic sigma "matrices" essential for the notion of twistors. These octonion units do not of course allow matrix representation unless one restricts to units in some quaternionic subspace of octonions. Space-time surfaces would be associative and thus have quaternionic tangent space at each point satisfying some additional conditions.

3.4.2 Strong form of holography

Strong form of holography (SH) following from general coordinate invariance (GCI) for space-times as surfaces states that the data assignable to string world sheets and partonic 2-surfaces allows to code for scattering amplitudes. The boundaries of string world sheets at the space-like 3-surfaces defining the ends of space-time surfaces at boundaries of causal diamonds (CDs) and the fermionic lines along light-like orbits of partonic 2-surfaces representing lines of generalized Feynman diagrams become the basic elements in the generalization of twistor diagrams (I will not use the attribute "Feynman" in precise sense, one could replace it with "twistor" or even drop away). One can assign fermionic lines massless in 8-D sense to flux tubes, which can also be braided. One obtains a fractal hierarchy of braids with strands, which are braids themselves. At the lowest level one has braids for which fermionic lines are braided. This fractal hierarchy is unavoidable and means generalization of the ordinary Feynman diagram. I have considered some implications of this hierarchy in [L18].

The precise formulation of strong form of holography (SH) is one of the technical problems in TGD. A comment in FB page of Gareth Lee Meredith led to the observation that besides the purely number theoretical formulation based on commutativity also a symplectic formulation in the spirit of non-commutativity of embedding space coordinates can be considered. One can however use only the notion of Lagrangian manifold and avoids making coordinates operators leading to a loss of General Coordinate Invariance (GCI).

3.4.3 The existence of WCW demands maximal symmetries

Quantum TGD reduces to the construction of Kähler geometry of infinite-D "world of classical worlds" (WCW), of associated spinor structure, and of modes of WCW spinor fields which are purely classical entities and quantum jump remains the only genuinely quantal element of quantum TGD. Quantization without quantization, would Wheeler say.

By its infinite-dimensionality, the mere mathematical existence of the Kähler geometry of WCW requires maximal isometries. Physics is completely fixed by the mere condition that its mathematical description exists. Super-symplectic and other symmetries of "world of classical worlds" (WCW) are in decisive role. These symmetry algebras have conformal structure and generalize and extend the conformal symmetries of string models (Kac-Moody algebras in particular). These symmetries give also rise to the hierarchy of Planck constants. The super-symplectic symmetries extend to a Yangian algebra, whose generators are polylocal in the sense that they involve products of generators associated with different partonic surfaces. These symmetries leave scattering amplitudes invariant. This is an immensely powerful constraint, which remains to be understood.

3.4.4 Quantum criticality

Quantum criticality (QC) of TGD Universe is a further principle. QC implies that Kähler coupling strength is mathematically analogous to critical temperature and has a discrete spectrum. Coupling constant evolution is replaced with a discrete evolution as function of p-adic length scale: sequence of jumps from criticality to a more refined criticality or vice versa (in spin glass energy landscape you at bottom of well containing smaller wells and you go to the bottom of smaller well). This implies that either all radiative corrections (loops) sum up to zero (QFT limit) or that diagrams containing loops correspond to the same scattering amplitude as tree diagrams so that loops can eliminated by transforming them to arbitrary small ones and snipping away moving the end points of internal lines along the lines of diagram (fundamental description).

Quantum criticality at the level of super-conformal symmetries leads to the hierarchy of Planck constants $h_{eff} = n \times h$ labelling a hierarchy of sub-algebras of super-symplectic and other conformal algebras isomorphic to the full algebra. Physical interpretation is in terms of dark matter hierarchy. One has conformal symmetry breaking without conformal symmetry breaking as Wheeler would put it.

3.4.5 Physics as generalized number theory, number theoretical universality

Physics as generalized number theory vision has important implications. Adelic physics is one of them. Adelic physics implied by number theoretic universality (NTU) requires that physics in real and various p-adic numbers fields and their extensions can be obtained from the physics in their intersection corresponding to an extension of rationals. This is also enormously powerful condition and the success of p-adic length scale hypothesis and p-adic mass calculations can be understood in the adelic context.

In TGD inspired theory of consciousness various p-adic physics serve as correlates of cognition and p-adic space-time sheets can be seen as cognitive representations, "thought bubbles". NTU is closely related to SH. String world sheets and partonic 2-surfaces with parameters (WCW coordinates) characterizing them in the intersection of rationals can be continued to space-time surfaces by preferred extremal property but not always. In p-adic context the fact that p-adic integration constants depend on finite number of pinary digits makes the continuation easy but in real context this need not be possible always. It is always possible to imagine something but not always actualize it!

3.4.6 Scattering diagrams as computations

Quantum criticality as possibility to eliminate loops has a number theoretic interpretation. Generalized Feynman diagram can be interpreted as a representation of a computation connecting given set X of algebraic objects to second set Y of them (initial and final states in scattering) (trivial example: $X = \{3,4\} \rightarrow 3 \times 4 = 12 \rightarrow 2 \times 6 \rightarrow \{2,6\} = Y$. The 3-vertices ($a \times b = c$) and their time-reversals represent algebraic product and co-product.

There is a huge symmetry: all diagrams representing computation connecting given X and Y must produce the same amplitude and there must exist minimal computation. This generalization of string model duality implies an infinite number of dualities unless the finite size of CD allows only a finite number of equivalent computations. These dualities are analogous to the dualities of super-string model, in particular mirror symmetry stating that same quantum physical situation does not correspond to a unique space-time geometry and topology (Calabi-Yau and its mirror represent the same situation). The task of finding this computation is like finding the simplest representation for the formula X=Y and the noble purpose of math teachers is that we should learn to find it during our school days. This generalizes the duality symmetry of old fashioned string models: one can transform any diagram to a tree diagram without loops. This corresponds to quantum criticality in TGD: coupling constants do not evolve. The evolution is actually there but discrete and corresponds to infinite number critical values for Kahler coupling strength analogous to temperature.

3.4.7 Reduction of diagrams with loops to braided tree-diagrams

1. In TGD pointlike particles are replaced with 3-surfaces and by SH by partonic 2-surfaces. The important implication of 3-dimensionality is braiding. The fermionic lines inside light-like orbits of partonic 2-surfaces can be knotted and linked - that is braided (this is dynamical braiding analogous to dance). Also the fermionic strings connecting partonic 2-surfaces at space-like 3-surfaces at boundaries of causal diamonds (CDs) are braided (space-like braiding).

   Therefore ordinary Feynman diagrams are not enough and one must allow braiding for tree diagrams. One can also imagine of starting from braids and allowing 3-vertices for their strands (product and co-product above). It is difficult to imagine what this braiding could mean. It is better to imagine braid and allow the strands to fuse and split (annihilation and pair creation vertices).
2. This braiding gives rise in the planar projection representation of braids to a generalization of non-planar Feynman diagrams. Non-planar diagrams are the basic unsolved problem of twistor approach and have prevented its development to a full theory allowing to construct exact expressions for the full scattering amplitudes (I remember however that Nima Arkani-Hamed *et al* have conjectured that non-planar amplitudes could be constructed by some procedure: they notice the role of permutation group and talk also about braidings (describable using covering groups of permutation groups)). In TGD framework the non-planar Feynman diagrams correspond to non-trivial braids for which the projection of braid to plane has crossing lines, say a and b, and one must decide whether the line a goes over b or vice versa.
3. An interesting open question is whether one must sum over all braidings or whether one can choose only single braiding. Choice of single braiding might be possible and reflect the failure of string determinism for Kähler action and it would be favored by TGD as almost topological quantum field theory (TQFT) vision in which Kähler action for preferred extremal is topological invariant.

3.4.8 Scattering amplitudes as generalized braid invariants

The last big idea is the reduction of quantum TGD to generalized knot/braid theory (I have talked also about TGD as almost TQFT). The scattering amplitude can be identified as a generalized braid invariant and could be constructed by the generalization of the recursive procedure transforming in a step-by-step manner given braided tree diagram to a non-braided tree diagram: essentially what Alexander the Great did for Gordian knot but tying the pieces together after cutting. At each step one must express amplitude as superposition of amplitudes associated with the different outcomes of splitting followed by reconnection. This procedure transforms braided tree diagram to a non-braided tree diagrams and the outcome is the scattering amplitude!

3.5 Tensor Networks and S-matrices

The concrete construction of scattering amplitudes has been the toughest challenge of TGD and the slow progress has occurred by identification of general principles with many side tracks. One of the key problems has been unitarity. The intuitive expectation is that unitarity should reduce to a local notion somewhat like classical field equations reduce the time evolution to a local variational principle. The presence of propagators have been however the obstacle for locally realized unitarity in which each vertex would correspond to unitary map in some sense.

TGD suggests two approaches to the construction of S-matrix.

1. The first approach is generalization of twistor program [L10]. What is new is that one does not sum over diagrams but there is a large number of equivalent diagrams giving the same outcome. The complexity of the scattering amplitude is characterized by the minimal diagram. Diagrams correspond to space-time surfaces so that several space-time surfaces give rise to the same scattering amplitude. This would correspond to the fact that the dynamics breaks classical determinism. Also quantum criticality is expected to be accompanied by quantum critical fluctuations breaking classical determinism. The strong form of holography would not

be unique: there would be several space-time surfaces assignable as preferred extremals to given string world sheets and partonic 2-surfaces defining "space-time genes".

2. Second approach relies on the number theoretic vision and interprets scattering amplitudes as representations for computations with each 3-vertex identifiable as a basic algebraic operation [L10]. There is an infinite number of equivalent computations connecting the set of initial algebraic objects to the set of final algebraic objects. There is a huge symmetry involved: one can eliminate all loops moving the end of line so that it transforms to a vacuum tadpole and can be snipped away. A braided tree diagram is left with braiding meaning that the fermion lines inside the line defined by light-like orbit are braided. This kind of braiding can occur also for space-like fermion lines inside magnetic flux tubes and defining correlate for entanglement. Braiding is the TGD counterpart for the problematic non-planarity in twistor approach.

Third approach involving local unitary as an additional key element is suggested by tensor networks relying on the notion of perfect entanglement discussed by Preskill *et al* [B44].

1. Tensor networks provide an elegant representation of holography mapping interior states isometrically (in Hilbert space sense) to boundary states or vice versa for selected subsets of states defining the code subspace for holographic quantum error correcting code. Again the tensor net is highly non-unique but there is some minimal tensor net characterizing the complexity of the entangled boundary state.

2. Tensor networks have two key properties, which might be abstracted and applied to the construction of S-matrix in zero energy ontology (ZEO): perfect tensors define isometry for any subspace defined by the index subset of perfect tensor to its complement and the non-unique graph representing the network. As far as the construction of Hilbert space isometry between local interior states and highly non-local entangled boundary states is considered, these properties are enough.

One cannot avoid the question whether these three constructions could be different aspects of one and same construction and that tensor net construction with perfect tensors representing vertices could provide and additional strong constraint to the long sought for explicit recipe for the construction of scattering amplitudes.

3.5.1 Objections

It is certainly clear from the beginning that the possibly existing description of S-matrix in terms of tensor networks cannot correspond to the perturbative QFT description in terms of Feynman diagrams.

1. Tensor network description relates interior and boundary degrees in holography by a isometry. Now however unitary matrix has quite different role. It could correspond to U-matrix relating zero energy states to each other or to the S-matrix relating to each other the states at boundary of CD and at the shifted boundary obtained by scaling. These scalings shifting the second boundary of CD and increasing the distance between the tips of CD define the analog of unitary time evolution in ZEO. The U-matrix for transitions associated with the state function reductions at fixed boundary of CD effectively reduces to S-matrix since the other boundary of CD is not affected.

   The only manner one could see this as holography type description would be in terms of ZEO in which zero energy states are at boundaries of CD and U-matrix is a representation for them in terms of holography involving the interior states representing scattering diagram in generalized sense.

2. The appearance of small gauge coupling constant tells that the entanglement between "states" in state spaces whose coordinates formally correspond to quantum fields is weak and just opposite to that defined by a perfect tensor. Quite generally, coupling constant might be the fatal aspect of the vertices preventing the formulation in terms of perfect entanglement.

   One should understand how coupling constant emerges from this kind of description - or disappears from standard QFT description. One can think of including the coupling constant to the definition of gauge potentails: in TGD framework this is indeed true for induced gauge fields. There is no sensical manner to bring in the classical coupling constants in the classical framework and the inverse of Kähler coupling strength appears only as multiplier of the Kähler action analogous to critical temperature.

More concretely, there are WCW spin degrees of freedom (fermionic degrees of freedom) and WCW orbital degrees of freedom involving functional integral over WCW. Fermionic contribution would not involve coupling constants whereas the functional integral over WCW involving exponential of vacuum functional could give rise to the coupling constants assignable to the vertices in the minimal tree diagram.

3. The decomposition $S = 1 + iT$ of unitary S-matrix giving unitarity as the condition $-i(T - T^{\dagger}) + T^{\dagger}T = 0$ reflects the perturbative thinking. If one has only isometry instead of unitary transformation, this decomposition becomes problematic since T and $T^{\dagger}$ whose some appears in the formula act in different spaces. One should have the generalization of Id as a "trivial" isometry. Alternatively, one should be able to extend the state space H_{in} by adding a tensor factor mapped trivially in isometry.

4. There are 3- and 4-vertices rather than only -say, 3-vertices as in tensor networks. For non-Abelian Chern-Simons term for simple Lie group one would have besides kinetic term only 3-vertex $Tr(A \wedge A \wedge A)$ defining the analog of perfect tensor entanglement when interpreted as co-product involving 3-D permutation symbol and structure constants of Lie algebra. Note also that for twistor Grassmannian approach the fundamental vertices are 3-vertices. It must be however emphasized that QFT description emerges from TGD only at the limit when one identifies gauge potentials as sums of induced gauge potentials assignable to the space-time sheets, which are replaced with single piece of Minkowski space.

5. Tensor network description does not contain propagators since the contractions are between perfect tensors. It is to make sense propagators must be eliminated. The twistorial factorization of massless fermion propagator suggest that this might be possible by absorbing the twistors to the vertices.

These reasons make it clear that the proposed idea is just a speculative question. Perhaps the best strategy is to look this crazy idea from different view points: the overly optimistic view developing big picture and the approach trying to debunk the idea.

3.5.2 The overly optimistic vision

With these prerequisites on one can follow the optimistic strategy and ask how tensor networks could the allow to generalize the notion of unitary S-matrix in TGD framework.

1. Tensor networks suggests the replacement of unitary correspondence with the more general notion of Hilbert space isometry. This generalization is very natural in TGD since one must allow phase transitions increasing the state space and it is quite possible that S-matrix represents only isometry: this would mean that $S^{\dagger}S = Id_{in}$ holds true but $SS^{\dagger} = Id_{out}$ does not even make sense. This conforms with the idea that state function reduction sequences at fixed boundary of causal diamonds defining conscious entities give rise evolution implying that the size of the state space increases gradually as the system becomes more complex. Note that this gives rise to irreversibility understandandable in terms of NMP [K57]. It might be even impossible to formally restore unitary by introducing formal additional tensor factor to the space of incoming states if the isometric map of the incoming state space to outgoing state space is inclusion of hyperfinite factors.

2. If the huge generalization of the duality of old fashioned string models makes sense, the minimal diagram representing scattering is expected to be a tree diagram with braiding and should allow a representation as a tensor network. The generalization of the tensor network concept to include braiding is trivial in principle: assign to the legs connecting the nodes defined by perfect tensors unitary matrices representing the braiding - here topological QFT allows realization of the unitary matrix. Besides fermionic degrees of freedom having interpretation as spin degrees of freedom at the level of "World of Classical Worlds" (WCW) there are also WCW orbital degrees of freedom. These two degrees of freedom factorize in the generalized unitarity conditions and the description seems much simpler in WCW orbital degrees of freedom than in WCW spin degrees of freedom.

3. Concerning the concrete construction there are two levels involved, which are analogous to descriptions in terms of boundary and interior degrees of freedom in holography. The level of fundamental fermions assignable to string world sheets and their boundaries and the level of physical particles with particles assigned to sets of partonic 2-surface connected by magnetic flux tubes and associated fermionic strings. One could also see the ends of causal diamonds as analogous to boundary degrees of freedom and the space-time surface as interior degrees of freedom.

The description at the level of fundamental fermions corresponds to conformal field theory at string world sheets.

1. The construction of the analogs of boundary states reduces to the construction of N-point functions for fundamental fermions assignable to the boundaries of string world sheets. These boundaries reside at 3-surfaces at the space-like space-time ends at CDs and at light-like 3-surfaces at which the signature of the induced space-time metric changes.
2. In accordance with holography, the fermionic N-point functions with points at partonic 2-surfaces at the ends of CD are those assignable to a conformal field theory associated with the union of string world sheets involved. The perfect tensor is assignable to the fundamental 4-fermion scattering which defines the microscopy for the geometric 3-particle vertices having twistorial interpretation and also interpretation as algebraic operation.

   What is important is that fundamental fermion modes at string world sheets are labelled by conformal weights and standard model quantum numbers. No four-momenta nor color quantum numbers are involved at this level. Instead of propagator one has just unitary matrix describing the braiding.
3. Note that four-momenta emerging in somewhat mysterious manner to stringy scattering amplitudes and mean the possibility to interpret the amplitudes at the particle level.

Twistorial and number theoretic constructions should correspond to particle level construction and also now tensor network description might work.

1. The 3-surfaces are labelled by four-momenta besides other standard model quantum numbers but the possibility of reducing diagram to that involving only 3-vertices means that momentum degrees of freedom effectively disappear. In ordinary twistor approach this would mean allowance of only forward scattering unless one allows massless but complex virtual momenta in twistor diagrams. Also vertices with larger number of legs are possible by organizing large blocks of vertices to single effective vertex and would allow descriptions analogous to effective QFTs.
2. It is highly non-trivial that the crucial factorization to perfect tensors at 3-vertices with unitary braiding matrices associated with legs connecting them occurs also now. It allows to split the inverses of fermion propagators into sum of products of two parts and absorb the halves to the perfect tensors at the ends of the line. The reason is that the inverse of massless fermion propagator (also when masslessness is understood in 8-D sense allowing M^4 mass to be non-vanishing) to be express as bilinear of the bi-spinors defining the twistor representing the four-momentum. It seems that this is absolutely crucial property and fails for massive (in 8-D sense) fermions.

3.5.3 Twistorial and number theoretic visions

Both twistorial and number theoretical ideas have given a strong boost to the development of ideas.

1. With experience coming from twistor Grassmannian approach, twistor approach is conjectured to allow an extension of super-symplectic and other superconformal symmetry algebras to Yangian algebras by adding a hierarchy of multilocal generators [L10]. The twistorial diagrams for $\mathcal{N} = 4$ SUSY can be reduced to a finite number and there is large number of equivalent diagrams. One expects that this is true also in TGD framework.

   Twistorial approach is extremely general and quite too demanding to my technical skills but its is a useful guideline. An important outcome of twistor approach is that the intermediate states are massless on-mass-shell states but with complex momenta. Does this generalize and could each vertex define unitary scattering event with complex four-momenta in possibly complexified Minkowski space? Or could even real momenta be possible for massive particles, which would be massless in 8-D sense thanks to the existence of octonionic tangent space structure of 8-D embedding space? And what is the role of the unique twistorial properties of M^4 and CP_2?

2. Number theoretical vision suggests that the scattering amplitudes correspond to sequences of algebraic operations taking inputs and producing outputs, which in turn serve as inputs for a neighboring node [L10]. The vertices form a diagram defining a network like structure defining kind of distributed computations leading from given inputs to given outputs. A computation leading from given inputs to given outputs is suggestive. There exists an infinite number of this kind of computations and there must be the minimal one which defines the complexity of the scattering. The maximally simplifying guess is that this diagram would correspond to a braided tree diagram. At space-time level these diagrams would correspond to different space-time surfaces defining same physics: this is because of holography meaning that only the ends of space-time surfaces at boundaries of CD matter.

   This vision generalizes of the old-fashioned stringy duality. It states that all diagrams can be reduced to minimal diagrams. This is achieved by by moving the ends of internal lines so that loops becomes vacuum tadpoles and can be snipped off. Tree diagrams must be however allowed to braid and outside the vertices the diagrams look like braids. Braids for which threads can split and glue together is the proper description for what the diagrams could be. Braiding would provide the counterpart for the non-planar twistor diagrams.

   The fermion lines inside the light-like 3-surfaces can get braided. Smaller partonic 2-surfaces can topologically condense at given bigger partonic 2-surface (electronic parton surface can topologically condense to nano-scopic parton surface) and the orbits of the condensed partonic 2-surfaces at the light-like orbit of the parton surface can get braided. This gives rise to a hierarchy of braids with braids.

3.5.4 Generalization of the notion of unitarity

The understanding of unitarity has been the most difficult issue in my attempts to understand S-matrix in TGD framework. When something turns out to be very difficult to understand, it might make sense to ask whether the definition of this something involves un-necessary assumptions. Could unitarity be this kind of notion?

The notion of tensor network suggests that unitarity can generalized and that this generalization allows the realization of unitarity in extremely simple manner using perfect tensors as building bricks of diagrams.

1. Both twistorial and number theoretical approaches define M-matrix and associated S-matrix as a map between the state spaces H_{in} and H_{out} assignable to the opposite boundaries of CD - say positive and negative energy parts of zero energy state. In QFT one has $H_{in} = H_{out}$ and the map would be Hilbert space unitary transformation satisfying $SS^\dagger = S^\dagger S = Id$.

2. The basic structure of TGD (NMP favoring generation of negentropic entanglement, the hierarchy of Planck constants, length scale hierarchies, and hierarchy of space-time sheets) suggests that the time evolution leads to an increasingly complex systems with higher-dimensional Hilbert space so that $H_{in} = H_{out}$ need not hold true but is replaced with $H_{in} \subset H_{out}$. This view is very natural since one must allow quantum phase transitions increasing the value of h_{eff} and the value of p-adic prime defining p-adic length scale.

   S-matrix would thus define isometric map $H_{in} \subset H_{out}$. Isometry property requires $U^\dagger U = Id_{in}$. If the inclusion of H_{in} to H_{out} is a genuine subspace of H_{out}, the condition $UU^\dagger = Id_{out}$ does not make sense anymore. This means breaking of reversibility and is indeed implied by the quantum measurement theory based on ZEO.

3. It would be at least formally possible to fuse all state spaces to single very large state space by replacing isometry $H_{in} \subset H_{out}$ with unitary map $H_{out} \rightarrow H_{out}$ by adding a tensor factor in which the map acts as identity transformation. This is not practical since huge amounts of redundant information would be introduced. Also the information about hierarchical structure essential for the idea of evolution would be lost. This hierarchical of inclusions should also be

crucial for understanding the construction of S-matrix or rather, the hierarchy of S-matrices of isometric inclusions including as a special case unitary S-matrices.

4. There is also a further intricacy, which might prevent the formal unitarization by the addition of an inert tensor factor. I have talked a lot about HFFs referring to hyper-finite factors of type II_1 (possibly also of type III_1) and their inclusions [K105]. The reason is that WCW spinors form a canonical representation for these von Neumann algebras.

   Could the isometries replacing unitary S-matrix correspond to inclusions of HFFs? In the recent interpretation the included factor (now H_{in}) corresponds to the degrees of freedom below measurement resolution. Certainly this does not make sense now. The interpretation in terms of finite measurement resolution need not however be the only possible interpretation and the interpretation in terms of measurement resolution might of course be wrong. Therefore one can ask whether the relation between H_{in} and H_{out} could be more complex than just $H_{out} = H_{in} \otimes H_1$ so that formal unitarization would fail.

3.5.5 Scattering diagrams as tensor networks constructed from perfect tensors

Preskill's tensor network construction [B44] realizes isometric maps as representations of holography and as models for quantum error correcting codes. These tensor networks have remarkable similarities with twistorial and number theoretical visions, which suggests that it could be used to construct scattering amplitudes. A further idea inspired by holography is that the description of scattering amplitudes in terms of fundamental fermions and physical particles are dual to each other.

1. In the construction of quantum error codes tensor network defines an isometric embedding of local states in the interior to strongly entangled non-local states at boundary. Their vertices correspond to tensors, which in the proposal of Preskill *et al* [B44] are perfect tensors such that one can take any m legs of the vertex and the tensor defines isometry from the state space of m legs to that of $n-m$ legs. When the number of indices is $2n$, the entanglement defined by perfect tensor between any n-dimensional subspace and its complement is maximal

   TGD framework maximal entanglement corresponds to negentropic entanglement with density matrix proportional to identity matrix. What is important that the isometry is constructed by composing local isometries associated with a network. Given isometry can be constructed in very many ways but there is some minimal realization.

2. The tensor networks considered in [B44] are very special since they are determined by tessellations of hyperbolic space H_2. This kind of tessellations of H_3 could be crucial for understanding the analog of condensed matter physics for dark matter and could appear in biology [L23]. What is crucial is that only the graph property and perfect tensor property matter as far as isometricity is considered so that it is possible to construct very general isometries by using tensor networks.

3.5.6 Eigenstates of Yangian co-algebra generators as a way to generate maximal entanglement?

Negentropically entangled objects are key entities in TGD inspired theory of consciousness and also of tensor networks, and the challenge is to understand how these could be constructed and what their properties could be. These states are diametrically opposite to unentangled eigenstates of single particle operators, usually elements of Cartan algebra of symmetry group. The entangled states should result as eigenstates of poly-local operators. Yangian algebras involve a hierarchy of poly-local operators, and twistorial considerations inspire the conjecture that Yangian counterparts of super-symplectic and other algebras made poly-local with respect to partonic 2-surfaces or end-points of boundaries of string world sheet at them are symmetries of quantum TGD [L22]. Could Yangians allow to understand maximal entanglement in terms of symmetries?

1. In this respect the construction of maximally entangled states using bi-local operator $Q^z = J_x \otimes J_y - J_x \otimes J_y$ is highly interesting since entangled states would result by state function. Single particle operator like J_z would generate un-entangled states. The states obtained as

eigenstates of this operator have permutation symmetries. The operator can be expressed as $Q^z = f^z_{ij} J^i \otimes J^j$, where f^A_{BC} are structure constants of SU(2) and could be interpreted as co-product associated with the Lie algebra generator J^z. Thus it would seem that unentangled states correspond to eigenstates of J^z and the maximally entangled state to eigenstates of co-generator Q^z. Kind of duality would be in question.

2. Could one generalize this construction to n-fold tensor products? What about other representations of SU(2)? Could one generalize from SU(2) to arbitrary Lie algebra by replacing Cartan generators with suitably defined co-generators and spin 1/2 representation with fundamental representation? The optimistic guess would be that the resulting states are maximally entangled and excellent candidates for states for which negentropic entanglement is maximized by NMP [K57].

3. Co-product is needed and there exists a rich spectrum of algebras with co-product (quantum groups, bialgebras, Hopf algebras, Yangian algebras). In particular, Yangians of Lie algebras are generated by ordinary Lie algebra generators and their co-generators subject to constraints. The outcome is an infinite-dimensional algebra analogous to one half of Kac-Moody algebra with the analog of conformal weight N counting the number of tensor factors. Witten gives a nice concrete explanation of Yangian [B26] for which co-generators of T^A are given as $Q^A = \sum_{i<j} f^A_{BC} T^B_i \otimes T^C_j$, where the summation is over discrete ordered points, which could now label partonic 2-surfaces or points of them or points of string like object (see `http://tinyurl.com/y727n8ua`). For a practically totally incomprehensible description of Yangian one can look at the Wikipedia article (see `http://tinyurl.com/y7heufjh`).

4. This would suggest that the eigenstates of Cartan algebra co-generators of Yangian could define an eigen basis of Yangian algebra dual to the basis defined by the totally unentangled eigenstates of generators and that the quantum measurement of poly-local observables defined by co-generators creates entangled and perhaps even maximally entangled states. A duality between totally unentangled and completely entangled situations is suggestive and analogous to that encountered in twistor Grassmann approach where conformal symmetry and its dual are involved. A beautiful connection between generalization of Lie algebras, quantum measurement theory and quantum information theory would emerge.

3.5.7 Two different tensor network descriptions

The obvious question is whether also unitary S-matrix of TGD could be constructed using tensor network built from perfect tensors. In ZEO the role of boundary would be taken by the ends of the space-time at upper and lower light-like boundaries of CD carrying the particles characterized by standard model quantum numbers. Strong form of holography would suggest that partonic surfaces and strings at the ends of CD provide information for the description of zero energy states and therefore of scattering amplitudes. The role of interior would be taken by the space-time surface - in particular the light-like orbits of partonic surfaces carrying the fermion lines identified as boundaries of string world sheets. Conformal field theory description would apply to fermions residing at string world sheets with boundaries at light-like orbits of partonic 2-surfaces.

In QFT Feynman diagrammatics one obtains a sum over diagrams with arbitrary numbers of loops. In both twistorial and number theoretic approach however only a finite number of diagrams with possibly complex on mass shell massless momenta are needed. If the vertices are however such that particles remain on-mass-shell but are allowed to have complex four-momenta then the integration over internal momenta (loops) is not present and tensor network description could make sense. This encourages the conjecture that tensor networks could be used to construct the scattering amplitudes in TGD framework.

What could perfect tensor property mean for the vertices identified as nodes of a tensor network? There are two levels to be considered: the geometric level identifying particles as 3-surfaces with net quantum numbers and the fermion level identifying particles as fundamental fermions at the boundaries of string world sheets.

1. At the geometric level vertices corresponds to light-like orbits of partonic 2-surfaces meeting at common end which is partonic 2-surface. This is 3-D generalization of Feynman diagram as a geometric entity. At the level of fermion lines associated with the light-like 3-surfaces one the basic interaction corresponds to the scattering of 2-fermions leading to re-sharing

of fermion lines between outgoing light-like 3-surfaces, which include also representations for virtual particles. One has 4-fermion vertex but not in the sense that it appears in the interaction of weak interactions at low energies.

Geometrically the basic vertex could be 3-vertex: $n > 3$-vertices are unstable against deformation to lower vertices. For 3-vertex perfect tensor property means that the tensor defining the vertex maps any 1-particle subspaces to 2-particle subspace isometrically. The geometric vertices define a network consisting of 3-D "lines" and 2-D vertices but one cannot tell what is within the 3-D lines and what happens in the 2-D nodes. The lines would consist of braided fundamental fermion lines and in nodes the basic process would be 2+2 scattering for fermions. In the case of 3-vertex momentum conservation would effectively eliminate the four-momentum and the state spaces associated with vertex would be effectively discrete. This is p-adically of utmost importance.

2. At the level of fundamental fermion lines in the interior of particle lines one would have 4-vertices and if a perfect tensor describes it, it gives rise to a unitary map of any 2-fermion subspace to its complement plus isometric maps of 1-fermion subspaces to 3-fermion subspaces. In this case momenta cannot act as labels of fermion lines for rather obvious reasons: the solution of the problem is that conformal weights label fundamental fermion lines

   The conservation of discrete quark and lepton numbers allows only vertices of type $qL \rightarrow qL$ and its variants obtained by crossing. In this case the isometries might allow realization. The isometries must be defined to take into account quark and lepton number conservation by crossing replacing fermion with antifermion. By allowing the states of Hilbert space in node to be both quarks and leptons, difficulties can be avoided.

Tensor network description in terms of fundamental fermions and CFT

Consider first fundamental fermions. What are the labels characterizing the states of fundamental fermions fermions propagating along the lines? There are two options: the labels are either conformal weights or four-momenta.

1. Since fermions corresponds to strings defining the boundaries of string world sheets and since strong form of holography implies effective 2-dimensionality also in fermion sector, the natural guess is that the conformal weights plus some discrete quantum numbers - standard model quantum numbers at least - are in question. The situation would be well-defined also p-adically for this option. In this case one can hope that conformal field theory at partonic 2-surface could define the fermionic 4-vertex more or less completely. There would be no need to assign propagators between different four-fermion vertices. The scattering diagram would define a composite formed from light-like 3-surfaces and one would have single isometry build from 4-fermion perfect tensors. There would be no integrations over internal momenta.

2. Second option is that fundamental fermions are labelled by four-momenta. The outgoing four-momenta in 4-vertices would not be completely fixed by the values of the incoming momenta and this extends the state space. Concerning p-adicization this integral is not desirable and this forces to consider seriously discrete labelling. The unitarity condition for 2+2 scattering would involve integral over 2-sphere. Four-fermion scattering must be unitary process in QFT so that this condition might be possible to satisfy. The problem would be how to fix this fundamental scattering matrix uniquely. This option does not look attractive number theoretically.

The most plausible option is that holography means that conformal field theory describes the scattering of fundamental fermions and QFT type description analogous to twistorial approach describes the scattering of physical fermions. If only 3-vertices are allowed, and if masslessness corresponds to masslessness in 8-D sense, one obtains non-trivial scattering vertices (for ordinary twistor approach all massless momenta would be collinear if real).

Tensor network description for physical particles

Could the twistorial description expected to correspond to the description in terms of particles allow tensor network description?

1. Certainly one must assign four-momenta to incoming *physical* particles - also fermions - but they correspond to pairs of wormhole contacts rather than fundamental fermions at the boundaries of string world sheets. It would be natural to assign four-momenta also to the virtual *physical* fermions appearing in the diagram and the geometric view about scattering would allow only 3-vertices so that momentum conservation would eliminate momentum degrees of freedom effectively. This would be a p-adically good news.

2. At the level of fundamental fermions entanglement is described as a tensor contraction of the CFT vertices. This locality is natural since the vertices are at null distance from each other. At QFT limit the entanglement between the ends of the line is characterized the propagator.

   One must get rid of propagators in order to have tensor network description. The inclusion of propagators to the fundamental tensor diagrams would break the symmetry between the legs of vertex since the propagator cannot be included to its both ends. Situation changes if one can represent the propagator as a bilinear of something more primitive and include the halves to the opposite ends of the line. Twistor representation of four-momentum indeed defines this kind of representation as a bilinear $p^{a\tilde{b}} = \lambda\tilde{\mu}^{\tilde{b}}$ of twistors λ and $\tilde{\mu}$. There is problem due to the diverging $1/p^2$ factor but residue integral eliminates this factor and one can write directly the fermionic propagator factors as $p^{a\tilde{b}}$.

3. In QFT description the perturbative expansion is in powers of coupling constant. If the reduction to braided tree diagrams analogous to twistor diagrams occurs, power g^{N-2} of coupling constant is expected to factorize as a multiplier of a tree diagram with N external legs. One should understand this aspect in the tensor net-work picture.

   For $\mathcal{N} = 4$ SUSY there is coupling constant renormalization. Similar prediction is expected from TGD. Coupling constant evolution is expected to be discrete and induced by the discrete evolution of Kähler coupling strength defined by the spectrum of its critical values. The conjecture is that critical values are naturally labelled by p-adic primes $p \simeq 2^k$, k prime, labelling p-adic length scales. Therefore one might hope that problems could be avoided.

These observations encourage the expectation that twistorial approach involving only 3-vertices allows to realize tensor network idea also at the level of physical particles. It might be essential that twistors can be generalized to 8-D twistors. Octonionic representation of gamma matrices might make this possible. Also the fact twistorial uniqueness of M^4 and CP_2 might be crucial.

Gauge theory follows as QFT limit of TGD so that one cannot in principle require that gauge theory vertices satisfy the isometricity conditions. Nothing however prevents from checking whether gauge theory limit might inherit this property.

1. For instance, could 3-vertices of Yang-Mills theory define isometric embedding of 1-particle states to 2 particle states? For a given gauge boson there should exist always a pair of gauge bosons, which can fuse to it. Consider a basis for Lie-algebra generators of the gauge group. If the generator T is such that there exists no pair $[A, B]$ with the property $[A, B] = T$, Jacobi identity implies that T must commute with all generators and one has direct sum of Lie algebras generated by T and remaining generators.

2. In the case of weak algebra $SU(2) \times U(1)$ the weak mixing of Y and I_3 might allow the isometric embeddings of type $1 \rightarrow 2$. Does this mean that Weinberg angle must be non-vanishing in order to have consistent theory? A realistic manner to get rid of the problem is to allow at QFT limit the lines to be also fermions so that also U(1) gauge boson can be constructed as fermion pair.

How the two tensor network descriptions would be related?

There are two descriptions for the zero energy states providing representation of scattering amplitudes: the CFT description in terms of fundamental fermions at the boundaries of string world sheets, and the description in terms of physical particles to which one can assign light-like 3-surfaces as virtual lines and total quantum numbers.

1. CFT description in terms of fundamental fermions in some aspects very simple because of its 2-dimensionality and conformal invariance. The description is in terms of physical particles

having light-like 3-surfaces carrying some total quantum numbers as correlates and is simpler in different sense. These descriptions should be related by an Hilbert space isometry.

2. The perfect tensor property for 4-fermion vertices makes fundamental fermion states analogous to physical states realizing logical qubits as highly entangled structures. Geometric description in terms of 3-surfaces is in turn analogous to the description in terms of logical qubits.

3. Holography-like correspondence between these descriptions of zero energy states (scattering diagrams) should exist. Physical particles should correspond to the level, at which resolution is smaller and which should be isometrically mapped to the strongly entangled level defined by fundamental fermions and analogous to boundary degrees of freedom (fundamental fermions *are* at the boundaries of string world sheets!).

   The map relating the two descriptions seems to exist. One can assign four-momenta to the legs of conformal four-point function as parameters so that one obtains a mapping from the states labelled by conformal weights to the states labelled by four-momenta! The appearance of 4-momenta from conformal theory is somewhat mysterious looking phenomenon but this duality makes it rather natural.

3.5.8 Taking into account braiding and WCW degrees of freedom

One must also take intro account braiding and orbital degrees of freedom of WCW. The generalization of tensor network to braided tensor network is trivial. Thanks to the properties of tensor network orbital and spinor degrees of freedom factorize so that also the treatment of WCW degrees of freedom seems to be possible.

What about braiding?

The scattering diagrams would be tree diagrams with braiding of fermionic lines along light-like 3-surfaces - dance of fundamental quarks and leptons at parquette defined by the partonic 2-surface one might say. Also space-like braiding at magnetic flux tubes at the ends of CD is possible and its time evolution between the ends of space-time surfaces defines 2-braiding which is generalization of the ordinary braiding but will not be discussed here. This gives rise to a hierarchy of braidings. One can talk about flux tubes within flux tubes and about light-like 3-surface within light-like 3-surfaces. The smaller light-like 3-surface would be glued by a wormhole contact to the larger one and contact could have Euclidian signature of induced metric.

How can one treat the braiding in the tensor network picture? The answer is simple. Braiding corresponds to an element of braid group and one can represent it by a unitary matrix as one does in topological QFT as one constructs knot invariants. In particular, the trace of this unitary matrix defines a knot invariant. The generalization of the tensor network is simple. One attaches to the links connecting two nodes unitary transformation defining a representation of the braid involved. Local variant of unitarity would mean isometricity at nodes and unitarity at links.

What about WCW degrees of freedom?

The above considerations are about fermions that its WCW spinor degrees of freedom and the space-time surface itself has been regarded as a fixed background. How can one take into account WCW degrees of freedom?

The scattering amplitude involves a functional integral over the 3-surfaces at the ends of CD. The functional integration over WCW degrees of freedom gives an expression depending on Kähler coupling strength α_K and determines the dependence on various gauge coupling strengths expressible in terms of α_K. This makes it possible to have the tensor network description in fermionic degrees of freedom without losing completely the dependence of the scattering amplitudes on gauge couplings. By strong form of holography the functional integral should reduce to that over partonic 2-surfaces and strings connecting them. Number theoretic discretization with a cutoff determined by measurement resolution forces the parameters characterizing the 2-surfaces to belong to an algebraic extension of rationals and is expected to reduce functional integral to a sum over discretized WCW so that it makes sense also in p-adic sectors [K80, K104].

A brief summary of quantum measurement theory in ZEO is necessary. The repeated state function reduction shifts active boundary A of CD and affects the states at it. The passive boundary of CD- call it P - and the states at it - remain unaffected. The repeated state function reductions leaving P unaffected and giving usually rise to Zeno effect, correspond now to the TGD counterpart of unitary time evolution by shifts between subsequent state function reductions. Call A and its shifted version A_{in} and A_{out} and the corresponding state spaces H_{in} and H_{out}. The unitary (or more generally isometric) S matrix represents this shift. This is the TGD counterpart of a unitary evolution of QFTs. S forms a building brick of a more general unitary matrix U acting in the space of zero energy states but U is not considered now.

Consider now the isometricity conditions.

1. Unitarity conditions generalized to isometricity conditions apply to S. Isometricity conditions $S^\dagger S = Id_{in}$ can be applied at A_{in}. The states appearing in the isometry conditions as initial and final states correspond to A_{in} and A_{out}. There is a trace over WCW spin indices (labels for many-fermion states) of H_{out} in the conditions $S^\dagger S = Id_{in}$. Isometricity conditions involve also an integral over WCW orbital degrees of freedom at both ends: these degrees of freedom are strongly correlated and for a strict classical determinism the correlation between the ends is complete. If the tensor network idea works, the summation over spinor degrees of freedom at A_{out} gives just a unit matrix in the spinor indices at A_{in} and leaves only the WCW orbital degrees of freedom in consideration. This factorization of spinor and orbital WCW degrees of freedom simplifies the situation dramatically.

2. One can express isometricity conditions for modes with $\Psi_{in,M}$ and $\Psi_{out,N}$ at A_{in} and A_{out}: this requires functional integration over 3-surfaces WCW at A_{in} and A_{out}. The conditions are formulated in terms of the labels - call them M_{in}, N_{in} - of WCW spinor modes at A_{in} including standard model quantum numbers and labels characterizing the states of supersymplectic and super-conformal representations. The trace is over the corresponding indices R_{out} at A_{out}. The WCW functional integrals in the generalized unitarity conditions are therefore over A_{in} and A_{out} and should give Kronecker delta $\sum_{R_{out}} S^\dagger_{M_{in}R_{out}} S_{R_{out}N_{in}} = \delta_{M_{in},N_{in}}$.

3. The simplest view would be that Kähler action with boundary conditions implies completely deterministic dynamics. The conditions expressing strong form of holography state that sub-algebras of super-symplectic algebra and related conformal algebras isomorphic to the entire algebra give rise to vanishing Noether charges. Suppose that these conditions posed at the ends of CD are so strong that they fix the time evolution of the space-time surface as preferred extremal completely when posed at either boundary. In this case the isometricity conditions would be so strong that the double functional integration appearing in the matrix product reduces to that at A_{in} and the isometricity conditions would state just the orthonormality of the basis of WCW spinor modes at A_{in}.

4. Quantum criticality and in particular, the hierarchy of Planck constants providing a geometric description for non-deterministic long range fluctuations, does not support this view. Also the fact that string world sheets connect the boundaries of CD suggests that determinism must be broken. The inner product defining the completeness of the WCW state basis in orbital degrees of freedom can be however generalized to a bi-local inner product involving functional integration over 3-surfaces at both A_{in} and A_{out}. There is however a very strong correlation so that integration volume at A_{out} is expected to be small. This also suggests that one can have only isometricity conditions.

3.5.9 How do the gauge couplings appear in the vertices?

Reader is probably still confused and wondering how the gauge couplings appear in the vertices from the functional integral over WCW degrees of freedom. In twistorial approach, the vanishing of loops in $\mathcal{N} = 4$ SYM theory gives just g^N, N the number of 3-vertices. Each vertex should give gauge coupling. Or equivalently, each propagator line connecting vertices should give α_K. The functional integral should give this factor for each propagator line. Generalization of conformal invariance is expected to give this picture.

To proceed some basic facts about N-point functions of CFTs are needed.

1. In conformal field theory the functional form of two-point function is completely fixed by conformal symmetry:

$$\begin{aligned} G^{(2)}(z_i,\overline{z}_i) &= \frac{C_{12}}{z_{12}^{2h}\overline{z}_{12}^{2\overline{h}}} \ , \\ z_{ij} &= z_i - z_j \ , \ \overline{z}_{ij} = \overline{z}_i - \overline{z}_j \ , \\ h_1 = h_2 = h &= h_a + ih_b \ , \ \overline{h} = \overline{h}_a + i\overline{h}_b \ . \end{aligned} \tag{3.5.1}$$

$h_1 = h_2 \equiv h$ and its conjugate $\overline{h}$ are conformal weights of conformal field and its conjugate. Note that the conformal weights of conformal fields Φ_1 and Φ_2 must be same. In TGD context C_{12} is expected to be proportional to α_K and this would give to each vertex g_K when couplings are absorbed into vertices.

2. The 3-point function for 3 conformal fields Φ_i, $i = 1, 2, 3$ is dictated by conformal symmetries apart from constant C_{123}:

$$G^{(3)}(z_i,\overline{z}_i) = C_{123} \times \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}} \times \frac{1}{\overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3} \overline{z}_{23}^{\overline{h}_2+\overline{h}_3-\overline{h}_1} \overline{z}_{31}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}} \ . \tag{3.5.2}$$

Here C_{123} should bef fixed by super-symplectic and related symmetries and determined the numerical coefficients various couplings when expressed in terms of g_K.

3. 4-point functions have analogous form

$$\begin{aligned} G^{(4)}(z_i,\overline{z}_i) &= f_{1234}(x,\overline{x}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \prod_{i<j} \overline{z}_{ij}^{-(\overline{h}_i+\overline{h}_j)+\overline{h}/3} \ , \\ h &= \sum_i h_i \ , \end{aligned} \tag{3.5.3}$$

but are proportional to an arbitrary function f_{1234} of conformal invariant $x = z_{12}z_{34}/z_{13}z_{24}$ and its conjugate.

If only 3-vertices appear/are needed for physical particles - as both twistorial and number theoretic approaches strongly suggest - the conformal propagators and vertices are fixed apart from constants C_{ijk}, which in turn should be fixed by the huge generalization of conformal symmetries. α_K emerges in the expected manner.

This picture seems to follow from first principles.

1. One can fix the partonic 2-surfaces at the boundaries of CD but there is a functional integral over partonic 2-surfaces defining the vertices: their deformations induce deformations of the legs. One can expand the exponent of Kähler action and in the lowest order the perturbation term is trilinear and non-local in the perturbations. This gives rise to 3-point function of CFT nonlocal in z_i. The functional integral over perturbations gives the propagators in legs proportional to α_K in terms of two point function of CFT. Note that the external propagator legs can be eliminated in S-matrix.

2. The cancellation of higher order perturbative corrections in WCW functional integral is required by the quantum criticality and means trivial coupling constant evolution for α_K and other coupling constants. Coupling constant evolution is discretized with values of α_K analogous to critical temperatures and should correspond to p-adic coupling constant evolution [L17].

3. This picture leaves a lot of details open. An integration over the values of z_i is needed and means a kind of Fourier analysis leading from complex domain. The analog of Fourier analysis would be for deformations of partonic 2-surface labelled by some natural labels. Conformal weights could be natural labels of this kind.

It is easy to get confused since there are several diagrammatics involved: the topological diagrammatics of 3-surfacse assignable to the physical particles with partonic 2-surfaces as vertices, the diagrammatics associated with the perturbative functional integral for the Kähler action, and the fermionic diagrammatics suggested to reduce to tensor network. The conjectures are as follows.

1. The "primary" vertices $G^{(n)}$, $n > 3$ assignable to single partonic 2-surface and coming from a functional integral for Kähler action vanishes. This corresponds to quantum criticality and trivial RG evolution.

2. $G^{(n)}$, $n > 3$ in the sense of topological diagrammatics without loops and involving n partonic 2-surfaces do not vanish. One can construct the analog of $G^{(4)}$ from two $G^{(3)}$:s at different partonic 2-surfaces and propagator defined by 2-point function connecting them as string diagram.

   Also topological variant of $G^{(4)}$ assignable to single partonic 2-surface can be constructed by allowing the 3-D propagator "line" to return back to the partonic 2-surface. This would correspond to an analog of loop. Similar construction applies to "primary" $G^{(n)}$,$n > 4$. In number theoretic vision these loops are eliminated as redundant representations so that one has only braided tree diagrams. Also twistor Grassmann approach supports this view.

To sum up, the tensor network description would apply to fermionic degrees of freedom. In bosonic degrees of freedom functional integral would give CFT picture with 3-vertex as the only "primary" vertex and from this twistorial and number theoretic visions follow via the super-symplectic symmetries of the vertex coefficients C_{ijk} extended to Yangian symmetries.

Chapter 4

About Twistor Lift of TGD

4.1 Introduction

The twistor lift of classical TGD [L22] is attractive physically but it is still unclear whether it satisfies all constraints. The basic implication of twistor lift would be the understanding of gravitational and cosmological constants. Volume term in action removes the infinite vacuum degeneracy of Kähler action but because of the extreme smallness of cosmological constant Λ playing the role of inverse of gauge coupling strength, the situation for nearly vacuum extremals of Kähler action in the recent cosmology is non-perturbative.

What is remarkable that twistor lift is possible only in zero energy ontology (ZEO) since the volume term would be infinite by infinite volume of space-time surface in ordinary ontology: by the finite size of causal diamond (CD) the space-time volume is however finite in ZEO. Furthermore, the condition that the destructive interference does not cancel vacuum functional implies Bohr quantization for the action in ZEO. The scale of CD corresponds naturally to the length scale $L_\Lambda = \sqrt{8\pi/\Lambda}$ defined by the cosmological constant.

One motivation for introducing the hierarchy of Planck constants [K35, **?**] was that the phase transition increasing Planck constant makes possible perturbation theory in strongly interacting system. Nature itself would take care about the converge of the perturbation theory by scaling Kähler coupling strength α_K to α_K/n, $n = h_{eff}/h$. This hierarchy might allow to construct gravitational perturbation theory as has been proposed already earlier. This would for gravitation to be quantum coherent in astrophysical and even cosmological scales.

In this chapter two options for the twistor lift are studied in detail.

1. Option I (the original option): The values of $\alpha_K(M^4)$ and $\alpha_K(CP_2)$ are widely different with $\alpha_K(M^4)$ being extremely large so that M^4 part of the 6-D Kähler action gives in dimensional reduction extremely small cosmological term. Allowing Kähler coupling strength $\alpha_K(CP_2)$ to correspond to zeros of zeta implies that for complex zeros the preferred extremals for $\alpha_K(M^4)$ having different phase are mimimal surface extremals of Kähler action so that the values of coupling constants do not matter and extremals depend on couplings only through the boundary conditions stating the vanishing of certain super-symplectic conserved charges. It has turned out that this option has several shortcomings. First of all, $\alpha_K(M^4) \neq \alpha_K(CP_2)$ looks like ad hoc assumption tailored to make cosmological constant small. Secondly, the decoupling between Kähler action and volume term implies separately conserved Noether charges which looks strange. Thirdly, for $\sqrt{g_4}$ instead of $\sqrt{|g_4|}$ in the volume element assumed hitherto, there is no charge transfer between Minkowski and Euclidian regions.
2. Option II: $\alpha_K(M^4) = \alpha_K(CP_2)$ is satisfied. Now entire action is identified as the cosmological term. A small effective value of cosmological constant is obtained if the Kähler action and volume term tend to cancel each other. Minimal surface extremals of Kähler action correspond naturally to asymptotic dynamics near the boundaries of CDs, where the analog of free geodesic motion as minimal surfaces is expected. For $\sqrt{|g_4|}$ option there is charge transfer between Minkowski and Euclidian regions.

The two options provide different generalizations of Chladni mechanism [K55] [L28, L29] (see "An Amazing Resonance Experiment" at `http://tinyurl.com/kcbmrzz`)to a "dynamics of avoidance". Both options have profound implications for the views about what happens in particle physics experiment and in quantum measurement, and for consciousness theory and for quantum biology. It is however clear that Option II is the favored one.

The need to understand the twistor lift leads to a critics of the formulation of the basic action principle and the outcome is a more elegant formulation with non-trivial physical consequences.

1. Dimensionless gauge field is obtained from dimension 2 induced Kähler form by division with constant R_1^2 with dimension two. This parameter defines a hidden coupling parameter in the action and the identification in terms of CP_2 radius made hitherto rather implicitly is probably reasonable but ad hoc. The simple idea is to use the induced Kähler form as basic object and formulate the action principle accordingly. This brings in the dimensional parameter $1/R_1^4$ compensating for the dimension of $\sqrt{g_4}$ in the action.

2. One ends up to a general formulation of both bosonic and fermionic action principles showing that the overall scaling factor of fermionic and bosonic actions - call it X, disappears from classical dynamics so that extremals have no explicit independence on X. This is crucial for number theoretical universality.

   Quantum Classical Correspondence (QCC) realized as the condition that classical Noether charges in Cartan algebra correspond to eigenvalues of quantal fermionic charges however breaks the invariance with respect to scalings of action via fermionic anticommutation relations which depend on the scaling factor. The new formulation leads to a unique guesses for the 6-D actions, their 4-D dimensionally reduced variants, and 2-D effective actions.

3. The formulation helps to realize that Number Theoretical Universality (NTU) requires that $\sqrt{|g_4|}$ option is the only possible one. Physically the need to have charge transfer between Euclidian and Minkowskian space-time regions implies the same result.

This leads to two different views about cosmological constant.

1. For Option I the explanation for dark energy is in terms of volume term of the action and small value of cosmological constant obeying p-adic coupling constant evolution as function of p-adic length scale. For Option II the cancellation of Kähler action and volume term would give rise to a small value of cosmological constant and its p-adic evolution.

2. Either $L_\lambda = \sqrt{8\pi/\Lambda}$ or the length L characterizing vacuum energy density as $\rho_{vac} = \hbar/L^4$ or both can obey p-adic length scale hypothesis as analogs of coupling constant parameters. The third option makes sense if the ratio R/l_P of CP_2 radius and Planck length is power of two: it can be indeed chosen to be $R/l_P = 2^{12}$ within measurement uncertainties. $L(now)$ corresponds to the p-adic length scale $L(k) \propto 2^{k/2}$ for $k = 175$, size scale of neuron and axon.

3. A microscopic explanation for the vacuum energy realizing strong form of holography (SH) is in terms of vacuum energy for radial flux tubes emanating from the source of gravitational field. The independence of energy from the value of $h_{eff}/h = n$ implies analog of Uncertainty Principle: the product Nn for the number N of flux tubes and the value of n defining the number of sheets of the covering associated with $h_{eff} = n \times h$ is constant. This picture suggests that holography is realized in biology in terms of pixels whose size scale is characterized by L rather than Planck length.

4. A interesting observation is that a fundamental length scale of biology - size scale of neuron and axon - would correspond to the p-adic length scale assignable to vacuum energy density characterized by cosmological constant and be therefore a fundamental physics length scale. An especially interesting result is that in the recent cosmology the size scale of a large neuron would be fundamental physical length scale determined by cosmological constant. This gives additional boost to the idea that biology and fundamental physics could relate closely to each other: the size scale of neuron would not be an accident but "determined in stars" and even beyond them!

4.2 More about twistor lift of Kähler action

The following piece of text was motivated by some observations relating to the twistor lift of Kähler action forcing a criticism of the earlier view about twistor lift.

The first observation was that the correct formulation of 6-D Kähler action in the framework of adelic physics implies that the *classical* physics of TGD does not depend on the overall scaling of Kähler action. This implies that the preferred extremals need not be minimal surface extremals of Kähler action. It is enough that they are so asymptotically - near the boundaries of CDs where they behave like free particles. This also nicely conforms with the physical idea that they are 4-D generalizations for orbits of particles in induced Kähler field.

The independence of the classical physics on the scale of the action inspires a detailed discussion of the number theoretic vision. Quantum Classical Correspondence (QCC) breaks the invariance with respect to the scalings via fermionic anti-commutation relations and Number Theoretical Universality (NTU) can fix the spectrum of values of the over-all scaling parameter of the action. One ends up to a condition guaranteeing NTU of the action exponential and finds an answer to the nagging question whether one should use $\sqrt{g_4}$ (imaginary in Minkowskian regions) or $\sqrt{|g_4|}$ in the action. Complex α_K allows $\sqrt{|g_4|}$ and NTU assuming that $1/\alpha_K = s$, $s = 1/2 + iy$ zero of Riemann zeta, implies $y = q\pi$, q rational as proposed also in [L17].

Second observation relates to cosmological constant. The proposed vision for the p-adic evolution of cosmological constant assumes that $\alpha_K(M^4)$ and $\alpha_K(CP_2)$ are different for the twistor lift. One however finds that single value of α_K is the natural choice. This destroys the original proposal for the p-adic length scale evolution of cosmological constant explaining why it is so small in cosmological scale.

The solution to the problem of the cosmological constant would be that the *entire* 6-D action decomposing to 4-D Kähler action and volume term is identified in terms of cosmological constant. The cancellation of Kähler electric contribution and remaining contributions would explain why the cosmological constant is so small in cosmological scales and also allows to understand p-adic coupling constant evolution of cosmological constant. One must however remain cautious: also the original proposal can be defended.

4.2.1 Kähler action contains overall scale as a hidden coupling parameter

The first observation leads to a more precise understanding of 6-D Kähler action relates to the induction procedure.

1. Kähler form has dimension two since its square gives metric: $J^2 = -g$. Gauge fields are however 2-forms, which are usually taken to be dimensionless (this requires that coupling constant g is included as multiplicative factor to gauge potential). Accordingly, I have assumed that induced Kähler form is obtained by diving Kähler form by $1/R^2$, R the radius of CP_2 identified as the radius of its geodesic sphere. One can however argue that the identification of the scaling factor is ad hoc since its value does not affect classical field equations.
2. What would happen if one induces the dimensional Kähler form as such? Kähler action density $L_K\sqrt{g_4}$ would have dimension of volume so that $1/\alpha_K$ must be replaced with $1/8\pi\alpha_K R_1^4$, where R_1 a fundamental coupling constant with dimension of length. This coupling however disappears from the classical field equations and in the recent adelic formulation also from quantum theory [L45].
3. For the 6-D twistor lift of Kähler action one must introduce an additional dimensional factor to get a dimensionless action. One has $R_1^4 \to R_1^4 R_0^2$, where R_0^2 has dimensions of area. The 4-D action density obtained from dimensional reduction for twistor sphere $S^2(X^4)$ assuming that the induced Kähler form for the sphere satisfies $J^4 = -g$ for $S^2(X^4)$ is proportional to

$$L = X \times (J \cdot J - 2)\sqrt{g_4} \quad , \quad X = \frac{1}{2\alpha_K}\frac{Area(S^2(X^4))}{S_0}\frac{1}{R_1^4} \quad , \quad S_0 = 4\pi R_0^2 \ . \tag{4.2.1}$$

   The shift of Kähler action density by -2 comes from $S^2(X^4)$ part of 6-D Kähler action.
4. From this form one can immediately see that the factor X in Eq. 4.2.1 disappears from field equations, and the functional form of preferred extremals has no dependence on coupling parameters! The quantum classical correspondence (QCC) stating that fermionic Noether charges in Cartan algebra have eigenvalues equals to their classical counterparts however implies this dependence.

Modified Dirac action and string world sheet action in the new formalism

What about the modified Dirac action related super-symmetrically to Kähler action in the new formalism? The 6-D formalism for the induced spinors doubles the number of spinor components and dimensional reduction must eliminate half of them to give something equivalent with the ordinary induced spinor structure. Chirality condition is the most plausible manner to achieve this. This answers the old question whether one could assume only leptonic spinors as fundamental spinors and construct quarks as some of anyonic leptons. This would require two chirality conditions and this is very probably not possible. The 6-D modified Dirac action can be written using the same rules as applied in 4-D case. The possible delicacies of the fermionic dimensional reduction require a separate discussion.

The 4-D dimensionally reduced part of 6-D modified Dirac action must reduce to the 4-D modified Dirac action associated with the full bosonic action. The modified gamma matrices Γ^α are expressible as contractions of the canonical momentum currents with embedding space gamma matrices (this applies also in $D = 6$). Therefore they are proportional to the dimensionless quantity $X\sqrt{g_4}$. Γ^α has dimension $1/L$ so that induced spinors must have dimension $L^{1/2}$. In the usual approach the dimension would be $1/L^{3/2}$.

With these conventions X apparently drop from the equations stating QCC as identity of eigenvalues of fermionic Noether charges and corresponding classical Noether charges in Cartan algebra. This not true. The anti-commutations for Ψ and time component J^0 of the canonical momentum density $J^\alpha = \partial L/\partial(\partial_\alpha\Psi) = \overline{\Psi}\Gamma^\alpha$ involve X and affect the scale of anti-commutation relations and therefore QCC. That the anti-commutations can be indeed realized under these dimensional constraints, requires a proof.

What about the spinors restricted to 2-D string world sheets and corresponding space-time action? Perhaps the most plausible option is that they do not appear at the fundamental level and appear only as the effective action suggested by SH. If this is the case, it is rather easy to guess the form of the bosonic and fermion 2-D effective actions. Their forms could be exactly the same as the form of 4-D actions. The only modification would be in the bosonic case the replacement of $1/R_1^4$ with $1/R_1^2$ to get the dimensions correctly! The bosonic action would dictate the fermionic action by above rules.

The bosonic string world sheet action would differ from the area action. The action density would be $XR_1^2(J \cdot J - 2)\sqrt{g}$ in complete analogy with the 4-D case. Two special cases deserve to be mentioned.

1. This action vanishes for string world sheets with $J \cdot J = 2$. This is the case if one has $J = M(M^4)$ and J is self-dual. This is true if string world sheet is the preferred plane M^2 defining the symplectic structure of M^4 (there is moduli space form them in order to gain Lorentz invariance and giving rise to sectors of WCW).

   Small deformations of this plane would give rise to strings with small string tension and be naturally relating to the small value of the cosmological constant. These strings should accompany long strings mediating gravitational interaction in long length scales. The small action would require large value of $h_{eff}/h = n = h_{gr}$ for the perturbation theory to work.

2. Second special case corresponds to Lagrangian surfaces for which $J(M^4) + J(CP_2)$ induced to string world sheet vanishes. One would have ordinary strings with area action. String tension would be determined by CP_2 size scale. The appearance of also light strings would distinguish between TGD and super string models.

Kähler action can contain also a topological instanton term affecting the field equations only via boundary condition. This term could induce to the string world sheet action a magnetic flux term reducing to a boundary term at the boundaries of string world sheets adding an interaction term to the usual action defined by word-line length. The outcome would be equation of motion for a point-like particle experiencing Kähler force. These topological terms give additional terms to corresponding modified Dirac equations.

It would seem that the new approach to action principle allows a more unified approach to the details of the variational principle in dimension $D = 4$ and allows also to deduce the general form of 6-D and 2-D effective action. It must be however made clear that one could have brane like hierarchy of structures already at fundamental level. Also in this case the new approach applies.

Action principle, quantum classical correspondence, and number theoretical universality

The above observations force to reconsider the interpretation of the action principle. Here the adelic physics based vision can be used as a guideline.

1. It is good to list the geometric parameters and coupling constant like parameters of TGD. CP_2 scale $R(CP_2)$ certainly appears in the theory. The radius of $S^2(M^4)$ makes l_P^2 a natural scale factor of M^4 metric. One can re-scale $J(M^4)$ and the M^4 part of the metric of $T(M^4)$ but not the entire metric.
2. $r = R_1/R(CP_2)$ can be seen as a dimensionless coupling constant like parameter and in principle quantum criticality allows it to have a spectrum values determined by the extension of rationals defining adeles. The QCC condition stating the quantized values of the fermionic Noether charges are equal to their classical counterparts having non-local expressions forces to consider the possibility that the value of R_1 can indeed vary and has value guaranteeing that QCC holds true. Also α_K has spectrum of values: one possible spectrum corresponds to the zeros of Riemann zeta [L17]. Even the number theoretically problematic exponent of action could belong to the extension with a suitable choice of R_1.

   This would allow to speak about the exponent of action and of Kähler function making sense also p-adically in the intersection of real and p-adic WCWs. Both action and its exponent should exist in the extension. This is true if the action is of form $q_1+q_2\pi$, q_i rational numbers. One might hope that a suitable choice of R_1 could make possible to realize QCC and this condition.

QCC and the value spectrum of R_1

Classical field equations do not depend at all on the value on the overall coefficient X of the action in Eq. 4.2.1. Also boundary conditions are independent of the scaling of X. Does this mean that one has projective invariance in the sense that the value of R_1 does not matter at all? No!

1. QCC for the Cartan algebra of fermionic and classical Noether charges gives meaning for the scale R_1. QCC states that the eigenvalues of the Cartan algebra charges are equal to the corresponding values of classical Noether charges. Since the normalization of quantal charges is fixed by the value of $\hbar$, this fixes the normalization of classical charges and thus the parameter R_1. If Ψ is taken dimensionless, the modified Dirac action can be taken to be proportional to factor $1/R_1^3$. Therefore R_1 has physical meaning. The above argument suggests that R_1 is fixed by quantum criticality and characterizes the extension of rationals.
2. Could one require that the values of classical charges belong to the extension of rationals defining the adeles in question? This condition involves in real context integral over 3-surface and is thus a non-local operation. How can one know, which 3-surfaces satisfy the condition? Is the choice of R_1 dictated by this condition so that it depends on the extension of rationals involved and obeys number theoretic coupling constant evolution?

   Note that classical Noether charges serve as WCW coordinates, and the interpretation would be the same as at space-time level: these special 3-surfaces would form a kind of cognitive representation analogous to that formed by the points of space-time surface with coordinates in extension. The quantization of these WCW coordinates would give a cognitive representation!
3. The action would be same for the symmetry related 3-surfaces and one could have WCW wave functions at the orbits of symmetries with coordinates which are conjugate variables for the quantized Noether charges. For the orbits of symmetry groups the allowed points in WCW would correspond to values of group parameters in the extension. Besides isometries and corresponding Kac-Moody algebras supersymplectic symmetry gives rise to this kind of wave functions. In case of four-momentum, the basic number theoretic conditions would be for rest masses.

   Strong form of holography (SH) could be realized by the reduction of both bosonic and fermionic action to an effective action restricted to string world sheets and partonic 2-surfaces. This option looks more attractive from the point of view of SH than fundamental action containing terms located at lower-dimensional surfaces.

Number theoretical universality and action exponential

In adelic physics number theoretical universality plays a key role.

1. Adelic physics leads to the proposal that the action exponentials appearing in the scattering amplitudes disappear. The normalization factor defined by functional integral of action exponential to which also the scattering amplitude is proportional would cancel them as in QFTs [L45].

   This would require that each maximum of Kähler function with respect to variations of 3-surface and having fixed topological scattering diagram defined by light-like partonic orbits and same action defines its own zero energy state as functional integral and these states can be freely superposed. One would not functionally integrate over different topological scattering diagrams: this would allow to interpret topological scattering diagram as a representation of computation.

2. At the level of scattering amplitudes - but not at the level of WCW geometry - the absence of exponents would allow to get rid of the grave difficulty posed by the fact that the exponent of Kähler action belongs to an extension of rationals only when powerful additional conditions are satisfied. The cancellation of exponents of action from scattering amplitudes looks compelling if one requires number theoretical universality since there are no practical means for checking that the exponent of action is in the extension of rationals for an arbitrary preferred extremal. Also the definition of the action as integral is problematic in p-adic context and the only possible means to define it seems to be in terms of algebraic continuations from the real sector.

   One can however argue against number theoretical extremism. Action exponentials are needed for the interpretation of the theory. Maxima of Kähler function, which also correspond to stationary phase correspond to the most probable 3-surfaces. Hence one can argue that the exponents should appear in the scattering amplitudes. Number theoretical cognition theorist could however argue that the points of WCW, which correspond to maxima have WCW coordinates in an extension of rationals and thus define cognitive representation at the level of WCW. Furthermore, one can argue that scattering amplitudes are not the entire physics. Kähler action and its exponent have real meaning independent of scattering amplitudes.

3. On the other hand, if the value of R_1 adjusts to guarantee that the action is of form

$$S = q_1 + iq_2\pi \quad . \tag{4.2.2}$$

   exponents can appear in the amplitudes and the standard approach allowing functional integral giving sum of several exponents makes sense. In this case the scattering amplitudes are proportional to X_i/X, $X = \sum_i X_i$, where X_i denotes action exponent for a particular maximum of action as function of WCW coordinates. Note however that the action itself is not number theoretically universal: only its exponent. This is completely analogous with the fact that angles do not make sense p-adically and one can speak about corresponding phases identified as roots of unity.

Number theoretical universality (NTU) allows two options to consider depending on whether the action exponentials can appear in the scattering amplitudes or not. In WCW geometry action and also its exponent certainly appear.

1. The elimination of exponents of 6-D action from the scattering amplitudes would be a huge simplification and make practical calculations possible. This kind of assumption is in practice made also in standard path integral approach as approximation. ZEO allows this and the interpretation is in terms of the notion of quantum phase of matter: different topologies for partonic 2-surfaces correspond to different phases and the localization to single phase for zero energy states is possible: space-time would be much more classical object than without localization. One must however remain critical: the value of R_1 depending on extension of rationals could allow to achieve QCC conditions.

2. If something is gained, something is also lost. The earlier arguments involving exponent of Kähler function are lost if the exponentials do not appear in scattering amplitudes. In

particular, the estimate for the value of gravitational coupling strength in terms of exponent of Kähler function and α_K (see the last section of [L57]) is lost if exponents do not appear anywhere. One can argue that this argument was actually lost already when the twistor lift was introduced and Planck length was transformed to a fundamental parameter appearing as scaling factor of M^4 Kähler form and metric.

There is a further challenge for the adelic physics. What could fix the value of the fundamental parameter $l_P^2/R^2(CP_2)$ (of order 10−7)? It seems that quantum criticality cannot help here. Both l_P^2 and R^2 appear in the induced metric of space-time surface and number theoretical universality for field equations demands that $l_P^2/R^2(CP_2)$ is a rational number. The p-adic evolution scenario of cosmological constant and empirical input for the cosmological constant gives $l_P^2/R^2(CP_2) = 2^{-12}$ [L24]. Why power of 2 which having unit p-adic norm for all odd primes and why just this power?

To sum up, a more precise adelic formulation of the classical action has allowed to detect a hitherto hidden scaling parameter in the action appearing as an additional coupling parameter depending on the extension of rationals, to understand better the number theoretical role of QCC, and allowed to answer a nagging question about whether to use metric determinant or its absolute value in the action assuming NTU for the exponential of action, and deduce the earlier conjecture for the zeros of zeta.

Answer to an old nagging question

Eq. 5.4.1 can be applied to the situation in which the extremal is known. For CP_2 type extremals volume and Kähler action (-4 times volume) are indeed known. Quite surprisingly, this suggest a solution to an old problem whether one should use $\sqrt{g_4}$ giving imaginary volume element in Minkowskian space-time regions or $\sqrt{|g_4|}$ used usually.

1. The action exponent

   $$e^{\frac{x}{2\alpha_K}} \quad , \quad x = \frac{6Vol(CP_2)}{R_1^4}$$

   is a number in an extension of rationals guaranteed if one has

   $$(1/2)Re(\frac{1}{\alpha_K}) \times x = q_1 \quad , \quad (1/2)Im(\frac{1}{\alpha_K}) \times x = q_2\pi \quad .$$

   .

2. Suppose that the volume integral uses volume element $\sqrt{g_4}$, which is imaginary in Minkowskian space-time regions and real in Euclidian regions. The motivation is that for real α_K the action exponential from Minkowskian space-time regions is phase as QFT picture demands.

   For $1/\alpha_K = is = i/2 + y$, s a complex zero of zeta, the phase of the action exponential coming from Minkowskian regions is proportional to iy and in a good approximation equal to $1/Re(\alpha_K)$. The conditions give $Vol(CP_2)/R_1^4 \propto \pi$ and $y = q$. Note that $Vol(CP_2)$ is proportional to π^2 so that the normalization volume R_1^4 would be proportional to π. Since $R_1^4 = q \times Vol(CP_2)$ is natural normalization factor one would have expected x to be rational. This does not look promising.

   That the zeros of zeta should be complex rationals is totally unexpected but would conform with the number theoretical universality. This would be of course very nice from TGD point of view strongly suggesting that zeros belong to some extension of rationals. I have proposed that the zeros of zeta appear as conformal weights in TGD framework [L17].

3. Suppose that the volume element is given by $\sqrt{|g_4|}$ as was done originally. If α_K is complex, the phase factor is obtained in any case. This option favours $1/\alpha_K = s$, s a complex zero of zeta. Eq. 5.4.1 would predict $Vol(CP_2)/R_1^4 = q$ and $y = q\pi$. These predictions conform with the physical intuition. I have proposed earlier [L17] that the exponents of imaginary parts for the zeros of zeta could correspond to roots of unity. Only the exponents of zeros of zeta would be number theoretically universal and continuable to the p-adic sectors.

To sum up, a more precise adelic formulation of the classical action has allowed to detect a hitherto hidden scaling parameter in the action appearing as an additional coupling parameter depending on the extension of rationals, to understand better the number theoretical role of QCC, and allowed to answer a nagging question about whether to use metric determinant or its absolute value in the action assuming NTU for the exponential of action, and deduce the earlier conjecture for the zeros of zeta.

There is however a further challenge for the adelic physics. What could fix the value of the fundamental parameter $l_P^2/R^2(CP_2)$ (of order 10−7)? It seems that quantum criticality cannot help here. Both l_P^2 and R^2 appear in the induced metric of space-time surface and number theoretical universality for field equations demands that $l_P^2/R^2(CP_2)$ is a rational number. The p-adic evolution scenario of cosmological constant and empirical input for the cosmological constant gives $l_P^2/R^2(CP_2) = 2^{-12}$ [L24]. Why power of 2 which having unit p-adic norm for all odd primes and why just this power?

4.2.2 The problem with cosmological constant

Second (unpleasant) observation was that the previous proposal for the twistor lift of Kähler action has an ad hoc feature.

Can the original proposal for the twistor lift of Kähler action be correct?

Consider first the unpleasant observation about cosmological constant.

1. α_K is also assumed to be complex and the conjecture [L17] has been that its values correspond to zeros of Riemann zeta. In the earlier proposal for twistor lift cosmological constant and α_K are assumed to obey independent p-adic evolutions, and cosmological constant was assumed to be real and to behave like $1/p$ as function of p-adic prime in p-adic length scale evolution so that its extreme smallness in cosmological scales could be understood [L22, L24].

   The motivation for the proposal was the decomposition $T(H) = T(M^4) \times T(CP_2)$ of the twistor space of H. It was argued that this allows to decompose the Kähler action of $T(H)$ to a sum of two parts with *different* values of α_K. For M^4 part the value of α_K, call it $\alpha_K(M^4)$, would be enormous and the resulting volume term in the dimensionally reduced 6-D Kähler action would have cosmological constant $\hbar/l_D^4$ as its coefficient: l_D would be of the order of the size about 10^{-4} meters of a large neuron in cosmological length scales.

2. If the value of $\alpha_K(M^4)$ is real or has different phase than $1/\alpha_K$, whose spectrum is proposed to correspond to zeros of zeta [L17], the action is complex, and one has separate field equations for real and imaginary part of action. The extremals would be minimal surface extremals of Kähler action. That all known extremals of Kähler action have this property was seen as a support for the hypothesis.

   The physically problematic aspect is that Kähler action and volume term effectively decouple. This would make sense asymptotically but looks strange as a general property [**?**] On the other hand, the independence of the extremals on coupling constants is a highly desirable outcome from the point of view of number theoretical universality.

3. The assumption about different Kähler coupling strengths admittedly looks somewhat ad hoc. If one assumes that also M^4 possesses Kähler form $J(M^4)$ [L47], and induced Kähler form corresponds to the sum $J(M^4) + J(M^2)$, universal value of α_K is the natural option. This assumption however allowed to understand the smallness of cosmological term in 4-D action and also the p-adic coupling constant evolution for the cosmological constant.

4. Also boundary conditions are problematic for this option. It would be highly desirable to have flow of classical Noether charges between Euclidian and Minkowskian space-time regions as a correlate for classical interactions between physical objects having Euclidian regions as space-time correlates (analogous to lines of scattering diagrams). The conditions stating the conservation of sums of complex Kähler and volume charges from Minkowskian and Euclidian regions however give 2+2 conditions if the phases of Kähler action and volume term are different and the metric determinant $\sqrt{g_4}$ is imaginary for Minkowskian regions. It is easy to see that Kähler and volume charges are conserved separately and that there is no charge transfer between Euclidian and Minkowskian regions. The alternative $\sqrt{|g_4|}$ allows the flow of real and imaginary charges between the two regions. One can however insist that the existence of two separate conserved energies should have been discovered long time ago.

What if one gives up the assumption $\alpha_K(M^4) \neq \alpha_K(CP_2)$?

1. The volume term would be also proportional to $1/\alpha_K$ so that the phases of both Kähler action and volume term would be identical. The pleasant surprise is that coupling constants disappear from the field equations altogether! It is not necessary to postulate minimal surface property of the preferred extremals anymore to guarantee number theoretical universality.

   Minimal surface property could be however asymptotic so that there would be no exchange of conserved quantities between these degrees of freedom. This would conform with the idea that incoming and outgoing particles are free and thus minimal surfaces as 4-D generalization of a geodesic line resulting when 4-D generalization of Abelian Maxwell force vanishes. Causal diamond (CD) would represent a region with the property that the extremals approach minimal surfaces at its boundary. One can loosely say that interactions are coupled on and off near the opposite boundaries of CD: CD corresponds to scattering volume.

   The vertices of topological diagrams defined by as 2-D intersections of the ends of orbits of partonic 2-surfaces - analogous to vertices of Feynman diagrams - would be also accompanied by transient regions, where there the motion of 3-surface is not geodesic. The results are extremely nice from the point of view of number theoretical universality.

2. Also in this case the charge transfer between Euclidian and Minkowskian regions is impossible if $\sqrt{g_4}$ defines volume element (imaginary in Minkowskian regions). $\sqrt{|g_4|}$ this is not the case. As found, also NTU favors this option.

3. The above result is extremely nice. What makes the shower cold is that one ends up with problems with cosmological constant since Kähler and volume terms in the action are of same order of magnitude. Also the proposed p-adic evolution scenario for the cosmological constant is lost. The only cure that I can imagine is that the entire 4-D action has interpretation as a cosmological term, and that a cancellation between Kähler action and volume term take place giving rise to a very small effective value of cosmological constant.

Can one understand the p-adic evolution of cosmological constant?

The above findings lead to a problem with cosmological constant.

1. If the cosmological constant corresponds to the volume term in the dimensionally reduced 6-D Kähler action with scaling factor $X = 1/2\alpha_K R_1^2 S_0$, one has from Eq. 4.2.1

$$\rho_{vac} = \frac{1}{l_D^4} = \frac{2}{\alpha_K R_1^4}\frac{Area(S^2(X^4))}{S_0} = \frac{\Lambda}{8\pi l_P^2} \quad . \tag{4.2.3}$$

   Here l_D corresponds to a length scale which is roughly the size 10^{-4} meters of large neuron for cosmological constant in cosmic scales. Also Kähler action would be extremely small. It would however seem that the ratio of these actions should be extremely small. The simplest solution corresponds to $\frac{Area(S^2(X^4))}{S_0} = 1$.

2. The Kähler action for CP_2 type extremal with light-like geodesic as M^4 projection the action would be

$$S = -3\frac{Vol(CP_2)}{l_D^4} \quad .$$

   The action has totally different order of magnitude than assumed earlier if R_1 corresponds to the value of cosmological constant. If one assumes $R_1 = R(CP_2)$, cosmological constant is enormous. Something seems to go wrong.

How could one overcome this problem?

1. Could l_D be small and imply large cosmological constant? Could the parameter $X = \frac{Area(S^2(X^4)}{S_0}$ be small and increase the effective size of l_D? Could the time-like signature for $S^2(M^4)$ allow this by reducing the value of $Area(S^2(X^4)$?

One can study the embedding of $S^2(X^4)$ to $S^2(M^4)$ and $S^2(CP_2)$ characterized by winding numbers n_1 and n_2. One can choose S_0 to be the area for the embedding with $n_1 = n_2 = 1$. This gives $\frac{Area(S^2(X^4)}{S_0} = (n_1X^2 - n_2)(X^2 - 1)$, $X^2 = (R^2(CP_2)/l_P^2)$ for time-like signature for $S^2(M^4)$. The condition $\frac{Area(S^2(X^4)}{S_0} = 1/p$ would give p-adic length scales but could be satisfied for finite number of primes p only. Second problem is that this would *not* affect the ratio of Kähler and volume contributions to the action.

2. Could effective cosmological constant correspond to the entire action so that Kähler would cancel the real cosmological term in cosmological scales?

   Could $J \cdot J - 2$ should become small in Minkowskian regions and be necessarily large in Euclidian regions? The positive Kähler electric contribution to the action should sum up to almost zero with the negative magnetic contribution and cosmological term. This cancellation should take place in cosmic scales at least and require long range induced Kähler electric fields. They are assumed to be present in the model for large voids. If M^4 Kähler form is present as CP breaking and some other arguments suggest [L47] [L24], it could give a large Kähler electric contribution in long scales if CP_2 contribution becomes small as one might expect.

   The values of 6-D Kähler action should have tendency to concentrate around values inversely proportional to prime p near power of 2 (also other small primes can be considered). The values of Kähler action for the maxima of Kähler function could have this property. This conjecture was made earlier in an attempt to understand gravitational constant in terms of p-adic length scale hypothesis and the exponent of Kähler action for CP_2 type extremals (see the last section of [L57]).

3. This interpretation would mean that for strings like objects having both vanishing induced M^4 and CP_2 parts of induced Kähler fields the action would be large and coming from cosmological constant in CP_2 scale, and one could at least formally say that the situation is perturbative. Strings could however carry non-vanishing and large M^4 parts of Kähler electric fields and the action could be small in this case.

4. I must be added that the interpretation of cosmological constant has varied during years. For the 4-D Kähler action the proposal was that cosmological constant corresponds to the magnetic part of Kähler action with magnetic tension responsible for the negative pressure. The twistor lift in turn led to ask whether Kähler action and volume term could provide alternative, dual ways to understand cosmological constant. For the recent option the small effective cosmological constant results from the cancellation of Kähler action and volume term.

The cautious conclusion would be following. If the 6-D Kähler action contains only single α_K, the cosmological constant is very large at short scales and for Euclidian space-time regions. The cancellation of Minkowskian Kähler electric contribution and Kähler magnetic action in 6-D sense however makes the effective value of cosmological very small. The solution of the problem of cosmological constant would be dynamical. The previous option for which Kähler action decomposes to M^4 and CP_2 parts with different values of $\alpha_K(M^4)$ and $\alpha_K(CP_2) \leq\leq \alpha_K(M^4)$ cannot be however excluded.

4.3 Twistor lift of TGD, hierarchy of Planck constant, quantum criticality, and p-adic length scale hypothesis

Kähler action is characterized by enormous vacuum degeneracy: any four-surface, whose CP_2 projection is Lagrangian sub-manifold of CP_2 having therefore vanishing induced Kähker form, defines a vacuum extremal. The perturbation theory around canonically imbedded M^4 in $M^4 \times CP_2$ defined in terms of path integral fails completely as also canonical quantization. This led to the construction of quantum theory in "world of classical worlds" (WCW) and to identification of quantum theory as classical physics for the spinor fields of WCW: WCW spinors correspond to fermionic Fock states. The outcome is 4-D spin glass degeneracy realizing non-determinism at classical space-time level [K45, K24, K106, K80].

The twistor lift of TGD is based on unique properties of the twistor spaces of M^4 and CP_2. Note that M^4 allows two notions of twistor space. The first one involves conformal compactification allowing only conformal equivalence class of metrics. Second one is equal to Cartesian product $M^4 \times S^2$ [B64] (see http://tinyurl.com/yb4bt74l). CP_2 has flag manifold $SU(3)/U(1) \times U(1)$ as twistor space having interpretation as the space for the choices for quantization axis of color hypercharge and isospin. Both these spaces Kähler structure (strictly speaking E^4 and S^4 allow it but the notion generalizes to M^4) and there are no others. Therefore TGD is unique both from standard model symmetries and twistorial considerations.

The existence of Kähler structure is a unique hint for how to proceed in the twistorial formulation of classical TGD. One must lift Kähler action to that in the twistor space of space-time surface having also S^2 as a fiber and identify the preferred extremals of this 6-D Kähler action as those of dimensionally reduced Kähler action, which is 4-D Kähler action plus volume term identifiable in terms of cosmological constant. As found, there are two options to consider.

1. Option I: The values of $\alpha_K(M^4)$ and $\alpha_K(CP_2)$ are widely different with $\alpha_K(M^4)$ being extremely large so that M^4 part of the 6-D Kähler action gives in dimensional reduction extremely small cosmological term. Allowing Kähler coupling strength $\alpha_K(CP_2)$ to correspond to zeros of zeta implies that for complex zeros the preferred extremals for $\alpha_K(M^4)$ having different phase are mimimal surface extremals of Kähler action so that the values of coupling constants do not matter and extremals depend on couplings only through the boundary conditions stating the vanishing of certain super-symplectic conserved charges. In this case the cosmological constant would correspond to running $\alpha_K(M^4)$ and would behave like $1/p$, p p-adic prime. This was the original proposal.

2. Option II: $\alpha_K(M^4) = \alpha_K(CP_2)$ is satisfied. A small effective value of cosmological constant is obtained if the Kähler action and volume term tend to cancel each other. In this case minimal surface extremals of Kähler action correspond naturally to asymptotic dynamics near the boundaries of CDs, where the analog of free geodesic motion as minimal surfaces is expected. In this case effective cosmological constant would correspond to the *entire action*: volume term and Kähler action receiving also M^4 contribution would cancel almost completely in cosmic scales.

   One can in fact argue that one cannot distinguish between Kähler and volume contributions to the action so that Option II remains the only possible one. Option I also breaks the symmetry between Kähler forms of M^4 and CP_2. It is natural that the induced Kähler form is the sum of both and appears in the Kähler action: hence $\alpha_K(M^4) = \alpha_K(CP_2)$.

Option I might be argued to be adhoc but at this moment it is not yet wise to select between these two options. The most conservative assumption is that the twistorial approach is only an alternative for the space-time formulation: in this formulation preferred extremal property might reduce to twistor space property.

Kähler action gives as fundamental constants the radius $R \simeq 2^{12} l_P$ of CP_2 serving as the TGD counterpart of the unification scale of GUTs and Kähler coupling strength α_K in terms of which gauge coupling strengths can be expressed. Twistor lift gives 2 additional dimensional constants. The radius of S^2 fiber of M^4 twistor space $M^4 \times S^2$ is essentially Planck length $l_P = \sqrt{G/\hbar}$, and the cosmological constant $\Lambda = 8\pi G\rho_{vac}$ defining vacuum energy density is dynamical in the sense that it allows p-adic coupling constant evolution as does also α_K.

For both Option I and II one can imagine two options for the p-adic coupling constant evolution of cosmological constant.

1. $\rho_{vac} = k_1 \times \hbar/L_p^4$, where $p \simeq 2^k$ characterizes a given level in the p-adic length scale hierarchy for space-time sheets. Here one can in principle allow $k_1 \neq 1$.

2. $\Lambda/8\pi = k_2/L_\Lambda^2 \propto \frac{1}{p_\Lambda^2}$. Also k_2 could differ from unity. Number theoretical universality suggests $k_1 = k_2 = 1$. The that here secondary p-adic length scale is assumed.

The first option seems more natural physically. During very early cosmology $\Lambda R^2/8\pi$ approaches l_P^2/R^2 for the first option, where $R \simeq 2^{12} l_P$ is the size scale of CP_2 so that one has $\Lambda R^2/8\pi \simeq 2^{-24} \simeq 6 \times 10^{-8}$ at this limit. Therefore perturbation theory would fail for Option I also in early cosmology near vacuum extremals. In the recent cosmology Λ is extremely small. Note that vacuum energy density would be always smaller than $\hbar/R^4$ and thus by a factor $(l_P/R)^4 \simeq 2^{-48} \simeq 3.6 \times 10^{-15}$ lower than in GRT based cosmology.

It it is good go recall that the earlier identification of the cosmological constant was in terms of the effective description for the magnetic energy density of the magnetic flux tubes. Magnetic tension would give rise to effective negative pressure. For Option II the cosmological constant would correspond to the entire action with magnetic and volume contributions slightly larger than Kähler electric contribution. For Option I it wold correspond to the volume term.

4.3.1 Twistor lift brings volume term back

Concerning volume term the situation changed as I introduced twistor lift of TGD. One could say that twistor lift forces cosmological constant. As already described, there are two options: Option I and Option II. The following arguments developed for Option I apply with small modifications also to Option II. The only difference is that the volume term has complex phase for complex α_K [L17] and effective cosmological constant follows from the compensation of Kähler and volume contributions.

1. The twistor lift of Kähler action is 6-D Kähler action for the twistor space $T(X^4)$ of space-time surface X^4. The analog of twistor structure would be induced from the product $T(M^4) \times T(CP_2)$, of twistor spaces $T(M^4) = M^4 \times S^2$ of M^4 [B64] and $T(CP_2) = SU(3)/U(1) \times U(1)$ of CP_2 having Kähler structure so that the induction of Kähler structure to $T(X^4)$ makes sense. Besides M^4 and CP_2 only the spaces E^4 and the S^4, which are variants of M^4 have twistor space with Kähler structure or analog of it. The induction conditions would imply dimensional reduction so that the 6-D Kähler action for the twistor lift would reduce to 4-D Kähler action plus volume term identifiable in terms of cosmological constant Λ.
2. 4-D Kähler action has Kähler coupling strength α_K as coupling parameter and volume term has coefficient $1/L^4$ identifiable in terms of cosmological constant

$$\frac{1}{L^4} \equiv \frac{\Lambda}{8\pi l_P^2} \quad .$$

$l_P = \sqrt{G/\hbar}$ would correspond to the radius of twistor sphere for M^4 and thus becomes fundamental length scale of twistorially lifted TGD besides radius of CP_2. Note that the radius of twistor sphere of CP_2 is naturally CP_2 radius.

L is in the role of coupling constant and expected to obey discrete p-adic coupling constant evolution $L \propto \sqrt{p}$, prime or prime near power of two if p-adic length scale hypothesis is accepted. In the recent cosmology L could correspond to the p-adic length scale $L(175) \simeq 40$ μm, the size of large neuron.

L $\simeq 40\mu$m corresponds to the energy scale $E = 1/L \simeq .031$ eV, which is thermal energy at temperature of 310 K (40 C) - the physiological temperature. A deep connection with quantum biology is suggestive. Also the energy scale defined by cell membrane potential is in this energy scale. This energy scale about 10 times smaller than the mass scale of neutrinos. Also $L_\Lambda = \sqrt{8\pi/\Lambda}$ would satisfy p-adic coupling constant evolution as already discussed. Now the p-adic length scale would be secondary p-adic length scale $L_\Lambda = L(2,p) = \sqrt{p} \times (R/l_P)$, l_P Planck length. p-Adic length scale hypothesis demands that R/l_P - the ratio for the radii of CP_2 and twistor sphere is power of 2. p-Adic mass calculations indeed allow this ratio can be indeed chosen to be equal to $R/l_P = 2^{12}$.

4.3.2 ZEO and twistor lift

The volume term, which I gave up 38 years ago, has creeped back to the theory! The infinite value of volume for space-time surfaces of infinite duration? This would not make the notion of vacuum functional poorly defined. Should one forget twistor lift because of this? No! ZEO saves the situation.

In ZEO given CD defines a sub-WCW consisting of space-time surfaces inside CD. This implies that the volumes for the M^4 projections of allowed space-time surfaces are smaller than CD volume having the order of magnitude $L^4(CD)$, $L(CD)$ is the temporal distance between the tips of CD (one has $c = 1$). I have also proposed that $L(CD)$ is quantized in multiples of integers, primes or primes near power of two so that the identification might make sense. $L(CD) = L$ is not possible due to the small value 40 μm of L but $L(CD) = L_\Lambda$ could make sense.

Stationary phase condition and ZEO

The preferred extremal property realizing SH poses extremely strong constraints on the value of total action and it should force the phase defined by action to be stationary so that interference effects would be practically absent. This argument assumes that the action exponentials indeed appear in the scattering amplitudes defined by the WCW spinor fields in ZEO. NTU however forces to challenge this assumption unless one assumes that action is quantized as $q_1 + iq_2\pi$: this might be achieved by the quantization of the overall scale factor X of the action. The construction of twistor scattering amplitude suggests that the cancellation of action exponentials might be indeed achieved. If the exponents are present, the question is how the stationarity of phase could be achieved.

1. The most general possibility is that the phase of the vacuum functional can be large but is localized around very narrow range of values. The imaginary part of the action S_{Im} for preferred extremals should be around values $S_{Im} = A_0 + n2\pi$. Standard Bohr orbitology indeed assumes the quantization of action in this manner. One could also argue that just the absence of destructive interference demands Bohr quantization of the action in the vacuum functional. Whether preferred extremal property indeed gives rise to this kind of Bohr quantization, is an open problem. The real exponent of the vacuum functional should in turn be large enough and positive values are favored. They are however bounded in ZEO because of the finite size of CDs.
2. To proceed further one must say something about the value spectrum of α_K. In the most general situation α_K is complex number: the proposal of [L17] is that the discrete p-adic coupling constant evolution for $1/\alpha_K$ corresponds to a complex zero $s = 1/2 + iy$ of Riemann zeta: also the trivial real zeros can be considered. For large values of y the imaginary part of y would determine $1/\alpha_K$ and $Re(s) = 1/2$ would be responsible for complex value of α_K. This makes sense since quantum TGD can be regarded formally as a complex square root of thermodynamics.
3. Denote by $S = S_{Re} + iS_{Im}$ the exponent of vacuum functional. For complex values of $1/\alpha_K$ S_{Im} and S_{Re} receive a contribution from both Euclidian and Minkowskian regions and a contribution also from the Minkowskian regions. For S_{Im} the contributions should obey the condition

$$S_{Im} = S_{Im}(M) + S_{Im}(E) \simeq A_0 + n2\pi \quad (4.3.1)$$

   to achieve constructive interference.

   For real parts the condition $S_{Re} = S_{Re}(M) + S_{Re}(E)$ must be small if negative. Large positive values of S_{Re} are favored. S_{Re} automatically selects the configurations, which contribute most and among these configurations the phase $exp(iS_{Im})$ must be stationary. The conditions for S_{Im} relate the values of action in the Euclidian and Minkowskian regions. If α_K is real, one has $S_{Im}(M) \simeq A_0 + n2\pi$ and $S_{Re}(E)$ small if negative and Euclidian and Minkowskian regions effectively decouple in the conditions. It seems that complex values of α_K are indeed needed.
4. $S_{Re}(E) = S_{Re}(M) + S_{Re}(E)$ receives a positive contribution from Euclidian regions. Minkowskian regions a contributions for complex value α_K. Both positive and negative contributions are present and the character of these contributions depends on sign of the imaginary part of α_K. Depending on the sign factor ± 1 of $Im(1/\alpha_K)$ Minkowskian regions give negative (positive) contribution from the space-time regions dominated by Kähler electric fields and positive (negative) contribution from the volume term and the regions dominated by Kähler magnetic field.

   The option "+" for which Kähler magnetic action and volume term give positive contribution to $S_{Re}(M)$ looks physically attractive. "+" option would have no problems in ZEO since the contribution to S_{Re} would be automatically positive but bounded by the finite size of CD: this would give a deep reason for the notion of CD (also the realization of super-symplectic symmetries gives it). For "-" option Minkowskian regions containing Kähler electric fields would be essential in order to obtain $S_{Re} > 0$: Kähler magnetic fields would not be favored and the unavoidable volume term would give wrong sign contribution to $S_{Re} > 0$.

The condition $S_{Im} \leq \pi/2$ is not realistic

One can look what the mere volume term contributes to S_{Im} assuming $S_{Im} \leq \pi/2$. Volume term dominates for near to vacuum extremals with a small Kähler action: in particular, for string like objects $X^2 \times S^2$, S^2 a homologically trivial geodesic sphere with vanishing induced Kähler form. It turns out that these conditions are not physically plausible and that $S_{Im} \simeq A_0 + n2\pi$ is the only realistic option.

1. Cosmological constant (parametrizable using the scale L) together with the finite size of CD gives a very stringent upper bound for the volume term of the action: $A = vol(X^4)/L^4$. The rough estimate is that for the largest CDs involved the volume action is not much larger than $L^4\pi/2$ in the recent cosmology. In the recent cosmology L would be only about 40 μm so that the bound is extremely strong! and suggests that $S_{Im} < \pi/2$ is not a realistic condition.
2. $L(CD) = L$ is certainly excluded. Can one have $L(CD) = L_\Lambda$? How can one achieve space-time volume not much larger that L^4 for space-time surfaces with duration $L(CD)$? Could magnetic flux tubes help! For the simplest string like objects $X^2 \times Y^2$, where $X^2 \subset M^4$ is minimal surface and Y^2 a 2-D surface (complex sub-manifold of CP_2) the volume action is essentially

$$Action = \frac{V}{l_P^2 L_\Lambda^2} = \frac{Area(X^2)}{L_\Lambda^2} \times \frac{Area(Y^2)}{l_P^2} \quad . \tag{4.3.2}$$

The conservative condition for the absence of destructive interference is roughly $Action < \pi/2$.

3. To get a more concrete idea about the situation one can use the parameterization

$$Area(string) = L(CD) \times L(string) \quad , \quad Area(Y^2) = x \times 4\pi R^2 \quad . \tag{4.3.3}$$

x is a numerical parameter, which can be quite large for deformations of cosmic strings with thick transversal M^4 projection. The condition for the absence of destructive interference is roughly

$$\frac{L(CD) \times L(string))}{L_\Lambda^2} \times x \times \frac{4\pi R^2}{l_P^2} < \frac{\pi}{2} \quad . \tag{4.3.4}$$

For $L(string) \ll L(CD)$ one can have space-time surfaces of temporal duration $L(CD) = L_\Lambda$. For these the condition reduces to

$$\begin{aligned} y \times x &< \pi \frac{l_P^2}{4\pi R^2} = 2^{-13}\pi \quad , \\ y &\equiv \frac{L(string)}{L_\Lambda} \quad . \end{aligned} \tag{4.3.5}$$

For deformations the transversal area of string like object can be also chosen to be considerably larger than the area of geodesic sphere. For flux tubes of length of order 1 AU the one have $y \sim 10^{-16}$. This would require $x \leq 10^{13}$. This would correspond to a radius $L(Y^2)$ about $10^6 R$ much smaller than required.

For $L(string) \sim L$ this would give $y \sim 10^{-31}$ giving $x \leq 10^{28}$ $L(Y^2) \leq 10^{14}R$, which corresponds to elementary particle scale. Still this fails to fit with intuitive expectations, which are of course inspired by the standard positive energy ontology.

4. One could try to invent mechanisms making volume term small. The required reduction would be enormous. This does look sensible. One can have vacuum extremals of Kähler action for which CP_2 projection is a geodesic line: $\Phi = \omega t$. The time component $g_{tt} = 1 - R^2\omega^2$ of the flat metric can be arbitrarily small so that the volume proportional to $\sqrt{gtt}$ can be arbitrarily small. One expects that this happens in early cosmology but as a general mechanism this is not plausible. Also very rapidly rotating string like objects with small area of string world sheet are in principle possible but do not represent a realistic option.

The cautious conclusion is that Bohr quantization $S_{Im} \simeq A_0 + n2\pi$ is the only sensible option. The hypothesis that the coupling constant evolution for $1/\alpha_K$ is given in terms of zeros of Riemann zeta seems to be consistent with this picture and correlates the values of actions in Minkowskian and Euclidian regions.

4.3.3 Hierarchy of Planck constants

One motivation (besides motivations from bio-electromagnetism and Nottale's work [E1]) for the hierarchy of Planck constants $h_{eff} = n \times h$ identified as gravitational Planck constants $\hbar_{gr} = GMm/v_0$ at the magnetic flux tubes mediating the gravitational interaction was that it effectively replaces the large coupling parameter GMm with dimensionless coupling $v_0/c < 1$. This assumes quantum coherence in even astrophysical length and time scales. For gauge interaction corresponding to gauge coupling g one $\hbar_g = Q_1 Q_2 \alpha/v_0$. Also Kähler coupling strength α_K to α_K/n and makes perturbation theory converging for large enough value of n.

The geometric interpretation for $h_{eff} = n \times h$ emerges if one asks how to make the action large for very large value of coupling parameter to guarantee convergence of functional integral.

1. The answer is simple: space-time surfaces are replaced with n-fold coverings of a space-space giving n-fold action and effectively scaling h to $h_{eff} = n \times h$ so that coupling strength scale down by $1/n$. The coverings would be singular in the sense that at the 3-D ends of space-time surface at the boundaries of causal diamond (CD) the sheets co-incide.
2. The branches of the space-time surface would be related by discrete symmetries. The symmetry group could be Galois group in number theoretic vision about finite measurement resolution realized in terms of what I call monadic or adelic geometries [L27] [L26].

   On the other hand, the twistor lift suggests that covering could be induced by the covering of the fiber $S^2(X^6)$ by the spheres $S^2(M^4 \times S^2)$ and the twistor space $S^2(SU(3)/U(1) \times U(1))$ defining fibers of twistor spaces of M^4 and CP_2. There would be gauge transformations transforming the light-like parton orbits to each other and the discrete set would consists of gauge equivalence classes. These two identifications for the symmetries could be equivalent.

$h_{eff} = h_{gr} = n \times h$ would make perturbation theory possible for the space-time surfaces near vacuum extremals. For far from vacuum extremals Kähler action dominates and one would have $h_{eff} = h_{g_K} = n \times h$. This picture would conform with the idea that gravitational interactions are mediated by massless extremals (MEs) topologically condensed at magnetic flux tubes obtained as deformations of string like objects $X^2 \times S^2_I$, S^2_I a homologically trivial geodesic sphere of CP_2. The other interactions could be mediated in the similar manner. The flux tubes would be deformations of $X^2 \times S^2_{II}$, S^2_{II} a homologically non-trivial sphere so that the flux tubes would carry monopole flux.

The enormously small value of cosmological constant would require large value of $h_{eff}/h = n$ explaining the huge value of h_{gr} whereas for other interactions the value of n would be much smaller. Since only the size of the action matters, this is true for both Option I and Option II. One can consider also variants of this working hypothesis. For instance, all long range interactions mediated by massless quanta could correspond to extremals for which cosmological constant is small.

What smallness requires depends on option. For Option I the reason is that very long homologically non-trivial magnetic flux tubes tend to have large energy (the energy goes as $1/S$) so that homologically trivial flux tubes having only vacuum energy are favored. For Option II the cancellation of Kähler action and volume term is necessary. The compensating Kähler electric action could come from the M^4 Kähler from $J(M^4)$. These flux tubes could be also homologically non-trivial

Quantum criticality would suggest that both homologically trivial and non-trivial phases are important. In TGD inspired quantum biology [K50] I have considered the possibility that structures with size scaled by $h_{eff}/h = n$ can transform to structures with $n = 1$ but p-adic length scale scaled up by n. Here n would be power of two by p-adic length scale hypothesis.

This would have interpretation in terms of quantum criticality. Homologically non-trivial string like objects with given string tension determined by Kähler action would be transformed to homologically trivial string like objects with the same string tension but determined by the cosmological constant term. This would give a condition on the value of the cosmological constant and thickness of flux tubes to be discussed later.

4.3.4 Magnetic flux tubes as mediators of interactions

The gravitational Planck constant $\hbar_{gr} = GMm/v_0$ [K85, K70, K71, ?] introduced originally by Nottale [E1] depends on the large central mass M and small mass m. This makes sense only if h_{gr} characterizes a magnetic flux tube connecting the two masses. Similar conclusion holds true for h_g. This leads to a picture in which mass M has involves a collection of radial flux tubes emanating radially from it. This assumption makes sense in many-sheeted space-time since the fluxes can go to the another space-time sheets through wormhole contacts associated also with elementary particles. For single-sheeted space-time one should have genuine magnetic charges.

This picture encourages a strongly simplified vision about how holography is realized. From center mass flux tubes emanate and in given size scale of the space-time sheet from by the flux tubes having say spherical boundary, the boundary is decomposed of pixels representing finite number of qubits. Each pixel receives one flux tube.

Vacuum energy for Options I and II

For Option I and magnetic flux tubes with vanishing Kähler form carry mere vacuum energy and are candidates for the mediators of long range interactions including gravitation. The homologically trivial flux tubes carry vacuum energy, which by flux conservation is proportional to $1/S$, where S is surface area. Long flux tubes are necessarily thick.

For Option II the thin magnetic flux tubes with vanishing induced Kähler form have very large tension and could be perturbative so that there would be no need for large values of $h_{eff}/h = n$. These flux tubes are expected to be short. The string world sheets mediating gravitational interaction should be long and have small string tension. They would naturally carry non-vanishing Kähler electric field in the direction of string (and flux tube).

1. Gravitational action (interaction energy from $J(M^4)$) and volume action (energy) would compensate to give a small cosmological constant forcing $h_{eff}/h = n$ hierarchy describing dark matter. Thus $J(M^4)$ crucial for understanding CP breaking and matter antimatter asymetry would be also crucial for the smallness of cosmological constant. This option looks physically rather attractive.
2. For flux tubes with vanishing induced $J(CP_2)$ the condition for cancellation would be $J{\cdot}J{-}2 \simeq 0$. The compensating Kähler field would be electric and would naturally due to $J(M^4)$ and also responsible for the gravitational field along flux tube at QFT limit. Compensation of actions giving a small and scale dependent cosmological constant requiring large $h_{eff}/h = n = h_{gr}/h$ is possible.
3. For flux tubes with Kähler magnetic tube carrying magnetic monopole flux the cancellation condition would $J(M^4) \cdot J(M^4) - 2 - J(CP_2) \cdot J(CP_2) \simeq 0$. The thickening of flux tubes weakening the value of $J(CP_2)$ behaving from flux conservation like $J(CP_2) \propto 1/S$, S the cross sectional area of the flux tube, should make approximate cancellation possible. Elementary particles would represent an example of structures formed by closed monopole flux tubes assignable with a pair of space-time sheets. Homologically non-trivial magnetic flux tubes with small string tension could explain the mysterious cosmic magnetic fields: homological non-triviality implies that no current is needed to create the fields.

Magnetic flux tubes as carriers of magnetic energy

The holographic picture leads to a picture about vacuum energy. The following arguments developed originally for Option I should apply to both options since it is enough that magnetic flux tubes have only low vacuum energy density. Possible delicacies relate to the fact that small Kähler action ($E^2 - B^2$) does not necessarily mean small Kähler energy. For Option II this situation is however not encountered.

1. Vacuum energy can be expressed as a sum of energies assignable to the flux tubes. Same applies to Kähler interaction energy. The contribution of individual flux tube is proportional to its length given by radius r of the large sphere considered. The total vacuum energy must be proportional to r^3 so that the number of flux tubes must be proportional to r^2. This implies that single flux tube correponds to constant area ΔS of the boundary sphere for given value

of cosmological constant. The natural guess is that ΔS is of the same order of magnitude as the area defined by the length scale defined L by the vacuum energy density $\rho_{vac} = \Lambda/8\pi G$ allowing parameterization $\rho_{vac} = k_1\hbar/L^4$.

2. In the recent cosmology one has $\hbar/L(now) \simeq .029$ eV, which equals roughly to $M/10$, where $M = \sum m(\nu_i) \simeq .032 \pm 0.081$ eV is the sum of the three neutrino masses. L is given as a geometric mean

$$L = \sqrt{L_\Lambda l_P} \simeq .42 \times 10^{-4}$$

meters of length scales $l_P = \sqrt{G/\hbar}$ and $L_\Lambda = (8\pi/\Lambda)^{1/2}$. $L(now)$ corresponds to the size scale of large neuron. This is perhaps not an accident.

The area of pixel must be of order $L^2(now)$ suggesting strongly a p-adic length scale assignable with neuron: maybe neuronal system wold realize holography. $L(151) = 10$ nm (cell length scale thickness) and $L(k) \propto \sqrt{p} \simeq 2^{k/2}$ gives the estimate $p \simeq 2^k$, $k = 175$: the p-adic length scale is 4 per cent smaller than $L(now)$.

3. The pixel area would be by a factor $L^2(now)/l_P^2$ larger than Planck length squared usually assumed to define the pixel size but would conform with the p-adic variant of Hawking-Bekenstein law in which p-adic length scale replaces Planck length [K66].

The value of the vacuum energy density for a given flux tube is proportional to the value of $h_{eff}/h = n$ by the multi-sheeted covering property. Vacuum energy cannot however depend on n. There are two ways to achieve this: local and global.

1. For the local option the energy of each flux tube would remain invariant under $h \rightarrow n \times h$ as would also the number N of flux tubes. This requires that the cross section S of the radial gravitational flux tube to which energy is proportional, scales down as S/n. This looks strange.

2. For the global option flux tubes are not changed but the number N of the radial flux tubes scales down as $N \propto 1/n$: one has $Nn = constant$. In the situation in which Kähler magnetic energy dominant local option demands $S \propto n$ and global option $N \propto 1/n$. Nn constant conditions brings in mind something analogous to Uncertainty Principle. The resolutions characterized by N and n are associated with complementary variables.

The global option applies to both homologically trivial and non-trivial options and is more promising.

Could the value of endogenous dark magnetic field relate to cosmological constant?

TGD development of inspired model for quantum biology was initiated by the observation [J2] that ELF em fields have non-trivial effects on the brain physiology and behavior of vertebrates [K75, K78]. Since the energies of ELF photons (with frequencies in EEG range) are many orders of magnitude below thermal energy, the proposal was that one has dark photons having $h_{eff}/h = n$ increasing the value of the energy $E = h_{eff}f$ of ELF photons above thermal energy, possibly even to the energies of bio-photons in visible and UV range identified as resulting in a phase transition reducing h_{eff} to its value for visible matter.

The effects appear at multiples of cyclotron frequencies of biologically important ions in endogenous ("dark") magnetic field of $B_{end} \simeq .2$ Gauss. This corresponds to magnetic length $1/\sqrt{eB}$ not far from the size of large neuron. Could this field strength correspond to the Kähler magnetic field assignable to the flux tubes carrying monopole magnetic field, whose strength is determined by the value of cosmological constant? This would give a direct connection between cosmology and biology!

1. In recent cosmology the value of B_K (more precisely, $g_K B_K$ using ordinary conventions) at criticality would be

$$B_K = \frac{\Phi_0}{4\pi}\frac{1}{L^2(175)} \quad .$$

B_K corresponds to the U(1) magnetic field in standard model and is therefore as such not the ordinary magnetic field. For S^2_{II} Kähler magnetic field is non-vanishing. If Z^0 field vanishes, classical em field (with e included as normalization factor) equals to $\gamma = 3J$, where J is K"ahler induced Kähler form (see [L4]. One has

$$B_K = \frac{eB_{em}}{3} \ . \tag{4.3.6}$$

2. An interesting question is whether one could identify physically the ordinary magnetic field assignable to the critical Kähler magnetic field.
   Earth's magnetic field $B_E = .5$ Gauss corresponds to magnetic length $L_B = \sqrt{\hbar e B} = 5 \mu$m. Endogenous magnetic field $B_{end} \simeq 2B_E/5$ explaining the findings of Blackman [J2] about the effects of ELF em fields on vertebrate brain in terms of cyclotron transitions corresponds to $L_B = 12.5\ \mu$m to be compared with the p-adic length scale $L(175) = 40\ \mu$m. Also these findings served as inspiration of $h_{eff} = n \times h$ hypothesis [K75, K74].
   I have assigned large Planck constant phases with the flux tubes of B_{end}, which have however remained somewhat mysterious entity. Could B_{end} correspond to quantum critical value of B_K and therefore relate directly to cosmology?
   One can check whether $B_K = eB_{end}/3$ holds true. The hypothesis would give

$$eB_{end} = \frac{1}{L_B^2} = 3 \times \frac{\Phi_0}{4\pi\hbar}\frac{1}{L^2(175)} \ .$$

   implying

$$r = \frac{L^2(175)}{L_B^2} = \frac{3\Phi_0}{4\pi\hbar} \ .$$

   The left hand side gives $r = 10.24$. For $\Phi_0 = 8\pi\hbar$ the right hand side gives $r = 6$. $B_E = .34$ Gauss left and right hand sides of the formula are identical.
3. One can wonder the proposed formulas might be exact for preferred extremals satisfying extremely powerful conditions to guarantee strong form of holography. This would require in both cases bundle structure with transversal cross section action as fiber. In the case of extremals of Kähler this would require that induce Kähler magnetic field is covariantly constant.

4.3.5 Two variants for p-adic length scale hypothesis for cosmological constant

There are two options for the dependence string tension Tandarea S of the cross section of the flux tube on p-adic length scale: either $L_\Lambda = \sqrt{8\pi/\Lambda}$ or $L = (\hbar/\rho_{vac})^{1/4}$ satisfies p-adic length scale hypothesis. The "boundary condition" is that the radius of flux tubes would be of the order of neutron size scale in recent cosmology.

1. $L(now) = L_p$ scaling gives

$$S = S(now)\frac{p(now)}{p} \tag{4.3.7}$$

   with $p_{now} \simeq 2^{175}$ by p-adic length scale hypothesis. $L(175)$ is by about 4 per cent smaller than the Compton length assignable to $\hbar/L(now) = .029$ eV.
   If one wants $L(now) = L(175)$ exactly, one must increase R by 4 per cent, which is allowed by p-adic mass calculations fixing the value of R only with 10 per cent accuracy. Indeed, the second order contribution in p-adic mass calculations is uncertain and the ratio of maximal and minimal values of R is $R_{max}/R_{min} = \sqrt{6/5} \simeq 1.1$.
   As already noticed, $L(now)$ corresponds to neutron size scale, which conforms with p-adic mass calculations since the radius of flux tubes would correspond to p-adic length scale. This option looks more natural and suggest a profound connection with biology and fundamental physics.

2. $L_\lambda \equiv \sqrt{8\pi/\Lambda}$ could be proportional to secondary p-adic length scale $L(2,p_\Lambda) \equiv \sqrt{p_\Lambda} L_{p_\Lambda}$. The scaling law

$$L_\Lambda \propto \frac{p_\Lambda(now)}{p_\Lambda} \tag{4.3.8}$$

gives

$$L_\Lambda^2(now) = \frac{8\pi}{\Lambda(now)} = (\frac{p}{p(now)})^2 \times \frac{L^4(now)}{l_P^2} \quad . \tag{4.3.9}$$

$L_\Lambda(now) \sim 50$ Gly (roughly the age of the Universe) holds true. Note that one has $S \propto \sqrt{p_{now}/p} S(now)$ and $T = T_{now}\sqrt{p/p_{now}}$.

$1/p$-dependence for the string tension T looks more natural in light of p-adic mass calculations. One must however notice that the $L = L(175)$ is 4 per cent small than $L(now)$.

The density of dark energy is uncertain by few per cent at least and one can ask whether $L(now) = L(175)$ could fix it. The change induced to ρ_{vac} by that of $L(now)$ is

$$\frac{\Delta\rho_{vac}}{\rho_{vac}} = -4\frac{\Delta L(now)}{L(now)}$$

and the reduction L by 4 per cent would reduce vacuum density by 16 per cent, which looks rather large change. The value of R can be determined by 10 per cent accuracy and the increase of R by four per cent is another manner to achieve $L(now) = L(175)$.

One can of course ask, whether both variants of p-adic length scale hypothesis could be correct. The reader night protest that this leads to the murky waters of p-adic numerology.

1. Could L_Λ be proportional to the secondary p-adic length scale $L(p,2) = \sqrt{p}L_p = 2^{k/2} \times L(k)$ associated with p characterizing L such that the proportionality constant is power of $\sqrt{2}$. The application of the condition defining L in terms of $L_\Lambda^2 = 8\pi/\Lambda$ gives

$$L_\Lambda^2 = \frac{L^4}{l_P^2} \quad .$$

Using $L_\Lambda = \sqrt{p_\Lambda} R$ and taking square roots, this gives

$$\sqrt{p_\Lambda} = pk^2 \quad , \quad k = \frac{R_{CP_2}}{l_P} \quad . \tag{4.3.10}$$

This conforms with the p-adic length scales hypothesis in its simplest form if k is power of $\sqrt{2}$.

2. The estimate from p-adic mass calculations for $r \equiv R(CP_2)/l_P$ is $r = 4.167 \times 10^3$ and is 2 per cent larger than 2^{12}. Could the $R(CP_2)/l_P = 2^{12}$ for the radii of CP_2 and M^4 twistorial sphere be an exact formula between fundamental length scales? As noticed, the second order contribution in p-adic mass calculations is uncertain by 10 per cent. This would allow the reduction of $R(CP_2)$ by 2 percent.

This looks an attractive option. The bad news is that the *increase* of $R(CP_2)$ by about 4 per cent to achieve $L(now) = L(175)$ is in conflict with its *reduction* by 2 per cent to achieve $R(CP_2)/l_P = 2^{12}$: this would reduce $L(175)$ by 2 per cent and increase ρ_{vac} by about 8 per cent. ρ_{vac} is however an experimental parameter depending on theoretical assumption and it value could allow this tuning. Therefore

$$\begin{aligned} \frac{R_{CP_2}}{l_P} &= 2^{12} \quad , \\ p_\Lambda &= 2^{48} \times p^2 \quad . \end{aligned} \tag{4.3.10}$$

is an attractive option fixing completely the value of $R(CP_2)/l_P$ and predicting relation between cosmological scale L_Λ and a fundamental scale in recent biology, which could be assigned to magnetic flux tubes assignable to axons. Note that for $k_{now} = 175$ the value of $k_\Lambda = k_{now} + 48$ is $k_\Lambda = 175 + 48 = 223$ which corresponds to p-adic length scale of 64 m.

3. Needless to say that one must be take these estimates with a big grain of salt. Number theoretical universality suggests that one might apply number theoretical constraints to fundamental constants like R, l_P, and Λ but one should be very critical concerning the values of empirical parameters such as ρ_{vac} depending on theoretical assumptions. Furthermore, p-adic length scale hypothesis is applied at the level of embedding space metric and one can ask whether it actually applies for the induced metric (Robertson-Walker metric now).

4.4 What happens for the extremals of Kähler action in twistor lift

As I started to work with TGD around 1977, I adopted path integral and canonical quantization as the first approaches. One of the first guesses for the action principle was 4-volume in induced metric giving minimal surfaces as preferred extremals. The field equations are a generalization of massless field equation and at least in the case of string models Hamiltonian formalism and second quantization is possible. The reason why for giving up this option was that for space-time surfaces of infinite duration the volume is infinite. This is not pleasant news concerning quantization since subtraction of exponent of infinite volume factor looked really ugly thing to do. At that time I did of course have no idea about ZEO and CDs.

For Kähler action there is however infinite vacuum degeneracy. All space-time surfaces with CP_2 projection, which is Lagrangian manifold (at most 2-dimensional) are vacuum extremals and canonical quantization fails completely. This implies classical non-determinism also for non-vacuum extremals obtained as small deformations of vacuum extremals. This feature seems to have nice implications such as 4-D spin glass degeneracy. It would however make WCW metric singular for nearly vacuum extremals.

The twistor lift brings volume term to the action. For option II there is also coupling between Kähler action and volume term but asymptotically one expects minimal surface extremals as analogs for free geodesic motion. The question is what happens to the known extremals of Kähler action, most of which are minimal surfaces.

4.4.1 The coupling between Kähler action and volume term

The addition of the volume term to Kähler action has very nice interpretation as a generalization of equations of motion for a world-line extended to a 4-D space-time surface. The field equations generalize in the same manner for 3-D light-like surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian, for 2-D string world sheets, and for their 1-D boundaries defining world lines at the light-like 3-surfaces. For 3-D light-like surfaces the volume term is absent. Either light-like 3-surface is freely choosable in which case one would have Kac-Moody symmetry as gauge symmetry or that the extremal property for Chern-Simons term fixes the gauge.

The condition that the dynamics based on Kähler action and volume term is number theoretically universal demands that coupling constants do not appear in it. This leaves only Option I ($\alpha_K(M^4) \neq \alpha_K(CP_2)$ with different phases) and option II ($\alpha_K(M^4) = \alpha_K(CP_2)$ with the same phase). This condition is taken as granted in the following.

The dynamics of twistor lift as a generalization of the dynamics of point like particle coupling to Maxwell field

Almost all the known non-vacuum extremals are minimal surface extremals of Kähler action [K12, K8] and it might well be that the preferred extremal property realizing SH quite generally demands this. CP_2 type vacuum extremals are also minimal surfaces if one assumes that the M^4 projection is light-like geodesic rather than only geodesic line.

The addition of the volume term could however make Kähler coupling strength a manifest coupling parameter also classically when the phases of Λ and α_K are same. Therefore quantum criticality for Λ and α_K would have a precise local meaning also classically in the interior of space-time surface. The equations of motion for a world line of U(1) charged particle would generalize to field equations for a "world line" of 3-D extended particle.

This is an attractive idea consistent with standard wisdom but for Option I one can invent strong objections against it.

1. The conjecture is that α_K has zeros of zeta as its spectrum of critical values [L17]. If so then all preferred extremals are minimal surface extremals of Kähler action for a real value of cosmological constant Λ possible for Option I ($\alpha_K(M^2)$ would be real). Hence the two actions decouple: this does not look nice. For Option II the phase is same and there is interaction between these degrees of freedom. One could of course force also the phase for Option I to be same.
2. All known non-vacuum extremals of Kähler action are minimal surfaces and the minimal surface vacuum extremals of Kähler action become non-vacuum extremals. This allows to consider the possibility that preferred extremals are minimal surface extremals of Kähler action so that the two dynamics apparently decouple. For Option II this makes sense since the solutions do not depend at all on the common over-all scaling factor of Kähler action and volume term. Minimal surface extremals are analogs for geodesics in the case of point-like particles: one might say that one has only gravitational interaction. This conforms with SH stating that gauge interactions at boundaries (orbits of partonic 2-surfaces and 2-surfaces at the ends of CD) correspond classically to the gravitational dynamics in the space-time interior.

   Note that at the boundaries of the string world sheets at light-like 3-surfaces the situation is different: one has equations of motion for geodesic line coupled to induce Kähler gauge potential and gauge coupling indeed appears classically as one might expect! For string world sheets one has only the topological magnetic flux term and minimal surface equation in string world sheet. Magnetic flux term gives the Kähler coupling at the boundary.
3. For Option I decoupling implied by extremal property of both real and imaginary parts of action would allow to realize number theoretical universality [K104] since the field equations would not depend on coupling parameters at all. For Option II same is achieved even without decoupling.
4. One can argue that the decoupling for Option I makes it impossible to understand coupling constant evolution. This need not be the case. The point is that the classical charges assignable to super-symplectic algebra are sums over contributions from Kähler action and volume term and therefore depend on the coupling parameters. Their vanishing conditions for sub-algebra and its commutator with entire algebra give boundary conditions on preferred extremals so that coupling constant evolution creeps in classically!

   Quantum classical correspondence realized as the condition that the eigenvalues of fermionic charge operators are equal to the classical charges brings in the dependence of quantum charges on coupling parameters. Since the elements of scattering matrix are expected to involve as building bricks the matrix elements of super-symplectic algebra and Kac-Moody algebra of isometry charges, one expects that discrete coupling constant evolution creeps in also quantally via the boundary conditions for preferred extremals.

Options I and II and Chladni mechanism

One can compare Options I and II.

1. For Option I the coupling between the two dynamics could be induced just by the condition that the space-time surface becomes an analog of geodesic line by arranging its interior so that the U(1) force vanishes! This would generalize Chladni mechanism (see `http://tinyurl.com/j9rsyqd`)!

   The interaction would be present but be based on going to the nodal surfaces! Also the dynamics of string world sheets is similar: if the string sheets carry vanishing W boson classical fields, em charge is well-defined and conserved. One would also avoid the problems produced by large coupling constant between the two-dynamics present already at the classical level. At quantum level the fixed point property of quantum critical couplings would be the counterparts for decoupling. This option however seems to be missing the transient phase preceding the Chladni configuration.

2. For Option II the coupling would be present during transient periods leading to decoupling. The alternative view is that the deviation from minimal surface and can act as a controller of the dynamics defined by the volume term providing a small push or pull now and then. Could this sensitivity relate to quantum criticality and to the view about morphogenesis relying on Chladni mechanism in which field patterns control the dynamics with charged flux tubes ending up to the nodal surfaces of (Kähler) electric field [L28]? Magnetic flux tubes containing dark matter would in turn control and serve as template for the dynamics of ordinary matter.

   Chladni mechanism would not be instantaneous but lead via transient phase to minimal surface extremals near either or both boundaries of CDs analogous to external particles in particle reaction. The space-time regions assignable to particle interaction vertices identified as 2-surfaces at which the ends of three 3-D light-like partonic orbits meet, would correspond to transient regions, where the coupling is present. This option looks clearly more realistic.

Admittedly Option II looks more attractive.

As an example one can consider a typical particle physics experiment. There are incoming and outgoing free particles moving along geodesics, these particles interact, and emanate as free particles from the interaction volume. This phenomenological picture does not follow from QFT but is put in by hand, in particular the idea about interaction couplings becoming non-zero is involved. Also the role of the observer remains poorly understood.

The motion of incoming and outgoing particles is analogous to free motion along geodesic lines with particles generalized to 3-D extended objects. For both options these would correspond to the preferred extremals in the complement of CD within larger CD representing observer or measurement instrument. Decoupling would take place. In interaction volume interactions are "coupled on" and particles interact inside the volume characterized by causal diamond (CD). What could be the TGD view translation of this picture?

1. For Option I one would still have decoupling and the interpretation would be in terms of twistor picture in which one always has also in the internal lines on mass shell particles but with complex four-momenta. In TGD framework the momenta would be always complex due to the contribution of Euclidian regions defining the lines of generalized scattering diagrams. Note however that the real and imaginary parts of the conserved charges are predicted to be proportional to each other. This result is obtained also in twistor approach from 8-D light-likeness and is crucial for twistorialization in TGD sense [L45]. As explained, coupling constant evolution can be understood also in this case and also classical dynamics depends on coupling parameters via the boundary conditions. There would be no counterpart for transitory period (interaction on) leading to the decoupled situation so that Option I is not attractive.
2. For Option II the transitory period would correspond to the coupling between the two classical dynamics in regions assignable to the vertices of topological scattering diagrams at which the ends of the parton orbits meet. Near the ends the dynamics would decouple and one would have the analog of free geodesic motion.

Second example comes from biology. The free geodesic line dynamics with vanishing $U(1)$ Kähler force indeed brings in mind the proposed generalization of Chladni mechanism generating nodal surfaces at which charged magnetic flux tubes are driven [K55] [L28, L29] . Chlandi mechanism could be seen as a basic mechanism behind morphogenesis.

1. For Option I the interiors of all space-time surfaces would be analogous to nodal surfaces and "big" state function reductions would correspond to transition periods between different nodal surfaces. The decoupling would be dynamics of avoidance and could highly analogous to Chladni mechanism.
2. For Option II transition period would correspond to a period during which nodal surfaces are formed.

It seems that Option II is favored by both SH, number theoretical universality, and generalization of Chladni mechanism to a dynamics of avoidance.

4.4.2 Twistor lift and the extremals of Kähler action

The addition of the volume term makes Kähler coupling strength a genuine coupling parameter also classically when the variation of Kähler action is non-vanishing. Therefore quantum criticality for Λ and α_K gets precise meaning also classically. The equations of motion for a worldline of U(1) charged particle generalize to field equations for a "world line" of 3-D extended particle.

The field equations generalize in the same manner for 3-D light-like surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian, for 2-D string world sheets, and for their 1-D boundaries defining world lines at the light-like 3-surfaces. For 3-D light-like surfaces the volume term is absent. Either light-like 3-surface is freely choosable in which case one would have Kac-Moody symmetry as gauge symmetry or that the extremal property for Chern-Simons term fixes the gauge.

What happens to the extremals of Kähler action?

What happens to the extremals of Kähler action when volume term is introduced?

1. The known non-vacuum extremals [K12, K8] such as massless extremals (topological light rays) and cosmic strings are minimal surfaces.
2. For $J(M^4) = 0$ these extremals remain extremals for both Option I and II and only the classical Noether charges receive an additional volume term. In particular, string tension is modified by the volume term. Homologically non-trivial cosmic strings are of form $X^2 \times Y^2$, where $X^2 \subset M^4$ is minimal surface and $Y^2 \subset CP_2$ is complex 2-surface and therefore also minimal surface.
3. For $J(M^4) \neq 0$ essential for obtaining small cosmological constant for Option II, the situation changes and minimal surface property is possible only under additional conditions. For instance, one can have minimal surfaces of form $X^2 \times Y^2 \subset M^4 \times Y^2$, where Y^2 is minimal surface in CP_2. X^2 cane be $M^2 \subset N^2 \times E^2$ defining the $J(M^4)$ giving $J(M^4) \cdot J(M^4) - 2 = 0$. X^2 can be also minimal surface, which is an analog of Lagrangian manifold for $J(M^4)$.
4. Vacuum degeneracy is lifted for both options. For $J(M^4) = 0$ vacuum extremals, which are minimal surfaces survive as extremals for both options. For $J(M^4) \neq 0$ the situation is more complex.

Vacuum extremals

For CP_2 type vacuum extremals [K12, K8] the roles of M^4 and CP_2 are changed. M^4 projection is light-like curve, and can be expressed as $m^k = f^k(s)$ with light-likeness conditions reducing to Virasoro conditions. These surfaces are isometric to CP_2 and have same Kähler and symplectic structures as CP_2 itself. What is new as compared to GRT is that the induced metric has Euclidian signature. The interpretation is as lines of generalized scattering diagrams. The addition of the volume term forces the random light-like curve to be light-like geodesic and the action becomes the volume of CP_2 in the normalization provided by cosmological constant. What looks strange is that the volume of any CP_2 type vacuum extremals equals to CP_2 volume but only the extremal with light-like geodesic as M^4 projection is extremal of volume term. A little calculation shows that for CP_2 type extremals the contribution of the volume term to the action would be completely negligible as compared to the Kähler action.

Consider next vacuum extremals, which have vanishing induced Kähler form and are thus have CP_2 projection belonging to at most 2-D Lagrangian manifold of CP_2 [K12, K8].

1. Vacuum extremals with 2-D projections to CP_2 and M^4 are possible and are of form $X^2 \times Y^2$, X^2 arbitrary 2-surface and Y^2 a Lagrangian manifold. Volume term forces X^2 to be a minimal surface and Y^2 is Lagrangian minimal surface unless the minimal surface property destroys the Lagrangian character.

   If the Lagrangian sub-manifold is homologically trivial geodesic sphere, one obtains string like objects with string tension determined by the cosmological constant alone.

Do more general 2-D Lagrangian minimal surfaces than geodesic sphere exist? For general Kähler manifold there are obstructions but for Kähler-Einstein manifolds such as CP_2, these obstructions vanish (see `http://tinyurl.com/gtkpya6`). The case of CP_2 is also discussed in the slides "On Lagrangian minimal surfaces on the complex projective plane" (see `http://tinyurl.com/jrhl6gy`). The discussion is very technical and demonstrates that Lagrangian minimal surfaces with all genera exist. In some cases these surfaces can be also lifted to twistor space of CP_2.

2. More general vacuum extremals have 4-D M^4 projection. Could the minimal surface condition for 4-D M^4 projection force a deformation spoiling the Lagrangian property? The physically motivated expectation is that string like objects give as deformations magnetic flux tubes for which string is thicknened so that it has a 2-D cross section. This would suggest that the deformations of string like objects $X^2 \times Y^2$, where Y^2 is Lagrangian minimal surface, give rise to homologically trivial magnetic flux tubes. In this case Kähler magnetic field would vanish but the spinor connection of CP_2 would give rise to induced magnetic field reducing to some U(1) subgroup of U(2). In particular, electromagnetic magnetic field could be present.

3. p-Adically Λ behaves like $1/p$ as also string tension. Could hadronic string tension be understood also in terms of cosmological constant in hadronic p-adic length scale for strings if one assumes that cosmological constant for given space-time sheet is determined by its p-adic length scale?

Maxwell phase

What might be called Maxwell phase which would correspond to small perturbations of M^4 is also possible for 4-D Kähler action. For the twistor lift the volume term makes this phase possible. Maxwell phase is highly interesting since it corresponds to the intuitive view about what QFT limit of TGD could be. The following arguments apply only for $J(M^4) = 0$.

1. The field equations are a generalization of massless field equations for fields identifiable as CP_2 coordinates and with a coupling to the deviation of the induced metric from M^4 metric. It representes very weak perturbation. Hence the linearized field equations are expected to be an excellent approximation. The general challenge would be however the construction of exact solutions. One should also understand the conditions defining preferred extremals and stating that most of symplectic Noether charges vanish at the ends of space-time surface about boundaries of CD.

2. Maxwell phase is the TGD analog for the perturbative phase of gauge theories. The smallness of the cosmological constant in cosmic length scales would make the perturbative approach useless in the path integral formulation. In TGD approach the path integral is replaced by functional integral involving also a phase but also now the small value of cosmological constant is a problem in long length scales. As proposed, the hierarchy of Planck constants would provide the solution to the problem.

3. The value of cosmological constant behaving like $\Lambda \propto 1/p$ as the function of p-adic prime could be in short p-adic length scales large enough to allow a converging perturbative expansion in Maxwellian phase. This would conform with the idea that Planck constant has its ordinary value in short p-adic length scales.

4. Does Maxwell phase allow extremals for which the CP_2 projection is 2-D Lagrangian manifold - say a perturbation of a minimal Lagrangian manifold? This perturbation could be seen also as an alternative view about thickened minimal Lagrangian string allowing also M^4 coordinates as local coordinates. If the projection is homologically trivial geodesic sphere this is the case. Note that solutions representable as maps $M^4 \to CP_2$ are also possible for homologically non-trivial geodesic sphere and involve now also the induced Kähler form.

5. The simplest deformations of canonically imbedded M^4 are of form $\Phi = k \cdot m$, where Φ is an angle coordinate of geodesic sphere. The induced metric in M^4 coordinates reads as $g_{kl} = m_{kl} - R^2 k_k k_l$ and is flat and in suitably scaled space-time coordinates reduces to Minkowski metric or its Euclidian counterpart. k_k is proportional to classical four-momentum assignable to the dark energy. The four-momentum is given by

$$P^k = A \times \hbar k^k \ , \quad A = \frac{Vol(X^3)}{L_\Lambda^4} \times \frac{1+2x}{1+x} \ , \quad x = R^2 k^2 \ .$$

Here k^k is dimensionless since the the coordinates m^k are regarded as dimensionless.

6. There are interesting questions related to the singularities forced by the compactness of CP_2. Eguchi-Hanson coordinates (r, θ, Φ, Ψ) [L4] (see `http://tinyurl.com/z86o5qk`) allow to get grasp about what could happen.

   For the cyclic coordinates Ψ and Φ periodicity conditions allow to get rid of singularities. One can however have n-fold coverings of M^4 also now.

   (r, θ) correspond to canonical momentum type canonical coordinates. Both of them correspond to angle variables ($r/\sqrt{1+r^2}$ is essentially sine function). It is convenient to express the solution in terms of trigonometric functions of these angle variables. The value of the trigonometric function can go out of its range $[-1, 1]$ at certain 3-surface so that the solution ceases to be well-defined. The intersections of these surfaces for r and θ are 2-D surfaces. Many-sheeted space-time suggests a possible manner to circumvent the problem by gluing two solutions along the 3-D surfaces at which the singularities for either variable appear. These surfaces could also correspond to the ends of the space-time surface at the boundaries of CD or to the light-like orbits of the partonic 2-surfaces.

   Could string world sheets and partonic 2-surfaces correspond to the singular 2-surfaces at which both angle variables go out of their allowed range. If so, 2-D singularities would code for data as assumed in strong form of holography (SH). SH brings strongly in mind analytic functions for which also singularities code for the data. Quaternionic analyticity which makes sense would indeed suggest that co-dimension 2 singularities code for the functions in absence of 3-D counterpart of cuts (light-like 3-surfaces?) [L22].

7. A more general picture might look like follows. Basic objects come in two classes. Surfaces $X^2 \times Y^2$, for which Y^2 is either homologically non-trivial complex minimal 2-surface of CP_2 of Lagrangian minimal surface. The perturbations of these two surfaces would also produce preferred extremals, which look locally like perturbations of M^4. Quaternionic analyticity might be shared by both solution types. Singularities force many-sheetedness and strong form of holography.

Astrophysical and cosmological solutions

Cosmological constant is expected to obey p-adic evolution and in very early cosmology the volume term becomes large. What are the implications for the vacuum extremals representing Robertson-Walker metrics having arbitrary 1-D CP_2 projection? [K12, K8, K86]. One can also ask what is the fate of spherically symmetric solutions of GRT providing a model of star.

Already the existing physical picture explaining $h_{gr}/hh_{eff}/h = n$ in terms of flux tubes mediating gravitational interactions suggests that Robertson-Walker metrics and spherically symmetric metrics are possible only at QFT limit. The presence of covariantly constant $J(M^4)$ breaking Lorentz symmetry and rotational symmetry makes this obvious. One could consider variants of $J(M^4)$ invariant under Lorentz group or some subgroup of Lorentz group but $J(M^4)$ would not be covariantly constant anymore. It is not clear when it makes sense to extend the moduli space for $J(M^4)$.

1. The TGD inspired cosmology involves primordial phase during a gas of cosmic strings in M^4 with 2-D M^4 projection dominates. The value of cosmological constant at that period could be fixed from the condition that homologically trivial and non-trivial cosmic strings have the same value of string tension. After this period follows the analog of inflationary period when cosmic strings condense are the emerging 4-D space-time surfaces with 4-D M^4 projection and the M^4 projections of cosmic strings are thickened. A fractal structure with cosmic strings topologically condensed at thicker cosmic strings suggests itself.

2. GRT cosmology is obtained as an approximation of the many-sheeted cosmology as the sheets of the many-sheeted space-time are replaced with region of M^4, whose metric is replaced with Minkowski metric plus the sum of deformations from Minkowski metric for the sheet. The vacuum extremals with 4-D M^4 projection and arbitrary 1-D projection could serve as an

approximation for this GRT cosmology. Note however that this representability is not required by basic principles.

3. For cosmological solutions with 1-D CP_2 projection minimal surface property forces the CP_2 projection to belong to a geodesic circle S^1. Denote the angle coordinate of S^1 by Φ and its radius by R. For the future directed light-cone M^4_+ use the Robertson-Walker coordinates $(a = \sqrt{m_0^2 - r_M^2}, r = ar_M, \theta, \phi)$, where (m^0, r_M, θ, ϕ) are spherical Minkowski coordinates. The metric of M^4_+ is that of empty cosmology and given by $ds^2 = da^2 - a^2 d\Omega^2$, where Ω^2 denotes the line element of hyperbolic 3-space identifiable as the surface $a = constant$.

   One can can write the ansatz as a map from M^4_+ to S^1 given by $\Phi = f(a)$. One has $g_{aa} = 1 \rightarrow g_{aa} = 1 - R^2(df/da)^2$. The field equations are minimal surface equations and the only non-trivial equation is associated with Φ and reads $d^2f/da^2 = 0$ giving $\Phi = \omega a$, where ω is analogous to angular velocity. The metric corresponds to a cosmology for which mass density goes as $1/a^2$ and the gravitational mass of comoving volume (in GRT sense) behaves is proportional to a and vanishes at the limit of Big Bang smoothed to "Silent whisper amplified to rather big bang" for the critical cosmology for which the 3-curvature vanishes. This cosmology is proposed to results at the limit when the cosmic temperature approaches Hagedorn temperature [K86].

4. The TGD counterpart for inflationary cosmology corresponds to a cosmology for which CP_2 projection is homologically trivial geodesic sphere S^2 (presumably also more general Lagrangian (minimal) manifolds are allowed). This cosmology is vacuum extremal of Kähler action. The metric is unique apart from a parameter defining the duration of this period serving as the TGD counterpart for inflationary period during which the gas of string like objects condensed at space-time surfaces with 4-D M^4 projection. This cosmology could serve as an approximate representation for the corresponding GRT cosmology.

   The form of this solution is completely fixed from the condition that the induced metric of $a = constant$ section is transformed from hyperbolic metric to Euclidian metric. It should be easy to check whether this condition is consistent with the minimal surface property. It seems that one cannot satisfy minimal surface equations.

5. For $J(M^4) \neq 0$ the spherical and Lorentz symmetries are lost and the only cosmological solution are light-cones $M^4_\pm$. Also the existence of stationary spherically symmetric minimal surface extremals is impossible for $J(M^4) \neq 0$. Spherically symmetric metrics and Robertson-Walker metric would serve only as long length scale approximations providing a statistical description of the gravitational interaction described microscopically in terms of a flux tube network.

4.4.3 Are minimal surface extremals of Kähler action holomorphic surfaces in some sense?

If the spectrum for the critical value of Kähler coupling strength is complex - say given by the complex zeros of zeta [L17] - the preferred extremals of Kähler action are minimal surfaces for Option I. For Option II they correspond to asymptotic solutions.

I have considered several ansätze for the general solutions of the field equations for the preferred extremals. One proposal is that preferred extremals as 4-surfaces of embedding space with octonionic tangent space structure have quaternionic tangent space or normal space (so called $M^8 - H$ duality [K91]). Second proposal is that preferred extremals can be seen as quaternion analytic [A90] surfaces [K80, L10] [L15]. Third proposal relies on a fusion of complex and hypercomplex structures to what I call Hamilton-Jacobi structure [K99, K8]. In Euclidian regions this would correspond to complex structure. Twistor approach [L22] suggests that the condition that the twistor lift of the space-time surface to a 6-D surface in the product of twistor spaces of M^4 and CP_2 equals to the twistor space of CP_2. This proposal is highly interesting since twistor lift works only fr $M^4 \times CP_2$. The intuitive picture is that the field equations are integrable and all these views might be consistent.

Preferred extremals of Kähler action as minimal surfaces would be a further proposal. Can one make conclusions about general form of solutions assuming that one has minimal surface extremals of Kähler action?

In $D = 2$ case minimal surfaces are holomorphic surfaces or they hyper-complex variants and the embedding space coordinates can be expressed as complex-analytic functions of complex coordinate or a hypercomplex analog of this. Field equations stating the vanishing of the trace $g\alpha\beta H^k_{\alpha\beta}$ if the second fundamental form $H^k_{\alpha\beta} \equiv D_\alpha\partial_\beta h^k$ are satisfied because the metric is tensor of type $(1, 1)$ and second fundamental form of type $(2, 0) \oplus (2, 0)$. Field equations reduce to an algebraic identity and functions involved are otherwise arbitrary functions. The constraint comes from the condition that metric is of form $(1, 1)$ as holomorphic tensor.

This raises the question whether this finding generalizes to the level of 4-D space-time surfaces and perhaps allows to solve the field equations exactly in coordinates generalizing the hypercomplex coordinates for string world sheet and complex coordinates for the partonic 2-surface.

Almost all the known non-vacuum extremals are minimal surface extremals of Kähler action [K12, K8] and it might well be that the preferred extremal property realizing SH quite generally demands this. CP_2 type vacuum extremals are also minimal surfaces if one assumes that the M^4 projection is light-like geodesic rather than only geodesic line. The common feature suggested already earlier to be common for all preferred extremals is the existence of generalization of complex structure.

1. For Minkowskian regions this structure would correspond to what I have called Hamilton-Jacobi structure [K99, K8]. The tangent space of the space-time surface X^4 decomposes to local direct sum $T(X^4) = T(X^2) \oplus T(Y^2)$, where the 2-D tangent places $T(X^2)$ and $T(Y^2)$ define an integrable distribution integrating to a decomposition $X^4 = X^2 \times Y^2$. The complex structure is generalized to a direct some of hyper-complex structure in X^2 meaning that there is a local light-like direction defining light-like coordinate u and its dual v. $Y2$ has complex complex coordinate $(w, \overline{w})$. Minkowski space M^4 has similar structure. It is still an open question whether metric decomposes to a direct sum of orthogonal metrics assignable to X^2 and Y^2 or is the most general analog of complex metric in question. g_{uv} and $g_{w\overline{w}}$ are certainly non-vanishing components of the induced metric. Metric could allow as non-vanishing components also g_{uw} and $g_{v\overline{w}}$. This slicing by pairs of surfaces would correspond to decomposition to a product of string world sheet and partonic 2-surface everywhere.

   In Euclidian regions ne would have 4-D complex structure with two complex coordinates (z, w) and their conjugates and completely analogous decompositions. In CP_2 one has similar complex structure and actually Kähler structure extending to quaternionic structure. I have actually proposed that quaternion analyticity could provide the general solution of field equations.

2. Assuming minimal surface property the field equations for Kähler action reduce to the vanishing of a sum of two terms. The first term comes from the variation with respect to the induced metric and is proportional to the contraction

   $$A = J^\alpha_{\ \gamma} J^{\gamma\beta} H^k_{\alpha\beta} \quad . \tag{4.4.1}$$

   Second term comes from the variation with respect to induced Kähler form and is proportional to

   $$B = j^\alpha P^k_{\ s} J^s_{\ l} \partial_\alpha h^l \quad . \tag{4.4.2}$$

   Here P^k_l is projector to the normal space of space-time surface and $j^\alpha = D_\beta J^{\alpha\beta}$ is the conserved Kähler current.

   For the known extremals j vanishes or is light-like (for massless extremals) in which case A and B vanish separately.

3. An attractive manner to satisfy field equations would be by assuming that the situation for 2-D minimal surface generalizes so that minimal surface equations are identically satisfied. Extremal property for Kähler action could be achieved by requiring that energy momentum tensor also for Kähler action is of type $(1, 1)$ so that one would have $A = 0$. This implies $j^\alpha\partial_\alpha s^k = 0$. This is true if j vanishes or is light-like as it is for the known extremals. In Euclidian regions one would have $j = 0$.

4. The proposed generalization is especially interesting in the case of cosmic string extremals of form $X^2 \times Y^2$, where $X^2 \subset M^4$ is minimal surface (string world sheet) and Y^2 is complex homologically non-trivial sub-manifold of CP_2 carrying Kähler magnetic charge. The generalization would be that the two transversal coordinates $(w, \overline{w})$ in the plane orthogonal to the string world sheet defining polarization plane depend holomorphically on the complex coordinates of complex surface of CP_2. This would transform cosmic string to flux tube.
5. There are also solutions of form $X^2 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2 with vanishing Kähler magnetic charge and their deformations with $(w, \overline{w})$ depending on the complex coordinates of Y^2 (see the slides "On Lagrangian minimal surfaces on the complex projective plane" at `http://tinyurl.com/jrhl6gy`). In this case Y^2 is not complex sub-manifold of CP_2 with arbitrary genus and induced Kähler form vanishes. The simplest choice for Y^2 would be as homologically trivial geodesic sphere. Because of its 2-dimensionality Y^2 has a complex structure defined by its induced metric so that solution ansatz makes sense also now.

4.5 About string like objects

String like objects and partonic 2-surfaces carry the information about quantum states and about space-time surfaces as preferred extremals if strong form of holography (SH) holds true. SH has of course some variants. The weakest variant states that fundamental information carrying objects are metrically 2-D. The light-like 3-surfaces separating space-time regions with Minkowskian and Euclidian signature of the induced metric are indeed metrically 2-D, and could thus carry information about quantum state.

The original observation was that string world sheets should carry vanishing W boson fields in order that the em charge for the modes of the induced spinor field is well-defined. This condition can be satisfied in certain situations also for the entire space-time surface. This raises several questions. What is the fundamental condition forcing the restriction of the spinor modes to string world sheets - or more generally, to a surface of given dimension?

Can one have an analog of brane hierarchy in which also higher-D objects can carry modes of induced spinor field [K84]. Or should one identify 2-surfaces in terms of effective action, which by SH allows to describe the dynamics in terms of 2-D data? Both options have their nice features.

4.5.1 Two options for fundamental variational principle

String world sheets and partonic 2-surfaces seems to be fundamental for TGD - especially so in the fermionic sector - but also the 4-D action seems to necessary and supersymmetry forces 4-D modified Dirac action too. The interpretation of the situation is far from obvious. One ends up to two options for the fundamental variational principle.

Option A: The *fundamental* action principle for space-time surfaces contains besides 4-D action also 2-D action assignable to string world sheets, whose topological part (magnetic flux) gives rise to a coupling term to Kähler gauge potentials assignable to the 1-D boundaries of string world sheets containing also geodesic length part. Super-symplectic symmetry demands that modified Dirac action has 1-, 2-, and 4-D parts: spinor modes would exist at both string boundaries, string world sheets, and space-time interior. A possible interpretation for the interior modes would be as generators of space-time super-symmetries [K84].

This option is not quite in the spirit of SH and string tension appears as an additional parameter. Also the conservation of em charge forces 2-D string world sheets carrying vanishing induced W fields and this is in conflict with the existence of 4-D spinor modes unless they satisfy the same condition. This looks strange.

Option B: Stringy action and its fermionic counterpart are effective actions only and justified by SH. In this case there are no problems of interpretation. SH requires only that the induced spinor fields at string world sheets determine them in the interior much like the values of analytic function at curve determine it in an open set of complex plane. At the level of quantum theory the scattering amplitudes should be determined by the data at string world sheets. If the induced W fields at string world sheets are vanishing, the mixing of different charge states in the interior of X^4 would not make itself visible at the level of scattering amplitudes!

If string world sheets are generalized Lagrangian sub-manifolds, only the induced em field would be non-vanishing and electroweak symmetry breaking would be a fundamental prediction. This however requires that M^4 has the analog of symplectic structure suggested also by twistorialization. This in turn provides a possible explanation of CP breaking and matter-antimatter asymmetry. In this case 4-D spinor modes do not define space-time super-symmetries.

The latter option conforms with number theoretically broken SH and would mean that the theory is amazingly simple. String world sheets together with number theoretical space-time discretization meaning small breaking of SH would provide the basic data determining classical and quantum dynamics. The Galois group of the extension of rationals defining the number-theoretic space-time discretization would act as a covering group of the covering defined by the discretization of the space-time surface, and the value of $h_{eff}/h = n$ would correspond to the dimension of the extension dividing the order of its Galois group. The phase transitions reducing $ord(G) \geq n$ would correspond to spontaneous symmetry breaking leading from Galois group to a subgroup H so that $ord(H)$ would divide $ord(G)$ and the new value of n would divide n.

The ramified primes of the extension would be preferred primes of given extension. The extensions for which the number of p-adic space-time surfaces representable also as a real algebraic continuation of string world sheets to preferred extrenal is especially large would be physically favored as also corresponding ramified primes. In other words, maximal number of p-adic imaginations would be realizable so that these extensions and corresponding ramified primes would be winners in the number-theoretic fight for survival. Whether this conforms with p-adic length scale hypothesis, remains an open question.

An attractive possibility is that this information is basically topological. For instance, the value of Planck constant $h_{eff} = n \times h$ would tell the number sheets of the singular covering defining this surface such that the sheets co-incide at partonic 2-surfaces at the ends of space-time surface at boundaries of CD. In the following some questions related to string world sheets are considered. The information could be also number theoretical. Galois group for the algebraic extension of rationals defining particular adelic physics would transform to each other the number theoretic discretizations of light-like 3-surfaces and give rise to covering space structure. The action is trivial at partonic 2-surfaces should be trivial if one wants singular covering: this would mean that discretizations of partonic 2-surfaces consist of rational points. $h_{eff}/h = n$ could in this case be a factor of the order of Galois group.

4.5.2 How to achieve low value of string tension?

String tension should be low for string world sheets in long scales. If string actions are effective actions (Option B), the same should be true for the string tensions of the magnetic flux tubes accompanying strings. Minimal surface property for string world sheets is natural. Let us consider only Option B in the following.

1. Could the analogs of Lagrangian sub-manifolds of $X^4 \subset M^4 \times CP_2$ satisfying $J(M^4) + J(CP_2) = 0$ define string world sheets and their variants with varying dimension? For Option I ($\alpha_K(M^4) \neq \alpha_K(CP_2)$) this could make sense if the flux tubes are homologically trivial. Homologically non-trivial (monopole) flux tubes should be thick enough to have small enough string tension, which is inversely proportional to the cross sectional area of the flux tube.

2. For Option II ($\alpha_K(M^4) = \alpha_K(CP_2)$) the action density is proportional to $J \cdot J - 2$ also for stringy action and this does not seem to make sense. Could the additional condition be $J(M^4) \cdot J(M^4) - 2 \sim 0$ holding true in 4-D sense for space-time regions with a small value of cosmological constant behaving like $1/p$, p preferred p-adic prime near power of 2. That low string tension and small cosmological constant would have the same origin, would be nice.

   The cancellation mechanism involving in an essential manner $J(M^4)$ would give rise to low mass strings and light hadron like particles and small cosmological constant instead of only high mass strings as in super string models. p-Adic thermodynamic for CP_2-mass excitations assignable to wormhole throats would determine elementary particle masses and long monopole flux tubes with small string tension connecting pairs of wormhole contacts would give stringy contribution to particle masses. In the case of hadrons this contribution from color magnetic flux tubes would dominate over quark masses. Clearly, Option II seems to conform with the existing picture about masses of elementary particles and hadrons.

4.5.3 How does the gravitational coupling emerge?

The appearance of $G = l_P^2$ has coupling constant remained for a long time actually somewhat of a mystery in TGD. l_P defines the radius of the twistor sphere of M^4 replaced with its geometric twistor space $M^4 \times S^2$ in twistor lift. G makes itself visible via the coefficients $\rho_{vac} = 8\pi\Lambda/G$ volume term but not directly and if preferred extremals are minimal surface extremals of Kähler action ρ_{vac} makes itself visible only via boundary conditions. How G appears as coupling constant?

Somehow the M^4 Kähler form should appear in field equations. $1/G$ could naturally appear in the string tension for string world sheets as string models suggest. p-Adic mass calculations identify the analog of string tension as something of order of magnitude of $1/R^2$ [K52]. This identification comes from the fact that the ground states of super-conformal representations correspond to embedding space spinor modes, which are solutions of Dirac equation in $M^4 \times CP_2$. This argument is rather convincing and allows to expect that the p-adic mass scale is not determined by string tension.

The problem is that the length of string like objects would be given by Planck length or CP_2 length if either of these pictures is the whole truth. One expects long gravitational flux tubes mediating gravitational interactions. The hypothesis $\hbar_{eff} = n\hbar = \hbar_{gr} = GMm/v_0$, where $v_0 < c$ is a parameter with dimensions of velocity, suggests that the string tension assignable to the flux tubes mediating gravitational interaction between masses M and m is apart from a numerical factor equal to Λ_{gr}^{-2}, where gravitational Compton length is $\Lambda_{gr} = h_{gr}/m = GM/v_0$ so that the length of the flux tubes is of order Λ_{gr}.

The problem is that the length of string like objects would be given by Planck length or CP_2 length if either of these pictures is the whole truth. One would like to have long gravitational flux tubes mediating gravitational interactions. Strong form of holography (SH) indeed suggests that stringy action appears as effective action expressing 4-D space-time action and modified Dirac action as 2-D actions assignable to string world sheets [L41] (see `http://tinyurl.com/zylrd7w`). This view would allow to understand the localization of spinor modes to string world sheets carrying vanishing W fields in terms as an effective description implying well-defineness of classical em charge and conservation of em charge at the level of scattering amplitudes. In fact that the introduction of the Kähler form $J(M^4)$ would allow to understand string world sheets as analogs of Lagrangian sub-manifolds.

4.5.4 Non-commutative embedding space and strong form of holography

Quantum group theorists have studied the idea that space-time coordinates are non-commutative and tried to construct quantum field theories with non-commutative space-time coordinates (see `http://tinyurl.com/z3m8sny`). My impression is that this approach has not been very successful. The non-commutativity is introduced by postulating the Minkowskian analog of symplectic form and $J(M^4)$ forced by Option II indeed is symplectic form. The loss of Lorentz invariance induced by $J(M^4)$ is the basic stumbling block. In TGD framework the moduli space for $J(M^4)$ emerges already when one introduces the moduli space for CDs. $J(M^4)$ would define quantization axis of energy (rest system) and quantization axis of spin. The nice features of $J(M^4)$ is that it could allow to understand CP breaking and matter antimatter asymmetry at fundamental level.

The analog of non-commutative space-time in TGD framework

In Minkowski space one introduces antisymmetry tensor J_{kl} and uncertainty relation in linear M^4 coordinates m^k would look something like $[m^k, m^l] = l_P^2 J^{kl}$, where l_P is Planck length. This would be a direct generalization of non-commutativity for momenta and coordinates expressed in terms of symplectic form J^{kl}.

1+1-D case serves as a simple example. The non-commutativity of p and q forces to use either p or q. Non-commutativity condition reads as $[p, q] = \hbar J^{pq}$ and is quantum counterpart for classical Poisson bracket. Non-commutativity forces the restriction of the wave function to be a function of p or of q but not both. More geometrically: one selects Lagrangian sub-manifold to which the projection of J_{pq} vanishes: coordinates become commutative in this sub-manifold. This condition can be formulated purely classically: wave function is defined in Lagrangian sub-manifolds to which the projection of J vanishes. Lagrangian manifolds are however not unique and this leads to problems in this kind of quantization. In TGD framework the notion of "World of Classical Worlds" (WCW) allows to circumvent this kind of problems and one can say that quantum theory is purely classical field theory for WCW spinor fields. "Quantization without quantization" would have Wheeler stated it.

General Coordinate Invariance (GCI) poses however a problem if one wants to generalize quantum group approach from M^4 to general space-time: linear M^4 coordinates assignable to Lie-algebra of translations as isometries do not generalize. In TGD space-time is surface in embedding space $H = M^4 \times CP_2$: this changes the situation since one can use 4 embedding space coordinates (preferred by isometries of H) also as space-time coordinates. The analog of symplectic structure J for M^4 makes sense and number theoretic vision involving octonions and quaternions leads to its introduction. Note that CP_2 has naturally symplectic form.

Could it be that the coordinates for space-time surface are in some sense analogous to symplectic coordinates (p_1, p_2, q_1, q_2) so that one must use either (p_1, p_2) or (q_1, q_2) providing coordinates for a Lagrangian sub-manifold. This would mean selecting a Lagrangian sub-manifold of space-time surface? Could one require that the sum $J_{\mu\nu}(M^4) + J_{\mu\nu}(CP_2)$ for the projections of symplectic forms vanishes and forces in the generic case localization to string world sheets and partonic 2-surfaces. In special case also higher-D surfaces - even 4-D surfaces as products of Lagrangian 2-manifolds for M^4 and CP_2 are possible: they would correspond to homologically trivial cosmic strings $X^2 \times Y^2 \subset M^4 \times CP_2$, which are not anymore vacuum extremals but minimal surfaces if the action contains besides Käction also volume term.

But why this kind of restriction? In TGD one has strong form of holography (SH): 2-D string world sheets and partonic 2-surfaces code for data determining classical and quantum evolution. Could this projection of $M^4 \times CP_2$ symplectic structure to space-time surface allow an elegant mathematical realization of SH and bring in the Planck length l_P defining the radius of twistor sphere associated with the twistor space of M^4 in twistor lift of TGD? Note that this can be done without introducing embedding space coordinates as operators so that one avoids the problems with general coordinate invariance. Note also that the non-uniqueness would not be a problem as in quantization since it would correspond to the dynamics of 2-D surfaces.

The analog of brane hierarchy at fundamental level or from SH?

The analog of brane hierarchy for the localization of spinors - space-time surfaces; string world sheets and partonic 2-surfaces; boundaries of string world sheets - is suggestive (note however that SH does not favour it). Could this hierarchy correspond to a hierarchy of Lagrangian sub-manifolds of space-time in the sense that $J(M^4) + J(CP_2) = 0$ is true at them? Boundaries of string world sheets would be trivially Lagrangian manifolds. String world sheets allowing spinor modes should have $J(M^4)+J(CP_2) = 0$ at them. The vanishing of induced W boson fields is needed to guarantee well-defined em charge at string world sheets and that also this condition allow also 4-D solutions besides 2-D generic solutions. As already found, for the physically favoured Option II the more plausible option is $J(M^4) \cdot J(M^4) - 2 \sim 0$ for space-time regions with small cosmological constant. Despite this one can discuss this idea.

This condition is physically obvious but mathematically not well-understood: could the condition $J(M^4) + J(CP_2) = 0$ force the vanishing of induced W boson fields? Lagrangian cosmic string type minimal surfaces $X^2 \times Y^2$ would allow 4-D spinor modes. If the light-like 3-surface defining boundary between Minkowskian and Euclidian space-time regions is Lagrangian surface, the total induced Kähler form Chern-Simons term would vanish. The 4-D canonical momentum currents would however have non-vanishing normal component at these surfaces. I have considered the possibility that TGD counterparts of space-time super-symmetries could be interpreted as addition of higher-D right-handed neutrino modes to the 1-fermion states assigned with the boundaries of string world sheets [K84].

Induced spinor fields at string world sheets could obey the "dynamics of avoidance" in the sense that *both* the induced weak gauge fields W, Z^0 and induced Kähler form (to achieve this U(1) gauge potential must be sum of M^4 and CP_2 parts) would vanish for the regions carrying induced spinor fields. They would couple only to the *induced em field (!)* given by the R_{12} part of CP_2 spinor curvature [L4] for $D = 2, 4$. For $D = 1$ at boundaries of string world sheets the coupling to gauge potentials would be non-trivial since gauge potentials need *not* vanish there. Spinorial dynamics would be extremely simple and would conform with the vision about symmetry breaking of electro-weak group to electromagnetic gauge group.

It seems relatively easy to construct am infinite family of Lagrangian string world sheets satisfying $J(M^4) + J(CP_2) = 0$ using generalized symplectic transformations of M^4 and CP_2 as Hamiltonian flows to generate new ones from a given Lagrangian string world sheets. One must pose minimal surface property as a separate condition. Consider a piece of M^2 with coordinates (t, z) and homologically non-trivial geodesic sphere S^2 of CP_2 with coordinates $(u = cos(\Theta), \Phi)$. One has $J(M^4)_{tz} = 1$ and $J_{u\Phi} = 1$. Identify string world sheet via map $(u, \Phi) = (kz, \omega t)$ from M^2 to S^2. The induced CP_2 Kahler form is $J(CP_2)_{tz} = k\omega$. $k\omega = -1$ guarantees $J(M^4)+J(CP_2) = 0$. The strings have necessarily finite length from $L = 1/k \leq z \leq L$. One can perform symplectic transformations of CP_2 and symplectic transformations of M^4 to obtain new string world sheets. In general these are not minimal surfaces and this condition would select some preferred string world sheets.

Number theoretic vision about the analog of brane hierarchy

An alternative - but of course not necessarily equivalent - attempt to formulate SH would be in terms of number theoretic vision. Space-time surfaces would be associative or co-associative depending on whether tangent space or normal space in embedding space is associative - that is quaternionic. These two conditions would reduce space-time dynamics to associativity and commutativity conditions. String world sheets and partonic 2-surfaces would correspond to maximal commutative or co-commutative sub-manifolds of embedding space. Commutativity (co-commutativity) would mean that tangent space (normal space as a sub-manifold of space-time surface) has complex tangent space at each point and that these tangent spaces integrate to 2-surface. SH would mean that data at these 2-surfaces plus number theoretic discretization of space-time surface would be enough to construct quantum states. Therefore SH would be thus slightly broken. String world sheet boundaries would in turn correspond to real curves of the complex 2-surfaces intersecting partonic 2-surfaces at points so that the hierarchy of classical number fields would have nice realization at the level of the classical dynamics of quantum TGD.

To sum up, one cannot exclude the possibility that $J(M^4)$ is present implying a universal transversal localization of embedding space spinor harmonics and the modes of spinor fields in the interior of X^4: this could perhaps relate to somewhat mysterious de-coherence interaction producing locality and to CP breaking and matter-antimatter asymmetry. The moduli space for M^4 Kähler structures proposed by number theoretic considerations would save from the loss of Poincare invariance and the number theoretic vision based on quaternionic and octonionic structure would have rather concrete realization. This moduli space would only extend the notion of WCW.

Chapter 5

Some Questions Related to the Twistor Lift of TGD

5.1 Introduction

During last couple years (I am writing this in the beginning of 2017) a kind of palace revolution has taken place in the formulation and interpretation of TGD. The notion of twistor lift and 8-D generalization of twistorialization have dramatically simplified and also modified the view about what classical TGD and quantum TGD are.

The notion of adelic physics suggests the interpretation of scattering diagrams as representations of algebraic computations with diagrams producing the same output from given input are equivalent. The simplest possible way to perform the computation corresponds to a tree diagram [L22]. As will be found, it is now possible to even propose explicit twistorial formulas for scattering formulas since the horrible problems related to the integration over WCW might be circumvented altogether.

From the interpretation of p-adic physics as physics of cognition, $h_{eff}/h = n$ could be interpreted dimension of extension dividing the the order of its Galois group. Discrete coupling constant evolution would correspond to phase transitions changing the extension of rationals and its Galois group. TGD inspired theory of consciousness is an essential part of TGD and the crucial Negentropy Maximization Principle in statistical sense follows from number theoretic evolution as increase of the order of Galois group for extension of rationals defining adeles.

In the sequel I consider the questions related to both classical and quantum aspects of twistorialization.

5.1.1 Questions related to the classical aspects of twistorialization

Classical aspects are related to the twistor lift of classical TGD replacing space-time surfaces with their twistor spaces realized as extremals of 6-D analog of Kähler action in the product $T(M^4) \times T(CP_2)$ of twistor space of M^4 and CP_2 such that twistor structure is induced. The outcome is 4-D Kähler action with volume term having interpretation in terms of cosmological constant. Hence the twistorialization has profound physical content rather than being mere alternative formulation for TGD.

1. What does the induction of the twistor structure really mean? What is meant with twistor space. For instance, is the twistor sphere for M^4 time-like or space-like. The induction procedure involves dimensional reduction forced by the condition that the projection of the sum of Kähler forms for the twistor spaces $T(M^4)$ and $T(CP_2)$ gives Kähler form for the twistor sphere of X^4. Better understanding of the details is required.
2. Can the analog of Kähler form $J(M^4)$ assignable to M^4 suggested by the symmetry between M^4 and CP_2 and by number theoretical vision appear in the theory? What would be the physical implications?

   The basic objection is the loss of Poincare invariance. This can be however avoided by introducing the moduli space for Kähler forms. This moduli space is actually the moduli space of causal diamonds (CDs) forced in any case by zero energy ontology (ZEO) and playing central role in the generalization of quantum measurement theory to a theory of consciousness and in the explanation of the relationship between geometric and subjective time [K57].

Why $J(M^4)$ would be needed? $J(M^4)$ corresponds to parallel constant electric and magnetic fields in given direction. Constant E and $B = E$ fix directions of quantization axes for energy (rest system) and spin. One implication is transversal localization of embedding space spinor modes: embedding space spinor modes are products of harmonic oscillator Gaussians in transversal degrees of freedom very much like quarks inside hadrons.

Also CP breaking is implied by the electric field and the question is whether this could explain the observed CP breaking as appearing already at the level of embedding space $M^4 \times CP_2$. The estimate for the mass splitting of neutral kaon and anti-kaon is of correct order of magnitude.

Whether stationary spherically symmetric metric as minimal surface allows a sensible physical generalization is a killer test for the hypothesis that $J(M^4)$ is covariantly constant. The question is basically about how large the moduli space of forms $J(M^4)$ can be allowed to be. The mere self duality and closedness condition outside the line connecting the tips of CD allows also variants which are spherically symmetric in either Minkowski coorinates or Robertson-Walker coordinates for light-cone.

3. How does gravitational coupling emerge at fundamental level? The first naive guess is obvious: string area action is scaled by $1/G$ as in string models. The objection is that p-adic mass calculations suggest that string tension is determined by CP_2 size R: the analog of string tension appearing in mass formula given by p-adic mass calculations would be by a factor about 10^{-8} smaller than that estimated from string tension. The discrepancy evaporates by noticing that p-adic mass calculations rely on p-adic thermodynamics at embedding space level whereas string world sheets appear at space-time level. Furthermore, if the action assignable to string world sheets is effective action expressing 4-D action in 2-D form as strong form of holography (SH) suggests string tension is expected to be function of the parameters appearing in the 4-D action.

4. Could one regard the localization of spinor modes to string world sheets as a localization to Lagrangian sub-manifolds of space-time surface having by definition vanishing induced Kähler form: $J(M^4) + J(CP_2) = 0$. Lagrangian sub-manifolds would be commutative in the sense of Poisson bracket? Could string world sheets be minimal surfaces satisfying $J(M^4) + J(CP_2) = 0$. The Lagrangian condition allows also more general solutions - even 4-D space-time surfaces and one obtains analog of brane hierarchy. Could one allow spinor modes also at these analogs of branes. Is Lagrangian condition equivalent with the original condition that induced W boson fields making the em charge of induced spinor modes ill-defined vanish and allowing also solution with other dimensions. How Lagrangian property relates to the idea that string world sheets correspond to complex (commutative) surfaces of quaternionic space-time surface in octonionic embedding space.

During the re-processing of the details related to twistor lift, it became clear that the earlier variant for the twistor lift [L24] contained an error. This led to much simpler view about twistor lift, to the conclusion that minimal surface extremals of Kähler action represent only asymptotic situation (external particles in scattering), and also to a re-interpretation for the p-adic evolution of the cosmological constant.

5.1.2 Questions related to the quantum aspects of twistorialization

Also the questions related to the quantum aspects of twistorialization of TGD are discussed.

1. There are several notions of twistor. Twistor space for M^4 is $T(M^4) = M^4 \times S^2$ [B64] (see `http://arxiv.org/pdf/1308.2820.pdf`) having projections to both M^4 and to the standard twistor space $T_1(M^4)$ often identified as CP_3. $T(M^4) = M^4 \times S^2$ is necessary for the twistor lift of space-time dynamics. CP_2 gives the factor $T(CP_2) = SU(3)/U(1) \times U(1)$ to the classical twistor space $T(H)$. The quantal twistor space $T(M^8) = T_1(M^4) \times T(CP_2)$ assignable to momenta. The possible way out is $M^8 - H$ duality relating the momentum space M^8 (isomorphic to the tangent space H) and H by mapping space-time associative and co-associative surfaces in M^8 to the surfaces which correspond to the base spaces of in H:

they construction would reduce to holomorphy in complete analogy with the original idea of Penrose in the case of massless fields.

2. The standard twistor approach has problems. Twistor Fourier transform reduces to ordinary Fourier transform only in signature (2,2) for Minkowski space: in this case twistor space is real RP_3 but can be complexified to CP_3. Otherwise the transform requires residue integral to define the transform (in fact, p-adically multiple residue calculus could provide a nice way to define integrals and could make sense even at space-time level making possible to define action).

   Also the positive Grassmannian requires (2,2) signature. In M^8-H relies on the existence of the decomposition $M^2 \subset M^2 = M^2 \times E^2 \subset M^8$. M^2 could even depend on position but $M^2(x)$ should define an integrable distribution. There always exists a preferred M^2, call it M_0^2, where 8-momentum reduces to light-like M^2 momentum. Hence one can apply 2-D variant of twistor approach. Now the signature is (1,1) and spinor basis can be chosen to be real! Twistor space is RP_3 allowing complexification to CP_3 if light-like complex momenta are allowed as classical TGD suggests!

3. A further problem of the standard twistor approach is that in M^4 twistor approach does not work for massive particles. In TGD all particles are massless in 8-D sense. In M^8 M^4-mass squared corresponds to transversal momentum squared coming from $E^4 \subset M^4 \times E^4$ (from CP_2 in H). In particular, Dirac action cannot contain anyo mass term since it would break chiral invariance.

   Furthermore, the ordinary twistor amplitudes are holomorphic functions of the helicity spinors λ_i and have no dependence on $\tilde{\lambda}_i$: no information about particle masses! Only the momentum conserving delta function gives the dependence on masses. These amplitudes would define as such the M^4 parts of twistor amplitudes for particles massive in TGD sense. The simplest 4-fermion amplitude is unique.

Twistor approach gives excellent hopes about the construction of the scattering amplitudes in ZEO. The construction would split into two pieces corresponding to the orbital degrees of freedom in "world of classical worlds" (WCW) and to spin degrees of freedom in WCW: that is spinors, which correspond to second quantized induced spinor fields at space-time surface (actually string world sheets- either at fundamental level or for effective action implied by strong form of holography (SH)).

1. At WCW level there is a perturbative functional integral over small deformations of the 3-surface to which space-time surface is associated. The strongest assumption is that this 3-surface corresponds to maximum for the real part of action and to a stationary phase for its imaginary part: minimal surface extremal of Kähler action would be in question. A more general but number theoretically problematic option is that an extremal for the sum of Kähler action and volume term is in question.

   By Kähler geometry of WCW the functional integral reduces to a sum over contributions from preferred extremals with the fermionic scattering amplitude multiplied by the ration X_i/X, where $X = \sum_i X_i$ is the sum of the action exponentials for the maxima. The ratios of exponents are however number theoretically problematic.

   Number theoretical universality is satisfied if one assigns to each maximum independent zero energy states: with this assumption $\sum X_i$ reduces to single X_i and the dependence on action exponentials becomes trivial! ZEO allow this. The dependence on coupling parameters of the action essential for the discretized coupling constant evolution is only via boundary conditions at the ends of the space-time surface at the boundaries of CD.

   Quantum criticality of TGD [**?**, K80, K104] demands that the sum over loops associated with the functional integral over WCW vanishes and strong form of holography (SH) suggests that the integral over 4-surfaces reduces to that over string world sheets and partonic 2-surfaces corresponding to preferred extremals for which the WCW coordinates parametrizing them belong to the extension of rationals defining the adele [L41]. Also the intersections of the real and various p-adic space-time surfaces belong to this extension.

2. Second piece corresponds to the construction of twistor amplitude from fundamental 4-fermion amplitudes. The diagrams consists of networks of light-like orbits of partonic two surfaces,

whose union with the 3-surfaces at the ends of CD is connected and defines a boundary condition for preferred extremals and at the same time the topological scattering diagram.

Fermionic lines correspond to boundaries of string world sheets. Fermion scattering at partonic 2-surfaces at which 3 partonic orbits meet are analogs of 3-vertices in the sense of Feynman and fermions scatter classically. There is no local 4-vertex. This scattering is assumed to be described by simplest 4-fermion twistor diagram. These can be fused to form more complex diagrams. Fermionic lines runs along the partonic orbits defining the topological diagram.

3. Number theoretic universality [K104] suggests that scattering amplitudes have interpretation as representations for computations. All space-time surfaces giving rise to the same computation wold be equivalent and tree diagrams corresponds to the simplest computation. If the action exponentials do not appear in the amplitudes as weights this could make sense but would require huge symmetry based on two moves. One could glide the 4-vertex at the end of internal fermion line along the fermion line so that one would eventually get the analog of self energy loop, which should allow snipping away. An argument is developed stating that this symmetry is possible if the preferred M_0^2 for which 8-D momentum reduces to light-like M^2-momentum having unique direction is same along entire fermion line, which can wander along the topological graph.

   The vanishing of topological loops would correspond to the closedness of the diagrams in what might be called BCFW homology. Boundary operation involves removal of BCFW bridge and entangled removal of fermion pair. The latter operation forces loops. There would be no BCFW bridges and entangled removal should give zero. Indeed, applied to the proposed four fermion vertex entangled removal forces it to correspond to forward scattering for which the proposed twistor amplitude vanishes.

To sum up, the twistorial approach leads to a proposal for an explicit construction of scattering amplitudes for the fundamental fermions. Bosons and fermions as elementary particles are bound states of fundamental fermions assignable to pairs of wormhole contacts carrying fundamental fermions at the throats. Clearly, this description is analogous to a quark level description of hadron. Yangian symmetry with multilocal generators is expected to crucial for the construction of the many-fermion states giving rise to elementary particles. The problems of the standard twistor approach find a nice solution in terms of M^8-H duality, 8-D masslessness, and holomorphy of twistor amplitudes in λ_i and their indepence on $\tilde{\lambda}_i$.

5.2 More details about the induction of twistor structure

The notion of twistor lift of TGD [L22] [L47] has turned out to have powerful implications concerning the understanding of the relationship of TGD to general relativity. The meaning of the twistor lift really has remained somewhat obscure. There are several questions to be answered. What does one mean with twistor space? What does the induction of twistor structure of $H=M^4\times CP_2$ to that of space-time surface realized as its twistor space mean?

5.2.1 What does one mean with twistor space?

The notion of twistor space has been discussed in [L22] from TGD point of view.

1. In the case of twistor space of M^4 the starting point of Penrose was the isomorphism between the conformal group of Spin(4,2) of 6-D Minkowski space $M^{4,2}$ and the group SU(2,2) acting on 2+2 complex spinors.

   6-D twistor space could be identified as 6-D coset space $SU(2,2)/SU(2,1)\times U(1)$. For E^6 this would give projective space $CP_3=SU(4)/SU(3)\times U(1)$ and in twistor Grassmann approach this definition is indeed used. It is thought that the problems caused by Euclidization are not serious.

2. One can think $SU(2,2)$ as 4×4 complex matrices with orthogonal complex row vector $Z_i=(Z_{i1},...,Z_{i4})$, and norms $(1,1,-1-1)$ in the metric $s^2=\sum\epsilon_i|z_i|^2$, $\epsilon_i\leftrightarrow(1,1,-1,-1)$. The sub-matrices defined by (Z_{k2},Z_{k3},Z_{k4}), $k=2,3,4$, can be regarded apart from normalization

elements of $SU(1,2)$. The column vector with components Z_{i1} with $Z_{11} = \sqrt{1+\rho^2}$, $\rho^2 = |Z_{21}|^2 - |Z_{31}|^2 - |Z_{41}|^2$ corresponds to a point of the twistor space. The S^2 fiber for given values of ρ and (Z_{31}, Z_{41}) could be identified as the space spanned by the values of Z_{21}. Note that S^2 would have time-like signature and the signature of twistor space would be (3,3), which conforms with the existence of complex structure. There would be dimensional democracy at this level.

3. The identification of 4-D base of the twistor space is unclear to me. The base space of the this twistor space should correspond to the conformal compactification M^4_c of M^4 having metric defined only apart from conformal scaling. The concrete realization M^4_c would be in terms of $M^{4,2}$ light-cone with points projectively identified. As a metric object this space is ill-defined and can appear only at the level of scattering amplitudes in conformally invariant quantum field theories in M^4.

4. Mathematicians define also a second variant of twistor space with S^2 fiber and this space is just $M^4 \times S^2$ [B64] (see `http://tinyurl.com/yb4bt74l`). This space has a well-defined metric and seems to be the only possible one for the twistor lift of classical TGD replacing space-time surfaces with their twistor spaces. Whether the signature of S^2 is time-like or space-like has remained an open question but time-like signature looks natural. The radius R_P of S^2 has been proposed to be apart from a numerical constant equal to Planck length l_P. Note that the isometry group is 9-D $SO(3,1) \times SU(2)$ rather than 15-D $SU(2,2)$. In TGD light-likeness in 8-D sense replaces light-likeness in 4-D sense: does this somehow replace the conformal symmetry group $SO(4,2)$ with $SO(3,1) \times SO(3)$? Could $SU(2)$ rotate the direction of spin quantization axis.

I must confess that I have found the notions of twistor and twistor sphere very difficult to understand. Perhaps this is not solely due to my restricted mathematical skills. Also the physics of twistors looks confusing to me.

The twistor space assignable to Minkowski space and corresponding twistor sphere have several meanings. Consider first the situation in standard framework.

1. One can define twistor space as complex 8-D space C^4. Given four-momentum corresponds however to projective line so that one can argue that twistor space is 6-D space $T_1(M^4) = CP_3 = SU(4)/SU(3) \times U(1)$ of projective lines of C^4 in C^4. One could also argue that one must take the signature of Minkowski space into account. $SU(2,2)$ acts as symmetries of twistor bilinear form and one would have $T_1(M^4) = SU(2,2)/SU(2,1) \times U(1)$. In this case twistor sphere could correspond to the projective line in C^4.

2. Incidence relations $\mu^{\dot{a}} = m^{a\dot{a}}\lambda_a$ relate M^4 points to those of twistor space. In the usual twistor formalism twistor sphere corresponds to the projective line of 8-D C^4. When m is not light-like, it corresponds to a matrix which is invertible and one can solve μ from λ and vice versa. The twistor spheres associated with m_1 and m_2 are said to intersect if $m_1 - m_2$ is a complex light-like vector defining a complexified light ray. One could identify twistor sphere of $T_1(M^4)$ as the Riemann sphere defined by these complex points and going to CP_3 one actually eliminates it altogether, which is somewhat unsatisfactory.

3. When m is light-like and thus expressible as $\mu = \lambda \otimes \tilde{\lambda}$ one has $\mu = \mu_0 + t\tilde{\lambda}$, t a complex number. One can say that one has a full Riemann sphere S^2 of solutions. There is also additional degeneracy due to the scaling of both λ and μ. For light-like M^4 points (say momenta) one obtains a Riemann sphere in 6-D twistor space. Which twistor sphere is the correct one: the sphere associated with all points of M^4 and 8-D twistor space or the sphere associated with light-like points of M^4 and 6-D twistor space?

Consider now the situation in TGD.

1. For the twistor lift of Kähler action lifting the dynamics of space-time surfaces to the dynamics of their twistor spaces, the twistor lift of M^4 corresponds to $T(M^4) = M^4 \times CP_2$. This might look strange but the proper mathematical definition of twistor space relies on double fibration involving both views about twistor space discussed in [B64] (see `http://tinyurl.com/yb4bt74l`). This double fibration would be crucially involved with $M^8 - H$ duality. The fiber space is $T(M^4) = M^4 \times CP_1$, where CP_1 corresponds to the projective sphere assignable

to complex spinors λ. This fiber is trivially projected both to M^4 and less trivially to a subset of 6-dimensional complex projective space $T_!(M^4) = CP_3$.

At space-time level $T(M^4)$ is the only correct choice since twistor space must have isometries of M^4. This choices brings into the dynamics Planck length essentially as the radius of S^2 and cosmological constant as volume term resulting in the dimensional reduction of 6-D Kähler action forced by twistor space property of 6-surface.

At the level of momentum space - perhaps the M^8 appearing in $M^8 - H$ duality identifiable as tangent space of H - the twistor space would correspond to twistor space assignable to momentum space and should relate to the ordinary twistor space $T_1(M^4)$ - whatever it is!

2. In M^8 picture the twistor space is naturally associated with preferred $M^2 \subset M^4$, where M^4 is quaternionic space. The moduli space of $M^2 \subset M^4$ for time direction assigned with real octonion, is parametrized by S^2 and a possible interpretation is as twistor sphere of $M^2 \times CP_1$. Interestingly, $M^2 \subset M^4$ is characterized by light-like vector together with its unique dual light-like vector.

   By restricting 4-D conformal invariance to 2-D situation, one finds that the twistor space becomes RP_3 but can be complexified to CP_3 to allowing complexified M^2 momenta. The signature (1,1) of M^2 and reality of spinor basis gives hopes of resolving the conceptual problems of the ordinary twistor approach. For the real spinor spinor pair (λ, μ) the solutions to the co-incidence relations real M^2 spinors but one can allowing their complex multiples.

3. $M^8 - H$ correspondence allows to map M^4 points to each other: this involves a choice of $M^4 \subset M^8$. $M^8 - H$ correspondence maps quaternionic (and co-quaternionic) surfaces in M^8 to preferred extremals of Kähler in H proposed to correspond to the base bases of of twistor bundles $T(X^4) \subset T(M^4) \times T(CP_2)$ constructible using holomorphic maps. One can thus argue that there should be also a correspondence between the twistor spaces $T(M^4)$ and $T_1(M^4)$ - the correspondence between the twistor spheres would be enough.

   The two M^4:s correspond to each other naturally. What is required is a map of twistorial spheres S^2 to each other. Suppose that the twistorial sphere of H corresponds to that assignable to the choice of $M^2 \subset M^8$ by a choice of quaternionic imaginary unit in M^4 of equivalently by a choice of a light-like vector n of M^2 plane. But by incidence relations the light-like vector n has twistor sphere CP_1 as a pre-image in complexified $T_1(M^2) = CP_3$ characterized by the shifts $\mu \to \mu + \tilde{\lambda}$. Therefore the two twistor spheres can be identified by mapping n of $S^2(T(M^4)$ to its counterpart of $T_1(M^2)$ isometrically.

It therefore seems that the double fibration is essential in TGD framework and the usual twistor space is assignable to the M^8 interpreted asthe space of complexified octonion momenta subject to the quaternionicity condition. Sharply defined transversed quaternionic momentum eigenstates in $E^2 \times E^4$ are replaced with wave functions in $T(CP_2)$ reducing locally to $CP_2 \times U(2)/U(1) \times U(1)$ with em charge identifiable as the analog of angular momentum for the wave functions in $CP_1 = U(2)/U(1) \times U(1)$. In $M^4 \times CP_2$ picture one has spinor modes labelled by electroweak quantum numbers.

5.2.2 Twistor lift of TGD

In TGD one replaces embedding space $H = M^4 \times CP_2$ with the product $T = T(M^4) \times T(CP_2)$ of their 6-D twistor spaces, and calls $T(H)$ the twistor space of H. For CP_2 the twistor space is the flag manifold $T(CP_2) = SU(3)/U(1) \times U(1)$ consisting of all possible choices of quantization axis of color isospin and hypercharge.

1. The basic idea is to generalize Penrose's twistor program by lifting the dynamics of space-time surfaces as preferred extremals of Kähler action to those of 6-D Kähler action in twistor space $T(H)$. The conjecture is that field equations reduce to the condition that the twistor structure of space-time surface as 4-manifold is the twistor structure induced from $T(H)$.

   Induction requires that dimensional reduction occurs effectively eliminating twistor fiber $S^2(X^4)$ from the dynamics. Space-time surfaces would be preferred extremals of 4-D Kähler action plus volume term having interpretation in terms of cosmological constant. Twistor lift would be more than an mere alternative formulation of TGD.

2. The reduction would take place as follows. The 6-D twistor space $T(X^4)$ has S^2 as fiber and can be expressed locally as a Cartesian product of 4-D region of space-time and of S^2. The signature of the induced metric of S^2 should be space-like or time-like depending on whether the space-time region is Euclidian or Minkowskian. This suggests that the twistor sphere of M^4 is time-like as also standard picture suggests.
3. Twistor structure of space-time surface is induced to the allowed 6-D surfaces of $T(H)$, which as twistor spaces $T(X^4)$ must have fiber space structure with S^2 as fiber and space-time surface X^4 as base. The Kähler form of $T(H)$ expressible as a direct sum

$$J(T(H)) = J(T(M^4)) \oplus J(T(CP_2))$$

induces as its projection the analog of Kähler form in the region of $T(X^4)$ considered.

There are physical motivations (CP breaking, matter antimatter symmetry, the well-definedness of em charge) to consider the possibility that also M^4 has a non-trivial symplectic/Kähler form of M^4 obtained as a generalization of ordinary symplectic/Kähler form [L47]. This requires the decomposition $M^4 = M^2 \times E^2$ such that M^2 has hypercomplex structure and E^2 complex structures.

This decomposition might be even local with the tangent spaces $M^2(x)$ and $E^2(x)$ integrating to locally orthogonal 2-surfaces. These decomposition would define what I have called Hamilton-Jacobi structure [K99]. This would give rise to a moduli space of M^4 Kähler forms allowing besides covariantly constant self-dual Kähler forms with decomposition (m^0, m^3) and (m^1, m^2) also more general self-dual closed Kähler forms assignable to integrable local decompositions. One example is spherically symmetric stationary self-dual Kähler form corresponding to the decomposition (m^0, r_M) and (θ, ϕ) suggested by the need to get spherically symmetric minimal surface solutions of field equations. Also the decomposition of Robertson-Walker coordinates to (a, r) and (θ, π) assignable to light-cone M^4_+ can be considered.

The moduli space giving rise to the decomposition of WCW to sectors would be finite-dimensional if the integrable 2-surfaces defined by the decompositions correspond to orbits of subgroups of the isometry group of M^4 or CD. This would allow planes of M^4, and radial half-planes and spheres of M^4 in spherical Minkowski coordinates and of M^4_+ in Robertson-Walker coordinates. These decomposition could relate to the choices of measured quantum numbers inducing symmetry breaking to the subgroups in question. These choices would chose a sector of WCW [K57] and would define quantum counterpart for a choice of quantization axes as distinct from ordinary state function reduction with chosen quantization axes.
4. The induced Kähler form of S^2 fiber of $T(X^4)$ is assumed to reduce to the sum of the induced Kähler forms from S^2 fibers of $T(M^4)$ and $T(CP_2)$. This requires that the projections of the Kähler forms of M^4 and CP_2 to $S^2(X^4)$ are trivial. Also the induced metric is assumed to be direct sum and similar conditions holds true.These conditions are analogous to those occurring in dimensional reduction.

Denote the radii of the spheres associated with M^4 and CP_2 as $R_P = kl_P$ and R and the ratio R_P/R by ϵ. Both the Kähler form and metric are proportional to R_p^2 resp. R^2 and satisfy the defining condition $J_{kr}g^{rs}J_{sl} = -g_{kl}$. This condition is assumed to be true also for the induced Kähler form of $J(S^2(X^4)$.

Let us introduce the following shorthand notations

$$\begin{array}{lll} S_1^2 = S^2(X^4) \ , & S_2^2 = S^2(CP_2) \ , & S_3^2 = S^2(M^4) \ , \\ J_i = \frac{J(S_i^2)}{R^2} \ , & g_i = \frac{g(S_i^2))}{R^2} \ . & \end{array} \tag{5.2.1}$$

This gives the following equations.

$$J_1 = J_2 + \epsilon J_3 \ , \quad g_1 = g_2 + \epsilon g_3 \ , \quad J_1 g_1 J_1 = -g_1 \ . \tag{5.2.2}$$

Projections to $S_1^2 = S^2(X^4)$ are assumed at r.h.s.. The product of the third equation is defined as tensor contraction and involves contravariant form of g.

5.2.3 Solutions to the conditions defining the twistor lift

Consider now solutions to the conditions defining the twistor lift.

1. The simplest solution type corresponds to the situation in which either S^2_2 (S^2_3) equals to S^2_1) and S^2_3 (S^2_2) projection of $T(X^4)$ is single point. In this case the conditions of Eq. are trivially satisfied. These two solutions could correspond to Euclidian and Minkowskian space-time regions. Also the solution for which twistor sphere degenerates to a point must be considered and form $J(M^4) = 0$ this would correspond to the reduction of dimensionally reduced action to Kähler action defining the original variant of TGD. Note that preferred extremals are conjectured to be minimal surfaces extremals of Kähler action always [L20].
2. One can consider also more general solutions. Depending on situation, one can use for $S^2(X^4)$ either the coordinates of S^2_2 or S^2_3. Let us choose S^2_2. One can of course change the roles of the spheres.

   Consider an ansatz for which the projections of J_2 and J_3 to S^2_1 are in constant proportionality to each other. This is guaranteed if the spherical coordinates ($u = cos(\Theta), \Phi$) of S^2_2 and S^2_3 are related by $(u(M^4), \Phi(M^4)) = (u(CP_2), n\Phi(CP_2))$ so that the map between the two spheres has winding number n. With this assumption one has

$$\begin{aligned} J_1 &= (1+\epsilon n)J_2 \ , \\ g_1 &= (1+\epsilon n^2)g_2 \ , \end{aligned} \tag{5.2.3}$$

   The third condition of Eq. 1 equation gives

$$(1+n\epsilon)^2 \quad = \quad (1+n^2\epsilon)^2 \ . \tag{5.2.4}$$

   This in turn gives

$$1+n\epsilon = \delta(1+n^2\epsilon) \ , \quad \delta = \pm 1 \ . \tag{5.2.5}$$

   The only solution for $\delta = +1$ is $n = 0$ or $n = 1$. For $\delta = -1$ there are no solutions.

   One has 3+1 different solutions corresponding to the degenerate solution $(n_1, n_2) = (0, 0)$ and 3 solutions with (n_1, n_2) equal $(1, 0)$, $(0, 1)$ or $(1, 1)$. The conditions are very stringent and it is not clear whether there are any other solutions.
3. The further conditions implying locally direct sum for g and J pose strong restrictions on space-time surfaces. The conjecture that the solutions of these conditions correspond to preferred extremals of 6-D Kähler action leads by dimensional reduction to the conclusion that the 4-D action contains besides 4-D Kähler action also a volume term coming from S^2 Kähler actions and giving rise to cosmological constant.

   What is of special interest is that for the degenerate solution the volume term vanishes, and one has mere 4-D Kähler action with induced Kähler form possibly containing also $J(M^4)$, which leads to a rather sensible cosmology having interpretation as infinite volume limit for causal diamond (CD) inside which space-time surfaces exist. This limit could be appropriate for QFT limit of TGD, which indeed corresponds to infinite-volume limit at which cosmological constant approaches zero.

What could be the physical interpretation of the solutions?

1. Physical intuition suggests that S^2_1 must be space-like for Euclidian signature of space-time region $[(n_1, n_2) = (1, 0)]$ and time-like for Minkowskian signature $[(n_1, n_2) = (0, 1)]$.
2. By quantum classical correspondence one can argue that the non-vanishing of space-time projection of $J(M^4)$ resp. $J(CP_2)$ is necessary to fix local quantization axis of spin resp. weak isospin. If so, then $n_1 = 1/0$ resp. $n_2 = 1/0$ would tell that the projection of $J(CP_2)$ resp. $J(M^2)$ is non-vanishing/vanishes. If both contributions vanish $[(n_1, n_2) = (0, 0)]$ one has generalized Lagrangian 4-surface, which would be vacuum extremal. The products of 2-D

Lagrangian manifolds for M^4 and CP_2 would be vacuum extremals. One can wonder whether there exist 4-surfaces representable as a graph of a map $M^4 \to CP_2$ such that the induced Kähler form vanishes. This picture allows only the embeddings of trivial Robertson-Walker cosmology as vacuum extremal of Kähler action since both M^4 contribution to Kähler action and volume term would be non-vanishing [$(n_1, n_2) = (0, 1)$].

5.2.4 Twistor lift and the reduction of field equations and SH to holomorphy

It has become clear that twistorialization has very nice physical consequences. But what is the deep mathematical reason for twistorialization? Understanding this might allow to gain new insights about construction of scattering amplitudes with space-time surface serving as analogs of twistor diatrams.

Penrose's original motivation for twistorilization was to reduce field equations for massless fields to holomorphy conditions for their lifts to the twistor bundle. Very roughly, one can say that the value of massless field in space-time is determined by the values of the twistor lift of the field over the twistor sphere and helicity of the massless modes reduces to cohomology and the values of conformal weights of the field mode so that the description applies to all spins.

I want to find the general solution of field equations associated with the Kähler action lifted to 6-D Kähler action. Also one would like to understand strong form of holography (SH). In TGD fields in space-time are are replaced with the embedding of space-time as 4-surface to H. Twistor lift imbeds the twistor space of the space-time surface as 6-surface into the product of twistor spaces of M^4 and CP_2. Following Penrose, these embeddings should be holomorphic in some sense.

Twistor lift $T(H)$ means that M^4 and CP_2 are replaced with their 6-D twistor spaces.

1. If S^2 for M^4 has 2 time-like dimensions one has 3+3 dimensions, and one can speak about hyper-complex variants of holomorphic functions with time-like and space-like coordinate paired for all three hypercomplex coordinates. For the Minkowskian regions of the space-time surface X^4 the situation is the same.
2. For $T(CP_2)$ Euclidian signature of twistor sphere guarantees this and one has 3 complex coordinates corresponding to those of S^2 and CP_2. One can also now also pair two real coordinates of S^2 with two coordinates of CP_2 to get two complex coordinates. For the Euclidian regions of the space-time surface the situation is the same.

Consider now what the general solution could look like. Let us continue to use the shorthand notations $S_1^2 = S^2(X^4); S_2^2 = S^2(CP_2); S_3^2 = S^2(M^4)$.

1. Consider first solution of type $(1,0)$ so that coordinates of S_2^2 are constant. One has holomorphy in hypercomplex sense (light-like coordinate $t-z$ and $t+z$ correspond to hypercomplex coordinates).
   (a) The general map $T(X^4)$ to $T(M^4)$ should be holomorphic in hyper-complex sense. S_1^2 is in turn identified with S_3^2 by isometry realized in real coordinates. This could be also seen as holomorphy but with different imaginary unit. One has analytical continuation of the map $S_1^2 \to S_3^2$ to a holomorphic map. Holomorphy might allows to achieve this rather uniquely. The continued coordinates of S_1^2 correspond to the coordinates assignable with the integrable surface defined by $E^2(x)$ for local $M^2(x) \times E^2(x)$ decomposition of the local tangent space of X^4. Similar condition holds true for $T(M^4)$. This leaves only $M^2(x)$ as dynamical degrees of freedom. Therefore one has only one holomorphic function defined by 1-D data at the surface determined by the integrable distribution of $M^2(x)$ remains. The 1-D data could correspond to the boundary of the string world sheet.
   (b) The general map $T(X^4)$ to $T(CP_2)$ cannot satisfy holomorphy in hyper-complex sense. One can however provide the integrable distribution of $E^2(x)$ with complex structure and map it holomorphically to CP_2. The map is defined by 1-D data.
   (c) Altogether, 2-D data determine the map determining space-time surface. These two 1-D data correspond to 2-D data given at string world sheet: one would have SH.
2. What about solutions of type $(0,1)$ making sense in Euclidian region of space-time. One has ordinary holomorphy in CP_2 sector.

(a) The simplest picture is a direct translation of that for Minkowskian regions. The map $S_1^2 \rightarrow S_2^2$ is an isometry regarded as an identification of real coordinates but could be also regarded as holomorphy with different imaginary unit. The real coordinates can be analytically continued to complex coordinates on both sides, and their imaginary parts define coordinates for a distribution of transversal Euclidian spaces $E_2^2(x)$ on X^4 side and $E^2(x)$ on M^4 side. This leaves 1-D data.

(b) What about the map to $T(M^4)$? It is possible to map the integrable distribution $E_2^2(x)$ to the corresponding distribution for $T(M^4)$ holomorphically in the ordinary sense of the word. One has 1-D data. Altogether one has 2-D data and SH and partonic 2-surfaces could carry these data. One has SH again.

3. The above construction works also for the solutions of type $(1,1)$, which might make sense in Euclidian regions of space-time. It is however essential that the spheres S_2^2 and S_3^2 have real coordinates.

SH thus would thus emerge automatically from the twistor lift and holomorphy in the proposed sense.

1. Two possible complex units appear in the process. This suggests a connection with quaternion analytic functions [L22] suggested as an alternative manner to solve the field equations. Space-time surface as associative (quaterionic) or co-associate (co-quaternionic) surface is a further solution ansatz.

   Also the integrable decompositions $M^2(x) \times E^2(x)$ resp. $E_1^2(x) \times E_2^2(x)$ for Minkowskian resp. Euclidian space-time regions are highly suggestive and would correspond to a foliation by string wold sheets and partonic 2-surfaces. This expectation conforms with the number theoretically motivated conjectures [K104].

2. The foliation gives good hopes that the action indeed reduces to an effective action consisting of an area term plus topological magnetic flux term for a suitably chosen stringy 2-surfaces and partonic 2-surfaces. One should understand whether one must choose the string world sheets to be Lagrangian surfaces for the Kähler form including also M^4 term. Minimal surface condition could select the Lagrangian string world sheet, which should also carry vanishing classical W fields in order that spinors modes can be eigenstates of em charge.

   The points representing intersections of string world sheets with partonic 2-surfaces defining punctures would represent positions of fermions at partonic 2-surfaces at the boundaries of CD and these positions should be able to vary. Should one allow also non-Lagrangian string world sheets or does the space-time surface depend on the choice of the punctures carrying fermion number (quantum classical correspondence)?

3. The alternative option is that any choice produces of the preferred 2-surfaces produces the same scattering amplitudes. Does this mean that the string world sheet area is a constant for the foliation - perhaps too strong a condition - or could the topological flux term compensate for the change of the area?

   The selection of string world sheets and partonic 2-surfaces could indeed be also only a gauge choice. I have considered this option earlier and proposed that it reduces to a symmetry identifiable as U(1) gauge symmetry for Kähler function of WCW allowing addition to it of a real part of complex function of WCW complex coordinates to Kähler action. The additional term in the Kähler action would compensate for the change if string world sheet action in SH. For complex Kähler action it could mean the addition of the entire complex function.

A couple of questions remain to be pondered.

1. In TGD the induced spinor structure need not be equivalent with the ordinary spinor structure. For instance, induced gamma matrices are not covariantly constant and spinors are embedding space spinors. Induced spinor structure saves also from problems. Induced spinor structure exists even when standard twistor structure fails to do so. Induced spinor structure is also unique unlike the ordinary spinor structure. A practical example relates to the difficulty of the lattice QCD as thermodynamics with periodic boundary conditions in a box: there are $2^4 = 16$ spinor structures.

   In the same way, there is no need to expect or require that the induced twistor structure reduces to ordinary one: it is enough to require that the S^2 bundle structure implied by the proposed dimensional reduction of 6-D surfaces to S^2 bundles having space-time surface as a base space takes place. This would simplify the construction in an essential manner.

2. Space-time surface can be identified as a section of twistor bundle. For physical reasons this section should not only exist but be global and unique. For general bundles this need not be the case. For non-trivial principal bundles one cannot find any sections. The tangent bundle of sphere does not allow a global everywhere non-vanishing section. Could some additional condition guarantee that the section exists and is unique? In algebraic geometry additional conditions such as holomorphy can fix the global section highly uniquely.

   Now the variational principle reducing the construction to finding of space-time surfaces as an extremal of dimensionally reduced Kähler action guarantees both the existence and uniqueness. This also gives the reason why for the twistor lift of Kähler action: one cannot only assume that the 6-surface equals to ordinary twistor bundle of some 4-surface since in this case the section need not be unique.

5.2.5 What about 2-D objects and fermions?

TGD involves also 2-D objects - partonic 2-surfaces and string world sheets in an essential manner and strong form of holography (SH) states that these objects carry the information about quantum states. This does not mean that the dynamics would reduce to that for string like objects since it is essential that these objects are sub-manifolds of space-time surface. String world sheets carry induced spinor fields and it seems that these are crucial for understanding elementary particles. There are several questions to be answered.

1. Are fermionic fields localized to 2-surfaces? The generalization superconformal symmetry fixing both the bosonic and fermion parts of the action requires that also the interior of space-time carries induced spinor field. Their interpretation is not quite clear: could they perhaps give rise to an additional supersymmetry induced by addition of interior fermions to the state?

   The condition of super-symmetry at the level of action fixes the analog of massless Dirac action uniquely for both string world sheets, partonic 2-surfaces in the interior of causal-diamond (CD), and for the interior of space-time surface. There is an infinite number of conserved super currents associated with the modes of the modified Dirac operator defining fermionic super generators. This leads to quantum classical correspondence stating that the eigenvalues of Cartan generators for the fermionic representations of Noether charges are equal to corresponding classical Noether charges defined by the space-time dynamics.

2. A long-standing question has been whether stringlike objects and partonic 2-surfaces are fundamental dynamical objects or whether they emerge only at the level of effective action. M^8-H duality [L37] suggests answer to this question.

   M^8-H duality states that space-time surfaces M^8 picture are associative in the sense that either tangent or normal space of space-time surface at any point is associative and therefore quaternionic. Number theoretic vision suggests that also 2-D objects are fundamental. Commutative sub-manifolds of space-time surfaces having induced quaternionic structure reducing to commutative (complex) structure are number theoretically very natural. Either the tangent space or normal space of 2-surface can be commutative and this gives rise to string world sheets and partonic 2-surfaces as duals of each other just as space-time surfaces have regions for which either tangent spaces or normal spaces are associative (these correspond to regions of space-time with Minkowskian *resp.* Euclidian signatures of the induced metric).

   Note that the reduction of the theory to mere string theory is not possible since partonic 2-surfaces have commutative normal space (partonic 2-surfaces) as part of the tangent space of space-time surface.

3. What action one should assign with the 2-D objects? The action should be assigned to string world sheets and partonic 2-surfaces representing vertices but the assignent of action with partonic 2-surfaces at the ends of CD does not look natural since they are in the role of initial values. The naïve first guess for the action is as area action. Fermionic action would be fixed uniquely in terms of modified gamma matrices reducing to induced gamma matrices.

Also space-time surfaces in the simplest scenario are minimal surfaces except for a discrete set of singular points at which there is energy transfer between Kähler action and volume term. Something similar should occur also in 2-D case: there must also second part in the action and transfer of Noether changes between the two parts in this set of points.

The singular points have an identification as point-like particles carrying fermion number and located at partonic 2-surfaces at boundaries of causal diamond (CD) or defining topological vertices so that a classical space-time correlates for twistor diagrams emerge.

Since particles in twistor approaches are associated with the ends of string boundaries at the ends of light-like orbits of partonic 2-surfaces at boundaries of causal diamond (CD), the exceptional points for both space-time surface and string world sheets would correspond to the intersections of string world sheets and partonic 2-surfaces defining also topological vertices.

Twistor lift provides a first principle approach to the action assignable to the 2-D surfaces.

1. The simplest possibility is that one has also now a Kähler action but now for 4-D space-time surface in the product of twistor spaces of M^4 and CP_2 dimensionally reduced to Cartesian product of twistor sphere S^2 and 2-D surface. The assignment of action to partonic 2-surface at the boundary of CD does not look feasible. 4-D Kähler action would be dimensionally reduced to 2-D form and area term.

2. Field equations contain two terms coming from the variation with respect to the induced metric and Kähler form respectively. The terms coming from the variation with respect to the metric vanish for minimal surfaces since energy momentum tensor is proportional to the induced metric. The term coming from the variation with respect to the induced Kähler form need not vanish for minimal surfaces unless there are additional conditions.

   The term is of the same form as in 4-D case, which case this term vanishes for holomorphic solutions and also for all known extremals. There are excellent reasons to expect that this is true also in 2-D case. It therefore seems that minimal surfaces are in question except for a discrete set of points as in 4-D case: this conforms with universality forced by quantum criticality stating that Kähler coupling constant disappears from dynamics except in this discrete set of points.

   In accordance with SH, this set of points at which the minimal surface property fails would define also the corresponding points for space-time surface itself. This singularity could mean breakdown of holomorphy, perhaps analogs of poles for analytic functions are in question. One cannot exclude the possibility that the boundaries of string world sheets defining orbits of fundamental fermions are analogous to cuts for holomorphic functions.

3. One might guess that 2-D minimal surfaces in space-time are also minimal surfaces in embedding space since the induction from space-time surface to 2-surface can be also thought of as an induction from embedding space. The variations for minimal surfaces inside space-time surface are more restricted so that this need not be the case. For holomorphic solutions the situation might change. SH in strongest form would therefore suggest that space-time as 4-D surface is determined by fixing the 2-D minimal surfaces in H and finding space-time surface containing them. A weaker condition would force to fix also the normal space of the minimal surface in space-time.

   This space-time surface need not always exist, and one of the key ideas about cognition [K65] is that in p-adic case the possibility of p-adic pseudo-constants allows the existence of p-adic space-time surfaces always but that in real case this is not always the case: what is imaginable is not necessarily realizable.

   At the level of M^8 the condition that the coefficients of a polynomial determining the space-time surface are in a fixed extension of rationals is very powerful requirement and might prevent SH. As a matter fact, SH becomes at the level of M^8 even stronger: discrete set of points naturally identifiable as the set of singular points and thus as poles and zeros of analytic function would determine the space-time surface. If fermion lines correspond to cuts, this super-strong form of SH would weaken. For polynomials considered in [L37] cuts are however not possible and they should be generated in the map from H to $M^4 \times CP_2$ for by allowing analytic functions instead of polynomials: this is quite possible in which case polynomials could define a a hierarchy of resolutions.

5.3 How does the twistorialization at embedding space level emerge?

An objection against twistorialization at embedding space level is that M^4-twistorialization requires 4-D conformal invariance and massless fields. In TGD one has towers of particle with massless particles as the lightest states. The intuitive expectation is that the resolution of the problem is that particles are massless in 8-D sense as also the modes of the embedding space spinor fields are.

To explain the idea, let us select a fixed decomposition $M^8 = M_0^4 \times E_0^4$ and assume that the momenta are complex - for motivations see below.

1. With inspiration coming from M^8-H duality [K91] suppose that for the allowed compositions $M^8 = M^4 \times E^4$ one has $M^4 = M_0^2 \times E^2$ with M_0^2 fixed, and corresponding to real octonionic unit and preferred imaginary unit. Obviously 8-D light-likeness for $M^8 = M_0^4 \times E_0^4$ reduces to 4-D light-likeness for a preferred choice of $M^8 = M^4 \times CP_2$ decomposition.
2. This suggests that in the case of massive M_0^4 momenta one can apply twistorialization to the light-like M^4-momentum and code the information about preferred M^4 by a point of CP_2 and about 8-momentum in $M^8 = M_0^4 \times E_0^4$ by an $SU(3)$ transformation taking M_0^4 to M^4. Pairs of twistors and $SU(3)$ transformations would characterize arbitrary quaternionic 8-momenta. 8-D masslessness gives however 2 additional conditions for the complex 8-momenta probably reducing $SU(3)$ to $SU(3)/U(1) \times U(1)$ - the twistor space of CP_2! This would also solve the basic problem of twistor approach created by the existence of massive particles.

The assumption of complex momenta in previous considerations might raise some worries. The space-time action of TGD is however complex if Kähler coupling strength is complex, and there are reasons to believe that this is the case. Both four-momenta and color quantum numbers - all Noether charges in fact - could be complex. A possible physical interpretation for complex momenta could be in terms of the natural width of states induced by the finite size of CD. Also in twistor Grassmannian approach one encounters complex but light-like four-momenta. Note that complex light-like space-time momenta correspond in general to massive real momenta. It is not clear whether it makes sense to speak about width of color quantum numbers: their reality would give additional constraint. The emergence of M^4 mass in this manner could be involved with the classical description for the emergence of the third helicity.

The observation that octonionic twistors make sense and their restriction to quaternionic twistors produce ordinary M^4 twistors provides an alternative view point to the problem. Also $M^8 - H$ duality proposed to map quaternionic 4-D surfaces in octonionic M^8 to (possibly quaternionic) 4-D surfaces in $M^4 \times CP_2$ is expected to be relevant. The twistor lift of $M^8 - H$ duality would give $T(M^8) - T(H)$ duality.

Twistor Grassmann approach [B29, B22, B20, B34, B36, B15] uses as twistor space the space $T_1(M^4) = SU(2,2)/SU(2,1) \times U(1)$ whereas the twistor lift of classical TGD uses $M^4 \times S^2$. The formulation of the twistor amplitudes in terms of SH using the data assignable to the 2-D surfaces - string world sheets and partonic 2-surfaces perhaps - identified as surfaces in $T(M^4) \times T(CP_2)$ requires the mapping of these twistor spaces to each other - the incidence relations of Penrose indeed realize this map.

5.3.1 $M^8 - H$ duality at space-time level

Twistors emerge as a description of massless particles with spin [B63] but are not needed for spin zero particles. Therefore one can consider first mere momenta.

1. Consider first space-time surfaces of M^8 with Minkowskian signature of the induced metric so that the tangent space is M^4. $M^8 - H$ duality [K91] implies that CP_2 points parameterize quaternionic sub-spaces M^4 of octonions containing fixed $M_0^2 \subset M^4$. Using the decomposition $1 + 1 + 3 + \overline{3}$ of complexified octonions to representations of $SU(3)$, it is easy to see that this space is indeed CP_2. M^4 correspond to the sub-space $1 + 1 + 2$ where 2 is $SU(2) \subset SU(3)$ doublet.

   CP_2 spinor mode would be spinor mode in the space of quaternionic sub-spaces $M^4 \subset M^8$ with $M_0^2 \subset M^4$ with real octonionic unit defining preferred time like direction and imaginary unit defining preferred spin quantization axis. M^8-H duality allows to map quaternionic 4-surfaces of $M^4 \supset M_0^2$ to 4-surfaces in H. The latter could be quaternionic but need not to.

2. For Euclidian signature of the induced metric tangent space is E^4. In this case co-associative surfaces are needed since the above correspondence make sense only if the tangent space corresponds to M^4. For instance, for CP_2 type exremals tangent space corresponds to E^4. M^4 and E^4 change roles. Also now the space of co-associative tangent spaces is CP_2 since co-associative tangent space is the octonionic orthogonal complement of the associative tangent space. One would have Euclidian variant of the associative case.

M^8-H correspondence raises the question whether the octonionic M^8 or $M^4\times CP_2$ represents the level, which deserves to be called fundamental. Or are they just alternative descriptions made possible by the quaternionicity of space-time surface in M^8 and quaternionic momentum space necessitating quaternionicity of the tangent space of X^4? In any case, one should demonstrate that the spectrum of states with $M^4\times E^4$ with quaternionic light-like 8-momenta is equivalent with the spectrum of states for $M^4\times CP_2$

5.3.2 Parametrization of light-like quaternionic 8-momenta in terms of $T(CP_2)$

The following argument shows that the twistor space $T(CP_2)$ emerges naturally from M^8-H correspondence for quaternionic light-like M^8 momenta.

1. Continue to assume a fixed decomposition $M^8 = M_0^4\times E_0^4$, and that for the allowed compositions $M^8 = M^4\times E^4$ one has $M^4 = M_0^2\times E^2$ with M_0^2 fixed. Light-like quaternionic 8-momentum in $M^8 = M_0^4\times E_0^4$ can be reduced to light-like M^4 momentum and vanishing E^4 momentum for some preferred $M^8 = M^4\times E^4$ decomposition.

   One can therefore describe the situation in terms of light-like M^4-momentum and $U(2)$ transformation (as it turns out) mapping this momentum to 8-D momentum in given frame and giving the M_0^4 and E_0^4 momenta. The alternative description is in terms M_0^4 massive momentum and the E_0^4 momentum. The space of light-like complex M^4 momenta with fixed M_0^2 part and non-vanishing E^2 part is given by CP_2 as also the space of quaternionic planes. Given quaternionic plane is in turn characterized by massless M^4-momentum.

2. The description of M^4-massive momentum should be based on twistor associated with the light-like M^4 momentum plus something describing the $SU(3)$ transformation leaving the preferred imaginary unit of M_0^2 un-affected. The transformations leaving unaffected the M^4 part of M^8-momentum coded by the $SU(2)$ doublet 2 of color triplet 3 in the color decomposition of complex 8-momentum $1+1+3+\overline{3}$ but acting on E^4 part $1+\overline{3}$ non-trivially correspond to $U(2)$ subgroup. $U(2)$ element thus codes for the E^4 part of the light-like momentum and $SU(3)$ code for quaternionic 8-momenta, which can be also massive. Massless and complex M^4 momenta are coded by $SU(3)/U(2) = CP_2$ as also the tangent spaces of Minkowskian space-time regions (by M^8-H duality).

   The complexity of particle 8-momenta -and more generally Noether charges - is not in conflict with the hermiticity of quantal Noether charges if total classical and quantal Noether charges are real (and equal by QCC). This would give rise to a kind of confinement condition applying to many-particle states. I have earlier proposed that single particle conformal weights are complex but that conformal confinement holds in the sense that the total conformal weights are real.

3. General complex quaternionic momenta with fixed M^4 part are parameterized by $SU(3)$. Complex light-like 8-momenta satisfy two additional constraints from light-likenes condition, and one expects the reduction of $SU(3)$ to $SU(3)/U(1)\times U(1)$ - the twistor space of CP_2. Therefore the light-like 8-momentum is coded by a twistor assignable to massless M^4-momentum by an point of $SU(3)/U(1)\times U(1)$ giving $T(M^4)\times T(CP_2)$.

By the previous arguments, the inclusion of helicities and electroweak charges gives twistor lift of M^8-H correspondence.

1. In the case of E^4 the helicities would correspond to two $SO(4)$ spins to be mapped to right and left-handed electroweak spins or weak spin and weak charges. Twistor space $T(CP_2)$

gives hopes about a unified description of color - and electro-weak quantum numbers in terms of partial waves in the space $SU(3)/U(1) \times U(1)$ for selections of quantization axes for color quantum numbers.

2. A possible problem relates to the particles massive in M^4 sense having more helicity states than massless particles. How can one describe the presence of additional helicities. Should one introduce the analog of Higgs mechanism providing the missing massless helicities? Quantum view about twistors describes helicity as a quantum number - conformal weight - of a wave function in the twistor sphere S^2. In the case of massive gauge bosons which would require the introduction of zero helicity as a spin 0 wave function in twistor space.
3. One should relate the description in terms of M^8 momenta to the description in terms of $M^4 \times CP_2$ color partial waves massless in 8-D sense. The number of partial waves for given CP_2 mass squared is finite and this should be the case for quaternionic E^4 momenta. How color quantum numbers determining the M^4 mass relate to complex E^4 momenta parameterized by $U(2)$ plus two constraints coming from complex light-likeness. The number of degrees of freedom is 2 for given $U(2)$ orbit and the quantization suggests dramatic reduction in the number of 8-momenta. This strongly suggests that it is only possible to talk about wave functions in the space of allowed E^4 momenta - that is in the twistor space $T(CP_2)$. Fixing the M^4-part of 8-momentum parameterized by a point of CP_2 leaves only a wave function in the fiber S^2.

The discussion leaves some questions to ponder.

1. $M^8 - H$ correspondence raises the question whether the octonionic M^8 or $M^4 \times CP_2$ represents the fundamental level. Or are they just alternative descriptions made possible by the quaternionicity of space-time surface in M^8 and quaternionic momentum space necessitating quaternionicity of the tangent space of X^4?
2. What about more general $SO(1,7)$ transformations? Are they needed? One could consider the possibility that $SO(1,7)$ acts in the moduli space of octonion structures of M^8. If so, then these additional moduli must be included. Otherwise given 8-D momenta have M_0^2 part fixed and orbit of given M^4 momentum is the smaller, the smaller the E^2 part of M^4 momentum is. It reduces to point if M^4 momentum reduces to M_0^2.

5.3.3 A new view about color, color confinement, and twistors

To my humble opinion twistor approach to the scattering amplitudes is plagued by some mathematical problems. Whether this is only my personal problem is not clear.

1. As Witten shows in [B29], the twistor transform is problematic in signature (1,3) for Minkowski space since the the bi-spinor μ playing the role of momentum is complex. Instead of defining the twistor transform as ordinary Fourier integral, one must define it as a residue integral. In signature (2,2) for space-time the problem disappears since the spinors μ can be taken to be real.
2. The twistor Grassmannian approach works also nicely for (2,2) signature, and one ends up with the notion of positive Grassmannians. Could it be that something is wrong with the ordinary view about twistorialization rather than only my understanding of it?
3. For M^4 the twistor space should be non-compact $SU(2,2)/SU(2,1)\times U(1)$ rather than $CP_3 = SU(4)/SU(3) \times U(1)$, which is taken to be. I do not know whether this is only about short-hand notation or a signal about a deeper problem.
4. Twistorilizations does not force SUSY but strongly suggests it. The super-space formalism allows to treat all helicities at the same time and this is very elegant. This however forces Majorana spinors in M^4 and breaks fermion number conservation in $D = 4$. LHC does not support $\mathcal{N} = 1$ SUSY. Could the interpretation of SUSY be somehow wrong? TGD seems to allow broken SUSY but with separate conservation of baryon and lepton numbers.

In number theoretic vision something rather unexpected emerges and I will propose that this unexpected might allow to solve the above problems and even more, to understand color and even color confinement number theoretically. First of all, a new view about color degrees of freedom emerges at the level of M^8.

1. One can always find a decomposition $M^8 = M_0^2 \times E^6$ so that the possibly complex light-like quaternionic 8-momentum restricts to M_0^2. The preferred octonionic imaginary unit represent the direction of imaginary part of quaternionic 8-momentum. The action of G_2 to this momentum is trivial. Number theoretic color disappears with this choice. For instance, this could take place for hadron but not for partons which have transversal momenta.

2. One can consider also the situation in which one has localized the 8-momenta only to $M^4 = M_0^2 \times E^2$. The distribution for the choices of $E^2 \subset M_0^2 \times E^2 = M^4$ is a wave function in CP_2. Octonionic $SU(3)$ partial waves in the space CP_2 for the choices for $M_0^2 \times E^2$ would correspond ot color partial waves in H. The same interpretation is also behind $M^8 - H$ correspondence.

3. The transversal quaternionic light-like momenta in $E^2 \subset M_0^2 \times E^2$ give rise to a wave function in transversal momenta. Intriguingly, the partons in the quark model of hadrons have only precisely defined longitudinal momenta and only the size scale of transversal momenta can be specified. This would of course be a profound and completely unexpected connection! The introduction of twistor sphere of $T(CP_2)$ allows to describe electroweak charges and brings in CP_2 helicity identifiable as em charge giving to the mass squared a contribution proportional to Q_{em}^2 so that one could understand electromagnetic mass splitting geometrically.

   The physically motivated assumption is that string world sheets at which the data determining the modes of induced spinor fields carry vanishing W fields and also vanishing generalized Kähler form $J(M^4)+J(CP_2)$. Em charge is the only remaining electroweak degree of freedom. The identification as the helicity assignable to $T(CP_2)$ twistor sphere is natural.

4. In general case the M^2 component of momentum would be massive and mass would be equal to the mass assignable to the E^6 degrees of freedom. One can however always find $M_0^2 \times E^6$ decomposition in which M^2 momentum is light-like. The naïve expectation is that the twistorialization in terms of M^2 works only if M^2 momentum is light-like, possibly in complex sense. This however allows only forward scattering: this is true for complex M^2 momenta and even in M^4 case.

   The twistorial 4-fermion scattering amplitude is however *holomorphic* in the helicity spinors λ_i and has no dependence on $\tilde{\lambda}_i$. Therefore carries no information about M^2 mass! Could M^2 momenta be allowed to be massive? If so, twistorialization might make sense for massive fermions!

M_0^2 momentum deserves a separate discussion.

1. A sharp localization of 8-momentum to M_0^2 means vanishing E^2 momentum so that the action of $U(2)$ would becomes trivial: electroweak degree of freedom would simply disappear, which is not the same thing as having vanishing em charge (wave function in $T(CP_2)$ twistorial sphere S^2 would be constant). Neither M_0^2 localization nor localization to single M^4 (localization in CP_2) looks plausible physically - consider only the size scale of CP_2. For the generic CP_2 spinors this is impossible but covariantly constant right-handed neutrino spinor mode has no electro-weak quantum numbers: this would most naturally mean constant wave function in CP_2 twistorial sphere.

   For the preferred extremals of twistor lift of TGD either M^4 or CP_2 twistor sphere can effectively collapse to a point. This would mean disappearence of the degrees of freedom associated with M^4 helicity or electroweak quantum numbers.

2. The localization to $M^4 \supset M_0^2$ is possible for the tangent space of quaternionic space-time surface in M^8. This could correlate with the fact that neither leptonic nor quark-like induced spinors carry color as a spin like quantum number. Color would emerge only at the level of H and M^8 as color partial waves in WCW and would require de-localization in the CP_2 cm coordinate for partonic 2-surface. Note that also the integrable local decompositions $M^4 = M^2(x)\times E^2(x)$ suggested by the general solution ansätze for field equations are possible.

3. Could it be possible to perform a measurement localization the state precisely in fixed M_0^2 always so that the complex momentum is light-like but color degrees of freedom disappear? This does not mean that the state corresponds to color singlet wave function! Can one say that the measurement eliminating color degrees of freedom corresponds to color confinement. Note that the subsystems of the system need not be color singlets since their momenta need not be

complex massless momenta in M_0^2. Classically this makes sense in many-sheeted space-time. Colored states would be always partons in color singlet state.

4. At the level of H also leptons carry color partial waves neutralized by Kac-Moody generators, and I have proposed that the pion like bound states of color octet excitations of leptons explain so called lepto-hadrons [K97]. Only right-handed covariantly constant neutrino is an exception as the only color singlet fermionic state carrying vanishing 4-momentum and living in all possible M_0^2:s, and might have a special role as a generator of supersymmetry acting on states in all quaternionic subs-spaces M^4.

5. Actually, already p-adic mass calculations performed for more than two decades ago [K52, K21, K64], forced to seriously consider the possibility that particle momenta correspond to their projections o $M_0^2 \subset M^4$. This choice does not break Poincare invariance if one introduces moduli space for the choices of $M_0^2 \subset M^4$ and the selection of M_0^2 could define quantization axis of energy and spin. If the tips of CD are fixed, they define a preferred time direction assignable to preferred octonionic real unit and the moduli space is just S^2. The analog of twistor space at space-time level could be understood as $T(M^4) = M^4 \times S^2$ and this one must assume since otherwise the induction of metric does not make sense.

What happens to the twistorialization at the level of M^8 if one accepts that only M_0^2 momentum is sharply defined?

1. What happens to the conformal group $SO(4,2)$ and its covering $SU(2,2)$ when M^4 is replaced with $M_0^2 \subset M^8$? Translations and special conformational transformation span both 2 dimensions, boosts and scalings define 1-D groups $SO(1,1)$ and R respectively. Clearly, the group is 6-D group $SO(2,2)$ as one might have guessed. Is this the conformal group acting at the level of M^8 so that conformal symmetry would be broken? One can of course ask whether the 2-D conformal symmetry extends to conformal symmetries characterized by hyper-complex Virasoro algebra.

2. Sigma matrices are by 2-dimensionality real (σ_0 and σ_3 - essentially representations of real and imaginary octonionic units) so that spinors can be chosen to be real. Reality is also crucial in signature $(2,2)$, where standard twistor approach works nicely and leads to 3-D real twistor space.

   Now the twistor space is replaced with the real variant of $SU(2,2)/SU(2,1)\times U(1)$ equal to $SO(2,2)/SO(2,1)$, which is 3-D projective space RP^3 - the real variant of twistor space CP_3, which leads to the notion of positive Grassmannian: whether the complex Grassmannian really allows the analog of positivity is not clear to me. For complex momenta predicted by TGD one can consider the complexification of this space to CP_3 rather than $SU(2,2)/SU(2,1)\times U(1)$. For some reason the possible problems associated with the signature of $SU(2,2)/SU(2,1) \times U(1)$ are not discussed in literature and people talk always about CP_3. Is there a real problem or is this indeed something totally trivial?

3. SUSY is strongly suggested by the twistorial approach. The problem is that this requires Majorana spinors leading to a loss of fermion number conservation. If one has $D = 2$ only effectively, the situation changes. Since spinors in M^2 can be chosen to be real, one can have SUSY in this sense without loss of fermion number conservation! As proposed earlier, covariantly constant right-handed neutrino modes could generate the SUSY but it could be also possible to have SUSY generated by all fermionic helicity states. This SUSY would be however broken.

   There is an delicacy involved. If $J(M^4)$ is present, the action of the gauge commutator $[D_k, D_l] = J_{kl}(M^4)$ on right-handed neutrino is non-vanishing and gives rise to the constant term $J^{kl}(M^4)\Sigma_{kl}$ appearing in the square of Dirac equation at embedding space level. Neutrino would become massive at embedding space level and also other states receive an additional contribution to mass squared. String world sheets can be however analogs of Lagrangian sub-manifolds so that $J(M^4)$ projected to them vanishes, and one can have massless right-handed neutrino. Also the right- or left M^4-handedness of operator $J^{kl}(M^4)\Sigma_{kl}$ makes it possible to annihilate the spinor mode at string world sheet. The physical interpretation of this picture is still unclear.

4. The selection of M_0^2 could correspond at space-time level to a localization of spinor modes to string world sheets. Could the condition that the modes of induced spinors at string world sheets are expressible using real spinor basis imply the localization? Whether this localization takes place at fundamental level or only for effective action being due to SH, is a question to be settled. The latter options looks more plausible.

To sum up, these observation suggest a profound re-evaluqtion of the beliefs related to color degrees of freedom, to color confinement, and to what twistors really are.

5.3.4 How do the two twistor spaces assignable to M^4 relate to each other?

Twistor Grassmann approach [B29, B22, B20, B34, B36, B15] uses as twistor space the space $T_1(M^4) = SU(2,2)/SU(2,1) \times U(1)$. Twistor lift of classical TGD uses $M^4 \times S^2$: this seems to be necessary since $T_1(M^4)$ does not allow M^4 as space-space. The formulation of the twistor amplitudes in terms of SH using the data assignable to the 2-D surfaces - string world sheets and partonic 2-surfaces perhaps - identifed as surfaces in $T(M^4) \times T(CP_2)$ is an attractive idea suggesting a very close correspondence with twistor string theory of Witten and construction of scattering amplitudes in twistor Grassmann approach.

One should be able to relate these two twistor spaces and map the twistor spaces $T(X^4)$ identified as surfaces in $T(H) = T(M^4) \times T(CP_2)$ to those in $T_1(H) = T_1(M^4) \times T(CP_2)$. This map is strongly suggested also by twistor string theory. This map raises hopes about the analogs of twistor Grassmann amplitudes based on introduction of $T(CP_2)$.

At least the projections of 2-surfaces to $T(M^4)$ should be mappable to those in $T_1(M^4)$. A stronger condition is that $T(M^4)$ is mappable to $T_1(M^4)$. Incidence relations for twistors $Z = (\lambda, \mu)$ assigning to given M^4 coordinates twistor sphere, are given by

$$\mu_{\dot{\alpha}} = m_{\alpha\dot{\alpha}}\lambda^{\alpha} \quad .$$

This condition determines a 2-D sub-space - complex light ray - of complexified Minkowski space M^4_c. Also complex scaling of Z determines the same sub-space. Therefore twistor sphere corresponds to a complex light ray M^4_c, whose points differ by a shift by a complex light-like vector (λ is null bi-spinor annihilated by light-like m).

Since twistor line (projective sphere) determines a point of M^4_c, two points of twistor sphere labelled by A and B are needed to determined m:

$$m_{\alpha\dot{\alpha}} = \frac{\lambda_{A,\alpha}\mu_{B,\dot{\alpha}}}{\langle\lambda_A\lambda_B\rangle\rangle} + \frac{\lambda_{B,\alpha}\mu_{A,\dot{\alpha}}}{\langle\lambda_B\lambda_A\rangle} \quad .$$

The solutions are invariant under complex scalings $(\lambda, \mu) \to k(\lambda, \mu)$. Therefore co-incidence relations allow to assign projective line - sphere S^2 - to a point of M^4 in $T(M^4)$. This sphere naturally corresponds to S^2 in $T(M^4) = M^4 \times S^2$. This allows to assign pairs $(m \times S^2)$ in $T(M^4)$ to spheres of $T_1(M^4)$ and one can map the projections of 2-surfaces to $T(M^4)$ to $T_1(M^4)$.

Thus one cannot assign M^4 point to single twistor but can map any pair of points at twistor sphere of $T_1(M^4)$ to the same point of M^4 in $T(M^4) = M^4 \times S^2$ and also identify the twistor sphere with S^2. Twistor spheres are labelled by the base space of $T_1(M^4)$ and therefore base space can be mapped to M^4.

Two M^4 points separated by light-like distance correspond to twistor spheres intersecting at one point as is clear from the fact that the difference $m_1 - m_2$ of the points annihilates the twistor λ. $T_1(M^4)$ is singular as fiber bundle over M^4 since the same point of fiber is projected to two different points of M^4.

Could one replace $T(M^4)$ with $T_1(M^4)$ by modifying the induction procedure suitable?

1. $T_1(M^4) = SU(2,2)/SU(2,1) \times U(1)$ has $SU(2,2)$ invariant metric and $SU(2,2)$ corresponds to the 15-D spin covering group of $SO(4,2)$ having $SO(3,1)$ as sub-group. What does one obtain if one induces the metric of the base space of $T_1(M^4)$ to M^4 via the above identification?

   The induced metric would depend on the choice of the base space, and one would have analog of gauge invariance since for a given point of the base the point of the fiber sphere can be chosen freely. A reasonable guess is that the induced metric is determined apart from conformal scaling. One could fix the gauge by - say - assuming that the S^2 point is constant but it is not clear whether this allows to get the flat M^4 metric with any choice.

2. If the twistor sphere of $T_1(M^4)$ has radius of order Planck length l_P, the overall scaling factor of the metric of $T_1(M^4)$ is of order l_P^2. Also the induced M^4 metric would have this scaling factor. For $T_1(M^4)$ one could not perform this scaling. This need not be a problem in $T(M^4)$ since one scale up the flat metric of M^4 by scaling the coordinates. This kind of scaling would in fact smooth out the possible deviations from flat M^4 metric very effectively. In any case, it seems that one must assume that embedding space corresponds to $T(M^4)$.

5.3.5 How could Planck length be actually equal to much larger CP_2 radius?!

The following argument stating that Planck length l_P equals to CP_2 radius R: $l_P = R$ and Newton's constant can be identified $G = R^2/\hbar_{eff}$. This idea looking non-sensical at first glance was inspired by an FB discussion with Stephen Paul King.

First some background.

1. I believed for long time that Planck length l_P would be CP_2 length scale R squared multiplied by a numerical constant of order $10^{-3.5}$. Quantum criticality would have fixed the value of l_P and therefore $G = l_P^2/\hbar$.

2. Twistor lift of TGD [L10, L24, L45, L58] led to the conclusion that that Planck length l_P is essentially the radius of twistor sphere of M^4 so that in TGD the situation seemed to be settled since l_P would be purely geometric parameter rather than genuine coupling constant. But it is not! One should be able to understand why the ratio l_P/R but here quantum criticality, which should determine only the values of genuine coupling parameters, does not seem to help.

   Remark: M^4 has twistor space as the usual conformal sense with metric determined only apart from a conformal factor and in geometric sense as $M^4 \times S^2$: these two twistor spaces are part of double fibering.

Could CP_2 radius R be the radius of M^4 twistor sphere, and could one say that Planck length l_P is actually equal to R: $l_P = R$? One might get $G = l_P^2/\hbar$ from $G = R^2/\hbar_{eff}$!

1. It is indeed important to notice that one has $G = l_P^2/\hbar$. $\hbar$ is in TGD replaced with a spectrum of $\hbar_{eff} = n\hbar_0$, where $\hbar = 6\hbar_0$ is a good guess [L25, L52]. At flux tubes mediating gravitational interactions one has

$$\hbar_{eff} = \hbar_{gr} = \frac{GMm}{v_0} \quad ,$$

   where v_0 is a parameter with dimensions of velocity. I recently proposed a concrete physical interpretation for v_0 [L50] (see `http://tinyurl.com/yclefxb2`). The value $v_0 = 2^{-12}$ is suggestive on basis of the proposed applications but the parameter can in principle depend on the system considered.

2. Could one consider the possibility that twistor sphere radius for M^4 has CP_2 radius R: $l_P = R$ after all? This would allow to circumvent introduction of Planck length as new fundamental length and would mean a partial return to the original picture. One would $l_P = R$ and $G = R^2/\hbar_{eff}$. $\hbar_{eff}/\hbar$ would be of $10^7 - 10^8$!

The problem is that $\hbar_{eff}$ varies in large limits so that also G would vary. This does not seem to make sense at all. Or does it?!

To get some perspective, consider first the phase transition replacing $\hbar$ and more generally $\hbar_{eff,i}$ with $\hbar_{eff,f} = h_{gr}$.

1. Fine structure constant is what matters in electrodynamics. For a pair of interacting systems with charges Z_1 and Z_2 one has coupling strength $Z_1Z_2e^2/4\pi\hbar = Z_1Z_2\alpha$, $\alpha \simeq 1/137$.

2. As shown in [K85, K70, K71, ?] one can also define gravitational fine structure constant α_{gr}. Only α_{gr} should matter in quantum gravitational scattering amplitudes. α_{gr} wold be given by

$$\alpha_{gr} = \frac{GMm}{4\pi\hbar_{gr}} = \frac{v_0}{4\pi} \ . \tag{5.3.1}$$

$v_0/4\pi$ would appear as a small expansion parameter in the scattering amplitudes. This in fact suggests that v_0 is analogous to α and a universal coupling constant which could however be subject to discrete number theoretic coupling constant evolution.

3. The proposed physical interpretation is that a phase transition $\hbar_{eff,i} \rightarrow \hbar_{eff,f} = h_{gr}$ at the flux tubes mediating gravitational interaction between M and m occurs if the perturbation series in $\alpha_{gr} = GMm/4\pi/\hbar$ fails to converge ($Mm \sim m_{Pl}^2$ is the naïve first guess for this value). Nature would be theoretician friendly and increase h_{eff} and reducing α_{gr} so that perturbation series converges again.

   Number theoretically this means the increase of algebraic complexity as the dimension $n = h_{eff}/h_0$ of the extension of rationals involved increases fron n_i to n_f [L37] and the number n sheets in the covering defined by space-time surfaces increases correspondingly. Also the scale of the sheets would increase by the ratio n_f/n_i.

   This phase transition can also occur for gauge interactions. For electromagnetism the criterion is that $Z_1Z_2\alpha$ is so large that perturbation theory fails. The replacement $\hbar \rightarrow Z_1Z_2e^2/v_0$ makes $v_0/4\pi$ the coupling constant strength. The phase transition could occur for atoms having $Z \geq 137$, which are indeed problematic for Dirac equation. For color interactions the criterion would mean that $v_0/4\pi$ becomes coupling strength of color interactions when α_s is above some critical value. Hadronization would naturally correspond to the emergence of this phase.

   One can raise interesting questions. Is v_0 (presumably depending on the extension of rationals) a completely universal coupling strength characterizing any quantum critical system independent of the interaction making it critical? Can for instance gravitation and electromagnetism are mediated by the same flux tubes? I have assumed that this is not the case. It it could be the case, one could have for $GMm < m_{Pl}^2$ a situation in which effective coupling strength is of form $(GmMm/Z_1Z_2e^2)(v_0/4\pi)$.

The possibility of the proposed phase transition has rather dramatic implications for both quantum and classical gravitation.

1. Consider first quantum gravitation. v_0 does not depend on the value of G at all! The dependence of G on $\hbar_{eff}$ could be therefore allowed and one could have $l_P = R$. At quantum level scattering amplitudes would not depend on G but on v_0. I was of course very happy after having found the small expansion parameter v_0 but did not realize the enormous importance of the independence on G! Quantum gravitation would be like any gauge interaction with dimensionless coupling, which is even small! This might relate closely to the speculated TGD counterpart of AdS/CFT duality between gauge theories and gravitational theories.

2. What about classical gravitation? Here G should appear. What could the proportionality of classical gravitational force on $1/\hbar_{eff}$ mean? The invariance of Newton's equation

$$\frac{d\overline{v}}{dt} = -\frac{GM\overline{r}}{r^3} \tag{5.3.2}$$

   under $h_{eff} \rightarrow xh_{eff}$ would be achieved by scaling $\overline{r} \rightarrow \overline{r}/x$ and $t \rightarrow t/x$. Note that these transformations have general coordinate invariant meaning as scalings of Minkowski coordinates of M^4 in $M^4 \times CP_2$. This scaling means the zooming up of size of space-time sheet by x, which indeed is expected to happen in $h_{eff} \rightarrow xh_{eff}$!

What is so intriguing that this connects to an old problem that I pondered a lot during the period 1980-1990 as I attempted to construct to the field equations for Kähler action approximate spherically symmetric stationary solutions [K99]. The naïve arguments based on the asymptotic behavior of the solution ansatz suggested that the one should have $G = R^2/\hbar$. For a long time indeed assumed $R = l_P$ but p-adic mass calculations [K52] and work with cosmic strings [K25] forced to conclude that this cannot be the case. The mystery was how $G = R^2/\hbar$ could be normalized to $G = l_P^2/\hbar$: the solution of the mystery is $\hbar \rightarrow \hbar_{eff}$ as I have now - decades later - realized!

5.3.6 Can the Kähler form of M^4 appear in Kähler action?

I have already earlier considered the question whether the analog of Kähler form assignable to M^4 could appear in Kähler action. Could one replace the induced Kähler form $J(CP_2)$ with the sum $J = J(M^4) + J(CP_2)$ such that the latter term would give rise to a new component of Kähler form both in space-time interior at the boundaries of string world sheets regarded as point-like particles? This could be done both in the Kähler action for the interior of X^4 and also in the topological magnetic flux term $\int J$ associated with string world sheet and reducing to a boundary term giving couplings to U(1) gauge potentials $A_\mu(CP_2)$ and $A_\mu(M^4)$ associated with $J(CP_2)$ and $J(M^4)$. The interpretation of this coupling is an interesting challenge.

Conditions on $J(M^4)$

What conditions one can pose on $J(M^4)$?

1. The simplest possibility is that $J(M^4)$ is covariantly constant and self-dual and satisfies $J^2(M^4) = -g(M^4)$ meaning that $J(M^4)$ *resp.* $g(M^4)$ represents imaginary *resp.* real unit. Hypercomplexity for M^2 would suggest the restriction $J^2(M^2) = g(M^2)$ and $J^2(E^2) = -g(E^2)$. Since complexified octonions are used, it is convenient to include imaginary unit to $J(M^2)$ so that one indeed obtains $J^2(M^4) = -g(M^4)$. $J(M^4)$ would define a global decomposition $M^4 = M^2 \times E^2$ in terms of parallel constant electric and magnetic fields of equal magnitude. CD with this variant of $J(M^4)$ would be naturally associated with planewave like radiative solutions.
2. One could however give up the covariant constancy. In this case spherically symmetric variants of $J(M^4)$ naturally associated with spherically symmetric stationary metric and possible analogs of Robertson-Walker metrics. $J(M^4)$ would be closed except at the world line connecting the tips of CD and carry identical magnetic and electric charges.
3. $J(M^4)$ would define Hamilton Jacobi-structure and an attractive idea is that the orthogonal 2-surfaces associated with the foliation of M^4 are orbits of a subgroup of Poincare group. This structure would characterize quantum measurement at the level of WCW and quantum measurement would involve selection of a sector of WCW characterized by $J(M^4)$ [K57].

   The most plausible assumption is that $J(M^4)$ is covariantly constant.

Objections against $J(M^4)$

Consider now the objections against introducing $J(M^4)$ to the Kähler action at embedding space level.

1. $J(M^4)$ would would break translational and Lorentz symmetries at the level of embedding space since $J(M^4)$ cannot be Lorentz invariant. For embedding space spinor modes this term would bring in coupling to the self-dual Kähler form in M^4. The simplest choice is $A = (A_t = z, A_z = 0, A_x = y, A_y = 0)$ defining decomposition $M^4 = M^2 \times E^2$. For Dirac equation in M^4 one would have free motion in preferred time-like (t,z)-plane plane M^2 in whereas in x- and y-directions (E^2 plane) would one have harmonic oscillator potentials due to the gauge potentials of electric and magnetic fields. One would have something very similar to quark model of hadron: quark momenta would have conserved longitudinal part and non-conserved transversal part. The solution spectrum has scaling invariance $\Psi(m^k) \rightarrow \Psi(\lambda m^k)$ so that there is no preferred scale and the transversal scales scale as $1/E$ and $1/k_x$.
2. Since $J(M^4)$ is not Lorentz invariant, Lorentz boosts would produce new $M^2 \times E^2$ decomposition (or its local variant). If one assumes above kind of linear gauge as gauge invariance suggests, the choices with fixed second tip of causal diamond (CD) define finite-dimensional

moduli space $SO(3,1)/SO(1,1) \times SO(2)$ having in number theoretic vision an interpretation as a choice of preferred hypercomplex plane and its orthogonal complement. This is the moduli space for hypercomplex structures in M^4 with the choices of origins parameterized by M^4. The introduction of the moduli space would allow to preserve Poincare invariance.

3. If one generalizes the condition for Kähler metric to $J^2(M^4) = -g(M^4)$ fixing the scaling of J, the coupling to $A(M^4)$ is also large and suggests problems with the large breaking of Poincare symmetry for the spinor modes of the embedding space for given moduli. The transversal localization by the self-dual magnetic and electric fields for $J(M^4)$ would produce wave packets in transversal degrees of freedom: is this physical?

   This moduli space is actually the moduli space introduced for causal diamonds (CDs) in zero energy ontology (ZEO) forced by the finite value of volume action: fixing of the line connecting the tips of CD the Lorentz boost fixing the position for the second tip of CD parametrizes this moduli space apart from division with the group of transformations leaving the planes M^2 and E^2 having interpretation a plane defined by light-like momentum and polarization plane associated with a given CD invariant.

4. Why this kind of symmetry breaking for Poincare invariance? A possible explanation proposed already earlier is that quantum measurement involves a selection of quantization axis. This choice necessarily breaks the symmetries and $J(M^4)$ would be an embedding space correlate for the selection of rest frame and quantization axis of spin. This conforms with the fact that CD is interpreted as the perceptive field of conscious entity at embedding space level: the contents of consciousness would be determined by the superposition of space-time surfaces inside CD. The choice of $J(M^4)$ for CD would select preferred rest system (quantization axis for energy as a line connecting tips of CD) via electric part of $J(M^4)$ and quantization axis of spin (via magnetic part of $J(M^4)$. The moduli space for CDs would be the space for choices of these particular quantization axis and in each state function reduction would mean a localization in this moduli space. Clearly, this reduction would be higher level reduction and correspond to a decision of experimenter.

To summarize, for $J(M^4) = 0$ Poincare symmetries are realized at the level of embedding space but obviously broken slightly by the geometry of CD. The allowance of $J(M^4) \neq 0$ implies that both translational and rotational symmetries are reduced for a given CD: the interpretation would be in terms of a choice of quantization axis in state function reduction. They are however lifted to the level of moduli space of CDs and exact in this more abstract sense. This is nothing new: already the introduction of ZEO and CDs force by volume term in action forced by twistor lift of TGD implies the same. Also the view about state function reduction requires wave functions in the moduli space of CDs. This is also essential for understanding how the arrow of geometric time is inherited from that of subjective time in TGD inspired theory of consciousness [K7, K49].

Situation at space-time level

What about the situation at space-time level?

1. The introduction of $J(M^4)$ part to Kähler action has nice number theoretic aspects. In particular, J selects the preferred complex and quaternionic sub-space of octonionic space of embedding space. The simplest possibility is that the Kähler action is defined by the Kähler form $J(M^4) + J(CP_2)$.

   Since M^4 and CP_2 Kähler geometries decouple it should be possible to take the counterpart of Kähler coupling strength in M^4 to be much larger than in CP_2 degrees of freedom so that M^4 Kähler action is a small perturbation and slowly varying as a functional of preferred extremal. This option is however not in accordance with the idea that entire Kähler form is induced.

2. Whether the proposed ansätze for general solutions make still sense is not clear. In particular, can one still assume that preferred extremals are minimal surfaces? Number theoretical vision strongly suggests - one could even say demands - the effective decoupling of Kähler action and volume term. This would imply the universality of quantum critical dynamics. The solutions would not depend at all on the coupling parameters except through the dependence on boundary conditions. The coupling between the dynamics of Kähler action and volume

term would come also from the conservation conditions at light-like 3-surfaces at which the signature of the induced metric changes.

3. At space-time level the field equations get more complex if the M^4 projection has dimension $D(M^4) > 2$ and also for $D(M^4) = 2$ if it carries non-vanishing induced $J(M^4)$. One would obtain cosmic strings of form $X^2 \times Y^2$ as minimal surface extremals of ordinary Kähler action or X^2 Lagrangian manifold of M^4 as also CP_2 type vacuum extremals and their deformations with M^4 projection Lagrangian manifold. Thus the differences would not be seen for elementary particle and string like objects. Simplest string worlds sheet for which $J(M^4)$ vanishes would correspond to a piece of plane M^2.

   M^4 is the simplest minimal surface extremal of Kähler action necessarily involving also $J(M^4)$. The action in this case vanishes identically by self-duality (in Euclidian signature self-duality does not imply this). For perturbations of M^4 such as spherically symmetric stationary metric the contribution of M^4 Kähler term to the action is expected to be small and the come mainly from cross term mostly and be proportional to the deviation from flat metric. The interpretation in terms of gravitational contribution from M^4 degrees of freedom could make sense.

4. What about massless extremals (MEs)? How the induced metric affects the situation and what properties second fundamental form has? Is it possible to obtain a situation in which the energy momentum tensor T^α and second fundamental form $H^k_{\alpha\beta}$ have in common components which are proportional to light-like vector so that the contraction $T^{\alpha\beta}H^k_{\alpha\beta}$ vanishes?

   Minimal surface property would help to satisfy the conditions. By conformal invariance one would expect that the total Kähler action vanishes and that one has $J^\alpha_{\ \gamma}J^{\gamma\beta} \propto ag^{\alpha\beta} + bk^\alpha k^\beta$. These conditions together with light-likeness of Kähler current guarantee that field equations are satisfied.

   In fact, one ends up to consider a generalization of MEs by starting from a generalization of holomorphy. Complex CP_2 coordinates ξ^i would be functions of light-like M^2 coordinate $u_+ = k \cdot m$, k light-like vector, and of complex coordinate w for E^2 orthogonal to M^2. Therefore the CP_2 projection would 3-D rather than 2-D now.

   The second fundamental form has only components of form $H^k_{u_+w}$, $H^k_{u_+\overline{w}}$ and H^k_{ww}, $H^k_{\overline{w}\overline{w}}$. The CP_2 contribution to the induced metric has only components of form Δg_{u_+w}, $\Delta g_{+\overline{w}}$, and $g_{\overline{w}w}$. There is also contribution $g_{u_+u_-} = 1$, where v is the light-like dual of u in plane M^2. Contravariant metric can be expanded as a power series for in the deviation $(\Delta g_{u_+w}, \Delta g_{u_+\overline{w}})$ of the metric from $(g_{u_+u_-}, g_{w\overline{w}})$. Only components of form g^{u_+,u_i} and $g^{w,\overline{w}}$ are obtained and their contractions with the second fundamental form vanish identically since there are no common index pairs with simultaneously non-vanishing components. Hence it seems that MEs generalize!

   I have asked earlier whether this construction might generalize for ordinary MEs. One can introduce what I have called Hamilton-Jacobi structure for M^4 consisting of locally orthogonal slicings by integrable 2-surfaces having tangent space having local decomposition $M^2_x \times E^2_x$ with light-like direction depending on point x. An objection is that the direction of light-like momentum depends on position: this need not be inconsistent with momentum conservation but would imply that the total four-momentum is not light-like anymore. Topological condensation for MEs and at MEs could imply this kind modification.

5. There is also a topological magnetic flux type term for string world sheet. Topological term can be transformed to a boundary term coupling classical particles at the boundary of string world sheet to CP_2 Kähler gauge potential (added to the equation for a light-like geodesic line). Now also the coupling to M^4 gauge potential would be obtained. The condition $J(M^4) + J(CP_2) = 0$ at string world sheets [L22] is very attractive manner to identify string world sheets as analogs of Lagrangian manifolds but does not imply the vanishing of the net $U(1)$ couplings at boundary since the induce gauge potentials are in general different.

   Also topological term including also M^4 Kähler magnetic flux for string world sheet contributes also to the modified Dirac equation since the gamma matrices are modified gamma matrices required by super-conformal symmetries and defined as contractions of canonical momentum densities with embedding space gamma matrices [K106]. This is true both in space-time interior, at string world sheets and at their boundaries. CP_2 (M^4) term gives a contribution proportional to CP_2 (M^4) gamma matrices.

At embedding space level transversal localization would be the outcome and a good guess is that the same happens also now. This is indeed the case for M^4 defining the simplest extremal. The general interpretation of M^4 Kähler form could be as a quantum tool for transversal dynamical localization of wave packets in Kähler magnetic and electric fields of M^4. Analog for decoherence occurring in transversal degrees of freedom would be in question. Hadron physics could be one application.

Testing the existence of $J(M^4)$

How to test the idea about $J(M^4)$?

1. It might be possible to kill the assumption that $J(M^4)$ is covariantly constant by showing that one does not obtain spherically symmetric Schwartschild type metric as a minimal surface extremal of generalized Kähler action: these extremals are possible for ordinary Kähler action [L20] [K17]. For the canonical embedding of M^4 field equations are satisfied since energy momentum tensor vanishes identically. For the small deformations the presence of $J(M^4)$ would reduce rotational symmetry to cylindrical symmetry.

   The question is basically about how large the moduli space of forms $J(M^4)$ can be allowed to be. The mere self duality and closedness condition outside the line connecting the tips of CD allows also variants which are spherically symmetric in either Minkowski coorinates or Robertson-Walker coordinates for light-cone.An attractive proposal is that the pairs of orthogonal 2-surface correspond to Hamilton-Jacobi structures for which the two surfaces are orbits of subgroups of Poincare group.

2. $J(M^4)$ could make its presence manifest in the physics of right-handed neutrino having no direct couplings to electroweak gauge fields. Mixing with left handed neutrino is however induced by mixing of M^4 and CP_2 gamma matrices. The transversal localization of right-handed neutrino in a background, which is a small deformation of M^4 could serve as an experimental signature.

3. CP breaking in hadronic systems is one of the poorly understood aspects of fundamental physics and relates closely to the mysterious matter-antimatter asymmetry. The constant electric part of self dual $J(M^4)$ implies CP breaking. I have earlier consider that Kähler electric fields could cause this breaking but now the electric field is not constant. Second possibility is that matter and antimatter correspond to different values of h_{eff} and are dark relative to each other. The question is whether $J(M^4)$ could explain the observed CP breaking as appearing already at the level of embedding space $M^4 \times CP_2$ and whether this breaking could explain hadronic CP breaking and matter anti-matter asymmetry. Could M^4 part of Kähler electric field induce different $h_{eff}/h = n$ for particles and antiparticles.

Kerr effect, breaking of T symmetry, and Kähler form of M^4

I encountered in Facebook a link to a very interesting article [D1] (see `http://tinyurl.com/h5lmplw`). Here is the abstract of the article.

We prove an instance of the Reciprocity Theorem that demonstrates that Kerr rotation, also known as the magneto-optical Kerr effect, may only arise in materials that break microscopic time reversal symmetry. This argument applies in the linear response regime, and only fails for nonlinear effects. Recent measurements with a modified Sagnac Interferometer have found finite Kerr rotation in a variety of superconductors. The Sagnac Interferometer is a probe for nonreciprocity, so it must be that time reversal symmetry is broken in these materials.

Magneto-optic Kerr effect (see `http://tinyurl.com/hef8xgv`) occurs when a circularly polarized light beam (plane wave) (often with normal incidence) reflects from a sample. For instance, reflected circular polarized beams suffers a phase change in the reflection: as if they would spend some time at the surface before reflecting. Linearly polarized light reflects as elliptically polarized light. In magneto-optic Kerr effect there are many options depending on the relative directions of the reflection plane (incidence is not normal in the general case so that one can talk about reflection plane) and magnetization.

Kerr angle θ_K is defined as 1/2 of the difference of these phase angle increments caused by reflection for oppositely circularly polarized plane wave beams. As the name tells, magneto-optic Kerr effect is often associated with magnetic materials. Kerr effect has been however observed also for high Tc superconductors and this has raised controversy. As a layman in these issues I can safely wonder whether the controversy is created by the expectation that there are no magnetic fields inside the super-conductor. Anti-ferromagnetism is however important for high Tc superconductivity. In TGD based model for high Tc superconductors the supracurrents would flow along pairs of flux tubes with the members of $S = 0$ ($S = 1$) Cooper pairs at parallel flux tubes carrying magnetic fields with opposite (parallel) magnetic fluxes. Therefore magneto-optic Kerr effect could be in question after all.

The author claims to have proven that Kerr effect in general requires breaking of microscopic time reversal symmetry. Time reversal symmetry breaking (TRSB) caused by the presence of magnetic field and in the case of unconventional superconductors is explained nicely at `http://tinyurl.com/jbabcjt`. Magnetic field is required. Magnetic field is generated by a rotating current and by right-hand rule time reversal changes the direction of the current and also of magnetic field. For spin 1 Cooper pairs the analog of magnetization is generated, and this leads to T breaking.

This result is very interesting from the point of TGD. The reason is that twistorial lift of TGD requires that embedding space $M^4 \times CP_2$ has Kähler structure in generalized sense [L24, L45]. M^4 has the analog of Kähler form, call it $J(M^4)$. $J(M^4)$ is assumed to be self-dual and covariantly constant as also CP_2 Kähler form, and contributes to the Abelian electroweak U(1) gauge field (electroweak hypercharge) and therefore also to electromagnetic field. By definition it satisfies $J^2(M^4) = -g(M^4)$ saying that it represents imaginary unit geometrically.

$J(M^4)$ implies breaking of Lorentz invariance since it defines decomposition $M^4 = M^2 \times E^2$ implying preferred rest frame and preferred spatial direction identifiable as direction of spin quantization axis. In zero energy ontology (ZEO) one has moduli space of causal diamonds (CDs) and therefore also moduli space of Kähler forms and the breaking of Lorentz invariance cancels. Note that a similar Kähler form is conjectured in quantum group inspired non-commutative quantum field theories and the problem is the breaking of Lorentz invariance.

What is interesting that the action of P,CP, and T on Kähler form transforms it from self-dual to anti-self-dual form and vice versa. If $J(M^4)$ is self-dual as also $J(CP_2)$, all these 3 discrete symmetries are broken in arbitrarily long length scales. On basis of tensor property of $J(M^4)$ one expects P: $(J(M^2), J(E^2) \rightarrow (J(M^2), -J(E^2)$ and T: $(J(M^2), J(E^2) \rightarrow (-J(M^2), J(E^2)$. Under C one has $(J(M^2), J(E^2) \rightarrow (-J(M^2), -J(E^2)$. This gives CPT: $(J(M^2), J(E^2) \rightarrow (J(M^2), J(E^2)$ as expected.

One can imagine several consequences at the level of fundamental physics.

1. One implication is a first principle explanation for the mysterious CP violation and matter antimatter asymmetry not predicted by standard model (see below).

2. A new kind of parity breaking is predicted. This breaking is separate from electroweak parity breaking and perhaps closely related to the chiral selection in living matter.

3. The breaking of T might in turn relate to Kerr effect if the argument of authors is correct. It could occur in high Tc superconductors in macroscopic scales. Also large $h_{eff}/h = n$ scaling up quantum scales in high Tc superconductors could be involved as with the breaking of chiral symmetry in living matter. Strontium ruthenate for which Cooper pairs are in $S = 1$ state is is indeed found to exhibit TRSB (for references and explanation see `http://tinyurl.com/jbabcjt`).

   In TGD based model of high Tc superconductivity [K76, K77] the members of the Cooper pair are at parallel magnetic flux tubes with the same spin direction of magnetic field. The magnetic fields and thus the direction of spin component in this direction changes under T causing TRSB. The breaking of T for $S = 1$ Cooper pairs is not spontaneous but would occur at the level of physics laws: the time reversed system finds itself experiences in the original self-dual $J(M^4)$) rather than in $(-J(M^2), J(E^2))$ demanded by T symmetry.

5.3.7 What causes CP violation?

CP violation and matter antimatter asymmetry involving it represent white regions in the map provided by recent day physics. Standard model does not predict CP violation necessarily accompanied by the violation of time reflection symmetry T by CPT symmetry assumed to be exact. The violation of T must be distinguished from the emergence of time arrow implies by the randomness associated with state function reduction.

CP violation was originally observed for mesons via the mixing of neutral kaon and antikaon having quark content $n\overline{s}$ and $\overline{n}s$. The lifetimes of kaon and antikaon are different and they transform to each other. CP violation has been also observed for neutral mesons of type $n\bar{b}$. Now it has been observed also for baryons Λ_b with quark composition u-d-b and its antiparticle (see http://tinyurl.com/zyk8w44). Standard model gives the Feynman graphs describing the mixing in standard model in terms of CKM matrix (see http://tinyurl.com/hvpz2su).

The CKM mixing matrix associated with weak interactions codes for the CP violation. More precisely, the small imaginary part for the determinant of CKM matrix defines the invariant coding for the CP violation. The standard model description of CP violation involves box diagrams in which the coupling to heavy quarks takes place. b quark gives rise to anomalously large CP violation effect also for mesons and this is not quite understood. Possible new heavy fermions in the loops could explain the anomaly.

Quite generally, the origin of CP violation has remained a mystery as also CKM mixing. In TGD framework CKM mixing has topological explanation in terms of genus of partonic 2-surface assignable to quark (sphere, torus or sphere with two handles). Topological mixings of U and D type quarks are different and the difference is not same for quarks and antiquarks. But this explains only CKM mixing, not CP violation.

Classical electric field - not necessary electromagnetic - prevailing inside hadrons could cause CP violation. So called instantons are basic prediction of gauge field theories and could cause strong CP violation since self-dual gauge field is involved with electric and magnetic fields having same strength and direction. That this strong CP violation is not observed is a problem of QCD. There are however proposals that instantons in vacuum could explain the CP violation of hadron physics (see http://tinyurl.com/zptbd4j).

What says TGD? I have considered this in [L47] and earlier blog posting (see http://tinyurl.com/hvzqjua).

1. M^4 and CP_2 are unique in allowing twistor space with Kähler structure (in generalized sense for M^4) [A54]. If the twistor space $T(M^4) = M^4 \times S^2$ having bundle projections to both M^4 and to the conventional twistor space CP_3, or rather its non-compact version) allows Kähler structure then also M^4 allow the generalized Kähler structure and the analog symplectic structure.

   This boils down to the existence of self-dual and covariantly constant U(1) gauge field $J(M^4)$ for which electric and magnetic fields E and B are equal and constant and have the same direction. This field is not dynamical like gauge fields but would characterize the geometry of M^4. $J(M^4)$ implies violation Lorentz invariance. TGD however leads to a moduli space for causal diamonds (CDs) effectively labelled by different choices of direction for these self-dual Maxwell fields. The common direction of E and B could correspond to that for spin quantization axis. $J(M^4)$ has nothing to do with instanton field. It should be noticed that also the quantum group inspired attempts to build quantum field theories for which space-time geometry is non-commutative introduce the analog of Kähler form in M^4, and are indeed plagued by the breaking of Lorentz invariance. Here there is no moduli space saving the situation.

2. The choice of quantization axis would therefore have a correlate at the level of "world of classical worlds" (WCW). Different choices would correspond to different sectors of WCW. The moduli space for the choices of preferred point of CP_2 and color quantization axis corresponds to the twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$ of WCW. One could interpret also the twistor space $T(M^4) = M^4 \times S^2$ as the space with given point representing the position of the tip of CD and the direction of the quantization axis of angular momentum. This choice requires a characterization of a unique rest system and the directions of quantization axis and time axes defines plane M^2 playing a key role in TGD approach to twstorialization [L24, L45].

3. The prediction would be CP violation for a given choice of $J(M^4)$. Usually this violation would be averaged out in the average over the moduli space for the choices of M^2 but in some situation this would not happen. Why the CP violation does not average out when there is CKM mixing of quarks? Why the parity violation due to the preferred direction is not compensated by C violation meaning that the directions of E and B fields would be exactly opposite for quarks and antiquarks. Could the fact that quarks are not free but inside hadron induce CP violation? Could a more abstract formulation say that the wave function in the moduli space for $J(M^4)$ (wave function for the choices of spin quantization axis!) is not CP symmetric and this is reflected in the CKM matrix.

4. An important delicacy is that $J(M^4)$ can be both self-dual and anti-self-dual depending on whether the magnetic and electric field have same or opposite directions. It will be found that reflection P and CP transform self-dual $J(M^4)$ to anti-self-dual one. If only self-dual $J(M^4)$ is allowed, one has both parity breaking and CP violations.

Can one understand the emergence of CP violation in TGD framework?

1. Zero energy state is pair of two positive and negative energy parts. Let us assume that positive energy part is fixed - one can call corresponding boundary of CD passive. This state corresponds to the outcome of state function reduction fixing the direction of quantization axes and producing eigenstates of measured observables, for instance spin. Single system at passive boundary is by definition unentangled with the other systems. It can consists of entangled subsystems hadrons are basic example of systems having entanglement in spin degrees of freedom of quarks: only the total spin of hadron is precisely defined.

   The states at the active boundary of CD evolve by repeated unitary steps by the action of the analog of S-matrix and are not anymore eigenstates of single particle observables but entangled. There is a sequence of trivial state function reductions at passive boundary inducing sequence of unitary time evolutions to the state at the active boundary of CD and shifting it. This gives rise to self as a generalized Zeno effect.

   Classically the time evolution of hadron corresponds to a superposition of space-time surfaces inside CD. The passive ends of the space-time surface or rather, the quantum superposition of them - is fixed. At the active end one has a superposition of 3-surfaces defining classical correlates for quantum states at the active end: this superposition changes in each unitary step during repeated measurements not affecting the passive end. Also time flows, which means that the distance between the tips of CD defining clock-time increases as the active boundary of CD shifts farther away.

2. The classical field equations for space-time surface follow from an action, which at space-time level is sum of Kähler action and volume term. If Kähler form at space-time surface is induced (projected to space-time surface) from $J = J(M^4) + J(CP_2)$, the classical time evolution is CP violating. CKM mixing is induced by different topological mixings for U and D type quarks (recall that 3 particle generations correspond to different genera for partonic 2-surfaces: sphere, torus, and sphere with two handles). $J(M^4)+J(CP_2)$ defines the electroweak $U(1)$ component of electric field so that $J(M^4)$ contributes to $U(1)$ part of em field and is thus physically observable.

3. Topological mixing of quarks corresponds to a superposition of time evolutions for the partonic 2-surfaces, which can also change the genus of partonic 2-surface defined as the number of handles attached to 2-sphere. For instance, sphere can transform to torus or torus to a sphere with two handles. This induces mixing of quantum states. For instance, one can say that a spherical partonic 2-surface containing quark would develop to quantum superposition of sphere, torus, and sphere with two handles. The sequence of state function reductions leaving the passive boundary of CD unaffected (generalized Zeno effect) by shifting the active boundary from its position after the first state function reduction to the passive boundary could but need not give rise to a further evolution of CKM matrix.

4. The determinant of CKM matrix is equal to phase factor by unitarity ($UU^\dagger = 1$) and its imaginary part characterizes CP breaking. The imaginary part of the determinant should be proportional to the Jarlskog invariant $J = \pm Im(V_{us}V_{cb}\overline{V}_{ub}\overline{V}_{cs})$ characterizing CP breaking of CKM matrix (see `http://tinyurl.com/kakxwl8`).

If the topological mixings are different for U and D type quarks, one obtains CKM mixing. How could the classical time evolution for quarks and for antiquarks as their CP transforms differ? To answer the question one must look how $J(M^4)$ transforms under C, P, T and CP.

1. $J(M^4) = (J_{0z}, J_{xy} = \epsilon J_{0z})$, $\epsilon = \pm 1$, characterizes hadronic space-time sheet (all space-time sheets in fact). Since $J(M^4)$ is tensor, P changes only the sign of J_{0z} giving $J(M^4) \rightarrow (-J_{0z}, J_{xy})$. Since C changes the signs of charges and therefore the signs of fields created by them, one expects $J(M^4) \rightarrow -J(M4)$ under C. CP would give $J(M^4) \rightarrow (J_{0z}, -J_{xy})$ transforming selfdual $J(M^4)$ to anti-selfdual $J(M^4)$. If WCW has no anti-self-dual sector, CP is violated at the level of WCW.

2. If CPT leaves $J(M^4)$ invariant, one must have $J(M^4) \rightarrow (J_{0z}, -J_{xy})$ under T rather than $J(M^4) \rightarrow (-J_{0z}, J_{xy})$. The anti-unitary character of T could correspond for additional change of sign under T. Otherwise CPT should act as $J(M^4) \rightarrow -J(M^4)$ and only $(CPT)^2$ would correspond to unity.

3. Same considerations apply to $J(CP_2)$ but the difference would be that induced $J(M^4)$ for space-time surfaces, which are small deformations of M^4 covariantly constant in good approximation. Also for string world sheets corresponding to small cosmological constant $J(M^4) \times J(M^4) - 2 \simeq 0$ holds true in good approximation and induced $J(M^4)$ at string world sheet is in good approximation covariantly constant. If the string world sheet is just M^2 characterizing $J(M^4)$ the condition is exact and was has Kähler electric field induced by $J(M^4)$ but no corresponding magnetic field. This would make the CP breaking effect large.

If CP is not violated, particles and their CP transforms correspond to different sectors of WCW with self dual and anti-self dual $J(M^4)$. If only self-dual sector of WCW is present then CP is violated. Also P is violated at the level of WCW and this parity breaking is different from that associated with weak interactions and could relate to the geometric parity breaking manifesting itself via chiral selection in living matter. Classical time evolutions induce different CKM mixings for quarks and antiquarks reflecting itself in the small imaginary part of the determinant of CKM matrix. CP breaking at the level of WCW could explain also matter-antimatter asymmetry. For instance, antimatter could be dark with different value of $h_{eff}/h = n$.

What is interesting that P is badly broken in long length scales as also CP. The same could be true for T. Could this relate to the thermodynamical arrow of time? In ZEO state function reductions to the opposite boundary change the direction of clock time. Most physicist believe that the arrow of thermodynamical time and thus also clock time is always the same. There is evidence that in living matter both arrows are possible. For instance, Fantappie has introduced the notion of syntropy as time reversed entropy [J3]. This suggests that thermodynamical arrow of time could correspond to the dominance of the second arrow of time and be due to self-duality of $J(M^4)$ leading to breaking of T. For instance, the clock time spend in time reversed phase could be considerably shorter than in the dominant phase. A quantitative estimate for the ratio of these times might be given some power of the ratio $X = l_P/R$.

5.3.8 Quantitative picture about CP breaking in TGD

One must specify the value of α_1 and the scaling factor transforming $J(CD)$ having dimension length squared as tensor square root of metric to dimensionless $U(1)$ gauge field $F = J(CD)/S$. This leads to a series of questions.

How to fix the scaling parameter S?

1. The scaling parameter relating $J(CD)$ and F is fixed by flux quantization implying that the flux of $J(CD)$ is the area of sphere S^2 for the twistor space $M^4 \times S^2$. The gauge field is obtained as $F = J/S$, where $S = 4\pi R^2(S^2)$ is the area of S^2.

2. Note that in Minkowski coordinates the length dimension is by convention shifted from the metric to linear Minkowski coordinates so that the magnetic field B_1 has dimension of inverse length squared and corresponds to $J(CD)/SL^2$, where L is naturally be taken to the size scale of CD defining the unit length in Minkowski coordinates. The $U(1)$ magnetic flux would the signed area using L^2 as a unit.

How $R(S^2)$ relates to Planck length l_P? l_P is either the radius $l_P = R$ of the twistor sphere S^2 of the twistor space $T = M^4 \times S^2$ or the circumference $l_P = 2\pi R(S^2)$ of the geodesic of S^2. Circumference is a more natural identification since it can be measured in Riemann geometry whereas the operational definition of the radius requires embedding to Euclidian 3-space.

How can one fix the value of $U(1)$ coupling strength α_1? As a guideline one can use CP breaking in K and B meson systems and the parameter characterizing matter-antimatter symmetry.

1. The recent experimental estimate for so called Jarlskog parameter characterizing the CP breaking in kaon system is $J \simeq 3.0 \times 10^{-5}$. For B mesons CP breading is about 50 times larger than for kaons and it is clear that Jarlskog invariant does not distinguish between different meson so that it is better to talk about orders of magnitude only.
2. Matter-antimatter asymmetry is characterized by the number $r = n_B/n_\gamma \sim 10^{-10}$ telling the ratio of the baryon density after annihilation to the original density. There is about one baryon 10 billion photons of CMB left in the recent Universe.

Consider now the identification of α_1.

1. Since the action is obtained by dimensional reduction from the 6-D Kähler action, one could argue $\alpha_1 = \alpha_K$. This proposal leads to unphysical predictions in atomic physics since neutron-electron $U(1)$ interaction scales up binding energies dramatically.

   $U(1)$ part of action can be however regarded a small perturbation characterized by the parameter $\epsilon = R^2(S^2)/R^2(CP_2)$, the ratio of the areas of twistor spheres of $T(M^4)$ and $T(CP_2)$. One can however argue that since the relative magnitude of $U(1)$ term and ordinary Kähler action is given by ϵ, one has $\alpha_1 = \epsilon \times \alpha_K$ so that the coupling constant evolution for α_1 and α_K would be identical.
2. ϵ indeed serves in the role of coupling constant strength at classical level. α_K disappears from classical field equations at the space-time level and appears only in the conditions for the super-symplectic algebra but ϵ appears in field equations since the Kähler forms of J resp. CP_2 Kähler form is proportional to $R^2(S^2)$ resp. $R^2(CP_2)$ times the corresponding $U(1)$ gauge field. $R(S^2)$ appears in the definition of 2-bein for $R^2(S^2)$ and therefore in the modified gamma matrices and modified Dirac equation. Therefore $\sqrt{\epsilon} = R(S^2)/R(CP_2)$ appears in modified Dirac equation as required by CP breaking manifesting itself in CKM matrix.

   NTU for the field equations in the regions, where the volume term and Kähler action couple to each other demands that ϵ and $\sqrt{\epsilon}$ are rational numbers, hopefully as simple as possible. Otherwise there is no hope about extremals with parameters of the polynomials appearing in the solution in an arbitrary extension of rationals and NTU is lost. Transcendental values of ϵ are definitely excluded. The most stringent condition $\epsilon = 1$ is also unphysical. $\epsilon = 2^{2r}$ is favoured number theoretically.

Concerning the estimate for ϵ it is best to use the constraints coming from p-adic mass calculations.

1. p-Adic mass calculations [K52] predict electron mass as

$$m_e = \frac{\hbar}{R(CP_2)\sqrt{5+Y}} \quad .$$

   Expressing m_e in terms of Planck mass m_P and assuming $Y = 0$ ($Y \in (0,1)$) gives an estimate for $l_P/R(CP_2)$ as

$$\frac{l_P}{R(CP_2)} \simeq 2.0 \times 10^{-4} \quad .$$

2. From $l_P = 2\pi R(S^2)$ one obtains estimate for ϵ, α_1, $g_1 = \sqrt{4\pi\alpha_1}$ assuming $\alpha_K \simeq \alpha \simeq 1/137$ in electron length scale.

$$\begin{aligned} \epsilon &= 2^{-30} \simeq 1.0 \times 10^{-9} \quad , \\ \alpha_1 &= \epsilon\alpha_K \simeq 6.8 \times 10^{-12} \quad , \\ g_1 &= \sqrt{4\pi\alpha_1} \simeq 9.24 \times 10^{-6} \quad . \end{aligned}$$

There are two options corresponding to $l_P = R(S^2)$ and $l_P = 2\pi R(S^2)$. Only the length of the geodesic of S^2 has meaning in the Riemann geometry of S^2 whereas the radius of S^2 has operational meaning only if S^2 is imbedded to E^3. Hence $l_P = 2\pi R(S^2)$ is more plausible option.

For $\epsilon = 2^{-30}$ the value of $l_P^2/R^2(CP_2)$ is $l_P^2/R^2(CP_2) = (2\pi)^2 \times R^2(S^2)/R^2(CP_2) \simeq 3.7 \times 10^{-8}$. $l_P/R(S^2)$ would be a transcendental number but since it would not be a fundamental constant but appear only at the QFT-GRT limit of TGD, this would not be a problem.

One can make order of magnitude estimates for the Jarlskog parameter J and the fraction $r = n(B)/n(\gamma)$. Here it is not however clear whether one should use ϵ or α_1 as the basis of the estimate

1. The estimate based on ϵ gives

$$J \sim \sqrt{\epsilon} \simeq 3.2 \times 10^{-5} \quad , \quad r \sim \epsilon \simeq 1.0 \times 10^{-9} \quad .$$

   The estimate for J happens to be very near to the recent experimental value $J \simeq 3.0 \times 10^{-5}$. The estimate for r is by order of magnitude smaller than the empirical value.

2. The estimate based on α_1 gives

$$J \sim g_1 \simeq 0.92 \times 10^{-5} \quad , \quad r \sim \alpha_1 \simeq .68 \times 10^{-11} \quad .$$

   The estimate for J is excellent but the estimate for r by more than order of magnitude smaller than the empirical value. One explanation is that α_K has discrete coupling constant evolution and increases in short scales and could have been considerably larger in the scale characterizing the situation in which matter-antimatter asymmetry was generated.

There is an intriguing numerical co-incidence involved. $h_{eff} = \hbar_{gr} = GMm/v_0$ in solar system corresponds to $v_0 \simeq 2^{-11}$ and appears as coupling constant parameter in the perturbative theory obtained in this manner [K85]. What is intriguing that one has $\alpha_1 = v_0^2/4\pi^2$ in this case. Where does the troublesome factor $(1/2\pi)^2$ come from? Could the p-adic coupling constant evolutions for v_0 and α_1 correspond to each other and could they actually be one and the same thing? Can one treat gravitational force perturbatively either in terms of gravitational field or $J(CD)$? Is there somekind of duality involved?

Atomic nuclei have baryon number equal the sum $B = Z+N$ of proton and neutron numbers and neutral atoms have $B = N$. Only hydrogen atom would be also $U(1)$ neutral. The dramatic prediction of $U(1)$ force is that neutrinos might not be so weakly interacting particles as has been thought. If the quanta of $U(1)$ force are not massive, a new long range force is in question. $U(1)$ quanta could become massive via $U(1)$ super-conductivity causing Meissner effect. As found, $U(1)$ part of action can be however regarded a small perturbation characterized by the parameter $\epsilon = R^2(S^2)/R^2(CP_2)$. One can however argue that since the relative magnitude of $U(1)$ term and ordinary Kähler action is given by ϵ, one has $\alpha_1 = \epsilon \times \alpha_K$.

Quantal $U(1)$ force must be also consistent with atomic physics. The value of the parameter α_1 consistent with the size of CP breaking of K mesons and with matter antimatter asymmetry is $\alpha_1 = \epsilon\alpha_K = 2^{-30}\alpha_K$.

1. Electrons and baryons would have attractive interaction, which effectively transforms the em charge Z of atom $Z_{eff} = rZ$, $r = 1 + (N/Z)\epsilon_1$, $\epsilon_1 = \alpha_1/\alpha = \epsilon \times \alpha_K/\alpha \simeq \epsilon$ for $\alpha_K \simeq \alpha$ predicted to hold true in electron length scale. The parameter

$$s = (1 + (N/Z)\epsilon)^2 - 1 = 2(N/Z)\epsilon + (N/Z)^2\epsilon^2$$

   would characterize the isotope dependent relative shift of the binding energy scale.

   The comparison of the binding energies of hydrogen isotopes could provide a stringent bounds of the value of α_1. For $l_P = 2\pi R(S^2)$ option one would have $\alpha_1 = 2^{-30}\alpha_K \simeq .68 \times 10^{-11}$ and $s \simeq 1.4 \times 10^{-10}$. s is by order of magnitude smaller than $\alpha^4 \simeq 2.9 \times 10^{-9}$ corrections from QED (see `http://tinyurl.com/kk9u4rh`). The predicted differences between the binding energy scales of isotopes of hydrogen might allow to test the proposal.

2. $B = N$ would be neutralized by the neutrinos of the cosmic background. Could this occur even at the level of single atom or does one have a plasma like state? The ground state binding

energy of neutrino atoms would be $\alpha_1^2 m_\nu/2 \sim 10^{-24}$ eV for $m_\nu = .1$ eV! This is many many orders of magnitude below the thermal energy of cosmic neutrino background estimated to be about 1.95×10^{-4} eV (see `http://tinyurl.com/ldu95o9`). The Bohr radius would be $\hbar/(\alpha_1 m_\nu) \sim 10^6$ meters and same order of magnitude as Earth radius. Matter should be $U(1)$ plasma. $U(1)$ superconductor would be second option.

5.4 About the interpretation of the duality assignable to Yangian symmetry

The $D = 4$ conformal generators acting on twistors have a dual representation in which they act on momentum twistors: one has dual conformal symmetry, which becomes manifest in this representation. These two separate symmetries extend to Yangian symmetry providing a powerful constraint on the scattering amplitudes.

In TGD the conformal Yangian extends to super-symplectic Yangian - actually, all symmetry algebras have a Yangian generalization with multi-locality generalized to multi-locality with respect to partonic 2-surfaces. The generalization of the dual conformal symmetry has remained obscure. In the following I describe what the generalization of the two conformal symmetries and Yangian symmetry would mean in TGD framework. I also propose an information theoretic duality between Euclidian and Minkowskian regions of space-time surface. I am not algebraist and apologize for the unavoidable inaccuracies.

5.4.1 Formal definition associated with Yangian

The notion of Yangian appears as two very different looking variants. The first variant can be found from Wikipedia (see `goo.gl/q1twRZ`) and second variant assignable to gauge theories can be found from [B26, B27].

Consider first the Wikipedia definition. The definition is in terms of quantum group notion in which the elements of matrix representing group element are made non-commuting operators.

1. The generators of Yangian algebra are labelled by an integer $n \geq -1$ with $n = -1$ generator identified as unit matrix. $n \geq 1$ generators generate the algebra and commutators with $n = 1$ generators preserving the weight allow to assign quantum numbers to them. From the Wikipedia article one learns that Yangian is generated by elements $t_{ij}^{(p)}$, $1 \leq i, j \leq N$, $p \geq 0$ of quantum matrices satisfy the relations

$$\left[t_{ij}^{(p+1)}, t_{kl}^{(q)}\right] - \left[t_{ij}^{(p)}, t_{kl}^{(q+1)}\right] = -(t_{kj}^{(p)} t_{il}^{(q)} - t_{kj}^{(q)} t_{il}^{(p)}) \quad . \tag{5.4.1}$$

   Note there are two operations involved: commutator and operator product. The formula here is not consistent with the formula used in Yang-Mills theories for the commutators between $m = 0$ generators and generators with generators having $n \in \{0, 1\}$, and it seems that this formula suggesting $m, n \to m + n - 1$ in commutator cannot hold true for the commutators with $m = 0$ generators.

   By defining $t_{ij}^{(-1)} = \delta_{ij}$ and setting

$$T(z) = \sum_{p \geq -1} t_{ij}^{(p)} z^{-p+1} \quad . \tag{5.4.2}$$

   $T(z)$ is thus a quantum matrix depending on the point of 2-D space.

2. Introduce R-matrix $R(z) = 1 + z^{-1}P$ acting on $C^N \otimes C^N$, where P is the operator permuting the tensor factors. This allows to write the defining relations as Yang-Baxter equation (see `http://tinyurl.com/gogn75s`):

$$R_{12}(z - w) T_1(z) T_2(w) = T_2(w) T_1(z) R_{12}(z - w) \quad . \tag{5.4.3}$$

R_{12}, which depends only on the difference $z - w$, performs the permutation of the generators $T_1(z)$ and $T_2(w)$.

Yangian is a Hopf algebra with co-multiplication Δ mapping $T(z)$ acting in V to operator acting in $V \otimes V$, co-unit ϵ and antipode s given by

$$(\Delta \otimes id)T(z) = T_{12}(z)T_{13}(z) \ , \quad (\epsilon \otimes id)T(z) = I \ , \quad (s \otimes id)T(z) = T(z)^{-1} \ . \quad (5.4.4)$$

Δ taking generator $T(z)$ acting in V to generator $\Delta(T) = T_{12}(z)$ acting in $V \otimes V$. Δ transforms a generator acting on single-particle states to a generator acting on 2-particles states.

3. The Yangian weight of the commutator of elements with weights m and n is $m + n - 1$ rather than $m + n$ as for Virasoro and Kac-Moody algebras. This means that generators with conformal weight 1 do not affect the conformal weight and Cartan algebra elements defining quantum numbers of generators have weight 1. For conformal algebras the Cartan algebra defining quantum numbers has conformal weight 0.

   For Virasoro algebra having integer valued conformal weights the scaling $L_0 = zd/dz$ appears as basic derivative operation and generators are products $L_n = z^n zd/dz$. By taking translation operator $T = d/dz$ as the derivative operator and writing $K_n = z^n d/dz$, the weight of commutator becomes $m + n - 1$. This is a trivial change. The map $u = exp(z)$ relates these two representations. That $n \leq 2$ appear in generators distinguishes the representations from Virasoro and Kac-Moody representations - note however that also for these algebras the generators with positive weight generate physical states.

What bothers me in this definition is that only the action of the generators with $p = 1$ leaves the weight unaffected whereas for the dual conformal symmetry generators with both $p = 0$ and $p = 1$ do this and define conformal symmetry and its dual.

5.4.2 Dual conformal symmetry in $\mathcal{N} = 4$ SUSY

Yangian symmetry appears also in gauge theories and the definition looks very different from the Wikipedia definition. In $\mathcal{N} = 4$ SUSY conformal symmetry (in 4-D sense) has two representations. There is a duality between two representations of conformal generators crucial for twistor Grassmannian approach [B26, B27] (see `http://tinyurl.com/n22lwuy`).

1. In the first representation conformal symmetry generators $J_a^{(0)}$ are local and act in the space of external momenta. This induces a local and linear action in twistor space.

2. The generators $J_a^{(1)}$ of the dual conformal symmetry act in a local manner in the space of region momenta and associated momentum twistor space whereas the action of $J_a^{(1)}$ is bi-local in the momentum space and corresponding twistor space.

   Region momenta can be assigned with a twistor diagram defined by a closed polygon of Minkowski space having region momenta (, which need not be light-like) as edges having external light-like momenta emitted at the corners. The dual of this representation is the representation in which the light-like external momenta summing up to zero form a closed polygon.

Yangian is generated by ordinary generators $J_a^{(0)}$ and bi-local dual generators $J_a^{(1)}$.

1. They satisfy the commutations

$$\left[J_a^{(0)}, J_b^{(1)}\right] = f_{ab}{}^{c} J_c^{(1)} \ . \quad (5.4.5)$$

   This condition is perfectly sensible physically but is not consistent with the above general consistency condition pf Eq. 5.4.1 from R-matrix requiring that the commutator has vanishing weight. Now the weights are additive in commutator.

2. The generators $J_a^{(1)}$ have an easy-to-guess representation:

$$J_a^{(1)} = f_a^{\ cb} \sum_{0\leq i<j\leq n} J_{ib}^{(0)} J_{jc}^{(0)} \tag{5.4.6}$$

making explicit the bi-locality. The commutators of these generators have also weight 1. This is consistent with the above general formula unlike the formula the commutators of generators with vanishing weight. Both generators form a closed sub-algebra of Yangian and this must be behind the possibility to represent $J_a^{(1)}$ locally.

3. Also so called Serre relations are satisfied. They look rather complex and look different from the relations associated with R-matrix.

$$\begin{aligned} X(a,b,c) &+ \epsilon(a,b,c)X(b,c,a) + \epsilon(c,a,b)X(c,a,b) = h\epsilon_{rm,tn}Y(l,m,n)f^l_{ar}f^m_{bs}f^n_{ct}f^{rst} \ , \\ X(a,b,c) &= \left[J_a^{(1)}, \left[J_b^{(1)}, J_c^{(0)}\right\}\right\} \ , \ Y(l,m,n) = \left\{J_l^{(0)}, J_m^{(0)}, J_n^{(0)}\right] \\ \epsilon(a,b,c) &= (-1)^{|a|(|b|+|c|)} \ , \ \epsilon_{rm,tn} = (-1)^{|r|m|+|t|n|} \ . \end{aligned} \tag{5.4.7}$$

Here the mixed brackets the $[.,\}$ denote the graded commutator, and $\{.,..]$ denotes the graded symmetrizer. h is a parameter characterizing the Yangian and should correspond to the parameter characterizing quantum group.

These conditions are sufficient to give a representation of graded Yangian if the tensor product $\mathcal{R}\otimes\overline{\mathcal{R}}$ of the representation $\mathcal{R}$ and its conjugate $\overline{\mathcal{R}}$ contains adjoint representation only once. The higher generators can be generate by applying co-product operation to the generators.

4. Both local and bi-local generators form two closed sub-algebras. This is not consistent with the consistency conditions of appearing in Wikipedia definition. The Wikipedia definition seems to be wrong for commutators of generators $[J_A^{(m)}, J_B^{(n)}]$ with weights $(m,n) \in \{(0,0),(0,1),(1,0)\}$.

5. Co-product Δ has representation

$$\begin{aligned} \Delta(J^A) &= J^A\otimes 1 + 1\otimes J^A \ , \\ \Delta(Q^A) &= Q^A\otimes 1 + 1\otimes Q^A + f^A_{BC}J^B\otimes J^C \ . \end{aligned} \tag{5.4.8}$$

The first formula is obvious. Single particle generator lifted to a tensor product is sum of the single particle generators acting on the tensor factors. When Q^A annihilates single spin representations, one obtains just the defining formula for the bi-local generators.

One could have a situation in which single particle states are actually many-particle states annihilated by Q^A and satisfying the condition that adjoint is contained only once in $\mathcal{R}\otimes\overline{\mathcal{R}}$. In TGD framework one might argue that this kind of effective single particle states could quite generally define bound states behaving like single particle states physically. One would obtain infinite hierarchy of this kind of states realizing concretely the vision about fractal hierarchy.

5.4.3 Possible TGD based interpretation of Yangian symmetries

In TGD partonic 2-surfaces replace point-like objects and multi-locality is with respect to these. The proposal is that the TGD counterpart of the Yangian algebra [B27] of gauge theories could act as symmetries of many-parton states characterized by n partonic 2-surfaces assignable to the same 3-D surface at the boundary of causal diamond (CD). What is remarkable that this symmetry would relate particle states with different particle numbers to each other unlike the usual single particle symmetries.

1. This condition forces the partons to form a bound state with partonic 2-surfaces having *space-like* separations. Note that the separations along orbits of wormhole throats at opposite ends of CD are space-like or light-like. This must be taken into account when correlation functions are calculated. In QFT there is no description of this kind and this could explain the general failure of QFT in the description of bound states already in QED, where Bethe-Salpeter equation predicts large numbers of non-existing states.
2. Yangian algebra involves complex (hypercomplex) coordinate z which could be associated with the boundaries of string world sheets connecting partonic surfaces at the same boundary (at opposite boundaries) of CD. One can also assign complex coordinate with partonic 2-surfaces and the braiding of fermionic lines would be described by the matrix R assignable to the Yangian. The Cartan algebra of local and bi-local string like operators define quantum numbers for states. That point-like and string-like operators generate the algebra conforms with the idea about tensor networks with nodes connected by edges.

   On can think that partonic 2-surfaces form a single connected unit consisting of partonic surfaces connected by boundaries of string world sheets assignable to the topological Feynman diagram defined by the light-like 3-surface defining the boundary between Euclidian and Minkowskian regions of the space-time surface.
3. The operation Δ for Yangian would assign to the generators acting on single parton states generators acting on 2-parton states. R_{12} would act as an exchange operation for parton states, which could reduces to many-fermion states at partonic 2-surfaces.
4. R_{12} can appear in many contexts in TGD. It can be associated with braiding of fermionic lines inside partonic orbits or magnetic flux tubes at the ends of space-time surfaces. It can be also associated with the fermionic lines in the preferred plane M^2 associated with twistor scattering amplitudes.

   From the twistorial point of view the preferred M^2 defined by light-like quaterionic 8-momentum is of special interest. M^2 identified as octonionic complex plane and its complexification brings in mind integrable field theories in M^2 allowing Yangian symmetry characterized by R-matrix. The scattering matrix is trivial for these field theories: scattering involves only a phase shift. In twistorial approach to TGD scattering is non-trivial. The R-matrix would be present also now and exchange the momentum projections in preferred M^2 plane. If the entire scattering diagram -apart from external lines corresponds to the same M^2, the braiding operation permutes also fermions at different partonic 2-surfaces located at the ends of string.

The possibility to localize the action of generators $J^{(1)}$ in momentum twistor representation leads to ask whether the stringy generators appearing TGD framework could allow local action using the analog of the space of region momenta. Could M^8-H duality [K91, K80] make this possible? At M^8 level the light-like momenta (in 8-D sense) would correspond to differences of region momenta assignable to strings connecting the partonic 2-surfaces. The 8-D region momenta should be quaternionic. They cannot be light-like as is easy to see.

The notion of region momentum and thus localization would make sense only in M^8, where the wave functions are completely localizable to quaternionic light-like momenta in M^8, whereas in H one has localization to light-like momenta only in preferred M^2 plus wave functions in the space of planes M^4 and in the space of transverse momenta in $E^2 \subset M^4$. This would suggest that M^8-H duality corresponds to the duality of twistor and momentum twistor representations.

What would be new that this duality would be realized also at the level of space-time surfaces. One would have associative/quaternionic space-time surfaces in M^8 and preferred extremals of dimensionally reduced Kähler action in H identifiable as 6-D holomorphic surfaces representing twistor spaces of space-time surfaces.

Note that M^8-H duality could be seen as a number-theoretic analog of spontaneous compactification. Non-perturbative effects would force a delocalization in the space of light-like 8-momenta in M^8 to give states having interpretation as wave functions in H. Nothing would happen to the topology of M^8. Only the state space would be compactified.

5.4.4 A new kind of duality of old duality from a new perspective?

M^8-H duality [K91, K80] maps the preferred extremals in H to those $M^4 \times CP_2$ and vice versa. The tangent spaces of an associative space-time surface in M^8 would be quaternionic (Minkowski) spaces.

In M^8 one can consider also co-associative space-time surfaces having associative *normal* space [K91]. Could the co-associative normal spaces of associative space-time surfaces in the case of preferred extremals form an integrable distribution therefore defining a space-time surface in M^8 mappable to H by M^8-H duality? This might be possible but the associative tangent space and the normal space correspond to the same CP_2 point so that associative space-time surface in M^8 and its possibly existing co-associative companion would be mapped to the same surface of H.

This dead idea however inspires an idea about a duality mapping Minkowskian space-time regions to Euclidian ones. This duality would be analogous to inversion with respect to the surface of sphere, which is conformal symmetry. Maybe this inversion could be seen as the TGD counterpart of finite-D conformal inversion at the level of space-time surfaces. There is also an analogy with the method of images used in some 2-D electrostatic problems used to reflect the charge distribution outside conducting surface to its virtual image inside the surface. The 2-D conformal invariance would generalize to its 4-D quaterionic counterpart. Euclidian/Minkowskian regions would be kind of Leibniz monads, mirror images of each other.

1. If strong form of holography (SH) holds true, it would be enough to have this duality at the informational level relating only 2-D surfaces carrying the holographic information. For instance, Minkowskian string world sheets would have duals at the level of space-time surfaces in the sense that their 2-D normal spaces in X^4 form an integrable distribution defining tangent spaces of a 2-D surface. This 2-D surface would have induced metric with Euclidian signature.

   The duality could relate either a) Minkowskian and Euclidian string world sheets or b) Minkowskian/Euclidian string world sheets and partonic 2-surfaces common to Minkowskian and Euclidian space-time regions. a) and b) is apparently the most powerful option information theoretically but is actually implied by b) due to the reflexivity of the equivalence relation. Minkowskian string world sheets are dual with partonic 2-surfaces which in turn are dual with Euclidian string world sheets.

   (a) Option a): The dual of Minkowskian string world sheet would be Euclidian string world sheet in an Euclidian region of space-time surface, most naturally in the Euclidian "wall neighbour" of the Minkowskian region. At parton orbits defining the light-like boundaries between the Minkowskian and Euclidian regions the signature of 4-metric is $(0,-1,-1,-1)$ and the induced 3-metric has signature $(0,-1,-1)$ allowing light-like curves. Minkowskian and Euclidian string world sheets would naturally share these light-like curves aas common parts of boundary.

   (b) Option b): Minkowskian/Euclidian string world sheets would have partonic 2-surfaces as duals. The normal space of the partonic 2-surface at the intersection of string world sheet and partonic 2-surface would be the tangent space of string world sheets so that this duality could make sense locally. The different topologies for string world sheets and partonic 2-surfaces force to challenge this option as global option but it might hold in some finite region near the partonic 2-surface. The weak form of electric-magnetic duality [K108] could closely relate to this duality.

   In the case of elementary particles regarded as pairs of wormhole contacts connected by flux tubes and associated strings this would give a rather concrete space-time view about stringy structure of elementary particle. One would have a pair of relatively long (Compton length) Minkowskian string sheets at parallel space-time sheets completed to a parallelepiped by adding Euclidian string world sheets connecting the two space-time sheets at two extremely short (CP_2 size scale) Euclidian wormhole contacts. These parallelepipeds would define lines of scattering diagrams analogous to the lines of Feynman diagrams.

This duality looks like new but as already noticed is actually just the old electric-magnetic duality [?]een from number-theoretic perspective.

5.5 TGD view about construction of twistor amplitudes

In the following TGD view about twistorialization and its relation to other visions about TGD is discussed. I start with a brief summary of twistor approach to scattering amplitudes and then describe the application of this approach TGD.

5.5.1 Some key ideas of the twistor Grassmann approach

In the following I summarize the basic technical ideas of twistor Grassmann approach. I am not a specialist. On the other hand, my views about twistorialization of TGD differ in many aspects about those applied in the twistorialization of gauge theories, and my own attention is directed towards the physical interpretation and mathematical consistency rather than calculational techniques.

Variants of twistor formalism

The reader can find details about twistors in the article of Witten [B29] and in the thesis of Trnka [B67] (see `http://tinyurl.com/zbj9ad7`).

1. Helicity spinor formalism assigns to light-like momentum pair of conjugate spinors $(\lambda_a, \tilde{\lambda}_{\dot{a}})$ transforming in conjugate representations of Lorentz group $SL(2,C)$. Light-like momentum is expressible as $p^k\sigma_k$ using Pauli sigma matrices and this gives the representation as matrix components $p^{a\dot{a}} = \lambda^a\lambda^{\dot{a}}$. The determinant of the matrix equals to $p^kp_k = 0$ since its rows are linearly dependent.

   One can introduce the bilinears $[\tilde{\lambda}_1, \tilde{\lambda}_2] = -[\tilde{\lambda}_2, \tilde{\lambda}_1]$ and $\langle\lambda_1, \lambda_2\rangle = -\langle\lambda_2, \lambda_1\rangle$ using the antisymmetric Lorentz invariant bilinear defined by permutation symbols ϵ^{ab} and $\epsilon^{\dot{a}\dot{b}}$. The inner product $p_1 \cdot p_2$ is expressible as $p_1 \cdot p_2 = \langle\lambda_1, \lambda_2\rangle[\tilde{\lambda}_1, \tilde{\lambda}_2]$.

   One could express also polarization vectors of massless bosons using pair $(\lambda, \tilde{\mu})$ of helicity spinors. There is however a more elegant approach available. The spinors $(t\lambda, \tilde{\lambda}/t)$ correspond to same momentum for all non-vanishing complex values of t. t represents an element of little group of Lorentz group leaving the helicity state invariant. The helicity dependence of the scattering amplitude is fixed by the transformation property under little group and coded to the weight under the scalings by t: $A(t_a\lambda, t_a^{-1}\tilde{\lambda}_a) = t_a^{-2h_a}A(\lambda, \tilde{\lambda})$. Thus the formalism allows very elegant description of spin and can be applied in SUSYs.

   For Minkowski signature (2,2) the spinors are real and this makes this signature preferred. Personally I see this as a basic problem of twistorialization. A possible TGD inspired solution of the problem is provided by the effective replacement of M^4 with M^2 with signature $(1,1)$ and thus allowing real spinors.

2. Twistors $(\lambda_a, \mu_{\dot{a}})$ are obtained by performing a twistor Fourier transform of scattering amplitude $A(\lambda, \tilde{\lambda})$ with respect to $\tilde{\lambda}$.

   At local level [B29] the twistor transform corresponds to Fourier transform

   $$\begin{array}{l} \tilde{\lambda}_{\dot{a}} \rightarrow i\frac{\partial}{\mu^{\dot{a}}} \ , \\ -i\frac{\partial}{\tilde{\lambda}^{\dot{a}}} \rightarrow \mu_{\dot{a}} \ . \end{array}$$

   The action of little group corresponds now to the scaling $(\lambda, \mu) \rightarrow t(\lambda, \mu)$ and does not affect the helicity state. For this reason twistors differing by complex scaling can be identified. The proper twistor space is CP_3 rather than C^4.

   The twistor transform of the amplitude transforms as $A(t_a\lambda, t_a\tilde{\lambda}_a) = t_a^{-2h_a-2}A(\lambda, \mu)$.

   In signature (2,2) the helicity spinors $(\lambda, \tilde{\lambda})$ are real so that the twistor Fourier transform reduces to an ordinary Fourier transform. In signature (1,3) the rigorous definition is rather challenging and is discussed by Penrose [B63]. One manner to define the transform is by using residue integral. Residue integral is also p-adically attractive.

   The incidence relation of Penrose given by

   $$\mu_{\dot{a}} = -x_{a\dot{a}}\lambda^a$$

   relates M^4 coordinates to λ, μ. By little group invariance entire complex twistor line corresponds to a given point of M^4.

   The twistor transform of plane wave allows to construct the twistor transform of momentum space wave function, and is given by $\delta^2(\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a)$, which is non-vanishing at complex light ray. Twistor Fourier transform in real Minkowski space is therefore non-vanishing at light ray and maps light rays to twistors.

If the incidence relation for given (λ, μ) is satisfied at two space-time points m_1, m_2, the difference $m_1 - m_2$ is a light-like vector since corresponding matrix has vanishing determinant. Two intersecting twistor lines correspond to M^4 points with light-like distance. This allows to develop geometric picture about twistor diagrams in which the external light-like momenta correspond to intersections of twistor lines assignable to the internal lines of graph.

3. Momentum twistors define a third basic notion. It is convenient to describe particle scattering with external light-like momenta in terms of a diagram in which the external momenta are assigned with the vertices of a polygon such that the lines carry possibly complex momenta. Clearly, the polygon like object is obtained by repeatedly adding light-like momenta to the polygon and since the sum of the external momenta vanishes, the polygon closes.

   The vertices of polygon correspond to intersections of twistor lines defining light-like momenta as differences of the momenta associated with the lines meeting at the vertex. One can assign to the complex momenta of internal lines twistors known as momentum twistors.

   Dual momentum twistor is a further variant of twistor concept being defined in terms of three adjacent momentum twistors contracting them with the 4-D permutation symbol defined in the representation of twistor as a point of C^4 [B67].

Leading singularities

Twistor Grassmann approach to planar loop amplitudes relies on the idea that the discontinuities associated with the singularities of the scattering amplitudes carry all information about the amplitudes. This of course holds true already for the tree diagrams having only poles as singularities.

The idea is same as in the case of analytic continuation: 1-D data at poles and cuts allows to construct the functions. This idea generalizes to functions of several variables and leads to a generalization of residue calculus. At space-time level strong form of holography (SH) relies on the same idea: the 3-D data determine 4-D dynamics and in TGD allowing strong form of holography 2-D data is almost enough.

The discontinuities assignable to singularities can have lower-dimensional singularities so that a hierarchical structure is obtained. The leading singularities are those for which maximal number of propagators are on mass shell and the diagram decomposes to a product of diagrams with virtual particle on mass shell. For one loop diagrams the maximal number of propagators is $N = 4$ corresponding to the fixing of four components of loop momentum. For L loops it is $N = 4L$.

Non-leading singularities have less than the maximal number of propagators on shell and this leaves integral over a subset of loop momenta. If the number of propagator is larger than $4L$, one can have kinematical singularities for some combinations of external momenta.

In the case of scattering amplitudes in twistor Grassmann formulation one encounters a similar situation. In twistor Grassmann approach one defines also the loop integrals in momentum space as residue integrals in the space of complexified momenta. If the functions involved are rational functions the residue integrals are well-defined.

One of the surprising findings is that the leading singularities of MHV loop amplitudes always proportional to tree amplitudes. Second finding is that for $\mathcal{N} = 4$ theory the leading singularities determine completely the scattering amplitudes [B67].

In TGD framework quantum criticality suggests that locally all loop corrections vanish and coupling constant evolution is discrete. This would mean that the only singularities correspond to poles of propagators and this indeed leads to diagrams in which internal lines have complex on mass shell momenta. If this vision is correct, this part of twistor Grassmann approach does not look relevant from TGD point of view.

BCFW recursion formula

The original form of BCFW recursion formula [B20] was derived for tree diagrams. The finding was that the diagrams can be decomposed to two pieces containing with a propagator line connecting them.

1. The proof of this result was rather simple in spinor helicity formalism and based on modification of two momenta p_k and p_n by BCFW shift:

$$\begin{aligned} p_k(z) &= \lambda_k(\tilde{\lambda}_k - z\tilde{\lambda}_n) \ , \\ p_n(z) &= (\lambda_n + z\lambda_k)\tilde{\lambda}_n) \ , \end{aligned} \tag{5.5.1}$$

   Obviously, the modification is induced by modifications $\tilde{\lambda}_k$ and λ_n. With some assumptions about asymptotic behaviour of scattering amplitude A, one can express the original amplitude $A = A(z=0)$ as residue integral

$$A(z=0) = \frac{1}{2\pi}\oint_C dz \frac{A(z)}{z} \ . \tag{5.5.2}$$

   Here C does not close any other poles than $z = 0$. This integral is the negative of the residue integral around the complement of the region closed by C.

2. It is assumed that poles are the only singularities in this region. Hence one can express $A(z)$ as sum of its poles

$$A(z) = \sum_i \frac{c_i}{z - z_i} \ . \tag{5.5.3}$$

3. With these assumptions the residue integral gives

$$A = A(0) = \frac{1}{2\pi}\sum_i \frac{c_i}{z_i} \ . \tag{5.5.4}$$

   This leads to the desired factorization with c_i reducing to a product of amplitudes and z_i identifiable as a complex pole for the propagator connecting the sub-diagrams in the decomposition.

In [B32] details of the BCFW shift in the general case are given. One assumes a more general shift $p_i \rightarrow \hat{p}_i = p_i + zr_i$ such that r_i are light-like, mutually orthogonal, orthogonal to p_i, and sum up to zero. The modified momenta are complex massless and sum up to zero. One can define $P_I = \sum_{i<I} p_i$ and $R_I = \sum_{i<I} r_i$. The shifted variant $\hat{P_I}^2 = P_I^2 + 2zP \cdot R_I$ is linear in z and vanishes for $z = z_I = -P_I^2/P_I \cdot R_I$. Z_I define the counterparts z_i. Performing the residue integral one obtains $A(0) = \frac{1}{2\pi}\sum_I \frac{c_I}{z_I}$.

This formula allows a recursive construction of tree diagrams by starting from the basic vertices of YM theory. BCFW recursion formula was later generalized to a recursion for the sum planar loops diagrams in terms of diagrams with lower number of loops [B32, B67].

Scattering amplitudes in terms of Yangian invariants defined as multiple residue integrals in Grassmannian manifolds

The generators of Yangian are ordinary conformal generators with conformal weight 0 and dual generators with conformal weight 1. The latter generators act in simple manner in momentum twistor space.

Twistor Grassmannian approach utilizing either twistors or momentum twistors allows to demonstrate that these both conformal symmetry and its dual are present.

The construction of Yangian invariants is summarize in [B67]. Grassmannian residues are Yangian invariants. Yangian transformation introduces total divergence and is exact if its integral vanishes. The operations producing new Yangian invariant can change n or k or both.

1. There are several relatively trivial ways to construct Yangian invariants. One can take the integrand of n-1-D invariant and formally interpret it as integrand of n-D invariant. One can integrate over one twistor variable so that n decreases by one unit.

Invariants can be multiplied. One can a merge invariants by identifying the twistors in the factors of the product. For instance, one can take the fundamental invariants defining 3-vertices and multiply them to build twistor box giving rise to four particles. One can also merge invariants by integrating over the identified invariants.

2. Inverse soft factor [B55] adds to the diagram expressed in terms of spinor helicity formalism one new particle but keeps k constant. Therefore this operation does cannot be applied in TGD where one has only fermions as external particles. The operation can be formulated as a linear shift for $\tilde{\lambda}_a$ and $\tilde{\lambda}_b$.

3. One can prove the BCFW recursion formula for tree diagrams [B20] by using a deformation of the twistor amplitude in helicity spinor formalism allowing to deduce the factorized formula of the amplitude, two adjacent external lines and deform the twistors λ and $\tilde{\lambda}$ in helicity spinor representation by performing the BCFW shift [B62].

   This deformation describes interaction between the external lines, and is essential in the construction of the scattering amplitudes using BCFW recursion. One takes the sum over the products of diagrams with left and right helicities obtained by putting internal particle on mass shell and adds BCFW bridge. BCFW allows to construct all tree amplitudes by starting from fundamental 3-particle amplitudes.

4. Entangled removal [B34, B67, B32] removing two external particles producing a loop in the sense of Feynman diagrammatics but residue of the pole of the propagator is possible and appears as part of the boundary operation for the diagrams. The resulting recursion formula allows to deduce loop corrections.

Twistor Grassmann diagrams are known to allow "moves" [B67, B33]. For instance, moves can be used to remove boxes: it is known that apart from scaling factors depending on momenta the diagrams are reducible to ordinary tree diagrams [B67] (`http://tinyurl.com/zbj9ad7`). This allows to consider the possibility that twistor trees could allow to construct all diagrams. Note however that the moves reducing the twistor diagram to a counterpart of tree diagram gives an overall multiplicative factor depending on momenta and helicities.

From TGD point the definition of loop integrals and Grassmannian integrals as residue integrals is of great potential importance. Scattering amplitudes should be number theoretically universal but in p-adic context the definition of definite integral is very difficult. Residue integral provides however a manner to define multiple residue integrals using only holomorphy and the notion of pole. This could be the deep reason for why one should be able to reduce loop integrals to residue integrals.

There is however a potential problem involved related to number theoretic universality. 2π does not exist p-adically in any reasonable sense (if one wants to define it one must introduce infinite-D extension of rationals by powers of 2π. One might hope that 2π cancels from the scattering amplitudes by normalization. Another possibility is that for an extension containing $exp(i2\pi/N)$ as the highest root of unity, one can define π approximately as $i\pi \equiv N \times (exp(i\pi/N) - 1)$. An alternative option is that only the analogs of tree diagrams having only poles as singularities are possible

Linearization of the twistorial representation of overall momentum delta function

An little but not insignificant technical detail [B34] is the linearization of the constraint expressing the overall momentum conservation by interpreting it as a condition in Grassmannian $G(k,n)$, where k is the number of negative helicities and n is the number of particles, and allowing to reduce integrations over $G(k,n)$ to those over $G(k-2, n-4)$.

Spinor helicity diagrams and twistor diagrams are proportional to a delta function expressing overall momentum conservation. Dropping twistor indices this delta function one reads as $\delta(\sum_k P_k) = \delta(\lambda_i \tilde{\lambda}_i)$. One can combine the 2 components of λ_i and $\tilde{\lambda}_i$ to form 2+2 n-component vectors and interpret momentum conservation as orthogonality conditions for the 2-planes spanned by λ_a and $\tilde{\lambda}_{\dot{a}}$ for $k > 2$. These plane spanned by 2 n-component λ vectors can be interpreted as 2 vectors in $G(k, n-k)$ defining rows of $G(k, n-k)$ matrix. $\tilde{\lambda}$ defines a similar plane in $G(n-k, k)$.

These conditions are equivalent with the condition that there exists in $G(k,n)$ a 2-D C and its $n-k$-dimensional orthogonal complement $\tilde{C}$ such that the 2-plane spanned by λ_a is orthogonal to $\tilde{C}$ and the two-plane spanned by $\tilde{\lambda}_{\dot{a}}$ is orthogonal to C. These conditions can be expressed as a product of delta functions $\delta(C \cdot \tilde{\lambda})$ and $\delta(\tilde{C} \cdot \lambda)$.

Since $G(k)$ acts as a "gauge symmetry" for $G(k,n)$, the first $k \times k$ block of the $k \times n$ matrix representing a point of C can be transformed to a unit matrix so that $k \times (n-k)$ variables remain.. Same can be carried out for the last $n \times (n-k)$ block of $\tilde{C}$ by $G(n)$ "gauge invariance" so that $(n-k) \times n$ variables remain. With these gauge choices the orthogonality conditions can be solved explicitly and corresponding integrations can be carried out. The integration over delta functions leaves $(k-2)(n-k-2)$ variables, the dimension of $G(k-2,n-4)$. $G(k,n)$ reduces to $G(k-2,n-4)$ by momentum conservation.

5.5.2 Basic vision behind scattering amplitudes

It is good to summarize the basic vision about TGD first.

Separation of WCW functional integral and fermionic dynamics

The works of Penrose and Witten have served as inspiration in the attempts to twistorialize TGD and led to the conjecture that the twistor lift of TGD is possible and means that space-time surfaces are replaced with their twistor spaces representable as 6-D surfaces in 12-D product of twistor spaces of M^4 and CP_2. What makes this idea so attractive is that S^4 and CP_2 are the only 4-D compact manifolds with Euclidian signature having twistor space with Kähler structure [A54]. TGD would be unique both from the existence of the lift of Kähler action to the product of twistor spaces of M^4 and CP_2!

What the twistor space of M^4 is, is however not at all clear. It can be defined in two ways: as the usual CP_3 very natural at the level of momentum space or as the trivial bundle $T(M^4) = M^4 \times S^2$ natural in the twistorialization at classical space-time level. Standard twistorialization has however problems.

1. There is problem associated with the signature. Twistorialization works best at signature $(2,2)$ for Minkowski space and gives rise to real projective space P^3.
2. Second problem is that CP_3 should be actually $SU(2,2)/SU(2,1) \times U(1)$. There is clearly something not so well understood.

In the number theoretic vision about TGD twistor space would be replaced with commutative hyper-complex $M_2 \subset M^4 \subset M^8$ and this space is just RP^3 and problems wth signature disappear since 2-D spinors can be chosen to have real basis. For complex momenta this extends to CP_3. Number theory would also justify the identification of geometric twistor sphere as $M^4 \times S^2$.

In TGD the dynamics of fields is replaced with that for 4-surfaces. Penrose's idea about generalization of holomorphy of field modes in twistor space generalizes to the holomorphy of the representation of 6-surface representing twistor bundle of space-time leads to a concrete ansatz for space-time surfaces as preferred extremals [L24] [L47].

SH leads to the proposal that the data determining space-surfaces are preferred extremals is given at 2-D surfaces and these 2-D surfaces bring in mind Witten's twistor strings [B29]. By SH the functional integral over them would correspond to that over WCW and twistor amplitudes asignable to given space-time surface would be constructed at fermionic level by the analog of twistor Grassmannian approach. This integral over 2-surfaces corresponds to the deviation of TGD from QFT in fixed background and cannot be equivalent with the introduction of twistor strings.

Adelic physics and scattering diagram as a representation of computation

Adelic physics [L41] suggested to provide quantum physical correlates also for cognition is in a central role. Adelic physics predicts the hierarchy $h_{eff} = n \times h$, where n as dimension of the extension is divisor of the order its Galois group identified in terms of dark matter regarded as a phase of ordinary matter. p-Adic physics and p-adic length scale hypothesis could be also understood.

The number theoretic universality of scattering amplitudes suggests that all loops vanish identically and the evolution of various couplings constants is discrete occurring by phase transitions changing the extension of rationals and values of various coupling parameters.

1. The vanishing of loops at the level of space-time action would mean that the loops associated with the functional integral defined by the action, which is sum of Kähler action and volume term. This vanishing would state essentially local quantum criticality as invariance of coupling parameters under local renormalization group evolution. One would obtain only a sum of action exponentials since Gaussian and metric determinants cancel each other in Kähler metric.

2. Exponents of Kähler action represent a number theoretical nightmare.

   (a) The functional integral expressions for scattering amplitudes are normalized by a functional integral for for the vacuum state. This implies that only the ratios X_i/X of the exponents X_i for the extrema and sum $\sum X_i$ appear in the amplitudes [L41] so that there are slightly better hopes of achieving number theoretic universality.

   (b) Number theoretical universality forces to imagine even more attractive option making sense in ZEO but not in standard ontology. If the amplitude is sum over the contributions normalized by corresponding exponentials X_i rather than $\sum X_i$, exponentials cancel altogether and the couplings constants appear only in boundary conditions. In this case one could speak of a basis of zero energy states assignable to various extrema of the action. The real part of the action is maximum and the the imaginary part of the action saddle point if preferred extrema are minimal surface extremals of Kähler action [L24]. Number-theoretical universality more or less forces this option.

3. An even stronger proposal is based on the idea that the TGD analogs of stringy diagrams. The lines of these diagrams correspond to light-like parton orbits carrying fermion lines and meeting at vertices which are partonic 2-surfaces. The proposal is that the topological diagrams involving analogs of loops represent algebraic computations so that all diagrams with given initial and final collection of algebraic objects are equivalent.

   If this is the case, all topological diagrams should reduce to topological tree diagrams by a generalization of the duality symmetry of the old-fashioned hadronic string model stating that the sum of s-channel resonances equals to the sum of t-channel exchanges and that these diagrams can be constructed as twistor Grassmann diagrams by allowing on mass shell fermions with complex momenta at internal lines. For external particles the momenta could be real and light-like in 8-D sense. A weaker condition is that real and imaginary parts of complex momenta 8-D momenta are separately light-like and orthogonal.

   One could indeed argue that one cannot allow loops of this kind since it would be impossible to decide which kind graph experimental scattering situation corresponds if all these graphs are different since one observes only the initial and final states. Therefore all scattering diagrams with same real particles in the final states correspond to identical scattering amplitudes.

   These diagrams would correspond to the same amplitude but it might be possible to perform a localization to any of them. p-Adically however the corresponding space-time surface would be different by p-adic non-determinism (the number theoretic discretization - cognitive representation - defined by the common points of reality and p-adicities as space-time surfaces would be different): one might say that the tree representation involves smallest cognitive representation and is therefore the shortest one.

   If the action exponentials X_i cancel from the scattering amplitudes, this option can indeed make sense. Otherwise it is extremely implausible since different contributions would have different vacuum weights.

4. If only the twistor analogs from tree diagrams in Feynman sense are allowed, the scattering amplitudes are rational functions of external momenta as strongly suggested by the number theoretic universality and by the requirement that the diagrams can be interpreted in terms of algebraic computations so that the simplest manner to do the computation corresponds to a tree diagram. Even tree diagrams in Feynman sense are planar so that one would get rid of the basic problem of the twistor approach to SUSY.

Quantum classical correspondence (QCC) states that scattering diagrams have classical counterparts in the sense that fermion lines correspond to the boundaries of string worlds sheets assignable to the light-like orbits of partonic 2-surfaces and topological 3.vertices correspond to 2-surfaces at which the ends of light-like orbits meet. This correlation is extremely restrictive and it is not at all clear whether it leaves room for loops.

In the most general case one would have a superposition of allowed space-time surfaces realizing scattering diagram with given initial and final quantum numbers identified as corresponding classical charges.

The idea about diagram as computation suggests that the simplest possible diagram - tree diagram - is realized together with the corresponding space-time topology. If diagrams with topological loops are possible this requires the existence of moves transforming diagrams to each other. This condition might be not consistent with the condition that the move acts on the space-time surface too. Very simple diagrammatics - even twistor tree diagrammatics - could follow from mere QCC.

Classical number fields and $M^8 - H$ duality

Quaternionicity and octonionicity is second central aspect of number theoretical vision.

1. The key concept is $M^8 - M^4 \times CP_2$ duality allowing to see space-time surfaces quaternionic surface in M^8 or as holomorphic surfaces in the twistor space $T(M^4) \times T(CP_2)$. This would realize SH. Physical states are characterized by quaternionic (possibly complexified-) octonion valued 8-momenta in accordance with the vision that tangent space Minkowskian region of space-time surface is quaternionic and contains preferred hyper-complex M^2, which can depend on point provided that tangent spaces $M^2(x)$ integrate to 2-D surface. This view leads to a new view about QCD color as octonionic color.

2. Twistor space reduces to that associated with M^2 and 2-D variant of conformal invariance corresponds to SO(2,2) and leads to the identification real projective space P^3 as twistor space. One can however complexify it to CP_3 since momenta are in general complex. The signature is (1,1) so that bi-spinors λ, $\tilde{\lambda}$ have real basis and twistor Fourier transform can be defined as ordinary Fourier transform. The reality of M^2 or induced spinors at string world sheets might allow to have SUSY without Majorana spinors.

   The reduction of external momenta to M^2 implies that real and imaginary parts are parallel and light-like. At classical level this poses strong conditions on preferred extremals. This does not require that color and electroweak quantum numbers are complex. The reason is that they emerge as labels of wave functions in twistor space $T(CP_2)$ representing wave functions in the moduli space of transversal E^2s with corresponding helicity identifiable as em charge.

   Localization of the light-like 8-momentum is possible to preferred M_0^2. Localization does not imply the disappearance of color wave function. The transversal E^2 momentum degrees of freedom however disappear. In the case of leptons and hadrons complete localization could be a good approximation but not in the case of quarks.

Elementary particles have fundamental fermions as building bricks

The assumption that the physics of elementary particles reduces at fundamental level to that of fundamental fermions has strong implications, when combined with the twistor Grassmann approach.

1. In TGD elementary particle would correspond to a pair of wormhole throats of wormhole connecting two space-time sheets with Minkowski signature. Wormhole itself would have Euclidian signature. Wormhole contacts would be connected by monopole flux tube with fermionic quantum numbers at the 4 wormhole throats defining the partonic 2-surfaces.

2. Fundamental vertices are associated with 2-surfaces at which light-like 3-surfaces carrying fermions and antifermions as string world sheet boundaries are glued together along their ends. Note that these surfaces are analogous to vertices of Feynman diagrams and singular as 4-surfaces but 3-surfaces are smooth unlike for stringy vertices.

3. Fermion lines correspond to the boundaries of string world sheets at the light-like orbits of partonic 2-surface at which the signature of the induced metric changes. At momentum space M^8 this picture should also make sense since space-time surfaces in M^8 and H would correspond to each other by $M^8 - H$ duality. At the level of M^8 the orbits of fermion lines could be seen as light-like geodesics along with twistor spheres move. At the edges of string world sheets they would intersect at single point and give rise to external massless particle.

4. The basic vertex is 4-fermion vertex in which fermions scatter classically and assignable to the 2-surface at which the ends of light-like 3-surfaces representing partonic orbits intersect. There would be no local 4-fermion vertex. Fermions would move as free particles in the background and the background would gives rise to the interaction between fermions at partonic vertices analogous to vertices of Feynman diagrams. This would automatically resolve possible problems caused by divergences and would be analogous to the vanishing of bosonic loops from WCW functional integration.
5. FFB couplings could be identified in terms of $FF(F\overline{F})$ couplings, where $F\overline{F}$ is associated with the same partonic orbit. These couplings would not be fundamental.

What could SUSY mean in TGD?

Extended super-conformal invariance is basic symmetry of TGD but it is not whether it possible to have SUSY (space-time supersymmetry) in TGD framework. Certainly the SUSY in question is not $\mathcal{N}=1$ SUSY since Majorana spinors are definitely excluded. $\mathcal{N}=2$ SUSY generated by right-handed neutrino and antineutrino can be however considered.

1. If one allows the boundaries of string world sheets carry fermion number bounded only by statistics (all spin-charge states for quarks and leptons would define maximal $\mathcal{N}$ for SUSY). This would allow local vertices for fermions and does not look like an attractive option unless SUSY manages to cancel the divergences.
2. SUSY could mean addition of fermions as separate lines to the orbits of wormhole throat. This SUSY would be broken and only approximately local. The question what the propagator for the many-fermion state at same string line is, is not quite obvious. SUSY would suggest propagator determined by the total spin of the state. I have also considered the possibility that the propagator is just the product of fermionic propagators acting on tensor power of single fermion spaces. The propagator behaves as $1/p^N$ for N fermion state and only for $N=1,2$ one would have the usual behavior. This option is not attractive.
3. SUSY could mean addition of right-handed neutrino or its antiparticle to the throat. The short range of weak interactions is explained by assuming that pair of right-handed neutrino and left-handed neutrino compensates the weak isospin at the second wormhole throat carrying quantum numbers of quark or lepton.

   Addition of right-handed neutrino or its antiparticle or both to a given boundary component could give rise to $\mathcal{N}=2$ SUSY. The breaking of SUSY could correspond to different p-adic length scales for spartners. Mass formula could be exactly the same and provided by p-adic thermodynamics. Why the p-adic mass scale would depend so much on the presence of covariantly constant ν_R having no color and ew interactions nor even gravitational interaction, remains to be understood. If the extensions of rationals are different for the members of SUSY multiplet, the corresponding preferred p-adic primes would be different and this could explain the widely different p-adic mass scales. One can of course ask the covariant constancy means that ν_R does not have any coupling to anything and its presence is undetectable.

5.5.3 Options for the construction of scattering amplitudes

There are several guidelines in the construction of scattering amplitudes.

1. SH in strongest form would mean that string word sheets and partonic 2-surfaces are all that is needed. In number theoretical vision also fixing the extension of rationals associated with the intersection of realities and p-adicities is needed and leads to a hierarchy of extensions which could realized discrete coupling constant evolution. SH would suggest that hybrids for analogs of string diagrams and Feynman diagrams code for the scattering amplitudes.
2. QCC suggests that the eigenvalues of the Cartan algebra generators of symmetries are equal to classical Noether charges. A weaker condition is that the eigenvalues of fermionic generators not affecting space-time surfaces are equal to the classical Noether charges. The generators have also bosonic parts acting in WCW.

   A prediction following from the condition that there is charge transfer between Euclidian and Minkowskian space-time regions is that the classical charges must be complex valued guaranteed if Kähler coupling strength as a spectrum of complex values. One proposal is that the spectrum of zeros of Riemann zeta determines if [L17]. This supports the twistorial view that momenta in the internal lines can be regarded as complex light-like on mass shell momenta.

3. QCC also suggests that scattering diagrams have space-time correlates. The lines of diagrams correspond to light-like orbits of partons at which the signature of induced metric changes. Vertices correspond to partonic 2-surfaces at which these 3-D lines meet. At fermion level fermion lines at partonic orbits correspond to boundaries of string world sheets.

This however leaves several alternative visions concerning the construction of scattering amplitudes.

What scattering diagrams are?

What does one mean with scattering diagrams is not at alle clear.

1. Are they counterparts of Feynman diagrams so that one would have a superposition of all space-time topologies corresponding to these diagrams? Probably not.

2. Or are they counterparts of twistor Grassmannian diagrams in which all particles are on mass shell but with possibly complex light-like quaternionic 8-momenta in $M^8 = M^4 \times E^4$ with $M^4 = M_0^2 \times E^2$. Why this option is interesting is that twistor Grassmann diagrams allow large number of moves reducing their number.

   This would translate to a conserved and massive longitudinal M^2-momentum; which for a special choice of M^2 is light-like, a wave function in the space of transversal E^2 momenta; color partial wave in the moduli space of E^2 planes for given M_0^2; and em charge describable as CP_2 helicity and allowing twistorialization.

   There is however a problem: the transverse E^6-momentum makes M^2 momentum massive and twistorialization fails. But what if the 8-momenta are real and in twistorial description M^2 momentum becomes complex but light-like. The square for the real part of M^2 momentum would be equal to the square of real E^6 momentum and twistor approach would apply! This map would be define the essence of M^2-twistorialization.

In ZEO one can interpret the construction of preferred extremals as a boundary value problem with ends of space-time surfaces at the boundaries of CD and the light-like orbits of partonic 2-surfaces defining a closed 3-surface and defining the scattering diagram as 3-D boundary. If so, it might be possible to construct rather large number of diagrams, even counterpartz of loop diagrams.

The situation would be analogous to the construction of soap films spanned by wires with wire network analogous to the network formed by the partonic orbits. Also an analogy with 4-D tensor network suggests strongly itself and scattering diagrams representing zero energy states would correxpond to the states of the tensor network.

The basic space-time vertex would be 3-vertex defined by partonic 2-surface.The basic fermionic vertex would be 4-fermion vertex in which fermions do not exchange gauge boson but interact classically at the 2-D vertex. All particles emerge as bound states of fundamental fermions at boundaries of string world sheets.

1. The basic view would be that M^2 momenta, and transversal momenta correspond to M^4-momenta. The moduli space for $M_0^2 \times E^2$ planes corresponds to CP_2 and color quantum numbers. M^2 helicities and electroweak quantum numbers would be coded to the weights twistor wave functions in twistor space if $M^2 \times CP_2$.

2. One approach to scattering amplitudes relies on symmetries. Twistor Grassmannian approach suggest strongly Yangian symmetry. The diagrams should be representations of multi-local Yangian algebra with basic algebra being that of the conformal group of M^4 restricted to M^2.

   This would give nicely real projective space RP^3 allowing to solve some problems of the standard twistor approach. In color degrees of freedom one would have color Yangian: hadrons could correspond to the multilocal generators created by multi-local Yangian generators. The E^2 degrees of freedom would correspond to states generated by Kac-Moody algebra and also now one could have Yangian algebra. The states for the representation of Yangian itself would be singlets.

Besides fermionic lines there are string world sheets. Infinite-D 2-D conformal group and Kac-Moody symmetries act as symmetries for string world sheets. The super-symplectic group would the isometry group of WCW and would give rise to conditions analogous to Super Virasoro conditions. These conditions would be satisfied by preferred extremals realizing number theoretic variant of SH. Also these symmetries would be extended to their Yangian versions naturally.

3. One can argue that classical field equations do not allow all possible diagrams. More precisely, for a given extension of rationals adelic physics allows only finite number diagrams and the extension induces a natural cutoff as minimal distance between points with coordinates in the extension representing intersection of reality and p-adicities [L41].

   The assumption that the end points of fermionic lines at partonic 2-surfaces at ends of CD and at the vertices carry fermions would give an immediate connection with the adelic physics. As the dimension of the extension increases, the number of the points in the intersection increases and more lines appear in the allowed diagrams. This would give rise to a discrete coupling constant evolution, hierarchy of Planck constants, and p-adic length scale hypothesis.

   Quantum criticality strongly suggests that coupling constant evolution is locally trivial and is discretized with discrete steps realized as phase transitions changing the extension. Galois group would be the fundamental number theoretic symmetry group acting on the intersection and its order would correspond to $h_{eff}/h = n$ allowing to realize the analogs of perturbative phases of gauge theories as perturbative phases.

4. The discreteness of coupling constant evolution demands that loop corrections vanish. This makes perfect sense for the functional integral over WCW. But what about fermionic degrees of freedom and topological counterparts of scattering diagrams, which very probably do not correspond to Feynman diagrams but could be analogous to twistor diagrams? For fermions there is actually no perturbation theory since effective 4-fermion vertices correspond to classical scattering of external fermions at partonic 2-surfaces defining the vertices. This is not a problem since thanks to h_{eff} guaranteeing the existence of perturbative expansion.

Three roads to follow

In ZEO construction of scattering amplitudes is basically a construction of zero energy states and one must be very cautious in applying QFT intuitions relying on positive energy ontology. One ends up to to a road fork.

Option I: Can one interpret the topological space-time diagrams as analogs of Feynman diagrams and assume that by quantum criticality the sum over the topological loops vanish? This option looks rather ad hoc.

Option II: Can one assume - with inspiration coming from adelic physics - that the number of these loops with fixed states at the boundaries of CD is finite and one just sums over these states with weights given by the exponential of the space-time action?

Here one encounters problems with number theoretical universality [L41]. One has superposition of vacuum exponentials over the diagrams and number theoretical universality demands that the ratio of given exponential to the sum is in the extension of rationals involved. This is very tough order - perhaps too tough.

Option III: Can one follow number theoretical vision suggesting that scattering diagrams correspond to computations in some sense [L22]. This leads to a new road fork.

1. Option IIIa): Could one generalize the old-fashioned string duality and require that there exist a huge symmetry allowing to transform the scattering diagrams using basic moves to tree diagrams? The basic moves would allow to shift the end of line past vertex and to remove self energy loop and hence the transformation to tree diagrams would become possible. Originally it was inspired by the idea that the vertices of the scattering diagram correspond to products and co-products in quantum algebra and that the condition involved can be interpreted as algebraic identities.

Twistor Grassmannian diagrams indeed allow moves allowing surprising simplification allowing to show that all loop corrections with a given number of loops sum up to something proportional to a tree diagram [B67].

The assumption that the states moving in the internal lines have light-like quaternionic M^8 momenta gives very strong constraints on the moves and it might well be that the moves are not possible in the general case. Even if the move is possible, the value of the action exponential can change so that this option seems to demand mathematical miracles. The proposed manner to achieve number theoretical universality however eliminates action exponentials.

The mathematical miracle might be made possible by the possibility to find preferred M_0^2 in which the 2-momentum of fermion line is light-like. If M_0^2 is constant along entire fermion line, it seems to be possible perform the gliding operation past vertices as will be found. Note that ach fermion can wander around the network formed by the partonic orbits.

Note that the different space-time surface realizing equivalent computations would be cognitively non-equivalent since the cognitive representation defined by the points in extension of rationals would be different. Optimum computation would have smallest number of points and would correspond to tree diagram.

2. Option IIIb): Should one sum over the possible diagrams so that one would have quantum superposition of computations. This is done for loop diagrams in twistor Grassmann approach. Infinite sum is however awkward number theoretically. Adelic vision suggests that the number of loops is finite. The action exponentials would not disappear from the scattering amplitudes and are very problematic from the point of view of number theoretical universality.

3. Option IIIc): Could one regard the light-like partonic orbits as part of the dynamical system - this is what effectively is done if they form part of connected 3-surface defining the topological scattering diagram - and assume that each such diagram corresponds to a different physical situation analogous to a computation?

   One can argue that one must be also able to localize the zero energy state to single computation by state function reduction [L46]! State function reduction to single diagram should be possible. A rather classical picture about space-time would emerge: one would have just a superposition of space-time surfaces with the same topology and same action apart from quantum fluctuations around the point which is maximum with stationary phase. One would also have color wave functions and momentum wave functions in cm degrees of freedom of partonic 2-surfaces as WCW degrees of freedom.

   The action exponential, which is very problematic from the point of view of number theoretic vision, would be cancelled from the functional integral since it is normalized by the action exponential. The dependence on coupling parameters is however visible in the boundary conditions at boundaries of CD stating the vanishing of most supersymplectic charges and identifying the remaining super-symplectic charges and also isometry charge with the fermionic counterparts.

   This picture would be extremely simple and would be analogous to that of integrable quantum field theories in which the integral over small fluctuations gives Gaussian determinant and action exponential (now Gaussian determinant is cancelled by the metric determinant coming the Kähler metric of WCW) [K80].

   One can argue that the absence of loops makes it impossible to have non-perturbative effects. This is not true in adelic physics. Recall that the original motivation for $h_{eff} = n \times h$ was that this phase is generated with perturbation theory ceases to converge [**?**]. The large value of h_{eff} scales down the coupling strengths proportional to $1/h_{eff}$ and perturbation theory works again.

It must be admitted that one must accept all these options. Number theoretical universality of scattering amplitudes would select $IIIa$) and the need to realize given topological diagram using complex enough extension of rationals supports Option IIIc). I believe that the large number of the options reflects my limited mathematical understanding of the situation a careful analysis of the general implications of the options allows to pinpoint the most feasible one.

5.5.4 About problems related to the construction of twistor amplitudes

The dream is to construct twistorially fermionic scattering amplitudes and this requires the identification of fermionic 4-vertex. There are however several conceptual problems to be solved.

Could M^2 momenta be massive?

The naïve objection against massive particles is that one loses the twistorial description both in M^4 sense and M^2 sense. Real quaternionic M^8 momenta are massless but the transversal momentum in E^6 degrees of freedom makes M^2 momenta and M^4 momenta for arbitrary choice of M^4 are massive, and one cannot describe the M^2 and M^4 momenta using the helicity spinor pair $(\lambda, \tilde{lambda})$. The beautiful formalism seems to be lost.

1. The naïve argument is however wrong in TGD framework where particles are massless in M^8 sense. This means that mass does not correspond to $\overline{\Psi}\Psi$ in Dirac action but to comes from E^4 momentum (CP_2 "momentum"). 8-D chiral symmetry is unbroken as required by separate conservation of lepton and baryon numbers. In preferred M_0^2 one can indeed make M^2-momentum light-like.
2. Furthermore, 4-fermion twistor amplitudes are *holomorphic* functions of λ_i . There is no dependence of $\tilde{\lambda}$ and therefore no information about light-likeness! Why this amplitude could not describe the scattering of fermions only apparently massive in TGD Universe? Note that the momentum conserving delta function depends on the masses of the particles so that mass-dependence would be purely kinematical and analogous to the dependence on transverse momentum squared. Note that this argument makes sense also for M^4 twistorialization. If this view is correct then twistors are something more profound than momenta.
3. For M^2 twistorialization end would end up to effective (2,2) signature favored by twistorialization. (1,1) signature of real M^2 becomes (2,2) signature for complexified M^2 and real twistor space RP^3 is replaced with CP_3. This looks attractive description. If this picture is correct, all the nice results such as the possibility to assume reduction of amplitudes to positive Grassmannian remain unaffected.

Momentum conservation and mass shell conditions in 4-vertex

What is the exact meaning of the mass shell condition?

1. $H = M^4 \times CP_2$ harmonics would suggest that it mass squared in M^4 is eigenvalue of spinor d'Alembertian plus possible super-conformal contribution from Super Virasoro algebra, which is integer valued in suitable units. M^4-momentum decomposes to longitudinal M_0^2 momentum and transversal E^2 momentum. Super Virasoro algebra in transversal degrees of freedom suggests quantization of E^2 mass squared in integer multiples of a basic unit.
2. The CP_2 part of wave function in H corresponds in M^8 to a wave function in the moduli space of transversal planes E^2 assignable to M_0^2 and is involved only if the deformations of M^4 (or equivalently E^2) are present.
3. In the preferred frame M_0^4 the wave function would be strictly localized in single point of CP_2 and have maximally uncertain color quantum numbers. This kind of localization does look feasible physically. For instance, for color singlet CP_2 wave function of right-handed neutrino there is no localization. For sharp localization of 8-momentum to M_0^2 both color degrees and transvervsal E^2 degrees of freedom would effectively disappear.
4. The wave function in transversal E^2 momentum space with interpretation in terms of transversal momentum distribution - this at least in the case of hadrons.
5. The physically motivated assumption is that string world sheets at which the data determining the modes of induced spinor fields carry vanishing W fields and also vanishing generalized Kähler form $J(M^4) + J(CP_2)$. Em charge would be the only remaining electroweak degree of freedom. The identification as the helicity assignable to $T(CP_2)$ twistor sphere looks therefore natural. Note that the contribution to mass squared would be proportional to Q_{em}^2 so that one would obtain the electroweak mass splitting automatically. This is true also for CP_2 spinor harmonics.

How plausible topological loops are?

Topological loops are associated with the networks formed from the orbits of partonic 2-surfaces meeting at their ends (this would define topological 3-vertex containing fermionic 4-vertex). The tree topologies would provide a nice space-time description of particle reactions but loops could be possible? The original vision about construction of WCW geometry indeed was that the space-time surfaces with fixed ends are unique.

In the original vision the non-determinism of Kähler action inspired the hypothesis that loops are possible but volume term removes to high extent this non-determinism. In the recent vision the fusion of 3-surfaces at the ends of CD with light-like parton orbits to single 3-surface as a boundary condition (analogous to a fixing of a frame for soap films) would define the scattering diagram classically. There is no reason why it could not contain topological loops. Option IIIa) assuming that one can transform the diagrams ot tree diagrams, is therefore attractive.

1. There are also conditions from space-time dynamics. Twistor graph topologies correlate with space-time topologies since fermion line are inside the parton orbits and at vertices the ends of the orbits meet. Topological vertices would be basically 3-vertices for partonic 2-surfaces. The fermion and anti-fermion lines associated with the effective boson exchange would be naturally associated with opposite throats of wormhole contact.

   By above argument one can in ZEO pose at space-time level conditions fixing the vertices and identify the graph topology as a topology of the network of light-like 3-surfaces defining the diagram as boundary of 3-surface defined by the union of the ends of space-time and by parton orbits forming a connected surface.
2. There is a further delicacy to be taken into account - measurement resolution coded by the extension of rationals involved. This might allow to interpret addition of loops as in quantum field theories: as a result of increased measurement resolution determined dynamically by the intersection of reality and p-adicities. Different computation yielding the same result would not be cognitively equivalent since these intersections would be different.
3. If this view is correct, one can obtain also loops but non-negativity of energy for a given arrow of time for quantum state would allow only loops resulting from the decay and re-fusion of partonic 2-surfaces. Tadpoles appearing in BCFW recursion formula are impossible if the energy is non-negative. One can of course ask whether the sign of energy could be also negative if complex four-momenta are allowed. If so, one could have also tadpoles classically.

Identification of the fundamental 4-fermion vertex

The fundamental 4-fermion vertex would not be local 4-fermion vertex but correspond to classical scattering at partonic 2-surface. This saves from the TGD counterparts of the problems of QFT approach produced by non-renormalizability.

What would be this 4-fermion vertex? Yangian invariance suggests that the classical interaction between fermions must be expressible in terms of fictive 3-vertex of SUSY theories describing classical interaction as exchange of a fictive boson. This leaves 3 options.

Option I: 4-fermion vertex could be fusion of two 3-vertices with complex massless 8-momenta in M^8 picture. For instance, the exchanged momentum could be complex massless momentum and external momenta real on-mass-shell momenta. This vertex does not have QFT counterpart as such.

Loops could be absent either in the strong sense twistorial loops are absent (Option Ia) or in the sense that corresponding Feynman diagrams contain no loops (Option Ib). In particular, formation of BCFW bridge would not be allowed for Option Ia). Given diagram would be twistorial tree diagram obtained by replacing the vertices of ordinary tree diagram with these 4-vertices with complex massless fermions in 8-D sense.

Option II: 4-fermion could be identified as BCFW bridge associated with a tree Feynman diagram describing an exchange of a fictive boson. This 4-vertex would be analogous to an exchange of ordinary boson and counterpart for a QFT tree diagram. One can even forget the presence of the fictive boson exchange and write the formula for the simplest Yangian invariant as a candidate for four-fermion vertex.

Option III: If one allows higher fermion numbers at the same line, it is also natural to allow branching of lines. This requires allowance of 3-vertex as branching of fermion line as analog of splitting of open string (now strings are actually closed if they continue to another space-time sheet through wormhole contact). The situation would resemble that in SUSY. One cannot completely exclude this possibility.

Consider now the construction of 4-fermion vertex in more detail.

1. The helicities of fermions are $h_i = \pm 1$ and the general conjecture for the 4-fermion twistorial scattering amplitude is the simplest possible holomorphic rational function in λ_i, which does not depend on $\tilde{\lambda}_i$, and satisfies the condition that the scaling $\lambda_i \to t\lambda_i$ introduces the scaling factor t^{-2}.
2. The rule is that fermions correspond to 2 positive powers of λ_i and antifermions to 2 negative powers in λ_i: schematically the $F_1F_2\bar{F}_3\bar{F}_4$ vertex is of form $\lambda_1^2\lambda_2^1/\lambda_3^2\lambda_4^2$ and constructible from $\langle\lambda_i, \lambda_j$. One can multiply any term in the expression of vertex by a rational function of for which the weights associated with λ_i vanish. Ratios $P_i(f)/P_j(f)$ of functions $P(f)$ obtained by via odd permutations P of the arguments λ_i of function

$$f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \langle\lambda_1, \lambda_2\rangle\langle\lambda_2, \lambda_3\rangle\langle\lambda_3, \lambda_4\rangle\langle\lambda_4, \lambda_1\rangle$$

3. invariant under 4 cyclic permutations. The number of these functions would be $4!/4 = 3! = 6$ corresponding to the 6 orbits of an odd permutation under the cyclic group Z_4. The simplest assumption is that these functions are not involved.

   The simplest guess for the 4-fermion scattering amplitude would be following:

$$T(F_1, F_2, \overline{F}_3, \overline{F}_4) = J \times \frac{\langle\lambda_1, \lambda_2\rangle^2}{\langle\lambda_3, \lambda_4\rangle^2} \quad . \tag{5.5.5}$$

   Charge conjugation would take the function to itse inverse. J is constant.

4. In 4-fermion vertex one has exchange of fictive boson and annihilation to fictive boson and the particles i, j in the vertex should contribute $\langle\lambda_i, \lambda_j\rangle$ to the scattering amplitudes.

Remarkably, this amplitude is holomorphic in λ_i and has no dependence on $\tilde{\lambda}_i$ and therefore carries no information about whether the momenta are light-like or not. It seems that one could allow massive fermions characterized by (λ_i, μ_i) and fermion masses would not be a problem! As already explained in TGD mass is not M^8-scalar and states are massless in 8-D sense: hence twistorialization should work!

One could construct more complex diagrams in very simple manner using these basic diagrams as building bricks just as in the twistor Grassmann approach. One could form product of diagrams A and B using merge operation [B67] identifying twistor variables Z_a and Z_b belonging to the two diagrams A and B to be fused.

For Option Ia) the diagram would represent repeated on mass shell 4-fermion scatterings but with of mass shell particles having complex momenta in 8-D sense. Real on mass shell particles would have massless but real 8-D momenta and physical polarizations.

The conservation of baryon and lepton numbers implies for all options that only $G(m, n = 2\times m)$ Grassmannians are needed. This simplifies considerably the twistor Grassmannian approach.

Why fermions as fundamental particles (to be distinguished from elementary particles in TGD) are so special?

1. The mass of the fundamental fermion is not visible in the holomorphic basic amplitude being visible only via momentum conserving delta function $\delta(\sum_i \lambda_i\tilde{\mu}_i)$. This property holds true also for more complex diagrams. Massivation does not require in TGD framework $\bar{\Psi}\Psi$ term in Dirac action since M^4-massive fermions are M^8-massless and have only chiral couplings in 8-D sense. Scalar coupling would also break separate baryon and lepton conservation. Mass term correspond to a momentum in $E^4 \subset M^4 \times E^4 = M^8$ degrees of freedom. Massivation without losing 8-D light-likeness is consistent with conformal symmetry and with 8-D twistor approach.
2. Fermions are exceptional in the sense that the number of helicities is same for both massive and massless fermions. In particular, 4-fermion amplitude has $k = n/2$ and positive Grassmannian $G(n/2, n)$ with special symmetry property that one can take either negative or

positive helicities in preferred role, could be important. For massless states with higher spin the number of helicities is 2 and maximal spin is $J_{max} = h_{max}/2$. For M^4 -massive states also the lower helicities $h_{max} - 2k$ are possible. The scattering amplitudes remain holomorphic.

3. For SUSY one would have all helicities $h(k) = h_{max} - k$ and the general form of amplitude could be written from the knowledge of $h(k)$. The number of fermions at the boundary of string world sheets could be maximal allowed by statistics. This would give SUSY in TGD sense but would require splitting of string boundaries: it is not clear whether this can be allowed. For light-like orbits of partonic 2-surface it has been assumed.

   Sparticles could correspond to states with higher fermion number at given partonic orbits. In this case one expects only approximate SUSY: the p-adic primes characterizing different SUSY states could be different. In adelic physics different p-adic prime could correspond to a different extension of rationals: one might say that the particles inside super-multiplets are at different levels in number theoretic evolution!

BCFW recursion formula as a consistency condition: BCFW homology

The basic consistency condition is that the boundary operation in the BCFW recursion formula gives zero so that the recursion formula can be solved without introducing sum over topological loops. The twistorial trees would have no boundaries but would not be boundaries and would be therefore closed in what might be called BCFW homology. Diagrams would correspond to closed forms.

Consider first the proposal assuming that all diagrams are equivalent with twistorial string diagrams with fermionic 4-vertex as the basic vertex. The boundary operation appearing in BCFW formula gives two terms [B34, B67, B32]. Recall that options I, II, and III correspond to twistorial diagrams without loops created by BCFW bridges, to twistor diagrams assignable to Feynman diagrams without loops, and to diagrams analogous to SUSY diagrams for which fermion lines carry also higher fermion number and can split.

1. The first term results as one BCFW bridge by contracting the three lines connecting the external particles to a larger diagram to a point in all possible ways. The non-vanishing of this term does not force loops in the sense of Feynman diagrams. For Option Ia) (no twistorial loops) there are no BCFW boxes to be reduced so that the outcome is zero.

   For option Ib) (no Feynman loops) a BCFW box diagram for which the two outward direct lines of the bridge are fictive, this operation makes sense and reduces the box to that describing the basic 4-fermion vertex. Same is true for the option II. For option III the operation would be essentially the same as in SUSY.

2. Second term corresponds to entangled removal of a fermion and anti-fermion and if it is non-vanishing, loops are unavoidable. This operation creates a closed fermionic loop to which several internal lines couple. By QCC the fermionic loop would be associated with a topological loop. One can argue that the topological tadpole loop must be closed time loop and that this is not possible since the sign of energy must change at the top and bottom of the loop, where the arrow of time changes: actually the energy should vanish. The same result would obtained if one requires that the energy identified as real part of complexified energy is non-negative for all on mass shell particles.

   Consider the 4-fermion vertex to which the fermionic tadpole loop is associated. Entangled removal gives for the members of a pair of external lines opposite momenta and helicities in twistor-diagrammatics. If so, there exist a vertex for which one fermion scatters in forward direction. Momentum conservation implies the same for the second fermion. One would obtain amplitude, which equals to unity rather than vanishing! Integration over four-momenta would give divergence. However, if the 4-momentum in the tadpole vanishes, the corresponding helicity spinor and also the amplitude vanishes. QCC indeed demands that fermionic loop corresponds to a time loop possible only only if the energy and by time-likeness also 3-momentum vanishes.

It seems that only the simplest option - Option Ia) - is consistent with the BCFW reduction formula. One can say that scattering diagrams are closed objects in the BCFW cohomology. Closedness condition might allow also topological loops, which are not tadpole loops: say decay of fermion to 3 fermions fusing back to the fermion.

Under what conditions fermionic self energy loop is removable?

Scattering diagram as a representation of computation demands that the fermionic "self energy" loop involving two external fermions gives free propagator. The situation in which the vertex contains only *light-like* complex momenta in M_0^2 can be considered as an example. In fact, one can always choose in M^8 the frame for given component of state in this manner.

1. The three fermion/antifermion internal lines in the loop would be light-like in complex 2-D sense as also external momentum. For external momenta $Re(p(M^2))$ would be light-like and orthogonal to light-like $Im(p(M^2))$: it is not clear whether $Im(p(M^2))$ vanishes.

   Light-likeness condition gives $Re(k)^2 - Im(k)^2 = 0$ and $Re(k) \cdot Im(k) = 0$, and $Re(k) = \pm Im(k)$ as a solution meaning that $Re(k)$ is proportional to a light-like vector $(1,1)$ or $(1-1)$. This applies to p, k_1,k_2, and $p - k_1 - k_2$. All these vectors are proportional to the same light-like vector in M^2.

   Apart from the degeneracy for sign factors the situation is equivalent with real 2-D case and one has from momentum conservation that the real parts of the virtual momenta are light-like and parallel and one has $Re(k_i) = \lambda_i p$ leaving two real parameters λ_i.

2. The only possible outcome from the integral is proportional to $D_F(p)$. The outcome is non-vanishing if the proportionality constant is proportional to $1/p^2$. This dependence should come from 4-fermion vertices. The integrand is proportional to the product $\lambda_1\lambda_2(1-\lambda_1-\lambda_2)$ and involves times the $D_F(p)$. Vertices give the inverses of these scaling factors. Since the outcome should be proportional to $1/D_F$ and lines are proportional to p^3, the 4- vertices should give a factor $1/p^2$ each.

   Assuming this one obtains integrand $1/(\lambda_1\lambda_2(1-(\lambda_1-\lambda_2)^2$. The integral over λ_i is of proportional to

   $$I = \int d\lambda_1 d\lambda_2/\lambda_1\lambda_2(1-\lambda_1-\lambda_2) \ .$$

   The ranges of integration are from $(-\infty, \infty)$.

   One can decompose the integral to four parts so that integration ranges are positive. The outcome is

   $$I = \int dlog(\lambda_1)dlog(\lambda 2)\left[\frac{1}{1-\lambda_1-\lambda_2}+\frac{1}{1+\lambda_1+\lambda_2}-\frac{1}{1+\lambda_1-\lambda_2}-\frac{1}{1-\lambda_1+\lambda_2}\right] \ .$$

   The change of variables $(u,v) = (\lambda_1+\lambda_2, \lambda_1-\lambda_2)$ transforms the integral to a product of integrals

   $$I = \int dudv\frac{1}{1-u^2}\int dv\frac{1}{1-v^2} \ .$$

   The interpretation as residue integral gives the outcome $I = (4\pi)^2$.

   Residue integration gives finite result for this integrals. One can worry about the singularity of the vertices for M_0^2 on mass shell momenta. The problem is that p is on mass shell so that the outcome from loop diverges. The outcome is $D_F$would be however finite.

Gliding conditions for 4-vertices

One can construct also loop diagrams with loops understood in twistorial sense. The interpretation of twistor diagram as computation requires that there exist moves reducing general loopy diagrams to tree diagrams. This requires that the vertices connected by a fermionic loop lines can be glided along fermion lines such that they become nearest neighbors and that these loops can be removed without affective the diagram.

If these diagrams are acceptable mathematically, moves reducing these loop diagrams to twistorial tree diagrams should exist. Could the basic rule be following?

1. One can glide the vertices past each other along fermion lines and reduce loops connecting points at different part of graph to the analogs of self-energy loops located at single fermion lines. These loops involve decay of fermion to 2 fermions and 1 anfifermion which then fuse to single fermion. All fermions are on mass shell in complex sense. The situation thus reduces to single fermion self energy loop if the gliding is possible always. Mass shell conditions could however prevent this.
2. To single fermion line one can assign D_F - the inverse of massless fermion propagator - having formal interpretation as a density matrix. The loop would not vanish but would give rise to a inverse of fermionic propagator so that the overall outcome should be just D_F. Is it possible to achieve this?

Under what conditions the gliding is possible?

1. Suppose that the 4-vertex V_1 is glided along fermion line past second 4-vertex V_1. V_1 corresponds to momenta $(P_{i,in}, P_{i1,in} - P, P_{i,1}, P_{i,2})$. The momentum $P_i = \sum_{k=1}^{2} P_{i,k}$ of 2 particles emanates from V_i so that the outgoing and incoming momenta are $P_{i,in} - P_i$, and $P_{i,in}$ $i = 1, 2$. Furthermore $P_{1,in} = P_{2,in} - P_2$. These complex momenta are on M^2 mass shell in the proposed sense.
2. Can one perform the gliding without changing the M_0^2-momenta $P_{i,1}$ and $P_{i,2}$? Gliding is possible if the on mass shell condition is satisfied also for $P_{2,in} - P_1 + P_2$ rather than only $P_{2,in} + P_2$. If the mass squared spectrum is integer valued in suitable units the condition reduces to the requirement that $2P_{2,in} \cdot P_1$ is real and integer valued.

   These conditions are independent of the conditions for $2P_{2,in} \cdot P_2$ coming from V_2, the conditions would correlate P_1 and P_2. The construction of the amplitude would involve non-local conditions on vertices rather than only momentum conservation and mass shell conditions at vertices as expected.

   M^2-momentum is however light-like for a special choice $M^2 = M_0^2$. If M_0^2 same along connected fermion lines, the gliding condition would make sense. M_0^2 is constant of motion along fermion line which can wander along the network formed by partonic orbits.

   In fact, M_0^2 must be same for all fermions in given vertex so that its is constant for all connected regions of fermionic part of the graph. Is there any hope of having non-trivial scattering amplitude or must all momenta be light-like and parallel in plane M_0^2? Tree diagrams certainly give rise to non-trivial scattering. One can also assign to all internal lines this kind of networks with M_0^2 that assignable to the internal line. It is quite possible that for general graphs allowing different M_0^2s in internal lines and loops, the reduction to tree graph is not possible.
3. The analogs of these conditions apply also to tree graphs. So that one must either sum over trees with different orderings of vertices or pose additional conditions on the M^2-momenta say the assumption that they are light-like and proportional to the same real momentum $(1, \pm 1)$ along the fermion line.

To conclude: if M_0^2 is constant of motion along the connected networks of fermion lines, the gliding conditions could be satisfied. Action exponentials do not produce trouble if one identifies the basis of zero energy states in such a way that every maximum of action gives its own separate amplitude (state) as also number theoretic universality demands. The most attractive option number theoretically is the option IIIa) assuming that localization of zero energy state to single computation is possible as quantum measurement: different localizations would have different intersections between reality and p-adicicities and would correspond to different computation sequences as cognitive processes. The idea that twistor diagrams are closed forms in the sense that tadpole diagrams vanish is also very attractive and natural in this framework.

Permutation as basic data for a scattering diagram

In twistor Grassmannian approach to $\mathcal{N} = 4$ SUSY the data determining the Yangian invariants defining the basic building bricks of the amplitudes can be constructed using two 3-vertices. For the first (second) kind of vertex the helicity spinors λ_i $(\tilde{\lambda}_i)$ are parallel that is $\lambda_1 \propto \lambda_2 \propto \lambda_3$ $(\overline{\lambda}_1 \propto \overline{\lambda}_2 \propto \overline{\lambda}_3)$ and can be chosen to be identical by complex scaling invariant: momentum conservation reduces to that for $\tilde{\lambda}_i$ (λ_i). The graphical notation for the two vertices is as a small white *resp.* black disk [B67, B32] (see Fig. 3.3.35 `http://tinyurl.com/zbj9ad7`).

There are two basic moves leaving the amplitude unaffected (see Fig. 3.3.38 at `http://tinyurl.com/zbj9ad7`). Merging symmetry implies that 4-vertices satisfy a symmetry analogous to the duality of old-fashioned hadron physics: an internal line connecting black (white) vertices as exchange in s-channel can be transformed to an exchange in t-channel: $1+2 \rightarrow 3+4 \equiv 1+3 \rightarrow 2+4$. Merging symmetry allows to transform the diagram into a form in which neighboring vertices have opposite colors. Square move symmetry follows from the cyclic symmetry of the 4-particle amplitude and means black↔white replacement in 4-vertex.

These two moves do not affect the permutation defining the diagram. A given diagram is represented as a disk with external lines ordered cyclically along its boundary. The permutation of the n external particles associated with the diagram is constructed from the two 3-particle diagrams is defined by the following rule.

Start from k:th point at boundary end and go to the left in each white vertex and to the right in each black vertex (see Fig. 3.3.35 at `http://tinyurl.com/zbj9ad7`*).*

This leads to a particle $P(k)$ and the outcome is a permutation $P : k \rightarrow P(k)$ charactering the twistor diagram.

Moves do not affect the permutation associated with the diagram and leave the amplitude unaffected. BCFW bridge can be interpreted as a permutation of two neighboring external lines and allows to generate non-equivalent diagrams.

This permutation symmetry generalizes to 4-D SUSY the role of permutations in 1+1-D integrable field theories, where the scattering S-matrix induces only a phase shift of the wave functions of identical particles. The scattering diagram depends only on the permutation of particles induced by the scattering event. Yang-Baxter relation expresses this . Scattering corresponds to particles passing by each other and diagram is drawn in M^2 plane.

1. In 1+1-D integrable theory 3+3 scattering reduces to 2 particle scatterings. This can be illustrated using world lines in M^2 plane (see the illustration of `http://tinyurl.com/gogn75s`). The particle 2 can be taken to be at rest and 1 and 3 move with opposite velocities. There are three 2-particle scatterings of i and j as crossings of world-lines of i and j (pass-by spatially): denote the crossing by ij.

   For the diagram on the left hand side one has crossings 12, 13 and 23 with this time order. For the second case one has crossings 23, 13, and 12 in this time order. Graphically YB relation (see the illustration of `http://tinyurl.com/gogn75s`) says that the scattering amplitude fo 3+3 scattering does not depend on whether the position of the stationary particle 2 is to the left or right from the point at which the second scattering occurs: the time order of scatterings 12 and 23 does not matter.

2. Mathematically the two-particle scatterings are described by operators $R_{12}(u)$, $R_{13}(u+v)$, and $R_{23}(v)$ representing basic braiding operation $ij \rightarrow ji$. u, $u+v$, and v are parameters characterizing the Lorentz boosts determining the velocities of particles. YB equation reads as

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \ .$$

   For a graphical illustration see `http://tinyurl.com/gogn75s`. The first and third R-matrices are permuted and the outcome is trivial. In pass-by interpretation YB equation states that the two ways to realize $123 \rightarrow 321$ give the same amplitude.

   Instead of pass-by one could assume a reconnection of the world lines at the intersection: world lines are split and future pieces are permuted and connected to the past pieces again. With this interpretation one has $123 \rightarrow 123$ (the illustration of Wikipedia article corresponds to this interpretation).

3. At the static limit $u, v \rightarrow 0$ YB equation gives rise to an identity satisfied by braiding matrices. The pass-by at this limit can be interpreted as permutation lifted to braiding (braid groups is covering group of permutation group).

2+2 vertices are fundamental in integrable theories in M^2. Also in TGD 2+2 vertices for fundamental fermions are proposed to be fundamental, and the effective reduction to M^2 is crucial in many respects and reflects $M^8 - CP_2$ duality and 8-D quaternionic light-likeness implying that 2+2 fermion vertices reduce to vertices in M^2. TGD could be an integrable theory able to circumvent the limitations of integrable QFTs in M^2.

1. How could the 2+2-fermionic scattering matrix relate to the R-matrix? In TGD framework the scattering involves momentum transfer even in M_0^2 frame: the parallel light-like M^2 momenta are rescaled in momentum conserving manner. Could R matrix appear as additional factor in the scattering? The earlier picture indeed is that the fermion lines at partonic orbits can experience braiding described by R-matrix at the static limit (string world sheet boundaries would braid!).
2. In TGD the scattering of 2 fermions could occur in two ways by classical interactions at partonic 2-surface. The world lines either cross each other or not. In M^2 the first contribution is planar and second one non-planar. Both options should contribute to the 4-fermion amplitude but this is not be visible in the proposed form of the amplitude. Does the proposed 4-fermion scattering amplitude allow this interpretation?

   In $\mathcal{N} = 4$ SUSY the addition of BCFW bridge would permute the two external particles. In TGD the introduction of BCFW bridge would force to have bosonic lines in the BCFW bridge. This is not possible. The only manner to have BCFW diagram is to allow SUSY perhaps realized as and addition right-handed neutrinos to the fermion lines but this would force to allow splitting of fermion lines requiring splitting of strings.
3. Annihilations of fermion-antifermion pairs to bosons are not possible in 1+1-D QFTs but in TGD topological 3-vertices allow them. Boson would correspond to the final $B \equiv F\overline{F}$ pair at same parton orbit. There are two ways to achieve the annihilation. In s-channel $F\overline{F} \rightarrow vacuum \rightarrow F\overline{F} \equiv B$ is possible. Both F_1 coming from past and F_2 from future scatter classically backwards in time to give $\overline{F}_1$ travelling back to past and $\overline{F}_2$ travelling back to future. In t-channel one can have braiding ($F\overline{F} \rightarrow \overline{F}F \equiv B$.

About unitarity for scattering amplitudes

The first question is what one means with S-matrix in ZEO. I have considered several proposals for the counterparts of S-matrix [K61].In the original U-matrix, M-matrix and S-matrix were introduced but it seems that U-matrix is not needed.

1. The first question is whether the unitary matrix is between zero energy states or whether it characterizes zero energy states themselves as time-like entanglement coefficients between positive and negative energy parts of zero energy states associated with the ends of CD. One can argue that the first option is not sensible since positive and negative energy parts of zero energy states are strongly correlated rather than forming a tensor product: the S-matrix would in fact characterize this correlation partially.

   The latter option is simpler and is natural in the proposed identification of conscious entity - self - as a generalized Zeno effect, that is as a sequence of repeated state function reductions at either boundary of CD shifting also the boundary of CD farther away from the second boundary so that the temporal distance between the tips of CD increases. Each shift of this kind is a step in which superposition of states with different distances of upper boundary from lower boundary results followed by a localization fixing the active boundary and inducing unitary transformation for the states at the original boundary.
2. The proposal is that the the proper object of study for given CD is M-matrix. M-matrix is a product for a hermitian square root of diagonalized density matrix ρ with positive elements and unitary S-matrix $S : M = \sqrt{\rho}S$. Density matrix ρ could be interpreted in this approach as a non-trivial Hilbert space metric. Unitarity conditions are replaced with the conditions $MM^\dagger = \rho$ and $M^\dagger M = \rho$. For the single step in the sequence of reductions at active boundary of CD one has $M \rightarrow MS(\Delta T)$ so that one has $S \rightarrow SS(\Delta T)$. $S(\Delta T)$ depends on the time interval ΔT measured as the increase in the proper time distance between the tips of CD assignable to the step.

What does unitarity mean in the twistorial approach?

1. In accordance with the idea that scattering diagrams is a representation for a computation, suppose that the deformations of space-time surfaces defining a given topological diagram as a maximum of the exponent of Kähler function, are the basic objects. They would define different quantum phases of a larger quantum theory regarded as a square root of thermodynamics in ZEO and analogous to those appearing also in QFTs. Unitarity would hold true for each phase separately.

   The topological diagrams would not play the role of Feynman diagrams in unitarity conditions although their vertices would be analogous to those appearing in Feynman diagrams. This would reduce the unitarity conditions to those for fermionic states at partonic 2-surfaces at the ends of CDs, actually at the ends of fermionic lines assigned to the boundaries of string world sheets.

2. The unitarity conditions be interpreted stating the orthonormality of the basis of zero energy states assignable with given topological diagram. Since 3-surfaces as points of WCW appearing as argument of WCW spinor field are pairs consisting of 3-surfaces at the opposite boundaries of CD, unitarity condition would state the orthonormality of modes of WCW spinor field. If might be even that no mathematically well-defined inner product assignable to either boundary of CD exists since it does not conform with the view provided by WCW geometry. Perhaps this approach might help in identifying the correct form of S-matrix.

3. If only tree diagrams constructed using 4-fermion twistorial vertex are allowed, the unitarity relations would be analogous to those obtained using only tree diagrams. They should express the discontinuity for T in $S = 1 + iT$ along unitary cut as $Disc(T) = TT^{\dagger}$. T and $T^{\dagger}$ would be T-matrix and its time reversal.

4. The correlation between the structure of the fermionic scattering diagram and topological scattering diagrams poses very strong restrictions on allowed scattering reactions for given topological scattering diagram. One can of course have many-fermion states at partonic 2-surfaces and this would allow arbitrarily high fermion numbers but physical intuition suggests that for given partonic 2-surface (throat of wormhole contact) the fermion number is only 0, 1, or perhaps 2 in the case of supersymmetry possibly generated by right-handed neutrino.

   The number of fundamental fermions both in initial and final states would be finite for this option. In quantum field theory with only masive particles the total energy in the final state poses upper bound on the number of particles in the final state. When massless particles are allowed there is no upper bound. Now the complexity of partonic 2-surface poses an upper bound on fermions.

   This would dramatically simplify the unitarity conditions but might also make impossible to satisfy them. The finite number of conditions would be in spirit with the general philosophy behind the notion of hyper-finite factor. The larger the number of fundamental fermions associated with the state, the higher the complexity of the topological diagram. This would conform with the idea about QCC. One can make non-trivial conclusions about the total energy at which the phase transitions changing the topology of space-time surface defined by a topological diagram must take place.

5.5.5 Criticism

One can criticize the proposed vision.

What about loops of QFT?

The idea about cancellation of loop corrections in functional integral and moves allowing to transform scattering diagrams represented as networks of partonic orbits meeting at partonic 2-surfaces defining topological vertices is nice.

Loops are however unavoidable in QFT description and their importance is undeniable. Photon-photon (see `http://tinyurl.com/lqhdujm`) scattering is described by a loop diagram in which fermions appear in box like loop. Magnetic moment of muon see `http://tinyurl.com/p7znfmd`) involves a triangle loop. A further, interesting case is CP violation for mesons (see `http://tinyurl.com/oop4apy`) involving box-like loop diagrams.

Apart from divergence problems and problems with bound states, QFT works magically well and loops are important. How can one understand QFT loops if there are no fundamental loops? How could QFT emerge from TGD as an approximate description assuming lengths scale cutoff?

The key observation is that QFT basically replaces extended particles by point like particles. Maybe loop diagrams can be "unlooped" by introducing a better resolution revealing the non-point like character of the particles. What looks like loop for a particle line becomes in an improved resolution a tree diagram describing exchange of particle between sub-lines of line of the original diagram. In the optimal resolution one would have the scattering diagrams for fundamental fermions serving as building bricks of elementary particles.

To see the concrete meaning of the "unlooping" in TGD framework, it is necessary to recall the qualitative view about what elementary particles are in TGD framework.

1. The fundamental fermions are assigned to the boundaries of string world sheets at the light-like orbits of partonic 2-surfaces: both fermions and bosons are built from them. The classical scatterings of fundamental fermions at the 2-D partonic 2-surface defining the vertices of topological scattering diagrams give rise to scattering amplitudes at the level of fundamental fermions and twistor lift with 8-D light-likeness suggests essentially unique expressions for the 4-fermion vertex.
2. Elementary particle is modelled as a pair of wormhole contacts (Euclidian signature of metric) connecting two space-time sheets with throats at the two sheets connected by monopole flux tubes. All elementary particles are hadronlike systems but at recent energies the substructure is not visible. The fundamental fermions at the wormhole throats at given space-time sheet are connected by strings. There are altogether 4 wormhole throats per elementary particle in the simplest model.

   Elementary boson corresponds to fundamental fermion and antifermion at opposite wormhole throats with very small size (CP_2 size). Elementary fermion has only single fundamental fermion at either throat. There is $\nu_L\overline{\nu}_R$ pair or its CP conjugate at the other end of the flux tube to neutralize the weak isospin. The flux tube has length of order Compton length (or elementary particle or of weak boson) gigantic as compared to the size of the wormhole contact.
3. The vertices of topological diagram involve joining of the stringy diagrams associated with elementary particles at their ends defined by wormhole contacts. Wormhole contacts defining the ends of partonic orbits of say 3 interacting particles meet at the vertex - like lines in Feynman diagram - and fundamental fermion scattering redistributes fundamental fermions between the outgoing partonic orbits.
4. The important point is that there are $2 \times 2 = 4$ ways for the wormhole contacts at the ends of two elementary particle flux tubes to join together. This makes a possible a diagrams in which particle described by a string like object is emitted at either end and glued back at the other end of string like object. This is basically tree diagram at the level of wormhole contacts but if one looks it at a resolution reducing string to a point, it becomes a loop diagram.
5. Improvement of the resolution reveals particles inside particles, which can scatter by tree diagrams. This allows to "unloop" the QFT loops. By increasing resolution new space-time sheets with smaller size emerge and one obtains "unlooped" loops in shorter scales. The space-time sheets are characterized by p-adic length scale and primes near powers of 2 are favored. p-Adic coupling constant evolution corresponds to the gradual "unlooping" by going to shorter and shorter p-adic length scales revealing smaller and smaller space-time sheets.

The loop diagrams of QFTs could thus be seen as a direct evidence of the fractal many-sheeted space-time and quantum criticality and number theoretical universality (NTU) of TGD Universe. Quantum critical dynamics makes the dynamics universal and this explains the unreasonable success of QFT models as far as length scale dependence of couplings constants is considered. The weak point of QFT models is that they are not able to describe bound states: this indeed requires that the extended structure of particles as 3-surfaces is taken into account.

Can action exponentials really disappear?

The disappearance of the action exponentials from the scattering amplitudes can be criticized. In standard approach the action exponentials associated with extremals determine which configurations are important. In the recent case they should be the 3-surfaces for which Kähler action is maximum and has stationary phase. But what would select them if the action exponentials disappear in scattering amplitudes?

The first thing to notice is that one has functional integral around a maximum of vacuum functional and the disappearance of loops is assumed to follow from quantum criticality. This would produce exponential since Gaussian and metric determinants cancel, and exponentials would cancel for the proposal inspired by the interpretation of diagrams as computations. One could in fact *define* the functional integral in this manner so that a discretization making possible NTU would result.

Fermionic scattering amplitudes should depend on space-time surface somehow to reveal that space-time dynamics matters. In fact, QCC stating that classical Noether charges for bosonic action are equal to the eigenvalues of quantal charges for fermionic action in Cartan algebra would bring in the dependence of scattering amplitudes on space-time surface via the values of Noether charges. For four-momentum this dependence is obvious. The identification of $h_{eff}/h = n$ as the dimension of the extension dividing the order of its Galois group would mean that the basic unit for discrete charges depends on the extension characterizing the space-time surface. Also the cognitive representations defined by the set of points for which preferred embedding space coordinates are in this extension. Could the cognitive representations carry maximum amount of information for maxima? For instance, the number of the points in extension be maximal. Could the maximum configurations correspond to just those points of WCW, which have preferred coordinates in the extension of rationals defining the adele? These 3-surfaces would be in the intersection of reality and p-adicities and would define cognitive representation.

These ideas suggest that the usual quantitative criterion for the importance of configurations could be equivalent with a purely number theoretical criterion. p-Adic physics describing cognition and real physics describing matter would lead to the same result. Maximization for action would correspond to maximization for information.

Irrespective of these arguments, the intuitive feeling is that the exponent of the bosonic action must have physical meaning. It is number theoretically universal if action satisfies $S = q_1 + iq_2\pi$. This condition could actually be used to fix the dependence of the coupling parameters on the extension of rationals [L24]. By allowing sum over several maxima of vacuum functional these exponentials become important. Therefore the above ideas are interesting speculations but should be taken with a big grain of salt.

5.6 Appendix: Some background about twistors

In the following I try to summarize my view about how the ideas related to the twistor approach to scattering amplitudes evolved. A readable summary of specialist about twistor approach is given in the article *Scattering amplitudes* of Elvang and Huang [B32]. Also the thesis *Grassmannian Origin of Scattering Amplitudes* of Trnka [B67] gives a good summary about the work done in association with Nima Arkani-Hamed. I am not a specialist and have not been endowed with practical calculations so that my representation considers only the basic ideas and their relationship to TGD. In the following I summarize my very partial view about the development of ideas.

5.6.1 The pioneering works of Penrose and Witten

The pioneering work of Penrose discussed in *The Central Programme of Twistor Theory* [B63] on twistors initiated the twistor program, which had already had applications in Yang-Mills theories int he description of instantons. The key vision is that massless field equations reduce to holomorphy in twistor formulation.

Witten's *Perturbative Gauge Theory As a String Theory In Twistor Space* [B29] in 2003 initiated the progress leading to dramatic understanding of the planar scattering amplitudes of $\mathcal{N} = 4$ SUSY and eventually to the notion of amplituhedron. The abstract gives some idea about the key ideas.

Perturbative scattering amplitudes in Yang-Mills theory have many unexpected properties, such as holomorphy of the maximally helicity violating amplitudes. To interpret these results, we Fourier transform the scattering amplitudes from momentum space to twistor space, and argue *that the transformed amplitudes are supported on certain holomorphic curves. This in turn is apparently a consequence of an equivalence between the perturbative expansion of* $\mathcal{N} = 4$ *super Yang-Mills theory and the D-instanton expansion of a certain string theory, namely the topological B model whose target space is the Calabi-Yau supermanifold* $CP_{3|4}$.

Witten's observation was that the twistor Fourier transform of the scattering amplitudes of YM theories seem to be localized at 2-dimensional complex surfaces of twistor space and this led him to propose that twistor string theory in the twistor space CP_3 *could allow to describe the scattering amplitudes. The basic problem of the twistor approach relates to space-time signature: all works nicely in signature (2,2), which suggests that something might be wrong in the basic assumptions.*

5.6.2 BCFW recursion formula

BCFW recursion was first derived for tree amplitudes and later generalized to planar loop diagrams.

1. *Twistor diagram recursion for all gauge-theoretic tree amplitudes* by Hodges [B8] in 2005 and *Direct Proof of Tree-Level Recursion Relation in Yang- Mills Theory* by Britto, Cachazo, Feng, and Witten [B20] in 2005 proposed at tree level a recursion formula for the tree level MHV amplitudes of Yang-Mills theory in twistor space.
2. *Scattering Amplitudes and BCFW Recursion in Twistor Space* By Mason and Skinner [B20] discussed BCFW recursion relations for tree diagrams of YM theories.
3. *The S-Matrix in Twistor Space* by Arkani-Hamed, Cachazo, Cheung and Kaplan [B35] in 2009 discussed NkMHV amplitudes with more than two negative helicities (MHV amplitudes have 2 negative helicities are are extremely simple).

This work is carried out in metric signature (2,2), where the twistor transform reduces to ordinary Fourier transform. The other signatures are problematic. Only planar diagrams are considered. *On-Shell Structures of MHV Amplitudes Beyond the Planar Limit* [B39] in 2014 of Arkani-Hamed *et al* consider the problem posed by the non-planar diagrams.

5.6.3 Yangian symmetry and Grassmannian

The discovery of dual super-conformal invariance is one of the key steps of progress. This symmetry means extension of the conformal algebra from space-time level to the level of twistor space so that the dual superconformal invariance acts also on so called momentum twistors assigned with the twistor diagram. These dual conformal symmetries extend to a Yangian algebra containing besides local generators also multilocal generators. The dual conformal generators are bi-local generators and have weight $n = 1$. The Yangian symmetry is completely general and expected to generalize.

In the following I list the abstracts of some important articles.

1. *Magic identities for conformal four-point integrals* by Drummond, Henn, Smirnov, and Sokatchev [B41] in 2006 initiated the development of ideas. The interpretation is as dual conformal invariance generator by the weight 1 generators of Yangian.

   We propose an iterative procedure for constructing classes of off-shell four-point conformal integrals which are identical. The proof of the identity is based on the conformal properties of a sub-integral common for the whole class. The simplest example are the so-called "triple scalar box" and "tennis court" integrals. In this case we also give an independent proof using the method of Mellin-Barnes representation which can be applied in a similar way for general off-shell Feynman integrals.

2. *Yangian symmetry of scattering amplitudes in* $\mathcal{N} = 4$ *super Yang-Mills theory* [B27] by Drummond, Henn, and Plefka in 2009 continued this work and discussed Yangian algebra as as a symmetry having besides local generators also multilocal generators.

   Tree-level scattering amplitudes in $\mathcal{N} = 4$ *super Yang-Mills theory have recently been shown to transform covariantly with respect to a "dual" superconformal symmetry algebra, thus extending the conventional superconformal symmetry algebra psu(2, 2|4) of the theory. In this paper we derive the action of the dual superconformal generators in on-shell superspace and extend the dual generators suitably to leave scattering amplitudes invariant. We then study the algebra of standard and dual symmetry generators and show that the inclusion of the dual* *superconformal generators lifts the psu(2,2|4) symmetry algebra to a Yangian. The non-local Yangian generators acting on amplitudes turn out to be cyclically invariant due to special properties of psu(2,2|4). The representation of the Yangian generators takes the same form as in the case of local operators, suggesting that the Yangian symmetry is an intrinsic property of planar* $\mathcal{N} = 4$ *super Yang-Mills, at least at tree level.*

3. *Dual Superconformal Invariance, Momentum Twistors and Grassmannians* [B59] by Mason and Skinner introduces momentum twistors and Grassmannians.

   Dual superconformal invariance has recently emerged as a hidden symmetry of planar scattering amplitudes in $\mathcal{N} = 4$ *super Yang-Mills theory. This symmetry can be made manifest by expressing amplitudes in terms of "momentum twistors", as opposed to the usual twistors that make the ordinary superconformal properties manifest. The relation between momentum twistors and on-shell momenta is algebraic, so the translation procedure does not rely on any choice of space-time signature. We show that tree amplitudes and box coefficients are succinctly generated by integration of holomorphic delta-functions in momentum twistors over cycles in a Grassmannian. This is analogous to, although distinct from, recent results obtained by Arkani-Hamed et al. in ordinary twistor space. We also make contact with Hodges' polyhedral representation of NMHV amplitudes in momentum twistor space.*

4. *A Duality For The S Matrix* [B34] in 2009 by Arkani-Hamed *et al* discusses also Yangian invariance and introduces central ideas in algebraic geometry: Grassmannians, higher-dimensional residue theorems, intersection theory, and the Schubert calculus.

   We propose a dual formulation for the S Matrix of $\mathcal{N} = 4$ *SYM. The dual provides a basis for the "leading singularities" of scattering amplitudes to all orders in perturbation theory, which are sharply defined, IR safe data that uniquely determine the full amplitudes at tree level and 1-loop, and are conjectured to do so at all loop orders. The scattering amplitude for n particles in the sector with k negative helicity gluons is associated with a simple integral over the space of k planes in n dimensions, with the action of parity and cyclic symmetries manifest. The residues of the integrand compute a basis for the leading singularities. A given leading singularity is associated with a particular choice of integration contour, which we explicitly identify at tree level and 1-loop for all NMHV amplitudes as well as the 8 particle* N^2MHV *amplitude. We also identify a number of 2-loop leading singularities for up to 8 particles. There are a large number of relations among residues which follow from the multi-variable generalization of Cauchy's theorem known as the "global residue theorem". These relations imply highly non-trivial identities guaranteeing the equivalence of many different representations of the same amplitude. They also enforce the cancellation of non-local poles as well as consistent infrared structure at loop level. Our conjecture connects the physics of scattering amplitudes to a particular subvariety in a Grassmannian; space-time locality is reflected in the topological properties of this space.*

5. *The All-Loop Integrand For Scattering Amplitudes in Planar* $\mathcal{N} = 4$ *SYM* [B36] by Arkani-Hamed *et al* in 2010.

   We give an explicit recursive formula for the all L-loop integrand for scattering amplitudes in $\mathcal{N} = 4$ *SYM in the planar limit, manifesting the full Yangian symmetry of the theory. This generalizes the BCFW recursion relation for tree amplitudes to all loop orders, and extends the Grassmannian duality for leading singularities to the full amplitude. It also provides a new physical picture for the meaning of loops, associated with canonical operations for removing particles in a Yangian-invariant way. Loop amplitudes arise from the "entangled" removal of pairs of particles, and are naturally presented as an integral over lines in momentum-twistor space. As expected from manifest Yangian-invariance, the integrand is given as a sum over non-local terms, rather than the familiar decomposition in terms of local scalar integrals with rational coefficients. Knowing the integrands explicitly, it is straightforward to express them in local forms if desired; this turns out to be done most naturally using a novel basis of chiral, tensor integrals written in momentum-twistor space, each of which has unit leading singularities. As simple illustrative examples, we present a number of new multi-loop results written in local form, including the 6- and 7-point 2-loop NMHV amplitudes. Very concise expressions are presented for all 2-loop MHV amplitudes, as well as the 5-point 3-loop MHV amplitude. The structure of the loop integrand strongly suggests that the integrals yielding the* *physical amplitudes are "simple", and determined by IR-anomalies. We briefly comment on extending these ideas to more general planar theories.*

5.6.4 Amplituhedron

The latest development in twistorial revolution was the notion of amplituhedron. Since I do not have intuitive understanding about amplituhedron and since amplituhedron does not have role in the twistorialization of TGD as I understand it now, I provide only abstracts about two articles to it.

1. *The Amplituhedron* [B15] by Arkani-Hamed and Trnka in 2013.

   Perturbative scattering amplitudes in gauge theories have remarkable simplicity and hidden infinite dimensional symmetries that are completely obscured in the conventional formulation of field theory using Feynman diagrams. This suggests the existence of a new understanding for scattering amplitudes where locality and unitarity do not play a central role but are derived consequences from a different starting point. In this note we provide such an understanding for $\mathcal{N}=4$ SYM scattering amplitudes in the planar limit, which we identify as "the volume" of a new mathematical object–the Amplituhedron–generalizing the positive Grassmannian. Locality and unitarity emerge hand-in-hand from positive geometry.

2. *Positive Amplitudes in the Amplituhedron* [B14] by Arkani-Hamed *et al* in 2014.

   The all-loop integrand for scattering amplitudes in planar $\mathcal{N}=4$ SYM is determined by an "amplitude form" with logarithmic singularities on the boundary of the amplituhedron. In this note we provide strong evidence for a new striking property of the superamplitude, which we conjecture to be true to all loop orders: the amplitude form is positive when evaluated inside the amplituhedron. The statement is sensibly formulated thanks to the natural "bosonization" of the superamplitude associated with the amplituhedron geometry. However this positivity is not manifest in any of the current approaches to scattering amplitudes, and in particular not in the cellulations of the amplituhedron related to on-shell diagrams and the positive Grassmannian. The surprising positivity of the form suggests the existence of a "dual amplituhedron" formulation where this feature would be made obvious. We also suggest that the positivity is associated with an extended picture of amplituhedron geometry, with the amplituhedron sitting inside a co-dimension one surface separating "legal" and "illegal" local singularities of the amplitude. We illustrate this in several simple examples, obtaining new expressions for amplitudes not associated with any triangulations, but following in a more invariant manner from a global view of the positive geometry.

Chapter 6

The Recent View about Twistorialization in TGD Framework

6.1 Introduction

The construction of scattering amplitudes is a dream that I have had since the birth of TGD for four decades ago. Various ideas have gradually emerged, some of them have turned out to be wrong, and some of them have survived. At this age I must admit that the dream about explicit algorithms that any graduate student could apply to construct the scattering amplitudes, would require a collective effort and probably will not be realized during my lifetime.

I have however identified a set of general powerful principles leading to a generalization of the recipes for constructing twistorial amplitudes and already now these principles suggest the possibility of rather concrete realizations. In the sequel several additional insights are developed in more detail. Some of them are discussed already earlier in the formulation of M^8-H duality [L37] in adelic framework [L42, L43] and in the chapters developing the TGD based generalization of twistor Grasmannian approach [L10, L22, L24, L45].

1. A proposal made already earlier [L45] is that scattering diagrams as analogs of twistor diagrams are constructible as tree diagrams for CDs connected by free particle lines. Loop contributions are not even well-defined in zero energy ontology (ZEO) and are in conflict with number theoretic vision. The coupling constant evolution would be discrete and associated with the scale of CDs (p-adic coupling constant evolution) and with the hierarchy of extensions of rationals defining the hierarchy of adelic physics.
2. Logarithms appear in the coupling constant evolution in QFTs. The identification of their number theoretic versions as rational number valued functions required by number-theoretical universality for both the integer characterizing the size scale of CD and for the hierarchy of Galois groups leads to an answer to a long-standing question what makes small primes and primes near powers of them physically special. The primes $p \in \{2,3,5\}$ indeed turn out to be special from the point of view of number theoretic logarithm.
3. The reduction of the scattering amplitudes to tree diagrams is in conflict with unitarity in 4-D situation. The imaginary part of the scattering amplitude would have discontinuity proportional to the scattering rate only for many-particle states with light-like total momenta. Scattering rates would vanish identically for the physical momenta for many-particle states.

   In TGD framework the states would be however massless in 8-D sense. Massless pole corresponds now to a continuum for M^4 mass squared and one would obtain the unitary cuts from a pole at $P^2=0$! Scattering rates would be non-vanishing only for many-particle states having light-like 8-momentum, which would pose a powerful condition on the construction of many-particle states. Single particle momenta cannot be however light-like for this kind of states unless they are parallel. They must be also complex as they indeed are already in classical TGD.

In fact, BCFW deformation $p_i \rightarrow p_i + zr_i$, $r_i \cdot r_j = 0$ creates at z-poles of the resulting amplitude pairs of zero energy states for which complex single particle momenta are not light-like but sum up to massless momentum. One can interpret these zero energy analogs of resonances, states inside CDs formed from massless external particles as they arrive to CD. This strong form of conformal symmetry has highly non-trivial implications concerning color confinement.

4. The key idea is number theoretical discretization [L42] in terms of "cognitive representations" as space-time time points with M^8-coordinates in an extension of rationals and therefore shared by both real and various p-adic sectors of the adele. Discretization realizes measurement resolution, which becomes an inherent aspect of physics rather than something forced by observed as outsider. This fixes the space-time surface completely as a zero locus of real or imaginary part of octonionic polynomial.

   This must imply the reduction of "world of classical worlds" (WCW) corresponding to a fixed number of points in the extension of rationals to a finite-dimensional discretized space with maximal symmetries and Kähler structure [K45, K24, K80].

   The simplest identification for the reduced WCW would be as complex Grassmannian - a more general identification would be as a flag manifold. More complex options can of course be considered. The Yangian symmetries of the twistor Grassmann approach known to act as diffeomorphisms respecting the positivity of Grassmannian and emerging also in its TGD variant would have an interpretation as general coordinate invariance for the reduced WCW. This would give a completely unexpected connection with supersymmetric gauge theories and TGD.

5. M^8 picture [L37] implies the analog of SUSY realized in terms of polynomials of super-octonions whereas H picture suggests that supersymmetry is broken in the sense that many-fermion states as analogs of components of super-field at partonic 2-surfaces are not local. This requires breaking of SUSY. At M^8 level the breaking could be due to the reduction of Galois group to its subgroup G/H, where H is normal subgroup leaving the point of cognitive representation defining space-time surface invariant. As a consequence, local many-fermion composite in M^8 would be mapped to a non-local one in H by $M^8 - H$ correspondence.

6.2 General view about the construction of scattering amplitudes in TGD framework

Before twistorial considerations a general vision about the basic principles of TGD and construction of scattering amplitudes in TGD framework is in order.

6.2.1 General principles behind S-matrix

Although explicit formulas for scattering amplitudes are probably too much to hope, one can try to develop a convincing general view about principles behind the S-matrix.

World of Classical Worlds

The first discovery was what I called the "world of classical worlds" (WCW) [K45, K24, K80] as a generalization of loop space allowing to replace path integral approach failing in TGD work. This led to a generalization of Einstein's geometrization program to an attempt to geometrize entire quantum physics. The geometry of WCW would be essentially unique from its mere existence since the existence of Riemann connection requires already in the case of loop spaces maximal isometries. Super-symplectic and super-conformal symmetries generalizing the 2-D conformal symmetries by replacing 2-D surfaces with light-like 3-surfaces (metrically 2-D!) would define the isometries.

Physical states would be classical spinor fields in the infinite-dimensional WCW and spinors at given point of WCW would be fermionic Fock states. Gamma matrices would be linear combinations of fermionic oscillator operators associated with the analog of massless Dirac equation at space-time surface determined by the variational principle whose preferred extremals the space-time surfaces are. Strong form of holography implied by strong form of general coordinate invariance would imply that it is enough to consider the restrictions of the induced spinor fields at string world sheets and partonic 2-surfaces (actually at discrete points at them defining the ends of boundaries of string world sheets) [K106, K80].

Zero Energy Ontology and generalization of quantum measurement theory to a theory of consciousness

The attempts to understand S-matrix led to the question about what does state function reduction really mean. This eventually led to the discovery of Zero Energy Ontology (ZEO) in which time=constant snapshot as a physical state is replaced with preferred extremal satisfying infinite number of additional gauge conditions [L46]. Temporal pattern becomes the fundamental entity: this conforms nicely with the view neuroscientists and computational scientists for whom behavior and program are basic notions. One can say that non-deterministic state function reduction replaces this kind time evolution with new one. One gets rid of the basic difficulty of ordinary quantum measurement theory.

Causal diamond (CD) is the basic geometric object of ZEO. The members of the state pair defining zero energy state - the analog of physical event characterized by initial and final states - have opposite total conserved quantum numbers and reside at the opposite light-like boundaries of CD being associated with 3-surfaces connected by a space-time surface, the preferred extremal. CDs form a fractal hierarchy ordered by their discrete size scale.

One ends up to a quite radical prediction: the arrow of time changes in "big" state function reduction changing the roles of active and passive boundaries of CD. The state function reductions occurring in elementary reactions represent an example of "big" state function reduction. The sequence of "small" state function reductions - analogs of so called weak measurements - defines self as a conscious entity having CD as embedding space correlate [L46].

In ZEO based view about WCW 3-surfaces X^3 are pairs of 3-surfaces at boundaries of CD connected by preferred extremals of the action principle. WCW spinors are pairs of fermionic Fock states at these 3-surfaces and WCW spinor fields are WCW spinors depending on X^3 . They satisfy the analog of massless Dirac equation which boils down to the analogs of Super Virasoro conditions including also gauge conditions for a sub-algebra of super-symplectic algebra. S-matrix describing time evolution followed by "small" state function reduction relates two WCW spinor fields of this kind.

Generalization of twistor Grassmannian approach to TGD framework

Twistorial approach generalizes from M^4 to $H = M^4 \times CP_2$. One possible motivation could be the fact that ordinary twistor approach describes only scattering of massless particles. In the proposed generalization particles are massless in 8-D sense and in general massive in 4-D sense [L10, L22, L24, L45].

1. The existence of twistor lift of Kähler action as 6-D analog of Kähler action fixes the choice of H uniquely: only M^4 and CP_2 allow twistor space with Kähler structure. The 12-D product of the twistor spaces of M^4 and CP_2 induces twistor structure for 6-D surface X^6 under additional conditions guaranteeing that the X^6 is twistor space of 4-D surface X^4 (S^2 bundle over X^4) - its twistor lift. The conjecture that 6-D Kähler action indeed gives rise to twistor spaces of X^4 as preferred extremals.
2. This conjecture is the analog for Penrose's original twistor representation of Maxwellian fields reducing dynamics of massless fields to homology. There is also an analogy with massless fields. Dimensional reduction of Kähler action occurs for 6-surfaces, which represent twistor spaces and the external particles entering CD would be minimal surfaces defining simultaneous preferred extremals of Kähler action satisfying infinite number of additional gauge conditions. Minimal surfaces indeed satisfy generalization of massless field equations. In the interior of CD defining interaction region there is a coupling to Kähler 4-force and one has analog of massless particle coupling to Maxwellian field.
3. 6-D Kähler action would give the preferred extremals via the analog of dimensional reduction essential for the twistor space property requiring that one has S^2 bundle over space-time surface. I have considered the generalization of the standard twistorial construction of scattering

amplitudes of $\mathcal{N} = 4$ SUSY to TGD context. In particular, the crucial Yangian invariance of the amplitudes holds true also now in both M^4 and CP_2 sectors.

4. Skeptic could argue that TGD generalization of twistors does not tell anything about the origin of the Yangian symmetry. During writing of this contribution I however realized that the hierarchy of Grassmannians realizing the Yangian symmetries could be seen as a hierarchy of reduced WCWs associated with the hierarchy of adeles defined by the hierarchy of extensions of rationals. The isometries of Grassmannian would emerge in the reduction of the isometry group of WCW to a finite-D isometry group of Grassmannian and would be caused by finite measurement resolution described number theoretically. Of course, one can consider also more general flag manifolds with Kähler property as candidates for the analogs of Grassmannians. I will represent the argument in more detail later.

   This could also relate to the postulated infinite hierarchy of hyper-finite factors of type II_1 (HFFs) [K105, K36] as a correlate for the finite measurement resolution with included sub-factor inducing transformations which act trivially in the measurement resolution used.

Remark: There is an amusing connection with empiria. Topologist Barbara Shipman observed that honeybee dance allows a description in terms of flag manifold $F = SU(3)/U(1) \times U(1)$, which is the space for the choices of quantization axes of color quantum numbers and also the twistor space in CP_2 degrees of freedom [A25]. This suggest that QCD type physics might make sense in macroscopic length scales. p-Adic length scale hypothesis and the predicted long range classical color gauge fields suggest a hierarchy of QCD type physics. One can indeed construct a TGD based model of honeybee dance with aconcrete interpretation and representation for the points of F at space-time level [L51].

$M^8 - H$ duality

$M^8 - H$ duality provides two equivalent ways to see the dynamics with either M^8 or $H = M^4 \times CP_2$ as embedding space [L37]. One might speak of number theoretic compactification which is a completely non-dynamical analog for spontaneous compactification.

1. In M^8 picture the space-time corresponds to a zero locus for either imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial ($RE(o)$ and $IM(o)$ are defined by the decomposition $o = RE(o) + I_4 IM(o)$, where I_4 is octonion unit orthogonal to quaternionic subalgebra). The dynamics is purely algebraic and ultra-local.

2. At the level of H the dynamics is dictated by variational principle and partial differential equations. Space-time surfaces are preferred extremals of the twistor lift of Kähler action reduced to a sum of 4-D Kähler action and volume term analogous to cosmological term in GRT. The equivalence of these descriptions gives powerful constraints and should follow from the infinite number of gauge conditions at the level of H associated with a sub-algebra of supersymplectic algebra implying the required dramatic reduction of degrees of freedom [K24, K80]. One has a hierarchy of these sub-algebras, which presumably relates to the hierarchy of HFFs and hierarchy of extensions of rationals.

   H picture works very nicely in applications. For instance, the notions of field body and magnetic body are crucial in all applications.

The notion of quaternionicity, which is a central element of $M^8 - H$ duality has a deep connection with causality which I have not noticed earlier. At the level of momentum space quaternionicity means that 8-momenta -, which by $M^8 - H$-duality correspond to 4-momenta at level of M^4 and color quantum numbers at the level of CP_2 - are quaternionic. Quaternionicity means that the time component of 8-momentum, which is parallel to real octonion unit, is non-vanishing. The 8-momentum itself must be time-like, in fact light-like. In this case one can always regard the momentum as momentum in some quaternionic sub-space. Causality requires a fixed sign for the time component of the momentum.

It must be however noticed that 8-momentum can be complex: also the 4-momentum can be complex at the level of $M \times CP_2$ already classically. A possible interpretation is in terms of decay width as part of momentum as it indeed is in phenomenological description of unstable particles.

Could one require that the quaternionic momenta form a linear space with respect to octonionic sum? This is the case if the energy - that is the time-like part parallel to the real octonionic unit - has a fixed sign. The sum of the momenta is quaternionic in this case since the sum of light-like momenta is in general time-like and in special case light-like. If momenta with opposite signs of energy are allowed, the sum can become space-like and the sum of momenta is co-quaternionic.

This result is technically completely trivial as such but has a deep physical meaning. Quaternionicity at the level of 8-momenta implies standard view about causality: only time-like or at most light-like momenta and fixed sign of time-component of momentum.

Adelic physics

The adelization of ordinary physics fusing real number based physics and various p-adic variants of physics in order to describe cognition.

1. Adelic physics [L42, L43] gives powerful number theoretic constraints when combined with M^8-H duality and leads to the vision about evolutionary hierarchy defined by extensions of rationals. The higher the level in the hierarchy, the higher the dimension n of the extension identified in terms of Planck constant $h_{eff}/h=n$ labelling the levels of dark matter hierarchy.
2. Adelic hypothesis allows to sharpen the strong form of holography to a statement that discrete cognitive representations consisting of a finite number of points identified as points of space-time surface with M^8 coordinates in the extension of rationals fixes the space-time surface itself. This dramatic reduction would be basically due to finite measurement resolution realized as an inherent property of dynamics. Cognitive representation in fact gives the WCW coordinates of the space-time surface in WCW! WCW reduces to a number theoretic discretization of a finite-dimensional space with Kähler structure and presumably maximal isometries.
3. In ZEO space-time surface becomes analogous to a computer program determined in terms of finite net of numbers! Of course, at the QFT limit of TGD giving standard model and GRT space-time is locally much more complex since one approximates the many-sheeted space-time with single slightly curved region of M^4. This is the price paid for getting rid (or losing) the topological richness of the many-sheeted space-time crucial for the understanding living matter and even physics in galactic scales.
4. Skeptic can argue that this discretization of WCW leads to the loss of WCW geometry based on real numbers. One can however consider also continuous values for the points of cognitive representations and assigning metric to the points of cognitive representation. Metric could be defined as kind of induced metric. One slices CD by parallel CDs by shift the CD along the axis connecting its tips. This allows to see the point of cognitive representation as point at one particular CD. One shifts slightly the point along its CD. Embedding space metric allows to deduce the infinitesimal line element ds^2 and to deduce the metric components. This allows a definition of differential geometry so that the analog of WCW metric makes sense as a hierarchy of finite-dimensional metrics for space-time surfaces characterize by the cognitive representations.

   The interpretation in real context would be in terms of finite measurement resolution and the hierarchy would correspond to a hierarchy of hyper-finite factors (HFFs) [K105, K36], whose defining property is that they allow arbitrarily precise finite-dimensional approximations. What would be new is that the hierarchy of extensions of rationals would define a hierarchy of discretizations and hierarchy of HFFs.

Thabove list involves several unproven conjectures, which I can argue to be intuitively obvious with the experience of four decades: I cannot of course expect that a colleague reading for the first time about TGD would share these intuitions.

6.2.2 Classical TGD

Classical TGD is now rather well understood both in both $H=M^4\times CP_2$ and M^8 pictures. Applications of classical TGD are in H picture and rather detailed phenomenology has emerged. M^8 picture has led to a rather precise vision about adelic physics and to understanding of finite measurement resolution.

Classical TGD in M^8 picture

Classical TGD in M^8 picture is discussed in [L37].

1. In M^8 picture one ends to an extremely simple number theoretic construction of space-time surfaces fixing only discrete or even finite number of space-time points to obtain space-time surface for a given extension of rationals. The reason is that space-time surfaces are zero loci for $RE(P)$ or $IM(P)$ of octonionic polynomials obtained by continuing real polynomial with coefficients in an extension of rationals to an octonionic polynomial.

   Needless to say, the hierarchy of algebraic extensions of rationals is what makes the dynamics at given level so simple. The coordinates of space-time surface as a point of WCW must be in the extension of rationals. As noticed, the points of space-time surface defining the cognitive representation determining the space-time surface serve as its natural WCW coordinates.

2. The highly non-trivial point is that no variational principle is involved with M^8 construction. Therefore it seems that neither WCW metric nor Kähler function is needed. If this is the case, the exponential of Kähler function definable as action exponential does not appear in scattering amplitudes and must disappear also at H-side from the scattering amplitudes.

3. Skeptic could argue that one loses general coordinate invariance in this approach. This is not true. Linear M^8 coordinates are the only possible option and forced already by symmetries. The choice octonionic and quaternionic structures fixes the linear M^8 coordinates almost uniquely since time direction is associated with real octonion unit and one spatial direction to special imaginary unit defining spin quantization axis. In algebraic approach identifying space-time surface as a zero locus of $RE(P)$ or $IM(P)$ these coordinates define space-time coordinates highly uniquely.

   Skeptic could also argue that number theoretic discretization implies reduction of the basic symmetry groups to their discrete sub-groups. This is true and one can argue that this loss of symmetry is due to the use of cognitive representations with finite resolution. Points with algebraic coordinates could be seen as a choices of representatives from a set of points, which are equivalent as far as measurement resolution is considered.

4. A physically important complication related to M^8 dynamics is the possibility of different octonionic and quaternionic structures. For instance, external particles arriving into CD correspond to different octonionic and quaternionic structures in general since Lorentz boost affects the octonionic structure changing the direction of time axis, which corresponds to the real octonionic unit. In color degrees of freedom one has wave function over different quaternionic structures: essentially color partial waves labelled by color quantum numbers [K52].

   One can apply Poincare transformations and color rotations (or transformation in sub-groups of these groups if one requires that the image points belong to the same extension) to the discrete cognitive representation defining space-time surface. The moduli spaces for these structures are essential for the understanding the standard Poincare and color quantum numbers and standard conservation laws in M^8 picture. Also the size scales of CDs define moduli as also Lorentz boosts leaving either boundary of CD unaffected.

Classical TGD in H picture

At the H side one action principle has partial differential equations and infinite number of gauge conditions associated with a sub-algebra of super-symplectic algebra selecting only extremely few preferred extremals of the action principle in terms of gauge conditions for a sub-algebra of super-symplectic algebra. This dynamics is conjectured to follow from the assumption that 6-D lift of space-time surface X^4 to a CP_1 bundle over X^4 is twistor space of X^4. This condition requires the analog of dimensional reduction since S^2 fiber is dynamically trivial.

For 6-D preferred extremals identifiable as twistor spaces of space-time surfaces the 6-D Kähler action in the product of twistor spaces of M^4 and CP_2 is assumed to dimensionally reduce to 4-D Kähler action plus volume term identifiable as the analog of cosmological constant term. This picture reproduces a description of scattering events highly analogous to that emerging in M^8. External particles correspond to minimal surfaces as analogs of free massless fields and all couplings disappear from the value of the action. The interior of CD corresponds to non-trivial coupling to Kähler 4-force which does not vanish. In M^8 picture one has associative and non-associative regions as counterparts of these regions.

What is remarkable is that the dynamics determined by partial differential equations plus gauge conditions would be equivalent with the number theoretic dynamics determined in terms of zero loci for real or imaginary parts of octonionic polynomials.

6.2.3 Scattering amplitudes in ZEO

The construction of scattering amplitudes even at the level of principle is far from well-understood. I have discussed rather concrete proposals for the twistorial construction but the feeling is that something is still missing [L10, L22, L24, L45]. This feeling might well reflect my quite too limited mathematical understanding of twistors and experience about practical construction of the scattering amplitudes. Later I will discuss possible identification of the missing piece of puzzle.

Consider first the general picture about the construction of scattering amplitudes suggested by ZEO inspired theory of quantum measurement theory defining also a theory of consciousness.

1. The portions of space-time surfaces outside CD correspond to external particles. They satisfy associativity conditions at M^8 side making possible to map them to minimal surfaces in $H = M^4 \times CP_2$ satisfying various infinite number of gauge conditions for a sub-algebra of super-symplectic algebra isomorphic with it.

   Remark: There is an additional condition requiring that associative tangent space or normal space contains fixed complex subspace of quaternions. It is not quite clear whether this condition can be generalized so that the distribution of these spaces is integrable.

   At both sides the dynamics of external particles is in a well-defined sense critical at both sides and does not depend at all on coupling constants.

2. Inside CDs associativity conditions break down in M^8 and one cannot map this spacetime region - call it X^4 - to H [L37]. It is however possible to construct counterpart of X^4 in H as a preferred extremal for the twistor lift of Kähler action by fixing the 3-surfaces at the boundaries of CD (boundary conditions). The dependence on couplings at the level of H would come from the vanishing conditions for classical Noether charges, which depend on coupling parameters.

3. If the two descriptions of the scattering amplitudes are equivalent, the dependence on coupling parameters in H should have a counterpart in M^8. Coupling constants making sense only at H side are expected to depend on the size scale of CD and on the extension of rationals defining the adele [L42, L43]. Coupling constants should be determined completely by the boundary values of Noether charges at the ends of space-time surface, and therefore by the 3-D ends of associative space-time regions representing external particles at M^8 side. This would suggest that coupling constants are functions of the coefficients of the polynomials and the points of cognitive representation.

Zero energy ontology and the life cycle of self

ZEO meant a decisive step in the understanding of quantum TGD since it solved the basic paradox of quantum measurement problem by forcing to realize that subjective and geometric time are not the same thing [L46].

1. Both the passive boundary of CD and the members of state pairs at it are unaffected during the sequence of state reductions analogous to weak measurements (see `http://tinyurl.com/zt36hpb`) defining self as a generalized Zeno effect. The members of state pairs associated with the active boundary change and the active boundary itself drifts farther away from the passive one in the sequence of "small" state function reductions.

   Also the space-time surfaces connecting passive and active boundaries change during the sequence of weak measurements. Only the 3-surfaces at the passive boundary are unaffected. Hence the geometric past relative to the active boundary changes during the life cycle of self. In positive energy ontology (PEO) this is not possible.

2. In "big" state function reduction the roles of passive and active boundary are changed and the arrow of time identifiable as the direction in which CD grows changes. In consciousness theory

"big" state function reduction corresponds to the death of self and subsequent re-incarnations as a self with an opposite arrow of geometric time.

3. In ZEO the life cycle of self corresponds to a sequence of steps. Single step begins with a unitary time evolution in which a superposition of states associated with CDs larger than the original CD emerges. Then follows the analog of weak measurement leading to a localization to a CD in the moduli space of CDs so that it has a fixed and in general larger size. A measurement of geometric time occurs and gives rise to an experience about the flow of time.

   This option would allow to identify the total S-matrix as a product of the S-matrices associated with various steps in spirit with the interpretation as a generalized Zeno effect.

   Remark: In the usual description one fixes the time interval to which one assigns the S-matrix. There is no division to steps giving rise to the experience of time flow.

4. The measurement of geometric time would be a partial measurement reducing more general unitary time evolution to a unitary time evolution in the standard sense. Can one generalize the notion of partial measurement to other observables so that one would still have unitary time evolution albeit in more restricted sense? Or should one consider giving up the unitary time evolution?

   These observables should commute with the observables having the states at passive boundary as eigenstates: otherwise the state at passive boundary would change. If this picture makes sense, the "big" reduction to the opposite boundary meaning the death of self would necessarily occur when all observables commuting with the eigen observables at the passive boundary have been measured. It could of course occur already earlier.

   Should one allow measurements of all observables commuting with the eigen observables at the passive boundary. This would lead to partial de-coherence of the zero energy state. In TGD inspired quantum biology this could allow to understand ageing as an unavoidable gradual loss of the quantum coherence.

More detailed interpretation of ZEO

There are several questions related to the detailed interpretation of ZEO. The intuitive picture is that inside CD representing self one has collection of sub-CDs representing sub-selves identified as mental images of self. On can loosely say, that sub-CDs represent mind. The sub-CDs are connected by on mass shell lines, which correspond to external particles - matter. Sub-CDs can also have sub-CDs and the hierarchy can have several levels.

The states at the boundaries of CD have opposite total quantum numbers. One can consider two interpretations.

1. In positive energy ontology (PEO) the notion of zero energy state could be seen only as an elegant manner to express conservation laws. This is done in QFT quite generally - also in twistor approach. Also the largest CD would have external particles emanating from its boundaries travelling to the geometric past and future. One would have however have only information about the interior of the CD possessed by conscious entity for which CD plus its sub-CDs (mental images) serve as correlates.

   In this picture the arrow of time is fixed since it must be same for all sub-CDs in order to void inconsistency with the basic idea about self as generalized Zeno effect realized as a sequence of weak measurements.

2. ZEO suggest a more radical interpretation. Zero energy state defines an event. There would be the largest CD defining self and sub-CDs would correspond to mental images. There would be no external particles emanating from the boundaries of the largest CD. In this framework it becomes possible to speak about the death of self as the first state function reduction to the opposite boundary changing the roles of active and passive boundaries of self.

This picture should be consistent with what we know about arrow of time and in TGD framework with the idea that the arrow of time can also change - in particular in living matter.

1. How would the standard arrow of time emerge in ZEO? One could see the emergence of the global arrow of geometric time as a process in which the size of the largest CD increases: the sub-CDs are forced to have the same arrow of time as the largest CD and cannot make state

function reductions on opposite boundary (die) independently of it. During evolution the size of the networks with the same arrow of geometric time increases and fixed arrow of geometric time is established in longer scales.

2. This picture cannot be quite correct. The applications of TGD inspired consciousness require that the mental images of self can have arrow of geometric time opposite to that of self. For instance, motor actions could be sensory perceptions in non-standard arrow of time. Memory could be communications with brain of geometric past - seeing in time direction - involving signals to geometric past requiring temporary reversals of the arrow of time at some level of self-hierarchy. Hence space-time regions with different arrows of time but forming a connected space-time surface ought to be possible.

   Many-sheeted space-time means a hierarchy of space-time sheets connected by what I call wormhole contacts having Euclidian signature of the induced metric. Space-time sheets at different levels of the hierarchy are not causally connected in the sense that one cannot speak of signal propagation in the regions of Euclidian signature. This suggests that the space-time sheets connected by wormhole contacts can have different arrows of geometric time and are associated with their own CDs.

   In this manner one would avoid the paradox resulting when sub-self - mental image - dies so that its passive boundary becomes active and the particles emanating from it end up to the passive boundary of CD, where no changes are allowed during the life cycle of self. If the particles emanating from time-reversed sub-self and up to boundaries of parallel CD, the problem is circumvented.

3. Wormhole contacts induce an interaction between Minkowskian space-time sheets that they connect. The interaction is not mediated by classical signals but by boundary conditions at the boundaries between Minkowskian regions and Euclidian wormhole contact. These two boundaries are light-like orbits of opposite wormhole throats (partonic 2-surfaces).

   In number theoretic picture the presence of wormhole contact is reflected in the properties set of points in extension of rationals defining the cognitive representation in turn defining the space-time surface. In particular, the points associated with wormhole contact have space-like distance although they are at opposite boundaries of CD and have time-like distance in the metric of embedding space. This kind of point pairs associated with wormhole contacts serve serve as a tell-tale signature for them.

6.3 The counterpart of the twistor approach in TGD

The analogs of twistor diagrams could emerge in TGD [L22, L45] in the following manner in ZEO.

1. Portions of space-time surfaces inside CDs would appear as analogs of vertices and the space-time surfaces connecting them as analogs of propagator lines. The "lines" connecting sub-CDs would carry massless on mass shell states but possibly with complex momenta analogous to those appearing in twistor diagrams. This is true also classically at level of H: the coupling constants appearing in the action defining classical dynamics - at least Kähler coupling strength - are complex so that also conserved quantities have also imaginary parts.

   Remark: At the level of M^8 one does not have action principle and cannot speak of Noether charges. Here the conserved charged are associated with the symmetries of the moduli spaces such as the moduli spaces for octonion and quaternion structures [L37]. The identification of the classical charges in Cartan algebra at H level with the quantum numbers labeling wave functions in moduli space at M^8 level could be seen as a realization of quantum classical correspondence.

2. At space-time level the vertices of twistor diagrams correspond to partonic 2-surfaces in the interior of given CD. In H description fermionic lines along the light-like orbits of partonic 2-surfaces scatter at partonic 2-surfaces. If each partonic 2-surface defining a vertex is surrounded by a sub-CD, these two views about TGD variants of twistor diagrams are unified. Sub-CD can of course contain more complex structures such as pair of wormhole contacts assignable to an elementary particle.

6.3.1 Could the classical number theoretical dynamics define the hard core of the scattering amplitudes?

The natural hope is that the simple picture about classical dynamics at the level of M^8 should have similar counterpart at the level of scattering amplitudes in M^8. The above arguments suggest that the scattering diagrams correspond to CDs connected by external particle lines representing on mass shell particles. These surfaces are associative at the level of M^8 and minimal surfaces at the level of H. This suggests that scattering amplitude for single CD serves as a building brick for scattering amplitudes: the rest would be "just kinematics" dictated by the enormous symmetries of WCW.

1. Everything in the construction should reduce to a hard core around which one would have integrations (or sums for number theoretic realization of finite measurement resolution) over various moduli characterizing the standard quantum numbers. Twistors for M^4 and CP_2 and the moduli for the choices of CDs should correspond to essentially kinematic contribution involving no genuine dynamics.
2. The scattering amplitudes should make sense in all sectors of adele. This poses powerful constraints on them. The exponential of Kähler function reducing to action exponential can in principle appear in the description at H-side but cannot be present at M^8 side. Therefore it should disappear also at the level of H.

   If the scattering amplitude at the level of H is sum over contributions with the same value of the action exponential, the exponentials indeed cancel and I have proposed that this condition holds true. In perturbative quantum field theory it holds practically always and in integrable theories is exact. This would mean enormous simplification since all information about the action principle in H would appear in the vanishing conditions for the Noether charges of the subalgebra of super-symplectic algebra at the ends of the space-time surface. These Noether changes indeed depend on the action principle and thus on coupling constants.
3. Could the hard core in the construction of the scattering amplitudes be just the choice of the cognitive representation as points if M^8 belonging to the algebraic extension defining the adele and determining space-time surface in terms of octonionic polynomial inside this CD defining the interaction region?

   The set of points of extension of rationals in the cognitive representation defines space-time surface and also its WCW coordinates. The restriction to a cognitive representation with given number of points in given extension of rationals would mean a reduction of WCW to a finite-dimensional sub-space.

   The first wild guess is that this space is Kähler manifold with maximal symmetries - just as WCW is. A further wild guess is that these reduced WCWs are Grassmannians and correspond to those appearing in the twistor Grassmannian approach. A more general conjecture is inspired by the vision that super-symplectic gauge conditions effectively reduce the super-symplectic algebra to a Kac-Moody algebra of a finite-dimensional Lie group - perhaps belonging to ADE hierarchy. The flag manifolds associated with these Lie groups define more general homogenous spaces as candidates for the reduced WCWs.
4. One must allow the action of Galois group and this gives several options for given set X of points in algebraic extension.
   (a) One can construct $X^4(X)$ in terms of octonionic polynomial and construct a representation of Galois group as superposition of space-time surfaces obtained from space-time surface by the action of Galois group on X giving rise to new sets $X_g = g(X)$.
   (b) One can also consider the action of Galois group on X and get larger set Y of points and construct single multi-sheeted surface $X^4(Y)$. This surface corresponds to Planck constant $h_{eff}/h = n$, where n is the dimension of algebraic extension.
   (c) One can also consider the actions of sub-groups of $H \subset Gal$ to X to get space-time surface with $h_{eff}/h = m$ dividing n. There are several options corresponding to representations for all sub-groups of Galois group. A hierarchy of symmetry breakings seems to be involved with unbroken symmetry associated with the largest value of h_{eff}/h.
5. In this picture the hard core would reduce to the classical number theoretical dynamics of space-time surface in M^8. The additional degrees of freedom would be due to the possibility of

different octonionic and quaternionic structures and choices of size scales and Lorentz boosts and translations of CDs. The symmetries would dictate the S-matrix in the moduli degrees of freedom: the dream is that this part of the dynamics reduces to kinematics, so to say.

The discrete coupling constant evolution would be determined by the hierarchy of extensions of rationals and by the hierarchy of p-adic length scales. The cancellation of radiative corrections in the sense of sub-CDs inside CDs could be achieved by replacing coupling constant evolution with its discrete counterpart.

If this dream has something to do with reality, the construction of scattering amplitudes would reduce to their construction in moduli degrees of freedom and here the generalization of twistorial approach relying on Yangian symmetry allowing to identify scattering amplitudes as Yangian invariants might "trivialize" the situation. It will be found that the Yangian symmetry could corresponds to general coordinate transformations for the reduced WCW forced by the restriction of the spacetime surfaces to those allowed by octonionic polynomials with coefficients in the extensiom of rationals.

6.3.2 Do loop contributions to the scattering amplitudes vanish in TGD framework?

In TGD scattering amplitudes interpreted as zero energy states would correspond at embedding space level to collections of space-time surfaces inside CDs analogous to vertices and connected by lines defined by the space-time surfaces representing on-mass-shell particles. One would have massless particles in 8-D sense. The quaternionicity of 8-momentum leads to $M^4 \times CP_2$ picture and CP_2 twistors should replace E^4 twistors of M^8 approach.

Why loop corrections should vanish?

There are several arguments suggesting that the loop contributions should vanish in TGD framework. This would give rise to a discrete coupling constant evolution analogous to a sequence of phase transitions between different critical coupling parameters. Amplitudes would be obtained as tree diagrams.

1. In ZEO it is far from clear what the basic operation defining the loop contribution could even mean. One would have zero energy state for which the members of added particle pair have opposite but momenta but the amplitude is superposition of states with varying momenta. Why should one allow zero energy states containing one particle which is not an eigenstate of momentum? This suggests that ZEO does not allow loop contributions at all: the distinction between PEO and ZEO would make itself visible in rather dramatic manner.

2. The restriction of the BCFW to tree diagrams is internally consistent since the loop term is identically vanishing in this case. The first term in the BCFW for diagram with l loops involves a factor with $l > 0$ loops which vanishes. In $l = 1$ case the second term is obtained from $(n+2, l-1=0)$ diagram by generating loop but this vanishes by assumption.

3. Number theoretic vision does not favor the decomposition of the amplitude to an infinite sum of amplitudes since this is expected to lead to the emergence of transcendental numbers and functions in the amplitude in conflict with the number theoretical universality.

   Loops indeed give logarithms and poly-logarithms of rational functions of external momenta in Grassmannian approach. This violates the number theoretical universality since the p-adic counterpart of logarithm exist only for the argument of form $x = 1 + O(p)$. This condition cannot hold true for all primes simultaneously.

Discrete coupling constant evolution suggests the vanishing of loops. One can imagine two alternative mechanisms for the vanishing of loop contributions. Either the loop contributions do not make sense at all in ZEO, or the sum of loop contributions for the critical values of coupling constants vanishes. The summing up of loop contributions to zero for critical values of couplings should happen for all values of external momenta and other quantum numbers: this does not look plausible.

General number theoretic ideas about coupling constant evolution

The discrete coupling constant evolution would be associated with the scale hierarchy for CDs and the hierarchy of extensions of rationals.

1. Discrete p-adic coupling constant evolution would naturally correspond to the dependence of coupling constants on the size of CD. For instance, I have considered a concrete but rather ad hoc proposal for the evolution of Kähler couplings strength based on the zeros of Riemann zeta [L17]. Number theoretical universality suggests that the size scale of CD identified as the temporal distance between the tips of CD using suitable multiple of CP_2 length scale as a length unit is integer, call it l. The prime factors of the integer could correspond to preferred p-adic primes for given CD.
2. I have also proposed that the so called ramified primes of the extension of rationals correspond to the physically preferred primes. Ramification is algebraically analogous to criticality in the sense that two roots understood in very general sense co-incide at criticality. Could the primes appearing as factors of l be ramified primes of extension? This would give strong correlation between the algebraic extension and the size scale of CD.

In quantum field theories coupling constants depend in good approximation logarithmically on mass scale, which would be in the case of p-adic coupling constant evolution replaced with an integer n characterizing the size scale of CD or perhaps the collection of prime factors of n (note that one cannot exclude rational numbers as size scales). Coupling constant evolution could also depend on the size of extension of rationals characterized by its order and Galois group.

In both cases one expects approximate logarithmic dependence and the challenge is to define "number theoretic logarithm" as a rational number valued function making thus sense also for p-adic number fields as required by the number theoretical universality.

1. Coupling constant evolution with respect to CD size scale

Consider first the coupling constant as a function of the length scale $l_{CD}(n)/l_{CD}(1) = n$.

1. The number $\pi(n)$ of primes $p \leq n$ behaves approximately as $\pi(n) = n/log(n)$. This suggests the definition of what might be called "number theoretic logarithm" as $Log(n) \equiv n/\pi(n)$. Also iterated logarithms such $log(log(x))$ appearing in coupling constant evolution would have number theoretic generalization.
2. If the p-adic variant of $Log(n)$ is mapped to its real counterpart by canonical identification involving the replacement $p \rightarrow 1/p$, the behavior can very different from the ordinary logarithm. $Log(n)$ increases however very slowly so that in the generic case one can expect $Log(n) < p_{max}$, where p_{max} is the largest prime factor of n, so that there would be no dependence on p for p_{max} and the image under canonical identification would be number theoretically universal.

   For $n = p^k$, where p is small prime the situation changes since $Log(n)$ can be larger than small prime p. Primes p near primes powers of 2 and perhaps also primes near powers of 3 and 5 - at least - seem to be physically special. For instance, for Mersenne prime $M_k = 2^k - 1$ there would be dramatic change in the step $M_k \rightarrow M_k + 1 = 2^k$, which might relate to its special physical role.
3. One can consider also the analog of $Log(n)$ as

$$Log(n) = \sum_p k_p Log(p) \quad ,$$

   where p^{k_i} is a factor of n. $Log(n)$ would be sum of number theoretic analogs for primes factors and carry information about them.

   One can extend the definition of $Log(x)$ to the rational values $x = m/n$ of the argument. The logarithm $Log_b(n)$ in base $b = r/s$ can be defined as $Log_b(x) = Log(x)/Log(b)$.
4. For $p \in \{2, 3, 5\}$ one has $Log(p) > log(p)$, where for larger primes one has $Log(p) < log(p)$. One has $Log(2) = 2 > log(2) = .693...$, $Log(3) = 3k/2 > log(3) = 1.099$, $Log(5) = 5/3 = 1.666.. > log(5) = 1.609$. For $p = 7$ one has $Log(7) = 7/4 \simeq 1.75 < log(7) \simeq 1.946$. Hence these primes and CD size scales n involving large powers of $p \in \{2, 3, 5\}$ ought to be physically

special as indeed conjectured on basis of p-adic calculations and some observations related to music and biological evolution [K65, K68, K78, K55].

In particular, for Mersenne primes $M_k = 2^k - 1$ one would have $Log(M_k) \simeq klog(2)$ for large enough k. For $Log(2^k)$ one would have $k \times Log(2) = 2k > log(2^k) = klog(2)$: there would be sudden increase in the value of $Log(n)$ at $n = M_k$. This jump in p-adic length scale evolution might relate to the very special physical role of Mersenne primes strongly suggested by p-adic mass calculations [K52].

5. One can wonder whether one could replace the $log(p)$ appearing as a unit in p-adic negentropy [K57] with a rational unit $Log(p) = p/\pi(p)$ to gain number theoretical universality? One could therefore interpret the p-adic negentropy as real or p-adic number for some prime. Interestingly, $|Log(p)|_p = 1/p$ approaches zero for large primes p (eye cannot see itself!) whereas $|Log(p)|_q = 1/|\pi(p)|_q$ has large values for the prime power factors q^r of $\pi(p)$.

2. The dependence of $1/\alpha_K$ on the extension of rationals

Consider next the dependence on the extension of rationals. The natural algebraization of the problem is to consider the Galois group of the extension.

1. Consider first the counterparts of primes and prime factorization for groups. The counterparts of primes are simple groups, which do not have normal subgroups H satisfying $gH = Hg$ implying invariance under automorphisms of G. Simple groups have no decomposition to a product of sub-groups. If the group has normal subgroup H, it can be decomposed to a product $H \times G/H$ and any finite group can be decomposed to a product of simple groups.

   All simple finite groups have been classified (see `http://tinyurl.com/jn44bxe`). There are cyclic groups, alternating groups, 16 families of simple groups of Lie type, 26 sporadic groups. This includes 20 quotients G/H by a normal subgroup of monster group and 6 groups which for some reason are referred to as pariahs.

2. Suppose that finite groups can be ordered so that one can assign number $N(G)$ to group G. The roughest ordering criterion is based on $ord(G)$. For given order $ord(G) = n$ one has all groups, which are products of cyclic groups associated with prime factors of n plus products involving non-Abelian groups for which the order is not prime. $N(G) > ord(G)$ thus holds true. For groups with the same order one should have additional ordering criteria, which could relate to the complexity of the group. The number of simple factors would serve as an additional ordering criterion.

   If its possible to define $N(G)$ in a natural manner then for given G one can define the number $\pi_1(N(G))$ of simple groups (analogs of primes) not larger than G. The first guess is that that the number $\pi_1(N(G))$ varies slowly as a function of G. Since Z_i is simple group, one has $\pi_1(N(G)) \geq \pi(N(G))$.

3. One can consider two definitions of number theoretic logarithm, call it Log_1.

$$\begin{aligned} &a) \quad Log_1(N(G)) = \frac{N(G)}{\pi_1(N(G))} \ , \\ &b) \quad Log_1(G) = \sum_i k_i Log_1(N(G_i)) \ , \quad Log_1(N(G_i)) = \frac{N(G_i)}{\pi_1(N(G_i))} \ . \end{aligned} \tag{6.3.1}$$

   Option a) does not provide information about the decomposition of G to a product of simple factors. For Option b) one decomposes G to a product of simple groups G_i: $G = \prod_i G_i^{k_i}$ and defines the logarithm as Option b) so that it carries information about the simple factors of G.

4. One could organize the groups with the same order to same equivalence class. In this case the above definitions would give

$$\begin{aligned} &a) \quad Log_1(ord(G)) = \frac{ord(G)}{\pi_1(ord(G))} < Log(ord(G)) \ , \\ &b) \quad Log_1(ord(G)) = \sum_i k_i Log(ord(G_i)) \ , \quad Log_1(ord(G_i)) = \frac{ord(G_i)}{\pi_1(ord(G_i))} \ . \end{aligned} \tag{6.3.2}$$

Besides groups with prime orders there are non-Abelian groups with non-prime orders. The occurrence of same order for two non-isomorphic finite simple groups is very rare (see http://tinyurl.com/ydd6uomb). This would suggests that one has $\pi_1(ord(G)) < ord(G)$ so that $Log_1(ord(G))/ord(G) < 1$ would be true.

5. For orders $n(G) \in \{2,3,5\}$ one has $Log_1(n(G)) = Log(n(G)) > log(n(G))$ so that the ordes $n(G)$ involving large factors of $p \in \{2,3,5\}$ would be special also for the extensions of rationals. S_3 with order 6 is the first non-abelian simple group. One has $\pi(S_3) = 4$ giving $Log(6) = 6/4 = 1.5 < log(6) = 1.79$ so that S_3 is different from the simple groups below it.

To sum up, number theoretic logarithm could provide answer to the long-standing question what makes Mersenne primes and also other small primes so special.

Considerations related to coupling constant evolution and Riemann zeta

I have made several number theoretic peculations related to the possible role of zeros of Riemann zeta in coupling constant evolution. The basic problem is that it is not even known whether the zeros of zeta are rationals, algebraic numbers or genuine transcendentals or belong to all these categories. Also the question whether number theoretic analogs of ζ defined for p-adic number fields could make sense in some sense is interesting.

1. Is number theoretic analog of ζ possible using $Log(p)$ instead of $log(p)$?

The definition of $Log(n)$ based on factorization $Log(n) \equiv \sum_p k_p Log(p)$ allows to define the number theoretic version of Riemann Zeta $\zeta(s) = \sum n^{-s}$ via the replacement $n^{-s} = exp(-log(n)s) \rightarrow exp(-Log(n)s)$.

1. In suitable region of plane number-theoretic Zeta would have the usual decomposition to factors via the replacement $1/(1-p^{-s}) \rightarrow 1/(1-exp(-Log(p)s)$. p-Adically this makes sense for $s = O(p)$ and thus only for a finite number of primes p for positive integer valued s: one obtains kind of cut-off zeta. Number theoretic zeta would be sensitive only to a finite number of prime factors of integer n.
2. This might relate to the strong physical indications that only a finite number of cognitive representations characterized by p-adic primes are present in given quantum state: the ramified primes for the extension are excellent candidates for these p-adic primes. The size scale n of CD could also have decomposition to a product of powers of ramified primes. The finiteness of cognition conforms with the cutoff: for given CD size n and extension of rationals the p-adic primes labelling cognitive representations would be fixed.
3. One can expand the regions of converge to larger p-adic norms by introducing an extension of p-adics containing e and some of its roots (e^p is automatically a p-adic number). By introducing roots of unity, one can define the phase factor $exp(-iLog(n)Im(s))$ for suitable values of $Im(s)$. Clearly, $exp(-ipIm(s))/\pi(p))$ must be in the extension used for all primes p involved. One must therefore introduce prime roots $exp(i/\pi(p))$ for primes appearing in cutoff. To define the number theoretic zeta for all p-adic integer values of $Re(s)$ and all integer values of $Im(s)$, one should allow all roots of unity ($ep(i2\pi/n)$) and all roots $e^{1/n}$: this requires infinite-dimensional extension.
4. One can thus define a hierarchy of cutoffs of zeta: for this the factorization of Zeta to a finite number of "prime factors" takes place in genuine sense, and the points $Im(s) = ik\pi(p)$ give rise to poles of the cutoff zeta as poles of prime factors. Cutoff zeta converges to zero for $Re(s) \rightarrow \infty$ and exists along angles corresponding to allowed roots of unity. Cutoff zeta diverges for $(Re(s) = 0, Im(s) = ik\pi(p))$ for the primes p appearing in it.

Remark: One could modify also the definition of ζ for complex numbers by replacing $exp(log(n)s)$ with $exp(Log(n)s)$ with $Log(n) = \sum_p k_p Log(p)$ to get the prime factorization formula. I will refer to this variant of zeta as modified zeta ($\tilde{\zeta}$) below. $\tilde{\zeta}$ would carry explicit number theoretic information via the dependence of its "prime factors" $1/(1 - exp(-Log(p)s))$.

2. Could the values of $1/\alpha_K$ be given as zeros of ζ or of $\tilde{\zeta}$

In [L17] I have discussed the possibility that the zeros $s = 1/2 + iy$ of Riemann zeta at critical line correspond to the values of complex valued Kähler coupling strength α_K: $s = i/\alpha_K$.

The assumption that p^{iy} is root of unity for some combinations of p and y [$log(p)y = (r/s)2\pi$] was made. This does not allow s to be complex rational. If the exponent of Kähler action disappears from the scattering amplitudes as $M^8 - H$ duality requires, one could assume that s has rational values but also algebraic values are allowed.

1. If one combines the proposed idea about the Log-arithmic dependence of the coupling constants on the size of CD and algebraic extension with $s = i/\alpha_K$ hypothesis, one cannot avoid the conjecture that the zeros of zeta are complex rationals. It is not known whether this is the case or not. The rationality would not have any strong implications for number theory but the existence irrational roots would have (see `http://tinyurl.com/y8bbnhe3`). Interestingly, the rationality of the roots would have very powerful physical implications if TGD inspired number theoretical conjectures are accepted.

   The argument discussed below however shows that complex rational roots of zeta are not favored by the observations [A61] about the Fourier transform for the characteristic function for the zeros of zeta. Rather, the findings suggest that the imaginary parts [L16] should be rational multiples of 2π, which does not conform with the vision that $1/\alpha_K$ is algebraic number. The replacement of $log(p)$ with $Log(p)$ and of 2π with is natural p-adic approximation in an extension allowing roots of unity however allows $1/\alpha_K$ to be an algebraic number. Could the spectrum of $1/\alpha_K$ correspond to the roots of ζ or of $\tilde{\zeta}$?

2. A further conjecture discussed in [L17] was that there is 1-1 correspondence between primes $p \simeq 2^k$, k prime, and zeros of zeta so that there would be an order preserving map $k \to s_k$. The support for the conjecture was the predicted rather reasonable coupling constant evolution for α_K. Primes near powers of 2 could be physically special because $Log(n)$ decomposes to sum of $Log(p)$:s and would increase dramatically at $n = 2^k$ slightly above them.

   In an attempt to understand why just prime values of k are physically special, I have proposed that k-adic length scales correspond to the size scales of wormhole contacts whereas particle space-time sheets would correspond to $p \simeq 2^k$. Could the logarithmic relation between L_p and L_k correspond to logarithmic relation between p and $\pi(p)$ in case that $\pi(p)$ is prime and could this condition select the preferred p-adic primes p?

3. The argument of Dyson for the Fourier transform of the characteristic function for the set of zeros of ζ

Consider now the argument suggesting that the roots of zeta cannot be complex rationals. On basis of numerical evidence Dyson [A61] (`http://tinyurl.com/hjbfsuv`) has conjectured that the Fourier transform for the characteristic function for the critical zeros of zeta consists of multiples of logarithms $log(p)$ of primes so that one could regard zeros as one-dimensional quasi-crystal.

This hypothesis makes sense if the zeros of zeta decompose into disjoint sets such that each set corresponds to its own prime (and its powers) and one has $p^{iy} = U_{m/n} = exp(i2\pi m/n)$ (see the appendix of [L16]). This hypothesis is also motivated by number theoretical universality [K104, L42].

1. One can re-write the discrete Fourier transform over zeros of ζ at critical line as

   $$f(x) = \sum_y exp(ixy)) \quad , \quad y = Im(s) \quad .$$

   The alternative form reads as

   $$f(u) = \sum_s u^{iy} \quad , \quad u = exp(x) \quad .$$

   $f(u)$ is located at powers p^n of primes defining ideals in the set of integers.

   For $y = p^n$ one would have $p^{iny} = exp(inlog(p)y)$. Note that $k = nlog(p)$ is analogous to a wave vector. If $exp(inlog(p)y)$ is root of unity as proposed earlier for some combinations of p and y, the Fourier transform becomes a sum over roots of unity for these combinations: this could make possible constructive interference for the roots of unity, which are same or at least have the same sign. For given p there should be several values of $y(p)$ with nearly the same value of $exp(inlog(p)y(p))$ whereas other values of y would interfere deconstructively.

For general values $y = x^n$ $x \neq p$ the sum would not be over roots of unity and constructive interference is not expected. Therefore the peaking at powers of p could take place. This picture does not support the hypothesis that zeros of zeta are complex rational numbers so that the values of $1/\alpha_K$ correspond to zeros of zeta and would be therefore complex rationals as the simplest view about coupling constant evolution would suggest.

Remark: Mumford has argued (`http://tinyurl.com/zemw27o`) that the Fourier transform should include also the trivial zeros at $s = -2, -4, -6...$ giving and exponentially small contributions and providing a slowly varying background to the Fourier transform.

2. What if one replaces $log(p)$ with $Log(p) = p/\pi(p)$, which is rational and thus ζ with $\tilde{\zeta}$? For large enough values of p $Log(p) \simeq log(p)$ finite computational accuracy does not allow distinguish $Log(p)$ from $log(p)$. For $Log(p)$ one could thus understand the finding in terms of constructive interference for the roots of unity if the roots of zeta are of form $s = 1/2 + i(m/n)2\pi$. The value of y cannot be rational number and $1/\alpha_K$ would have real part equal to y proportional to 2π which would require infinite-D extension of rationals. In p-adic sectors infinite-D extension does not conform with the finiteness of cognition.

   Remark: It is possible to check by numerical calculations whether the locus of complex zeros of $\tilde{\zeta}$ is at line $Res(2) = 1/2$. If so, then Fourier transform would make sense. One can also check whether the peaks at $nlog(p)$ are shifted to $nLog(p)$: for $p = 2$ one would have $Log(2) = 2 > log(2)$. The positions of peaks should shift to the right for $p = 2, 3, 5$ and to the left for $p > 5$. This should be easy to check by numerical calculations.

3. Numerical calculations have however finite accuracy, and allow also the possibility that y is algebraic number approximating rational multiple of 2π in some natural manner. In p-adic sectors would obtain the spectrum of y and $1/\alpha_K$ as algebraic numbers by replacing 2π in the formula $is = \alpha_K = i/2 + q \times 2\pi$, $q = r/s$, with its approximate value:

$$2\pi \rightarrow sin(2\pi/n)n = i\frac{n}{2}(exp(i2\pi/n) - exp(-i2\pi/n))$$

   for an extension of rationals containing n:th of unity. Maximum value of n would give the best approximation. This approximation performed by fundamental physics should appear in the number theoretic scattering amplitudes in the expressions for $1/\alpha_K$ to make it algebraic number.

   y can be approximated in the same manner in p-adic sectors and a natural guess is that $n = p$ defines the maximal root of unity as $exp(i2\pi/p)$. The phase $exp(ilog(p)y)$ for $y = qsin(2\pi/n(y))$, $q = r/s$, is replaced with the approximation induced by $log(p) \rightarrow Log(p)$ and $2\pi \rightarrow sin(2\pi/n)n$ giving

$$exp(ilog(p)y) \rightarrow exp(iq(y)sin(2\pi/n(y))\frac{p}{\pi(p)}) \ .$$

   If s in $q = r/s$ does not contain higher powers of p, the exponent exists p-adically for this extension and can can be expanded in positive powers of p as

$$\sum_n i^n q^n sin(2\pi/p)^n (p/\pi(p))^n \ .$$

   This makes sense p-adically.

   Also the actual complex roots of ζ could be algebraic numbers:

$$s = i/2 + q \times sin(\frac{2\pi}{n(y)})n(y) \ .$$

   If the proposed correlation between p-adic primes $p \simeq 2^k$, k prime and zeros of zeta predicting a reasonable coupling constant evolution for $1/\alpha_K$ is true, one can have naturally, $n(y) = p(y)$, where p is the p-adic prime associated with y: the accuracy in angle measurement would increase with the size scale of CD. For given p there could be several roots y with same $p(y)$ but different $q(y)$ giving same phases or at least phases with same sign of real part.

   Whether the roots of $tilde\zeta$ are algebraic numbers and at critical line $Re(s) = 1/2$ is an interesting question.

Remark: This picture allows many variants. For instance, if one assumes standard zeta, one could consider the possibility that the roots y_p associated with p and giving rise to constructive interference are of form $y = q \times (Log(p)/log(p)) \times sin(2\pi/p)p$, $q = r/s$.

4. Could functional equation and Riemann hypothesis generalize?

It is interesting to list the elementary properties of the $\tilde{\zeta}$ before trying to answer to the questions of the title.

1. The replacement $log(n) \rightarrow Log(n) \equiv sum_p k_p Log(p)$ implies that $\tilde{\zeta}$ codes explicitly number theoretic information. Note that $Log(n)$ satisfies the crucial identity $Log(mn) = Log(m) + Log(n)$. $\tilde{\zeta}$ is an analog of partition function with rational number valued $Log(n)$ taking the role of energy and $1/s$ that of a complex temperature. In ZEO this partition function like entity could be associated with zero energy state as a "square root" of thermodynamical partition function: in this case complex temperatures are possible. $|\tilde{\zeta}|^2$ would be the analog of ordinary partition function.
2. Reduction of $\tilde{\zeta}$ to a product of "prime factors" $1/[1 - exp(-Log(p)s)]$ holds true by $Log(n) \equiv sum_p k_p Log(p)$, $Log(p) = p/\pi(p)$.
3. $\tilde{\zeta}$ is a combination of exponentials $exp(-Log(n)s)$, which converge for $Re(s) > 0$. For ζ one has exponentials $exp(-log(n)s)$, which also converge for $Re(s) > 0$: the sum $\sum n^{-s}$ does not however converge in the region $Re(s) < 1$. Presumably $\tilde{\zeta}$ fails to converge for $Re(s) \leq 1$. The behavior of terms $exp(-Log(n)s)$ for large values of n is very similar to that in ζ.
4. One can express ζ o in terms of η function defined as

   $$\eta(s) = \sum (-1)^n n^{-s} \quad .$$

   The powers $(-1)^n$ guarantee that η converges (albeit not absolutely) inside the critical strip $0 < s < 1$.

   By using a decomposition of integers to odd and even ones, one can express ζ in terms of η:

   $$\zeta = \frac{\eta(s)}{(-1 + 2^{-s+1})} \quad .$$

   This definition converges inside critical strip. Note the pole at $s = 1$ coming from the factor.

   One can define also $\tilde{\eta}$:

   $$\tilde{\eta}(s) = \sum (-1)^n e^{-Log(n)s} \quad .$$

   The formula relating $\tilde{\zeta}$ and $\tilde{\eta}$ generalizes: 2^{-s} is replaced with $exp(-2s)$ ($Log(2) = 2$):

   $$\tilde{\zeta} = \frac{\tilde{\eta}(s)}{-1 + 2exp^{-2s}} \quad .$$

   This definition $\tilde{\zeta}$ converges in the critical strip $Re(s) \in (0,1)$ and also for $Re(s) > 1$. $\tilde{\zeta}(1-s)$ converges for $Re(s) < 1$ so that in $\tilde{\eta}$ representation both converge.

   Note however that the poles of ζ at $s = 1$ has shifted to that at $s = log(2)/2$ and is below $Re(s) = 1/2$ line. If a symmetrically posioned pole at $s = 1 - log(2)/2$ is not present in $\tilde{\eta}$, functional equation cannot be true.
5. $Log(n)$ approaches $log(n)$ for integers n not containing small prime factors p for which $\pi(n)$ differs strongly from $p/log(p)$. This suggests that allowing only terms $exp(-Log(n)s)$ in the sum defining $\tilde{\zeta}$ not divisible by primes $p < p_{max}$ might give a cutoff $\tilde{\zeta}^{cut,p_{max}}(s)$ behaving very much like ζ from which "prime factors" $1/(1 - exp(-Log(p)s)$, $p < p_{max}$ are dropped of. This is just division of $\tilde{\zeta}$ by these factors and at least formally, this does not affect the zeros of $\tilde{\zeta}$. Arbitrary number of factors can be droped. Could this mean that $\tilde{\zeta}^{cut}$ has same or very nearly same zeros as ζ at critical line? This sounds paradoxical and might reflect my sloppy thinking: maybe the lack of the absolute implies that the conclusion is incorrect.

The key questions are whether $\tilde{\zeta}$ allows a generalization of the functional equation $\xi(s) = \xi(1-s)$ with $\xi(s) = \frac{1}{2}s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s)$ and whether Riemann hypothesis generalizes. The derivation of the functional equation is quite a tricky task and involves integral representation of ζ .

1. One can start from the integral representation of ζ true for $s > 0$.

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t+1} dt \ , \ Re(s) > 0 \ .$$

deducible from the expression in terms of $\eta(s)$. The factor $1/(1+e^t)$ can be expanded in geometric series $1/(1+e^t) = \sum (-1)^n exp(nt)$ converning inside the critical strip. One formally performs the integrations by taking nt as an integration variable. The integral gives the result $\sum (-1)^n/n^z)\Gamma(s)$.

The generalization of this would be obtained by a generalization of geometric series:

$$1/(1+e^t) = \sum (-1)^n exp(nt) \rightarrow \sum (-1)^n e^{exp(Log(n))t}$$

in the integral representation. This would formally give $\tilde{\zeta}$: the only difference is that one takes $u = exp(Log(n))t$ as integration variable.

One could try to prove the functional equation by using this representation. One proof (see `http://tinyurl.com/yak93hyr`) starts from the alternative expression of ζ as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{e^t-1} dt \ , \ Re(s) > 1 \ .$$

One modifies the integration contour to a contour C coming from $+\infty$ above positive real axis, circling the origin and returning back toc$+\infty$ below the real axes to get a modified representation of ζ:

$$\zeta(s) = \frac{1}{2isin(\pi s)\Gamma(s)} \int_1^\infty \frac{(-w)^{s-1}}{e^w-1} dw \ , \ Re(s) > 1 \ .$$

One modifies the C further so that the origin is circled around a square with vertices at $\pm(2n+1)\pi$ and $\pm i(2n+1)\pi$.

One calculates the integral the integral along C as a residue integral. The poles of the integrand proportional to $1/(1-e^t)$ are at imaginary axis and correspond to $w = ir2\pi$, $r \in Z$. The residue integral gives the other side of the functional equation.

2. Could one generalize this representation to the recent case? One must generalize the geometric series defined by $1/(e^w - 1)$ to $-\sum e^{exp(Log(n))w}$. The problem is that one has only a generalization of the geometric series and not closed form for the counterpart of $1/(exp(w)-1)$ so that one does not know what the poles are. The naïve guess is that one could compute the residue integrals term by term in the sum over n. An equally naïve guess would be that for the poles the factors in the sum are equal to unity as they would be for Riemann zeta. This would give for the poles of n:th term the guess $w_{n,r} = r2\pi/exp(Log(n)$, $r \in Z$. This does not however allow to deduce the residue at poles.Note that the poles of $\tilde{\eta}$ at $s = log(2)/2$ suggests that functional equation is not true.

There is however no need for a functional equation if one is only interested in $F(s) \equiv \tilde{\zeta}(s) + \tilde{\zeta}(1-s)$ at the critical line! Also the analog of Riemann hypothesis follows naturally!

1. In the representation using $\tilde{\eta}$ $F(s)$ converges at critical striple and is *real*(!) at the critical line $Re(s) = 1/2$ as follows from the fact that $1 - s = \overline{s}$ for $Re(s) = 1/2$! Hence $F(s)$ is expected to have a large number of zeros at critical line. Presumably their number is infinite, since $F(s)^{cut,p_{max}}$ approaches $2\zeta^{cut,p_{max}}$ for large enough p_{max} at critical line.

2. One can define a different kind of cutoff of $\tilde{\zeta}$ for given n_{max}: $n < n_{max}$ in the sum over $e^{-Log(n)s}$. Call this cutoff $\tilde{\zeta}^{cut,n_{max}}$. This cutoff must be distinguished from the cutoff $\tilde{\zeta}^{cut,p_{max}}$ obtained by dropping the "prime factors" with $p < pmax$. The terms in the cutoff

are of the form $u^{\sum k_p p/\pi(p)}$, $u = exp(-s)$. It is analogous to a polymomial but with fractional powers of u. It can be made a polynomial by a change of variable $u \rightarrow v = exp(-s/a)$, where a is the product of all $\pi(p)$:s associated with all the primes involved with the integers $n < n_{max}$.

One could solve numerically the zeros of $\zeta(\tilde{s}) + \zeta(\tilde{s})$ using program modules calculating $\pi(p)$ for a given p and roots of a complex polynomial in given order. One can check whether also all zeros of $\zeta(\tilde{s}) + \zeta(\tilde{s})$ might reside at critical line.

3. One an define also $F(s)^{cut,n_{max}}$ to be distinguished from $F(s)^{cut,p_{max}}$. It reduces to a sum of terms $exp(-Log(n)/2)cos(-Log(n)y)$ at critical line, $n < n_{max}$. Cosines come from roots of unity. $F(s)$ function is not sum of rational powers of $exp(-iy)$ unlike $\zeta(\tilde{s})$. The existence of zero could be shown by showing that the sign of this function varies as function of y. The functions $cos(-Log(n)y)$ have period $\Delta y = 2\pi/Log(n)$. For small values of n the exponential terms $exp(-Log(n)/2)$ are largest so that they dominate. For them the periods Δy are smallest so that one expected that the sign of both $F(s)$ and $F(s)^{cut,n_{max}}$ varies and forces the presence of zeros.

   One could perhaps interpret the system as quantum critical system. The rather large rapidly varying oscillatory terms with $n < n_{max}$ with small $Log(n)$ give a periodic infinite set of approximate roots and the exponentially smaller slowly varying higher terms induce small perturbations of this periodic structure. The slowly varying terms with large $Log(n)$ become however large near the $Im(s) = 0$ so that here the there effect is large and destroys the period structure badly for small root of $\hat{\zeta}$.

Is the vanishing of the loop corrections consistent with unitarity?

Skeptic could argue that the vanishing of loop corrections is not consistent with unitarity. The following argument however shows that the fact that momenta in TGD framework are 8-D light-like momenta could save the situation. If not only single particle states but also *many-particle states* have light-like 8-momenta, the discontinuity of the amplitude at pole $P^2(M^8) = 0$ implies the discontinuity of the amplitude as function of $s \equiv P^2(M^4)$ along s-axis.

Minkowskian contribution to mass squared would essentially the sum of conformal (stringy) contribution from vibrational degrees of freedom and color contribution from CP_2 degrees of freedom. This suggests a weak form of color confinement: many-particle states could have vanishing color hyper charge and isospin but the eigenvalue value of color Casimir operator would be non-vanishing.

To get more concrete view about the situation the reader is encouraged to study the slides of Jaroslav Trnka explaining BCFW recursion formula [B50] (see `http://tinyurl.com/pqjzffj`) or the article [B32] of Elvang and Huang (see `http://tinyurl.com/y9rhbzhk`).

1. Unitarity condition $SS^\dagger = Id$ for S-matrix $S = 1 + iT$ gives $i(T - T^\dagger) = TT^\dagger$. For forward scattering the physical interpretation is that the discontinuity of $-2Im(T) = i(T - T^\dagger)$ in forward scattering as a function of total mass s above kinematical threshold along real axis is essentially the total scattering rate.

2. For a given tree amplitude, which is rational function, one replaces external momenta p_i with $\hat{p}_i = p_i + zr_i$. r_i real, light-like and orthogonal to each other and their sum vanishes. This gives on mass shell scattering amplitude with complex light-like momenta satisfying conservation conditions.

3. One can consider any non-trivial subset I of momenta and for this set one has $\hat{P}_I^2 = P_i^2 + 2zP \cdot R_I$, where one has $P_I = \sum_i p_i$ and $R_I = \sum_i r_i$. This gives

$$\hat{P}_I^2 = -P_I^2 \frac{(z - z_I)}{z_I} \quad , \quad z_I = \frac{P_I^2}{2P_I \cdot R_I} \quad .$$

   The poles of the modified amplitude $\hat{A}_n(z)$ come from the propagators at $\hat{P}_I^2 = 0$ and correspond to the points $z = z_I$.

4. From the modified scattering amplitude $\hat{A}_n(z)$ one can obtain the original scattering amplitude by performing a residue integral for $\hat{A}_n(z)/z$ along a curve enclosing the poles z_I. This gives

$$A_n = \hat{A}_n(z=0) + \sum_{z_I} Res_{z=z_I}(\frac{\hat{A}_n(z)}{z}) + B_n \quad .$$

B_n comes from the possible pole at $z = \infty$ and is often assumed to vanish. If so, the amplitude factorizes into a sum of products

$$Res_{z=z_I}\frac{\hat{A}_n(z)}{z} = \sum_I \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) \quad .$$

The amplitudes appearing in the product are for modified complex momenta.

The vanishing of loop corrections thus implies that the product terms $\hat{A}_L(1/P^2)\hat{A}_R$ in the BCFW formula give rational functions having no cuts just as the number theoretical vision demands. The discontinuities of the imaginary part of the amplitude are at poles and reduce to the products $\hat{A}_L\hat{A}_R$ with complex on-mass- shell light-like momenta as unitarity demands.

For forward scattering the discontinuity would be essentially positive definite total scattering rate. It would be however non-vanishing only at $P^2 = 0$ so that scattering rate could be non-vanishing only for $P^2 = 0$! This does not make sense in 4-D physics. Is it possible to overcome this difficulty in TGD framework?

1. The first thing to notice is that classical TGD predicts complex Noether charges since for instance Kähler coupling strength has imaginary part. This would suggest that the momenta of incoming particles could be complex. Could complex value of $P(M^4) \equiv P$ implying

$$P^2 = Re(P)^2 - Im(P)^2 + i2Re(P) \cdot Im(P) = 0$$

save the situation? The condition requires that $Re(P)$ and $Im(P)$ are light-like and parallel so that one would obtain only light-like four-momenta as total M^4 momenta.

2. However, in TGD light-likeness holds true in 8-D sense for single particle states: this led to the proposed generalization of twistor approach allowing particles to be massive in 4-D sense. $M^8 - H$ duality allows to speak about light-like M^8 momenta satisfying quaternionicity condition. The wave functions in CP_2 degrees of freedom emerge from momentum wave functions in M^8 degrees of freedom respecting quaternionicity. The condition $P^2(M^8) = 0$ implies that $Re[P(M^8)]$ and $Im[P(M^8)]$ are light-like and parallel. $Im[P(M^8)]$ can be arbitrarily small. One has also $Re[P(M^4)]^2 = Re[P(E^4)]^2$ and $Im[P(M^4)]^2 = Im[P(E^4)]^2$.

3. Could one pose the condition $P^2(M^8) = 0$ also on *many-particle states* or only to the many-particle states appearing as complex massless poles in the BCFW conditions? Kind of strong form of conformal invariance would be in question: not only single-particle states but also many-particle states would be massless in 8-D sense. Now $s = Re[P(M^4))]^2 = Re[P(E^4))]^2$ could have a continuum of values. The discontinuity along s-axis required by unitarity would would emerge from the discontinuity due to the pole at $P^2(M^8) = 0$! Hence 8-dimensional light-likeness in strong sense would be absolutely essential for having vanishing loop corrections together with non-vanishing scattering rates!

Here one must be however extremely careful.

1. In BCFW approach the expression of residue integral as sum of poles in the variable z associated with the amplitude obtained by the deformation $p_i \rightarrow p_i + zr_i$ of momenta ($\sum r_i = 0$, $r_i \cdot r_j = 0$) leads to a decomposition of the tree scattering amplitude to a sum of products of amplitudes in resonance channels with complex momenta at poles. The products involve $1/P^2$ factor giving pole and the analog of cut in unitary condition. Proof of tree level unitarity is achieved by using complexified momenta as a mere formal trick and complex momenta are an auxiliary notion. The complex massless poles are associated with groups I of particles whereas the momenta of particles inside I are complex and non-light-like.

2. Could BCFW deformation give a description of massless bound states massless particles so that the complexification of the momenta would describe the effect of bound state formation on the single particle states by making them non-light-like? This makes sense if one assumes

that all 8-momenta - also external - are complex. The classical charges are indeed complex already classically since Kähler coupling strength is complex [L17]. A possible interpretation for the imaginary part is in terms of decay width characterizing the life-time of the particle and defining a length of four-vector.

3. The basic question in the construction of scattering amplitudes is what happens inside CD for the external particles with light-like momenta. The BCFW deformation leading to factorization suggests an answer to the question. The factorized channel pair corresponds to two CDs inside which analogs of M and $N-M$ particle bound states of external massless particles would be formed by the deformation $p_i \rightarrow p_i + zr_i$ making particle momenta non-light-like. The allowed values of z would correspond to the physical poles. The factorization of BCFW scattering amplitude would correspond to a decomposition to products of bound state amplitudes for pairs of CDs. The analogs of bound states for zero energy states would be in question. BCFW factorization could be continued down to the lowest level below which no factorization is possible.

4. One can of course worry about the non-uniqueness of the BCFW deformation. For instance, the light-like momenta r_i must be parallel ($r_i = \lambda_i r$) but the direction of r is free. Also the choice of λ_i is free to a high extent. BCFW expression for the amplitude as a residue integral over z is however unique. What could this non-uniqueness mean?

   Suppose one accepts the number theoretic vision that scattering amplitudes are representations for sequences of algebraic manipulations. These representations are bound to be highly non-unique since very many sequences can connect the same initial and final expressions. The space-time surface associated with given representation of the scattering amplitude is not unique since each computation corresponds to different space-time surface. There however exists a representation with maximal simplicity.

   Could these two kinds of non-uniqueness relate?

It is indeed easy to see that many-particle states with light-like single particle momenta cannot have light-like momenta unless the single-particle momenta are parallel so that in non-parallel case one must give up light-likeness condition also in complex sense.

1. The condition of light-likeness in complex sense allows the vanishing of real and imaginary mass squared for individual particles

$$Im(p_i) = \lambda_i Re(p_i) \ , \quad (Re(p_i))^2 = (Im(p_i))^2 = 0 \ . \tag{6.3.3}$$

   Real and imaginary parts are parallel and light-like in 8-D sense. All λ_i have same sign and p_i has positive or negative time component depending on whether positive or negative energy part of zero energy state is in question.

2. The remaining two conditions come from the vanishing of the real and imaginary parts of the total mass squared:

$$\sum_{i\neq j} Re(p_i)\cdot Re(p_j) - Im(p_i)\cdot Im(p_j) = 0 \ , \quad \sum_{i\neq j} Re(p_i)\cdot Im(p_j) = 0 \ . \tag{6.3.4}$$

   By using proportionality of $Im(p_i)$ and $Re(p_i)$ one can express the conditions in terms of the real momenta

$$\sum_{i\neq j}(1-\lambda_i\lambda_j) Re(p_i)\cdot Re(p_j) = 0 \ , \quad \sum_{i\neq j} \lambda_j Re(p_i)\cdot Re(p_j) = 0 \ . \tag{6.3.5}$$

   For positive/negative energy part of zero energy state the sign of time component of momentum is fixed and therefore λ_i have fixed sign. Suppose that λ_i have fixed sign. Since the inner products $p_i \cdot p_j$ of time-like vectors with fixed sign of time compomemet are all positive or negative the second term can vanish only if one has $p_i \cdot p_j = 0$. If the sign of λ_i can vary, one can satisfy the condition linear in λ_i but not the first condition as is easy to see in 2-particle case.

3. States with light-like parallel 8-momenta are allowed and one can ask whether this kind of states might be realized inside magnetic flux tubes identified as carriers of dark matter in TGD sense. The parallel light-like momenta in 8-D sense would give rise to a state analogous to super-conductivity. Could this be true also for quarks inside hadrons assumed to move in parallel in QCD based model. This also brings in mind the earlier intuitive proposal that the momenta of fermions and antifermions associated with partonic 2-surfaces must be parallel so that the propagators for the states containing altogether n fermions and antifermions would behave like $1/(p^2)^{n/2}$ and would not correspond to ordinary particles.

These arguments are formulated in M^8 picture. What could this mean in $M^4 \times CP_2$ picture?

1. The intuitive expectation is that $Re[P(E^4)]^2$ corresponds to the eigenvalue Λ of CP_2 d'Alembertian so that the higher the momentum, the larger the value of Λ. CP_2 d'Alembertian would be essentially the M^4 mass squared of the state. This would allow vanishing color quantum numbers Y and I_3 but force symmetry breaking $SU(3) \to SU(2) \times U(1)$. This picture is not quite accurate: also the vibrational degrees of freedom contribute to the mass squared what might be called stringy contribution.
2. Could the geometry of CP_2 induce this symmetry breaking? For instance, Kähler gauge potential depends on the $U(2)$ invariant "radial" coordinate of CP_2 and is invariant only under $U(2)$ rotations and changes by gauge transformation in other color rotations. Could one assign the symmetry breaking to the choice of color quantization axes boiling down at the classical level to the fixing of CP_2 Kähler function would?

   One would have color confinement in weak sense: in QCD picture physical states correspond to color singlet representations. This is certainly very strong statement in a sharp conflict with the standard view about color confinement. It would make sense in TGD framework, where color as a spin like quantum number is replaced with angular momentum like quantum number. One could say that macroscopic systems perform macroscopic color rotation. The model for the honeybee dance [L51] conforms with this view and actually led to the proposal for a modification of cosmic string type extremals $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$ by putting Y^2 in 2-D rigid body color rotation along both time axis and spatial axis of the string world sheet X^2.
3. This picture raises again the old question about the relationship of color and electroweak quantum numbers in TGD framework. Could one regard electroweak quantum numbers as a spin related to color group $SU(3)$ just as one can relate ordinary spin with Lorentz transformations? Color quantum numbers of say quarks would be analogous to orbital angular momentum. The realization of the action of the electroweak $U(2)_{ew}$ on CP_2 spinors indeed involves also geometric color rotation affecting the gauge potentials in the general case and $U(2)_{ew}$ can be identified as holonomy group of CP_2 spinor connection and sugroup of $SU(3)$. One could also see electroweak symmetry breaking as a further symmetry breaking $U(2) \to U(1) \times U(1)$ assignable with the flag manifold $SU(3)/U(1) \times U(1)$ parameterizing different choices of color quantization axes and having interpretation as CP_2 twistor space.

Remark: Number theoretic vision means that the quaternionic M^8-momenta are discrete with components having values in the extension of rationals. $P^2(M^4)$ becomes discrete if one poses $P^2(M^8) = 0$ condition for all states. The values of discontinuity of $Im(T)$ correspond now to a discrete sequence of poles along s-axis approximating cut. At the continuum limit this discrete sequence of poles becomes cut. Continuum limit would correspond to a finite measurement resolution in which one cannot distinguish the poles from each other.

6.3.3 Grassmannian approach and TGD

Grassmannian approach has provided besides technical progress deeper views about twistorialization and also led to the understanding of the Yangian symmetry.

Grassmannian twistorialization - or what I understand about it

The twistorialization of the scattering amplitudes works for planar amplitudes in massless theories and involves the following ingredients.

1. All scattering amplitudes are expressible in terms of on-mass-shell scattering amplitudes with massless on-mass-shell particles in complex sense.
2. The scattering amplitude is sum over contributions with varying number of loops. BCFW recursion relation allows to construct scattering amplitudes from their singularities using 3-particle amplitudes as building brick amplitudes. There are two types of singularities.

   For the first type of singularity one has on-shell internal line and one obtains a sum over all possible decompositions of the scattering amplitude to a product of on-mass-shell scattering amplitudes multiplied by delta function for momentum squared of the internal line. Second type of singularity corresponds to the so called forward limit and is obtained from $(n+2,k)$ amplitude by contracting two added adjacent particles to form a loop so that their momenta are opposite and integrating over the momentum.
3. The singular term is algebraically analogous to an exterior derivative of the scattering amplitude and can be integrated explicitly: the integration adds BCFW bridge to the both terms such that the forward limit loop in the second term is under the bridge. The outcome is BCFW formula for l-loop amplitude with n external particles with k negative helicities consisting of these two terms.

Twistor Grassmannian approach expresses the on mass shell scattering amplitudes appearing as building bricks as residue integrals over Grassmannian $Gr(n,k)$, where n is the number of particles and k is the number of negative helicities. The Grassmannian approach is described in a concise form in the slides by Jaroslav Trnka [B50] (see `http://tinyurl.com/pqjzffj`).

1. The construction of the on-mass-shell scattering amplitudes appearing in BCFW formula as residue integrals in Grassmannians follows by expressing the momentum conserving delta functions in twistor description in terms of auxiliary variables serving as coordinates of Grassmannian $G(n,k,C)$ for the on mass shell tree amplitude with n external particles having k negative helicities. Grassmannian has dimension $d=(n-k)k$ and can be identified as the space of k-planes - or equivalently $n-k$-planes in C^N. Grasmannian has a representation as homogenous space $G(n,k,C) = U(n)/U(n-k) \times U(k)$ having $SU(n)$ as the group of isometries. For $k=1$ one obtains projective space which is also symmetric space (allowing reflection along geodesic lines as isometries).
2. Grassmannians emerge as an auxiliary construct, and the multiple residue integral over Grassmannian gives sum of residues so that the introduction of Grassmannians might look like unnecessary complication. The selection of points of Grassmannian for given external quantum numbers by residue integral given at the same time the value of the amplitude might however have some deeper meaning.

   The construction involves standard mathematics, which is however new for physicists. For instance, notions such as Plücker coordinates, Schubert cells and cell decomposition appear. One can relate to each other various widely different looking expressions for the amplitudes as being associated with different cell decompositions of Grassmannian. The singularities of the integrand of the scattering amplitude defined as a multiple residue integral over $G(k,n)$ define a hierarchy of Schubert cells.
3. The so called positive Grassmannian [B33] defines a subset of singularities appearing in the scattering amplitudes of $\mathcal{N}=4$ SUSY. The points of positive Grassmannian $Gr_+(k,n)$ are representable as $k\times n$ matrices with positive $k\times k$ determinants. The singularities correspond to the boundaries of $Gr_+(k,n)$ with some $k\times k$ determinants vanishing. For tree diagrams the singularities correspond to poles appearing in the factorized term of the BCFW decomposition of the scattering amplitude. The positivity conditions hold true also for the twistors representing external particles.
4. Positivity conditions guarantee the convexity of the integration region determined by the C-matrix as point of $Gr_+(k,n)$ appearing in the conditions dictating the integration region.

   To better understand the meaning of positivity one can first consider triangle call it T - as a representation of positive Grassmannian $Gr_+(1,3) = P^2_+$. Any interior points of T can be regarded as center of mass for suitable positive masses at the vertices of the triangle. These conditions generalizes to the case of general polygons, which must be convex. If the number of vertices of the polygon is larger than 3, convexity is not automatically satisfied, and requires additional conditions.

This description generalizes to Grassmannians $Gr_+(k,n)$. Masses define the analog of C-matrix as element of $Gr_+(k,n)$ appearing in the twistor approach and the vertices of the triangle are analogous to the twistors associated with external particles combining to form a point of $Gr(4,n)$. Positivity condition is generalized to the condition that $k\times k$ minors of the $k\times n$ matrix are positive.

5. Also the twistors associated with the external particles must satisfy analogs of the positivity conditions. This involves the replacement of $Gr(4,n)$ associated with twistors of the external particles with $Gr_+(k+4,n)$. The additional k components of the twistors are Grassman numbers and determined by the superparts of the twistors (see the slides of Trnka at `http://tinyurl.com/pqjzffj`. I must admit that I did not understand this.

6. Residue integral can be defined in terms of what is called canonical form Ω - analog of volume form - having logarithmic singularities at the boundaries of the $Gr_+(k,n)$. Hence one can perform a reduction of the residue integral to a sum of integrals over $G(k,k+4)$ instead of $G(k,n)$ (actually not so surprising since the residue integrals give as outcome the residues at discrete points!).

   This leads to a reduction of the residue integral over $Gr_+(k,n)$ to a sum of lower dimensional residue integrals over triangulation defined by $Gr_+(k,k+4)$ represented as surfaces of $Gr_+(k,n)$ glued together along sides. The geometric analog would be decomposition of polygon to a union of triangles.

   This simplifies the situation dramatically [B67, B50, B33] and leads to the notion of amplituhedron [B15, B14]. What is so remarkable, is the simplicity of the expressions for all-loop amplitudes and the fact that positivity implies locality and unitarity for $\mathcal{N}=4$ SUSY.

7. It should be possible to construct Ω explicitly having the desired singularities which would be in TGD framework poles with $P^2(M^8)=P^2(M^4\times CP_2)=0$ if the proposed realization of unitary makes sense? Could one just assumes that Ω vanishes for that part of the boundary of $Gr_+(k,n)$, which gives loop singularities? Could these points $Gr_+(k,n)$ be transcendental and excluded for this reason?

If loop corrections are vanishing as ZEO strongly suggests, only tree amplitudes are needed. Therefore it is appropriate to summarize what I have managed to understand about the construction of the tree amplitudes with general value of k in the amplituhedron approach.

1. The notion of amplituhedron relies on the mapping of $G(k,n)$ to $G_+(k,k+m)$ $n\geq k+m$. Actually a map from $G(k,n)\times G(k+4,n)\rightarrow G_+(k,k+m)$ is in question. $m=4$ identifiable as the apparent dimension of twistor space without projective identification giving the actual dimension $d=3$. n is the number of external particles and k the number of negative helicities.

   The value of m is $m=4$ and follows from the conditions that amplitudes come out correctly. The constraint $Y=C\cdot Z$, where Y corresponds to point of $G_+(k,k+4)$ and Z to the point of $G(k+4,n)$ performs this mapping, which is clearly many-to one. One can decompose integral over $G_+(k,n)$ to integrals over positive regions $G_+(k,k+4)$ intersecting only along their common boundary portions. The decomposition of a convex polygon in plane to triangles represent the basic example of this kind of decomposition. Obviously there are several decompositions of this kind.

2. Each decomposition defines a sum of contributions to the scattering amplitude involving integration of a projectively invariant volume form over the positive region in question. The form has a logarithmic singularity at the boundaries of the integration region but spurious singularities cancel so that only the contribution of the genuine boundary of $G_+(k,k+4)$ remains. There are additional delta function constraints fixing the integral completely in real case.

3. In complex case one has residue integral. The proposed generalization to the complex case is by analytic continuation. TGD inspired proposal is that the positivity condition in the real case is generalized to the condition that the positive coordinates are replaced by complex coordinates of hyperbolic space representable as upper half plane or equivalently as the unit disk obtained from upper half plane by exponential mapping $w=exp(iz)$. The measure $d\alpha/\alpha$

would correspond to $dz = dw/w$. If taken over boundary circle labelled by discrete phase factors $exp(i\phi)$ given by roots of unity the integral would be numerically a discrete Riemann sum making no sense p-adically but residue theorem could allow to avoid the discretizaton and to define the p-adic variant of the integral by analytic continuation. These conditions would be completely general conditions on various projectively invariant moduli involved.

4. One must extend the bosonic twistors Z_a of external particles by adding k coordinates. This extension looks very difficult to understand intuitively. Somewhat surprisingly, these coordinates are anti-commutative super-coordinates expressible as linear combinations of fermionic parts of super-twistor using coefficients, which are also Grassmann numbers. Integrating over these one ends up with the standard expression of the amplitude using canonical integration measure for the regions in the decomposition of amplituhedron. An interesting question is whether the addition of k-dimensional anti-commutative parts to Z_a expressible in terms of super-coordinates is only a trick or whether it could have some physical interpretation.

Grassmannians as reduced WCWs?

Grassmannians appear as auxiliary spaces in twistor approach. Could Grassmannians and the procedure assigning to external momenta and helicities discrete set of points of Grassmannian and scattering amplitude have some concrete interpretation in TGD framework?

1. The points of cognitive representation define WCW coordinates for space-time surface. For a fixed number of points in cognitive representation WCW is effectively replaced with a finite-dimensional reduced WCW. These points would naturally correspond to the points defining ends of fermionic lines at partonic 2-surfaces. WCW has Kähler metric with Euclidian signature. This could be true also for its reduction.

2. The experience with twistorialization suggests that these spaces could be simply Grassmannians $Gr(n, r, C)$ consisting or r-dimensional complex planes of n-dimensional complex space representable as coset spaces $U(n)/U(n-r) \times U(r)$ appearing as auxiliary spaces in the construction of twistor amplitudes.

   Note that the correlation between quantum states and geometry would be present since n corresponds to the number of external particles and r to those with negative helicity in ordinary twistor Grassmann approach. In TGD framework discretized variants of these spaces corresponding to the extension of rationals used would appear. Yangian symmetries could correspond to general coordinate transformations for the reduced WCW acting as gauge symmetry. These transformations act as diffeomorphisms for so called positive Grassmannians also in the standard twistorialization. If the reduced WCWs indeed correspond to twistor Grassmannians, one would have a completely unexpected connection with supersymmetric QFTs.

3. The reduction of WCW to a finite dimensional Kähler manifold suggests that also WCW spinors become ordinary spinors for Kähler manifold so that gamma matrices form a finite-D fermionic oscillator operator algebra. WCW has maximal symmetries and it would not be surprising if also the finite-D Kähler manifold would possess maximal symmetries. Note that WCW gamma matrices together with isometry generators of WCW give rise to a super-symplectic algebra involving a generalization of 2-D conformal invariance replacing 2-D surfaces with light-like 3-surfacs.

4. The interpretation of supersymmetry would be different from the standard one. Kähler structure implies that $\mathcal{N}$ is even and Majorana spinors are absent and both baryon and lepton number can be conserved separately. The ordinary fermionic oscillator algebra is a Clifford algebra and could be interpreted in terms of a broken supersymmetry.

Also more general flag manifolds than Grassmannians can be considered. If these spaces are homogenous spaces they have maximal isometries. They should have also Kähler structure. Compactness looks also a highly desirable property. The gauge conditions for the subalgebra of super-symplectic algebra state that the sub-algebra and its commutator with the entire algebra annihilate physical states and give rise to vanishing classical Noether charges. This would effectively reduce the super-symplectic algebra to a finite-D Lie group or Kac-Moody algebra of a finite-dimensional Lie group - perhaps belonging to the ADE hierarchy as the hierarchy of inclusions of HFFs as an alternative correlate for the realization of finite measurement resolution suggests. The flag manifolds associated with these Lie groups define more general homogenous spaces as candidates for the reduced WCWs.

Interpretation for Grassmannian residue integrations

The identification of Grassmannians (or possibly more general spaces) as reduced WCWs would give a genuine physical interpretation for the Grassmannian integrations as residue integrations over reduced WCW. What looks mysterious and maybe even frustrating is that the outcome of the entire process is sum over discrete residues: what does this mean?

1. The residue integration is only over a surface of reduced WCW with dimension equal to one half of that of WCW. One has integrand, which depends on the external quantum numbers coded in terms of twistors and on coordinates of reduced WCW. The residue integration is analogous to summation over amplitude associated with space-time surfaces coded by different cognitive representations.
2. One can argue that a continuous residue integral over Grassmannian is not consistent with the number theoretic discretization. The outcome is however discrete set of space-time surfaces labelled by cognitive representations as points of Grassmannian. Of the points in question are in the extension and if this is equivalent with the corresponding property for the coordinates of Grassmannian, there should be no problems. The restriction of external momenta to the extension of rationals might guarantee this.
3. The full multiple residue integral leaves only pole contributions, which correspond to a discrete collection of space-time surfaces (at least the set of space-time surfaces obtained by the action of Galois group), that is discrete set of points of reduced WCW. It seems that the entire residue integration is just a way to realize quantum classical correspondence by associating to the external quantum numbers space-time surfaces and corresponding cognitive representations - and of course, also the scattering amplitude.
4. One can also ask whether the positivity of Grassmannian might relate to the fact that p-adic numbers as ordinary integers are always non-negative (most of them infinite). The positivity might be necessary in order to have number theoretic universality. If the minors associated with the C-matrix serve as coordinates for $Gr_+(k,n)$ they could be interpreted also as p-adic numbers. If they are allowed to be negative, one encounters problems since p-adic numbers are not well-ordered and one cannot say whether p-adic number is negative or positive.

Posible description of SUSY and its breaking in TGD framework

Although twistor description make sense also in the absence of supersymmetry, super-symmetry is an essential part of the elegance of the Grassmannian approach. For the ordinary SUSY one has gluons and their superpartners characterized in terms of super-twistors. In TGD one has two pictures [L37, L45].

1. At the level H fermions as fundamental particles are described in terms of second quantized induced spinor fields, whose oscillator operators can be used to build gamma matrices for WCW [K106, K80]. In TGD universe all known elementary particles would be composites of fundamental fermions represented as lines at the light-like orbits of partonic 2-surfaces (wormhole throats) and ordinary elementary particles involve a pair of wormhole contacts with throats containing these fermion lines. It is assumed that the fermions are at different points: this allows to avoid problems due to infinities.

   In the proposed generalization of twistor approach $2 \to 2$ fermion scattering in the classical fields at partonic 2-surface would define the basic $2 \to 2$-vertex replacing 3-vertices of twistorial SUSY. Essentially one has only two-vertices describing the redistribution of fermions at partonic 2-surface between orbits of the partonic 2-surfaces meeting at it. This is different from $\mathcal{N} = 4$ SUSY [L22]. If one allows completely local multi-fermion states at the level of H one cannot avoid fermionic contact interactions.

   The many-fermion states associated with partonic 2-surfaces would define the analogs of super-multiplets. One can wonder whether a SUSY type description could exist as a limit when the partonic 2-surface is approximated with single point so that also positions of fermions are approximated as single point. SUSY would be only approximate.

2. At the level of M^8 I have proposed the use of polynomials P of super-octonion serving as analogs of super-gluon fields to construct scattering amplitudes [L37]. This allows geometric description of all particles using super-multiplets. Each monomial of theta parameters would give rise to its own space-time surface by the condition that either $IM(P)$ or $RE(P)$ vanishes for the corresponding polynomial P. This condition would reduce the components of super-field to algebraic surfaces.

   There is however an important difference from H picture. The members of super-multiplet defined by P correspond to the coefficients of monomials of theta parameters having interpretation as analogs of oscillator operators. Super-partners would be in this sense point-like objects unlike in H approach, where this can hold true only approximately.

Could H- and M^8 pictures be equivalent and could one understand the breaking of SUSY in this framework?

1. $M^8 - H$ correspondence as a map of associative space-time regions from M^8 to minimal surfaces in H makes sense for the external particles and thus at boundaries of CDs. It assigns to a point of the partonic 2-surface $X^2 \subset X^4 \subset M^8$ the quaternionic tangent space of X^4 at it characterized by a point of CP_2. M^4 point is mapped to itself. There is additional condition requiring that quaternionic tangent space contains fixed complex sub-space but this is not relevant now.

2. Could this map be one-to-many so that super-field component describing purely many-fermion state would be mapped to several points at the image of X^2 in H describing multi-local many-fermion state? This is possible if the points in M^8 are singular in the sense that the action of a normal subgroup H of Galois group Gal leaves the point invariant so that Gal reduces to Gal/H: symmetry breaking takes place.

   The tangent spaces of the degenerate points are however different and are mapped to different points of CP_2 in $M^8 - H$ correspondence making sense at boundaries of CDs but not in their interiors. One would have several fermions with same M^4 coordinates but different CP_2 coordinates and the outcome would be many-fermion state. In the case of 2-fermion state the different values of CP_2 coordinates would be associated with the opposite throats of a wormhole contact whose orbit defines light-like 3-surface. Could light-likeness inducing the reduction of the metric dimension of the tangent space from 4 to 3 somehow induce also this degeneration?

3. Could symmetry breaking as a degeneration of Gal action to that for Gal/H take place for the conditions defining the 4-surfaces associated with the higher components of super-octonion and induce the breaking of SUSY at the level of M^8 manifesting as the non-locality of the fermion state at the level of H? This degeneration would be a typical manifestation of quantum criticality: criticality in general means co-incidence of two roots.

6.3.4 Summary

Since the contribution means in well-defined sense a breakthrough in the understanding of TGD counterparts of scattering amplitudes, it is useful to summarize the basic results deduced above as a polished answer to a Facebook question.

There are two diagrammatics: Feynman diagrammatics and twistor diagrammatics.

1. Virtual state is an auxiliary mathematical notion related to Feynman diagrammatics coding for the perturbation theory. Virtual particles in Feynman diagrammatics are off-mass-shell.

2. In standard twistor diagrammatics one obtains counterparts of loop diagrams. Loops are replaced with diagrams in which particles in general have complex four-momenta, which however light-like: on-mass-shell in this sense. BCFW recursion formula provides a powerful tool to calculate the loop corrections recursively.

3. Grassmannian approach in which Grassmannians $Gr(k,n)$ consisting of k-planes in n-D space are in a central role, gives additional insights to the calculation and hints about the possible interpretation.

4. There are two problems. The twistor counterparts of non-planar diagrams are not yet understood and physical particles are not massless in 4-D sense.

In TGD framework twistor approach generalizes.

1. Massless particles in 8-D sense can be massive in 4-D sense so that one can describe also massive particles. If loop diagrams are not present, also the problems produced by non-planarity disappear.

2. There are no loop diagrams- radiative corrections vanish. ZEO does not allow to define them and they would spoil the number theoretical vision, which allows only scattering amplitudes, which are rational functions of data about external particles. Coupling constant evolution - something very real - is now discrete and dictated to a high degree by number theoretical constraints.

3. This is nice but in conflict with unitarity if momenta are 4-D. But momenta are 8-D in M^8 picture (and satisfy quaternionicity as an additional constraint) and the problem disappears! There is single pole at zero mass but in 8-D sense and *also many-particle states* have vanishing mass in 8-D sense: this gives all the cuts in 4-D mass squared for all many-particle state. For many-particle states not satisfying this condition scattering rates vanish: these states do not exist in any operational sense! This is certainly the most significant new discovery in the recent contribution.

   BCFW recursion formula for the calculation of amplitudes trivializes and one obtains only tree diagrams. No recursion is needed. A finite number of steps are needed for the calculation and these steps are well-understood at least in 4-D case - even I might be able to calculate them in Grassmannian approach!

4. To calculate the amplitudes one must be able to explicitly formulate the twistorialization in 8-D case for amplitudes. I have made explicit proposals but have no clear understanding yet. In fact, BCFW makes sense also in higher dimensions unlike Grassmannian approach and it might be that the one can calculate the tree diagrams in TGD framework using 8-D BCFW at M^8 level and then transform the results to $M^4 \times CP_2$.

What I said above does yet contain anything about Grassmannians.

1. The mysterious Grassmannians $Gr(k, n)$ might have a beautiful interpretation in TGD: they could correspond at M^8 level to reduced WCWs which is a highly natural notion at $M^4 \times CP_2$ level obtained by fixing the numbers of external particles in diagrams and performing number theoretical discretization for the space-time surface in terms of cognitive representation consisting of a finite number of space-time points.

   Besides Grassmannians also other flag manifolds - having Kähler structure and maximal symmetries and thus having structure of homogenous space G/H - can be considered and might be associated with the dynamical symmetries as remnants of super-symplectic isometries of WCW.

2. Grassmannian residue integration is somewhat frustrating procedure: it gives the amplitude as a sum of contributions from a finite number of residues. Why this work when outcome is given by something at finite number of points of Grassmannian?!

   In M^8 picture in TGD cognitive representations at space-time level as finite sets of points of space-time determining it completely as zero locus of real or imaginary part of octonionic polynomial would actually give WCW coordinates of the space-time surface in finite resolution.

   The residue integrals in twistor diagrams would be the manner to realize quantum classical correspondence by associating a space-time surface to a given scattering amplitude by fixing the cognitive representation determining it. This would also give the scattering amplitude.

   Cognitive representation would be highly unique: perhaps modulo the action of Galois group of extension of rationals. Symmetry breaking for Galois representation would give rise to supersymmetry breaking. The interpretation of supersymmetry would be however different: many-fermion states created by fermionic oscillator operators at partonic 2-surface give rise to a representation of supersymmetry in TGD sense.

6.4 New insights about quantum criticality for twistor lift inspired by analogy with ordinary criticality

Quantum criticality (QC) is one of the basic ideas of TGD. Zero energy ontology (ZEO) is second key notion and leads to a theory of consciousness as a formulation of quantum measurement theory making observer part of the quantum system in terms of notion of self identified as a generalized Zeno effect or analog for a sequence of weak measurements, and solving the basic paradox of standard quantum measurement theory, which one usually tries to avoid by introducing some "interpretation".

ZEO allows to see quantum theory could be seen as "square root" of thermodynamics. It occurred to me that it would be interesting to apply this vision in the case of quantum criticality to perhaps gain additional insights about its meaning. We have a picture about criticality in the framework of thermodynamics: what would be the analogy in ZEO based interpretation of Quantum TGD? Could it help to understand more clearly the somewhat poorly understood views about the notion of self, which as a quantum physical counterpart of observer becomes in ZEO a key concept of fundamental physics?

The basic ingredients involved are discrete coupling constant evolution, zero energy ontology (ZEO) implying that quantum theory is analogous to "square root" of thermodynamics, self as generalized Zeno effect as counterpart of observer made part of the quantum physical system, $M^8 \leftrightarrow M^4 \times CP_2$ duality, and quantum criticality. A further idea is that vacuum functional is analogous to a thermodynamical partition function as exponent of energy $E = TS - PV$.

The correspondence rules are simple. The mixture of phases with different 3-volumes per particle in a critical region of thermodynamical system is replaced with a superposition of space-time surfaces of different 4-volumes assignable to causal diamonds (CDs) with different sizes. Energy E is replaced with action S for preferred extremals defining Kähler function in the "world of classical worlds" (WCW). S is sum of Kähler action and 4-volume term, and these terms correspond to entropy and volume in the generalization $E = TS - PV \rightarrow S$. P resp. T corresponds to the inverse of Kähler coupling strength α_K resp. cosmological constant Λ. Both have discrete spectrum of values determined by number theoretically determined discrete coupling constant evolution. Number theoretical constraints force the analog of micro-canonical ensemble so that S as the analog of E is constant for all 4-surfaces appearing in the quantum superposition. This implies quantization rules for Kähler action and volume, which are very strong since α_K is complex.

This kind of quantum critical zero energy state is created in unitary evolution created in single step in the process defining self as a generalized Zeno effect. This unitary process implying time de-localization is followed by a weak measurement reducing the state to a fixed CD so that the clock time identified as the distance between its tips is well-defined. The condition that the action is same for all space-time surfaces in the superposition poses strong quantization conditions between the value of Kähler action (Kähler coupling strength is complex) and volume term proportional to cosmological constant. The outcome is that after sufficiently large number of steps no space-time surfaces satisfying the conditions can be found, and the first reduction to the opposite boundary of CD must occur - self dies. This is the classical counterpart for the fact that eventually all state function reduction leaving the members of state pairs at the passive boundary of CD invariant are made and the first reduction to the opposite boundary remains the only option.

The generation of magnetic flux tubes provides a way to satisfy the constancy conditions for the action so that the existing phenomenology as well as TGD counterpart of cyclic cosmology as re-incarnations of cosmic self follows as a prediction. This picture allows to add details to the understanding of the twistor lift of TGD at classical level and allows an improved understanding of the p-adic length scale evolution of cosmological constant solving the standard problem caused by the huge value of Λ. The sign of Λ is predicted correctly.

This picture generalizes to the twistor lift of TGD and cosmology provides an interesting application. One ends up with a precise model for the p-adic coupling constant evolution of the cosmological constant Λ explaining the positive sign and smallness of Λ in long length scales as a cancellation effect for M^4 and CP_2 parts of the Kähler action for the sphere of twistor bundle in dimensional reduction, a prediction for the radius of the sphere of M^4 twistor bundle as Compton length associated with Planck mass (2π times Planck length), and a prediction for the p-adic coupling constant evolution for Λ and coupling strength of M^4 part of Kähler action giving also insights to the CP breaking and matter antimatter asymmetry. The observed two values of Λ could correspond to two different p-adic length scales differing by a factor of $\sqrt{2}$.

6.4.1 Some background

Some TGD background is needed to understand the ideas proposed in the sequel.

Discrete coupling constant evolution

The most obvious implication is discrete coupling constant evolution in which the set of values for coupling constants is discrete and analogous to the set of the critical values of temperature [L58] (see http://tinyurl.com/y9hlt3rp). Zeros of Riemann Zeta or its slight modification suggest themselves as the spectrum for the Kähler coupling strength. This discrete coupling constant evolution requires that loop corrections vanish. This vision is realized concretely in the generalization of the twistorial approach to the construction of scattering amplitudes [L58].

Non-manifest unitarity is the basic problem of the twistor Grassmann approach. A generalization of the BCFW formula without the loop corrections gives scattering amplitudes satisfying unitary constraints. The needed cuts are be replaced by sequences of massless poles in 8-D sense and cuts approximate these sequences (consider electrostatic analogy in which line charge approximates a discrete sequences of poles). The replacement cuts with sequences of poles is forced by the number theoretic discretization of momenta so that they belong to an extension of rationals defining the adele [L42] (see http://tinyurl.com/ycbhse5c).

Non-planar loop diagrams are a chronic problem of twistor approach since there is no general rule loop integrations allowing to combine them neatly. Also this problem disappears now.

M^8-H duality plays key role in the twistorial approach [L37] (see http://tinyurl.com/yd43o2n2). In the ordinary twistor approach all momenta are light-like so that it does not apply to massive particles. TGD solves this problem: at M^8 level one has quaternionic light-like 8-D momenta, which correspond to massive 4-D momenta in M^8 picture. In $H=M^4\times CP_2$ picture ground states of super-conformal representations are constructed in terms of spinor harmonics of in $M^4\times CP_2$, which are products plane-waves characterized by massive 4-momenta and color wave functions associated with massless Dirac equation in H. Also the analog of Dirac equation for the induced spinor fields at space-time surface is massless [K106] (see http://tinyurl.com/yc2po5gf).

ZEO and self as generalized Zeno effect

ZEO allows to see self as generalized Zeno effect [L46](see http://tinyurl.com/ycxm2tpd).

1. Generalized Zeno effect can be regarded as a sequence of “small” state function reductions analogous to weak measurements performed at active boundary of causal diamond (CD). In usual Zeno effect the state is unaffected under repeated measurements: now the same is true at passive boundary of CD whereas the members of state pairs at the active boundary change. The unitary evolutions followed by these evolutions leave thus passive boundary and states at it invariant whereas active boundary shifts farther away from the passive boundary and the members of state pairs at it are affected. This gives rise to the experienced flow of time.

   The change of states is characterized unitary S-matrix. Each unitary evolution involves delocalization in the space of CDs so that one has quantum superposition of CDs with sizes not smaller than the CD to which the state was localized at previous reduction. This gives rise to a steady increase of clock time defined as the distance between the tips of CD. Self dies and reincarnates as a self with opposite direction of clock time when the first unitary evolution at the passive boundary followed by a weak measurement at it takes place. Self dies when all observables leaving the states at passive boundary invariant are measured. There are no choices to be made anymore.

2. Quantum TGD as “square root ” of thermodynamics means that the partition function of thermodynamics is replaced by its “square root” defined by the vacuum functional identified as exponent of Kähler function of “world of classical worlds” (WCW). Kähler function is analogous to energy $E=TS-PV$ in thermodynamics with T replaced with the inverse

of complex Kähler coupling strength and P with cosmological constant, which have discrete spectrum of values.

One has the analog of micro-canonical ensemble for which only states with given energy are possible. Now the action (Kähler function) is same for the space-time surfaces assignable to the zero energy states involved. This condition allows to get rid of the exponentials defining the vacuum functional otherwise appearing in the scattering amplitudes. This condition is strongly suggested by number theoretic universality for which these exponentials are extremely troublesome since both the exponent and exponential should belong to the extension of rationals used.

This implies a huge simplification in the construction of the amplitudes [L37] (see http://tinyurl.com/yd43o2n2) because finite measurement resolution effectively replaces space-time surfaces with their cognitive representation defined by a discrete set of space-time points with embedding space coordinates in the extension of rationals defining the adele.This representation codes for the space-time surface if it corresponds to zero locus of real or imaginary part (in quaterionic sense) of an octonionic polynomial with real coefficients. WCW coordinates are given by the cognitive representation and are discrete. One is led to enumerative algebraic geometry.

M^8-H duality

M^8-H duality [L37] (see http://tinyurl.com/yd43o2n2) states that the purely algebraic dynamics determined by the vanishing of real or imaginary part for octonionic polynomial is dual to the dynamics dictated by partial differential equations for an action principle.

1. There are two options for how to identify M^8 counterparts of space-time surfaces in terms roots of four polynomials defining real or imaginary part of an octonionic polynomial obtained as a continuation of real polynomial.
   (a) One can allow all roots $x+iy$ and project them to M^4 or M^8 from M_c^8. One can decompose these surfaces to regions with associative (quaternionic) tangent space or normal space and they are analogous to external particles of a twistor diagram entering CD and to interaction regions in which associativity does not hold true and which correspond to interiors of CD. One can criticizes the projection as somewhat adhoc process.
   (b) It became later clear that one can also consider space-time surface as Minkowskian real regions so that the projection to a sub-space $M^4 \subset M_c^8$ of complexified octonions is invariant under the conjugation $i \to -i, I_k \to -I_k$, where I_k are quaternionic units. M_c^4 parts of space-time coordinates would be form $m = m^0 + iI_k m^k$, m^0, m^k real. This conditions need not or even cannot be posed on E_c^4 coordinates since M^8-H duality assigns to the tangent space of space-time surface a CP_2 point irrespective of whether the point is in M_c^8 or M^8.
2. At the level of H external particles correspond to minimal surfaces, which are also extremals of Kähler action and in accordance with the number theoretical universality and quantum criticality do not depend on the coupling parameters at all. They are obtained by a map taking the 4-surfaces in M^8 to those in H. These conditions should be equivalent with the condition that the 6-D surfaces X^6 in 12-D twistor space of H define twistor bundles of space-time surfaces X^4.
3. The space-time regions in the interiors of CDs are not minimal surfaces so that Kähler action and volume term couple dynamically and coupling parameters characterize the extremals. The analog is motion of point like particle in the Maxwell field defined by induced Kähler form: this is generalize to the motion of 3-D object with purely internal Kähler field and that associated with wormhole contacts and mediating interaction with larger and smaller space-time sheets.

   In these regions the map mediating M^8-H duality does not exist since one cannot label the tangent spaces of space-time surface by points of CP_2. The non-existence of this map is due to the failure of either associativity of tangent space or normal space at M^8 level. The initial values at boundaries of CD for the incoming preferred extremals however allows to fix the time evolution in the interior of CD. This is essentially due to the infinite number of gauge conditions for the super-symplectic algebra.

It has later turned out [L58] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

Quantum criticality

Quantum criticality is a further key notion of TGD and was originally motivated by the idea that Kähler coupling strength must be unique in order that the theory is unique.

1. The first implication of quantum criticality is quantization of various coupling strengths as analogs of critical temperature and of other critical parameters such as pressure. This quantization is required also by number theoretical universality in the adelic approach: coupling constant parameters must belong to the extension of rationals used.
2. Second implication of quantum criticality is a huge generalization of conformal symmetries to their 4-D analogs. The key observation is that 3-D light-like surfaces allow a generalization of conformal invariance to get the Kac-Moody algebra associated with the isometries of H (at least) as symmetries. In the case of boundary of CD this leads to what I call supersymplectic invariance: the symplectic transformations of the two components of $\delta CD \times CP_2$ act as isometries of WCW. This algebra allows a fractal hierarchy of sub-algebras isomorphic to the algebra itself and gauge conditions state that this kind of sub-algebra and its commutator with the entire algebra annihilate physical states and classical Noether charges for them vanish [L58] (see `http://tinyurl.com/y9hlt3rp`). By quantum classical correspondence (QCC) the eigenvalues of quantum charges are equal to the classical Noether charges in Cartan algebra of supersymplectic algebra.
3. The third implication is the understanding of preferred extremals in $H = M^4 \times CP_2$ and their counterparts at the level of M^8. Associativity condition at the level of M^8 satisfied by the spacetime surfaces representing external particles arriving into CD corresponds to quantum criticality posing conditions on the coefficients of octonionic polynomials. The space-time regions inside CD the space-time surfaces do not satisfy associativity conditions and are not critical.
4. TGD as "square root" of thermodynamics idea suggests a fourth application of quantum criticality. This analogy might allow a better understanding of self as Zeno effect. This application will be studied in the sequel.

6.4.2 Analogy of the vacuum functional with thermodynamical partition function

Consider first the thermodynamical view about criticality. I have discussed criticality from slightly different perspective in [L54] (see `http://tinyurl.com/ydhknc2c`).

1. Thermodynamical states in critical region, where phases with different densities - say liquid and gas - are present serves as a basic example. This situation is actually a problem of the approach relying on partition function as van der Waals equation predicting 3 different densities for the density of molecules as function of pressure and temperature. Cusp catastrophe gives a view about situation: number density n is behavior variable and P and T are the control variables.
2. The experimental fact is that the density is constant as function of volume V for fixed temperature T whereas van der Waals predicts dependence on V. The phase corresponding to the middle sheet of the cusp is not at all present and the portions of liquid and gas phases vary. Maxwell's rules (area rule and lever rule) allow to solve the problem plaguing actually all approaches based on partition function. Lever rule assumes that there are actually two

kinds of "elements" present. Molecules are the first element but what the second element could be? TGD identification is as magnetic tubes [L54].

3. In the more general case in which the catastrophe is more general than cusp and has more sheets, two or more phases with different volumes are present and their volumes and possibly other behavior variables analogous to volume vary at criticality.

4. If one applies criticality in stronger sense by requiring that the function which has extremum as function of n at the surface represented by cusp catastrophe has same value at different sheets of the cusp, only the boundary line of the cusp having V-shaped projection in (p, T)-plane remains.

Generalization of thermodynamical criticality to TGD context

The generalization of this picture to TGD framework replaces the mixture of thermodynamical phases with different volumes with quantum superposition of space-time surfaces with different 4-volumes assignable to CDs with different quantized sizes (by number theoretical constraints).

1. Vacuum functional, which is exponent of Kähler function of WCW expressible as Kähler action for its preferred extremal, can be regarded as a complex "square root" of thermodynamical partition function Z meaning that its real valued modulus squared is analogous to partition function [L10, L22, L24, L45].

   Action S, whose value for preferred extremal defines Kähler function of WCW serves as the analog of energy assumed to have expression $E = PV - TS$, which is not generally true but implied by the condition that E is homogenous as function of conjugate variable pairs P, V and T, S. The analogs of P and T correspond to coupling constant parameters. Pressure p is replaced with the coefficient of volume term in action - essentially cosmological constant. T is replaced with the coefficient $1/\alpha_K$ of Kähler action representing entropy (or negentropy depending on situation).

   Remark: Note that T corresponds now to $1/\alpha_K$ rather than α_K analogous to temperature when Kähler action S_K is regarded as analog of energy E rather than entropy S.

2. Quantum criticality in the sense of ZEO is the counterpart for the criticality in thermodynamics. The mixture of thermodynamical phases with different 3-volumes is replaced with quantum superposition of zero energy states with 4-surface having same action S but different 4-volumes assignable to different CDs. Critical system consists of several phases with same values of coupling parameters α_K and Λ but different 4-volume.

   There is also a number theoretic constraint identifiable as the counterpart of the constant energy condition defining micro-canonical ensemble. The exponent of action S must cancel from the scattering amplitudes to avoid serious existence problems in the p-adic sectors of adele associated with given extension of rationals. Criticality means thus that $exp(S)$ has same value for all preferred extremals involved. Real parts are same for all of them and imaginary parts of the action exponential are fixed modulo multiple of 2π. The analog in the case of van der Waals equation of state that the allowed states are associated with the boundary of the projection of the cusp catastrophe to (p, T) plane.

   Critical quantum states are superpositions of space-time surfaces with different 4-volumes associated with CDs with quantized size scales (distance between tips) and are generated by unitary evolution. The value of time as size of CD (distance between its tips) is not well-defined in these states.

   Remark: Quantum critical states are "timeless" as meditative practices would express it.

   This kind of superposition is created by unitary evolution operator at each step in the sequence of unitary evolutions followed by a state function reduction measuring clock time as the distance between the tips of CD. Localization to single CD is the outcome and only superposition with same time-scale and same S but possibly different 4-volumes.

3. The condition that action is same is very strong and applies to both real and imaginary parts of action (α_K is complex). The proposal [L17, L58] (see `http://tinyurl.com/yas6ofhv` and `http://tinyurl.com/y9hlt3rp`) is that the coupling constant evolution as p-adic length scale $p \simeq 2^k$, k prime corresponds to zero of Riemann ζ for $1/\alpha_K$ or is proportional to it by

rational multiplier q. For $q = 1$ $Re(1/\alpha_K)$ analogous to the ordinary temperature would be equal to $Re(s) = 1/2$ for the zeros at the critical line and imaginary parts would correspond to the imaginary parts $Im(s)$ of the zeros. Constancy of the action S would boil down to the conditions

$$Re(S_K) + Re(S_{vol}) = constant \ \ , \ \ Im(S_K) + Im(S_{vol}) = constant \ \ mod \ \ 2\pi \ \ . \tag{6.4.1}$$

Note that the condition for imaginary part is a typical quantization condition.

4-volume can can have arbitrary large values but for S_K this is probably not the case - this already by the quantization conditions. Hence one expects that there is some maximal possible volume for preferred extremals and thus maximal distance between the tips of CDs involved.

When the zero energy state is a superposition of only space-time surfaces with this maximal volume, further unitary evolutions are not possible and the first state function reduction to the opposite boundary of CD happens (death of self and reincarnation with opposite direction of clock time). Self has finite lifetime! This would be the classical correlate for the situation in which no quantum measurements leaving invariant the members of state pairs at the passive boundary of CD are possible.

The constancy of $Re(S)$

How the cancellation of real part of $\Delta(Re(S_K)) + \Delta(Re(S_{vol}))$ could take place?

1. The physical picture is that the time evolution giving rise to self starts from flux tube dominated phase obtained in the first state function reduction to the opposite boundary of CD and that also asymptotically one obtains flux tube dominated phase again but the flux tubes are scaled up. This is the TGD view about quantum cosmology as a sequences of selves and of their time reversals [K86] [L21] (see `http://tinyurl.com/y7fmaapa`). This picture suggests that the generation of magnetic flux tubes allows to satisfy the $\Delta Re(S_K) + \Delta Re(S_{vol}) = 0$ condition: in Minkowskian regions the change magnetic part of $\Delta Re(S_K)$ tends to cancel $\Delta Re(S_{vol})$ whereas the electric part is of the same sign. Therefore magnetic flux tubes are favored.

   If the sign of the volume term is negative the exponential defining the vacuum functional decreases with volume. If the relative sign of S_K and S_{vol} is negative, the magnetic part of the action is positive. The generation of flux tubes generates positive magnetic action ΔS_K helping to cancel the change ΔS_{vol}.

   The additional conditions coming from the imaginary parts are analogous to semiclassical quantization conditions.

2. The proposed picture can be realized by a proper choice of the relative signs of volume term and Kähler action term. The relative sign comes automatically correct for a positive value of cosmological constant Λ. For this choice the total action density is

$$L_{tot} = (L_K + \frac{\Lambda}{8\pi G})\sqrt{g_4} \ \ . \tag{6.4.2}$$

   This choice gives positive vacuum energy density associated with the volume term.

3. The density of Kähler action associated with CP_2 degrees of freedom is

$$L_{K,CP_2} = -\frac{1}{4g^2}J^{\mu\nu}J_{\mu\nu} \ \ . \tag{6.4.3}$$

   The action is proportional to $E^2 - B^2$ in Minkowskian regions and magnetic term has sign opposite to that of volume term so that these terms can compensate with the condition guaranteeing constant action. The overall sign of action in the exponent can be chosen so that the exponential vanishes for large volumes. This suggests that the volume term is negative in the vacuum functional (Kähler function as negative of the action for preferred extremal). Euclidian regions, where CP_2 part of Kähler action is of form $B^2 + E^2$ and tends to cancel the volume term.

4. There is also Kähler action in M^4 degrees of freedom. In twistor lift dimensional reduction occurs for 6-D Kähler action and M^4 part and CP_2 part contribute to Kähler action. The S^2 parts of these actions must give rise to a cosmological constant decreasing like the inverse of p-adic length scale squared. This is achieved if the Kähler contributions have opposite signs so that M^4 contribution has a non-standard sign. This is possible if M^4 Kahler form is proportional to imaginary unit and M^4 Kähler coupling strength contains additional scaling factor.

   The induced Kähler form must be sum of the M^4 parts and CP_2 parts and also the action must be sum of M^4 and CP_2 parts. This is achieved if the charge matrices of these two Kähler forms are orthogonal (the trace of their product vanishes). Since CP_2 part couples to both 1 and Γ_9 giving rise to Kähler charges proportional to 1 for quarks and 3 for leptons having opposite chiralities, the corresponding charges would be proportional to 3 for quarks and -1 for leptons.

   The imaginary unit multiplying M^4 Kähler form disappears in action and field equations and one obtains

$$L_K = -\frac{1}{4g_K^2}(\epsilon^2 J^2(M^4) + J^2(CP_2) \ , \quad (6.4.4)$$

   where ϵ is purely imaginary so that one has $\epsilon^2 < 0$. Since the fields are induced, negative sign for M^4 Kähler action is not expected to lead to difficulties if M^4 term is small.

Some examples are in order.

1. For cosmic string extremals Kähler action is multiple of volume action. The condition that the two actions cancel would give a constraint between Λ and α_K. Net string tension would be reduced from the value determined by CP_2 scale to a rather small value. This need not occur generally but might be true for very short p-adic length scales, where Λ is large as required by the large value of string tension associated with Kähler action. For thickened cosmic strings (magnetic flux tubes) the value of string tension assignable to Kähler action is reduced and the condition can be satisfied for smaller values of Λ.

2. For CP_2 type extremals assignable to wormhole contacts serving as basic building bricks of elementary particles the action would be finite for all size scales of CD. Both magnetic and electric contribution to the action are of same sign. For Euclidian regions with 4-D space-time projection with so strong electric field that it changes the signature of the induced metric the same is true.

3. One can ask whether blackhole interiors as Euclidian regions correspond to these Euclidian space-time sheets or to highly tangled magnetic flux tubes with length considerably longer than Schwartschild radius for which cancellation also can occur (see `http://tinyurl.com/ydhknc2c`). Both pictures are consistent in many-sheeted space-time: magnetic flux tube tangle could topologically condense to a space-time sheet with Euclidian signature. Cancellation cannot last for ever so that also blackholes are unstable against big state function reduction changing the arrow of time. Blackhole evaporation might relate to this instability.

The constancy of $Im(S)$ modulo 2π

If cosmological constant is real, the condition for the constancy of imaginary part of ΔS modulo 2π applies only to the case of S_K and implies that ΔS_K is fixed modulo 2π in the superposition of space-time surfaces. If zeros of ζ [L17] (see `http://tinyurl.com/yas6ofhv`) or its modification $Zeta$) [L58]) (see `http://tinyurl.com/y9hlt3rp`) give the spectrum of $1/\alpha_K$ the value of $\Delta S_{K,red} = \int Tr(J^2)dV$ is given as multiples of $2\pi n/y$, where y is imaginary part for a zero of zeta. The constancy of $Re(S)$ implies that the 4-volume ΔV is quantized as multiples of $2\pi n/\Lambda$. These conditions bring in mind semiclassical quantization of the action in multiples of $\hbar$.

It however turns out that twistor lift forces same phase for M^4 and CP_2 parts of the Kähler action so that the quantization condition for volume is lost. The reason is that $1/\alpha_K(M^4)$ and $1/\alpha_K(CP_2)$ are proportional to

$$\frac{1}{\alpha_{K,6}} = \frac{1}{\alpha_{K,4}R^2} \ , \tag{6.4.5}$$

where R^2 has dimensions of length squared.

6.4.3 Is the proposed picture consistent with twistor lift of Kähler action?

Is it possible to realize the cancellation of real parts of ΔS_{vol} and ΔS_K (modulo 2π for imaginary part) for the twistor lift of Kähler action? Does the sign of the cosmological constant Λ come out correctly (wrong sign of Λ is the probably fatal problem of M-theory)? Can one understand the p-adic evolution of the cosmological constant Λ implying that Λ becomes small in long p-adic length scales and thus solving the key problem related to Λ?

Dimensional reduction of the twistor lift

The condition that the induction of the product of twistor bundles of M^4 and CP_2 to the space-time surface gives the twistor bundle of the space-time surface is conjectured to determine the dynamics of the space-time surfaces. A generalization of 4-D Kähler action to 6-D Kähler action is proposed to give this dynamics, and to dimensionally reduce to a sum of Kähler actions associated with M^4 and CP_2 Kähler forms plus cosmological term.

1. Twistor bundles are sphere bundles. For the extremals of 6-D Kähler action dimensional reduction takes place since 6-D extremals must be twistor bundle of corresponding space-time surface. Therefore S^2 degrees of freedom are frozen and become non-dynamical.

   One could say that the spheres appearing as fibers of twistor bundles of M^4 and CP_2 are identified in the embedding map. The simplest correspondence between $S^2(M^4)$ and $S^2(CP_2)$ identifies (θ_1, ϕ_1) for $S^2(M^4)$ with (θ_2, ϕ_2) for $S^2(CP_2)$. This means that $S^2(X^6)$ is mapped in the same manner to $S^2(M^4)$ and $S^2(CP_2)$.

   One can imagine also correspondence with n-fold winding based on the identification $(\theta_1, \phi_1) = (\theta_2, n\phi_2)$. The area of $S^2(M^4)$ are becomes n-fold and the S^2 part of the Kähler action using θ_2 as coordinate transforms as $S_K(S^2(M^4)n = 1) \rightarrow S_K(S^2(M^4)n) = n^2 S_K(S^2(M^4))$. $n = 1$ is the most plausible option physically.

2. What the proposed general vision implies for cosmological constant as a sum of $S^2(M^4)$ and $S^2(CP_2)$ parts of 6-D Kähler action giving in dimensional reduction 4-D volume term responsible for the cosmological constant and 4-D Kähler action. If the charge matrices of M^4 and CP_2 parts of Kähler form are orthogonal one can induce Kähler form. If the coupling to M^4 Kähler form is imaginary, M^4 and CP_2 contributions to the total Kähler action have opposite signs. M^4 and CP_2 parts have opposite signs of magnetic terms and the sign of CP_2 magnetic part is opposite to the volume term.

3. The dimensionally reduced action is obtained by integrating the 6-D Kähler action over S^2 fiber. The integration gives the area $A(S^2)$ of the S^2 fiber, which in the metric induced from the spheres of twistor space of X^4 is given by

$$A(S^2) = (1+r^2)4\pi R^2(S^2(CP_2)) \ , \quad r = \frac{R(S^2(CP_2))}{R(S^2(M^4))} \ . \tag{6.4.6}$$

   The very natural but un-checked assumption is that the radius of $S^2(CP_2)$ equals to the radius $R(CP_2)$ of the geodesic sphere of CP_2:

$$R(S^2(CP_2)) = R(CP_2) \ . \tag{6.4.7}$$

One obtains

$$L = -\frac{1}{16\pi\alpha_{K,6}} \left[J^2(CP_2) + \epsilon^2 J^2(M^4) + J^2(S^2(CP_2)) + \epsilon^2 J^2(S^2(M^4))\right] A(S^2) \ . \quad (6.4.8)$$

The immediate conclusion is that the phases of Kähler action and volume term are same so that the quantization condition for imaginary part of the action is not obtained.

4. The Kähler coupling strengths $\alpha_K(CP_2)$ and $\alpha_K(M^4)$ can be read from the first term

$$\begin{aligned} \frac{1}{\alpha_K(CP_2)} &= \frac{1}{\alpha_{K,4}4\pi(1+r^2)}\frac{R^2(CP_2)}{R^2} \ , \\ \frac{1}{\alpha_K(M^4)} &= \frac{\epsilon^2}{\alpha_K(CP_2)} \ . \end{aligned} \quad (6.4.9)$$

One can choose the factor R^2 to be the area of S^2 by suitably renormalizing $1/\alpha_K$. This would give simpler expression

$$\begin{aligned} \frac{1}{\alpha_K(CP_2)} &= \frac{1}{\alpha_{K,4}} \ , \\ \frac{1}{\alpha_K(M^4)} &= \frac{\epsilon^2}{\alpha_K(CP_2)} \ . \end{aligned} \quad (6.4.10)$$

5. One can deduce constraints on the value of the ϵ^2 from the smallness of the contributions of the corresponding $U(1)$ gauge potential to the ordinary Coulomb potential affecting the energies of atoms by a coupling proportional to mass number A rather than Z as for Coulomb potential. This allows to distinguish between isotopes. This gives very stringent bounds on ϵ^2. I have earlier derived an upper bound treating this term as a perturbation and by considering the contribution to the Coulomb energy of hydrogen atom [L33] (see `http://tinyurl.com/y8xcem2d`). One obtains $\epsilon^2 \leq 10^{-10}$. The upper bound is also the size scale of CP breaking induced by M^4 part and characterizes also matter-antimatter asymmetry.

Cosmological constant

Consider next the prediction for the cosmological constant term.

1. The S^2 parts of the actions have constant values. The natural normalization of Kähler form of $J(S^2(X))$, $X = M^4, CP_2$ is as $J^2 = -2$. This a convention is the overall scale of normalization can be chosen freely by rescaling $1/\alpha_{K,4}$. Taking into account the fact that index raising is carried out by induced metric one finds that the cosmological term given the sum of M^4 and CP_2 contributions to S^2 part of Kähler action multiplied by $A(S^2)$

$$\Lambda = \frac{1}{16\pi\alpha_K}\frac{2}{(1+r^2)R^2(CP_2)}(1+\frac{\epsilon^2}{r^4}) \ . \quad (6.4.11)$$

If ϵ is imaginary one can achieve the cancellation giving rise to small cosmological constant.

2. The empirical condition on cosmological constant (see `https://en.wikipedia.org/wiki/Cosmological_constant`) can be expressed in terms of critical mass density corresponding to flat 3-space as

$$\begin{aligned} &\Lambda = 3\Omega_\Lambda H^2 \ , \quad \Omega \simeq .691 \ , \\ &H = \frac{da}{dt}a \qquad \frac{da}{dt} = \frac{1}{\sqrt{g_{aa}}} \ . \end{aligned} \quad (6.4.12)$$

Here a corresponds to the proper time for the light-cone M^4_+ and t for the proper time for the space-time surface, which is Lorentz invariant under the Lorentz group leaving the boundary δM^4_+.

From this one obtains a condition for allowing to get idea about the discrete evolution of Λ with p-adic length scale occurring in jumps:

$$1+\frac{\epsilon^2}{r^4} = 24\pi\alpha_K(1+r^2)R^2(CP_2)\times\Omega_\Lambda H^2 \quad . \tag{6.4.13}$$

In an excellent approximation one must have $\epsilon \simeq r^2$, $r = R(M^4)/(CP_2)$. One can consider two obvious guesses. One has either $R(M^4) = L_{Pl} = \sqrt{G}$ - that is Planck length - or one has the Compton length associated with Planck mass given by $R(M^4) = 2\pi l_{Pl}$. The first option gives in reasonable approximation $r = 2^{-11}$ and $\epsilon^2 = r^4 = 2^{-44} \sim .6\times 10^{-13}$. The second option gives $\epsilon^2 \simeq .9\times 10^{-10}$. This values corresponds roughly to the CP_2 breaking parameter and matter-antimatter asymmetry and M^4 part of the Kähler action indeed gives rise to CP_2 breaking. I have earlier derived an upper bound for ϵ by demanding that the Kähler U(1) forces does not give rise to observable effects in the energy levels of hydrogen atom. The upper bound is of the same magnitude as the estimate for ϵ^2 for the Compton scale option.

3. If one accepts p-adic length scale hypothesis $L_p \propto \sqrt{p}$, $p \simeq 2^k$ [K62], one expects $\Lambda(k) \propto 1/L(k)^2$ [L24] (see http://tinyurl.com/ybrhguux). How to achieve this? The only possibility is that the parameter ϵ^2 is subject to coupling constant evolution. One would have for the cosmological constant

$$\Lambda(k) \propto \frac{\epsilon^2}{r^4} - 1 \propto \frac{1}{L^2(k)} \propto 2^{-k} \quad . \tag{6.4.14}$$

This would suggest for the 2-adic coupling constant evolution of ϵ the expression

$$\epsilon^2 = -r^4(1-X) \quad , \quad X = 24\pi\alpha_K(1+r^2)R^2(CP_2)\times\Omega_\Lambda H^2 = q\times 2^{-k} \quad . \tag{6.4.15}$$

where q is rational number. Note that from p-adic length scale hypothesis one has $2^{-k} \propto 1/L^2(k)$. One can consider also p-adic primes near powers of small prime in which case one obtains different evolution.

4. For Ω_Λ constant this would predict quantization of Hubble constant as $\Omega_\Lambda H^2 \propto 1/L(k)^2$ determined by naïve scaling dimension. The ratio of Hubble constants for two subsequent scales would be $H(k)/H(k+1) = \sqrt{2}$ if Ω is constant. The observed - and poorly understood - variation of Hubble constant from cosmological studies and distance ladder studies is in the range $50-73.2$ km/s/Mpc. Cosmological studies correspond to longer scales so that the smaller value of H is consistent with the decrease of H. The ratio of these upper and lower bounds is $1.46 < \sqrt{2} \simeq 1.141$ (see http://tinyurl.com/yd6m8sca and http://tinyurl.com/ycr4ffm4).

   Remark: The uncertainty in the value of Hubble constant is reflected as uncertainty in the distances D deduced from cosmic redshift $z \simeq HD/c$. This is taken into account in the definition of cosmological distant unit $h^{-1}Mpc$, where h is in the range $.5-.75$ corresponding to a scale factor 1.5 rather near to $\sqrt{2}$.

5. Piecewise constant evolution means that acceleration parameter is positive since constant value of H gives

$$\frac{d^2a}{dt^2} = \frac{(da/dt)^2}{a} = aH^2 > 0 \quad . \tag{6.4.16}$$

If the phase transitions reducing H by factor 1/2 occur at $a(k) = 2^{k/2}a_0$, one has

$$\frac{d^2a}{dt^2} \propto 2^{-k/2} \quad . \tag{6.4.17}$$

Acceleration would be reduced gradually with rate determined by its naïve scaling dimension.

Solution of Hubble constant discrepancy from the length scale dependence of cosmological constant

One can critize this proposal. The recent best values of the Hubble constant are 67.0 km/s/Mpc and and 73.5 km/s/Mpc and their ratio is about 1.1 rather than $\sqrt{2}$. Therefore the hypothesis that H satisfies p-adic length scale hypothesis might be too strong. In the following a proposal in which the variation of H could be due to the variation of cosmological constant Λ satisfying p-adic length scale hypothesis is discussed.

The discrepancy of the two determinations of Hubble constant has led to a suggestion that new physics might be involved (see `http://tinyurl.com/yabszzeg`).

1. Planck observatory deduces Hubble constant H giving the expansion rate of the Universe from CMB data something like 360,000 y after Big Bang, that is from the properties of the cosmos in long length scales. Riess's team deduces H from data in short length scales by starting from galactic length scale and identifies standard candles (Cepheid variables), and uses these to deduce a distance ladder, and deduces the recent value of $H(t)$ from the redshifts.
2. The result from short length scales is 73.5 km/s/Mpc and from long scales 67.0 km/s/Mpc deduced from CMB data. In short length scales the Universe appears to expand faster. These results differ too much from each other. Note that the ratio of the values is about 1.1. There is only 10 percent discrepancy but this leads to conjecture about new physics: cosmology has become rather precise science!

TGD could provide this new physics. I have already earlier considered this problem but have not found really satisfactory understanding. The following represents a new attempt in this respect.

1. The notions of length scale are fractality are central in TGD inspired cosmology. Many-sheeted space-time forces to consider space-time always in some length scale and p-adic length scale defined the length scale hierarchy closely related to the hierarchy of Planck constants $h_{eff}/h_0 = n$ related to dark matter in TGD sense. The parameters such as Hubble constant depend on length scale and its value differ because the measurements are carried out in different length scales.
2. The new physics should relate to some deep problem of the recent day cosmology. Cosmological constant Λ certainly fits the bill. By theoretical arguments Λ should be huge making even impossible to speak about recent day cosmology. In the recent day cosmology Λ is incredibly small.
3. TGD predicts a hierarchy of space-time sheets characterized by p-adic length scales (Lk) so that cosmological constant Λ depends on p-adic length scale $L(k)$ as $\Lambda \propto 1/GL(k)^2$, where $p \simeq 2^k$ is p-adic prime characterizing the size scale of the space-time sheet defining the sub-cosmology. p-Adic length scale evolution of Universe involve as sequence of phase transitions increasing the value of $L(k)$. Long scales $L(k)$ correspond to much smaller value of Λ.
4. The vacuum energy contribution to mass density proportional to Λ goes like $1/L^2(k)$ being roughly $1/a^2$, where a is the light-cone proper time defining the "radius" $a = R(t)$ of the Universe in the Robertson-Walker metric $ds^2 = dt^2 - R^2(t)d\Omega^2$. As a consequence, at long length scales the contribution of Λ to the mass density decreases rather rapidly.

   Must however compare this contribution to the density ρ of ordinary matter. During radiation dominated phase it goes like $1/a^4$ from $T \propto 1/a$ and form small values of a radiation dominates over vacuum energy. During matter dominated phase one has $\rho \propto 1/a^3$ and also now matter dominates. During predicted cosmic string dominated asymptotic phase one has $\rho \propto 1/a^2$ and vacuum energy density gives a contribution which is due to Kähler magnetic energy and could be comparable and even larger than the dark energy due to the volume term in action.
5. The mass density is sum $\rho_m + \rho_d$ of the densities of matter and dark energy. One has $\rho_m \propto H^2$. $\Lambda \propto 1/L^2(k)$ implies that the contribution of dark energy in long length scales is considerably smaller than in the recent cosmology. In the Planck determination of H it is however assumed that cosmological constant is indeed constant. The value of H in long length scales is underestimated so that also the standard model extrapolation from long to short length scales gives too low value of H. This is what the discrepancy of determinations of H performed in two different length scales indeed demonstrate.

A couple of remarks are in order.

1. The twistor lift of TGD [L10, L24] [L55] suggests an alternative parameterization of vacuum energy density as $\rho_{vac} = 1/L^4(k_1)$. k_1 is roughly square root of k. This gives rise to a pair of short and long p-adic length scales. The order of magnitude for $1/L(k_1)$ is roughly the same as that of CMB temperature T: $1/L(k_1) \sim T$. Clearly, the parameters $1/T$ and R correspond to a pair of p-adic length scales. The fraction of dark energy density becomes smaller during the cosmic evolution identified as length scale evolution with largest scales corresponding to earliest times. During matter dominated era the mass density going like $1/a^3$ would to dominate over dark energy for small enough values of a. The asymptotic cosmology should be cosmic string dominated predicting $1/GT^2(k)$. This does not lead to contradiction since Kähler magnetic contribution rather than that due to cosmological constant dominates.
2. There are two kinds of cosmic strings: for the other type only volume action is non-vanishing and for the second type both Kähler and volume action are non-vanishing but the contribution of the volume action decreases as function of the length scale.

6.5 Further comments about classical field equations in TGD framework

In the sequel some remarks about field equations defining space-time surfaces in TGD framework are made.

First three dualities at the level of field equations are discussed. These dualities are rather obvious but extremely important concerning the physical interpretation of TGD.

The earlier proposal that external particles correspond to minimal surfaces is strengthened. Also the interaction regions would correspond to minimal surfaces. The strongest condition would be that the minimal surface property break down at reaction vertices only associated with partonic 2-surfaces defining the 2-D counterparts of vertices: this would mean physical exchange of classical conserved charges between volume part of the action and Kähler action just at these points. This condition might be too strong.

The strongest condition could mean strengthening of the strong form of holography to $M^4 \times CP_2$ counterpart of the proposed number theoretic holography based on the notion of cognitive representation at the level of M^8 [L37] and also justification for the proposed construction of twistor Grassmannian variants of scattering amplitudes involving also data at a discrete set of points [L58].

6.5.1 Three dualities at the level of field equations

The basic field equations of TGD allow several dualities. There are 3 of them at the level of basic field equations (and several other dualities such as $M^8 - M^4 \times CP_2$ duality).

1. The first duality is the analog of particle-field duality. The spacetime surface describing the particle (3-surface of $H = M^4 \times CP_2$ instead of point-like particle) corresponds to the particle aspect whereas the fields inside it geometrized in terms of sub-manifold geometry correspond to the field aspect. Particle orbit serves as wave guide for field, one might say.
2. Second duality is particle-spacetime duality. Particle identified as 3-D surface means that particle orbit is space-time surface glued to a larger space-time surface by topological sum contacts. It depends on the scale used, whether it is more appropriate to talk about particle or of space-time.
3. The third duality is hydrodynamics- massless field theory duality. Hydrodynamical equations state local conservation of Noether currents. Field equations indeed reduce to local conservation conditions of Noether currents associated with the isometries of H. One the other hand, these equations have interpretation as non-linear geometrization of massless wave equation with coupling to Maxwell fields. This realizes the ultimate dream of theoretician: symmetries dictate the dynamics completely. This is expected to be realized also at the level of scattering amplitudes and the generalization of twistor Grassmannian amplitudes could realize this in terms of Yangian symmetry.

Hydrodynamics-wave equations duality generalizes to the fermionic sector and involves super-conformal symmetry.

1. What I call modified gamma matrices Γ^α are obtained as contractions of the partial derivatives of the action defining space-time surface with respect to the gradients of embedding space coordinate with embedding space gamma matrices [K106]. The divergence $D_\alpha\Gamma^\alpha$ vanishes by field equations for the space-time surface and this is necessary for the internal consistency the Dirac equation ($\overline{\Psi}$ satisfies essentially the same equation as Ψ). Γ^α reduce to ordinary ones if the space-time surface is M^4 and one obtains ordinary massless Dirac equation.

2. Modified Dirac equation [K106] expressess conservation of super current and actually infinite number of super currents obtained by contracting second quantized induced spinor field with the solutions of modified Dirac. This corresponds to the super-hydrodynamic aspect. On the other hand, modified Dirac equation corresponds to fermionic analog of massless wave equation.

6.5.2 Are space-time surfaces minimal surfaces everywhere except at 2-D interaction vertices?

If one starts from the analogy with complex analysis, the natural hypothesis would be that singular surfaces are co-dimension 2 surfaces - string world sheets and partonic 2-surfaces, which are at the ends of space-time surfaces and define topological reaction vertices. Light-like 3-surfaces as partonic orbits would be formally analogous to cuts of analytic function.

One can argue [L67] that the singular surface defines a sub-manifold giving a deltafunction like contribution to the action density and that one can assign conserved quantities to this surface. This requires that the singular contributions to the energy momentum tensor and canonical momentum currents as spacetime vectors are parallel to the singular surface. There must be one time-like or light-like direction and singular points do not satisfy this condition. There can be however an exchange of conserved charged between Kähler and volume degrees of freedom for the singular surfaces [L67]. One can also consider the possibility that the exchange is non-vanishing at singular points only. This option, which is perhaps non-realistic would be the strongest and will be discussed below.

String boundaries represent orbits of fundamental point-like fermions located at 3-D light-like surfaces which represent orbits of partonic 2-surfaces. String world sheets are minimal surfaces and correspond to stringy objects associated with say hadrons. There are also degrees of freedom associated with space-time interior. One have objects of various dimension which all are minimal surfaces. Modified Dirac equation extends the field equations to supersymmetric system and assigns fermionic degrees of freedom to these minimal surfaces of varying dimension.

From the physics point of view, the singular surfaces are analogous to carriers of currents acting as point- and string-like sources of massless field equations (more general option allows also string world sheets as carriers of currents).

The action S determining space-time surfaces as preferred extremals follows from twistor lift [L10, L45, L24, L58] and equals to the sum of volume term Vol multiplied by the TGD counterpart of cosmological constant and Kähler action S_K. The field equation is a geometric generalization of d'Alembert (Laplace) equation in Minkowskian (Eucidian) regions of space-time surface coupled with induced Kähler form analogous to Maxwell field. Generalization of equations of motion for particle by replacing it with 3-D surface is in question and the orbit of particle defines a region of space-time surface.

1. Zero energy ontology (ZEO) suggests that the external particles arriving to the boundaries of given causal diamond (CD) are like free massless particles and correspond to minimal surfaces as a generalization of light-like geodesic. This dynamic reduces to mere algebraic conditions and there is no dependence on the coupling parameters appearing in S. In contrast to this, in the interaction regions inside CDs there could be a coupling between Vol and S_K due to the non-vanishing divergences of energy momentum currents associated with the two terms in action cancelling each other.

2. Similar algebraic picture emerges from $M^8 - H$ duality [L37] at the level of M^8 and from what is known about preferred extremals of S assumed to satisfy infinite number of super-

symplectic gauge conditions at the 3-surfaces defining the ends of space-time surface at the opposite boundaries of CD.

At M^8 side of M^8-H duality associativity is realized as quaternionicity of either tangent or normal space of the space-time surface. The condition that there is 2-D integral distribution of sub-spaces of tangent spaces defining a distribution of complex planes as subspaces of octonionic tangent space implies the map of the space-time surface in M^8 to that of H. Given point m_8 of M^8 is mapped to a point of $M^4\times CP_2$ as a pair of points (m_4,s) formed by $M^4\subset M^8$ projection m_4 of m_8 point and by CP_2 point s parameterizing the tangent space or the normal space of $X^4\subset M^8$.

Remark: The assumption about integrable distribution of $M^2(x)$ defining string world sheet in M^4 might be too general: $M^2x)$ could not depend on x.

If associativity or even the condition about the existence of the integrable distribution of 2-planes fails, the map to $M^4\times CP_2$ is lost. One could cope with the situation since the gauge conditions at the boundaries of CD would allow to construct preferred extremal connecting the 3-surfaces at the boundaries of CD if this kind of surface exists at all. One can however wonder whether giving up the map $M^8\rightarrow H$ is necessary.

3. Number theoretic dynamics in M^8 involves no action principle and no coupling constants, just the associativity and the integrable distribution of complex planes $M^2(x)$ of complexified octonions. This suggests that also the dynamics at the level of H involves coupling constants only via boundary conditions. This is the case for the minimal surface solutions suggesting that M^8-H duality maps the surfaces satisfying the above mentioned conditions to minimal surfaces. The universal dynamics conforms also with quantum criticality.

4. One can argue that the dependence of field equations on coupling parameters of S leading to a perturbative series in coupling parameters in the interior of the space-time surface inside CD spoils the extremely beautiful purely algebraic picture about the construction of solutions of field equations using conformal invariance assignable to quantum criticality. Classical perturbation series is also in conflict with the vision that the TGD counterparts twistorial Grassmannian amplitudes do not involve any loop contributions coming as powers of coupling constant parameters [L58].

To sum up, both M^8-H duality, number theoretic vision, quantum criticality, twistor lift of TGD reducing dynamics to the condition about the existence of induced twistor structure, and the proposal for the construction of twistor scattering amplitudes suggest an extremely simple picture about the situation. The divergences of the energy momentum currents of Vol and S_K would be non-vanishing delta function type singularities only at discrete points at partonic 2-surfaces defining generalized vertices so that minimal surface equations would hold almost everywhere as the original proposal indeed stated.

1. The fact that all the known extremals of field equations for S are minimal surfaces conforms with the idea. This might be due to the fact that these extremals are especially easy to construct but could be also true quite generally apart from singular points. The divergences of the energy momentum currents associated with S_K and Vol vanish separately: this follows from the analog of holomorphy reducing the field equations to purely algebraic conditions.

It is essential that Kähler current j_K vanishes or is light-like so that its contraction with the gradients of the embedding space coordinates vanishes. Second condition is that in transversal degrees of freedom energy momentum tensor is tensor of form (1,1) in the complex sense and second fundamental form consists of parts of type (1,1) and (-1-1). In longitudinal degrees of freedom the trace H^k of the second fundamental form $H^k_{\alpha\beta}=D_\beta\partial_\alpha h^k$ vanishes.

2. Minimal surface equations are a non-linear analog of massless field equation but one would like to have also the analog of massless particle. The 3-D light-like boundaries between Minkowskian and Euclidian space-time regions are indeed analogs of massless particles as are also the string like word sheets, whose exact identification is not yet fully understood. In any case, they are crucial for the construction of scattering amplitudes in TGD based generalization of twistor Grassmannian approach. At M^8 side these points could correspond to singularities at which Galois group of the extension of rationals has a subgroup leaving the point invariant. The points at which roots of polynomial as function of parameters co-incide would serve as an analog.

The intersections of string world sheets with the orbits of partonic 2-surface are 1-D light-like curves X_L^1 defining fermion lines. The twistor Grassmannian proposal [L58] is that the ends of the fermion lines at partonic 2-surfaces defining vertices provide the information needed to construct scattering amplitudes so that information theoretically the construction of scattering amplitudes would reduce to an analog of quantum field theory for point-like particles.

3. Number theoretic vision discretizes coupling constant evolution: the values of coupling constants are labelled by parameters of extension of rationals and p-adic primes. This implies that twistor scattering amplitudes for given discrete values of coupling constants involve no radiative corrections [L58]: the construction of twistor Grassmannian amplitudes would be extremely simple. Note that infinite perturbation series would break the expression of scattering amplitudes as rational functions with coefficients int he extension of rationals defining the adele [L42, L43]. The cuts for the scattering amplitudes would be replaced by sequences of poles. This is unavoidable also because there is number theoretical discretization of momenta from the condition that their components belong to an extension of rationals defining the adele.

What could the reduction of cuts to poles for twistorial scattering amplitudes at the level of momentum space [L58] mean at space-time level?

1. Poles of an analytic function are co-dimension 2 objects. d'Alembert/Laplace equations holding true in Minkowskian/Euclidian signatures express the analogs of analyticity in 4-D case. Co-dimension 2 rule forces to ask whether partonic 2-surfaces defining the vertices and string world sheets could serve analogs of poles at space-time level? In fact, the light-like orbits X_L^3 of partonic 2-surfaces allow a generalization of 2-D conformal invariance since they are metrically 2-D so that X_L^3 and string world sheets could serve in the role of poles.

   X_L^3 could be seen as analogs of orbits of bubbles in hydrodynamical flow in accordance with the hydrodynamical interpretations. Particle reactions would correspond to fusions and decays of these bubbles. Strings would connect these bubbles and give rise to tensor networks and serve as space-time correlates for entanglement. Reaction vertices would correspond to common ends for the incoming and outgoing bubbles. They would be analogous to the lines of Feynman diagram meeting at vertex: now vertex would be however 2-D partonic 2-surface.

2. What can one say about the singularities associated with the light-like orbits of partonic 2-surfaces? The divergence of the Kähler part T_K of energy momentum current T is proportional to a sum of contractions of Kähler current j_K with gradients ∇h^k of H coordinates. j_K need not be vanishing: it is enough that its contraction with ∇h^k vanishes and this is true if j_K is light-like. This is the case for so called massless extremals (MEs). For the other known extremals j_K vanishes.

   Could the Kähler current j_K be light-like and non-vanishing and singular at X_L^3 and at string world sheets? This condition would provide the long sought-for precise physical identification of string world sheets. This would also induce to the modified Dirac action a 2-D contribution. Minimal surface equations would hold true also at these two kinds of surfaces apart from possible singular points. Even more: j_K could be non-vanishing and thus also singular only at the 1-D intersections X_L^1 of string world sheets with X_L^3 - I have called these curves fermionic lines.

   What it means that j_K is singular - that is has 2-D delta function singularity at string world sheets? j_K is defined as divergence of the induced Kähler form J so that one can use the standard definition of derivative to define j_K at string world sheet as the limiting value $j_K^\alpha = (Div_{+-}J)^\alpha = lim_{\Delta x^n \rightarrow 0}(J_+^{\alpha n} - J_-^{\alpha n})/\Delta x^n$, where x^n is a coordinate normal to the string world sheet. If J is discontinuous, this gives rise to a singular current located at string world sheet. This current should be light like to guarantee that energy momentum currents are divergenceless. If J is not light-like, it gives rise to isometry currents with non-vanishing divergence at string world sheet. This is guaranteed if the isometry currents $T^{\alpha A}$ are continuous through the string world sheet.

3. If the light-like j_K at partonic orbits is localized at fermionic lines X_L^1, the divergences of isometry currents could be non-vanishing and singular only at the vertices defined at partonic 2-surfaces at which fermionic lines X_L^1 meet. The divergences $DivT_K$ and $DivT_{Vol}$ would

be non-vanishing only at these vertices. They should of course cancel each other: $DivT_K = -DivT_{Vol}$.

4. $DivT_K$ should be non-vanishing and singular only at the intersections of string world sheets and partonic 2-surfaces defining the vertices as the ends of fermion lines. How to translate this statement to a more precise mathematical form? How to precisely define the notions of divergence at the singularity?

   The physical picture is that there is a sharing of conserved isometry charges of the incoming partonic orbit $i = 1$ determined T_K between 2 outgoing partonic orbits labelled by $j = 2, 3$. This implies charge transfer from $i = 1$ to the partonic orbits $j = 2, 3$ such that the sum of transfers sum up to the total incoming charge. This must correspond to a non-vanishing divergence proportional to delta function. The transfer of the isometry charge for given pair i, j of partonic orbits that is $Div_{i\to j}T_K$ must be determined as the limiting value of the quantity $\Delta_{i\to j}T_K^{\alpha,A}/\Delta x^\alpha$ as Δx^α approaches zero. Here $\Delta_{i\to j}T_K^{\alpha,A}$ is the difference of the components of the isometry currents between partonic orbits i and j at the vertex. The outcome is proportional delta function.

5. Similar description applies also to the volume term. Now the trace of the second fundamental form would have delta function singularity coming from $Div_{i\to j}T_K$. The condition $Div_{i\to j}T_K = -Div_{i\to j}T_{Vol}$ would bring in the dependence of the boundary conditions on coupling parameters so that space-time surface would depend on the coupling constants in accordance with quantum-classical correspondence. The manner how the coupling constants make themselves visible in the properties of space-time surface would be extremely delicate.

This picture conforms with the vision about scattering amplitudes at both M^8 and H sides of $M^8 - H$ duality.

1. M^8 dynamics based on algebraic equations for space-time surfaces [L37] leads to the proposal that scattering amplitudes can be constructed using the data only at the points of space-time surface with M^8 coordinates in the extension of the rationals defining the adele [L43, L42]. I call this discrete set of points cognitive representation with motivations coming from TGD inspired theory of consciousness [K65].

2. At H side the information theoretic interpretation would be that all information needed to construct scattering amplitudes would come from points at which the divergences of the energy momentum tensors of S_K and Vol are non-vanishing and singular.

Both pictures would realize extremely strong form of holography, much stronger than the strong form of holography that stated that only partonic 2-surfaces and string world sheets are needed.

6.6 Still about twistor lift of TGD

Twistor lift of TGD led to a dramatic progress in the understanding of TGD but also created problems with previous interpretation. The new element was that Kähler action as analog of Maxwell action was replaced with dimensionally reduced 6-D Kähler action decomposing to 4-D Kähler action and volume term having interpretation in terms of cosmological constant.

One can of course ask whether the resulting induced twistor structure is acceptable. Certainly it is not equivalent with the standard twistor structure. In particular, the condition $J^2 = -g$ is lost. In the case of induced Kähler form at X^4 this condition is also lost. For spinor structure the induction guarantees the existence and uniqueness of the spinor structure, and the same applies also to the induced twistor structure being together with the unique properties of twistor spaces of M^4 and CP_2 the key motivation for the notion.

There are some potential problems related to the definition of Kähler function. The most natural identification is as 6-D dimensionally reduced Kähler action.

1. WCW metric must be Euclidian - that positive definite. Since it is defined in terms of second partial derivatives of the Kähler function with respect to complex WCW coordinates and their conjugates, the preferred extremals must be completely stable to guarantee that this quadratic form is positive definite. This condition excludes extremals for which this is

not the case. There are also other identifications for the preferred extremal property and stability condition would is a obvious additional condition. Note that at quantum criticality the quadratic form would have some vanishing eigenvalues representing zero modes of the WCW metric.

2. Vacuum functional of WCW is exponent of Kähler function identified as negative of Kähler action for a preferred extremal. The potential problem is that Kähler action contains both electric and magnetic parts and electric part can be negative. For the negative sign of Kähler action the action must remain bounded, otherwise vacuum functional would have arbitrarily large values. This favours the presence of magnetic fields for the preferred extremals and magnetic flux tubes are indeed the basic entities of TGD based physics.

3. One can ask whether the sign of Kähler action for preferred extremals is same as the overall sign of the diagonalized Kähler metric: this would exclude extremals dominated by Kähler electric part of action or at least force the electric part be so small that WCW metric has the same overall signature everywhere.

If one accepts the proposal that the preferred extremals are minimal surfaces (the known extremals are), extremal property is satisfied for both 4-D Kähler action and volume term separately except at finite set of singular points at which there is transfer of conserved charges between the two degrees of freedom. In this principle this would allow the identification of Kähler function as either 4-D Kähler function or 4-D volume term (actually magnetic S^2 part of 6-D Kähler action). This option looks however rather ad hoc.

6.6.1 Is the cosmological constant really understood?

The interpretation of the coefficient of the volume term as cosmological constant has been a longstanding interpretational issue and caused many moments of despair during years. The intuitive picture has been that cosmological constant obeys p-adic length scale scale evolution meaning that Λ would behave like $1/L_p^2 = 1/p \simeq 1/2^k$ [L24].

This would solve the problems due to the huge value of Λ predicted in GRT approach: the smoothed out behavior of Λ would be $\Lambda \propto 1/a^2$, a light-cone proper time defining cosmic time, and the recent value of Λ - or rather, its value in length scale corresponding to the size scale of the observed Universe - would be extremely small. In the very early Universe - in very short length scales - Λ would be large.

A simple solution of the problem would be the p-adic length scale evolution of Λ as $\Lambda \propto 1/p$, $p \simeq 2^k$. The flux tubes would thicken until the string tension as energy density would reach stable minimum. After this a phase transition reducing the cosmological constant would allow further thickening of the flux tubes. Cosmological expansion would take place as this kind of phase transitions (for a mundane application of this picture see [K37]).

This would solve the basic problem of cosmology, which is understanding why cosmological constant manages to be so small at early times. Time evolution would be replaced with length scale evolution and cosmological constant would be indeed huge in very short scales but its recent value would be extremely small.

I have however not really understood how this evolution could be realized! Twistor lift seems to allow only a very slow (logarithmic) p-adic length scale evolution of Λ [L57]. Is there any cure to this problem?

1. The magnetic energy decreases with the area S of flux tube as $1/S \propto 1/p \simeq 1/2^k$, where $\sqrt{p}$ defines the transversal length scale of the flux tube. Volume energy (magnetic energy associated with the twistor sphere) is positive and increases like S. The sum of these has minimum for certain radius of flux tube determined by the value of Λ. Flux tubes with quantized flux would have thickness determined by the length scale defined by the density of dark energy: $L \sim \rho_{vac}^{-1/4}$, $\rho_{dark} = \Lambda/8\pi G$. $\rho_{vac} \sim 10^{-47}$ GeV^4 (see `http://tinyurl.com/k4bwlzu`) would give $L \sim 1$ mm, which would could be interpreted as a biological length scale (maybe even neuronal length scale).

2. But can Λ be very small? In the simplest picture based on dimensionally reduced 6-D Kähler action this term is not small in comparison with the Kähler action! If the twistor spheres of

M^4 and CP_2 give the same contribution to the induced Kähler form at twistor sphere of X^4, this term has maximal possible value!

The original discussions in [L10, L24] treated the volume term and Kähler term in the dimensionally reduced action as independent terms and Λ was chosen freely. This is however not the case since the coefficients of both terms are proportional to $(1/\alpha_K^2)S(S^2)$, where $S(S^2)$ is the area of the twistor sphere of 6-D induced twistor bundle having space-time surface as base space. This are is same for the twistor spaces of M^4 and CP_2 if CP_2 size defines the only fundamental length scale. I did not even recognize this mistake.

The proposed fast p-adic length scale evolution of the cosmological constant would have extremely beautiful consequences. Could the original intuitive picture be wrong, or could the desired p-adic length scale evolution for Λ be possible after all? Could non-trivial dynamics for dimensional reduction somehow give it? To see what can happen one must look in more detail the induction of twistor structure.

1. The induction of the twistor structure by dimensional reduction involves the identification of the twistor spheres S^2 of the geometric twistor spaces $T(M^4) = M^4 \times S^2(M^4)$ and of T_{CP_2} having $S^2(CP_2)$ as fiber space. What this means that one can take the coordinates of say $S^2(M^4)$ as coordinates and embedding map maps $S^2(M^4)$ to $S^2(CP_2)$. The twistor spheres $S^2(M^4)$ and $S^2(CP_2)$ have in the minimal scenario same radius $R(CP_2)$ (radius of the geodesic sphere of CP_2. The identification map is unique apart from $SO(3)$ rotation R of either twistor sphere possibly combined with reflection P. Could one consider the possibility that R is not trivial and that the induced Kähler forms could almost cancel each other?

2. The induced Kähler form is sum of the Kähler forms induced from $S^2(M^4)$ and $S^2(CP_2)$ and since Kähler forms are same apart from a rotation in the common S^2 coordinates, one has $J_{ind} = J + RP(J)$, where R denotes a rotation and P denotes reflection. Without reflection one cannot get arbitrary small induced Kähler form as sum of the two contributions. For mere reflection one has $J_{ind} = 0$.

   Remark: It seems that one can do with reflection if the Kähler forms of the twistor spheres are of opposite sign in standard spherical coordinates. This would mean that they have have opposite orientation.

   One can choose the rotation to act on (y, z)-plane as $(y, z) \rightarrow (cy + sz, -sz + cy)$, where s and c denote the cosines of the rotation angle. A small value of cosmological constant is obtained for small value of s. Reflection P can be chosen to correspond to $z \rightarrow -z$. Using coordinates $(u = cos(\Theta), \Phi)$ and their primed counterparts and by writing the reflection followed by rotation explicitly in coordinates (x, y, z) one finds $u' = -cu - s\sqrt{1-u^2}sin(\Phi)$, $\Phi' = arctan[(su/\sqrt{1-u^2}cos(\Phi) + ctan(\Phi)]$. In the lowest order in s one has $u' = -u - s\sqrt{1-u^2}sin(\Phi)$, $\Phi' = \Phi + scos(\Phi)(u/\sqrt{1-u^2})$.

3. Kähler form J^{tot} is sum of unrotated part $J = du \wedge d\Phi$ and $J' = du' \wedge d\Phi'$. J' equals to the determinant $\partial(u', \Phi')/\partial(u, \Phi)$. A suitable spectrum for s could reproduce the proposal $\Lambda \propto 2^{-k}$ for Λ. The S^2 part of 6-D Kähler action equals to $(J^{tot}_{\theta\phi})^2/\sqrt{g_2}$ and in the lowest order proportional to s^2. For small values of s the integral of Kähler action for S^2 over S^2 is proportional to s^2.

   One can write the S^2 part of the dimensionally reduced action as $S(S^2) = s^2F^2(s)$. Very near to the poles the integrand has $1/[sin(\Theta) + O(s)]$ singularity and this gives rise to a logarithmic dependence of F on s and one can write: $F = F(s, log(s))$. In the lowest order one has $s \simeq 2^{-k/2}$, and in improved approximation one obtains a recursion formula $s_n(S^2, k) = 2^{-k/2}/F(s_{n-1}, \log(s_{n-1})$ giving renormalization group evolution with k replaced by anomalous dimension $k_{n,a} = k + 2log[F(s_{n-1}, log(s_{n-1})]$ differing logarithmically from k.

4. The sum $J + RP(J)$ defining the induced Kähler form in $S^2(X^4)$ is covariantly constant since both terms are covariantly constant by the rotational covariance of J.

5. The embeddings of $S^2(X^4)$ as twistor sphere of space-time surface to both spheres are holomorphic since rotations are represented as holomorphic transformations. Also reflection as $z \rightarrow 1/z$ is holomorphic. This in turn implies that the second fundamental form in complex coordinates is a tensor having only components of type $(1, 1)$ and $(-1, -1)$ whereas metric and energy momentum tensor have only components of type $(1, -1)$ and $(-1, 1)$. Therefore

all contractions appearing in field equations vanish identically and $S^2(X^4)$ is minimal surface and Kähler current in $S^2(X^4)$ vanishes since it involves components of the trace of second fundamental form. Field equations are indeed satisfied.

6. The solution of field equations becomes a family of space-time surfaces parameterized by the values of the cosmological constant Λ as function of S^2 coordinates satisfying $\Lambda/8\pi G = \rho_{vac} = J \wedge (*J)(S^2)$. In long length scales the variation range of Λ would become arbitrary small.

7. If the minimal surface equations solve separately field equations for the volume term and Kähler action everywhere apart from a discrete set of singular points, the cosmological constant affects the space-time dynamics only at these points. The physical interpretation of these points is as seats of fundamental fermions at partonic 2-surface at the ends of light-like 3-surfaces defining their orbits (induced metric changes signature at these 3-surfaces). Fermion orbits would be boundaries of fermionic string world sheets.

   One would have family of solutions of field equations but particular value of Λ would make itself visible only at the level of elementary fermions by affecting the values of coupling constants. p-Adic coupling constant evolution would be induced by the p-adic coupling constant evolution for the relative rotations R combined with reflection for the two twistor spheres. Therefore twistor lift would not be mere manner to reproduce cosmological term but determine the dynamics at the level of coupling constant evolution.

8. What is nice that also $\Lambda = 0$ option is possible. This would correspond to the variant of TGD involving only Kähler action regarded as TGD before the emergence of twistor lift. Therefore the nice results about cosmology [K86] obtained at this limit would not be lost.

6.6.2 Does p-adic coupling constant evolution reduce to that for cosmological constant?

One of the chronic problems if TGD has been the understanding of what coupling constant evolution could be defined in TGD.

1. The notion of quantum criticality is certainly central. The continuous coupling constant evolution having no counterpart in the p-adic sectors of adele would contain as a sub-evolution discrete p-adic coupling constant evolution such that the discrete values of coupling constants allowing interpretation also in p-adic number fields are fixed points of coupling constant evolution.

   Quantum criticality is realized also in terms of zero modes, which by definition do not contribute to WCW metric. Zero modes are like control parameters of a potential function in catastrophe theory. Potential function is extremum with respect to behavior variables replaced now by WCW degrees of freedom. The graph for preferred extremals as surface in the space of zero modes is like the surface describing the catastrophe. For given zero modes there are several preferred extremals and the catastrophe corresponds to the regions of zero mode space, where some branches of co-incide. The degeneration of roots of polynomials is a concrete realization for this.

   Quantum criticality would also mean that coupling parameters effectively disappear from field equations. For minimal surfaces (generalization of massless field equation allowing conformal invariance characterizing criticality) this happens since they are separately extremals of Kähler action and of volume term.

   Quantum criticality is accompanied by conformal invariance in the case of 2-D systems and in TGD this symmetry extends to its 4-D analogas isometries of WCW.

2. In the case of 4-D Kähler action the natural hypothesis was that coupling constant evolution should reduce to that of Kähler coupling strength $1/\alpha_K$ inducing the evolution of other coupling parameters. Also in the case of the twistor lift $1/\alpha_K$ could have similar role. One can however ask whether the value of the 6-D Kähler action for the twistor sphere $S^2(X^4)$ defining cosmological constant could define additional parameter replacing cutoff length scale as the evolution parameter of renormalization group.

3. The hierarchy of adeles should define a hierarchy of values of coupling strengths so that the discrete coupling constant evolution could reduce to the hierarchy of extensions of rationals and be expressible in terms of parameters characterizing them.

4. I have also considered number theoretical existence conditions as a possible manner to fix the values of coupling parameters. The condition that the exponent of Kähler function should exist also for the p-adic sectors of the adele is what comes in mind as a constraint but it seems that this condition is quite too strong.

   If the functional integral is given by perturbations around single maximum of Kähler function, the exponent vanishes from the expression for the scattering amplitudes due to the presence of normalization factor. There indeed should exist only single maximum by the Euclidian signature of the WCW Kähler metric for given values of zero modes (several extrema would mean extrema with non-trivial signature) and the parameters fixing the topology of 3-surfaces at the ends of preferred extremal inside CD. This formulation as counterpart also in terms of the analog of micro-canonical ensemble (allowing only states with the same energy) allowing only discrete sum over extremals with the same Kähler action [L56].

5. I have also considered more or less ad hoc guesses for the evolution of Kähler coupling strength such as reduction of the discrete values of $1/\alpha_K$ to the spectrum of zeros of Riemann zeta or actually of its fermionic counterpart [L17]. These proposals are however highly ad hoc.

As I started once again to consider coupling constant evolution I realized that the basic problem has been the lack of explicit formula defining what coupling constant evolution really is.

1. In quantum field theories (QFTs) the presence of infinities forces the introduction of momentum cutoff. The hypothesis that scattering amplitudes do not depend on momentum cutoff forces the evolution of coupling constants. TGD is not plagued by the divergence problems of QFTs. This is fine but implies that there has been no obvious manner to define what coupling constant evolution as a continuous process making sense in the real sector of adelic physics could mean!

2. Cosmological constant is usually experienced as a terrible head ache but it could provide the helping hand now. Could the cutoff length scale be replaced with the value of the length scale defined by the cosmological constant defined by the S^2 part of 6-D Kähler action? This parameter would depend on the details of the induced twistor structure. It was shown above that if the moduli space for induced twistor structures corresponds to rotations of S^2 possibly combined with the reflection, the parameter for coupling constant restricted to that to $SO(2)$ subgroup of $SO(3)$ could be taken to be taken $s = sin(\epsilon)$.

3. RG invariance would state that the 6-D Kähler action is stationary with respect to variations with respect to s. The variation with respect to s would involve several contributions. Besides the variation of $1/\alpha_K(s)$ and the variation of the $S(^2)$ part of 6-D Kähler action defining the cosmological constant, there would be variation coming from the variations of 4-D Kähler action plus 4-D volume term . This variation vanishes by field equations. As matter of fact, the variations of 4-D Kähler action and volume term vanish separately except at discrete set of singular points at which there is energy transfer between these terms. This condition is one manner to state quantum criticality stating that field equations involved no coupling parameters.

   One obtains explicit RG equation for α_K and Λ having the standard form involving logarithmic derivatives. The form of the equation would be

$$\frac{dlog(\alpha_K)}{ds} = -\frac{S(S^2)}{S_K(X^4) + S(S^2)} \frac{dlog(S(S^2))}{ds} \quad . \tag{6.6.1}$$

   The equation contains the ratio $S(S^2)/(S_K(X^4) + S(S^2))$ of actions as a parameter. This does not conform with idea of micro-locality. One can however argue that this conforms with the generalization of point like particle to 3-D surface. For preferred extremal the action is indeed determined by the 3 surfaces at its ends at the boundaries of CD. This implies that the construction of quantum theory requires the solution of classical theory.

   In particular, the 4-D classical theory is necessary for the construction of scattering amplitudes. and one cannot reduce TGD to string theory although strong form of holography states that the data about quantum states can be assigned with 2-D surfaces. Even more: $M^8 - H$ correspondence implies that the data determining quantum states can be assigned with discrete set of points defining cognitive representations for given adel This set of points depends on the preferred extremal!

4. How to identify quantum critical values of α_K? At these points one should have $dlog(\alpha_K)/ds = 0$. This implies $dlog(S(S^2)/ds = 0$, which in turn implies $dlog(\alpha_K)/ds = 0$ unless one has $S_K(X^4) + S(S^2) = 0$. This condition would make exponent of 6-D Kähler action trivial and the continuation to the p-adic sectors of adele would be trivial. I have considered also this possibility [L57].

   The critical values of coupling constant evolution would correspond to the critical values of S and therefore of cosmological constant. The basic nuisance of theoretical physics would determine the coupling constant evolution completely! Critical values are in principle possible. Both the numerator $J^2_{u\Phi}$ and the numerator $1/\sqrt{det(g)}$ increase with ϵ. If the rate for the variation of these quantities with s vary it is possible to have a situation in which the one has

$$\frac{dlog(J^2_{u\Phi})}{ds} = -\frac{dlog(\sqrt{det(g)})}{ds} \quad . \tag{6.6.2}$$

5. One should demonstrate that the critical values of s are such that the continuation to p-adic sectors of the adele makes sense. For preferred extremals cosmological constant appears as a parameter in field equations but does not affect the field equations expect at the singular points. Singular points play the same role as the poles of analytic function or point charges in electrodynamics inducing long range correlations. Therefore the extremals depend on parameter s and the dependence should be such that the continuation to the p-adic sectors is possible.

   A naïve guess is that the values of s are rational numbers. Above the proposal $s = 2^{-k/2}$ motivated by p-adic length scale hypothesis was considered but also $s = p^{-k/2}$ can be considered. These guesses might be however wrong, the most important point is that there is that one can indeed calculate $\alpha_K(s)$ and identify its critical values.

6. What about scattering amplitudes and evolution of various coupling parameters? If the exponent of action disappears from scattering amplitudes, the continuation of scattering amplitudes is simple. This seems to be the only reasonable option. In the adelic approach [L42] amplitudes are determined by data at a discrete set of points of space-time surface (defining what I call cognitive representation) for which the points have M^8 coordinates belong to the extension of rationals defining the adele.

   Each point of $S^2(X^4)$ corresponds to a slightly different X^4 so that the singular points depend on the parameter s, which induces dependence of scattering amplitudes on s. Since coupling constants are identified in terms of scattering amplitudes, this induces coupling constant evolution having discrete coupling constant evolution as sub-evolution.

The following argument suggests a connection between p-adic length scale hypothesis and evolution of cosmological constant but must be taken as an ad hoc guess: the above formula is enough to predict the evolution.

1. p-Adicization is possible only under very special conditions [L42], and suggests that anomalous dimension involving logarithms should vanish for $s = 2^{-k/2}$ corresponding to preferred p-adic length scales associated with $p \simeq 2^k$. Quantum criticality in turn requires that discrete p-adic coupling constant evolution allows the values of coupling parameters, which are fixed points of RG group so that radiative corrections should vanish for them. Also anomalous dimensions Δk should vanish.

2. Could one have $\Delta k_{n,a} = 0$ for $s = 2^{-k/2}$, perhaps for even values $k = 2k_1$? If so, the ratio c/s would satisfy $c/s = 2^{k_1} - 1$ at these points and Mersenne primes as values of c/s would be obtained as a special case. Could the preferred p-adic primes correspond to a prime near to but not larger than $c/s = 2^{k_1} - 1$ as p-adic length scale hypothesis states? This suggest that we are on correct track but the hypothesis could be too strong.

3. The condition $\Delta d = 0$ should correspond to the vanishing of dS/ds. Geometrically this would mean that $S(s)$ curve is above (below) $S(s) = xs^2$ and touches it at points $s = x2^{-k}$, which would be minima (maxima). Intermediate extrema above or below $S = xs^2$ would be maxima (minima).

6.6.3 Appendix: Explicit formulas for the evolution of cosmological constants

What is needed is induced Kähler form $J(S^2(X^4)) \equiv J$ at the twistor sphere $S^2(X^4) \equiv S^2$ associated with space-time surface. $J(S^2(X^4))$ is sum of Kähler forms induced from the twistor spheres $S^2(M^4)$ and $S^2(CP_2)$.

$$J(S^2(X^4) \equiv J = P[J(S^2(M^4)) + J(S^2(CP_2))] \ , \tag{6.6.3}$$

where P is projection taking tensor quantity T_{kl} in $S^2(M^4) \times S^2(CP_2)$ to its projection in $S^2(X^4)$. Using coordinates y^k for $S^2(M^4)$ or $S^(CP_2)$ and x^μ forS^2, P is defined as

$$P : T_{kl} \to T_{\mu\nu} = T_{kl}\frac{\partial y^k}{\partial x^\mu}\frac{\partial y^l}{\partial x^\nu} \ . \tag{6.6.4}$$

For the induced metric $g(S^2(X^4)) \equiv g$ one has completely analogous formula

$$g = P[g(J(S^2(M^4)) + g(S^2(CP_2))] \ . \tag{6.6.5}$$

The expression for the coefficient K of the volume part of the dimensionally reduced 6-D Kähler action density is proportional to

$$L(S^2) = J^{\mu\nu}J_{\mu\nu}\sqrt{det(g)} \ . \tag{6.6.6}$$

(Note that $J_{\mu\nu}$ refers to S^2 part 6-D Kähler action). This quantity reduces to

$$L(S^2) = (\epsilon^{\mu\nu}J_{\mu\nu})^2\frac{1}{\sqrt{det(g)}} \ . \tag{6.6.7}$$

where $\epsilon^{\mu\nu}$ is antisymmetric tensor density with numerical values +,-1. The volume part of the action is obtained as an integral of K over S^2:

$$S(S^2) = \int_{S^2} L(S^2) = \int_{-1}^{1} du \int_0^{2\pi} d\Phi \frac{J_{u\Phi}^2}{\sqrt{det(g)}} \ . \tag{6.6.8}$$

$(u,\Phi) \equiv (cos(\Theta,\Phi)$ are standard spherical coordinates of S^2) varying in the ranges $[-1,1]$ and $[0,2\pi]$.

This the quantity that one must estimate.

General form for the embedding of twistor sphere

The embedding of $S^2(X^4) \equiv S^2$ to $S^2(M^4) \times S^2(CP_2)$ must be known. Dimensional reduction requires that the embeddings to $S^2(M^4)$ and $S^2(CP_2)$ are isometries. They can differ by a rotation possibly accompanied by reflection

One has

$$(u(S^2(M^4)), \Phi(S^2(M^4)) = (u(S^2(X^4), \Phi(S^2(X^4)) \equiv (u,\Phi) \ ,$$

$$\left[u(S^2(CP_2)), \Phi(S^2(CP_2))\right] \equiv (v,\Psi) = RP(u,\Phi)$$

where RP denotes reflection P following by rotation R acting linearly on linear coordinates (x,y,z) of unit sphere S^2). Note that one uses same coordinates for $S^2(M^4)$ and $S^2(X^4)$. From this action one can calculate the action on coordinates u and Φ by using the definite of spherical coordinates.

The Kähler forms of $S^2(M^4)$ resp. $S^2(CP_2)$ in the coordinates $(u = cos(\Theta),\Phi)$ resp.(v,Ψ) are given by $J_{u\Phi} = \epsilon = \pm 1$ resp. $J_{v\Psi} = \epsilon = \pm 1$. The signs for $S^2(M^4)$ and $S^2(CP_2)$ are same or opposite. In order to obtain small cosmological constant one must assume either

1. $\epsilon = -1$ in which case the reflection P is absent from the above formula (RP $\rightarrow$ R).
2. $\epsilon = 1$ in which case P is present. P can be represented as reflection $(x,y,z) \rightarrow (x,y,-z)$ or equivalently $(u,\Phi) \rightarrow (-u,\Phi)$.

Rotation R can represented as a rotation in (y,z)-plane by angle ϕ which must be small to get small value of cosmological constant. When the rotation R is trivial, the sum of induced Kähler forms vanishes and cosmological constant is vanishing.

6.6.4 Induced Kähler form

One must calculate the component $J_{u\Phi}(S^2(X^4)) \equiv J_{u\Phi}$ of the induced Kähler form and the metric determinant $det(g)$) using the induction formula expressing them as sums of projections of M^4 and CP_2 contributions and the expressions of the components of $S^2(CP_2)$ contributions in the coordinates for $S^2(M^4)$. This amounts to the calculation of partial derivatives of the transformation R (or RP) relating the coordinates (u,Φ) of $S^2(M^4)$ and to the coordinates (v,Ψ) of $S^2(CP_2)$.

In coordinates (u,Φ) one has $J_{u\Phi}(M^4) = \pm 1$ and similar expression holds for $J(v\Psi)S^2(CP_2)$. One has

$$J_{u\Phi} = 1 + \frac{\partial(v,\Psi)}{\partial(u,\Phi)} \quad . \tag{6.6.9}$$

where right-hand side contains the Jacobian determinant defined by the partial derivatives given by

$$\frac{\partial(v,\Psi)}{\partial(u,\Phi)} = \frac{\partial v}{\partial u}\frac{\partial \Psi}{\partial \Phi} - \frac{\partial v}{\partial \Phi}\frac{\partial \Psi}{\partial u} \quad . \tag{6.6.10}$$

Induced metric

The components of the induced metric can be deduced from the line element

$$ds^2(S^2(X^4) \equiv ds^2 = P[ds^2(S^2(M^4)) + ds^2(S^2(CP_2))] \quad .$$

where P denotes projection. One has

$$P(ds^2(S^2(M^4))) = ds^2(S^2(M^4)) = \frac{du^2}{1-u^2} + (1-u^2)d\Phi^2 \quad .$$

and

$$P[ds^2(S^2(CP_2))] = P[\frac{(dv)^2}{1-v^2} + (1-v^2)d\Psi^2] \quad ,$$

One can express the differentials $(dv, d\Psi)$ in terms of $(du, d\Phi)$ once the relative rotation is known and one obtains

$$P[ds^2(S^2(CP_2))] = \frac{1}{1-v^2}[\frac{\partial v}{\partial u}du + \frac{\partial v}{\partial \Phi}d\Phi]^2 + (1-v^2)[\frac{\partial \Psi}{\partial u}du + \frac{\partial \Psi}{\partial \Phi}d\Phi]^2 \quad .$$

This gives

$$\begin{array}{l} P[ds^2(S^2(CP_2))] \\ = [(\frac{\partial v}{\partial u})^2\frac{1}{1-v^2} + (1-v^2)(\frac{\partial \Psi}{\partial u})^2]du^2 \\ +[(\frac{\partial v}{\partial \Phi})^2\frac{1}{1-v^2} + (\frac{\partial \Psi}{\partial \Phi})^2 1-v^2]d\Phi^2 \\ +2[\frac{\partial v}{\partial u}\frac{\partial v}{\partial \Phi}\frac{1}{(1-v^2)} + \frac{\partial \Psi}{\partial u}\frac{\partial \Psi}{\partial \Phi}(1-v^2)]dud\Phi \quad . \end{array}$$

From these formulas one can pick up the components of the induced metric $g(S^2(X^4)) \equiv g$ as

$$g_{uu} = \frac{1}{1-u^2} + (\frac{\partial v}{\partial u})^2 \frac{1}{1-v^2} + (1-v^2)(\frac{\partial \Psi}{\partial u})^2] \ ,$$

$$g_{\Phi\Phi} = 1-u^2 + (\frac{\partial v}{\partial \Phi})^2 \frac{1}{1-v^2} + (\frac{\partial \Psi}{\partial \Phi})^2 (1-v^2) \quad (6.6.11)$$

$$g_{u\Phi} = g_{\Phi u} = \frac{\partial v}{\partial u}\frac{\partial v}{\partial \Phi}\frac{1}{(1-v^2)} + \frac{\partial \Psi}{\partial u}\frac{\partial \Psi}{\partial \Phi}(1-v^2) \ .$$

The metric determinant $det(g)$ appearing in the integral defining cosmological constant is given by

$$det(g) = g_{uu}g_{\Phi\Phi} - g_{u\Phi}^2 \ . \quad (6.6.12)$$

Coordinates (v,Ψ) in terms of (u,Φ)

To obtain the expression determining the value of cosmological constant one must calculate explicit formulas for (v,Ψ) as functions of (u,Φ) and for partial derivations of (v,Ψ) with respect to (u,Φ).

Let us restrict the consideration to the RP option.

1. P corresponds to $z \rightarrow -z$ and to

$$u \rightarrow -u \ . \quad (6.6.13)$$

2. The rotation R $(x,y,z) \rightarrow (x',y',z')$ corresponds to

$$x' = x, \quad y' = sz + cy = su + c\sqrt{1-u^2}sin(\Phi) \ , \quad z' = v = cu - s\sqrt{1-u^2}sin(\Phi) \quad (6.6.14)$$

Here one has $(s,c) \equiv (sin(\epsilon), cos(\epsilon)$, where ϵ is rotation angle, which is extremely small for the value of cosmological constant in cosmological scales.

From these formulas one can pick v and $\Psi = arctan(y'/x)$ as

$$v = cu - s\sqrt{1-u^2}sin(\Phi) \quad \Psi = arctan[\frac{su}{\sqrt{1-u^2}}cos(\Phi) + tan(\Phi)] \ . \quad (6.6.15)$$

3. RP corresponds to

$$v = -cu - s\sqrt{1-u^2}sin(\Phi) \quad \Psi = arctan[-\frac{su}{\sqrt{1-u^2}}cos(\Phi) + tan(\Phi)] \ . \quad (6.6.16)$$

Various partial derivatives

Various partial derivates are given by

$$\frac{\partial v}{\partial u} = -1 + s\frac{u}{\sqrt{1-u^2}}sin(\Phi) \ ,$$

$$\frac{\partial v}{\partial \Phi} = -s\frac{u}{\sqrt{1-u^2}}cos(\Phi) \ ,$$

$$\frac{\partial \Psi}{\partial \Phi} = (-s\frac{u}{\sqrt{1-u^2}}sin(\Phi) + c)\frac{1}{X} \ , \quad (6.6.17)$$

$$\frac{\partial \Psi}{\partial u} = \frac{scos(\Phi)(1+u-u^2)}{(1-u^2)^{3/2}}\frac{1}{X} \ ,$$

$$X = cos^2(\Phi) + [-s\frac{u}{\sqrt{1-u^2}} + csin(\Phi)]^2 \ .$$

Using these expressions one can calculate the Kähler and metric and the expression for the integral giving average value of cosmological constant. Note that the field equations contain S^2 coordinates as external parameters so that each point of S^2 corresponds to a slightly different space-time surface.

Calculation of the evolution of cosmological constant

One must calculate numerically the dependence of the action integral S over S^2 as function of the parameter $s = sin(\epsilon)$). One should also find the extrema of S as function of s.

Especially interesting values are very small values of s since for the cosmological constant becomes small. For small values of s the integrand (see Eq. 6.6.8) becomes very large near poles having the behaviour $1/\sqrt{g} = 1/(sin(\Theta) + O(s))$ coming from $\sqrt{g}$ approaching that for the standard metric of S^2. The integrand remains finite for $s \neq 0$ but this behavior spoils the analytic dependence of integral on s so that one cannot do perturbation theory around $s = 0$. The expected outcome is a logarithmic dependence on s.

In the numerical calculation one must decompose the integral over S^2 to three parts.

1. There are parts coming from the small disks D^2 surrounding the poles: these give identical contributions by symmetry. One must have criterion for the radius of the disk and the natural assumption is that the disk radius is of order s.
2. Besides this one has a contribution from S^2 with disks removed and this is the regular part to which standard numerical procedures apply.

One must be careful with the expressions involving trigonometric functions which give rise to infinite if one applies the formulas in straightforward manner. These infinities are not real and cancel, when one casts the formulas in appropriate form inside the disks.

1. The limit $u \to \pm 1$ at poles involves this kind of dangerous quantities. The expression for the determinant appearing in $J_{u\Phi}$ remains however finite and $J^2_{u\phi}$ vanishes like s^2 at this limit. Also the metric determinant $1/\sqrt{g}$ remains finite expect at $s = 0$.
2. Also the expression for the quantity X in $\Psi = arctan(X)$ contains a term proportional to $1/cos(\Phi)$ approaching infinity for $\Phi \to \pi/2, 3\pi/2$. The value of $\Psi = arc(tan(X)$ remains however finite and equal to $\pm\Phi$ at this limit depending on on the sign of us.

Concerning practical calculation, the relevant formulas are given in Eqs. 6.6.7, 6.6.8, 6.6.9, 6.6.10, 6.6.11, 6.6.12, and 6.6.17.

The calculation would allow to test the conjectures already discussed.

1. There indeed exist extrema satisfying thus $dS/ds = 0$.
2. These extrema correspond to $s = 2{-}k$ or more generally $s = p^{-k}$. This conjecture is inspired by p-adic length scale hypothesis.
3. A further conjecture is that for certain integer values of integer k the integral $S(S^2)$ of Eq. 6.6.8 is of form $S(S^2) = xs^2$ for $s = 2^{-k}$, where x is a universal numerical constant.

   This would realize the idea that p-adic length scales realized as scales associated with cosmological constant correspond to fixed points of renormalization group evolution implying that radiative corrections otherwise present cancel. In particular, the deviation from $s = 2^{-d/2}$ would mean anomalous dimension replacing $s = 2^{-d/2}$ with $s^{-(d+\Delta d)/2}$ for $d = k$ the anomalies dimension Δd would vanish.
4. The condition $\Delta d = 0$ should be equivalent with the vanishing of the dS/ds. Geometrically this means that $S(s)$ curve is above (below) $S(s) = xs^2$ and touches it at points $s = x2^{-k}$, which would be minima (maxima). Intermediate extrema above or below $S = xs^2$ would be maxima (minima).

6.7 More about the construction of scattering amplitudes in TGD framework

The construction of scattering amplitudes in TGD framework has been a longstanding problem, and I have considered several proposals - perhaps the most realistic proposal relies on the generalization of twistor Grassmann approach to TGD context [L58]. These approaches have however suffered from their ad hoc character.

One reason for the slow progress might be the fact that I have not conditioned Feynman diagrams into my spine: I have intentionally avoided this in the fear that it would prevent genuine thinking. Second reason is that TGD is really different and my mathematical skills are rather limited. For instance, in TGD classical theory is an exact part of quantum theory and particles are replaced with 3-surfaces: there is no hope of starting from Lagrangian with simple non-linearities and writing Feynman rules and deducing beta functions.

There are several questions waiting for an answer. How to achieve unitarity? What it is to be a particle in classical sense? Can one identify TGD analogs of quantum fields? Could scattering amplitudes have interpretation as Fourier transforms of n-point functions for the analogs of quantum fields?

Unitarity is certainly the issue #1 and in the sequel almost trivial solution to unitarity problem is proposed. Also quantum classical correspondence is discussed.

6.7.1 Some background

Supersymplectic algebra

Let us collect what I think is known in TGD framework.

1. The "world of classical worlds" (WCW) [K80] geometry does not exist without maximal group of isometries and WCW is assumed to possess super-symplectic algebra (SSA) assignable to light-cone boundary (boundaries of causal diamonds (CDs)) as isometries. Also Kac-Moody algebras for isometries of embedding space realized at the light-like partonic orbits serving as boundaries between Euclidian and Minkowskian regions of space-time surface are expected to be of key importance (for p-adic mass calculations applying these symmetries see [K52].

   SSA has a fractal hierarchy of isomorphic sub-algebras and the proposal is that one has hierarchy of criticalities such that sub-SSA and its commutator with SSA annihilate the physical states so that SSA effectively reduces to a finite-D Lie-algebra generating the physical states. Sub-SSA takes the role of gauge algebra and one could say that it represents finite measurement resolution. This hierarchy would correspond to a hierarchies of inclusions of von Neumann algebras known as hyper-finite factors of type II_1 [K105, K36].

   It seems obvious to me that the scattering amplitudes should allow a formulation in terms of SSA effectively reducing to finite-D Lie-algebra of corresponding Kac-Moody algebra plus Kac-Moody algebras associated with embedding space isometries.

   Remark: Conformal weights of SSA associated with the radial light-like coordinate are non-negative so that one has analogy with Yangian algebra. The TGD variant of twistor Grassmann approach [L45] [L58] strongly suggests that SSA extends to Yangian having multi-local generators with locus corresponding to partonic 2-surface.

2. There are both classical and fermionic Noether charges associated with SSA and the Kac-Moody algebras [K24, K106, K80]. Quantum-classical correspondence (QCC) suggests that the eigenvalues for Cartan algebra Noether charges in the fermionic representation correspond to bosonic charges assignable to the dimensionally reduced Kähler action. One obtains also fermionic super-charges in 1-1 correspondence with the modes of the induced spinor field. Super-charges are very much like oscillator operators creating or annihilating fermions and there is a temptation to think that these fermionic SSA and Kac-Moody charges take the role of operators creating fermionic and bosonic states.

   One could think of constructing many-particle states at both boundaries of causal diamond (CD) by decomposing SSA to Cartan algebra and to parts acting like creation and annihilation operators. States would be created by the generators acting like oscillator operators.

   The time evolution dictated by preferred extremals and corresponding modified Dirac equation would transform initial states at boundary A of CD to final states at boundary B. This time evolution is determined by preferred extremal property and by modified Dirac equation [K106]. Time evolution is not obtained by exponentiating quantum Hamiltonian as in QFT approach. The existence of infinite-D SSA of Noether changes should make it possible to prove unitarity.

General argument for unitarity

The argument for unitarity is very general and based on zero energy ontology (ZEO). Causal diamond (CD) containing space-time surfaces having ends at its opposite boundaries is central for

ZEO. Zero energy states are quantum superpositions of space-time surfaces, which are preferred extremals of dimensionally reduced 6-D Kähler action decomposing to 4-D Kähler action and volume term. CD has two boundaries: the active boundary (B) and passive boundary (A) and space-time surfaces as preferred extremals have ends at these boundaries [L46].

In ZEO one has two kinds of state function reductions.

1. At the active boundary (B) one has "small" state function reductions as counterparts of weak measurements following unitary time evolutions shifting the active boundary B farther from passive boundary A in statistical sense. During each unitary time evolution there is a de-localization with respect to the distance between the tips of CD followed by localization serving also as time measurement. This would yield the correlation between experienced time as sequence of these weak measurements and geometric time identified as distance between the tips of CD.

   Also measurements of observables commuting with the observables, whose eigenstates the states at boundary A are, are possible. Passive boundary (A) and the members of zero energy states associated with it do not change, and this gives rise to what one might call generalized Zeno effect.

   S-matrix would correspond to the evolution between two weak measurements for the states at the active boundary of CD and expected to be unitary. At passive boundary of CD and states at it would not be affected. The time evolution in the fermionic sector would be induced by the modified Dirac equation. Now one can express the states at new active boundary in terms of those at old active boundary and one would obtain unitary S-matrix by expressing the final states in terms of the state basic for the original boundary.

2. In "big" state function reduction the roles of passive boundary A and active boundary B are changed. The states at B are superpositions of states in the state basis for SSA. Unitary S-matrix would be obtained by expressing these states in terms of SSA basis.

Unitarity does not seem to be a problem since the conservation of Cartan charges for SSA in the fermionic representation would not allow breaking of unitarity. The time evolution would be induced by the preferred extremal property and modified Dirac equation.

Scattering amplitudes would involve an integration over positions of particles meaning that instead of single 4-surface one would have large number of them contributing to single scattering amplitude. Different position would correspond to different values of zero modes not contributing to WCW metric. Number theoretical vision [L42, L43] demands that the exponent of action is same for all of these surfaces: with inspiration coming from the idea about quantum TGD as square root of thermodynamics, I have indeed proposed [L56] this quantum analog of micro-canonical ensemble (for which energy is constant) as a way to get rid of difficulties in the realization of number theoretical universality. The number theoretically cumbersome action exponents would cancel out from the scattering amplitudes.

6.7.2 Does 4-D action generate lower-dimensional terms dynamically?

The original proposal was that the action defining the preferred extremals is 4-D Kähler action. Later it became obvious that there must be also 2-D string world sheet term present and probably also 1-D term associated with string boundaries at partonic 2-surfaces. The question has been whether these lower-D terms in the action are primary of generated dynamically. By superconformal symmetry the same question applies to the fermionic part of the action. The recent formulation based on the twistor lift of TGD contains also volume term but the question remains the same.

Quantum criticality would be realized as a minimal surface property realized by holomorphy in suitably generalized sense [L63, L57]. The reason is that the holomorphic solutions of minimal surface equations involve no coupling parameters as the universality of the dynamics at quantum criticality demands.

Minimal surface equation would be true apart from possible singular surfaces having dimension $D = 2, 1, 0$. $D = 2$ corresponds to string world sheets and partonic 2-surfaces. If there are 0-D singularities they would be associated with the ends of orbits of partonic 2-surfaces at boundaries of causal diamond (CD). Minimal surfaces are solutions of non-linear variant of massless d'Alembertian having as effective sources the singular surfaces at which d'Alembertian equation fails. The analogy with gauge theories is highly suggestive: singular surfaces would act as sources of massless field.

Strings world sheets seem to be necessary. The basic question is whether the singular surfaces are postulated from the beginning and there is action associated with them or whether they emerge dynamical from 4-D action. One can consider two extreme options.

Option I: There is an explicit assignment of action to the singular surfaces from the beginning. A transfer of Noether charges between space-time interior and string world sheets is possible. This kind of transfer process can take place also between string world sheets and their light-like boundaries and happens if the normal derivatives of embedding space coordinates are discontinuous at the singular surface.

Option II: No separate action is assigned with the singular surfaces. There could be a transfer of Noether charges between 4-D Kähler and volume degrees of freedom at the singular surfaces causing the failure of minimal surface property in 4-D sense. But could singular surfaces carry Noether currents as 2-D delta function like densities?

This is possible if the discontinuity of the normal derivatives generates a 2-D singular term to the action. Conservation laws require that at string world sheets energy momentum tensor should degenerate to a 2-D tensor parallel to and concentrated at string world sheet. Only 4-D action would be needed - this was actually the original proposal. Strings and particles would be essentially edges of space-time - this is not possible in GRT. Same could happen also at its boundaries giving rise to point like particles. Super-conformal symmetry would make this possible also in the fermionic sector.

For both options the singular surfaces would provide a concrete topological picture about the scattering process at the level of single space-time surface and telling what happens to the initial state. The question is whether Option I actually reduces to Option II. If the 2-D term is generated to 4-D action dynamically, there is no need to postulate primary 2-D action.

Can Option II generate separate 2-D action dynamically?

The following argument shows that Option II with 4-D primary action can generate dynamically 2-D term into the action so that no primary action need to be assigned with string world sheets.

1. Dimensional hierarchy of surfaces and strong form of holography

String world sheets having light-like boundaries at the light-like orbits of partonic 2-surfaces are certainly needed to realize strong form of holography [K106]. Partonic 2-surfaces emerge automatically as the ends of the orbits of wormhole contacts.

1. There could (but neet not) be a separate terms in the primary action corresponding to string world sheets and their boundaries. This hierarchy bringing in mind branes would correspond to the hierarchy of classical number fields formed by reals, complex numbers, quaternions (space-time surface), and octonions (embedding space in M^8-side of M^8 duality). The tangent - or normal spaces of these surfaces would inherit real, complex, and quaternionic structures as induced structure. The number theoretic interpretation would allow to see these surfaces as images of those surfaces in M^8 mapped to H by $M^8 - H$ duality. Therefore it would be natural to assign action to these surfaces.
2. This makes in principle possible the transfer of classical and quantum charges between space-time interior and string world sheets and between from string world sheets to their light-like boundaries. TGD variant of twistor Grassmannian approach [L45, L58] relies on the assumption that the boundaries of string world sheets at partonic orbits carry quantum numbers. Quantum criticality realized in terms of minimal surface property realized holomorphically is central for TGD and one can ask whether it could play a role in the definition of S-matrix and identification of particles as geometric objects.
3. For preferred extremals string world sheets (partonic 2-surfaces) would be complex (co-complex) manifolds in octonionic sense. Minimal surface equations would hold true outside string world sheets. Conservation of various charges would require that the divergences of canonical momentum currents at string world sheet would be equal to the discontinuities of the normal components of the canonical momentum currents in interior. These discontinuities

would correspond to discontinuities of normal derivatives of embedding space coordinates and are acceptable. Similar conditions would hold true at the light-like boundaries of string world sheets at light-like boundaries of parton orbits. String world sheets would not be minimal surfaces and minimal surface property for space-time surface would fail at these surfaces.

Quantum criticality for string world sheets would also correspond to minimal surface property. If this is realized in terms of holomorphy, the field equations for Kähler and volume parts at string world sheets would be satisfied separately and the discontinuities of normal components for the canonical momentum currents in the interior would vanish at string world sheets.

4. The idea about asymptotic states as free particles would suggest that normal components of canonical momentum currents are continuous near the boundaries of CD as boundary conditions at least. The same must be true at the light-like boundaries of string world sheets. Minimal surface property would reduce to the property of being light-like geodesics at light-like partonic 2-surface. If this is not assumed, the orbit is space-like. Even if these conditions are realized, one can imagine the possibility that at string world sheets 4-D minimal surface equation fails and there is transfer of charges between Kähler and volume degrees of freedom (Option II) and therefore breaking of quantum criticality.

   If the exchange of Noether charges vanishes everywhere at string world sheets and boundaries, one could argue that they represent independent degrees of freedom and that TGD reduces to string model. The proposed equation for coupling constant evolution however contains a coefficients depending on the total action so that this would not be the case.

5. Assigning action to the lower-D objects requires additional coupling parameters. One should be able to express these parameters in terms of the parameters appearing in 4-D action (α_K and cosmological constant). For string sheets the action containing cosmological term is 4-D and Kähler action for $X^2 \times S^2$, where S^2 is non-dynamical twistor sphere is a good guess. Kähler action gets contributions from X^2 and S^2. If the 2-D action is generated dynamically as a singular term of 4-D action its coupling parameters are those of 4-D action.

6. There is a temptation to interpret this picture as a realization of strong form of holography (SH) in the sense that one can deduce the space-time surfaces by using data at string world sheets and partonic 2-surfaces and their light-like orbits. The vanishing of normal components of canonical momentum currents would fix the boundary conditions.

   If double holography $D = 4 \rightarrow D = 2 \rightarrow D = 1$ were satisfied it might be even possible to reduce the construction of S-matrix to the proposed variant of twistor Grassmann approach. This need not be the case: p-adic mass calculations rely on p-adic thermodynamics for the excitions of massless particles having CP_2 mass scale and it would seem that the double holography can makes sense for massless states only.

   In M^8-picture [L37] the information about space-time surface is coded by a polynomial defined at real line having coefficients in an extension of rationals. This real line for octonions corresponds to the time axis in the rest system rather than light-like orbit as light-like boundary of string world sheet.

2. Stringy quantum criticality?

The original intuition [L63] was that there are canonical momentum currents between Kähler and volume degrees of freedom at singular surfaces but no transfer of canonical momenta between interior and string world sheets nor string world sheets and their boundaries. Also string world sheets would be minimal surfaces as also the intuition from string models suggests. Could also the stringy quantum criticality be realized?

1. Some embedding space coordinates h^k must have discontinuous partial derivatives in directions normal to the string world sheet so that 3-surface has 1-D edge along fermionic string connecting light-like curves at partonic 2-surfaces in both Minkowskian and Euclidian regions. A closed highly flattened rectangle with long and short edges would be associated with closed monopole flux tube in the case of wormhole contact pairs assigned with elementary particles. 3-surfaces would be “edgy” entities and space-time surfaces would have 2-D and 1-D edges. In condensed matter physics these edges would be regarded as defects.

2. Quantum criticality demands that the dynamics of string world sheets and of interior effectively decouple. Same must take place for the dynamics of string world sheets and their boundaries. Decoupling allows also string world sheets to be minimal surfaces as analogs of complex surfaces whereas string world sheet boundaries would be light-like (their deformations are always space-like) so that one obtains both particles and string like objects.

3. By field equations the sums for the divergences of stringy canonical momentum currents and the corresponding singular parts of these currents in the interior must vanish. By quantum criticality in interior the divergencespf Kähler and volume terms vanish separately. Same must happen for the sums in case of string world sheets and their boundaries. The discontinuity of normal derivatives implies that the contribution from the normal directions to the divergence reduces to the sum of discontinuities in two normal directions multiplied by 2-D delta function. Thid contribution is in the general case equal to the divergence of corresponding stringy canonical momentum current but must vanish if one has quantum criticality also at string world sheets and their boundaries.

   The separate continuity of Kähler and volume parts of canonical momentum currents would guarantee this but very probably implies the continuity of the induced metric and Kähler form and therefore of normal derivatives so that there would be no singularity. However, the condition that total canonical momentum currents are continuous makes sense, and indeed implies a transfer of various conserved charges between Kähler action and volume degrees of freedom at string world sheets and their boundaries in normal directions as was conjectured in [L63].

4. What about the situation in fermionic degrees of freedom? The action for string world sheet X^2 would be essentially of Kähler action for $X^2 \times S^2$, where S^2 is twistor sphere. Since the modified gamma matrices appearing in the modified Dirac equation are determined in terms of canonical momentum densities assignable to the modified Dirac action, there could be similar transfer of charges involved with the fermionic sector and the divergences of Noether charges and super-charges assignable to the volume action are non-vanishing at the singular surfaces. The above mechanism would force decoupling between interior spinors and string world sheets spinors also for the modified Dirac equation since modified gamma matrices are determined by the bosonic action.

   Remark: There is a delicacy involved with the definition of modified gamma matrices, which for volume term are proportional to the induced gamma matrices (projections of the embedding space gamma matrices to space-time surface). Modified gamma matrices are proportional to the contractions $T^{\alpha}_k\Gamma^k$ of canonical momentum densities $T^{\alpha_k} = \partial L/\partial(\partial_\alpha h^k)$ with embedding space gamma matrices Γ^k. To get dimension correctly in the case of volume action one must divide away the factor $\Lambda/8\pi G$. Therefore fermionic super-symplectic currents do not involve this factor as required.

It remains an open question whether the string quantum criticality is realized everywhere or only near the ends of string world sheets near boundaries of CD.

3. String world sheet singularities as infinitely sharp edges and dynamical generation of string world sheet action

The condition that the singularities are 2-D string world sheets forces 1-D edges of 3-surfaces to be infinitely sharp.

Consider an edge at 3-surface. The divergence from the discontinuity contains contributions from two normal coordinates proportional to a delta function for the normal coordinate and coming from the discontinuity. The discontinuity must be however localized to the string rather than 2-surface. There must be present also a delta function for the second normal coordinate. Hence the value of also discontinuity must be infinite. One would have infinitely sharp edge. A concrete example is provided by function $y = |x|^\alpha$ $\alpha < 1$. This kind of situation is encountered in Thom's catastrope theory for the projection of the catastrophe: in this case one has $\alpha = 1/2$. This argument generalizes to 3-D case but visualization is possible only as a motion of infinitely sharp edge of 3-surface.

Kähler form and metric are second degree monomials of partial derivatives so that an attractive assumption is that $g_{\alpha\beta}$, $J_{\alpha\beta}$ and therefore also the components of volume and Kähler energy momentum tensor are continuous. This would allow $\partial_{n_i} h^k$ to become infinite and change sign at the discontinuity as the idea about infinitely sharp edge suggests. This would reduce the continuity conditions for canonical momentum currents to rather simple form

$$T^{n_i n_j} \Delta \partial_{n_j} h^k = 0 \ . \tag{6.7.1}$$

which in turn would give

$$T^{n_i n_j} = 0 \tag{6.7.2}$$

stating that canonical momentum is conserved but transferred between Kähler and volume degrees of freedom. One would have a condition for a continuous quantity conforming with the intuitive view about boundary conditions due to conservation laws. The condition would state that energy momentum tensor reduces to that for string world sheet at the singularity so that the system becomes effectively 2-D. I have already earlier proposed this condition.

The reduction of 4-D locally to effectively 2-D system raises the question whether any separate action is needed for string world sheets (and their boundaries)? The generated 2-D action would be similar to the proposed 2-D action. By super-conformal symmetry similar generation of 2-D action would take place also in the fermionic degrees of freedom. I have proposed also this option already earlier. This would mean that Option II is enough.

The following gives a more explicit analysis of the singularities. The vanishing on the discontinuity for the sum of normal derivative gives terms with varying degree of divergence. Denote by n_i resp. t_i the coordinate indices in the normal resp. tangent space. Suppose that some derivative $\partial_{n_i} h^k$ become infinite at string. One can introduce degree n_D of divergence for a quantity appearing as part of canonical momentum current as the degree of the highest monomial consisting of the diverging derivatives $\partial_{n_i} h^k$ appearing in quantity in question. For the leading term in continuity conditions for canonical momentum currents of total action one should have $n_D = 2$ to give the required 2-D delta function singularity.

- $\partial_{n_i} h^k$ has $n_D \leq 1$. If it is also discontinuous - say changes sign - one has $n_D = 2$ for $\Delta \partial_{n_i} h^k$ in direction n_i.
- One has $n_D(g_{t_i t_j}) = 0$, $n_D(g_{t_i n_j}) = 1$, $n_D(g_{n_i n_i}) = 2$ and $n_D(g_{n_i n_j}) = 1$ or 2 for $i \neq j$. One has $n_D(g) = 4$ ($g = det(g_{\alpha\beta})$). For contravariant metric one gas $n_D(g^{t_i t_j}) = 0$ and $n_D(g^{n_i j}) = n_D(g^{n_i n_j}) = -2$ as is easy to see from the formula for $g^{\alpha\beta}$ in terms of cofactors.
- Both Kähler and volume terms in canonical momentum current are proportional to $\sqrt{g}$ with $n_D(\sqrt{g}) = 2$ having leading term proportional to 2-determinant $\sqrt{det(g_{n_i n_j})}$. In Kähler action the leading term comes from tangent space part J_{ij} and has $n_D = -1$ coming from the partial derivative. The remaining parts involving $J_{t_i n_j}$ or $J_{n_i n_j}$ have $n_D < 0$.
- Consider the behavior of the contribution of volume term to the canonical momentum currents. For $g^{n_i t_j} \partial_{t_j} h^k \sqrt{g}$ one has $n_D = 0$ so that this term is finite. For $g^{n_i n_j} \partial_{n_j} h^k \sqrt{g}$ one has $n_D \leq 1$ and this term can be infinite as also its discontinuity coming solely from the change of sign for $\partial_{n_j} h^k$. If $\partial_{n_j} h^k$ is infinite and changes sign, one can have $n_D = 2$ as required by 2-D delta function singularity.

  The continuity condition for the canonical momentum current would state the vanishing of $n_D = 2$ discontinuity but would not imply separate vanishing of discontinuity for Kähler and volume parts of canonical momentum currents - this in accordance with the idea about canonical momentum transfer. If the sign of partial derivative only changes the coefficient of the partial derivative must vanish so that the condition reduces to the condition $T^{n_i n_j} = 0$ already given for the components of the total energy momentum tensor, which would be continuous by the above assumption.

4. A connection with Higgs vacuum expectation?

What about the physical interpretation of the singular divergences of the isometry currents J_A of the volume action located at string world sheet?

1. The divergences of J_A are proportional to the trace of the second fundamental form H formed by the covariant derivatives of gradients $\partial_\alpha h^k$ of H-coordinates in the interior and vanish. The singular contribution at string world sheets is determined by the discontinuity of the isometry current J_A and involves only the first derivatives $\partial_\alpha h^k$.
2. One of the first questions after ending up with TGD for 41 years ago was whether the trace of H in the case of CP_2 coordinates could serve as something analogous to Higgs vacuum expectation value. The length squared for the trace has dimensions of mass squared. The discontinuity of the isometry currents for $SU(3)$ parts in $h = u(2)$ and its complement t, whose complex coordinates define $u(2)$ doublet. $u(2)$ is in correspondence with electroweak algebra and t with complex Higgs doublet. Could an interpretation as Higgs or even its vacuum expectation make sense?
3. p-Adic thermodynamics explains fermion masses elegantly (understanding of boson masses is not in so good shape) in terms of thermal mixing with excitations having CP_2 mass scale and assignable to short string associated with wormhole contacts. There is also a contribution from long strings connecting wormhole contacts and this could be important for the understanding of weak gauge boson masses. Could the discontinuity of isometry currents in t determine this contribution to mass. Edges/folds would carry mass.
4. The non-singular part of the divergence multiplying 2-D delta function has dimension 1/length squared and the square of this vector in CP_2 metric has dimension of mass squared. Could the interpretation of the discontinuity as Higgs expectation make sense? If so, Higgs expectation would vanish in the space-time interior.

   Could the interior modes of the induced spinor field - or at least the interior mode of right-handed neutrino ν_R having no couplings to weak or color fields - be massless in 8-D or even 4-D sense? Could ν_R and $\overline{\nu}_R$ generate an unbroken $\mathcal{N} = 2$ SUSY in interior whereas inside string world sheets right-handed neutrino and antineutrino would be eaten in neutrino massivation and the generators of $\mathcal{N} = 2$ SUSY would be lost somewhat like charged components of Higgs!

   If so, particle physicists would be trying to find SUSY from wrong place. Space-time interior would be the correct place. Would the search of SUSY be condensed matter physics rather than particle physics?

Summarizing the recent view about elementary particles

It is interesting to see how elementary particles and their basic interaction vertices could be realized in this framework.

1. In TGD framework particle would correspond to pair of wormhole contact associated with closed magnetic flux tube carrying monopole flux. Strongly flattened rectangle with Minkowskian flux tubes as long edges with length given by weak scale and Euclidian wormhole contacts as short edges with CP_2 radius as lengths scale is a good visualization. 3-particle vertex corresponding to the replication of this kind of flux tube rectangle to two rectangles would replace 3-vertex of Feynman graph. There is analogy with DNA replication. Similar replication is expected to be possible also for the associated closed fermionic strings.
2. Denote the wormhole contacts by A and B and their opposite throats by A_i and B_i, $i = 1, 2$. For fermions A_1 can be assumed to carry the electroweak quantum numbers of fermion. For electroweak bosons A_1 and A_2 (for instance) could carry fermion and anti-fermion, whose quantum numbers sum up to those of ew gauge boson. These "corner fermions" can be called *active*.

   Also other distributions of quantum numbers must be considered. Fermion and anti-fermion could in principle reside at the same throat - say A_1. One can however assume that second wormhole contact, say A has quantum numbers of fermion or weak boson (or gluon) and second contact carries quantum numbers screening weak isospin.
3. The model assumes that the weak isospin is neutralized in length scales longer than the size of the flux tube structure given by electro-weak scale. The screening fermions can be called *passive*. If the weak isospin of $W^\pm$ boson is neutralized in the scale of flux tube, 2 $\nu_L\overline{\nu}_R$ pairs are needed (lepton number for these pairs must vanish) for W^-. For Z $\nu_L\overline{\nu}_R$ and $\overline{\nu}_L nu_R$ are needed. The pairs of passive fermions could reside in the interior of flux tube, at string world

sheet or at its corners just like active fermions. The first extreme is that the neutralizing neutrino-antineutrino pairs reside in interior at the opposite long edges of the rectangular *flux tube*. Second extreme is that they are at the corners of rectangular *closed string*.

4. Rectangular closed string containing active fermion at wormhole A (say) and with members of isospin neutralizing neutrino-antineutrino pair at the throats of B serves as basic units. In scales shorter than string length the end A would behave like fermion with weak isospin. At longer scales physical fermion would be hadron like entity with vanishing isospin and one could speak of confinement of weak isospin.

   From these physical fermions one can build gauge bosons as bound states. Weak bosons and also gluons would be pairs of this kind of fermionic closed strings connecting wormhole contacts A and B. Gauge bosons (and also gravitons) could be seen as composites of string like physical fermions with vanishing net isospin rather than those of point like fundamental fermions.

5. The decay of weak boson to fermion-antifermion pair would be flux tube replication in which closed strings representing physical fermion and anti-fermion continue along different copies of flux tube structure. The decay of boson to two bosons - say $W \rightarrow WZ$ - by replication of flux tube would require creation of a pair of physical fermionic closed strings representing Z. This would correspond to a V-shaped vertex with the edge of V representing closed fermionic closed string turning backwards in time. In decays like $Z \rightarrow W^+W^-$ two closed fermion strings would be created in the replication of flux tube. Rectangular fermionic string would turns backwards in time in the replication vertex and the rectangular strings of Z would be shared between W^+ and W^-.

This mesonlike picture about weak bosons as bound states of fermions sounds complex as compared with standard model picture. On the other hand only the spinor fields assignable to single fermion family are present.

A couple of comments concerning this picture are in order.

1. M^8 duality provides a different perspective. In M^8 picture these vertices could correspond to analogs of local 3 particle vertices for octonionic superfield, which become nonlocal in the map taking $M^8 = M^4 \times CP_2$ surfaces to surfaces in $H = M^4 \times CP_2$. The reason is that M^4 point is mapped to M^4 point but the tangent space at E^4 point is mapped to a point of CP_2. If the point in M^8 corresponds to a self-intersection point the tangent space at the point is not unique and point is mapped to two distinct points. There local vertex in M^8 would correspond to non-local vertex in H and fermion lines could just begin. This would mean that at H-level fermion line at moment of replication and V-shaped fermion line pair beginning at different point of throat could correspond to 3-vertex at M^8 level.

2. The 3-vertex representing replication could have interpretation in terms of quantum criticality: in reversed direction of time two branches of solution of classical field equations would coincide.

Gravitation as a square of gauge interaction

I encountered in FB a link to an interesting popular article (see `http://tinyurl.com/y5r4glgg`) about theoretical physicist Henrik Johansson who has worked with supergravity in Wallenberg Academy. He has found strong mathematical evidence for a new duality. Various variants of super quantum gravity support the view that supersymmetric quantum theories of gravitation can be seen as a double copy of a gauge theory. One could say that spin 2 gravitons are gluons with color charge replaced with spin. Since the information about charges disappears, gluons can be understood very generally as gauge bosons for given gauge theory, not necessarily QCD.

The article of C. D. White [B24] (see `https://arxiv.org/pdf/1708.07056.pdf`) entitled "The double copy: gravity from gluons" explains in more detail the double copy duality and also shows that it relates in many cases also exact classical solutions of Einsteins equations and YM theories. One starts from L-loop scattering amplitude involving products of kinematical factors n_i and color factors c_i and replaces color factors with extra kinematical factors $\tilde{n}_i$. The outcome is an L-loop amplitude for gravitons.

As if gravitation could be regarded as a gauge theory with polarization and/or momenta identified giving rise to effective color charges. This is like taking gauge potential and giving it additional index to get metric tensor. This naïve analogy seems to hold true at the level of scattering amplitudes and also for many classical solutions of field equations. Could one think that gravitons as states correspond to gauge singlets formed from two gluons and having spin 2? Also spin 1 and spin 0 states would be obtained and double copies involve also them.

TGD view about elementary particles indeed predicts that gravitons be regarded in certain sense pairs of gauge bosons. Consider now gravitons and assume for simplicity that spartners of fundamental fermions - identifiable as local multi-fermion states allowed by statistics - are not involved: this does not change the situation much [L77]. Graviton's spin 2 requires 2 fermions and 2 anti-fermions: fermion or anti-fermion at each throat. For gauge bosons fermion and anti-fermion at two throats is enough. One could therefore formally see gravitons as pairs of two gauge bosons in accordance with the idea about graviton is a square of gauge boson.

The fermion contents of the monopole flux tube associated with elementary particle determines quantum numbers of the flux tube as particle and characterizes corresponding interaction. The interaction depends also on the charges at the ends of the flux tube. This leads to a possible interpretation for the formation of bound states in terms of flux tubes carrying quantum numbers of particles.

1. These long flux tubes can be arbitrarily long for large values of $\hbar_{eff} = n \times \hbar_0$ assigned to the flux tube. A plausible guess for for the expression of $\hbar$ in terms of $_0$ is as $\hbar = 6 \times \hbar_0$ [L25, L52]. The length of the flux tube scales like $\hbar_{eff}$.

2. Nottale [E1] proposed that it makes sense to speak about gravitational Planck constant h_{gr}. In TGD this idea is generalized and interpreted in framework of generalized quantum theory [K85, K70, K9]. For flux tubes assignable to gravitational bound states along which gravitons propagate, one would have $\hbar_{eff} = \hbar_{gr} = GMm/v_0$, where $v_0 < c$ is parameter with dimensions of velocity. One could write interaction strength as

$$GMm = v_0 \times \hbar_{gr} \quad .$$

3. $\hbar_{gr}$ obtained from this formula must satisfy $\hbar_{gr} > \hbar$. This generalizes to other interactions. For instance, one has one would have

$$Z_1 Z_2 e^2 = \frac{v_0 \hbar_{em}}{}$$

for electromagnetic flux tubes in the case that ones $h_{em} > \hbar$. The interpretation of the velocity parameter v_0 is discussed in [K9].

One could even turn the situation around and say that the value of $\hbar_{eff}$ fixes the interaction strength. $\hbar_{eff}$ would depend on fermion content and thus of virtual particle and also on the masses or other charges at the ends of the flux tube. The longer the range of the interaction, the larger the typical value of $\hbar_{eff}$.

4. The interpretation could be in terms long length scale quantum fluctuations at quantum criticality. Particles generate U-shaped monopole flux tubes with varying length proportional to $\hbar_{gr}$. If these U-shaped flux tubes from two different particles find each other, they reconnect to flux tube pairs connecting particles and give rise to interaction. What comes in mind is tiny curious and social animals studying their environment.

5. I have indeed proposed this picture in biology: the U-shaped flux tubes would be tentacles with which bio-molecules (in particular) would be scanning their environment. This scanning would be the basic mechanism behind immune system. It would also make possible for bio-molecules to find each in molecular crowd and provide a mechanism of catalysis. Could this picture apply completely generally? Would even elementary particles be scanning their environment with these tentacles?

6. Could one interpret the flux tubes as analogs of virtual particles or could they replace virtual particles of quantum field theories? The objection is that flux tubes would have time-like momenta whereas virtual particle analogs would have space-like momenta. The interpretation makes sense

only if the associated momenta are between space-like and time-like that is light-like so that flux tube would correspond to mass shell particle. But this is the case in twistor approach to gauge theories also in TGD [L77] (see `http://tinyurl.com/y62no62a`).

Perhaps the following interpretation is more appropriate. Flux tubes are accompanied by strings and string world sheets can be interpreted as stringy description of gravitation and other interactions.

Kähler calibrations: an idea before its time?

While updating book introductions I was surprised to find that I had talked about so called calibrations of sub-manifolds as something potentially important for TGD and later forgotten the whole idea! A closer examination however demonstrated that I had ended up with the analog of this notion completely independently later as the idea that preferred extremals are minimal surfaces apart form 2-D singular surfaces, where there would be exchange of Noether charges between Kähler and volume degrees of freedom.

1. The original idea that I forgot too soon was that the notion of calibration (see `http://tinyurl.com/y3lyead3`) generalizes and could be relevant for TGD. A calibration in Riemann manifold M means the existence of a k-form ϕ in M such that for any orientable k-D sub-manifold the integral of ϕ over M equals to its k-volume in the induced metric. One can say that metric k-volume reduces to homological k-volume.

   Calibrated k-manifolds are minimal surfaces in their homology class, in other words their volume is minimal. Kähler calibration is induced by the k^{th} power of Kähler form and defines calibrated sub-manifold of real dimension $2k$. Calibrated sub-manifolds are in this case precisely the complex sub-manifolds. In the case of CP_2 they would be complex curves (2-surfaces) as has become clear.

2. By the Minkowskian signature of M^4 metric, the generalization of calibrated sub-manifold so that it would apply in $M^4 \times CP_2$ is non-trivial. Twistor lift of TGD however forces to introduce the generalization of Kähler form in M^4 (responsible for CP breaking and matter antimatter asymmetry) and calibrated manifolds in this case would be naturally analogs of string world sheets and partonic 2-surfaces as minimal surfaces. Cosmic strings are Cartesian products of string world sheets and complex curves of CP_2. Calibrated manifolds, which do not reduce to Cartesian products of string world sheets and complex surfaces of CP_2 should also exist and are minimal surfaces.

   One can also have 2-D calibrated surfaces and they could correspond to string world sheets and partonic 2-surfaces which also play key role in TGD. Even discrete points assignable to partonic 2-surfaces and representing fundamental fermions play a key role and would trivially correspond to calibrated surfaces.

3. Much later I ended up with the identification of preferred extremals as minimal surfaces by totally different route without realizing the possible connection with the generalized calibrations. Twistor lift and the notion of quantum criticality led to the proposal that preferred extremals for the twistor lift of Kähler action containing also volume term are minimal surfaces. Preferred extremals would be separately minimal surfaces and extrema of Kähler action and generalization of complex structure to what I called Hamilton-Jacobi structure would be an essential element. Quantum criticality outside singular surfaces would be realized as decoupling of the two parts of the action. May be all preferred extremals be regarded as calibrated in generalized sense.

   If so, the dynamics of preferred extremals would define a homology theory in the sense that each homology class would contain single preferred extremal. TGD would define a generalized topological quantum field theory with conserved Noether charges (in particular rest energy) serving as generalized topological invariants having extremum in the set of topologically equivalent 3-surfaces.

   It is interesting to recall that the original proposal for the preferred extremals as absolute minima of Kähler action has transformed during years to a proposal that they are absolute minima of volume action within given homology class and having fixed ends at the boundaries of CD.

4. The experience with CP_2 would suggest that the Kähler structure of M^4 defining the counterpart of form ϕ is unique. There is however infinite number of different closed self-dual Kähler forms of M^4 defining what I have called Hamilton-Jacobi structures. These forms can have subgroups of Poincare group as symmetries. For instance, magnetic flux tubes correspond to given cylindrically symmetry Kähler form. The problem disappears as one realizes that Kähler structures characterize families of preferred extremals rather than M^4 itself.

If the notion of calibration indeed generalizes, one ends up with the same outcome - preferred extremals as minimal surfaces with 2-D string world sheets and partonic 2-surfaces as singularities - from many different directions.

1. Quantum criticality requires that dynamics does not depend on coupling parameters so that extremals must be separately extremals of both volume term and Kähler action and therefore minimal surfaces for which these degrees of freedom decouple except at singular 2-surfaces, where the necessary transfer of Noether charges between two degrees of freedom takes place at these. One ends up with string picture but strings alone are of course not enough. For instance, the dynamical string tension is determined by the dynamics for the twistor lift.
2. Almost topological QFT picture implies the same outcome: topological QFT property fails only at the string world sheets.
3. Discrete coupling constant evolution, vanishing of loop corrections, and number theoretical condition that scattering amplitudes make sense also in p-adic number fields, requires a representation of scattering amplitudes as sum over resonances realized in terms of string world sheets.
4. In the standard QFT picture about scattering incoming states are solutions of free massless field equations and interaction regions the fields have currents as sources. This picture is realized by the twistor lift of TGD in which the volume action corresponds to geodesic length and Kähler action to Maxwell action and coupling corresponds to a transfer of Noether charges between volume and Kähler degrees of freedom. Massless modes are represented by minimal surfaces arriving inside causal diamond (CD) and minimal surface property fails in the scattering region consisting of string world sheets.
5. Twistor lift forces M^4 to have generalize Kähler form and this in turn strongly suggests a generalization of the notion of calibration. At physics side the implication is the understanding of CP breaking and matter anti-matter asymmetry.
6. M^8-H duality requires that the dynamics of space-time surfaces in H is equivalent with the algebraic dynamics in M^8. The effective reduction to almost topological dynamics implied by the minimal surface property implies this. String world sheets (partonic 2-surfaces) in H would be images of complex (co-complex sub-manifolds) of $X^4 \subset M^8$ in H. This should allows to understand why the partial derivatives of embedding space coordinates can be discontinuous at these edges/folds but there is no flow between interior and singular surface implying that string world sheets are minimal surfaces (so that one has conformal invariance).

The analogy with foams in 3-D space deserves to be noticed.

1. Foams can be modelled as 2-D minimal surfaces with edges meeting at vertices. TGD space-time could be seen as a dynamically generated foam in 4-D many-sheeted space-time consisting of 2-D minimal surfaces such that also the 4-D complement is a minimal surface. The counterparts for vertices would be light-like curves at light like orbits of partonic 2-surfaces from which several string world sheets can emanate.
2. Can one imagine something more analogous to the usual 3-D foam? Could the light-like orbits of partonic 2-surfaces define an analog of ordinary foam? Could also partonic 2-surfaces have edges consisting of 2-D minimal surfaces joined along edges representing strings connecting fermions inside partonic 2-surface?

   For years ago I proposed what I called as symplectic QFT (SQFT) as an analog of conformal QFT and as part of quantum TGD [K18]. SQFT would have symplectic transformations as symmetries, and provide a description for the symplectic dynamics of partonic 2-surfaces. SQFT involves an analog of triangulation at partonic 2-surfaces and Kähler magnetic fluxes associated with them serve as observables. The problem was how to fix this kind of network.

Partonic foam could serve as a concrete physical realization for the symplectic network and have fundamental fermions at vertices. The edges at partonic 2-surfaces would be space-like geodesics. The outcome would be a calibration involving objects of all dimensions $0 \leq D \leq 4$ - a physical analog of homology theory.

6.7.3 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [L10]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A54]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

M^8-H duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L57].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.
2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L42]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?
3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?
4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.
5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t - m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

Number-theoretic approach to unitarity

Twistorialization leads to the proposal that cuts in the scattering amplitudes are replaced with sums over poles, and that also many-particle states have discrete momentum and mass squared spectrum having interpretation in terms of bound states. Gravitation would be the natural physical reason for the discreteness of the mass spectrum and in string models it indeed emerges as "stringy" mass spectrum. The situation is very similar to that in dual resonance models, which were precedessors of super string theories.

Number theoretical discretization based on the hierarchy of extensions of rationals defining extensions of p-adic number fields gives rise to cognitive representatations as discrete sets of space-time surface and discretization of 4-momenta and S-matrix with discrete momentum labels. In number theoretic discretization cuts reduce automatically to sequences of poles. Whether this discretization is an approximation reflecting finite cognitive resolution or whether finite cognitive representation is a property of physical states reflecting itself as a condition that various parameters characterizing them belong to the extension considered, remains an open question.

One can approach the unitarity conditions also number theoretically. In the discretization forced by the extension of rationals the amplitudes are defined between states having a discrete spectrum of 4-momenta. Unitarity condition reduces to a purely algebraic condition involving only sums. In these conditions the Dirac delta functions associated with the mass squared of the resonances are replaced with Kronecker deltas.

1. For given extension of rationals the unitary conditions are purely algebraic equations

$$i(T_{mn}+\overline{T}_{nm}) = \sum_r T_{mr}\overline{T}_{nr} = T_{mn}\overline{T}_{nn} + T_{mm}\overline{T}_{mn} + \sum_{r\neq m,n} T_{mr}\overline{T}_{nr} \quad .$$

where T_{mn} belongs the extension. Complex imaginary unit i corresponds to that appearing in the extension of octonions in M^8-H duality [L37].

2. In the forward direction $m=n$ one obtains

$$2Im(T_{mm}) = Re(T_{mm})^2 + Im(T_{mm})^2 + P_m \quad , \quad P_m = \sum_{r\neq m} T_{mr}\overline{T}_{mr} \quad .$$

P_m represents total probability for non-forward scattering.

3. One can think of solving $Im(T_{mm})$ algebraically from this second order polynomial in the lowest order approximation in which $T_{mn}=0$ for $m\neq n$. This gives

$$2Im(T_{mm}) = 1+\sqrt{1-P_m-Re(T_{mm})^2} \quad .$$

Reality requires $1-Re(T_{mm})^2-P_m \geq 0$ giving

$$Re(T_{mm})^2+P_m \leq 1 \quad .$$

This condition is identically true by unitarity since probability for scattering cannot be larger than 1.

Besides this the real root must belong to the original extension of rationals. For instance, if the extension of rationals is trivial, the quantity $1-P_m-Re(T_{mm})^2$ must be a square of rational y giving $1-P_m = y^2+Re(T_{mm})^2$. In the case of extension y is replaced with a number in the extension. I am not enough of number theorist to guess how powerful this kind of number theoretical conditions might be. In any case, the general ansatz for S is a unitary matrix in extension of rationals and this kind of matrices form a group so that there is no hope about unique solution.

4. One could think of iterative solution of the conditions by assuming in the zeroth order approximation $T_{mn}=0$ for $m\neq n$ giving $Re(T_{mm})^2+Im(T_{mm})^2=1$ reducing to $cos^2(\theta)+sin^2(\theta)=1$. For trivial extension of rationals θ would correspond to Pythagorean triangle.

For non-diagonal elements of T_{mn} one would obtain at the next step the conditions

$$i(T_{mn}+\overline{T}_{nm}) = T_{mn}\overline{T}_{nn} + T_{mm}\overline{T}_{nm} \quad .$$

This gives a 2 linear equations for T_{mn}.

5. These conditions are not enough to give unique solution. Time reversal invariance gives additional conditions and might help in this respect. T invariance is slightly broken but CPT symmetry could replace T symmetry in the general situation.

   Time reversal operator T (to be not confused with T_{mn} above) is anti-unitary operator and one has $S^\dagger = T(S)$. In wave mechanics one can show that T-invariant S-matrix and thus also T-matrix is symmetric: $S = S^T$. The matrices of this kind do not form a group so that the conditions can be very powerful.

   Combined with the above equations symmetry gives

   $$2Im(T_{mn}) = T_{mn}\overline{T}_{nn} + T_{mm}\overline{T}_{mn} \ .$$

   The two conditions for T_{mn} in principle fix it completely in this order.

   One obtains from the real part of the equation

   $$2Im(T_{mn}) = Re[T_{mn}\overline{T}_{nn} + T_{mm}\overline{T}_{mn}] \ .$$

   The vanishing of the imaginary part gives

   $$Im[T_{mn}\overline{T}_{nn} + T_{mm}\overline{T}_{mn}] = 0 \ .$$

   giving a linear relation between the real and imaginary parts of T_{mn}. No new number theoretical conditions emerge. This relation requires that real and imaginary parts belong to the extension.

6. At higher orders one must feed the resulting ansatz to the unitarity conditions for the diagonal elements T_{nn}. One can hope that the lowest order ansatz leads to rather unique solution by iteration of the unitarity conditions. In higher order conditions the higher order corrections appear linearly so that no new number theoretic conditions emerge at higher orders.

   Physical picture suggests that the S-matrices could be obtained by an iterative procedure. Since infinitely long procedure very probably leads out of the extension, one can ask whether the procedure should stop after finite steps. This property would pose an additional conditions to the S-matrix.

7. Diagonal matrices are solutions to the conditions and for then the diagonal elements are roots of unity in the extension of rationals considered. The automorphisms $S_d \rightarrow US_dU^{-1}$ produce new S-matrices and if the unitary matrix U is orthogonal real matrix in algebraic extension satisfying therefore $UU^T = 1$, the condition $S = S^T$ is satisfied. There are therefore a large number of solutions.

   S-matrices diagonalizable in the extension are not the only solutions. The diagonalization of a unitary matrix $S = S^T$ in general gives a diagonal S-matrix, for which the roots of unity in general do not belong to the extension. Also the diagonalizating matrix fails to be in the extension. This non-diagonalizability might have deep physics content and explain why the physically natural state basis is not the one in which S-matrix is diagonal. In the case of density matrix it would guarantee stability of entanglement.

To sum up, number theoretic conditions could give rise to highly unique discrete S-matrices, when CPT symmetry can be formulated purely algebraically and be combined with unitarity. CPT symmetry might not however allow formulation in terms of automorphisms of diagonal unitary matrices analogous to orthogonal transformations.

6.7.4 Summary

It seems that unitarity of S-matrix reduces to the existence of maximal group of WCW isometries. The conservation of charges implies conservation of probability and unitarity.

Disjoint 3-surfaces and also those topologically condensed at larger space-time sheets would have interpretation as topological representations of particles in this approach. The special role of the partonic orbits suggests holography in the sense that these orbits have particle interpretation. Similar holography would make sense true for string world sheets and their boundaries. Action could therefore contain parts associated with $D = 2$ and $D = 1$ surfaces so that oscillator operators associated with these would be involved in the construction of states.

The transfer of quantum numbers from space-time interior to string world sheets could take place in interaction regions for Option I for which one assigns action to singular surfaces identified as surfaces having complex or real tangent space at M^8 level. The transfer would naturally vanish near the boundaries of CD. Same applies to the transfer from string world sheets to their boundaries. For Option II two the string world sheets would not carry Noether currents and only minimal surface property could fail at these surfaces: therefore this option is not realistic. Also for Option I there could be breaking of minimal surface property in this sense and the discontinuity of normal component for Noether currents would imply it automatically.

When this picture is combined with the twistor Grassmannian inspired view about scattering amplitudes using the constraints coming from quantum criticality, discreteness of the coupling constant evolution, and the existence of amplitudes as rational functions with coefficients in a extension of rationals allowing p-adic variants, one ends up to a picture in which amplitudes reduces to sums over resonances - this was just what was assumed in Veneziano model besides s-t duality.

This picture does not conform with QFT picture in superstring framework, where one has single large string tension so that poles cannot be approximated by cuts for low energies. In TGD framework this can be the case since string tension has spectrum reducing to that for cosmological constant. Since momenta are already classically predicted to be complex, resonance poles have finite width and one can in principle understand also unitarity. Therefore twistorialization in TGD framework leads to string models, and strings are indeed an essential part of twistorialization in TGD framework.

6.8 Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals

Number theorist Minhyong Kim has speculated about very interesting general connection between number theory and physics [A62, A72] (see `http://tinyurl.com/y86bckmo`). The reading of a popular article about Kim's work revealed that number theoretic vision about physics provided by TGD has led to a very similar ideas and suggests a concrete realization of Kim's ideas [L71]. The identification of points of algebraic surface with coordinates, which are rational or in extension of rationals, gives rise to what one can call identification problem. In TGD framework the embedding space coordinates for points of space-time surface belonging to the extension of rationals defining the adelic physics in question are common to reals and all extensions of p-adics induced by the extension. These points define what I call cognitive representation, whose construction means solving of the identification problem.

Cognitive representation defines discretized coordinates for a point of "world of classical worlds" (WCW) taking the role of the space of spaces in Kim's approach. The symmetries of this space are proposed by Kim to help to solve the identification problem. The maximal isometries of WCW necessary for the existence of its Kähler geometry provide symmetries identifiable as symplectic symmetries. The discrete subgroup respecting extension of rationals acts as symmetries of cognitive representations of space-time surfaces in WCW, and one can identify symplectic invariants characterizing the space-time surfaces at the orbits of the symplectic group.

This picture could be applied to the construction of scattering amplitudes with finite cognitive precision in terms of cognitive representations and their orbits under subgroup S_D of symplectic group respecting the extension of rationals defining the adele. One could pose to S_D the additional condition that it leaves the value of action invariant: call this group $S_{D,S}$: this would define what I have called micro-canonical ensemble (MCE).

The obvious question is whether the simplest zero energy states could correspond to single orbit of S_D or whether several orbits are required. For the more complex option zero energy states would be superposition of states corresponding to several orbits of S_D with coefficients constructed of symplectic invariants. The following arguments lead to the conclusion that MCE and single orbit orbit option are non-realistic, and raise the question whether the orbits of S_D could combine to an orbit of its Yangian analog. A generalization of the formula for scattering amplitudes in terms of n-point functions emerges and somewhat surprisingly one finds that the unitarity is an automatic consequence of state orthonormalization in zero energy ontology (ZEO).

6.8.1 Zero energy states

The degrees of freedom at WCW level can be divided to zero modes, which do not contribute to WCW metric and correspond to symplectic invariants and to dynamical degrees of freedom which correspond to the orbits of symplectic group of $\delta M^4_\pm \times CP_2$. The assumption is that symplectic group indeed acts as isometries. The general proposal for the state construction in continuum case should have a discrete analog. There are good reasons to hope that the zero energy states in the degrees of freedom corresponding to the orbits of the discrete variant S_D of the symplectic grop are analogous to spherical harmonics and are dictated completely by symmetry considerations.

Quantum superposition of space-time surfaces - preferred extremals - defines zero energy state. The natural question is whether zero energy state could correspond to single orbit of S_D or whether several of them are needed.

1. Preferred extremal is fixed more or less uniquely by its ends, which are 3-surfaces at the opposite light-like boundaries of CD. The interpretation is in terms of holography forced also by general coordinate invariance requiring that one must be able to assign to a given 3-surface a unique space-time surface at which general coordinate transformations act. In ZEO 3-surface means union of 3-surface at opposite ends of CD.

   The idea about preferred extremals as analogs of Bohr orbits suggests that the 3-surface at the either end determines the 3-surface at the opposite end highly uniquely. The proposal that preferred extremals are minimal surfaces apart from singular 2-surfaces identifiable as string world sheet, means that they are separately extremals of both Kähler action and volume term supports this expectation as also the condition that sub-algebra of symplectic group Lie algebra isomorphic to it gives rise to vanishing Noether charges and also the Noether charges associated with its commutator with the full algebra vanish.

   The condition that the zero energy state at the active boundary of CD is superposition of many-particle states with different particle number in topological sense suggests that this is not the case.

   Even stronger form of holography would be that the data at string world sheets and partonic 2-surfaces determines the preferred extremal completely. In number theoretic vision one can consider even stronger number theoretic holography: if octonionic polynomials code for the space-time surfaces as $M^8 - H$ holography suggests [L37], cognitive representation consisting of discrete set of points with M^8 coordinates in extension of rationals would determine the preferred extremals.

2. Also fermionic degrees of freedom at the ends are involved. Quantum classical correspondence (QCC) states that the classical charges in Cartan sub-algebra of symmetries are equal to the eigenvalues of quantal charges constructible in terms of fermionic oscillator operator algebra. Many-fermion states would correspond to preferred extremals and the fermionic statistics requires that one has superposition over corresponding 4-surfaces. The state at second end of CD is quantum entangled, and fermionic statistics suggests entanglement at both ends.

Symplectic isometries have subgroup with parameters in the extension of rationals defining the adele: call this subgroup S_D. Denote the subgroup of S_D leaving action invariant by $S_{D,S}$. The representations of S_D (or possibly $S_{D,S}$) are expected to be important concerning the construction of scattering amplitudes and on basic of zero energy state property one expects that the action of S_D ($S_{D,S}$) on the opposite ends of space-time surface compensate each other for zero energy states.

A reasonable looking question is whether simplest zero energy states could corresponds to single orbit of S_D. One expects that the number of points defining the cognitive representation is same for all preferred extremals at its orbit. There are several questions to be answered.

1. The existence of preferred extremals connecting given 3-surface with fixed topological particular number to 3-surface at the second end of CD having varying topological particle number

looks rather plausible. Topological particle number can be identified either as number of disjoint 3-surfaces and number of disjoint partonic 2-surfaces carrying fermions.

Can single orbit of S_D contain space-time surfaces with varying topological particle number at the other end of CD? If not , one must allow some minimal number of orbits of S_D in the definition of minimal zero energy state. This option looks the most realistic one.

2. What is the precise definition of cognitive representation?

3. Micro-canonical ensemble (MCE) hypothesis states that action is same for all space-time surfaces appearing in zero energy state. Can this hypothesis be consistent with the presence of many-particle states with different topological particle number? CP_2 type extremals represent particles and have non-vanishing actions. Also the action of symplectic group in general changes the Kähler action although the action is constant at co-dimension 1 surface of WCW so that the subgroup $S_{D,S}$ should act at this surface. It would seem that one must allow the variation of action and this is a challenge for number theoretic universality since the number theoretically non-universal part of action exponentials must be common to all space-time surfaces involves and must cancel in S-matrix.

What does one mean with cognitive representation? Is single orbit of S_D enough? Can one assume MCE? These are the key questions to be considered.

6.8.2 The action of symplectic isometries on cognitive representations

The action of S_D on cognitive representation defining the adele is straightforward. It is not however quite clear how to identify the cognitive representation.

1. Cognitive representation in question corresponds to a set of points of space-time surface with M^8 coordinates in extension of rationals defining the adele (a stronger condition is that also $M^4 \times CP_2$ coordinates satisfy the same condition).

2. Does cognitive representation contain only the points at the ends of CD, either end, or also interior points? Or does cognitive representation consists of singular points at which non-trivial subgroup of Galois group leaves the point invariant? The singular points could correspond to fundamental fermions at partonic 2-surfaces.

   Remark: If the fermionic lines are light-like geodesic they would correspond as cognitive representations exceptionally informative and easy ones containing infinite number of points of extensions essentially the number line defined by the extension. This raises the question whether the simplest string world sheets identifiable as planes M^2 could be the most interesting singularities of preferred extremals identified as singular minimal surfaces. Canonical embedding of M^4 is also cognitively easy.

   The condition that the actions of symplectic group at opposite boundaries of CD compensate each other makes sense only if one restricts the cognitive representations at either boundary of CD. This would exclude interior points.

   Could one allow also points in the interior of space-time surface by generalizing the view about symplectic invariance of zero energy state? For instance, could the partonic 2-surface defining vertices in the interior contain points of the cognitive representation. Does the allowance of the points of cognitive representation in interior mean giving up strict determinism and does the variational principle with volume term allow it (mere 4-D Kähler action allows huge vacuum degeneracy).

3. When does the point of cognitive representation correspond to a fundamental fermion? I have proposed [L37] that this is the case if the point is critical in number theoretical sense meaning that there is subgroup of Galois group leaving it invariant: the sheets corresponding to different elements of Galois sub-group would co-incide at critical point. The number of singular points and thus number of fundamental fermions might vary.

4. Could the number of singular points vary for the 4-surfaces at the orbit so that the number of fundamental fermions would vary too? Could this allow to have superposition of many-particle states as active part of the zero energy state? This does not seem plausible since the number of points of cognitive representations must be S_D invariant. Several orbits of S_D seem to be required.

The role of Galois group of extension of rationals must be important.

1. Galois group act do not affect space-time surface but only inside the cognitive representation. Galois group can also have subgroup leaving invariant given point. A possible interpretation is as number theoretic correlate for fundamental fermion.
2. A natural hypothesis is that the sub-group of symplectic group leaving the cognitive representation invariant acts as Galois group. A goo analogous for Galois group is provide by the rotation group $SO(3)$ serving as isotropy group of time-like 4-momentum having vanishing 3-momentum in the rest system. For induced representations $SO(3)$ acts in spin degrees of freedom. In the recent case Galois group could act in number theoretic spin degrees of freedom. Could the action of Galois group be physically non-trivial. For instance, could the ordinary symmetries be represented as Galois transformations in fermionic degrees of freedom?

Symplectic invariants characterize the representation and Kähler fluxes for M^4 and CP_2 Kähler forms define this kind of invariants. Also higher fluxes are possible. The general state as superposition of states associated with the over orbits of S_D would have functions of these invariants as coefficients.

6.8.3 Zero energy states and generalization of micro-canonical ensemble

The space-time surfaces in micro-canonical ensemble (MCE) [L56] would have same action so that Kähler function would be constant. It is interesting to discuss this hypothesis in light of the idea that simplest zero energy state corresponds to a finite set of orbits of $S_{D,S}$.

Is micro-canonical ensemble consistent with zero energy state- S_D orbit correspondence?

The assumption that action is constant at the orbit is not problematic. Kähler function must vary in order to give rise to non-trivial Kähler metric. Kähler function is however constant at co-dimension 1 surfaces of WCW. For instance, the Kähler function of CP_2 is function of the radial coordinate invariant under subgroups invariant under $U(2)$ but not under $SU(3)$.

1. The simplest variant of MCE is that single space-time surface is involved. The action of $S_{D,S}$ would be essentially trivial - zero momentum would be more familiar Minkowski analogy. One would get rid of the action exponentials: this would solve the problems related to number theoretical universality caused by the fact that the exponential need not exist in various p-adic number fields.
2. A more realistic hypothesis is that $S_{D,S}$ has several 4-surfaces at its orbit. If the number of surfaces is N the sum of action exponentials is N-fold and the exponential disappears from the S-matrix elements in analogy with what happens in the full theory without discretization by cancellation of the exponential strong suggested by what happens in QFTs.

MCE has however problems.

1. It is not at all clear whether one can make restriction to a subgroup preserving the action. To gain some perspective, not that in the case of CP_2 this would mean restriction to $r = constant$ surface of CP_2 and this is not possible. In the case of rotation group this would mean restriction to sphere.

   Physically it is also obvious that one should allow in the zero energy state all 4-surfaces which are allowed by the conditions posed by preferred extremal property and there seems no good reason to prevent final states with varying particle topological particle number.
2. Also the standard view about S-matrix suggests at active boundary of CD a superposition of final states with different topological particle numbers having different number disjoint 3-surfaces or same number of disjoint 3-surfaces but varying number of partonic 2-surfaces. That the action of S_D changes the number of the disjoint 3-surfaces is in conflict with naïve intuitions but one must remember that number theoretic discretization loses information about connectedness.

3. If the zero energy state has at the active boundary 3-surfaces with a varying topological particle number identified as a number of CP_2 type extremals with unique maximal action, one expects that action exponential is not constant along the orbit of S_D. If the subgroup of S_D, call it $S_{D,S}$, preserves the value of the action, one must allow orbits of S_D with varying value of action. This would give superposition MCEs. Action preserving subgroup would be analogous to the little group of Poincare group preserving the momentum of particle. As notice, also several orbits of S_D must be allowed.

   The conclusions seems to be that MCE is physically non-realistic.

Can one generalize micro-canonical ensemble to single orbit of S_D?

Suppose that the orbit of S_D contains many-particle states having in final state varying particle numbers measured as number disjoint 3-surfaces or partonic 2-surfaces. Is there any hope of understanding these many-particle states in terms of single representation of S_D?

1. The orbit of S_D must have 4-surfaces with varying value of action. This is possible if the action exponentials differ by a multiplicative rational number so that the number theoretically problematic part cancels out from the S-matrix since it appears in both denominator and numerator of the expression defining S-matrix element.
2. That cognitive representations at the orbit would have same number of points at all points of orbits is intuitively in conflict with varying topological particle number. If Galois group has a subgroup of order $m > 1$ acting trivially on points representing fundamental fermions, the number of points in the representation is effectively reduced since m points are replaced by 1 point. This could allows to have a varying particle numbers identified as the number of points of cognitive representation.

   If CP_2 type extremals in the final state serve as correlates for particles, one should understand how their addition is possible. Their addition to the state would require that some non-degenerate points of representation become degenerate. If the number N points is large, it is quite possible to have rather large number of fundamental fermions in the final state. The degeneration of these points would give rise to fermions. There is however an upper bound which also comes from infrared cutoff for energy.
3. It is not clear whether S_D can transform to each other points with different value of m. The problem is that idea that S_D maps some points to single point is in conflict with the idea that S_D action is bijective. It seems that this idea simply fails.

The conclusion seems to be that one must allow several orbits on basis of purely classical picture and QCC suggesting the possibility of finals states with varying topological particles number.

Could ZEO allow to understand the possibility of particle creation and annihilation?

The idea about quantum superposition of states with varying particle number in topological sense is natural if one believes in QFT based intuition. Just for fun one can ask whether ZEO could provide a loophole.

In ZEO "self" corresponds to a sequence of unitary time evolutions changing the state at active boundary. The active boundary itself becomes de-localized. "Small" state function reduction induces localization of the active boundary. This means measurement of clock time as temporal distance between CDs. The time increment ΔT between subsequent values of clock time varies, and one expects that particle number changes in each unitary evolution. The big state function reduction occurs at some time T, the lifetime of self, and one can assume that the value of T varies statistically.

Could one think that the particle number in topological is actually well-defined after each small reduction? The ensemble of detected particle reactions providing the data allowing to deduce the cross sections. Could the variation of intervals ΔT and the variation for the duration T gives rise to a variation of detected particle numbers in the final state. If this is the case the unitary time evolutions and "small" state function reductions would be very "classical". If so ZEO would simplify dramatically the structure of S-matrix.

To make this mechanism more detailed, one can add the existing wisdom about CP_2 type extremals as building bricks of particles.

1. The action is expected to depend on particle number and different numbers of CP_2 type extremals assignable to which fundamental fermions are assigned correspond to different values of actions. This is not a problem now since would not have have superposition over states with different number of CP_2 type extremals and even micro-canonical ensemble could make sense.
2. The addition of particle to the final state during the unitary evolution taking the active boundary farther away from the passive boundary would correspond to a creation of CP_2 type extremal. Simplest mechanism is 3-vertex defined by partonic 2-surface at which CP_2 type extremal replicates. The outgoing lines in the analogs of twistor diagrams would be unstable against replication. Replication is suggested to be universal process in TGD and the replication of magnetic body (MB) would induce DNA replication in TGD inspired quantum biology.
3. A possible interpretation would be in terms of quantum criticality. CP_2 type extremals would be unstable against decay. One could also interpret the analog of twistor diagram as a sequence of algebraic operations.

In this framework the scattering rates would be determined by a hierarchy of S-matrices labelled by different values of total durations $T_n \sum_{k=1}^{n} \Delta T_k$ for a sequence of unitary evolution followed by time localization. In standard picture they would correspond to single infinitely long time evolution. It would not be surprising if this difference could exclude the proposal as unrealistic.

Could one regard zero energy state involving several orbits of S_D as an orbit of Yangian analog of S_D?

QCC suggest strongly that one must allow zero energy states, which correspond to several orbits of S_D. An interesting possibility is that these orbits could be integrated to a representation of a larger group. What suggests itself is the possibly existing Yangian variant of S_D in which the group action is not local anymore even at the level of WCW. The Yangian of projective transformations of M^4 indeed appears in twistor Grassmannian approach and gives rise to huge symmetries behind the success of twistor Grassmannian approach. I have proposed that super-symplectic variant of Grassmannian indeed exists [L10, L45, L24, L58].

6.8.4 How to construct scattering amplitudes?

Lubos Motl (see `http://tinyurl.com/y5lndpn3`) told about two new hep-th papers, by Pate, Raclariu, and Strominger (see `http://tinyurl.com/yxqx237b`) and by Nandan, Schreiber, Volovich, Zlotnikov (see `http://tinyurl.com/y642yspf`) related to a new approach to scattering amplitudes based on the replacement of the quantum numbers associated with Poincare group labelling particles appearing in the scattering amplitudes with quantum numbers associated with the representations of Lorentz group.

Why I got interested was that in zero energy ontology (ZEO) the key object is causal diamond (CD) defined as intersection of future and past directed M^4 light-cones with points replaced with CP_2. Space-time surfaces are inside CD and have ends at its light-like boundaries. The Lorentz symmetries associated with ithe boundaries of CD could be more natural than Poincare symmetry, which would emerge in the integration over the positions of CDs of external particles arriving to the opposite light-like boundaries of the big CD defining the scattering region where preferred extremal describing the scattering event resides.

I did my best to understand the articles and - of course relate these ideas to TGD, where the construction of scattering amplitudes is the basic challenge. My technical skills are too limited for to meet this challenge at the level of explicit formulas but I can try to understand the physics and mathematics brought in by TGD.

While playing with more or less crazy and short-lived ideas inspired by the reading of the articles I finally realized that there is perhaps no point in starting from quantum field theories. TGD is not quantum field theory and I must start from TGD itself.

In TGD framework the picture inspired by adelic physics [L43, L42] is roughly following.

1. Cognitive representations realizing number theoretic universality of adelic physics consist of points of embedding space with coordinates in the extension of rationals. The number of points is typically finite. Cognitive representation should contain as subset the points associated with n-point functions, which are essentially correlation functions.

   Fundamental fermions are building bricks of elementary particles, and a good guess is that fundamental fermions correspond to singular points for which the action of subgroup of Galois group of extension is trivial so that several points collapse together.

2. One must sum over the orbits of a subgroup S_D of symplectic group of light-cone boundary acting as isometries of both boundaries of CD. S_D consists of isometries with parameters in the extension of rationals defining the adele. All orbits needed to represent the pairs of initial and final 3-surfaces at the boundaries of CD allowed by the action principle must be realized so that single orbit very probably is not enough.

3. Correlations code for the quantum dynamics. In quantum field theories quantum fluctuations of fields at distinct points of space-time correlate and give rise to n-point functions expressible in terms of propagators and vertices: massless fields and conformal fields define the basic example. Operator algebra or path integral describes them mathematically.

   In TGD correlations between embedding space points belonging to the space-time surface result from classical deterministic dynamics: the points of 3-surface at opposite boundaries of CD are not independent.

   This dynamics is non-linear geometric analog for the dynamics of massless fields: space-time sheets as preferred extremals are indeed minimal surfaces with string world sheets appearing as singularities. Minimal surface property is forced by the volume action implied by the twistor lift and having interpretation in terms of cosmological constant. The correlation between points at the same boundary of CD are expected to be independent since these 3-surfaces chosen rather freely as analogs of boundary values for fields.

   Fermionic dynamics governed by modified Dirac action is dictated completely by super-symplectic and super-conformal symmetries. Second quantization of fermions at space-time level is necessary to realized WCW spinor structure: WCW gamma matrices are linear combinations of fermionic oscillator operators.

4. This suggests that the attempts to guess the conformal field theory producing the correlation functions makes things much more complex than they actually are. It should be possible to understand how these correlations emerge from the classical dynamics of space-time surfaces.

As the first brave guess one could try to calculate directly the correlations of spinor harmonics of embedding space assigned with these points.

1. Sum over the symplectic orbits of cognitive representations must be involved as also vacuum expectation values in the fermionic sector for fermionic fields which must appear in vertices for external particles. At the level of cognitive representations anti-commutators for oscillator operators involve Kronecker deltas so that one has discretized variant of second quantization.

2. This could be achieved by expanding the restriction $\Psi^A_{|X^3}$ of the embedding space harmonic Ψ^A restricted to 3-surface at end of space-time surface as sum of modes Ψ_n of the induced spinor field. This would be counterpart for the induction procedure. One can assign to singular points bilinear of type $\overline{\Psi}^A_{|X^3} D^{\leftrightarrow} \Psi$, where Ψ is second quantized induced spinor field expressible as sum over its modes multiplied by oscillator operators. D is modified Dirac operator. This gives as vacuum expectations propagators connecting fermions vertices at the opposite ends of space-time surface.

3. A more concrete picture must rely on a concrete model for elementary particles. Elementary particles have as building bricks pair of wormhole contacts with fermion lines at the light-like orbits of the throats at which the signature of the metric changes from Minkowskian to Euclidian. Particle is necessarily a pair of two wormhole contacts and flux tube connects them at both space-time sheets and forms a closed flux tube carrying monopole flux.

   All particles consist of fundamental fermions and anti-fermions: for instance gauge bosons involve fermion and anti-fermion responsible for the quantum numbers at the opposite throats of second wormhole contact. Second wormhole contact involves neutrino pair neutralizing electroweak isospin in scales longer than the size of the flux tube structure.

4. The topological counterpart of 3-vertex appearing in Feynman diagram corresponds to a replication of this kind of 3-surface highly analogous to bio-replication. In replication vertex, there is no singularity of 3-surface analogous to that appearing in the vertices of stringy diagrams but space-time surface is singular just like 1-D manifold is singular for at vertex of Feynman diagram.

   These singular replicating 3-surfaces and the partonic 2-surfaces give rise to the counterparts of interaction vertices. Fermionic 4-vertex is impossible and fermion lines can only be re-shared between outgoing partonic orbits. This is however not enough as will be found. It will be found that also the creation of fermion pair as effective turning of fermion lines entering along "upper" wormhole throat and turning back at Euclidian wormhole throat and continuing along the orbit of "lower" wormhole throat must be possible.

To see how this conclusion emerges consider the following problem. One should obtain also emission of bosons identified as fermion pairs from fermion line. One has incoming fermion and outgoing fermion and fermion pair describing boson which represents gauge boson or graviton with vanishing B and L. Fermionic 4-vertex is not allowed since this would bring in divergences.

1. The appearance of a sub-CD assignable to the partonic 2-surface is possible but does not solve the problem considered. There would be incoming fermion line at lower boundary and 1 fermion line and fermion and anti-fermion line associated with the boson at the "upper" boundary. There would be non-locality in the scale of the partonic 2-surface and sub-CD meaning that the lines can end to vacuum. Now one would encounter the same difficulty but only in shorter scale.

2. Could one say that fermion line turns backwards in time? A line turning back could be described as an annihilation of fermion pair to vacuum carrying classical gauge field, which is standard process. In QFT picture this would be achieved if a bilinear $\overline{\Psi}D\Psi$ is allowed in the vertex where annihilation takes place. Not in TGD: fermionic action vanishes identically by field equations expressing essentially the conservation of fermion current and various super currents obtained as contractions fermion field with modes.

   Could fermion-anti-fermion pair creation occur at singular points associated with partonic surfaces representing the turning of fermion line backwards in time. This looks still too singular.

   Rather, the turning backwards in time should mean that a fermion line arriving from future along the orbit of "upper" throat (say) goes through Euclidian wormhole throat and continues along the orbit of "lower" throat back to future than making discontinuous turn-around. Euclidian regions of space-time surface representing one key distinction between GRT and TGD would thus be absolutely essential for the generalized scattering diagrams. An exchange of momentum with classical field would be Feynman diagrammatic manner to say this.

   New oscillator operator pairs emerge at the partonic vertices and would correspond to the above described turn-around for fermion line at wormhole contact. Fermion pairs present at the "lower" boundary of CD could also disappear.

3. The anti-commutation relations fermions are modified due to the presence of vacuum gauge fields so that the anti-commutator of fermionic creation operators $a_m^\dagger$ and anti-fermionic creation operators $b_n^\dagger$ is non-vanishing. A proper formulation of the fermionic anti-commutation relations at the ends of space-time surface is needed and in discretization provided by cognitive representation this should be relatively straightforward.

   One can imagine that although standard anti-commutation relations at the lower end of space-time surface hold true, the time evolution of Ψ in the presence of vacuum gauge potentials implies that the vacuum expectations $\langle vac|a_m^\dagger b_n^\dagger|vac\rangle$ are non-vanishing. This would require that for instance $b_n^\dagger$ and a_n are mixed.

There are still questions to be answered.

1. Is the first guess enough? It is not as becomes clear after a thought about the continuum limit. The WCW degrees of freedom are described at continuum limit in terms of super-symplectic algebra (SSA) acting on ground state are neglected. Embedding space spinor modes characterizee only the ground staes of these representations. These degrees of freedom are essential already in elementary particle physics [K52].

Sub-algebra SSA_m of SSA with conformal weights coming as m-multiples of those of SSA and its commutator with SSA annihilate the physical states, and one obtains a hierarchy. How to describe these states in the discretization? The natural possibility are the representations of S_D such that $(S_D)_m$ and the subgroup generated by the commutator algebra are represented trivially. One has non-trivial $(S_D)_m$ representations at both ends of WCW such that the action of S_D on the tensor product acts trivially.

There are also fermionic degrees of freedom. The challenge is to identify among other things WCW gamma matrices as fermionic super charges and it would be nice if all charges were Noether charges. The simplest guess is that the algebra generated by fermionic Noether charges Q^A for symplectic transformations $h^k \rightarrow h^k + j^{Ak}$ assumed to induce isometries of WCW and Noether supercharges Q_n and their conjugates for the shifts $\Psi \rightarrow \Psi + \epsilon u_n$, where u_n is a solution of the modified Dirac equation, is enough.

The commutators $\Gamma^A_n = [Q^A, Q_n]$ are super-charges labelled by (A, n). One would like to identify them as gamma matrices of WCW. The problem is that they are labelled by (A, n) whereas isometry generators are labelled by A only. There should be one-one correspondence. Do all supercharges Γ^A_n except Γ^A_0 corresponding to $u_0 = constant$ annihilate the physical states so that one would have 1-1 correspondence. This would be analogous to what happens quite generally in super-conformal algebras.

The generators of this fermionic algebra could be used to generate more general states. One should also construct the discretized versions of the generators as sums over points of the cognitive representation at the ends of space-time surface. Note that this requires tangent space data.

2. What about the conservation of four-momentum and other conservation laws? This can be handled by quantum classical correspondence (QCC). The momentum and color labels defined by fermionic quantum numbers in Cartan algebra can be assumed to be equal to the corresponding classical Noether charges for particle-like space-time surfaces entering to CD. The technical problem is that if one knows only the discretization - even with tangent space data - one does not know the values of these charges! It might be that $M^8 - H$ correspondence in which M^8 side fixes space-time surfaces as roots for real or imaginary parts of octonionic polynomials from the data at discrete set of points is needed.

3. ZEO means deviations from ordinary description. S_D invariance of zero energy state forces sum over the 4-surfaces of the orbit with identical coefficients. Symplectic invariance implies time-like entanglement. One can describe this in terms of hermitian square root Ψ of density matrix satisfying $\Psi^\dagger \Psi = \rho$. The coefficients of different orbits need not be same and allows description in terms of dynamical density matrix. If there is Yangian symmetry also this entanglement is analogous to the entanglement due to statistics.

   Surprisingly - and somewhat disappointingly after decades of attempts to understand unitarity in TGD - unitarity is trivial in ZEO since state basis is defined essentially by the rows of matrices and orthogonality conditions their orthogonality and therefore unitarity. More concretely, for single state at the passive end state function normalization to unity defined by inner product as sum over 3-surfaces at active end would give conservation of probability. Orthogonality of the state basis with inner product as sum over surfaces passive boundary gives orthogonality for the coefficients defining rows of a matrix and therefore unitarity. In the case that single orbit or even several of them defines the states one obtains the same result.

   What then guarantees the orthogonality of zero energy states? In ordinary quantum mechanics the property of being eigenstates of some hermitian operator guarantees orthogonality. In TGD zero energy states would be solutions of the analog of massless Dirac equation in WCW consisting of pairs of 3-surfaces with members at the ends of preferred extremals inside CD. This generalizes Super Virosoro conditions of superconformal theories and would provide the orthonormal state basis.

The outcome would be amazingly simple. There would be no propagators, no vertices, just spinor harmonics of embedding assigned with these $n = n_1 + n_2$ points at the boundaries of CD, and summation over the orbits of the symplectic group. All these mathematical objects would emerge from classical dynamics. The sum over the orbits for chosen spinor harmonics would produce n-point functions, vertices and propagators. It is difficult to imagine anything simpler and quantum classical correspondence would be complete.

6.9 Minimal surfaces: comparison of the perspectives of mathematician and physicist

The popular article "*Math Duo Maps the Infinite Terrain of Minimal Surfaces*" (see `http://tinyurl.com/yyetb7c7`) was an exceptional representative of its species. It did not irritate the reader with non-sense hype but gave very elegant and thought provoking representation of very abstract ideas in mathematics.

6.9.1 Progress in the understanding of closed minimal surfaces

The article tells about the work of mathematicians Fernando Coda Marques and Andre Neves based on a profound and - as they tell - extremely hard-to-understand work of Jon Pitts forgotten by mathematics community. It is comforting that at least in mathematics good work is eventually recognized.

The results of Marques and Neves are about minimal hyper-surfaces imbedded in various spaces with dimension varying between 3 and 7 and clearly extremely general. These spaces have varying topologies and are called "shapes" in the popular article.

Some examples of minimal surfaces

To begin it is good to have some examples about minimal surfaces.

1. For mathematician any lower-dimensional manifold in some embedding space is surface, even 1-D curve! Simplest minimal surfaces are indeed 1-D geodesic lines. In flat 3-space they are straight lines of infinite length but at the surface of sphere they are big circles.
2. Soap films are 2-D minimal surfaces spanned by frames and familiar for every-one. Frame is necessary for having minimal surface, which does not collapse to a point or extend to infinity and possibly self-intersect.

   Why minimal surfaces are not nice closed surfaces of finite size not intersecting themselves is due to the fact that the equations for minimal surface state the vanishing of the sum of external curvatures defined by the trace of so called second fundamental form defined by the covariant derivatives of tangent vectors of the minimal surface.

   One can say that for 2-D minimal surface the external curvatures in 2 orthogonal directions at given point of surface are of opposite sign. Surface looks locally like saddle rather than sphere. In n-dimensional case the sum of n principal curvatures - eigenvalues of second fundamental form as matrix- sum up to zero for each normal direction: more general saddle.

   In flat embedding space this implies the saddle property always but in curved space it might happen that the covariant derivatives replacing the ordinary derivatives in the definition of second fundamental form - having interpretation as generalized acceleration - can change the situation and the question is whether non-flat closed embedding space could contain closed minimal surfaces.

   Indeed, in compact spaces with non-flat metric minimal surface can be closed and there is a century old theorem by Birkhoff stating that sphere has always at least one closed geodesic independent of metric. In the case of ordinary sphere this geodesic is big circle, the equator. In complex projective space CP_2 there is infinite number of 2-D minimal surfaces which are closed: geodesic spheres are the simplest examples.
3. A good example about a non-closed 1-D surface is generic geodesic in torus with points labelled by two angles (ϕ_1, ϕ_2) in flat metric. The geodesic lines are of form $\phi_1 = \alpha\phi_2$). For non-rational value of α the curve winds the torus infinitely many times and has infinite length. For $\alpha = m/n$ the curve winds m times around second non-contractible circles and n times around the second one. Note that now the geodesic line is absolute minimum: this is caused by the non-contractibility. It can be only shifted in both directions so that the minimum has 2-D degeneracy.

4. In spaces allowing Kähler structure - means that imaginary unit i satisfying $i^2 = -1$ has a representation as antisymmetric tensor - any complex algebraic surfaces representable as root for a set of polynomials, whose number is smaller than complex dimension of the space, is a minimal surface. This huge variety of minimal surfaces is due to the presence of complex structure.

What does minimal surface property mean?

Consider now what minimal surface property really means.

1. Strictly speaking, minimal surfaces are stationary with respect to the *local* variations of volume only. This is practically always true for physical variational principles defined by an action. For a great circle at sphere the minimality of length with respect to small variations is easy to understand by drawing to see what this variation means. With respect to non-local variations meaning shift toward North or East the area decreases so that one has maximum! This leads to the term Minimax principle used by Jon Pitts and his followers as a powerful guideline.

   In fact, minimal surfaces can be both minima and maxima for volume simultaneously. The general extremum as solution of equations defined by a general action principle is saddle. Minimum with respect to some variables and maximum with respect to others and minimal surfaces are this kind of objects in the general case.

2. There is a deep connection with Morse theory in topology (see `http://tinyurl.com/ych4chg9`). Morse function gives information about the topology of space. Morse function is a continuous monotonously increasing function from the space to real line and its extrema provide information about the topology of the space. Morse function can be seen as a kind of height function, a particular coordinate for the space.

   The height as z-coordinate for torus imbedded in 3-space gives a classical example of height function. As z varies on obtains 1-D intersections of torus. The minimum of z corresponds to a single point, above it one has circle, then circle decomposes to 2 circles at lower saddle, and circles fuse back to circle at upper saddle, which becomes a point at maximum. Therefore the extrema of heigh function tell about how the topology of the cross section of the torus varies with height: point-circle-2 circles-circle-point. The area of surface serves as a Morse function and minimal points are analogous to the points of the torus at which cross section changes its topology.

   A good guess is that the volume of the surface serves as a Morse function and thus gives information about the topology of rather abstract infinite-dimensional space: the space of surfaces. Minimal surfaces would be analogous to the critical points of height function at torus: points at which the cross section changes its topology.

3. Minimax property states the fact that minimal surfaces are in in generic situation saddle points in the space of surfaces. There would be a strange correspondence. The points of minimal surfaces are locally saddles in the finite-dimensional embedding space H and minimal surfaces represent saddle points in the finite-dimensional space of surfaces in H. This strange local-global correspondence bringing in mind holography might be behind a general principle: saddle property could have representations at two levels: points of the surface and points of the space of surfaces.

   Are minimal surfaces a rare exception or could it be that for a general action principle the extremals are saddles locally and that the space of all field configurations (not only extremals) contains the extremals as saddle points?

   Remark: Minimal surfaces might be very special and related to what corresponds in physics to criticality implying that the dynamics in certain sense universal. The space of surfaces corresponds in TGD as the space of 3-surfaces and is analogous to Wheeler's superspace consisting of 3-metrics. By holography forced by 4-D general coordinate invariance 3-surfaces in question must be in one-one correspondence with 4-D surfaces identified as space-time surfaces. I have christened this space world of classical worlds (WCW). Space-time surfaces are 4-D minimal surfaces in 8-D $H = M^4 \times CP_2$ but possessing lower dimensional singularities having interpretation as orbits of string like objects and point like particles. Minimal surface property would be a correlate for quantum criticality so that minimal surface would be very special.

The question and the answer

The question that Marquez and Neves posed to themselves was under which conditions compact space allows a closed minimal surface not intersecting itself or whether all candidates intersect themselves or have infinite volume. In fact, Marquez and Neves restricted the consideration to hyper-surfaces. A possible good reason for this is that there is only one field like dynamical degree of freedom for co-dimension 1 - the coordinate in the normal direction- and this is expected to simplify the situation considerably. From the tone of the article - "-hyper" has been dropped away - one has a temptation to guess that the results are much more general.

The basic result of Marques and Neves was rather astonishing. In almost all closed spaces with dimension between 3 and 7 there exists an infinite series of imbedded *closed* minimal hyper-surfaces (embedding means that there are no self-intersections). No frames needed! The irony was that they could not prove their result for spaces with roundest metrics (no bumps making metric positively curved, which in turn helps to have minimal surface property without local saddle property). Song however generalized this result to apply for arbitrary closed embedding spaces [A22] (see `http://tinyurl.com/yycbw4lx`).

What helped in the proof was a surprising result by Marques, Neves, and Liokumovich that the volume for these minimal hyper-surfaces depends on the volume of the compact embedding space only [A70] (see `http://tinyurl.com/y59pdawj`)!

This dependence suggests that these closed minimal hyper-surfaces manage to visit a dense set of points of the embedding space without intersecting themselves: in this manner they could "measure" the volume. Marques, Neves and Irie show that there is infinite set of imbedded minimal hyper-surfaces in spaces of dimension $3 \leq n \leq 7$ intersecting any given ball of the embedding space [A56] (see `http://tinyurl.com/y3u3bvnc`). Even more, these minimal surfaces tend to fill space in some sense evenly.

A natural guess inspired by Minimax Principle is that minimals surfaces correspond to saddle points for the volume as functional of surface defining Morse function. The volume is analogous to action in TGD framework.

Two remarks are in order.

1. As noticed, the popular article says that these results hold for minimal surfaces. The articles however restrict the consideration to minimal hyper-surfaces.
2. The theorem about the dependence of volume of hyper-surface on the volume of embedding space was inspired by a result proven by Weyl for the high frequencies of drum defined as a boundary of some space: these frequencies depend on the volume of the space, not on the shape of drum! One can understand this intuitively by the fact that high frequency vibrations correspond to short wave lengths and therefore depend only on the *local* properties of the space and not on the global topology. The dependence on volume comes from boundary conditions at the boundaries of the volume.

   In the case of minimal hyper-surfaces the analogy would suggests that the addition of details to the minimal hyper-surface corresponds to the increase of the frequency for drum. Boundary conditions for drum would be replaced by the compactness of the embedding space leading to the quantization of the volume analogous to that for frequency.
3. The infinite geodesic on flat torus described above is a rough analog for omni-presence although it is not closed. Also complex surfaces in CP_2 defined as zero loci of polynomials of complex coordinates (ξ^1, ξ^2) modified to contain irrational powers of ξ^i could define this kind of omni-present surfaces having however infinite area. There is however infinite number of minimal surfaces defined by complex polynomials, which are closed but not omni-present.

6.9.2 Minimal surfaces and TGD

In TGD framework surfaces satisfying minimal surface equations almost everywhere - play a central role.

Space-time surfaces as singular minimal surfaces

From the physics point this is not surprising since minimal surface equations are the geometric analog for massless field equations.

1. The boundary value problem in TGD is analogous to that defining soap films spanned by frames: space-time surface is thus like a 4-D soap film. Space-time surface has 3-D ends at the opposite boundaries of causal diamond of M^4 with points replaced with CP_2: I call this 8-D object just causal diamond (CD). Geometrically CD brings in mind big-bang followed by big crunch.

   These 3-D ends are like the frame of a soap film. This and the Minkowskian signature guarantees the existence of minimal surface extremals. Otherwise one would expect that the non-compactness does not allow minimal surfaces as non-self-intersecting surfaces.

2. Space-time is a 4-surface in 8-D $H = M^4 \times CP_2$ and is a minimal surface, which can have 2-D or 1-D singularities identifiable as string world sheets having 1-D singularities as light-like orbits - they could be geodesics of space-time surface.

   Remark: I considered in [L53] the possibility that the minimal surface property could fail only at the reaction vertices associated with partonic 2-surfaces defining the ends of string world sheet boundaries. This condition however seems to be too strong. It is essential that the singular surface defines a sub-manifold giving deltafunction like contribution to the action density and that one can assign conserved quantities to this surface. This requires that the singular contributions to energy momentum tensor and canonical momentum currents as spacetime vectors are parallel to the singular surface. Singular points do not satisfy this condition.

   String boundaries represent orbits of fundamental point-like fermions located at 3-D light-like surfaces which represent orbits of partonic 2-surfaces. String world sheets are minimal surfaces and correspond to stringy objects associated with say hadrons. There are also degrees of freedom associated with space-time interior. One have objects of various dimension which all are minimal surfaces. Modified Dirac equation extends the field equations to supersymmetric system and assigns fermionic degrees of freedom to these minimal surfaces of varying dimension.

   From the physics point of view, the singular surfaces are analogous to carriers of currents acting as point- and string-like sources of massless field equations.

3. Geometrically string world sheets are analogous to folds of paper sheet. Space-time surfaces are extremals of an action which is sum of volume term having interpretation in terms of cosmological constant and what I call Kähler action - analogous to Maxwell action. Outside singularities one has minimal surfaces stationary with respect to variations of both volume term and Kähler action - note the analogy with free massless field. At singularities there is an exchange of conserved quantities between volume and Kähler degrees of freedom analogous to the interaction of charged particle with electromagnetic field. One can see TGD as a generalization of a dynamics of point-like particle coupled to Maxwell field by making particle 3-D surface.

4. The condition that the exchange of conserved charges such as four-momentum is restricted to lower-D surfaces realizes preferred extremal property as a consequence of quantum criticality demanding a universal dynamics independent of coupling parameters [L63]. Indeed, outside the singularities the minimal surfaces dynamics has no explicit dependence on coupling constants provided local minimal surface property guarantees also the local stationarity of Kähler action.

   Preferred extremal property has also other formulations. What is essential is the generalization of super-conformal symmetry playing key role in super string models and in the theory of 2-D critical systems so that field equations reduce to purely algebraic conditions just like for analytic functions in 2-D space providing solutions of Laplace equations.

5. TGD provides a large number of specific examples about closed minimal surfaces [K8]. Cosmic strings are objects, which are Cartesian products of minimal surfaces (string world sheets) in M^4 and of complex algebraic curves (2-D surfaces). Both are minimal surfaces and extremize also Kähler action. These algebraic surfaces are non-contractible and characterized by homology charge having interpretation as Kähler magnetic charge. These surfaces are genuine minima just like the geodesics at torus.

   CP_2 contains two kinds of geodesic spheres, which are trivially minimal surfaces. The reason is that the second fundamental form defining as its trace the analogs of external curvatures in the normal space of the surfaces vanishes identically. The geodesic sphere of the first kind is non-contractible minimal surface and absolute minimum. Geodesic spheres of second kind is contractible and one has Minimax type situation.

These geodesic spheres are analogous to 2-planes in flat 3-space with vanishing external curvatures. For a generic minimal surface in 3-space the principal curvatures are non-vanishing and sum up to zero. This implies that minimal surfaces look locally like saddles. For 2-plane the curvatures vanish identically so that saddle is not formed.

Kähler action as Morse function in the space of minimal surfaces

It was found that surface volume could define a Morse function in the space of surfaces. What about the situation in TGD, where volume is replaced with action which is sum of volume term and Kähler action [L58, L57, L63]?

Morse function interpretation could appear in two ways. The first possibility is that the action defines an analog of Morse function in the space of 4-surfaces connecting given 3-surfaces at the boundaries of CD. Could it be that there is large number of preferred extremals connecting given 3-surfaces at the boundaries of CD? This would serve as analogy for the existence of infinite number of closed surfaces in the case of compact embedding space. The fact that preferred extremals extremize almost everywhere two different actions suggests that this is not the case but one must consider also this option.

1. The simplest realization of general coordinate invariance would allow only single preferred extremal but I have considered also the option for which one has several preferred extremals. In this case one encounters problem with the definition of Kähler function which would become many-valued unless one is ready to replace 3-surfaces with its covering so that each preferred extremal associated with the given 3-surface gives rise to its own 3-surface in the covering space. Note that analogy with the definition of covering space of say circle by replacing points with the set of homologically equivalence classes of closed paths at given point (rotating arbitrary number of times around circle).
2. Number theoretic vision [K104, L22] suggests that these possibly existing different preferred extremals are analogous to same algebraic computation but performed in different ways or theorem proved in different ways. There is always the shortest manner to do the computation and an attractive idea is that the physical predictions of TGD do not depend on what preferred extremal is chosen.
3. An interesting question is whether the "drum theorem" could generalize to TGD framework. If there exists infinite series of preferred extremals which are singular minimal surfaces, the volume of space-time surface for surfaces in the series would depend only on the volume of the CD containing it. The analogy with the high frequencies and drum suggests that the surfaces in the series have more and more local details. In number theoretic vision this would correspond to emergence of more and more un-necessary pieces to the computation. One cannot exclude the possibility that these details are analogs for what is called loop corrections in quantum field theory.
4. If the action defines Morse action, the preferred extremals give information about its topology. Note however that the requirement that one has extremum of both volume term and Kähler action almost everywhere is an extremely strong additional condition and corresponds physically to quantum criticality.

   Remark: The original assumption was that the space-time surface decomposes to critical regions which are minimal surfaces locally and to non-critical regions inside which there is flow of canonical momentum currents between volume and Kähler degrees of freedom. The stronger hypothesis is that this flow occurs at 2-D and 1-D surfaces only.

Kähler function as Morse function the space of 3-surfaces

The notion of Morse function can make sense also in the space of 3-surfaces - the world of classical worlds which in zero energy ontology consists of pairs of 3-surfaces at opposite boundaries of CD connected by preferred extremal of Kähler action [K24, K80, L58, L57]. Kähler action for the preferred extremal is assumed to define Kähler function defining Kähler metric of WCW via its second derivates $\partial_K \partial_{\overline{L}} K$. Could Kähler function define a Morse function?

1. First of all, Morse function must be a genuine function. For general Kähler metric this is not the case. Rather, Kähler function K is a section in a $U(1)$ bundle consisting of patches transforming by real part of a complex gradient as one moves between the patches of the bundle. A good example is CP_2, which has non-trivial topology, and which decomposes to 3 coordinate patches such that Kähler functions in overlapping patches are related bythe analog of U(1) gauge transformation.

   Kähler action for preferred extremal associated with given 3-surface is however uniquely defined unless one includes Chern-Simons term which changes in U(1) gauge transformation for Kähler gauge potential of CP_2.

2. What could one conclude about the topology of WCW if the action for preferred extremal defines a Morse function as a functional of 3-surface? This function cannot have saddle points: in a region of WCW around saddle point the WCW metric depending on the second derivatives of Morse function would not be positive definite, and this is excluded by the positivity of Hilbert space inner product defined by the Kähler metric essential for the unitarity of the theory. This would suggest that the space of 3-surfaces has very simple topology if Kähler function.

   This is too hasty conclusion! WCW metric is expected to depend also on zero modes, which do not contribute to the WCW line element. What suggests itself is bundle structure. Zero modes define the base space and dynamical degrees of freedom contributing to WCW line element as fiber. The space of zero modes can be topologically complex.

There is a fascinating open problem related to the metric of WCW.

1. The conjecture is that WCW metric possess the symplectic symmetries of $\Delta M^4_+ \times CP_2$ as isometries. In infinite dimensional case the existence of Riemann/Kähler geometry is not at all obvious as the work of Dan Freed demonstrated in the case of loops spaces [A37], and the maximal group of isometries would guarantee the existence of WCW Kähler geometry. Geometry would be determined by symmetries alone and all points of the space would be metrically equivalent. WCW would be an infinite-dimensional analog of symmetric space.

2. Isometry group property does not require that symplectic symmetries leave Kähler action, and even less volume term for preferred extremal, invariant. Just the opposite: if the action would remain invariant, Kähler function and Kähler metric would be trivial!

3. The condition for the existence of symplectic isometries must fix the ratio of the coefficients of Kähler action and volume term highly uniquely. The physical interpretation is in terms of quantum criticality realized mathematically in terms of the symplectic symmetry serving as analog of ordinary conformal symmetry characterizing 2-D critical systems. Note that at classical level quantum criticality realized as minimal surface property says nothing outside singular surfaces since the field equations in this regions are algebraic. At singularities the situation changes. Note also that the minimal surface property is a geometric analog of masslessness which in turn is a correlate of criticality.

4. Twistor lift of TGD [?]eads to a proposal for the spectra of Kähler coupling strength and cosmological constant allowed by quantum criticality [L57]. What is surprising that cosmological constant identified as the coefficient of the volume term takes the role of cutoff mass in coupling constant evolution in TGD framework. Coupling constant evolution discretizes in accordance with quantum criticality which must give rise to infinite-D group of WCW isometries. There is also a connection with number theoretic vision in which coupling constant evolution has interpretation in terms of extensions of rationals [K104, L42, L37].

Can one apply the mathematical results about closed minimal surfaces to TGD?

The general mathematical thinking involved with the new results is applied also in TGD as should be clear from the above. But can one apply the new mathematical results described above to TGD? Unfortunately not as such. There are several reasons for this.

1. The dimension of $H = M^4 \times CP_2$ is $D = 8 > 7$. M^4 is non-compact and also the signature of M^4 metric is Minkowskian rather than Euclidian. Could one apply these results to special kinds of 4-surface such as stationary surfaces $M^1 \times X^3$, $X^3 \subset E^3 \times CP_2$. No: the problem is that E^3 is non-compact.
2. In TGD one does not consider closed space-time surfaces but analogs of soap films spanned by a frame defined by the 3-surfaces at the opposite ends of CD. Note that the singular surfaces of dimension $D = 2, 1$ are analogous to frames with boundaries at the ends of space-time surface.
3. In TGD framework preferred extremal property requires that space-time surface is both minimal surface and extremal of Kähler action outside singularities. This is known to be the case for all known extremals. This poses very strong conditions on extremals and seems to mean the existence of a generalization of Kähler structure and conformal invariance to 4-D situation. This drops a large number of minimal surface extremals from consideration
4. Minimal surfaces filling space evenly do not have any reasonable physical interpretation. Maybe this could be used to argue that one must have $D = 8$ and that signature must be Minkowskian in order to have soap films rather than closed minimal surfaces.

   What about E^4 with Euclidian signature instead of M^4 and closed space-time surfaces in analogy with Euclidian field theories? Would the projections of closed minimal 4-surfaces in $E^4 \times CP_2$ which are also extremals of Kähler action reduce to a point in E^4 and complex 2-surfaces in CP_2: Euclidianized TGD would degenerate to an Euclidian version of string model. Also in $H = S^4 \times CP_2$ the situation might be same since the property of being extremal of Kähler action is very powerful. It is however essential that also M^4 has analog of Kähler structure: S^4 does not have it although it allows twistor structure so that this options drops out.
5. Can one apply the results of Marques, Neves and others about hyper-surfaces to TGD? What comes in mind is a minimal 4-surface, which is a Cartesian product of geodesic line $M^1 \subset M^4$ and 3-D hyper-surface $X^3 \subset CP_2$ visiting all points of CP_2 and having a finite volume. If the action would contain only the volume term, this extremal would be possible. The action however contains Kähler action and this very probably excludes this extremal.

Chapter 7

TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors

7.1 Introduction

There are two mysterious looking correspondences involving ADE groups. McKay correspondence between McKay graphs characterizing tensor products for finite subgroups of $SU(2)$ and Dynkin diagrams of affine ADE groups is the first one. The correspondence between principal diagrams characterizing inclusions of hyper-finite factors of type II_1 (HFFs) with Dynkin diagrams for a subset of ADE groups and Dynkin diagrams for affine ADE groups is the second one.

I have considered the interpretation of McKay correspondence in TGD framework already earlier [K105, K36] but the decision to look it again led to a discovery of a bundle of new ideas allowing to answer several key questions of TGD.

1. Asking questions about $M^8 - H$ duality at the level of 8-D momentum space [L37] led to a realization that the notion of mass is relative as already the existence of alternative QFT descriptions in terms of massless and massive fields suggests (electric-magnetic duality). Depending on choice $M^4 \subset M^8$, one can describe particles as massless states in $M^4 \times CP_2$ picture (the choice is M^4_L depending on state) and as massive states (the choice is fixed M^4_T) in M^8 picture. p-Adic thermal massivation of massless states in M^4_L picture can be seen as a universal dynamics independent mechanism implied by ZEO. Also a revised view about zero energy ontology (ZEO) based quantum measurement theory as theory of consciousness suggests itself.

2. Hyperfinite factors of type II_1 (HFFs) [K105, K36] and number theoretic discretization in terms of what I call cognitive representations [L57] provide two alternative approaches to the notion of finite measurement resolution in TGD framework. One obtains rather concrete view about how these descriptions relate to each other at the level of 8-D space of light-like momenta. Also ADE hierarchy can be understood concretely.

3. The description of 8-D twistors at momentum space-level is also a challenge of TGD. 8-D twistorializations in terms of octo-twistors (M^4_T description) and $M^4 \times CP_2$ twistors (M^4_L description) emerge at embedding space level. Quantum twistors could serve as a twistor description at the level of space-time surfaces.

7.1.1 McKay correspondence in TGD framework

Consider first McKay correspondence in more detail.

1. McKay correspondence states that the McKay graphs characterizing the tensor product decomposition rules for representations of discrete and finite sub-groups of $SU(2)$ are Dynkin diagrams for the affine ADE groups obtained by adding one node to the Dynkin diagram of ADE group. Could this correspondence make sense for any finite group G rather than only discrete subgroups of $SU(2)$? In TGD Galois group of extensions K of rationals can be any finite group G. Could Galois group take the role of G?

2. Why the subgroups of $SU(2)$ should be in so special role? In TGD framework quaternions and octonions play a fundamental role at M^8 side of M^8-H duality [L37]. Complexified M^8 represents complexified octonions and space-time surfaces X^4 have quaternionic tangent or normal spaces. $SO(3)$ is the automorphism group of quaternions and for number theoretical discretizations induced by extension K of rationals it reduces to its discrete subgroup $SO(3)_K$ having $SU(2)_K$ as a covering. In certain special cases corresponding to McKay correspondence this group is finite discrete group acting as symmetries of Platonic solids. Could this make the Platonic groups so special? Could the semi-direct products $Gal(K) \triangleleft SU(2)_K$ take the role of discrete subgroups of $SU(2)$?

7.1.2 HFFs and TGD

The notion of measurement resolution is definable in terms of inclusions of HFFs and using number theoretic discretization of X^4. These definitions should be closely related.

1. The inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs with index $\mathcal{M} : \mathcal{N} < 4$ are characterized by Dynkin diagrams for a subset of ADE groups. The TGD inspired conjecture is that the inclusion hierarchies of extensions of rationals and of corresponding Galois groups could correspond to the hierarchies for the inclusions of HFFs. The natural realization would be in terms of HFFs with coefficient field of Hilbert space in extension K of rationals involved.

   Could the physical triviality of the action of unitary operators $\mathcal{N}$ define measurement resolution? If so, quantum groups assignable to the inclusion would act in quantum spaces associated with the coset spaces $\mathcal{M}/\mathcal{N}$ of operators with quantum dimension $d = \mathcal{M} : \mathcal{N}$. The degrees of freedom below measurement resolution would correspond to gauge symmetries assignable to $\mathcal{N}$.

2. Adelic approach [L42] provides an alternative approach to the notion of finite measurement resolution. The cognitive representation identified as a discretization of X^4 defined by the set of points with points having H (or at least M^8 coordinates) in K would be common to all number fields (reals and extensions of various p-adic number fields induced by K). This approach should be equivalent with that based on inclusions. Therefore the Galois groups of extensions should play a key role in the understanding of the inclusions.

How HFFs could emerge from TGD?

1. The huge symmetries of "world of classical words" (WCW) could explain why the ADE diagrams appearing as McKay graphs and principal diagrams of inclusions correspond to affine ADE algebras or quantum groups. WCW consists of space-time surfaces X^4, which are preferred extremals of the action principle of the theory defining classical TGD connecting the 3-surfaces at the opposite light-like boundaries of causal diamond $CD = cd \times CP_2$, where cd is the intersection of future and past directed light-cones of M^4 and contain part of $\delta M^4_{\pm} \times CP_2$. The symplectic transformations of $\delta M^4_{+} \times CP_2$ are assumed to act as isometries of WCW. A natural guess is that physical states correspond to the representations of the super-symplectic algebra SSA.

2. The sub-algebras SSA_n of SSA isomorphic to SSA form a fractal hierarchy with conformal weights in sub-algebra being n-multiples of those in SSA. SSA_n and the commutator $[SSA_n, SSA]$ would act as gauge transformations. Therefore the classical Noether charges for these sub-algebras would vanish. Also the action of these two sub-algebras would annihilate the quantum states. Could the inclusion hierarchies labelled by integers $.. < n_1 < n_2 < n_3....$

with n_{i+1} divisible by n_i would correspond hierarchies of HFFs and to the hierarchies of extensions of rationals and corresponding Galois groups? Could n correspond to the dimension of Galois group of K.

3. Finite measurement resolution defined in terms of cognitive representations suggests a reduction of the symplectic group SG to a discrete subgroup SG_K, whose linear action is characterized by matrix elements in the extension K of rationals defining the extension. The representations of discrete subgroup are infinite-D and the infinite value of the trace of unit operator is problematic concerning the definition of characters in terms of traces. One can however replace normal trace with quantum trace equal to one for unit operator. This implies HFFs and the hierarchies of inclusions of HFFs [K105, K36]. Could inclusion hierarchies for extensions of rationals correspond to inclusion hierarchies of HFFs and of isomorphic sub-algebras of SSA?

Quantum spinors are central for HFFs. A possible alternative interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group. This has also application in TGD inspired theory of consciousness [K36]: the idea is that the truth value of Boolean statement is fuzzy. At the level of quantum measurement theory this would mean that the outcome of quantum measurement is not anymore precise eigenstate but that one obtains only probabilities for the appearance of different eigenstate. One might say that probability of eigenstates becomes a fundamental observable and measures the strength of belief.

7.1.3 New aspects of $M^8 - H$ duality

$M^8 - H$ duality ($H = M^4 \times CP_2$) [L37] has become one of central elements of TGD. $M^8 - H$ duality implies two descriptons for the states.

1. $M^8 - H$ duality assumes that space-time surfaces in M^8 have associative tangent- or normal space M^4 and that these spaces share a common sub-space $M^2 \subset M^4$, which corresponds to complex subspace of octonions (also integrable distribution of $M^2(x)$ can be considered). This makes possible the mapping of space-time surfaces $X^4 \subset M^8$ to $X^4 \subset H = M^4 \times CP_2$) giving rise to $M^8 - H$ duality.

2. $M^8 - H$ duality makes sense also at the level of 8-D momentum space in one-one correspondence with light-like octonions. In $M^8 = M^4 \times E^4$ picture light-like 8-momenta are projected to a fixed quaternionic $M^4_T \subset M^8$. The projections to $M^4_T \supset M^2$ momenta are in general massive. The group of symmetries is for E^4 parts of momenta is $Spin(SO(4)) = SU(2)_L \times SU(2)_R$ and identified as the symmetries of low energy hadron physics.

   $M^4 \supset M^2$ can be also chosen so that the light-like 8-momentum is parallel to $M^4_L \subset M^8$. Now CP_2 codes for the E^4 parts of 8-momenta and the choice of M^4_L and color group $SU(3)$ as a subgroup of automorphism group of octonions acts as symmetries. This correspond to the usual description of quarks and other elementary particles. This leads to an improved understanding of $SO(4) - SU(3)$ duality. A weaker form of this duality $S^3 - CP_2$ duality: the 3-spheres S^3 with various radii parameterizing the E^4 parts of 8-momenta with various lengths correspond to discrete set of 3-spheres S^3 of CP_2 having discrete subgroup of $U(2)$ isometries.

3. The key challenge is to understand why the MacKay graphs in McKay correspondence and principal diagrams for the inclusions of HFFs correspond to ADE Lie groups or their affine variants. It turns out that a possible concrete interpretation for the hierarchy of finite subgroups of $SU(2)$ appears as discretizations of 3-sphere S^3 appearing naturally at M^8 side of $M^8 - H$ duality. Second interpretation is as covering of quaternionic Galois group. Also the coordinate patches of CP_2 can be regarded as piles of 3-spheres and finite measurement resolution. The discrete groups of $SU(2)$ define in a natural way a hierarchy of measurement resolutions realized as the set of light-like M^8 momenta. Also a concrete interpretation for Jones inclusions as inclusions for these discretizations emerges.

4. A radically new view is that descriptions in terms of massive and massless states are alternative options leads to the interpretation of p-adic thermodynamics as a completely universal

massivation mechanism having nothing to do with dynamics. The problem is the paradoxical looking fact that particles are massive in H picture although they should be massless by definition. The massivation is unavoidable if zero energy states are superposition of massive states with varying masses. The M^4_L in this case most naturally corresponds to that associated with the dominating part of the state so that higher mass contributions can be described by using p-adic thermodynamics and mass squared can be regarded as thermal mass squared calculable by p-adic thermodynamics.

5. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory. 4-D space-time surfaces correspond to roots of octonionic polynomials $P(o)$ with real coefficients corresponding to the vanishing of the real or imaginary part of $P(o)$.

   These polynomials however allow universal roots, which are not 4-D but analogs of 6-D branes and having topology of S^6. Their M^4 projections are time =constant snapshots $t = r_n, r_M \leq r_n$ 3-balls of M^4 light-cone (r_n is root of $P(x)$). At each point the ball there is a sphere S^3 shrinking to a point about boundaries of the 3-ball.

   What suggests itself is following "braney" picture. 4-D space-time surfaces intersect the 6-spheres at 2-D surfaces identifiable as partonic 2-surfaces serving as generalized vertices at which 4-D space-time surfaces representing particle orbits meet along their ends. Partonic 2-surfacew would define the space-time regions at which one can pose analogs of boundary values fixing the space-time surface by preferred extremal property. This would realize strong form of holography (SH): 3-D holography is implied already by ZEO.

   This picture forces to consider a modification of the recent view about ZEO based theory of consciousness. Should one replace causal diamond (CD) with light-cone, which can be however either future or past directed. "Big" state function reductions (BSR) meaning the death and re-incarnation of self with opposite arrow of time could be still present. An attractive interpretation for the moments $t = r_n$ would be as moments assignable to "small" state function reductions (SSR) identifiable as "weak" measurements giving rise to sensory input of conscious entity in ZEO based theory of consciousness. One might say that conscious entity becomes gradually conscious about its roots in increasing order. The famous question "What it feels to be a bat" would reduce to "What it feels to be a polynomial?"! One must be however very cautious here.

7.1.4 What twistors are in TGD framework?

The basic problem of the ordinary twistor approach is that the states must be massless in 4-D sense. In TGD framework particles would be massless in 8-D sense. The meaning of 8-D twistorialization at space-time level is relatively well understood but at the level of momentum space the situation is not at all so clear.

1. In TGD particles are massless in 8-D sense. For M^4_L description particles are massless in 4-D sense and the description at momentum space level would be in terms of products of ordinary M^4 twistors and CP_2 twistors. For M^4_T description particles are massive in 4-D sense. How to generalize the twistor description to 8-D case?

   The incidence relation for twistors and the need to have index raising and lowering operation in 8-D situation suggest the replacement of the ordinary l twistors with either with octo-twistors or non-commutative quantum twistors.

2. I have assumed that what I call geometric twistor space of M^4 is simply $M^4 \times S^2$. It however turned out that one can consider standard twistor space CP_3 with metric signature (3,-3) as an alternative. This option reproduces the nice results of the earlier approach but the philosophy is different: there is no fundamental length scale but the hierarchy of causal diamonds (CDs) predicted by zero energy ontology (ZEO) gives rise to the breaking of the exact scaling invariance of M^8 picture. This forces to modify $M^8 - H$ correspondence so that it involves map from M^4 to CP_3 followed by a projection to hyperbolic variant $CP_{2,h}$ of CP_2. Note that also the original form of $M^8 - H$ duality continues to make sense and results from the modification by projection from $CP_{3,h}$ to M^4 rather than $CP_{2,h}$.

   M^4 in H would not be be replaced with conformally compactified M^4 (M^4_{conf}) but conformally compactified cd (cd_{conf}) for which a natural identification is as CP_2 with second complex coordinate replaced with hypercomplex coordinate. The sizes of twistor spaces of cd_{conf} using CP_2 size as unit would reflect the hierarchy of size scales for CDs. The consideration on the twistor space of M^8 in similar picture leads to the identification of corresponding twistor space as HP_3 - quaternionic variant of CP_3: the counterpart of CD_8 would be HP_2.

3. Octotwistors can be expressed as pairs of quaternionic twistors. Octotwistor approach suggests a generalization of twistor Grassmannian approach obtained by replacing the bi-spinors with complexified quaternions and complex Grassmannians with their quaternionic counterparts. Although TGD is not a quantum field theory, this proposal makes sense for cognitive representations identified as discrete sets of spacetime points with coordinates in the extension of rationals defining the adele [L42] implying effective reduction of particles to point-like particles.
4. The outcome of octo-twistor approach together with $M^8 - H$ duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor Grassmannian approach. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. By $M^8 - H$ duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

What about super-twistors in TGD framework?

1. The parallel progress in the understanding SUSY in TGD framework [L74] in turn led to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.
2. The great surprise from physics point of view is that in fermionic sector only quarks are allowed by $SO(1,7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of embedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.
3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

What about the interpretation of quantum twistors? They could make sense as 4-D spacetime description analogous to description at space-time level. Now one can consider generalization of the twistor Grassmannian approach in terms of quantum Grassmannians.

7.2 McKay correspondence

Consider first McKay correspondence from TGD point of view.

7.2.1 McKay graphs

McKay graps are defined in the following manner. Consider group G which is discrete but not necessarily finite. If the group is finite there is a finite number of irreducible representations χ_I. Select preferred representation V - usually V is taken to be the fundamental representation of G and form tensor products $\chi_I \otimes V$. Suppose irrep χ_J appears n_{ij} times in the tensor product

$\chi_I \otimes \chi_0$. Assign to each representation χ_I dot and connect the dots of χ_I and χ_J by n_{ij} arrows. This gives rise to MacKay graph.

Consider now the situation for finite-D groups of $SU(2)$. 2-D $SU(2)$ spinor representation as a fundamental representation is essential for obtaining the identification of McKay graphs as Dynkin diagrams of simply laced affine algebras having only single line connecting the roots (the angle between positive roots is 120 degrees) (see `http://tinyurl.com/z48d92t`).

1. For $SU(2)$ representations one has the basic rule $j_1 - 1/2 \leq j \leq j_1 + 1/2$ for the tensor product $j_1 \otimes 1/2$. This rule must be broken for finite subgroups of $SU(2)$ since the number of representations if finite so that branching point appears in McKay graph. In branching point the decomposition of $j_1 \otimes 1/2$ decomposes to 3 lower-dimensional representations of the finite subgroup takes place.
2. Simply lacedness means that given representation appears only once in $chi_I \otimes V$, when V is 2-D representation as it can be for a subgroup of $SU(2)$. Additional exceptional properties is the absence of loops ($n_{ii} = 0$) and connectedness of McKay graph.
3. One can consider binary icosahedral group (double covering of icosahedral group A_5 with order 60) as an example (for the McKay graph see `http://tinyurl.com/y2h55jwp`). The representations of A_5 are $1_A, 3_A, 3'_B, 4_A, 5_A$, where integer tells the dimension. Note that $SO(3)$ does not allow 4-D representation. For binary icosahedral group one obtains also the representations $2_A, 2'_B, 4_B, 6_A$. The McKay graph of E_8 contains one branching point in which one has the tensor product of 6-D and 2-D representations 6_A and 2_A giving rise to $5_A \oplus 3_C \oplus 4_B$.

McKay graphs can be defined for any finite group and they are not even unions of simply laced diagrams unless one has subgroups of $SU(2)$. Still one can wonder whether McKay correspondence generalizes from subgroups of $SU(2)$ to all finite groups. At first glance this does not seem possible but there might be some clever manner to bring in all finite groups.

The proposal has been that this McKay correspondence has a deeper meaning. Could simply laced affine ADE algebras, ADE type quantum algebras, and/or corresponding finite groups act as symmetry algebras in TGD framework?

7.2.2 Number theoretic view about McKay correspondence

Could the physical picture provided by TGD help to answer the above posed questions?

1. Could one identify discrete subgroups of $SU(2)$ with those of the covering group $SU(2)$ of $SO(3)$ of quaternionic automorphisms defining the continuous analog of Galois group and reducing to a discrete subgroup for a finite resolution characterized by extension K of rationals. The tensor products of 2-D spinor representation of these discrete subgroups $SU(2)_K$ would give rise to irreps appearing in the McKay graph.
2. In adelic physics [L42] extensions K of rationals define an evolutionary hierarchy with effective Planck constant $h_{eff}/h_0 = n$ identified as the dimension of K. The action of discrete and finite subgroups of various symmetry groups can be represented as Galois group action making sense at the level of X^4 [L37] for what I have called cognitive representations. By $M^8 - H$ duality also the Galois group of quaternions and its discrete subgroups appear naturally.

   This suggests a possible generalization of McKay correspondence so that it would apply to all finite groups G. Any finite group G can appear as Galois group. The Galois group $Gal(K)$ characterizing the extension of rationals induces in turn extensions of p-adic number fields appearing in the adele. Could the representation of G as Galois group of extension of rationals allow to generalize McKay correspondence?

Could the following argument inspired by these observations make sense?

1. $SU(2)$ is identified as spin covering of the quaternionic automorphism group. One can define the subgroups in matrix representation of $SU(2)$ based on complex numbers. One can replace complex numbers with the extension of rationals and speak of group $SU(2)_K$ identified as a discrete subgroup of $SU(2)$ having in general infinite order.

   The discrete finite subgroups $G \subset SU(2)$ appearing in the standard McKay correspondence correspond to extensions K of rationals for which one has $G = SU(2)_K$. These special extensions can be identified by studying the matrix elements of the representation of G and include the discrete groups Z_n acting as rotation symmetries of the Platonic solids. For instance, for icosahedral group Z_2,Z_3 and Z_5 are involved and correspond to roots of unity.

2. The semi-direct product $Gal \triangleleft SU(2)_K$ with group action

$$(gal_1, g_1)(gal_2, g_2) = (gal_1 \circ gal_2, g_1(gal_1 g_2))$$

makes sense. The action of $Gal \triangleleft SU(2)_K$ in the representation is therefore well-defined. Since all finite groups G can appear as Galois groups, it seems that one obtains extension of the McKay correspondence to semi-direct products involving all finite groups G representable as Galois groups.

3. A good guess is that the number of representations of $SU(2)_K$ involved is infinite if $SU(2)_K$ has infinite order. For $\tilde{A}_n$ and $\tilde{D}_n$ the branching occurs only at the ends of the McKay graph. For E_k, $k = 6, 7, 8$ the branching occurs in middle of the graph (see `http://tinyurl.com/y2h55jwp`). What happens for arbitrary G. Does the branching occur at all? One could argue that if the discrete subgroup has infinite order, the representation can be completed to a representation of $SU(2)$ in terms of real numbers so that the McKay graphs must be identical.

4. A concrete realization of ADE type Kac-Moody algebras is proposed. It relies on the group algebra of $Gal(K) \triangleleft SU(2)_K$ and free field representation of ADE type Kac-Moody algebra identifying the free scalar fields in Kac-Moody Cartan algebra as group algebra elements defined by the traces of representation matrices (characters).

5. A possible interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group [K36]. TGD inspired theory of consciousness is a possible application.

   Also the notion of quantum twistor [L78] can be considered. In TGD particles are massless in 8-D sense and in general massive in 4-D sense but 4-D twistors are needed also now so that a modification of twistor approach is needed. The incidence relation for twistors suggests the replacement of the usual twistors with non-commutative quantum twistors.

7.3 ADE diagrams and principal graphs of inclusions of hyperfinite factors of type II$_1$

Dynkin diagrams for ADE groups and corresponding affine groups characterize also the inclusions of hyperfinite factors of type II$_1$ (HFFs) [K36].

7.3.1 Principal graphs and Dynkin diagrams for ADE groups

1. If the index $\beta = \mathcal{M} : \mathcal{N}$ of the Jones inclusion satisfies $\beta < 4$, the affine Dynkin diagrams of $SU(n)$ (discrete symmetry groups of n-polygons) and E_7 (symmetry group of octahedron and cube) and $D(2n + 1)$ (symmetries of double 2n+1-polygons) are not allowed.

2. Vaughan Jones [A87] (see `http://tinyurl.com/y8jzvogn`) has speculated that these finite subgroups could correspond to quantum groups as kind of degenerations of Kac-Moody groups. Modulo arithmetics defined by the integer n defining the quantum phase suggests itself strongly. For $\beta = 4$ one can construct inclusions characterized by extended Dynkin diagram and any finite sub-group of $SU(2)$. In this case affine ADE hierarchy appear as principal graphs characterizing the inclusions. For $\beta < 4$ the finite measurement resolution could reduce affine algebra to quantum algebra.

3. The rule is that for odd values of n defining the quantum phase the Dynkin diagram does not appear. If Dynkin diagrams correspond to quantum groups, one can ask whether they allow only quantum group representations with quantum phase $q = exp(i\pi/n)$ equal to even root of unity.

7.3.2 Number theoretic view about inclusions of HFFs and preferred role of $SU(2)$

Consider next the TGD inspired interpretation.

1. TGD suggests the interpretation in terms of representations of $Gal(K(G)) \triangleleft G$ for finite subgroups G of $SU(2)$, where $K(G)$ would be an extension associated with G. This would generalize to subgroups of $SU(2)$ with infinite order in the case of arbitrary Galois group. Quantum groups have finite number of representations in 1-1-correspondence with terms of finite-D representations of G. Could the representations of $Gal(K(G)) \triangleleft G$ correspond to the representations of quantum group defined by G?

   This would conform with the vision that there are two ways to realize finite measurement resolution. The first one would be in terms of inclusions of hyper-finite factors. Second would be in terms cognitive representations defining a number theoretic discretization of X^4 with embedding space coordinates in the extension of rationals in which Galois group acts.

   In fact, also the discrete subgroup of infinite-D group of symplectic transformations of $\Delta M^4_+ \times CP_2$ would act in the cognitive representations and this suggests a far reaching implications concerning the understanding of the cognitive representations, which pose a formidable looking challenge of finding the set of points of X^4 in given extension of rationals [L70].

2. This brings in mind also the model for bio-harmony in which genetic code is defined in terms of Hamiltonian cycles associated with icosahedral and tetrahedral geometries [L13, L61]. One can wonder why the Hamiltonian cycles for cubic/octahedral geometry do not appear. On the other hand, according to Vaughan the Dynkin diagram for E_7 is missing from the hierarchy of so principal graphs characterizing the inclusions of HFFs for $\beta < 4$ (a fact that I failed to understand). Could the genetic code directly reflect the properties of the inclusion hierarchy?

How would the hierarchies of inclusions of HFFs and extensions of rationals relate to each other?

1. I have proposed that the inclusion hierarchies of extensions K of rationals accompanied by similar hierarchies of Galois groups $Gal(K)$ could correspond to a fractal hierarchy of sub-algebras of hyperfinite factor of type II_1. Quantum group representations in ADE hierarchy would somehow correspond to these inclusions. The analogs of coset spaces for two algebras in the hierarchy define would quantum group representations with quantum dimension characterizing the inclusion.

2. The quantum group in question would correspond to a quantum analog of finite-D group of $SU(2)$ which would be in completely unique role mathematically and physically. The infinite-D group in question could be the symplectic group of $\delta M^4_+ \times CP_2$ assumed to act as isometries of WCW. In the hierarchy of Galois groups the quantum group of finite group $G \subset SU(2)$ would correspond to Galois group of an extension K.

3. The trace of unit matrix defining the character associated with unit element is infinite for these representations for factors of type I. Therefore it is natural to assume that hyper-finite factor of type II_1 for which the trace of unit matrix can be normalized to 1. Sub-factors would have trace of projector with trace smaller than 1.

4. Do the ADE diagrams for groups $Gal(K(G)) \triangleleft G$ indeed correspond to quantum groups and affine algebras? Why the finite groups should give rise to affine/Kac-Moody algebras? In number theoretic vision a possible answer would be that depending on the value of the index β of inclusion the symplectic algebra reduces in the number theoretic discretization by gauge conditions specifying the inclusion either to Kac-Moody group ($\beta = 4$) or to quantum group ($\beta < 4$).

What about subgroups of groups other than $SU(2)$? According to Vaughan there has been work about inclusion hierarchies of $SU(3)$ and other groups and it seems that the results generalize and finite subgroups of say $SU(3)$ appear. In this case the tensor products of finite sub-groups McKay graphs do not however correspond to the principal graphs for inclusions. Could the number theoretic vision come in rescue with the replacement of discrete subgroup with Galois group and the identification of $SU(2)$ as the covering for the Galois group of quaternions?

7.3.3 How could ADE type quantum groups and affine algebras be concretely realized?

The questions discussed are following. How to understand the correspondence between the McKay graph for finite group $G \subset SU(2)$ and ADE (affine) group Dynkin diagram for $\beta < 4$ ($\beta = 4$)? How the nodes of McKay grap representing the irreps of finite group can correspond to the positive roots of a Dynkin diagram, which are essentially vectors defined by eigenvalues of Cartan algebra generators for complexified Lie-algebra basis.

The first thing that comes in mind is the construction of representation of Kac-Moody algebra using scalar fields labelled by Cartan algebra generators (see `http://tinyurl.com/y9lkeelk`): these representations are discussed by Edward Frenkel [A38]. The charged generators of Kac-Moody algebra in the complement of Cartan algebra are obtained by exponentiating the contractions of the vector formed by these scalar fields with roots to get $\alpha\cdot\Phi = \alpha_i\Phi^i$. The charged field is represented as a normal ordered product $:exp(i\alpha\cdot\Phi):$. If one can assign to each irrep of G a scalar field in a natural manner one could achieve this.

Neglect first the presence of the group algebra of $Gal(K(G)) \triangleleft G$. The standard rule for the dimensions of the representations of finite groups reads as $\sum_i d_I^2 = n(G)$. For double covering of G one obtains this rule separately for integer spin representations and half-odd integers spin representations. An interesting possibility is that this could be interpreted in terms of supersymmetry at the level of group algebra in which representation of dimension d_I appears d_I times.

The group algebra of G and its covering provide a universal manner to realize these representations in terms of a basis for complex valued functions in group (for extensions of rationals also the values of the functions must belong to the extension).

1. Representation with dimension d_I occurs d_I times and corresponds to $d_I \times d_I$ representation matrices D^I_{mn} of representation χ_I, whose columns and rows provide representations for left- and right-sided action of G. The tensor product rules for the representations χ_I can be formulated as double tensor products. For basis states $|J,n\rangle$ in χ_I and $|J,n\rangle$ in χ_J one has

$$|I,m\rangle_\otimes|J,n\rangle = c^{K,p}_{I,m|J,n}|K,p\rangle \quad ,$$

where $c^{K,p}_{J,n|J,n}$ are Glebch-Gordan coefficients.

2. For the tensor product of matrices D^I_{mn} and D^J_{mn} one must apply this rule to both indices. The orthogonality properties of Glebsch-Gordan coefficients guarantee that the tensor product contains only terms in which one has same representation at left- and right-hand side. The orthogonality rule is

$$\sum_{m,n} c^{K,p}_{I,m|J,n}c^{K,q}_{I,r|J,s} \propto \delta_{K,L} \quad .$$

3. The number of states is $n(G)$ whereas the number $I(G)$ of irreps corresponds to the dimension of Cartan algebra of Kac-Moody algebra or of quantum group is smaller. One should be able to pick only one state from each representation D^I.

   The condition that the state X of group algebra is invariant under automorphism gXg^{-1} implies that the allowed states as function in group algebra are traces $Tr(D^I)(g)$ of the representation matrices. The traces of representation matrices indeed play fundamental role as automorphism invariants. This suggests that the scalar fields Φ_I in Kac-Moody algebra correspond to Hilbert space coefficients of $Tr(D^I)(g)$ as elements of group algebra labelled by the representation. The exponentiation of $\alpha\cdot\Phi$ would give the charged Kac-Moody algebra generators as free field representation.

4. For infinite sub-groups $G \subset SU(2)$ $d(G)$ is infinite. The traces are finite also in this case if the dimensions of the representations involved are finite. If one interprets the unit matrix as a function having value 1 in entire group $Tr(Id)$ diverges. Unit dimension for HFFs provide a more natural notion of dimension $d = n(G)$ of group algebra $n(G)$ as $d = n(G) = 1$. Therefore HFFs would emerge naturally.

It is easy to take into account $Gal(K(G))$. One can represent the elements of semi-direct product $Gal(K(G)) \triangleleft G$ as functions in $Gal(K(G)) \times G$ and the proposed construction brings in also the tensor products in the group algebra of $Gal(K(G))$. It is however essential that group algebra elements have values in K. This brings in tensor products of representations Gal and G and the number of representations is $n(Gal) \times n(G)$. The number of fields Φ_I as also the number of Cartan algebra generators of ADE Lie algebra increases from $I(G)$ to $I(Gal) \times I(G)$. The reduction of the extension of coefficient field for the Kac-Moody algebra from complex numbers to K splits the Hilbert space to sectors with smaller number of states.

7.4 $M^8 - H$ duality

The generalization of the standard twistor Grassmannian approach to TGD remains a challenge for TGD and one can imagine several approaches. $M^8 - H$ duality is essential for these approaches and will be discussed in the sequel.

The original form of $M^8 - H$ duality assumed $H = M^4 \times CP_2$ but quite recently it turned out that one could replace the twistor space of M^4 identified as $M^4 \times S^2$ with $CP_{3,h}$, which is hyperbolic variant of CP_3. This option forces to replace H with $H = CP_{2,h} \times CP_2$. $M^8 - H$ duality would consist of a map of M^4 point to corresponding twistor sphere in $CP_{3,h}$ and its projection to $CP_{2,h}$. This option will be discussed in the section about twistor lift of TGD.

7.4.1 $M^8 - H$ duality at the level of space-time surfaces

$M^8 - H$ duality [L37] relates two views about space-time surfaces X^4: as algebraic surfaces in complexified octonionic M^8 and as minimal surfaces with singularities in $H = M^4 \times CP_2$.

1. Octonion structure at the level of M^8 means a selection of a suitable decomposition $M^8 = M^4 \times E^4$ in turn determining $H = M^4 \times CP_2$. Choices of M^4 share a preferred 2-plane $M^2 \subset M^4$ belonging to the tangent space of allowed space-time surfaces $X^4 \subset M^8$ at various points. One can parameterize the tangent space of $X^4 \subset M^8$ with this property by a point of CP_2. Therefore X^4 can be mapped to a surface in $H = M^4 \times CP_2$: one M^8-duality. One can consider also the possibility that the choice of M^2 is local but that the distribution of $M^2(x)$ is integrable and defines string world sheet in M^4. In this case $M^2(x)$ is mapped to same $M^2 \subset H$.
2. Since 8-momenta p_8 are light-like one can always find a choice of $M^4_L \subset M^8$ such that p_8 belongs to M^4_L and is thus light-like. The momentum has in the general case a component orthogonal to M^2 so that M^4_L is unique by quaternionicity: quaternionic cross product for tangent space quaternions gives the third imaginary quaternionic unit. For a fixed M^4, call it M^4_T, the M^4 projections of momenta are time-like. When momentum belongs to M^2 the choices is non-unique and any $M^4 \subset M^2$ is allowed.
3. Space-time surfaces $X^4 \subset M^8$ have either quaternionic tangent- or normal spaces. Quantum classical correspondence (QCC) requires that charges in Cartan algebra co-incide with their classical counters parts determined as Noether charges by the action principle determining X^4 as preferred extremal. Parallelity of 8-momentum currents with tangent space of X^4 would conform with the naïve view about QCC. It does not however hold true for the contributions to four-momentum coming from string world sheet singularities (string world sheet boundaries are identified as carriers of quantum numbers), where minimal surface property fails.

An important aspect of $M^8 - H$ duality is the description of space-time surfaces $X^4_c \subset M^8_c$ as roots for the "real" or "imaginary" part in quaternionic sense of complexified-octonionic polynomial with real coefficients: these options correspond to complexified-quaternionic tangent - or normal spaces. The real space-time surfaces would be naturally obtained as "real" parts with respect to i of their complexified counterparts by projection from M^8_c to M^4_c. One could drop the subscripts "$_c$" but in the sequel they are kept.

Remark:O_c,O_c,C_c,R_c will be used in the sequel for complexifications of octonions, quaternions, etc.. number fields using commuting imaginary unit i appearing naturally via the roots of real polynomials.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions.

1. Space-time surface is identified as a 4-D root for a H_c-valued "imaginary" or "real" part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. For $P(x) = x^n +..$ ordinary roots are algebraic integers. The 4-D space-time surface is projection of this surface from M^8_c to M^8.

   The tangent space of space-time surface and thus space-time surface itself contains a preferred $M^2_c \subset M^4_c$ or more generally, an integrable distribution of tangent spaces $M^2_c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in R_c sense.

   X^2c can be fixed by posing to the non-vanishing Q_c-valued part of octonionic polynomial condition that the C_c valued "real" or "imaginary" part in C_c sense for this polynomial vanishes. M^2_c would be the simplest solution but also more general complex sub-manifolds $X^2_c \subset M^4_c$ are possible. In general one would obtain book like structures as collections of several string world sheets having real axis as back.

   By assuming that R_c-valued "real" or "imaginary" part of the polynomial at this 2-surface vanishes. one obtains preferred M^1_c or E^1_c containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_{\rightarrow} C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

   Remark: Also M^4_c appears as a special solution for any polynomial P. M^4_c seems to be like a universal reference solution with which to compare other solutions. M^4_c would intersect all other solutions along string world sheets X^2_c. Also this would give rise to a book like structures with 2-D string world sheet representing the back of given book. The physical interpretation of these book like structures remains open in both cases.

   I have proposed that string world sheets as singularities correspond to 2-D folds of space-time surfaces at which the dimension of the quaternionic tangent space degenerates from 4 to 2 [L66] [K8]. This interpretation is consistent with the identification as a book like structure with 2-pages. Also 1-D real and imaginary manifols could be interpreted as folds or equivalently books with 2 pages.

2. Associativity condition for tangent-/normal space is second essential condition and means that tangent - or normal space is quaternionic. The conjecture is that the identification in terms of roots of polynomials guarantees this and one can formulate this as rather convincing argument [L38, L39, L40].

One cannot exclude rational functions and or even real analytic functions in the sense that Taylor coefficients are octonionically real (propotional to octonionic real unit). Number theoretical vision - adelic physics [L42], suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexifiation of octonions. i appears also in the roots defining complex extensions of rationals.

1. In general the zero loci for imaginary or real part are 4-D but the 7-D light-cone δM^8_+ of M^8 with tip at the origin of coordinates is an exception [L37]. At δM^8_+ the octonionic coordinate o is light-like and one can write $o = re$, where 8-D time coordinate and radial coordinate are related by $t = r$ and one has $e = (1 + e_r)/\sqrt{2}$ such that one as $e^2 = e$.

   Polynomial $P(o)$ can be written at δM^8_+ as $P(o) = P(r)e$ and its roots correspond to 6-spheres S^6 represented as surfaces $t_M = t = r_N$, $r_M = \sqrt{r_N^2 - r_E^2} \leq r_N$, $r_E \leq r_N$, where the value of Minkowski time $t = r = r_N$ is a root of $P(r)$ and r_M denotes radial Minkowski coordinate. The points with distance r_M from origin of $t = r_N$ ball of M^4 has as fiber 3-sphere with radius $r = \sqrt{r_N^2 - r_E^2}$. At the boundary of S^3 contracts to a point.

2. These 6-spheres are analogous to 6-D branes in that the 4-D solutions would intersect them in the generic case along 2-D surfaces X^2. The boundaries $r_M = r_N$ of balls belong to the boundary of M^4 light-cone. In this case the intersection would be that of 4-D and 3-D surface, and empty in the generic case (it is however quite not clear whether topological notion of "genericity" applies to octonionic polynomials with very special symmetry properties).

3. The 6-spheres $t_M = r_N$ would be very special. At these 6-spheres the 4-D space-time surfaces X^4 as usual roots of $P(o)$ could meet. Brane picture suggests that the 4-D solutions connect the 6-D branes with different values of r_n.

   The basic assumption has been that particle vertices are 2-D partonic 2-surfaces and light-like 3-D surfaces - partonic orbits identified as boundaries between Minkowskian and Euclidian regions of space-time surface in the induced metric (at least at H level) - meet along their 2-D ends X^2 at these partonic 2-surfaces. This would generalize the vertices of ordinary Feynman diagrams. Obviously this would make the definition of the generalized vertices mathematically elegant and simple.

   Note that this does not require that space-time surfaces X^4 meet along 3-D surfaces at S^6. The interpretation of the times t_n as moments of phase transition like phenomena is suggestive. ZEO based theory of consciousness suggests interpretation as moments for state function reductions analogous to weak measurements ad giving rise to the flow of experienced time.

4. One could perhaps interpret the free selection of 2-D partonic surfaces at the 6-D roots as initial data fixing the 4-D roots of polynomials. This would give precise content to strong form of holography (SH), which is one of the central ideas of TGD and strengthens the 3-D holography coded by ZEO alone in the sense that pairs of 3-surfaces at boundaries of CD define unique preferred extremals. The reduction to 2-D holography would be due to preferred extremal property realizing the huge symplectic symmetries and making $M^8 - H$ duality possible as also classical twistor lift.

   I have also considered the possibility that 2-D string world sheets in M^8 could correspond to intersections $X^4 \cap S^6$? This is not possible since time coordinate t_M constant at the roots and varies at string world sheets.

   Note that the compexification of M^8 (or equivalently octonionic E^8) allows to consider also different variants for the signature of the 6-D roots and hyperbolic spaces would appear for $(\epsilon_1, \epsilon_i, .., \epsilon_8)$, $epsilon_i = \pm 1$ signatures. Their physical interpretation - if any - remains open at this moment.

5. The universal 6-D brane-like solutions S^6_c have also lower-D counterparts. The condition determining X^2 states that the C_c-valued "real" or "imaginary" for the non-vanishing Q_c-valued "real" or "imaginary" for P vanishes. This condition allows universal brane-like solution as a restriction of O_c to M^4_c (that is CD_c) and corresponds to the complexified time=constant hyperplanes defined by the roots $t = r_n$ of P defining "special moments in the life of self" assignable to CD. The condition for reality in R_c sense in turn gives roots of $t = r_n$ a hypersurfaces in M^2_c.

7.4.2 $M^8 - H$ duality at the level of momentum space

$M^8 - H$ duality occurs also at the level of momentum space and has different meaning now.

1. At M^8 level 8-momenta are quaternionic and light-like. The choices of $M^4_L \supset M^2$ for which 8-momentum in M^4_L, are parameterized by CP_2 parameterizing also the choices of tangent or normal spaces of $X^4 \subset M^8$ at space-time level. This maps M^8 light-like momenta to light-like M^4_L momenta and to CP_2 point characterizing the M^4 and depending on 8-momentum. One can introduce CP_2 wave functions expressible in terms of spinor harmonics and generators of of a tensor product of Super-Virasoro algebras.

2. For a fixed choice M^4_T momenta in general time-like and the E^4 component of 8-momentum has value equal to mass squared. E^4 momenta are points of 3-sphere so that $SO(3)$ harmonics with $SO(4)$ symmetry could parametrize the states. The quantum numbers are $M^4_T \supset M^2$ momenta with fixed mass and the two angular momenta with identical values for S^3 harmonics, which correspond to the quantum states of a spherical quantum mechanical rigid body, and are given by the matrix elements $D^j_{m,n}$ $SU(2)$ group elements ($SO(4)$ decomposes to $SU(2)_L) \times SU(2)_R$ acting from left and right).

   This picture suggests what one might call $SO(4) - SU(3)$ duality at the level of momentum space. There would be two descriptions of states: as massless states with $SU(3)$ symmetry and massive states with $SO(4)$ symmetry.

3. What about the space formed by the choices of the space of the light-like 8-momenta? This space is the space for the choices of preferred M^2 and parameterized by the 6-D (symmetric space $G_2/SU(3)$, where $SU(3) \subset G_2$ leaving complex plane M^2 invariant is subgroup of quaternionic automorphic group $G(2)$ leaving octonionic real unit defining the rest system invariant. This space is moduli space for octonionic structures each of which defines family of space-time surfaces. 8-D Lorent transformations produce even more general octonionic structures. The space for the choices of color quantization axes is $SU(3)/U(1) \times U(1)$, the twistor space of CP_2.

Do M^4_L and M^4_T have analogs at the space-time level?

As found, the solutions of octonionic polynomials consisting of 4-D roots and special 6-D roots coming as 6-sphere S^6 s at 7-D light-cone of M^8. The roots at $t = r$ light-cone boundary are given by the roots $r = r_N$ of the polynomial $P(t)$ and correspond to M^4 slices $t_M = r_N, r_M \leq r_N$. At point r_M S^3 fiber as radius $r(S^3) = \sqrt{r_N^2 - r_M^2}$ and contracts to a point at its boundaries.

Could M^4_L and M_T have analogies at the space-time level?

1. The sphere S^3 associated M^4_T could have counterpart at the level of space-time description. The momenta in M^4_T would naturally be mapped to momenta in the section $t = r_n$ in this case the S^3:s of different mass squared values would naturally correspond to S^3:s assignable to the points of the balls $t = r_n$ and code for mass squared value.

   The counterpart of M^4_L should correspond to light-cone boundary but what does CP_2 correspond? Could the pile of S^3 associated with $t = r_n$ correspond also to CP_2. Could this be the case if there is wormhole contact carrying monopole flux at the origin so that the analog for the replacement of 3-sphere at $r_{CP_2} = \infty$ with homologically non-trivial 2-sphere would be realized?

2. Does the 6-sphere as a root polynomial have counterpart in H? The image should be consistent with $M^8 - H$ duality and correspond to a fixed structure depending on the root r_n only. Since S^3 associated with the E^4 momenta reduces to a point for M^4_L, the natural guess is that S^6 reduces to $t = r_n, 0 \leq r_M \leq r_n$ surface in H.

$S^3 - CP_2$ duality

$S^3 - CP_2$ duality at the level of quantum numbers suggest strongly itself. What does this require? One can approach the problem from two different perspectives.

1. The first approach would be that the representations of $SU(3)$ and $SO(4)$ groups somehow correspond to each other: one could speak of $SU(3) - SO(4)$ duality [K91, K104]. The original form of this duality was this. The color symmetries of quark physics at high energies would be dual to the $SO(4) = SU(2)_L \times SU(2)_R$ symmetries of the low energy hadron physics. Since the physical objects are partons and hadrons formed from the one cannot expect the duality to hold true at the level of details for the representations, and the comparison of the representations makes this clear.

2. The second approach relies on the notion of cognitive representation meaning discretization of CP_2 and S^3 and counting of points of cognitive representations providing discretization in terms of M^8 or H points belonging to the extension of rationals considered. In this case it is more natural to talk about $S^3 - CP_2$ duality.

   The basic observation is that the open region $0 \leq r < \infty$ of CP_2 in Eguchi-Hanson coordinates with r labeling 3-spheres $S^3(r)$ with finite radius can be regarded as pile of $S^3(r)$. In discretization one would have discrete pile of these 3-spheres with finite number of points in the extension of rationals. They points of given S^3 could be related by isometries in special cases.

How $S^3 - CP_2$ duality at the level of light-like M^8 momenta could emerge?

1. Consider first the situation in which one chooses $M^4 \supset M^2$ sub-spaces so that momentum projection to it is light-like. For cognitive representation the choices of $M^4 \supset M^2$ correspond to ad discrete set of points of CP_2 and thus points in the pile of S^3 with discrete radii since all

E^4 parts of momenta with fixed length squared to zero in this choice and each E^4 momentum with fixed lengthand thus identifiable as discrete point of S^3 would correspond to one choice. All these choices would give rise to a pile of S^3:s identifiable as set $0 \leq r < \infty$ of CP_2. The number of CP_2 points would be same as total number of points in the pile of discrete S^3s. This is what $S^3 - CP_2$ duality would say.

Remark: The volumes of CP_2 and S^3 with unit radius are $8\pi^2$ and $2\pi^2$ so that ration is rational number.

2. Consider now the situation for M^4_T so that one has non-vanishing M^4 mass squared equal to E^4 mass squared, having discretized values. The E^4 would momenta correspond to points for a pile of discretized S^3 and thus to the points of CP_2 by above argument. One would have $S^3 - CP_2$ correspondence also now as one indeed expects since the two ways to see the situation should be equivalent.

3. In the space of light-like M^8 momenta E^8 momenta could naturally organize into representations of finite discrete subgroups of $SU(2)$ appearing in McKay correspondence with ADE groups: the groups are cyclic groups, dihedral groups, and the isometry groups associated with tetrahedron, octahedron (cube) and icosahedron (dodecahedron) (see `http://tinyurl.com/yyyn9p95`).

4. Could a concrete connection with the inclusion hierarchy of HFFs be based on increasing momentum resolution realized in terms of these groups at sphere S^3. Notice however that for cyclic and dihedral groups the orbits are circles and pairs of circles for dihedral groups so that the discretization looks too simple and is rotationally asymmetric. Discretization should improve as n increases.

   One can of course ask why C_n and D_n with single direction of rotation axes would appear? Could it be that the directions of rotation axis correspond to the directions defined by the vertices of the 5 Platonic solids. Or could the orbits of fixed axis under the 5 Platonic orbits be allowed. Also this looks still too simple.

   Could the discretization labelled by n_{max} be determined by the product of the groups up to n_{max} and define a group with infinite order. One can consider also groups defined by subsets $\{n_1, n_2...n_3\}$ and these a pair of sequences with larger sequence containing the smaller one could perhaps define an inclusion. The groups C_n and D_n allow prime decomposition in obvious manner and it seems enough to include to the product only the groups C_p and D_p, where p is prime as generators so that one would have set $\{p_1, ...p_n\}$ of primes labelling these groups besides the Platonic groups. The extension of rationals used poses a cutoff on the number of groups involved and on the group elements representable since since too high roots of unity resulting in the multiplication of C_{p_i} and D_{p_j} do not belong to the extension.

   At the level of momentum space the hierarchy of finite discrete groups of $SU(2)$ would define the notion measurement resolution. The discrete orbits of $SU(2) \times U(1)$ at S^3 would be analogous to tessellations of sphere S^2 known as Platonic solids at sphere S^2 and appearing in the ADE correspondence assignable to Jones inclusions as description of measurement resolution. This would also explain also why Z_2 coverings of the subgroups of $SO(3)$ appear in ADE sequence.

   This picture is probably not enough for the needs of adelic physics [L42] allowing all extensions of rationals. Besides roots of unity only the roots of small integers $2, 3, 5$ associated with the geometry of Platonic solids would be included in these discretizations. One could interpret these discretizations in terms of subgroups of discrete automorphism groups of quaternions. Also the extensions of rationals are probably needed.

Could $S^3 - CP_2$ duality make sense at space-time level? Consider the space-time analog for the projection of M^8 momenta to fixed M^4_T.

1. Suppose that the 3-surfaces determining the space-time surfaces as algebraic surfaces in $X^4 \subset M^8$ are given at the surfaces $t = r_N, r_M \leq r_N$ and have a 3-D fiber which should be surface in CP_2. On can assign to each point of this ball $S^3(r_M)$ with radius going to zero at $r_M = r_N$. One has pile of $S^3(r_M)$ which corresponds to the region $0 \leq r < \infty$ of CP_2. This set is discretized. Suppose that the discretization is like momentum discretization. If so, the points would correspond to points of CP_2. It is not however clear why the discretization should be so symmetric as in momentum discretization.

2. The initial values could be chosen by choosing discrete set of points in this pile of S^3:s and this would give rise to a discrete set of points of CP_2 fixing tangent or normal plane of X^4 at these points. One should show that the selection of a point of S^6 at each point indeed determines quaternionic tangent or normal plane plane for a given polynomial $P(o)$ in M^8.

   It would seem that this correspondence need not hold true.

7.4.3 M^8-H duality and the two ways to describe particles

The isometry groups for $M^4\times CP_2$ is $P\times SU(3)$ (P for Poincare group). The isometry group for $M^8=M^4\times E^4$ with a fixed choice of M^4 breaks down to $P\times SO(4)$. A further breaking by selection $M^4\subset M^2$ of preferred octonionic complex plane M^2 necessary in the algebraic approach to space-time surfaces $X^4\subset M^8$ brings in preferred rest system and reduces the Poincare symmetry further. At the space-time level the assumption that the tangent space of X^4 contains fixed M^2 or at least integral distribution of $M^2(x)\subset M^4$ is necessary for M^8-H duality [L37].

The representations $SO(4)$ and $SU(3)$ could provide alternative description of physics so that one would have what I have called $SO(4)-SU(3)$ duality [K91]. This duality could manifest in the description of strong interaction physics in terms of hadrons and quarks respectively (conserved vector current hypothesis and partially conserved axial current hypothesis based on $Spin(SO(4))=SU(2)\times SU(2)_R$. The challenge is to understand in more detail this duality. This could allow also to understand better how the two twistor descriptions might relate.

$SO(4)-SU(3)$ duality implies two descriptions for the states and scattering amplitudes.

Option I: One uses projection of 8-momenta to a fixed $M^4_T\supset M^2$.

Option II: One assumes that $M^4_L\supset M^2$ is defines the frame in which quaternionic octonion momentum is parallel to M^4_L: this M^4_L depends on particle state and describes this dependence in terms of wave function in CP_2.

Option I: fixed $M^4_T\supset M^2$

For Option I the description would be in terms of a *fixed* $M^4_T\subset M^8=M^4_T\times E^4$ and $M^2\subset M^4_T$ fixed for both options. For given quaternionic light-like M^8 momentum one would have projection to M^4_T, which is in general massive. E^4 momentum would have same the length squared by light-likeness.

De-localization M^4_T mass squared equal to $p^2(M^4_T)=m^2$ in E^4 can be described in terms of $SO(4)$ harmonics at sphere having $p^2(E^4)=m^2$. This would be the description applied to hadrons and leptons and particles treated as massive particles. Particle mass would be due to the fixed choice of M^4_T. What dictates this choice is an interesting question. An interesting question is how these descriptions relate to QFT Higgs mechanism as (in principle) alternative descriptions: the choice of fixed M^4_T could be seen as analog for the generation of vacuum expectation of Higgs selecting preferred direction in the space of Higgs fields.

Option II: varying $M^4_L\supset M^2$

For Option II the description would use $M^4_L\supset M^2$, which is *not fixed* but chosen so that it contains light-like M^8 momentum. This would give light-like momentum in M^4_L identifiable as quaternionic sub-space of complexified octonions.

1. One could assign to the state wave function function for the choices of M^4 and by quaternionicity of 8-momenta this would correspond to a state in super-conformal representation with vanishing M^4_L mass: CP_2 point would code the information about E^4 component light-like 8-momentum. This description would apply to the partonic description of hadrons in terms of massless quarks and gluons.

2. For this option one could use the product of ordinary M^4 twistors and CP_2 twistors. One challenge would be the generalization of the twistor description to the case of CP_2 twistors.

p-Adic particle massivation and ZEO

The two pictures about description of light-like M^8 momenta do not seem to be quite consistent with the recent view about TGD in which H-harmonics describe massivation of massless particles. What looks like a problem is following.

1. The resulting particles are massive in M^4. But they should be massless in $M^4 \times CP_2$ description. The non-vanishing mass would suggest the correct description in terms of Option I for which the description in terms of E^4 momenta with length equal to mass and thus identifiable as points of S^3. Momentum space wave functions at S^3 - essentially rigid body wave functions given by representation matrices of $SU(2)$ could characterize the states rather than CP_2 harmonic.
2. The description based on CP_2 color partial waves however works and this would favor Option II with vanishing M^4 mass. What goes wrong?

To understand what might be involved, consider p-adic mass calculations.

1. The massivation of physical fermion states includes also the action of super-conformal generators changing the mass. The particles are originally massless and p-adic mass squared is generated thermally and mapped to real mass squared by canonical identification map.

   For CP_2 spinor harmonics mass squared is of order CP_2 mass squared and thus gigantic. Therefore the mass squared is assumed to contain negative tachyonic ground state contribution due to the negative half-odd integer valued conformal weight $h_{vac} < 0$ of vacuum. The origin of this contribution has remained a mystery in p-adic thermodynamics but it makes possible to construct massless states. h_{vac} cancels the spinorial contributions and the contribution from positive conformal weights of super-conformal generators so that the particle states are massless before thermalization. This would conform with the idea of using varying M^4_L and thus CP_2 description.
2. What does the choice of M^4 mean in terms of super-conformal representations? Could the mysterious vacuum conformal weight h_{vac} provide a description for the effect of the needed $SU(3)$ rotation of M^4 from standard orientation on super-conformal representation. The effect would be very simple and in certain sense reversal to the effect of Higgs vacuum expectation value in that it would cancel mass rather than generate it.

   An important prediction would be that heavy states should be absent from the spectrum in the sense that mass squared would be p-adically of order $O(p)$ or $O(p^2)$ (in real sense of order $O(1/p)$ or $O(1/p^2)$). The trick would be that the generation of h_0 as a representation of SU(3) rotation of M^4 makes always the dominating contribution to the mass of the state vanishing.

   Remark: If the canonical identification I mapping the p-adic mass integers to their real numbers is of the simplest form $m = \sum_n x_n p^n \rightarrow I(m) = \sum_n x_n p^{-n}$, it can happen that the image of rational m/n with p-adic norm not larger than 1 represented as p-adic integer by expanding it in powers of p, can be near to the maximal value of p and the mass of the state can be of order CP_2 mass - about 10^{-4} Planck masses. If the canonical identification is defined as $m/n \rightarrow I/(m)/I(n)$ the image of the mass is small for small values of m and n.
3. Unfortunately, it is easy to get convinced that this explanation of h_{vac} is not physically attractive. Identical mass spectra at the level of M^8 and H looks like a natural implication of $M^8 - H$-duality. $SU(3)$ rotation of M^4 in M^8 cannot however preserve the general form of $M^4 \times CP_2$ mass squared spectrum for the M^4 projections of M^8 momenta at level of M^8.

   Remark: For $H = M^4 \times CP_2$ the mass squared in given representation of Super-conformal symmetries is given as a sum of CP_2 mass squared for the spinor harmonic determining the ground state and of a Virasoro contribution proportional to a non-negative integer. The masses are required to independent of electroweak quantum numbers.

One can imagine two further identifications for the origin of h_{vac}.

1. Take seriously the possibility of complex momenta allowed by the complexification of M^8 by commuting imagine unit i and also suggested by the generalization of the twistorialization. The real and imaginary parts of light-like complex 8-momenta $p_8 = p_{8,Re} + ip_{8,Im}$ are orthogonal to each other: $p_{8,Re} \cdot p_{8,Im} = 0$ so that both real and imaginary parts of p_8 are

light-like: $p_{8,Re}^2 = p_{8,Im}^2 = 0$. The M^4 mass squared can be written has $m^2 = m_{Re}^2 - m_{Im}^2$ with $h_{vac} \propto -m_{Im}^2$. The representations of Super-conformal algebra would be labelled by $h_{vac} \propto m_{Im}^2$.

Could the needed wrong sign contribution to CP_2 mass squared mass make sense? CP_2 type extremals having light-like geodesic as M^4 projection are locally identical with CP_2 but because of light-like projection they can be regarded as CP_2 with a hole and thus non-compact. Boundary conditions allow analogs of CP_2 harmonics for which spinor d'Alembertian would have complex eigenvalues.

Does quantum-classical correspondence allow complex momenta: can the classical four-momenta assignable to 6-D Kähler action be complex? The value of Kähler coupling strength allows the action to have complex phase but parts with different phases are not allowed. Could the imaginary part to 4-momentum could come from the CP_2 type extremal with Euclidian signature of metric?

2. Second - not necessarily independent - idea relies on the observation that in TGD one has besides the usual conformal algebra acting on complex coordinate z also its analog acting on the light-like radial coordinate r of light-cone boundary. At light-cone boundary one has both super-symplectic symmetries of $\Delta M_+^4 \times CP_2$ and extension of super-conformal symmetries of sphere S^2 to analogs of conformal symmetries depending on z and r and it seems that one must chose between these two options. Also the extension of ordinary Kac-Moody algebra acts at the light-like orbits of partonic 2-surfaces.

   There are two scaling generators: the usual $L_0 = zd/dz$ and the second generator $L_{0,1} = ird/dr$. For $L_{0,1}$ at light-cone boundary powers of z^n are replaced with $(r/r_0)^{ik} = exp(iku)$, $u = log(r/r_0))$. Could it be that mass squared operator is proportional to $L_0 + L_{0,1}$ having eigenvalues $h = n - k$? The absence of tachyons requires $h \geq 0$. Could k characterize given Super-Virasoro representation? Could $k \geq 0$ serve as an analog of positive energy condition allowing to analytically continue $exp(iku)$ to upper u-plane? How to interpret this continuation?

   The 3-D generalization of super-symplectic symmetries at light-cone boundary and extended Ka-Moody symmetries at partonic 2-surfaces should be possible in some sense. Could the continuation to the upper u-plane correspond to the continuation of the extended conformal symmetries from light-cone boundary to future light-one and from light-partonic 2-surfaces to space-time interior?

Why p-adic massivation should occur at all? Here ZEO comes in rescue.

1. In ZEO one can have superposition of states with different 4-momenta, mass values and also other charges: this does not break conservation laws. How to fix M^4 in this case? One cannot do it separately for the states in superposition since they have different masses. The most natural choices is as the M^4 associated with the dominating contribution to the zero energy state. The outcome would be thermal massivation described excellently by p-adic thermodynamics [K52]. Recently a considerable increase in the understanding of hadron and weak boson masses took place [L79].

2. In ZEO quantum theory is square root of thermodynamics in a well-defined formal sense, and one can indeed assign to p-adic partition function a complex square root as a genuine zero energy state. Since mass varies, one must describe the presence of higher mass excitations in zero energy state as particles in M^4 assigned with the dominating part of the state so that the observed particle mass squared is essentially p-adic thermal expectation value over thermal excitations. p-Adic thermodynamics would thus describe the fact that the choice of M_L^4 cannot not ideal in ZEO and massivation would be possible only in ZEO.

3. Current quarks and constituent quarks are basic notions of hadron physics. Constituent quarks with rather large masses appear in the low energy description of hadrons and current quarks in high energy description of hadronic reactions. That both notions work looks rather paradoxical. Could massive quarks correspond to M_T picture and current quarks to M_L^4 picture but with p-adic thermodynamics forced by the superposition of mass eigenstates with different masses.

The massivation of ordinary massless fermion involves mixing of fermion chiralities. This means that the $SU(3)$ rotation determined by the dominating component in zero energy state must induce this mixing. This should be understood.

7.4.4 $M^8 - H$ duality and consciousness

$M^8 - H$ duality is one of the key ideas of TGD and one can ask whether it has implications for TGD inspired theory of consciousness and it indeed forces to challenge the recent ZEO based view about consciousness [L46] .

Objections against ZEO based theory of consciousness

Consider first objections against ZEO based view about consciousness.

1. ZEO (zero energy ontology) based view about conscious entity can be regarded as a sequence of "small" state function reductions (SSRs) identifiable as analogs of so called weak measurements at the active boundary of causal diamond (CD) receding reduction by reduction farther away from the passive boundary, which is unchanged as also the members of state pairs at it. One can say that weak measurements commute with the observables, whose eigenstates the states at passive boundary are. This asymmetry assigns arrow of time to the self having CD as embedding space correlate. "Big" state function reductions (BSRs) would change the roles of boundaries of CD and the arrow of time. The interpretation is as death and re-incarnation of the conscious entity with opposite arrow of time.

   The question is whether quantum classical correspondence (QCC) could allow to say something about the time intervals between subsequent values of temporal distance between weak state function reductions.

2. The questionable aspect of this view is that $t_M = constant$ sections look intuitively more natural as seats of quantum states than light-cone boundaries forming part of CD boundaries. The boundaries of CD are however favoured by the huge symplectic symmetries assignable to the boundary of M^4 light-cone with points replaced with CP_2 at level of H. These symmetries are crucial or the existence of the geometry of WCW ("world of classical worlds").

3. Second objection is that the size of CD increases steadily: this nice from the point of view of cosmology but the idea that CD as correlate for a conscious entity increases from CP_2 size to cosmological scales looks rather weird. For instance, the average energy of the state assignable to either boundary of CD would increase. Since zero energy state is a superposition of states with different energies classical conservation law for energy does not prevent this [L75]: essentially quantal effect due to the fact that the zero energy states are not exact eigenstates of energy could be in question. In BSRs the energy would gradually increase. Admittedly this looks strange and one must be keen for finding more conventional options.

4. Third objection is that re-incarnated self would not have any "childhood" since CD would increase all the time.

One can ask whether $M^8 - H$ duality and this braney picture has implications for ZEO based theory of consciousness. Certain aspects of $M^8 - H$ duality indeed challenge the recent view about consciousness based on ZEO (zero energy ontology) and ZEO itself.

1. The moments $t = r_n$ defining the 6-branes correspond classically to special moments for which phase transition like phenomena occur. Could $t = r_n$ have a special role in consciousness theory?

   (a) For some SSRs the increase of the size of CD reveals new $t = r_n$ plane inside CD. One can argue that these SSRS define very special events in the life of self. This would not modify the original ZEO considerably but could give a classical signature for how many ver special moments of consciousness have occurred: the number of the roots of P would be a measure for the lifetime of self and there would be the largest root after which BSR would occur.

   (b) Second possibility is more radical. One could one think of replacing CD with single truncated future- or past-directed light-cone containing the 6-D universal roots of P up to some r_n defining the upper boundary of the truncated cone? Could $t = r_n$ define a sequence of moments of consciousness? To me it looks more natural to assume that they are associated with very special moments of consciousness.

2. For both options SSRs increase the number of roots r_n inside CD/truncated light-one gradually and thus its size? When all roots of $P(o)$ would have been measured - meaning that the largest value r_{max} of r_n is reached -, BSR would be unavoidable.

   BSR could replace $P(o)$ with $P_1(r_1 - o)$: r_1 must be real and one should have $r_1 > r_{max}$. The new CD/truncated light-cone would be in opposite direction and time evolution would be reversed. Note that the new CD could have much smaller size size if it contains only the smallest root r_0. One important modification of ZEO becomes indeed possible. The size of CD after BSR could be much smaller than before it. This would mean that the re-incarnated self would have "childhood" rather than beginning its life at the age of previous self - kind of fresh start wiping the slate clean.

   One can consider also a less radical BSR preserving the arrow of time and replacing the polynomial with a new one, say a polynomial having higher degree (certainly in statistical sense so that algebraic complexity would increase).

Could one give up the notion of CD?

A possible alternative view could be that one the boundaries of CD are replaced by a pair of two $t = r_N$ snapshots $t = r_0$ and $t = r_N$. Or at least that these surfaces somehow serve as correlates for mental images. The theory might allow reformulation also in this case, and I have actually used this formulation in popular lectures since it is easier to understand by laymen.

1. Single truncated light-cone, whose size would increase in each SSR would be present now since the spheres correspond to balls of radius r_n at times r_n. If $r_0 = 0$, which is the case for $P(o) \propto o$, the tip of the light-cone boundary is one root. One cannot avoid association with big bang cosmology. For $P(0) \neq r_0$ the first conscious moment of the cosmology corresponds to $t = r_0$. One can wonder whether the emergence of consciousness in various scales could be described in terms of the varying value of the smallest root r_0 of $P(o)$.

   If one allows BSR:s this picture differs from the earlier one in that CDs are replaced with alternation of light-cones with opposite directions and their intersections would define CD.

2. For this option the preferred values of t for SSRs would naturally correspond to the roots of the polynomial defining $X^4 \subset M^8$. Moments of consciousness as state function reductions would be due to collisions of 4-D space-time surfaces X^4 with 6-D branes! They would replace the sequence of scaled CD sizes. CD could be replaced with light-one and with the increasing sequence $(r_0, ...r_n)$ of roots defining the ticks of clock and having positive and negative energy states at the boundaries r_0 and r_n.

3. What could be the interpretation for BSRs representing death of a conscious entity in the new variant of ZEO? Why the arrow of time would change? Could it be because there are no further roots of $P(o)$? The number of roots of $P(o)$ would give the number of small state function reductions?

   What would happen to $P(o)$ in BSR? The vision about algebraic evolution as increase of the dimension for the extension of rationals would suggest that the degree of $P(o)$ increases as also the number of roots if all complex roots are allowed. Could the evolution continue in the same direction or would it start to shift the part of boundary corresponding to the lowest root in opposite direction of time. Now one would have more roots and more algebraic complexity so that evolutionary step would occur.

   In the time reversal one would have naturally $t_{max} \geq r_{n_{max}}$ for the new polynomial $P(t - t_{max})$ having $r_{n_{max}}$ as its smallest root. The light-cone in M^8 with tip at $t = t_{max}$ would be in opposite direction now and also the slices $t - t_{max} = r'_n$ would increase in opposite direction! One would have two light-cones with opposite directions and the $t = r_n$ sections would replace boundaries of CDs. The reborn conscious entity would start from the lowest root so that also it would experience childhood.

This option could solve the argued problems of the previous scenario and give concrete connection with the classical physics in accordance with QCC. On the other hand, a minimal modification of original scenario combined with $M^8 - H$ duality with moments $t = r_n$ as special moments in the life of conscious entity allows also to solve these problems if the active boundary of CD is interpreted as boundary beyond which classical signals cannot contribute to perceptions.

What could be the minimal modification of ZEO based view about consciousness?

What would be the minimal modification of the earlier picture? Could one *assume* that CDs serve as embedding space correlates for the perceptive field?

1. Zero energy states would be defined as before that is in terms of 3-surfaces at boundaries of CD: this would allow a realization of huge symmetries of WCW and the active boundary A of CD would define the boundary of the region from which self can receive classical information about environment. The passive boundary P of CD would define the boundary of the region providing classical information about the state of self. Also now BSR would mean death and reincarnation with an opposite arrow of time. Now however CD would shrink in BSR before starting to grow in opposite time direction. Conscious entity would have "childhood".

2. If the geometry of CD were fixed, the size scale of the $t = r_n$ balls of M^4 would first increase and then start to decrease and contract to a point eventually at the tip of CD. One must however remember that the size of $t = r_n$ planes increases all the time as also the size of CD in the sequences of SSRs. Moments $t = r_n$ could represent special moments in the life of conscious entity taking place in SSRs in which $t = r_n$ hyperplane emerges inside CD with increased size. The recent surprising findings challenging the Bohrian view about quantum jumps [L62] can be understood in this picture [L62].

3. $t = r_n$ planes could also serve as correlates for memories. As CD increases at active boundary new events as $t = r_n$ planes would take place and give rise to memories. The states at $t = r_n$ planes are analogous to seats of boundary conditions in strong holography and the states at these planes might change in state function reductions - this would conform with the observations that our memories are not absolute.

To sum up, the original view about ZEO seems to be essentially correct. The introduction of moments $t = r_n$ as special moments in the life of self looks highly attractive as also the possibility of wiping the slate clear by reduction of the size of CD in BSR.

7.5 Could standard view about twistors work at space-time level after all?

While asking what super-twistors in TGD might be, I became critical about the recent view concerning what I have called geometric twistor space of M^4 identified as $M^4 \times S^2$ rather than CP_3 with hyperbolic metric. The basic motivations for the identification come from M^8 picture in which there is number theoretical breaking of Poincare and Lorentz symmetries. Second motivation was that M^4_{conf} - the conformally compactified M^4 - identified as group $U(2)$ [B9] (see `http://tinyurl.com/y35k5wwo`) assigned as base space to the standard twistor space CP_3 of M^4, and having metric signature (3,-3) is compact and is stated to have metric defined only modulo conformal equivalence class.

As found in the previous section, TGD strongly suggests that M^4 in $H = M^4 \times CP_2$ should be replaced with hyperbolic variant of CP_2, and it seems to me that these spaces are not identical. Amusingly, $U(2)$ and CP_2 are fiber and base in the representation of $SU(3)$ as fiber space so that the their identification does not seem plausible.

On can however ask whether the selection of a representative metric from the conformal equivalence class could be seen as breaking of the scaling invariance implied also by ZEO introducing the hierarchy of CDs in M^8. Could it be enough to have M^4 only at the level of M^8 and conformally compactified M^4 at the level of H? Should one have $H = cd_{conf} \times CP_2$? What cd_{conf} would be: is it hyperbolic variant of CP_2?

7.5.1 Getting critical

The only way to make progress is to become very critical now and then. These moments of almost despair usually give rise to a progress. At this time I got very critical about the TGD inspired identification of twistor spaces of M^4 and CP_2 and their properties.

Getting critical about geometric twistor space of M^4

Let us first discuss the recent picture and how to modify it so that it is consistent with the hierarchy of CDs. The key idea is that the twistor space and its base space represents CD so that one obtains scale hierarchy of twistor spaces as a realization of broken scale invariance giving rise to the p-adic length scale hierarchy.

1. I have identified the twistor space of M^4 simply as $T(M^4) = M^4 \times S^2$. The interpretation would be at the level of octonions as a product of M^4 and choices of M^2 as preferred complex sub-space of octonions with S^2 parameterizing the directions of spin quantization axes. Real octonion axis would correspond to time coordinate. One could talk about the space of of light-like directions. Light-like vector indeed defines M^2. This view could be defended by the breaking of both translation and Lorentz invariance in the octonionic approach due to the choice of M^2 and by the fact that it seems to work.

   Remark: $M^8 = M^4 \times E^4$ is complexified to M_c^8 by adding a commuting imaginary unit i appearing in the extensions of rationals and ordinary M^8 represents its particular sub-space. Also in twistor approach one uses often complexified M^4.

2. The objection is that it is ordinary twistor space identifiable as CP_3 with (3,-3) signature of metric is what works in the construction of twistorial amplitudes. CP_3 has metric as compact space and coset space. Could this choice of twistor space make sense after all as geometric twistor space?

   Here one must pause and recall that the original key idea was that Poincare invariance is symmetry of TGD for $X^4 \subset M^4 \times CP_2$. Now Poincare symmetry has been transformed to a symmetry acting at the level of M^8 in the moduli space of octonion structures defined by the choice of the direction of octonionic real axis reducing Poincare group to $T \times SO(3)$ consisting of time translations and rotations. Fixing of M^2 reducrs the group to $T \times SO(2)$ and twistor space can be seen as the space for selections of quantization axis of energy and spin.

3. But what about the space H? The first guess is $H = M^4_{conf} \times CP_2$. According to [B9] (see `http://tinyurl.com/y35k5wwo`) one has $M^4_{conf} = U(2)$ such that $U(1)$ factor is time- like and $SU(2)$ factor is space-like. One could understand $M^4_{conf} = U(2)$ as resulting by addition and identification of metrically 2-D light-cone boundaries at $t = \pm\infty$. This is topologically like compactifying E^3 to S^3 and gluing the ends of cylinder $S^3 \times D^1$ together to the $S^3 \times S^1$.

   The conformally compactified Minkowski space M^4_{conf} should be analogous to base space of CP_3 regarded as bundle with fiber S^2. The problem is that one cannot imagine an analog of fiber bundle structure in CP_3 having $U(2)$ as base. The identification $H = M^4_{conf} \times CP_2$ does not make sense.

4. In ZEO based breaking of scaling symmetry it is CD that should be mapped to the analog of M^4_{conf} - call it cd_{conf}. The only candidate is $cd_{conf} = CP_2$ with one hypercomplex coordinate. To understand why one can start from the following picture. The light-like boundaries of CD are metrically equivalent to spheres. The light-like boundaries at $t = \pm\infty$ are identified as in the case of M^4_{conf}. In the case of CP_2 one has 3 homologically trivial spheres defining coordinate patches. This suggests that cd_{conf} is simply CP_2 with second complex coordinate made hypercomplex. M^4 and E^4 differ only by the signature and so would do cd_{conf} and CP_2.

   The twistor spheres of CP_3 associated with points of M^4 intersect at point if the points differ by light-like vector so that one has and singular bundle structure. This structure should have analog for the compactification of CD. CP_3 has also bundle structure $CP_3 \rightarrow CP_2$. The S^2 fibers and base are homologically non-trivial and complex analogs of mutually orthogonal line and plane and intersect at single point. This defines the desired singular bundle structure via the assignment of S^2 to each point of CP_2.

The M^4 points must belong to the interior of cd and this poses constraints on the distance of M^4 points from the tips of cd. One expects similar hierarchy of cds at the level of momentum space.

5. In this picture $M^4_{conf} = U(2)$ could be interpreted as a base space for the space of CDs with fixed direction of time axis identified as direction of octonionic real axis associated with various points of M^4 and therefore of M^4_{conf}. For Euclidian signature one would have base and fiber of the automorphism sub-group $SU(3)$ regarded as $U(2)$ bundle over CP_2: now one would have CP_2 bundle over $U(2)$. This is perhaps not an accident, and one can ask whether these spaces could be interpreted as representing local trivialization of $SU(3)$ as $U(2) \times CP_2$. This would give to metric cross terms between $U(2)$ and CP_2.

6. The proposed identification can be tested by looking whether it generalizes. What the twistor space for entire M^8 would be? $cd = CD_4$ is replaced with CD_8 and the discussion of the preceding chapter demonstrated that the only possible identification of the twistor space is now is as the 12-D hyperbolic variant of HP_3 whereas $CD_{8,conf}$ would correspond to 8-D hyperbolic variant of HP_2 analogous to hyperbolic variant of CP_2.

The outcome of these considerations is surprising.

1. One would have $T(H) = CP_3 \times F$ and $H = CP_{2,H} \times CP_2$ where $CP_{2,H}$ has hyperbolic metric with metric signature $(1, -3)$ having M^4 as tangent space so that the earlier picture can be understood as an approximation. This would reduce the construction of preferred extremals of 6-D Kähler action in $T(H)$ to a construction of polynomial holomorphic surfaces and also the minimal surfaces with singularities at string world sheets should result as bundle projection. Since $M^8 - H$ duality must respect algebraic dynamics the maximal degree of the polynomials involved must be same as the degree of the octonionic polynomial in M^8.

2. The hyperbolic variant Kähler form and also spinor connection of hyperbolic CP_2 brings in new physics beyond standard model. This Kähler form would serve as the analog of Kähler form assigned to M^4 earlier, and suggested to explain the observed CP breaking effects and matter antimatter asymmetry for which there are two explanations [L74].

Some comments about the Minkowskian signature of the hyperbolic counterparts of CP_3 and CP_2 are in order.

1. Why the metric of CP_3 could not be Euclidian just as the metric of F? The basic objection is that propagation of fields is not possible in Euclidian signature and one completely loses the earlier picture provided by $M^4 \times CP_2$. The algebraic dynamics in M^8 picture can hardly replace it.

2. The map assigning to the point M^4 a point of CP_3 involves Minkowskian sigma matrices but it seems that the Minkowskian metric of CP_3 is not explicitly involved in the construction of scattering amplitudes. Note however that the antisymmetric bi-spinor metric for the spin 1/2 representation of Lorentz group and its conjugate bring in the signature. $U(2,2)$ as representation of conformal symmetries suggests $(2,2)$ signature for 8-D complex twistor space with 2+2 complex coordinates representing twistors.

   The signature of CP_3 metric is not explicitly visible in the construction of twistor amplitudes but analytic continuations are carried out routinely. One has also complexified M^4 and M^8 and one could argue that the problems disappear. In the geometric situation the signatures of the subspaces differ dramatically. As already found, analytic continuation could allow to define the variants of twistor spaces elegantly by replacing a complex coordinate with a hyperbolic one.

   Remark: For E^4 CP_3 is Euclidian and if one has $E^4_{conf} = U(2)$, one could think of replacing the Cartesian product of twistor spaces with $SU(3)$ group having $M^4_{conf} = U(2)$ as fiber and CP_2 as base. The metric of $SU(3)$ appearing as subgroup of quaternionic automorphisms leaving $M^4 \subset M^8$ invariant would decompose to a sum of M^4_{conf} metric and CP_2 metric plus cross terms representing correlations between the metrics of M^4_{conf} and CP_2. This is probably mere accident.

$M^8 - H$ duality and twistor space counterparts of space-time surfaces

It seems that by identifying $CP_{3,h}$ as the twistor space of M^4, one could develop $M^8 - H$ duality to a surprisingly detailed level from the conditions that the dimensional reduction guaranteed by the identification of the twistor spheres takes place and the extensions of rationals associated with the polynomials defining the space-time surfaces at M^8- and twistor space sides are the same. The reason is that minimal surface conditions reduce to holomorphy meaning algebraic conditions involving first partial derivatives in analogy with algebraic conditions at M^8 side but involving no derivatives.

1. The simplest identification of twistor spheres is by $z_1 = z_2$ for the complex coordinates of the spheres. One can consider replacing z_i by its Möbius transform but by a coordinate change the condition reduces to $z_1 = z_2$.
2. At M^8 side one has either $RE(P) = 0$ or $IM(P) = 0$ for octonionic polynomial obtained as continuation of a real polynomial P with rational coefficients giving 4 conditions (RE/IM denotes real/imaginary part in quaternionic sense). The condition guarantees that tangent/normal space is associative.

   Since quaternion can be decomposed to a sum of two complex numbers: $q = z_1+Jz_2$ $RE(P) = 0$ correspond to the conditions $Re(RE(P)) = 0$ and $Im(RE(P)) = 0$. $IM(P) = 0$ in turn reduces to the conditions $Re(IM(P)) = 0$ and $Im(IM(P)) = 0$.
3. The extensions of rationals defined by these polynomial conditions must be the same as at the octonionic side. Also algebraic points must be mapped to algebraic points so that cognitive representations are mapped to cognitive representations. The counterparts of both $RE(P) = 0$ and $IM(P) = 0$ should be satisfied for the polynomials at twistor side defining the same extension of rationals.
4. $M^8 - H$ duality must map the complex coordinates $z_{11} = Re(RE)$ and $z_{12} = Im(RE)$ ($z_{21} = Re(IM)$ and $z_{22} = Im(IM)$) at M^8 side to complex coordinates u_{i1} and u_{i2} with $u_{i1}(0) = 0$ and $u_{i2}(0) = 0$ for $i = 1$ or $i = 2$, at twistor side.

   Roots must be mapped to roots in the same extension of rationals, and no new roots are allowed at the twistor side. Hence the map must be linear: $u_{i1} = a_iz_{i1}+b_iz_{i2}$ and $u_{i2} = c_iz_{i1}+d_iz_{i2}$ so that the map for given value of i is characterized by SL(2,Q) matrix $(a_i, b_i; c_i, d_i)$.
5. These conditions do not yet specify the choices of the coordinates (u_{i1}, u_{i2}) at twistor side. At CP_2 side the complex coordinates would naturally correspond to Eguchi-Hanson complex coordinates (w_1, w_2) determined apart from color $SU(3)$ rotation as a counterpart of $SU(3)$ as sub-group of automorphisms of octonions.

   If the base space of the twistor space $CP_{3,h}$ of M^4 is identified as $CP_{2,h}$, the hyper-complex counterpart of CP_2, the analogs of complex coordinates would be (w_3, w_4) with w_3 hypercomplex and w_4 complex. A priori one could select the pair (u_{i1}, u_{i2}) as any pair $(w_{k(i)}, w_{l(i)})$, $k(i) \neq l(i)$. These choices should give different kinds of extremals: such as CP_2 type extremals, string like objects, massless extremals, and their deformations.

String world sheet singularitees and world-line singularities as their light-like boundaries at the light-like orbits of partonic 2-surfaces are conjectured to characterize preferred extremals as surfaces of H at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom so that the extremal is not simultaneously an extremal of both Kähler action and volume term as elsewhere. What could be the counteparts of these surfaces in M^8?

1. The interpretation of the pre-images of these singularities in M^8 should be number theoretic and related to the identification of quaternionic imaginary units. One must specify two non-parallel octonionic imaginary units e^1 and e^2 to determine the third one as their cross product $e^3 = e^1 \times e^2$. If e^1 and e^2 are parallel at a point of octonionic surface, the cross product vanishes and the dimension of the quaternionic tangent/normal space reduces from $D = 4$ to $D = 2$.
2. Could string world sheets/partonic 2-surfaces be images of 2-D surfaces in M^8 at which this takes place? The parallelity of the tangent/normal vectors defining imaginary units e_i, $i = 1, 2$ states that the component of e_2 orthogonal to e_1 vanishes. This indeed gives 2 conditions in the space of quaternionic units. Effectively the 4-D space-time surface would degenerate into

2-D at string world sheets and partonic 2-surfacesa as their duals. Note that this condition makes sense in both Euclidian and Minkowskian regions.

3. Partonic orbits in turn would correspond surfaces at which the dimension reduces to D=3 by light-likeness - this condition involves signature in an essential manner - and string world sheets would have 1-D boundaries at partonic orbits.

Getting critical about implicit assumptions related to the twistor space of CP_2

One can also criticize the earlier picture about implicit assumptions related the twistor spaces of CP_2.

1. The possibly singular decomposition of F to a product of S^2 and CP_2 would has a description similar to that for CP_3. One could assign to each point of CP_2 base homologically non-trivial sphere intersecting it orthogonally.

2. I have assumed that the twistor space $T(CP_2) = F = SU(3)/U(1) \times U(1)$ allows Kaluza-Klein type metric meaning that the metric decomposes to a sum of the metrics assignable to the base CP_2 and fiber S^2 plus cross terms representing interaction between these degrees of freedom. It is easy to check that this assumption holds true for Hopf fibration $S^3 \rightarrow S^2$ having circle $U(1)$ as fiber (see `http://tinyurl.com/qbvktsx`). If Kaluza-Klein picture holds true, the metric of F would decompose to a sum of CP_2 metric and S^2 metric plus cross terms representing correlations between the metrics of CP_2 and S^2.

3. One should demonstrate that $F = SU(3)/U(1) \times U(1)$ has metric with the expected Kaluza-Klein property. One can represent $SU(3)$ matrices as products XYZ of 3 matrices. X represents a point of base space CP_2 as matrix, Y represents the point of the fiber $S^2 = U(2)/U(1) \times U(1)$ of F in similar manner as $U(2)$ matrix, and the Z represents $U(1) \times U(1)$ element as diagonal matrix [B9](see `http://tinyurl.com/y6c3pp2g`).

   By dropping $U(1) \times U(1)$ matrix one obtains a coordinatization of F. To get the line element of F in these coordinates one could put the coordinate differentials of $U(1) \times U(1)$ to zero in an expression of $SU(3)$ line element. This should leave sum of the metrics of CP_2 and S^2 with constant scales plus cross terms. One might guess that the left- and righ-invariance of the $SU(3)$ metric under $SU(3)$ implies KK property.

Also CP_3 should have the KK structure if one wants to realize the breaking of scaling invariance as a selection of the scale of the conformally compactified M^4. In absence of KK structure the space-time surface would depend parametrically on the point of the twistor sphere S^2.

7.5.2 The nice results of the earlier approach to M^4 twistorialization

The basic nice results of the earlier picture should survive in the new picture.

1. Central for the entire approach is twistor lift of TGD replacing space-time surfaces with 6-D surfaces in 12-D $T(M^4) \times T(CP_2)$ having space-time surfaces as base and twistor sphere S^2 as fiber. Dimensional reduction identifying twistor spheres of $T(M^4)$ an $T(CP_2)$ and makes these degrees of freedom non-dynamical.

2. Dimensionally reduced action 6-D Kähler action is sum of 4-D Kähler action and a volume term coming from S^2 contribution to the induced Kähler form. On interpretation is as a generalization of Maxwell action for point like charge by making particle a 3-surface.

   The interpretation of volume term is in terms of cosmological constant. I have proposed that a hierarchy of length scale dependent cosmological constants emerges. The hierarchy of cosmological constants would define the running length scale in coupling constant evolution and would correspond to a hierarchy of preferred p-aic length scales with preferred p-adic primes identified as ramified primes of extension of rationals.

3. The twistor spheres associated $M^4 \times S^2$ and F were assumed to have same radii and most naturally same Euclidian signature: this looks very nice since there would be only single fundamental length equal to CP_2 radius determining the radius of its twistor sphere. The vision to be discussed would be different. There would be no fundamental scale and length

scales would emerge through the length scale hierarchy assignable to CDs in M^8 and mapped to length scales for twistor spaces.

The identification of twistor spheres with same radius would give only single value of cosmological constant and the problem of understanding the huge discrepancy between empirical value and its naïve estimate would remain. I have argued that the Kähler forms and metrics of the two twistor spheres can be rotated with respect to each other so that the induced metric and Kähler form are rotated with respect to each other, and the magnetic energy density assignable to the sum of the induced Kähler forms is not maximal.

The definition of Kähler forms involving preferred coordinate frame would gives rise to symmetry breaking. The essential element is interference of real Kähler forms. If the signatures of twistor spheres were opposite, the Kähler forms differ by imaginary unit and the interference would not be possible.

Interference could give rise to a hierarchy of values of cosmological constant emerging as coefficient of the Kähler magnetic action assignable to $S^2(X^4)$ and predict length scale dependent value of cosmological constant and resolve the basic problem related to the extremely small value of cosmological constant.

4. One could criticize the allowance of relative rotation as adhoc: note that the resulting cosmological constant becomes a function depending on S^2 point. For instance, does the rotation really produce preferred extremals as minimal surfaces extremizing also Kähler action except at string world sheets? Each point of S^2 would correspond to space-time surface X^4 with different value of cosmological constant appearing as a parameter. Moreover, non-trivial relative rotation spoils the covariant constancy and $J^2(S^2) = -g(S^2)$ property for the S^2 part of Kähler form, and that this does not conform with the very idea of twistor space.

5. One nice implication would be that space-time surfaces would be minimal surfaces apart from 2-D string world sheet singularities at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom. One can also consider the possibility that the minimal surfaces correspond to surfaces give as roots of 3 polynomials of hypercomplex coordinate of M^2 and remaining complex coordinates.

   Minimal surface property would be direct translation of masslessness and conform with the twistor view. Singular surfaces would represent analogs of Abelian currents. The universal dynamics for minimal surfaces would be a counterpart for the quantum criticality. At M^8 level the preferred complex plane M^2 of complexified octonions would represent the singular string world sheets and would be forced by number theory.

   Masslessness would be realized as generalized holomorphy (allowing hyper-complexity in M^2 plane) as proposed in the original twistor approach but replacing holomorphic fields in twistor space with 6-D twistor spaces realized as holomorphic 6-surfaces.

7.5.3 ZEO and twistorialization as ways to introduce scales in M^8 physics

M^8 physics as such has no scales. One motivation for ZEO is that it brings in the scales as sizes of causal diamonds (CDs).

ZEO generates scales in M^8 physics

Scales are certainly present in physics and must be present also in TGD Universe.

1. In TGD Universe CP_2 scale plays the role of fundamental length scale, there is also the length scale defined by cosmological constant and the geometric mean of these two length scales defining a scale of order 10^{-4} meters emerging in the earlier picture and suggesting a biological interpretation.

   The fact that conformal inversion $m^k \to R^2 m^k/a^2$, $a^2 = m^k m_k$ is a conformal transformation mapping hyperboloids with $a \geq R$ and $a \leq R$ to each other, suggests that one can relate CP_2 scale and cosmological scale defined by Λ by inversion so that cell length scale would define one possible radius of cd_{conf}.

2. In fact, if one has $R(cd_{conf}) = x \times R(CP_2)$ one obtains by repeated inversions a hierarchy $R(k) = x^k R$ and for $x = \sqrt{p}$ one obtains p-adic length scale hierarchy coming as powers of $\sqrt{p}$,

which can be also negative. This suggests a connection with p-adic length scale hypothesis and connections between long length scale and short length scale physics: they could be related by inversion. One could perhaps see Universe as a kind of Leibnizian monadic system in which monads reflect each other with respect to hyperbolic surfaces $a = constant$. This would conform with the holography.

3. Without additional assumptions there is a complete scaling invariance at the level of M^8. The scales could come from the choice of 8-D causal diamond CD_8 as intersection of 8-D future and past directed light-cones inducing choice of cd in M^4. CD serves as a correlate for the perceptive field of a conscious entity in TGD inspired theory of consciousness and is crucial element of zero energy ontology (ZEO) allowing to solve the basic problem of quantum measurement theory.

Twistorial description of CDs

Could the map of the surfaces of 4-surfaces of M^8 to $cd_{conf} \times CP_2$ by a modification of $M^8 - H$ correspondence allow to describe these scales? If so, compactification via twistorialization and $M^8 - H$ correspondence would be the manner to describe these scales as something emergent rather than fundamental.

1. The simplest option is that the scale of cd_{conf} corresponds to that of CD_8 and CD_4. One should also understand what CP_2 scale corresponds. The simplest option is that CP_2 scale defines just length unit since it is difficult to imagine how this scale could appear at M^8 level. cd_{conf} scale squared would be multiple or CP_2 scale squared, say prime multiple of it, and assignable to ramified primes of extension of rationals. Inversions would produce further scales. Inversion would allow kind of hologram like representation of physics in long length scales in arbitrary short length scales and vice versa.
2. The compactness of cd_{conf} corresponds to periodic time assignable to over-critical cosmologies starting with big bang and ending with big crunch. Also CD brings in mind over-critical cosmology, and one can argue that the dynamics at the level of cd_{conf} reflects the dynamics of ZEO at the level of M^8.

Modification of H and $M^8 - H$ correspondence

It is often said that the metric of M^4_{conf} is defined only modulo conformal scaling factor. This would reflect projectivity. One can however endow projective space CP_3 with a metric with isometry group $SU(2,2)$ and the fixing of the metric is like gauge choice by choosing representative in the projective equivalence class. Thus CP_3 with signature (3,-3) might perhaps define geometric twistor space with base cd_{conf} rather than M^4_{conf} very much like the twistor space $T(CP_2) = F = SU(3)/U(1) \times U(1)$ at the level. Second projection would be to M^4 and map twistor sphere to a point of M^4. The latter bundle structure would be singular since for points of M^4 with light-like separation the twistor spheres have a common point: this is an essential feature in the construction of twistor amplitudes.

New picture requires a modification of the view about H and about $M^8 - H$ correspondence.

1. H would be replaced with $cd_{conf} \times CP_2$ and the corresponding twistor space with $CP_3 \times F$. $M^8 - H$ duality involves the decomposition $M^2 \subset M^4 \subset M^8 = M^4 \times CP_2$, where M^4 is quaternionic sub-space containing preferred place M^2. The tangent or normal space of X^4 would be characterized by a point of CP_2 and would be mapped to a point of CP_2 and the point of CP_2 - or rather point plus the space S^2 or light-like vectors characterizing the choices of M^2 - would mapped to the twistor sphere S^2 of CP_3 by the standard formulas.

   $S^2(cd_{conf})$ would correspond to the choices of the direction of preferred octonionic imaginary unit fixing M^2 as quantization axis of spin and $S^2(CP_2)$ would correspond to the choice of isospin quantization axis: the quantization axis for color hyperspin would be fixed by the choice of quaternionic $M^4 \subset M^8$. Hence one would have a nice information theoretic interpretation.
2. The M^4 point mapped to twistor sphere $S^2(CP_3)$ would be projected to a point of cd_{conf} and define $M^8 - H$ correspondence at the level of M^4. This would define compactification

and associate two scales with it. Only the ratio $R(cd_{conf})/R(CP_2)$ matters by the scaling invariance at M^8 level and one can just fix the scale assignable to $T(CP_2)$ and call it CP_2 length scale.

One should have a concrete construction for the hyperbolic variants of CP_n.

1. One can represent Minkowski space and its variants with varying signatures as sub-spaces of complexified quaternions, and it would seem that the structure of sub-space must be lifted to the level of the twistor space. One could imagine variants of projective spaces CP_n, $n = 2, 3$ as and HP_n, $n = 2, 3$. They would be obtained by multiplying imaginary quaternionic unit I_k with the imaginary unit i commuting with quaternionic units. If the quaternions λ involved with the projectivization $(q_1, ..., q_n) \equiv \lambda(q_1, ..., q_n)$ are ordinary quaternions, the multiplication respects the signature of the subspace. By non-commutativity of quaternions one can talk about left- and right projective spaces.
2. One would have extremely close correspondence between M^4 and CP_2 degrees of freedom reflecting the $M^8 - H$ correspondence. The projection $CP_3 \rightarrow CP_2$ for E^4 would be replaced with the projection for the hyperbolic analogs of these spaces in the case of M^4. The twistor space of M^4 identified as hyperbolic variant of CP_3 would give hyperbolic variant of CP_2 as conformally compactified cd. The flag manifold $F = SU(3)/U(1) \times U(1)$ as twistor space of CP_2 would also give CP_2 as base space.

The general solution of field equations at the level of $T(H)$ would correspond to holomorphy in general sense for the 6-surfaces defined by 3 vanishing conditions for holomorphic functions - 6 real conditions. Effectively this would mean the knowledge of the exact solutions of field equations also at the level of H: TGD would be an integrable theory. Scattering amplitudes would in turn constructible from these solutions using ordinary partial differential equations [L74].

1. The first condition would identify the complex coordinates of $S^2(cd_{conf})$ and $S^2(CP_2)$: here one cannot exclude relative rotation represented as a holomorphic transformation but for $R(cd_{conf}) \gg R(CP_2)$ the effect of the rotation is small.
2. Besides this there would be vanishing conditions for 2 holomorphic polynomials. The coordinate pairs corresponding to $M^2 \subset M^4$ would correspond to hypercomplex behavior with hyper complex coordinate $u = \pm t - z$. t and z could be assigned with $U(1)$ fibers of Hopf fibrations $SU(2) \rightarrow S^2$.
3. The octonionic polynomial $P(o)$ of degree $n = h_{eff}/h_0$ with rational coefficients fixes the extension of rationals and since the algebraic extension should be same at both sides, the polynomials in twistor space should have same degree. This would give enormous boos concerning the understanding of the proposed cancellation of fermionic Wick contractions in SUSY scattering amplitudes forced by number theoretic vision [L74].

Possible problems related to the signatures

The different signatures for the metrics of the twistor spheres of cd_{conf} and CP_2 can pose technical problems.

1. Twistor lift would replace X^4 with 6-D twistor space X^6 represented as a 6-surface in $T(M^4) \times T(CP_2)$. X^6 is defined by dimensional reduction in which the twistor spheres $S^2(cd_{conf})$ and $S^2(CP_2)$ are identified and define the twistor sphere $S^2(X^4)$ of X^6 serving as a fiber whereas space-time surface X^4 serves as a base. The simplest identification is as $(\theta, \phi)_{S^2(M^4)} = (\theta, \phi)_{S^2(CP_2)}$: the same can be done for the complex coordinates $z_{S^2(M^4_{conf})} = z_{S^2(CP_2))}$). An open question is whether a Möbius transformation could relate the complex coordinates. The metrics of the spheres are of opposite sign and differ only by the scaling factors $R^2(cd_{conf})$ and $R^2(CP_2)$.
2. For cd_{conf} option the signatures of the 2 twistor spheres would be opposite (time-like for cd_{conf}). For $R(cd_{conf})/R(CP_2) = 1$. $J^2 = -g$ is the only consistent option unless the signature of space is not totally positive or negative and implies that the Kähler forms of the two twistor spheres differ by i. The magnetic contribution from $S^2(X^4)$ would give rise to an infinite value of cosmological constant proportional to $1/\sqrt{g_2}$, which would diverge

$R(cd_{conf})/R(CP_2) = 1$. There is however no need to assume this condition as in the original approach.

7.5.4 Hierarchy of length scale dependent cosmological constants in twistorial description

At the level of M^8 the hierarchy of CDs defines a hierarchy of length scales and must correspond to a hierarchy of length scale dependent cosmological constants. Even fundamental scales would emerge.

1. If one has $R(cd_{conf})/R(CP_2) >> 1$ as the idea about macroscopic cd_{conf} would suggest, the contribution of $S^2(cd_{conf})$ to the cosmological constant dominates and the relative rotation of metrics and Kähler form cannot affect the outcome considerably. Therefore different mechanism producing the hierarchy of cosmological constants is needed and the freedom to choose rather freely the ratio $R(cd_{conf})/R(CP_2)$ would provide the mechanism. What looked like a weakness would become a strength.
2. $S^2(cd_{conf}$ would have time-like metric and could have large scale. Is this really acceptable? Dimensional reduction essential for the twistor induction however makes $S^2(cd_{conf})$ non-dynamical so that time-likeness would not be visible even for large radii of $S^2(cd_{conf})$ expected if the size of cd_{conf} can be even macroscopic. The corresponding contribution to the action as cosmological constant has the sign of magnetic action and also Kähler magnetic energy is positive. If the scales are identical so that twistor spheres have same radius, the contributions to the induced metric cancel each other and the twistor space becomes metrically 4-D.
3. At the limit $R(cd_{conf}) \to RCP_2)$ cosmological constant coming from magnetic energy density diverges for $J^2 = -G$ option since it is proportional to $1/\sqrt{g_2}$. Hence the scaling factors must be different. The interpretation is that cosmological constant has arbitrarily large values near CP_2 length scale. Note however that time dependence is replaced with scale dependence and space-time sheets with different scales have only wormhole contacts.

It would seem that this approach could produce the nice results of the earlier approach. The view about how the hierarchy of cosmological constants emerges would change but the idea about reducing coupling constant evolution to that for cosmological constant would survive. The interpretation would be in terms of the breaking of scale invariance manifesting as the scales of CDs defining the scales for the twistor spaces involved. New insights about p-adic coupling constant evolution emerge and one finds a new "must" for ZEO. $H = M^4 \times CP_2$ picture would emerge as an approximation when cd_{conf} is replaced with its tangent space M^4. The consideration of the quaternionic generalization of twistor space suggests natural identification of the conformally compactified twistor space as being obtained from CP_2 by making second complex coordinate hyperbolic. This need not conform with the identification as $U(2)$.

7.6 How to generalize twistor Grassmannian approach in TGD framework?

One should be able to generalize twistor Grassmannian approach in TGD framework. The basic modification is replacement of 4-D light-like momenta with their 8-D counterparts. The octonionic interpretation encourages the idea that twistor approach could generalize to 8-D context. Higher-dimensional generalizations of twistors have been proposed but the basic problem is that the index raising and lifting operations for twistors do not generalize (see `http://tinyurl.com/y24lkwce`).

1. For octonionic twistors as pairs of quaternionic twistors index raising would not be lost working for M_T option and light-like M^8 momenta can be regarded sums of M_T^4 and E^4 parts as also twistors. Quaternionic twistor components do not commute and this is essential for incidence relation requiring also the possibility to raise or lower the indices of twistors. Ordinary complex twistor Grassmannians would be replaced with their quaternionic countparts. The twistor space as a generalization of CP_3 would be 3-D quaternionic projective space $T(M^8) = HP_3$ with Minkowskian signature (6,6) of metric and having real dimension 12 as one might expect.

Another option realizing non-commutativity could be based on the notion of quantum twistor to be also discussed.

2. Second approach would rely on the identification of $M^4 \times CP_2$ twistor space as a Cartesian product of twistor spaces of M^4 and CP_2. For this symmetries are not broken, $M^4_L \subset M^8$ depends on the state and is chosen so that the projection of M^8 momentum is light-like so that ordinary twistors and CP_2 twistors should be enough. $M^8 - H$ relates varying M^4_L based and M^4_T based descriptions.

3. The identification of the twistor space of M^4 as $T(M^4) = M^4 \times S^2$ can be motivated by octonionic considerations but might be criticized as non-standard one. The fact that quaternionic twistor space HP_3 looks natural for M^8 forces to ask whether $T(M^4) = CP_3$ endowed with metric having signature (3,3) could work in the case of M^4. In the sequel also a vision based on the identification $T(M^4) = CP_3$ endowed with metric having signature (3,3) will be discussed.

7.6.1 Twistor lift of TGD at classical level

In TGD framework twistor structure is generalized [L10, L45, L24, L58]. The inspiration for TGD approach to twistorialization has come from the work of Nima Arkani-Hamed and colleagues [B36, B26, B27, B32, B67, B38, B15]. The new element is the formulation of twistor lift also at the level of classical dynamics rather than for the scattering amplitudes only [L10, L24, L45, L58].

1. The 4-D light-like momenta in ordinary twistor approach are replaced by 8-D light-like momenta so that massive particles in 4-D sense become possible. Twistor lift of TGD takes places also at the space-time level and is geometric counterpart for the Penrose's replacement of massless fields with twistors. Roughly, space-time surfaces are replaced with their 6-D twistor spaces represented as 6-surfaces. Space-time surfaces as preferred extremals are minimal surfaces with 2-D string world sheets as singularities. This is the geometric manner to express masslessness. X^4 is simultaneously also extremal of 4-D Kähler action outside singularities: this realizes preferred extremal property.

2. One can say that twistor structure of X^4 is induced from the twistor structure of $H = M^4 \times CP_2$, whose twistor space $T(H)$ is the Cartesian product of geometric twistor space $T(M^4) = M^4 \times CP_1$ and $T(CP_2) = SU(3)/U(1) \times U(1)$. The twistor space of M^4 assigned to momenta is usually taken as a variant of CP_3 with metric having Minkowski signature and both CP_1 fibrations appear in the more precise definition of $T(M^4)$. Double fibration [B64] (see `http://tinyurl.com/yb4bt74l`) means that one has fibration from $M^4 \times CP_1$ - the trivial CP_1 bundle defining the geometric twistor space to the twistors space identified as complex projective space defining conformal compactification of M^4. Double fibration is essential in the twistorialization of TGD [L22].

3. The basic objects in the twistor lift of classical TGD are 6-D surfaces in $T(H)$ having the structure of twistor space in the sense that they are CP_1 bundles having X^4 as base space. Dimensional reduction to CP_1 bundle effectively eliminates the dynamics in CP_1 degrees of freedom and its only remnant is the value of cosmological constant appearing as coefficient of volume term of the dimensionally reduced action containing also 4-D Kähler action. Cosmological term depends on p-adic length scales and has a discrete spectrum [L58, L57].

CP_1 has also an interpretation as a projective space constructed from 2-D complex spinors. Could the replacement of these 2-spinors with their quantum counterparts defining in turn quantum CP_1 realize finite quantum measurement resolution in M^4 degrees of freedom? Projective invariance for the complex 2-spinors would mean that one indeed has effectively CP_1.

7.6.2 Octonionic twistors or quantum twistors as twistor description of massive particles

For M^4_T option the particles are massive and the one encounters the problem whether and how to generalize the ordinary twistor description.

7.6.3 Basic facts about twistors and bi-spinors

It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality (see `http://tinyurl.com/y6bnznyn`).

1. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle \lambda, \mu \rangle &= \epsilon_{ab} \lambda^a \mu^b \ , \\ \left[\tilde{\lambda}, \tilde{\mu} \right] &= \epsilon_{a'b'} \tilde{\lambda}^{a'} \tilde{\mu}^{b'} \ , \\ p \cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu} \right] \ , \ (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ . \end{aligned} \tag{7.6.1}$$

2. Spinor indices are lowered and raised using antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda} \right]} \ , \ \text{negative helicity} \ . \end{aligned} \tag{7.6.2}$$

In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

3. What makes 4-D twistors unique is the existence of the index raising and lifting operations using ϵ tensors. In higher dimensions they do not exist and this causes difficulties. For octonionic twistors with quaternionic components possibly only in $D = 8$ the situation changes.

To get a very rough idea about twistor Grassmannian approach idea, consider tree amplitudes of $\mathcal{N} = 4$ SUSY as example and it is convenient to drop the group theory factor $Tr(T_1 T_2 \cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{7.6.3}$$

When the sign of the helicities is changed $\langle .. \rangle$ is replaced with [..].

An essential point in what follows is that the amplitudes are expressible in terms of the antisymmetric bi-linears $\langle \lambda_i, \lambda_j \rangle$ making sense also for octotwistors and identifiable as quaternions rather than octonions.

$M^8 - H$ duality and two alternative twistorializations of TGD

$M^8 - H$ duality suggests two alternative twistorializations of TGD.

1. The first approach would be in terms of M^8 twistors suggested by quaternionic light-lineness of 8-momenta. M^8 twistors would be Cartesian products of M^4 and E^4 twistors. One can imagine a straightforward generalization of twistor scattering amplitudes in terms of generalized Grassmannian approach replacing complex Grassmannian with quaternionic Grassmannian, which is a mathematically well-defined notion.
2. Second approach would rely on $M^4 \times CP_2$ twistors, which are products of M^4 twistors and CP_2 twistors: this description works nicely at classical space-time level but at the level of momentum space the problem is how to describe massivation of M^4 momenta using twistors.

Why the components of twistors must be non-commutative?

How to modify the 4-D twistor description of light-like 4-momenta so that it applies to massive 4-momenta?

1. Twistor consists of a pair $(\mu_{\dot{\alpha}}, \lambda^{\alpha})$ of bi-spinors in conjugate representations of $SU(2)$. One can start from the 4-D incidence relations for twistors

$$\mu_{\dot{\alpha}} = p_{\alpha\dot{\alpha}}\lambda^{\alpha} \ .$$

Here $p_{\alpha\dot{\alpha}}$ denotes the representation of four-momentum $p^k\sigma_k$. The antisymmetric permutation symbols $\epsilon^{\alpha\beta}$ and its dotted version define antisymmetric "inner product" in twistor space. By taking the inner product of μ with itself, one obtains the commutation relation $\mu_1\mu_2 - \mu_2\mu_1 = 0$, which is consistent with right-hand side for massless particles with $p_kp^k = 0$.

2. In TGD framework particles are massless only in 8-D sense so that the right hand side in the contraction is in general non-vanishing. In massive case one can replace four-momentum with unit vector. This requires

$$\langle \mu 1, \mu_2 \rangle = \mu_1\mu_2 - \mu_2\mu_1 \neq 0 \ .$$

The components of 2-spinor become non-commutative.

This raises two questions.

1. Could the replacement of complex twistors by quaternionic twistors make them non-commutative and allow massive states?
2. Could non-commutative quantum twistors solve the problem caused by the light-likeness of momenta allowing 4-D twistor description?

Octotwistors or quantum twistors?

One should be able to generalize twistor amplitudes and twistor Grassmannian approach to TGD framework, where particles are massless in 8-D sense and massive in 4-D sense. Could twistors be replaced by octonionic or quantum twistors.

1. One can express mass squared as a product of commutators of components of the twistors λ and $\tilde{\lambda}$, which is essentially the conjugate of λ:

$$p \cdot p = \langle \lambda, \lambda \rangle \left[\tilde{\lambda}, \tilde{\lambda}\right] \ . \tag{7.6.4}$$

This operator should be non-vanishing for non-vanishing mass squared. Both terms in the product vanish unless commutativity fails so that mass vanishes. The commutators should have the quantum state as its eigenstate.

2. Also 4-momentum components should have well-defined values. Four-momentum has expression $p^{aa'} = \lambda^a\tilde{\lambda}^{a'}$ in massless case. This expression should generalized to massive case as such. Eigenvalue condition and reality of the momentum components requires that the components $p^{aa'}$ are commuting Hermitian operators.

In twistor Grassmannian approach complex but light-like momenta are possible as analogs of virtual momenta. Also in TGD framework the complexity of Kähler coupling strength allows to consider complex momenta. For twistor lift they however differ from real momenta only by a phase factor associated with the $1/\alpha_K$ associated with 6-D Kähler action.

Remark: I have considered also the possibility that states are eigenstates only for the longitudinal M^2 projection of 4-momentum with quark model of hadrons serving as a motivation.

(a) Could this equation be obtained in massive case by regarding λ^a and $\tilde{\lambda}^{a'}$ as commuting octo-spinors and their complex conjugates? Octotwistors would naturally emerge in the description at embedding space level. I have already earlier considered the notion of octotwistor [K90] [L37]).

(b) Or could it be obtained for quantum bi-spinors having same states as eigenstates. Could quantum twistors as generalization of the ordinary twistors correspond to the reduction of the description from the level of M^8 or H to at space-time level so that one would have 4-D twistors and massive particles with 4-momentum identifiable as Noether charge for the action principle determining preferred extremals? I have considered also the notion of quantum spinor earlier [K36, K61, K54, K2, K102].

3. In the case of quantum twistors the generalization of the product of the quantities $\langle\lambda_i, \lambda_{i+1}\rangle$ appearing in the formula should give rise to c-number in the case of quantum spinors. Can one require that the quantities $\langle\lambda_i, \lambda_{i+1}\rangle$ or even $\langle\lambda_i, \lambda_j\rangle$ are c-numbers simultaneously? This would also require that $\langle\lambda, \lambda\rangle$ is non-vanishing c-number in massive case: also incidence relation suggest this condition. Could one think λ as an operator such that $\langle\lambda, \lambda\rangle$ has eigenvalue spectrum corresponding to the quantities $\langle\lambda_i, \lambda_{i+1}\rangle$ appearing in the scattering amplitude?

7.6.4 The description for M^4_T option using octo-twistors?

For option I with massive M^4_T projection of 8-momentum one could imagine twistorial description by using M^8 twistors as products of M^4_T and E^4 twistors, and a rather straightforward generalization of standard twistor Grassmann approach can be considered.

Could twistor Grassmannians be replaced with their quaternionic variants?

The first guess would simply replace $Gr(k, n)$ with $Gr(2k, 2n)$ 4-D twistors 8-D twistors. From twistor amplitudes with quaternionic M^8-momenta one could construct physical amplitudes by going from 8-momentum basis to the 4-momentum- basis with wave functions in irreps of $SO(3)$. Life is however not so simple.

1. The notion of ordinary twistor involves in an essential manner Pauli matrices σ_i satisfying the well-known anti-commutation relations. They should be generalized. In fact, σ_0 and $\sqrt{-1}\sigma_i$ can be regarded as a matrix representation for quaternionic units. They should have analogs in 8-D case.

   Octonionic units ie_i indeed provide this analog of sigma matrices. Octonionic units for the complexification of octonions allow to define incidence relation and representation of 8-momenta in terms of octo-spinors. They do not however allow matrix representation whereas time-like octonions allow interpretation as quaternion in suitable bases and thus matrix representation. Index raising operation is essential for twistors and makes dimension $D = 4$ very special. For naïve generalizations of twistors to higher dimensions this operation is lost (see `http://tinyurl.com/y24lkwce`).

2. Could one avoid multiplication of more than two octo-twistors in Grassmann amplitudes leading to difficulties with associativity. An important observation is that in the expressions for the twistorial scattering amplitudes only products $\langle\lambda_i, \lambda_j\rangle$ or $[\tilde{\lambda}_i, \tilde{\lambda}_{i+1}]$ but not both occur. These products are associative even if the spinors are replaced by quaternionic spinors.

   These operations are antisymmetric in the arguments, which suggests cross product for quaternions giving rise to imaginary quaternion so that the product of objects would give rise to a product of imaginary quaternions. This might be a problem since a large number of terms in the product would approach to zero for random imaginary quaternions.

   An ad hoc guess would be that scattering probability is proportional to the product of amplitude as product $\langle\lambda_i, \lambda_j\rangle$ and its "hermitian conjugate" with the conjugates $[\tilde{\lambda}_i, \tilde{\lambda}_{i+1}]$ in the reverse order (this does not affect the outcome) so that the result would be real. Scattering amplitude would be more like quaternion valued operator. Could one have a formulation of quantum theory or at least TGD view about quantum theory allowing this?

3. If ordinary massless 4-momenta correspond to quaternionic sigma matrices, twistors can be regarded as pairs of 2-spinors in matrix representation. Octonionic 8-momenta should correspond to pairs of 4-spinors. As already noticed, octonions do not however allow matrix representation! Octonions for a fixed decomposition $M^8 = M^4 \times E^4$ can be however decomposed to linear combination of two quaternions just like complex numbers to a combination of real numbers. These quaternions would have matrix representation and quaternionic analogs

of twistor pair $(\mu, \tilde{\lambda})$. One could perhaps formulate the generalization of twistor Grassmann amplitudes using these pairs. This would suggest replacement of complex bi-spinors with complexified quaternions in the ordinary formalism. This might allow to solve problems with associativity if only $\langle \lambda_i, \lambda_j \rangle$ or $[\tilde{\lambda}_i, \tilde{\lambda}_{i+1}]$ appear in the amplitudes.

4. The argument in the momentum conserving delta function $\delta(\lambda_i \tilde{\lambda}_i)$ should be real so that the conjugation with respect to i would not change the argument and non-commutativity would not be problem. In twistor Grassmann amplitudes the argument $C \cdot Z$ of delta momentum conserving function is linear in the components of complex twistor Z. If complex twistor is replaced with quaternionic twistor, the Grassmannian coordinates C in delta functions $\delta(C \cdot Z)$ must be replaced with quaternionic one.

The replacement of complex Grassmannians $Gr_C(k, n)$ with quaternionic Grassmannians $Gr_H(k, n)$ is therefore highly suggestive. Quaternionic Grassmannians (see `http://tinyurl.com/y23jsffn`) are quotients of symplectic Lie groups $Gr_H(k, n) = U_n(H)/(U_r(H) \times U_{n-r}(H))$ and thus well-defined. In the description using $Gl_H(k, n)$ matrices the matrix elements would be quaternions and $k \times k$ minors would be quaternionic determinants.

Remark: Higher-D projective spaces of octonions do not exist so that in this sense dimension $D = 8$ for embedding space would be maximal.

Twistor space of M^8 as quaternionic projective space HP_3?

The simplest Grassmannian corresponds to twistor space and one can look what one obtains in this case. One can also try to understand how to cope with the problems caused by Minkowskian signature.

1. In previous section it was found that the modification of H to $H = cd_{conf} \times CP_2$ with $cd_{conf} = CP_{2,h}$ identifiable as CP_2 with Minkowskian signature of metric is strongly suggestive.

2. For E^8 quaternionic twistor space as analog of CP_3 would be its quaternionic variant HP_3 with expected dimension $D = 16 - 4 = 12$. Twistor sphere would be replaced with its quaternionic counterpart $SU(2)_H/U(1)_H$ having dimension 4 as expected. $CD_{8,conf}$ as conformally compactified CD_8 must be 8-D. The space HP_2 has dimension 8 and is analog of CP_2 appearing as analog of base space of CP_3 identified as conformally compactified 4-D causal diamond cd_{conf}. The quaternionic analogy of $M^4_{conf} = U(2)$ identified as conformally compactified M^4 would be $U(2)_H$ having dimension $D = 10$ rather than 8.

   HP_3 and HP_2 might work for E^8 but it seems that the 4-D analog of twistor sphere should have signature (2,-2) whereas base space should have signature (1,-7). Some kind of hyperbolic analogs of these spaces obtained by replacing quaternions with their hypercomplex variant seem to be needed. The same receipe in the twistorialization of M^4 would give cd_{conf} as analog of CP_2 with second complex coordinate made hyperbolic. I have already considered the construction of hyperbolic analogs of CP_2 and CP_3 as projective spaces. These results apply to HP_2 and HP_3.

3. What about octonions? Could one define octonionic projective plane OP_2 and its hyperbolic variants corresponding to various sub-spaces of M^8? Euclidian OP_2 known as Cayley plane exists as discovered by Ruth Moufang in 1933. Octonionic higher-D projective spaces and Grassmannians do not however exist so that one cannot assign OP_3 as twistor spaces.

Can one obtain scattering amplitudes as quaternionic analogs of residue integrals?

Can one obtain complex valued scattering amplitudes (i commuting with octonionic units) in this framework?

1. The residue integral over quaternionic C-coordinates should make sense, and pick up the poles as vanishing points of minors. The outcome of repeated residue integrations should give a sum over poles with complex residues.

2. Residue calculus requires analyticity. The problem is that quaternion analyticity based on a generalization of Cauchy-Riemann equations allows only linear functions. One could define quaternion (and octonion) analyticity in restricted sense using powers series with real coefficients (or in extension involving i commuting with octonion units). The quaternion/octonion

analytic functions with real coefficients are closed with respect to sum and product. I have used this definition in the proposed construction of algebraic dynamics for in $X^4 \subset M^8$ [L37].

3. Could one define the residue integral purely algebraically? Could complexity of the coefficients (i) force complex outcome: if pole q_0 is not quaternionically real the function would not allow decompose to $f(q)/(q-q_0)$ with f allowing similar Taylor series at pole. If so, then the formulas of Grassmannian formalism could generalize more or less as such at M^8 level and one could map the predictions to predictions of $M^4 \times CP_2$ approach by analog of Fourier transform transforming these quantum state basis to each other.

This option looks rather interesting and involves the key number theoretic aspects of TGD in a crucial manner.

7.6.5 Do super-twistors make sense at the level of M^8?

By $M^8 - H$ duality [L37] there are two levels involved: M^8 and H. These levels are encountered both at the space-time level and momentum space level. Do super-octonions and super-twistors make sense at M^8 level?

1. At the level of M^8 the high uniqueness and linearity of octonion coordinates makes the notion of super-octonion natural. By $SO(8)$ triality octonionic coordinates (bosonic octet 8_0), octonionic spinors (fermionic octet 8_1), and their conjugates (anti-fermionic octet 8_{-1}) would for triplet related by triality. A possible problem is caused by the presence of separately conserved B and L. Together with fermion number conservation this would require $\mathcal{N}=4$ or even $\mathcal{N}=4$ SUSY, which is indeed the simplest and most beautiful SUSY.
2. At the level of the 8-D momentum space octonionic twistors would be pairs of two quaternionic spinors as a generalization of ordinary twistors. Super octo-twistors would be obtained as generalization of these.

The progress in the understanding of the TGD version of SUSY [L74] led to a dramatic progress in the understanding of super-twistors.

1. In non-twistorial description using space-time surfaces and Dirac spinors in H, embedding space coordinates are replaced with super-coordinates and spinors with super-spinors. Theta parameters are replaced with quark creation and annihilation operators. Super-coordinate is a super-polynomial consisting of monomials with vanishing total quark number and appearing in pairs of monomial and its conjugate to guarantee hermiticity.

   Dirac spinor is a polynomial consisting of powers of quark creation operators multiplied by monomials similar to those appearing in the super-coordinate. Anti-leptons are identified as spartners ofquarks identified as local 3-quark states. The multi-spinors appearing in the expansions describe as such local many-quark-antiquark states so that super-symmetrization means also second quantization. Fermionic and bosonic states assignable to H-geometry interact since super-Dirac action contains induced metric and couplings to induced gauge potentials.
2. The same recipe works at the level of twistor space. One introduces twistor super-coordinates analogous to super-coordinates of H and M^8. The super YM field of $\mathcal{N}=4$ SUSY is replaced with super-Dirac spinor in twistor space. The spin degrees of freedom associated with twistor spheres S^2 would bring in 2 additional spin-like degrees of freedom.

   The most plausible option is that the new spin degrees are frozen just like the geometric S^2 degrees of freedom. The freezing of bosonic degrees of freedom is implied by the construction of twistor space of X^4 by dimensional reduction as a 6-D surface in the product of twistor spaces of M^4 and CP_2. Chirality conditions would allow only single spin state for both spheres.
3. Number theoretical vision implies that the number of Wick contractions of quarks and anti-quarks cannot be larger than the degree of the octonionic polynomial, which in turn should be same as that of the polynomials of twistor space giving rise to the twistor space of space-time surface as 6-surface. The resulting conditions correspond to conserved currents identifiable as Noether currents assignable to symmetries.

Also Grassmannian is replaced with super-Grassmannian and super-coordinates as matrix elements of super matrices are introduced.

1. The integrand of the Grassmannian integral defining the amplitude can be expanded in Taylor series with respect to theta parameters associated with the super coordinates C as rows of super $G(k,n)$ matrix.

2. The delta function $\delta(C,Z)$ factorizing into a product of delta functions is also expanded in Taylor series to get derivatives of delta function in which only coordinates appear. By partial integration the derivatives acting on delta function are transformed to derivatives acting on integrand already expanded in Taylor series in theta parameters. The integration over the theta parameters using the standard rules gives the amplitudes associated with different powers of theta parameters associated with Z and from this expression one can pick up the scattering amplitudes for various helicities of external particles.

The super-Grassmannian formalism is extremely beautiful but one must remember that one is dealing with quantum field theory. It is not at all clear whether this kind of formalism generalizes to TGD framework, where particle are 3-surfaces [L37]. The notion of cognitive representation effectively reducing 3-surfaces to a set of point-like particles strongly suggests that the generalization exists.

The progress in understanding of $M^8 - H$ duality throws also light to the problem whether SUSY is realized in TGD and what SUSY breaking does mean. It seems now clear that sparticles are predicted and SUSY remains in the simplest scenario exact but that p-adic thermodynamics causes thermal massivation: unlike Higgs mechanism, this massivation mechanism is universal and has nothing to do with dynamics. This is due to the fact that zero energy states are superpositions of states with different masses. The selection of p-adic prime characterizing the sparticle causes the mass splitting between members of super-multiplets although the mass formula is same for all of them.

The increased undestanding of what twistorialization leads to an improved understanding of what twistor space in TGD could be. It turns out that the hyperbolic variant $CP_{3,h}$ of the standard twistor space CP_3 is a more natural identification than the earlier $M^4 \times S^2$ also in TGD framework but with a scale corresponding to the scale of CD at the level of M^8 so that one obtains a scale hierarchy of twistor spaces. Twistor space has besides the projection to M^4 also a bundle projection to the hyperbolic variant $CP_{2,h}$ of CP_2 so that a remarkable analogy between M^4 and CP_2 emerges. One can formulate super-twistor approach to TGD using the same formalism as will be discussed in this article for the formulation at the level of H. This requires introducing besides 6-D Kähler action and its super-variant also spinors and their super-variants in super-twistor space. The two formulations are equivalent apart from the hierarchy of scales for the twistor space. Also M^8 allows analog of twistor space as quaternionic Grassmannian HP_3 with signature (6,6). What about super- variant of twistor lift of TGD? consider first the situation before the twistorialization.

1. The parallel progress in the understanding SUSY in TGD framework [L74] leads to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.

2. In fermionic sector only quarks are allowed by $SO(1,7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of embedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.

3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

Super-counterpart of twistor lift using the proposed formalism

The construction of super-coordinates and super-spinors [L74] suggests a straightforward formulation of the super variant of twistor lift . One should only replace the super-embedding space and super-spinors with super-twistor space and corresponding super-spinors and formulate the theory using 6-D super-Kähler action and super-Dirac equation and the same general prescription for constructing S-matrix. Dimensional reduction should give essentially the 4-D theory apart from the variation of the radius of the twistor space predicting variation of cosmological constant. The size scale of CD would correspond to the size scale of the twistor space for M^4 and for CP_2 the size scale would serve as unit and would not vary.

The first step is the construction of ordinary variant of Kähler action and modified Dirac action for 6-D surfaces in 12-D twistor space.

1. Replace the spinors of H with the spinors of 12-D twistor space and assume only quark chirality. By the bundle property of the twistor space one can express the spinors as tensor products of spinors of the twistor spaces $T(M^4)$ and $T(CP_2)$. One can express the spinors of $T(M^4)$ tensor products of spinors of M^4 - and S^2 spinors locally and spinors of $T(CP_2)$ as tensor products of CP_2 - and S^2 spinors locally. Chirality conditions should reduce the number of 2 spin components for both $T(M^4)$ and $T(CP_2)$ to one so that there are no additional spin degrees of freedom.

   The dimensional reduction can be generalized by identifying the two S^2 fibers for the preferred extremals so that one obtains induced twistor structure. In spinorial sector the dimensional reduction must identify spinorial degrees of freedom of the two S^2s by the proposed chirality conditions also make them non-dynamical. The S^2 spinors covariantly constant in S^2 degrees of freedom.

2. Define the spinor structure of 12-D twistor space, define induced spinor structure at 6-D surfaces defining the twistor space of space-time surface. Define the twistor counterpart of the analog of modified Dirac action using same general formulas as in case of H.

Construct next the super-variant of this structure.

1. Introduce second quark oscillator operators labelled by the points of cognitive representation in 12-D twistor space effectively replacing 6-D surface with its discretization and having quantized quark field q as its continuum counterpart. Replace the coordinates of the 12-D twistor space with super coordinates h_s expressed in terms of quark and anti-quark oscillator operators labelled by points of cognitive representation, and having interpretation as quantized quark field q restricted to the points of representation.
2. Express 6-D Kähler action and Dirac action density in terms of super-coordinates h_s. The local monomials of q appear in h_s and therefore also in the expansion of super-variants of modified gamma matrices defined by 6-D ähler action as contractions of canonical momentum currents of the action density L_K with the gamma matrices of 12-D twistor space. In super-Kähler action also the local composites of q giving rise to currents formed from the local composites of 3-quarks and antiquarks and having interpretation as leptons and anti-leptons occur - leptons would be therefore spartners of squarks.
3. Perform super-expansion also for the induced spinor field q_s in terms of monomials of q. $q_s(q)$ obeys super-Dirac equation non-linear in q. But also q should satisfy super-Dirac action as an analog of quantized quark field and non-linearity indeed forces also q to have has super-expansion. Thus both quark field q and super-quark field q_s both satisfy super-Dirac equation.

   The only possibility is $q_s = q$ stating fixed point property under $q \rightarrow q_s$ having interpretation in terms of quantum criticality fixing the values of the coefficients of various terms in q_s and in the super-coordinate h_s having interpretation as coupling constants. One has quantum criticality and discrete coupling constant evolution with respect to extension of rationals characterizing adelic physics.
4. Super-Dirac action vanishes for its solutions and the exponent of super-action reduces to exponent of super-Kähler action, whose matrix elements between positive and negative energy parts of zero energy states give S-matrix elements.

   Super-Dirac action has however an important function: the derivatives of quark currents appearing in the super-Kähler action can be transformed to a linear strictly local action of super spinor connection ($\partial_\alpha \to A_{\alpha,s}$ effectively). Without this lattice discretization would be needed and cognitive representation would not be enough.

To sum up, the super variants of modified gamma matrices of the 6-surface would satisfy the condition $D_{\alpha,s}\Gamma_s^\alpha = 0$ expressing preferred extremal property and guaranteeing super-hermicity of D_s. q_s would obey super-Dirac equation $D_s q_s = 0$. The self-referential identification $q = q_s$ would express quantum criticality of TGD.

7.7 Could one describe massive particles using 4-D quantum twistors?

The quaternionic generalization of twistors looks almost must. But before this I considered also the possibility that ordinary twistors could be generalized to quantum twistors to describe particle massivation. Quantum twistors could provide space-time level description, which requires 4-D twistors, which cannot be ordinary M^4 twistors. Also the classical 4-momenta, which by QCC would be equal to M^8 momenta, are in general massive so that the ordinary twistor approach cannot work. One cannot of course exclude the possibility that octo-twistors are enough or that M^8_L description is equivalent with space-time description using quantum twistors.

7.7.1 How to define quantum Grassmannian?

The approach to twistor amplitude relies on twistor Grassmann approach [B29, B22, B20, B34, B36, B15] (see `http://tinyurl.com/yxllwcsn`). This approach should be replaced by replacing Grassmannian $GR(K,N) = Gl(n,C)/Gl(n-m,C) \times Gl(m,C)$ with quantum Grassmannian.

naïve approach to the definition of quantum Grassmannian

Quantum Grassmannian is a notion studied in mathematics and the approach of [A67] (see `http://tinyurl.com/y5q6kv6b`) looks reasonably comprehensible even for physicist. I have already earlier tried to understand quantum algebras and their possible role in TGD [K11]. It is however better to start as ignorant physicist and proceed by trial and error and find whether mathematicians have ended up with something similar.

1. Twistor Grassmannian scattering amplitudes involving k negative helicity gluons involve product of $k \times k$ minors of an $k \times n$ matrix C taken in cyclic order. C defines $k \times n$ coordinates for Grassmannian $Gr(k,n)$ of which part is redundant by the analogs of gauge symmetries $Gl(n-m,C) \times Gl(m,C)$. Here n is the number of external gluons and k the number of negative helicity gluons. The $k \times k$ determinants taken in cyclic order appear in the integrand over Grassmannian. Also the quantum variants of these determinants and integral over quantum Grassmannian should be well-defined and residue calculus gives hopes for achieving this.
2. One should define quantum Grassmannian as algebra according to my physicist's understanding algebra can be defined by starting from a free algebra generated by a set of elements - now the matrix elements of quantum matrix. One poses on these elements relations to get the algebra considered. What could these conditions be in the recent case.
3. A natural condition is that the definition allows induction in the sense that its restriction to quantum sub-matrices is consistent with the general definition of $k \times n$ quantum matrices. In particular, one can identify the columns and rows of quantum matrices as instances of quantum vectors.
4. How to generalize from 2×2 case to $k \times n$ case? The commutation relations for neighboring elements of rows and columns are fixed by induction. In 4×4 corresponding to M^4 twistors one would obtain for $(a_1, ..., a_4)$. $a_i a_{i+1} = q a_{i+1} a_i$ cyclically ($k = 1$ follows $k = 4$).

   What about commutations of a_i and a_{i+k}, $k > 1$. Is there need to say anything about these commutators? In twistor Grassmann approach only connected $k \times k$ minors in cyclic order appear. Without additional relations the algebra might be too large. One could argue that the simplest option is that one has $a_i a_{i+k} = q a_{i+k} a_i$ for k odd $a_i a_{i+k} = q^{-1} a_{i+k} a_i$ for k even. This is required from the consistency with cyclicity. These conditions would allow to define also sub-determinants, which do not correspond to connected $k \times k$ squares by moving the elements to a a connected patch by permutations of rows and columns.

5. What about elements along diagonal? The induction from 2×2 would require the commutativity of elements along right-left diagonals. Only commutativity of the elements along left-right diagonal be modified. Or is the commutativity lost only along directions parallel to left-right diagonal? The problem is that the left-right and right-left directions are transformed to each other in odd permutations. This would suggest that only even permutations are allowed in the definition of determinant

6. Could one proceed inductively and require that one obtains the algebra for 2×2 matrices for all 2×2 minors? Does this apply to all 2×2 minors or only to connected 2×2 minors with cyclic ordering of rows and columns so that top and bottom row are nearest neighbors as also right and left column. Also in the definition of 3×3 determinant only the connected developed along the top row or left column only 2×2 determinants involving nearest neighbor matrix elements appear. This generalizes to $k \times k$ case.

It is time to check how wrong the naïve intuition has been. Consider 2×2 matrices as simple example. In this case this gives only 1 condition ($ad - bc = -da + cb$) corresponding to the permutation of rows or columns. Stronger condition suggested by higher-D case would be $ad = da$ and $bc = cb$. The definition of 2×2 in [A67] however gives for quantum 2-matrices $(a, b; c, d)$ the conditions

$$\begin{array}{ll} ac = qca \ , & bd = qda \ , \\ ab = qba \ , cd = qdc \ , & \\ bc = cb \ , & ad - da = (q - q^{-1})bc \ . \end{array} \tag{7.7.1}$$

The commutativity along left-right diagonal is however lost for $q \neq 1$ so that quantum determinant depends on what row or column is used to expand it. The modification of the commutation relations along rows and columns is what one might expect and wants in order to achieve non-commutativity of twistor components making possible massivation in M^4 sense.

The limit $q \rightarrow 1$ corresponds to non-trivial algebra in general and would correspond to $\beta = 4$ for inclusions of HFFs expected to give representations of Kac-Moody algebras. At this limit only massless particles in 4-D sense are allowed. This suggests that the reduction of Kac-Moody algebras to quantum groups corresponds to symmetry breaking associated with massivation in 4-D sense.

Mathematical definition of quantum Grassmannian

It would seem that the proposed approach is reasonable. The article [A85] (see `http://tinyurl.com/yycflgrd`) proposing a definition of quantum determinant explains also the basic interpretation of what the non-commutativity of elements of quantum matrices does mean.

1. The first observation is that the commutation of the elements of quantum matrix corresponds to braiding rather than permutation and this operation is represented by R-matrix. The formula for the action of braiding is

$$R^{ab}_{cd} t^c_e t^d_f = t^a_d t^b_c R^{cd}_{ef} \ . \tag{7.7.2}$$

Here R-matrix is a solution of Yang-Baxter equaion and characterizes completely the commutation relations between the elements of quantum matrix. The action of braiding is obtained by applying the inverse of R-matrix from left to the equation. By iterating the braidings of nearest neighbors one can deduce what happens in the braiding exchanging quantum matrix elements which are not nearest neighbors. What is nice that the R-matrix would fix the quantum algebra, in particular quantum Grassmannian completely.

2. In the article the notion of quantum determinant is discussed and usually the definition of quantum determinant involves also the introduction of metric g^{ab} allowing the raising of the indices of the permutation symbol. One obtains formulas relating metric and R-matrix and restricting the choice of the metric. Note however that if ordinary permutation symbol is used there is no need to introduce the metric.

The definition quantum Grassmannian proposed does not involve hermitian conjugates of the matrices involved. One can define the elements of Grassmannian and Grassmannian residue integrals without reference to complex conjugation: could one do without hermitian conjugates? On the other hand, Grassmannians have complex structure and Kähler structure: could this require hermitian conjugates and commutation relations for these?

7.7.2 Two views about quantum determinant

If one wants to define quantum matrices in $Gr(k,n)$ so that quantal twistor-Grassmann amplitudes make sense, the first challenge is to generalize the notion of $k \times k$ determinant.

One can consider two approaches concerning the definition of quantum determinant.

1. The first guess is that determinant should not depend on the ordering of rows or columns apart from the standard sign factor. This option fails unless one modifies the definition of permutation symbol.
2. The alternative view is that permutation symbol is ordinary and there is dependence on the row or column with respect to which one develops. This dependence would however disappear in the scattering amplitudes. If the poles and corresponding residues associated with the $k \times k$-minors of the twistor amplitude remain invariant under the permutation, this is not a problem. In other words, the scattering amplitudes are invariant under braid group. This is what twistor Grassmann approach implies and also TGD predict.

For the first option quantum determinant would be braiding invariant. The standard definition of quantum determinant is discussed in detail in [A85] (see `http://tinyurl.com/yycflgrd`).

1. The commutation of the elements of quantum matrix corresponds to braiding rather than permutation and as found, this operation is represented by R-matrix.
2. Quantum determinant would change only by sign under the braidings of neighboring rows and columns. The braiding for the elements of quantum matrix would compensate the braiding for quantum permutation symbol. Permutation symbol is assumed to be q-antisymmetric under braiding of any adjacent indices. This requires that permutation $i_k \leftrightarrow i_{k+1}$ regarded as braiding gives a contraction of quantum permutation symbol $\epsilon_{i_1,...1_k}$ with $R^{ij}_{i_k i_{k+1}}$ plus scaling by some normalization factor λ besides the change of sign.

$$\epsilon_{a_1...a_k a_{k+1}...a_n} = -\lambda \epsilon_{a_1...ij...a_n} R^{ji}_{a_k a_{k+1}} \quad . \tag{7.7.3}$$

   The value of λ can be calculated.
3. The calculation however leads to the result that quantum determinant $\mathcal{D}$ satisfies $\mathcal{D}^2 = 1$! If the result generalizes for sub-determinants defined by $k \times k$-minors (, which need not be the case) would have determinants satisfying $\mathcal{D}^2 = 1$, and the idea about vanishing of $k \times k$-minor essential for getting non-trivial twistor scattering amplitude as residue would not make sense.

It seems that the braiding invariant definition of quantum determinant, which of course involves technical assumptions) is too restrictive. Does this mean that the usual definition requiring development with respect to preferred row is the physically acceptable option? This makes sense if only the integral but not integrand is invariant under braidings. Braiding symmetry would be analogous to gauge invariance.

7.7.3 How to understand the Grassmannian integrals defining the scattering amplitudes?

The beauty of the twistor Grassmannian approach is that the residue integrals over quantum $Gr(k,n)$ would reduce to sum over poles (or possibly integrals over higher-D poles). Could residue calculus provide a manner to integrate q-number valued functions of q-numbers? What would be the minimal assumptions allowing to obtain scattering amplitudes as c-numbers?

Consider first what the integrand to be replaced with its quantum version looks like.

1. Twistor scattering amplitudes involve also momentum conserving delta function expressible as $\delta(\lambda_a\tilde{\lambda}^a)$. This sum and - as it seems - also the summands should be c-numbers - in other words one has eigenstates of the operators defining the summands.
2. By introducing Grassmannian space $Gr(k,n)$ with coordinates $C_{\alpha,i}$ (see `http://tinyurl.com/yxllwcsn`), one can linearize $\delta(\lambda_a\tilde{\lambda}^a)$ to a product of delta functions $\delta(C\cdot Z)=\delta(C\cdot\tilde{\lambda})\times\delta(C^{\perp}\cdot\lambda)$ (I have not written the delta function is Grassmann parameters related to super coordinates). Z is the n-vector formed by the twistors associated with incoming particles.

   The $4\times k$ components of $C_{\alpha,k}Z^k$ should be c-numbers at least when they vanish. One should define quantum twistors and quantum Grassmannian and pose the constraints on the poles.

How to achieve the goal? Before proceeding it is good to recall the notion of non-commutative geometry (see `http://tinyurl.com/yxrcr8xv`). Ordinary Riemann geometry can be obtained from exterior algebra bundle, call it E. The Hilbert space of square integrable sections in E carries a representation of the space of continuous functions $C(M)$ by multiplication operators. Besides this there is unbounded differential operator D, which so called signature operator and defined in terms of exterior derivative and its dual: $D=d+d^*$. This spectral triple of algebra, Hilbert space, and operator D allows to deduce the Riemann geometry.

The dream is that one could assign to non-commutative algebras non-commutative spaces using this spectral triple. The standard q-p quantization is example of this: one obtains now Lagrange manifolds as ordinary commutative manifolds.

Consider now the situation in the case of quantum Grassmannian.

1. In the recent case the points defining the poles of the function - it might be that the eventual poles are not a set of discrete points but a higher-dimensional object - would form the commutative part of non-commutative quantum space. In this space the product of quantum minors would become ordinary number as also the argument $C\cdot Z$ of the delta function. This commutative sub-space would correspond to a space in which maximum number of minors vanish and residues reduce to c-numbers.

   Thus poles of the integrand of twistor amplitude would correspond to eigenstates for some $k\times k$ minors of Grassmannian with a vanishing eigenvalue. The residue at the pole at given step in the recursion pole by pole need not be c-number but the further residue integrals should eventually lead to a c-number or c-number valued integrand.
2. The most general option would be that the conditions hold true only in the sense that some $k\times k$ minors for $k\geq 2$ are c-numbers and have a vanishing eigenvalue but that smaller minors need not have this property. Also $C_{\alpha,k}Z^k$ should be c-numbers and vanish. Residue calculus would give rise to lower-D integrals in step-wise manner.

   The simplest and most general option is that one can speak only about eigenvalues of $k\times k$ minors. At pole it is enough to have one minor for which eigenvalue vanishes whereas other minors could remain quantal. In the final reduction the product of all non-vanishing $k\times k$ minors appearing in cyclic order in the integrand should have a well-defined c-number as eigenvalue. Does this allow the appearance of only cyclic minors.

   A stronger condition would be that all non-vanishing minors reduce to their eigenvalues. Could it be that only the n cyclic minors can commute simultaneously and serve as analogs of q-coordinates in phase space? The complex dimension of $G_C(n,k)$ is $d=(n-k)k$. If the space spaced by minors corresponds to Lagrangian manifold with real dimension not larger than d, one has $k\leq d=(n-k)k$. This gives $k\leq n/2(1+\sqrt{1-2/n})$ For $k=2$ this gives $k\leq n/2$. For $n\rightarrow\infty$ one has $k\leq n/2+1$. For $k>n/2$ one can change the roles of positive and negative helicities. It has been found that in certain sense the Grassmannian contributing to the twistor amplitude is positive.

   The notion of positivity found to characterize the part of Grassmannian contributing to the residue integral and also the minors and the argument of delta function [B33](see `http://tinyurl.com/yd9tf2ya`) would suggest that it is also real sub-space in some sense and this finding supports this picture.

   The delta function constraint forcing $C\cdot Z$ to zero must also make sense. $C\cdot Z$ defines $k\times 6$ matrix and also now one must consider eigenvalues of $C\cdot Z$. Positivity suggest reality also now. Z adds $4\times n$ degrees of freedom and the number $6\times k$ of additional conditions is smaller than $4 \times n$. $6k \leq 4 \times n$ combined with $k \leq n/2$ gives $k \leq n/2$ so that the conditions seems to be consistent.

3. The c-number property for the cyclic minors could define the analog of Lagrangian manifold for the phase space or Kähler manifold. One can of course ask, whether Kähler structure of $Gr(k,n)$ could generalize to quantum context and give the integration region as a sub-manifold of Lagrangian manifold of $Gr(k,n)$ and whether the twistor amplitudes could reduce to integral over sub-manifold of Lagrangian manifold of ordinary $Gr(k,n)$.

To sum up, I have hitherto thought that TGD allows to get rid of the idea of quantization of coordinates. Now I have encountered this idea from totally unexpected perspective in an attempt to understand how 8-D masslessness and its twistor description could relate to 4-D one. Grassmannians are however very simple and symmetric objects and have natural coordinates as $k \times n$ matrices interpretable as quantum matrices. The notion of quantum group could find very concrete application as a solution to the basic problem of the standard twistor approach. Therefore one can consider the possibility that they have quantum counterparts and at least the residue integrals reducing to c-numbers make sense for quantum Grassmannians in algebraic sense.

Chapter 8

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

8.1 Introduction

This article deals with two questions.

1. The ideas related to topological quantum computation [L113] suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of Hilbert space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum. I have considered this kind of idea already earli [K69].

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Could one replace the roots labelling states with group representations? Or could even the coefficient field for the state space be replaced with a ring of representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums of algebraic numbers in quantum-classical correspondence interpreted as a kind of category theoretic morphism, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k,C)$, $k = 2,3,4$ [A46, A45] to those of $SL(n,C)$. Is there a deep connection between finite subgroups of $SL(n,C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

8.1.1 Could one generalize arithmetics by replacing sum and product with direct sum and tensor product?

In the model for topological quantum computation (TQC) [B11, B10] quantum states in the representations of groups are replaced with entire representations (anyons). One can argue that this helps to guarantee stability: this generalization could be regarded as error correction code. In TGD, these representations would correspond to irreps of Galois groups or of discrete subgroups of the covering group for automorphisms of quaternions. Also discrete subgroups of $SL(2,C)$ assignable naturally to the tessellations of H^3 can be considered.

Tensor product $\otimes$ and direct sum $\oplus$ are commutative operations and very much like operations of ordinary arithmetics. One can also speak of positive integer multiples of representation. The algebras of irreps of various algebraic structures generated by $\oplus$ and $\otimes$ are applied quite generally in mathematics and especially so in gauge theories and conformal field theories and are known as fusion algebras (`https://cutt.ly/TLU3hvJ`) and quivers (`https://cutt.ly/xLU3zrM`).

Could the replacement of the roots of the EDD of the ADE group with representations of the finite subgroup of $SL(2,C)$ associated with the diagram make sense? The trivial representation would correspond to an additional node and lead to an extended Dynkin diagram (EDD).

Could one regard the irreps as quantum roots of an ordinary monic polynomial so that the ordinary algebraic numbers would have representation as state spaces? Could one obtain the full root diagram by a generalization of the Weyl group operation as reflection of root with respect to root? The first guess is that the isotropy group Gal_I of a root acts as a subgroup of Gal defines the polynomial, which gives the roots replaced by irreps and that Gal itself acts in the same role as the Weyl group.

McKay graph characterizes the rules for the tensor product compositions for the irreps of a finite group G, in particular Galois group. There is an excellent description of McKay graphs on the web (see `https://cutt.ly/zLzoAwF`). The article describes first the special McKay graphs for finite subgroups of $SL(2,C)$ and their geometric interpretation in terms of the geometry of Platonic solids and their denerate versions as regular polygons and shows that they turn out to correspond to EDDs for ADE type Lie algebras. Also general McKay graphs are considered.

8.1.2 McKay graphs and McKay correspondence

The McKay graphs are a special case of quiver diagrams (`https://cutt.ly/xLU3zrM`) and code for the tensor product decomposition rules for the irreps of finite groups [A71, A55].

For a general finite group, McKay graphs can be constructed in the following way. Consider any finite group G and its irreducible representations (irreps) ξ_i and assign to ξ_i vertices. Select one irrep V and assign also to it a vertex. For all tensor products $\xi_i \otimes V$ and decompose them to a direct sum of irreps ξ_j. If ξ_j is contained to $V \otimes \xi_i$ a_{ij} times, draw a_{ij} directed arrows connecting vertex i to vertex j. One obtains a weighted, directed graph with incidence matrix a_{ij}. Adjacency matrix plays a central role in graph theory.

McKay correspondence is only one of the mysteries related to MacKay graphs for finite subgroups of $SL(k,C)$, $k = 2,3,4$ and presumably also $k > 4$ [A46, A45]. The MacKay graphs correspond to EDDs for ADE type Lie groups having interpretations as Dynkin diagrams for ADE type affine algebras.

The classification of singularities of complex surfaces represents another example of McKay correspondence.

1. ADE Dynkin diagrams provide a classification of Kleinian singularities of complex surfaces having real dimension 4 and satisfying a polynomial equation $P(z_1,z_2,z_3) = 0$ with $P(0,0,0) = 0$ so that the singularity is at origin [A55] (`https://cutt.ly/5LQPyhy`). The finite subgroups of $SL(2,C)$ naturally appear as symmetries of the singularities at origin.

2. In the TGD framework, this kind of complex surfaces could correspond to surfaces with an Euclidean signature of induced metric as 4-surfaces in $E^2 \times CP_2 \subset M^4 \times CP_2$. What I call CP_2 type extremals have light-like M^4 projection as deformations of the canonically imbedded CP_2. These surfaces could correspond to deformations of CP_2 type extremals. One can ask whether one could assign ADE type affine algebras as affine algebras with these singularities.

8.2 Could the arithmetics based on direct sum and tensor product for the irreps of the Galois group make sense and have physical meaning?

The idea about the generalization of the mathematical structures based on integer arithmetics with arithmetics replacing $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$ raises a bundle of questions. This idea makes sense also for the finite subgroups of $SU(2)$ defining the covering group of quaternion automorphism having a role similar to that of the Galois group.

What motivates this proposal is that the extensions of rationals and their Galois groups are central in TGD. Polynomials P with integer coefficients are proposed to determine space-time surfaces by M^8-H duality in terms of holography based on the realization of dynamics in M^8 in terms of roots of P having interpretation as mass shells. Holography is realized in terms of the condition that the normal space of the space-time surface going through the mass shells has associative normal space [L82, L83].

8.2.1 Questions

The following questions and considerations are certainly very naive from the point of view of a professional mathematician and the main motivation for the mathematical self ridicule is that there are fascinating physical possibilities involved.

The basic question is whether $\otimes$ and $\oplus$ can give rise to quantum variants of rings of integers and even algebraic integers defined in terms of quantum roots of ordinary polynomial equations and could one even generalize the notion of number field: do quantum variants of extensions of rationals, finite fields, and p-adic number fields make sense?

Recall that also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L43, L42] [K62, K43, K44].

Quantum polynomials

To build extensions of rationals, one must have polynomials. The notion of polynomial playing central role in M^8-H duality [L82, L83], or rather the notion of a root of polynomial, generalizes.

1. Polynomials would look exactly like ordinary monic polynomials, with the real unit replaced with identity representation but their quantum roots would be expressible as direct sums of irreps associated with a given extension of rationals.
2. One would obtain roots as direct sums of the generators of the extension which could correspond to irreps of the isotropy group Gal_I of Galois group Gal. McKay graph would define the multiplication rules for the tensor products appearing in the polynomial whose coefficients would be quantum counterparts of ordinary (positive) integers.
3. Also a generalization of an imaginary unit could make sense for p-adic ring and finite fields as a root of a polynomial. Note that $\sqrt{-1}$ can exist for p-adic number fields. Also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L43, L42] [K62, K43, K44].

Does one obtain additive and multiplicative group structures, rings, and fields?

Could one give to the space spanned by irreps a structure of ring or even field?

1. Could one replace algebraic integers of the ordinary extension of rationals with direct sums of the n_C irreps of Galois group G, where n_C is the number of classes of G? Note that the dimensions n_i of irreps satisfy the formula $\sum n_i^2 = n_C$.

   If $\oplus$ corresponds to $+$ for ordinary integers, only non-negative integers can appear as coefficients so that one would have semigroups with respect to both $\oplus$ and $\otimes$.
2. The inverse with respect to $\oplus$ requires that negative multiples of quantum integers make sense. This is possible in p-adic topology: the number -1 would correspond to the quantum part of the integer $(p-1)\sum_{\oplus} p^{\oplus n}$. The summands in this expression would have p-adic norms p^{-n}. This allows to define also the negatives of other roots playing the role of generator of the quantum extension of rationals.
3. Is even the quantum analog of a number field possible? If one requires multiplicative inverse, only the finite field option remains under consideration since the quantum variant of $1/p^k$ does not make sense since one has $p \equiv= 0$. If one requires group structure for only $\oplus$, quantum p-adics remain under consideration.

Can one map the numbers of quantum extensions of rationals the numbers of ordinary extensions?

Concerning the physical interpretation, it would be important to map the quantum variants of algebraic integers to their real counterparts. Mathematicians might talk of some kind of category theoretical correspondence.

1. Since the same polynomial would have ordinary roots and quantum roots, the natural question is whether the quantum roots can be mapped to the ordinary roots.
2. If the quantum roots correspond to roots of the Dynkin diagram as quantum numbers in quantum extension of rationals, it should be able to map all quantum roots of the ADE type affine algebra to ordinary roots. This requires that sums with respect to $\oplus$ correspond to sums with respect to $+$: additivity of quantum numbers would hold true at both levels and one would have category theoretic correspondence as algebraic isomorphisms.

   Note that Galois confinement means that 4-momenta and other quantum numbers of states are integer valued, when one uses the momentum scale defined by causal diamond (CD). This means that they would correspond to $\oplus$ multiples of trivial representation of the Galois group acting as Weyl group.
3. What about the tensor products of roots appearing in the McKay graph? Can one require that the products with respect to $\otimes$ correspond to products with respect to $\times$. Only $\otimes$ does appear in the generation of the quantum roots of a given KM algebra representation.

   What about quantum variants of quantum states? If the quantum variants of p-adic integers or finite fields appear also as a coefficient field of quantum states, one can always express the coefficients as direct sums of quantum roots and map these sums to sums of ordinary polynomial roots, that is algebraic numbers. Extensions of rationals can appear as coefficient fields for Hilbert spaces.

   If one assumes that only quantum variants of p-adic numbers with a finite number of the pinary digits and their negatives are possible, they can be mapped to numbers in algebraic extension. One could overcome the problems related to the definition of inner product when finite field or p-adic numbers define the coefficient field for Hilbert state.
4. For generalized finite fields, the notions of vector space and matrix algebra, hermiticity and unitarity, and eigenvalue problem could be generalized. For instance, eigenvalues of a Hermitian operator could be just real numbers. Also a relatively straightforward looking generalization of group theory can be imagined, and would be obtained by replacing the elements of the matrix group with the elements of a generalized finite field.

8.2.2 Could the notion of quantum arithmetics be useful in the TGD framework?

These ideas might find an application in TGD.

1. The quantum generalization of the notion of rationals, p-adic number fields, and finite fields could be defended as something more than a mere algebraic game. In particular, in TGD the ramified primes of extension of rationals correspond to physically important p-adic primes, especially the largest ramified prime of the extension. Algebraic prime is a generalization of the notion of ordinary prime. Also its generalization could make sense and give rise to the notion of quantum prime.

   Unfortunately, the extension of finite field F_p induced by a given extension of rationals does not exist for the ramified primes appearing as divisors in the discriminant determined by the product of root differences.

   Could the generalization of the notion of finite field save the situation? Topological quantum computations (TQC) relying on Galois representations as counterparts for anyons would mean an increase of the abstraction level replacing numbers of algebraic extension with representations of Galois group as their cognitive representations.

   One can assign also to the possibly unique monic polynomial P_c defining the n_c-dimensional extension, a discriminant, call it D_c. For the primes dividing the discriminant D of P but not D_c, the quantum counterpart of the finite-field extension could make sense.

2. In TGD, the roots of polynomials define 3-D mass and energy shells in M^8 in turn defining holographic data defining 4-D surface in M^8 mapped to space-time surfaces in H by M^8-H duality. Could one consider quantum variants of the polynomial equations defining space-time surfaces by holography in the generalized extensions of rationals based on representations of Galois groups?

   Could monic polynomials define quantum variants of 4-surfaces or at least of discretizations of hyperbolic spaces H^3 as 3-D sections of 4-surface in M^8 defined as roots of polynomial P and containing holographic data as cognitive representation? Mass shells would be mapped by M^8-H duality to light-cone proper time hyperboloids in H.

   The interiors of 4-surfaces in M^8 would contain very few points of cognitive representation as momentum components in the extension of rationals defined by the polynomial P. Mass shells and their H images would be different and represent a kind of cognitive explosion. The presence of fermions (quarks) at the points of cognitive representation of given mass shells would make them active.

3. Could the transition from the classical to a quantum theory, which also describes cognition, replace discrete classical mass shells as roots of a polynomial in M^8 with roots with direct sums of irreps of the Galois group?

   This idea would conform with category theoretic thinking which leaves the internal structure of the basic object, such as point, open. That points of cognitive representations would be actually irreducible representations of the Galois groups would reveal a kind of cognitive hidden variables and quantum cognition.

These ideas are now completely new. I have earlier considered the possibility that points could have an infinite complex internal structure and that the "world of classical worlds" could be actually M^8 or H with points having this structure [K89]. I have also considered the possibility that Hilbert spaces could have arithmetic structure based on $\otimes$ and $\oplus$ with Hilbert spaces with prime dimension defining the primes [K69].

"Do not quantize" has been my motto for all these years but in this framework, it might be possible to talk about quantization of cognition as a deformation of number theory obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary numbers with representations of Galois group. Perhaps this quantization could apply to cognition.

8.3 What could lurk behind McKay correspondence?

The appearance of EDDs in so many contexts having apparently no connection with affine algebras is an almost religious mystery and one cannot avoid the question of whether there is a deep connection between some finite groups G, in particular finite subgroups of $SL(n,C)$, and affine algebras. In the TGD framework M^8-H duality relates number theoretic and differential geometric views about physics and the natural question whether it could provide some understanding of this mystery.

M^8-H duality also suggests how to understand the Langlands correspondence: during years I have tried to understand Langlands correspondence [A40, A39] from the TGD perspective [K47, L26].

8.3.1 McKay correspondence

There is an excellent article of Khovanov [A71] describing the details of McKay correspondence for the discrete subgroups of $SL(2,C)$ (`https://cutt.ly/1LQDqce`). There is also an article "McKay correspondence" by Nakamura about various aspects of McKay correspondence [A55] (`https://cutt.ly/5LQPyhy`).

1. Consider finite subgroups G of $SL(2,C)$. The McKay graph for the tensor products of what is called canonical (faithful) 2-D representation V of G with irreps ξ_i of G corresponds to an extended Dynkin diagram with one node added to a Dynkin diagram. Note that V need not be always irreducible.

The constraints on the graph come from the conditions for the dimension $d = 2d_j$ of the tensor product $V \otimes \xi_i$ satisfies $2d_i = \sum_j a_{ij} d_j$, where the sum is over all vertices directed away from the vertex i. If arrows in both directions are present, there is no arrow. This implies that the dimensions d_j associated with the vertex have G.C.D equal to 1.

2. Dynkin diagram in turn describes the minimal set of roots from which the roots of Lie algebra can be generated by repeated reflections with respect to roots. EDDs can be assigned to affine algebras and for them the eigenvalues of the adjacency matrix are not larger than 2. The maximum of the eigenvalues measures the complexity of the graph.

3. The Weyl group characterizes the symmetries of the root diagram and is generated by reflections of roots with respect to other roots. The Dynkin diagram contains a minimal number of roots needed to generate all roots by reflections as Weyl orbits of the roots of the Dynkin diagram. The action of the Weyl group leads away from the Dynkin diagram since otherwise this set of roots would not be minimal.

   The number of lines characterizes the angle between the roots i and j. For ADE groups $a_{ij} = 1$ codes for angle of 120 degrees $2\pi/3$, $a_{ij} = 2$ corresponds to 135 degrees, and $a_{ij} = 3$ to 150 degrees. $a_{ij} = 0$ means either angle π or $\pi/2$. In the general case, there are 2-valent and 3-valent nodes depending on the number of oriented lines emerging from the node.

   For instance, in the case of a triangle group with 6 elements with irreps $1, 1_1, 1_2$. The canonical representation to 2-D reducible representation decomposes to $1_1 + 1_2$ so that there are 3 vertices involved corresponding to 1_1 and 1_2 and 1. It is easy to see that the adjacency matrix is symmetric and gives rise to an EDD with 3 vertices. From the corresponding Dynkin diagram, representing 2 neighboring roots of the root diagram one obtains the entire root diagram by repeated reflections having 6 roots characterizing the octet representation of A_2 ($SU(3)$).

4. What kind of McKay graphs are associated with other than canonical 2-D representations in the case of rotation groups? Every representation of G belongs to some minimal tensor power $V^{\otimes k}$ and one can study the MacKay diagrams assignable to $V^{\otimes k}$. It is easy to see that the number of paths connecting vertices i and j in the McKay graph $M^k(V)$ for $V^{\otimes k}$ can be understood in terms of the McKay graph $M(V)$ for V. The paths leading from i to j are all k-edged paths along $M(V)$ leading from i to j.

   The symmetry of the adjacency matrix A implies that forth and back movement along $M(V)$ is possible. The adjacency matrix has the same number of nodes and equals the $k:th$ power A^k of A so that extended ADE type Dynkin diagrams are not in question.

8.3.2 Questions

McKay correspondence raises a series of questions which I have discussed several times from the TGD point of view several times [L36, L77, L76]. In the following these questions are discussed by introducing the possibility of quantum arithmetics and cognitive representations as new elements.

Why would $SL(2,C)$ be so special?

$SL(2,C)$ is in a very special role in McKay correspondence. Of course, also the finite subgroups of other groups could have a special role and it is actually known that $SL(n,C)$ $n < 5$ are in the same role, which suggests that all groups $SL(n,C)$ have this role [A46, A45].

Why? In the TGD framework, a possible reason for the special role of $SL(2,C)$ acts as the double covering group of the isometries of the mass shell $H^3 \subset M^4 \subset M^8$ and its counterpart in $M^4 \times CP_2$ obtained by $M^8 - H$ correspondence. $SL(2,C)$ has also natural action on the spinors of H. The finite subgroups relate naturally to the tessellations of the mass shell H^3 leaving the basic unit of tessellation invariant.

The tessellations could naturally force the emergence of ADE type affine algebras as dynamical symmetries in the TGD framework. In fact, the icosa-tetrahedral tessellation plays a key role in the proposed model of the genetic code based on Hamiltonian cycles at icosahedron and tetrahedron [L98].

Why does the faithful representation have a special role?

The mathematical reason for the special role of the faithful canonical representation V is that its tensor powers contain all irreps of the finite group: the tensor product structure for other choices of V can be deduced from that for canonical representations. It is known that any irrep V, which is faithful irrep of G, generates the fusion algebra.

However, this kind of irrep might fail to exist. If G has a normal subgroup H and the irrep χ has H as kernel then the powers of χ contain only the irreps of G/H. In the article "McKay Connectivity Properties of McKay Quivers" by Hazel Brown [A50] (`https://arxiv.org/pdf/2003.09502.pdf`) it was shown that the number of connected components of the McKay quiver is the number of classes of the G, which are contained in H. For instance, the classes associated with the center of G are such (Z_n for $SL(n, C)$).

For simple groups this does not happen but in the case of Galois groups assignable to composite polynomials one has a hierarchy of normal subgroups and this kind of situation can occur since the number of classes of G contained in normal subgroups can be non-vanishing.

2-D representation is also in a special role physically in the TGD framework, the ground states of affine representation correspond to a 2-D spinor representation since quarks are the fundamental particles.

The irreps of the affine representation are obtained as tensor products of the irrep associated with the affine generators with it. Cognitive representations imply a unique discretization and this forces discrete subgroups of $SL(2, C)$ and implies that the irreps of $SL(2, C)$ decompose to irreps of a discrete subgroup. Therefore the quivers for their tensor products appear naturally.

Electroweak gauge group $U(2)$ corresponds to the holonomy group $U(2)$ for CP_2 and for $SU(2)_w$ the McKay correspondence holds true. Also the isometry group $SU(3)$ of CP_2 is assumed to appear as affine algebra. Discretization due to cognitive representations in M^8 induces discretization in H and CP_2. The replacement of $SU(3)$ with its discrete subgroups would decompose irreps for $SU(3)$ to irreps of $SU(3)$. $SL(3, C)$ allows analog of McKay correspondence [A46] so that also the finite subgroups of $SU(3)$ allow it.

What about McKay graphs for more general finite groups?

The obvious question concerns the generality of McKay correspondence. What finite groups and therefore corresponding Galois groups correspond to representations of affine type algebras.

In the general case, the McKay graphs look very different from Dynkin diagrams. The article "Spectral measures for G_2" of Evans and Pugh [A35] (`https://cutt.ly/hLQO7HE`) is of special interest from the TGD point of view since G_2 is the automorphism group of octonions. G_2 however naturally reduces to $SU(3)$ corresponding to color isometries in H. The article discusses in detail McKay graphs for the finite subgroups of G_2. These finite subgroups correspond to those for $SU(2) \times SU(2)$ and $SU(3)$ plus 7 other groups. The McKay graphs for the latter groups contain loops are very complex and contain loops.

What can one say about finite groups, which allow McKay correspondence.

1. ADE diagrams are known to classify the following three finite simple groups, the derived group F'_{24} of the Fischer F_{24}, the Baby monster B and the Monster M are related with E_6, E_7 and E_8 respectively [A55] (`https://cutt.ly/5LQPyhy`). In the TGD framework, this finding inspires the question whether these groups could appear as Galois groups of some polynomial and give rise to E_6, E_7 and E_8 as dynamical symmetries.

   In the TGD framework, one can ask whether also the above mentioned simple groups could appear as Galois groups. What is fascinating that monster would relate to icosahedron and dodecahedron: icosahedron and tetrahedron play key role in TGD inspired model of genetic code, in particular in the proposal that it relates to tetra-icosahedral tessellation of hyperbolic space H^3 [L98].

2. The article [A91](`https://cutt.ly/jLQPgkQ`) mentioned the conjecture that the tensor product structure for the finite subgroups of $SU(3)$ could relate to the integrable characters for some representations of affine algebra associated with $SU(3)$. This encourages the conjecture that this is true also for $SU(n)$.

In TGD, this inspires the question whether finite Galois groups representable as subgroups of $SU(3)$ could give rise to corresponding affine algebras as dynamical symmetries of TGD.

3. Butin and Perets demonstrated McKay correspondence in the article "Branching law for finite subgroups of $SL(3,C)$ and McKay correspondence" [A46] (https://cutt.ly/CLQPvp2) for finite subgroups of $SL(3,C)$ in the sense that branching law defines a generalized Cartan matrix. In the article "Branching Law for the Finite Subgroups of SL(4,C) and the Related Generalized Poincare Polynomials" [A45] (https://cutt.ly/mLQPQnT) shows that the same result holds true for $SL(4,C)$, which suggests that it is true for all $SL(n,C)$.

   A generalization to finite subgroups of $SL(n,C)$ is a natural guess. Therefore Galois groups with this property could be assigned with affine algebras characterized by the generalized Cartan matrices and could correspond to physically very special kind of extensions of rationals,

8.3.3 TGD view about McKay correspondence

The key idea is that one replaces quantum numbers representable as sums of the roots of Lie algebra with representations of the isotropy group of Galois group which is same as a finite subgroup of say $SL(2,C)$ and that Galois groups acts as Weyl group. The Weyl group codes for the differential geometric notion of symmetry realized by Lie groups and Galois group codes for the number theoretic view of symmetry. This correspondence would represent a facet of the duality between number theory and differential geometry.

Quantum roots as direct sums of irreps

Consider first the correspondence between quantum roots (or more generally weights defined as dual space of roots) and ordinary roots (weights) as quantum numbers.

1. The representations of finite group G (say subgroup of $SL(2,C)$) represented by the isotropy group Gal_I of Galois group for a given root, would appear as labels of states rather than as counterparts of states. Galois group Gal itself would act as Weyl group on the roots.

2. Quantum numbers as labels of quantum states would be replaced with representations of Gal_I. The additivity of quantum numbers would correspond to the additivity of representations with respect to $\oplus$. Tensor product for the representations would be analogous to multiplication of quantum quantum numbers so that they would form an algebra. An abstraction or cognitive representation would be in question. Since the roots of the Dynkin diagram correspond to roots of a monic polynomial, one could map them to ordinary algebraic numbers. Same applies to the root of affine representations.

Could also the quantal version of the coeffient field of the state space make sense?

Could also the coefficient field of state space be replaced with a quantum variant of p-adic numbers or of finite field?

1. Here one encounters a technical problem that is encountered already at the level of ordinary p-adics and finite fields. Inner products are bilinear. If norm squared is defined as a sum for the squares of the coefficients of the state in the basis of n states, the non-well-ordered character of p-adics implies that one can have states for which this sum vanishes in p-adic and finite fields.

   In the p-adic case, allowance of only finite number of non-vanishing binary digits for the coefficients might help and would conform with the idea about finite measurement resolution as a pinary cutoff. One could even allow negatives of integers with finite number of pinary digits if the p-adic quantum integers are mapped to the real counterparts.

2. There is also a problem associated with the normalization factors of the states, which cannot be p-adic integers in general. Overall normalization does not however matter so that this problem might be circumvented.

Physical predictions would require the map of the quantum integers to real ones. The fact that quantum integers are $\oplus$ sums of quantum roots of ordinary monic polynomials, makes this possible. The irreps appearing as coefficients of states would be mapped to ordinary algebraic numbers and the normalization of the states could be carried out at the level of the ordinary algebraic numbers.

What about negative multiples of quantum roots

If the quantum roots of a polynomial correspond to irreps of the Galois group, one encounters a technical problem with negative multiples of quantum roots.

1. The negatives of positive roots correspond to -1 multiples of irreps. This does not make sense in ordinary arithmetics. p-Adically -1 corresponds to $(p-1)(1+p+p^2+...)$ and would correspond to infinite $\oplus$-multiple of root but decompose to p^n multiples to which one can assign norm p^{-k} so that the sum converges: $-\xi_i = (p-1)(Id \oplus pId \oplus p^2 Id \oplus ...)\xi_i$.

   One has finite measurement resolution so that the appearance of strictly infinite sums is highly questionable. Should one consider only finite sums of positive roots and their negatives but how should one deal with the negatives?

   Could the creation operators labelled by negative roots correspond to annihilation operators with positive roots as in the case of super-Virasoro and affine algebras. Note that if one restricts to ordinary integers at the level of algebra as one must to for supersymplectic and Yangian algebras, one must consider only half-algebras with generators, which have only non-negative conformal weights. This does not make sense for ordinary affine generators.

2. The most plausible solution of the problem relies on the proposed categorical correspondence between quantum roots and ordinary roots as roots of the same monic polynomial. One could map the quantum roots and their direct summands to sums of ordinary roots and this would make sense also for the negatives of positive roots with a finite number of summands. It would be essential that p-adic integers correspond to finite ordinary integers and to their negatives and are mapped to numbers in an extension of rationals. As found, this map would also allow us to circumvent the objections against the quantum variant of the state space.

3. Could zero energy ontology (ZEO) come to the rescue? In zero energy ontology creation and annihilation operators are assigned with the opposite boundaries of causal diamond (CD). Could one assign the negative conformal weights and roots with the members of state pairs located at the opposite boundary of CD?

   This works for the Virasoro and affine generators but this kind of restriction is unphysical in the case of eigenvalues of L_z with both signs? Why would opposite values of L_z be assigned to opposite boundaries of CD?

Wheels and quantum arithmetics

Gary Ehlenberg gave a link to a Wikipedia article telling of Wheel theory (`https://cutt.ly/RZnUB5y`). Wheel theory could be very relevant to the TGD inspired idea about quantum arithmetics.

I understood that Wheel structure is special in the sense that division by zero is well defined and multiplication by zero gives a non-vanishing result. The wheel of fractions, discussed in the Wikipedia article as an example of wheel structure, brings into mind a generalization of arithmetics and perhaps even of number theory to its quantum counterpart obtained by replacing + and - with direct sum $\oplus$ and tensor product $\otimes$ for irreps of finite groups with trivial representation as multiplicative unit: Galois group is the natural group in TGD framework.

Could wheel structure provide a more rigorous generalization of the notions of the additive and multiplicative inverse of the representation in order to build quantum counterparts of rationals, algebraic numbers and p-adics and their extensions?

1. One way to achieve this is to restrict consideration to the quantum analogs of finite fields $G(p,n)$: + and x would be replaced with $\oplus$ and $\otimes$ obtained as extensions by the irreps of the

Galois group in TGD picture. There would be quantum-classical correspondence between roots of quantum polynomials and ordinary monic polynomials.

2. The notion of rational as a pair of integers (now representations) would provide at least a formal solution of the problem, and one could define non-negative rationals.

   p-Adically one can also define quite concretely the inverse for a representation of form $R = 1 \oplus O(p)$ where the representation $O(p)$ is proportional to p (p-fold direct sum) as a geometric series.

3. Negative integers and rationals pose a problem for ordinary integers and rationals: it is difficult to imagine what direct sum of -n irreps could mean.

   The definition of the negative of representation could work in the case of p-adic integers: $-1 = (p-1) \otimes (1 \oplus p * 1 \oplus p^2 * 1 \oplus ...)$ would be generalized by replacing 1 with trivial representation. Infinite direct sum would be obtained but it would converge rapidly in p-adic topology.

4. Could $1/p^n$ make sense in the Wheel structure so that one would obtain the quantum analog of a p-adic number field? The definition of rationals as pairs might allow this since only non-negative powers of p need to be considered. p would represent zero in the sense of Wheel structure but multiplication by p would give a non-vanishing result and also division with p would be well-defined operation.

Galois group as Weyl group?

The action of the Weyl group as reflections could make sense in the quantum arithmetics for quantum variants of extensions of p-adics and finite fields. The generalized Cartan matrix $C_{ij} = d\delta_{ij} - n_{ij}$, where n_{ij} is the number of lines connecting the nodes i and j and d is the dimension of V, is indeed well-defined for any finite group and has integer valued coefficients so that Weyl reflection makes sense also in quantum case.

Can one identify the Weyl group giving the entire root diagram number theoretically? The natural guess is $Gal = W$: Gal would define the Weyl group giving the entire root diagram from the Dynkin diagram by reflections of the roots of the EDD. One can assign to Gal an extension defined by a monic polynomial P with Galois group Gal.

How the group defining the McKay graph is represented?

How the group G defining the McKay graph is represented? The irreps of G should have natural realization and the quarks at mass shells would provide these representations.

One can consider two options. The first option is based on the isotropy group G_I of $Gal = W$ leaving a given root invariant. Second option is based on the finite subgroup of $SU(2)$ as a covering group of quaternion automorphisms.

1. The subgroup $Gal_I \subset Gal$ acting as an isotropy group of a given root of Gal would naturally define the EDD since the action of $Gal = W$ would not leave its nodes as irreps of Gal_I invariant.

   The root diagram should be the orbit of the EDD under $Gal = W$. The irreps of the EDD would correspond to the roots of a monic polynomial P_I associated with Gal_I and having $n_c + 1$ quantum roots. The quantum roots would be in the quantum extension defined by a monic polynomial P for Gal so that the action of Gal on EDD would be well-defined and non-trivial.

2. In the TGD framework, the mass mass squared values assignable to the monic polynomial representing the EDD correspond to different mass squared values. There is no deep reason for why the irreps of Gal_I could not correspond to different mass squared values and in the TGD framework the symmetry breaking $Gal \rightarrow Gal_I$ is the analog for the symmetry breaking in the Higgs mechanism.

   In the recent case this symmetry breaking would be associated with $Gal_I \rightarrow Gal_{I,I}$ and imply that quantum roots correspond to different mass squared values. At the level of affine algebra this could mean symmetry breaking since the different roots would not have different mass squared values.

If Gal acts as a Weyl group, the McKay graph associated with Gal_I corresponds to the EDD. Gal_I is a subgroup of Gal so that the action of $Gal = Weyl$ on the quantum roots of the monic polynomial P_I would be non-trivial and natural. Could Gal_I be a normal subgroup in which case Gal/Gal_I would be a group and one would have a composite polynomial $P = Q \circ P_1$? This cannot be true generally: for instance for A_p, p prime and E_6 the W is simple. For E^7 and E^8 W is a semidirect product.

3. There is an additional restriction coming from the fact that Gal_I does not affect the rational parts of the 4-momenta. Is it possible to have construct irreps for a finite subgroup of $SL(2,C)$ or even $SL(n,C)$ using many quark states at a given mass shell? The non-rational part of 4-momentum corresponds to the "genuinely" virtual part of virtual momentum and for Galois confined states only the rational parts contribute to the total 4-momentum. Could one say that these representations are possible but only for the virtual states which do not appear as physical states: cognition remains physically hidden.

The very cautious, and perhaps over-optimistic conclusion, would be that only Galois groups, which act as Weyl groups, can give rise to affine algebras as dynamical symmetries. For this option, one would obtain cognitive representations for the isotropy groups of all Galois groups. For Galois groups acting as Weyl groups, EDDs could define cognitive representations of affine algebras. Also cognitive representations for finite subgroups of $SL(n,C)$ and groups like Monster would be obtained.

For the second option in which the subgroup G of quaternionic automorphisms affecting the real parts of 4-momenta is involved. This representation would be possible only for the subgroups of $SL(2,C)$. In this case one would have 3 different groups $Gal = W$, Gal_I and G rather than $Gal = W$ and Gal_I.

1. Quaternionic automorphisms are analogous to the Galois group and one can ask whether the finite subgroups G of quaternionic automorphisms could be directly involved with cognitive representations. This would give McKay correspondence for $SL(2,C)$ only. The quaternionic automorphism would affect the rational part of the 4-momentum in an extension of rationals unlike the Galois group which leaves it invariant. The irrep of G would be realized as many-quark states at a fixed mass shell. Different irreps would correspond to different masses having interpretation in terms of symmetry breaking.

2. Also now one would consider the extension defined by the roots of a monic polynomial P having Galois group $Gal = W$ associated with the corresponding EDD. P_I would give quantum roots defining the Dynkin diagram and define the mass squared values assignable to irreps of G.

3. The situation would differ from the previous one in that the action of G_I on irreps would be replaced by the action of G. Indeed, since G_I leaves the rational part of the 4-momentum invariant, G_I cannot represent G as a genuine subgroup of rotations.

4. The roots would correspond to irreps of a subgroup G of quaternionic automorphisms, which would affect the 4-momenta with a given mass shell and define an irrep of G. Different roots of P would define the mass shells and irreps of G associated with EDD as a McKay graph.

Information about Weyl groups of ADE groups

The Wikipedia article about Coxeter groups (`https://en.wikipedia.org/wiki/Coxeter_group#Properties`), which include Weyl groups, lists some properties of finite irreducible Coxter groups and contains information about Weyl groups. This information might be of interest in the proposed realization as a Galois group.

- $W(A_n) = S_{n+1}$, which is the maximal Galois group associated with a polynomial of degree $n+1$.

- $W(D_n) = Z_2^{n-1} \rtimes S_n$.
- $W(E_6)$ is a unique simple group of order 25920.
- $W(E^7)$ is a direct product of a unique simple group of order 2903040 with Z_2.
- $W(E_8)$ acts as an orthogonal group for F_2 linear automorphisms preserving a norm in Ω/Z_2, where Ω is E_8 lattice (`https://mathoverflow.net/questions/230120/the-weyl-group-of-e8-versus-o-82/230130#230130`)
- $W(B_n) = W(C_n) = Z_2^n \rtimes S_n$.
- $W(F_4)$ is a solvable group of order 1152 , and is isomorphic to the orthogonal group $O_4(F_3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the field F_3.
- $W(G_2) = D_6 = Z_2 \rtimes Z_6$.

Candidates for symmetry algebras of WCW, inclusions of hyperfinite factors, and Galois groups acting as Weyl groups

TGD allows several candidates for the symmetry algebras acting in WCW. The intuitive guess is that the isometries and possibly also symplectic transformations of the light-cone boundary $\delta M^4_+ \times CP_2$ define isometries of WCW whereas holonomies of H induce holonomies of WCW.

1. In TGD, supersymplectic algebra SSA could replace affine algebras of string models.
2. By the metric 2-dimensionality of the light-cone boundary δM^4_+, one can assign to it an infinite-dimensional conformal group of sphere S^2 in well-defined sense local with respect to the complex coordinate z of S^2. These transformations can be made local with respect to the light-like coordinate r of δM^4_+. Also a S^2-local radial scaling making these transformations isometries is possible. This is possible only for M^4 and makes it unique.

   Whether SSA or this algebra or both act as isometries of WCW is not clear: see the more detailed discussion in the Appendix of [L110].
3. One can assign this kind of hierarchy also to affine algebras assignable to the holomies of H and Virasoro algebras and their super counterparts. The geometric interpretation of these algebras would be as analogs of holonomy algebras, which serve at the level of H as the counterparts of broken gauge symmetries: isometries would correspond to non-broken gauge symmetries.

All these algebras, refer to them collectively by A, define inclusion hierarchies of sub-algebras A_n with the radial conformal weights given by n-ples of the weights of A.

1. I have proposed that the hierarchy of inclusions of hyperfinite factors of type II_1 to which one could perhaps assign ADE hierarchy could correspond to the hierarchies of subalgebras assignable to SSA and labelled by integer n: the radial conformal weights would be multiples of n. Only non-negative values of n would be allowed.
2. For a given hierarchy A_n, one has $n_1 \mid n_2 \mid$ ....., where $\mid$ means "divides". At the n:th level of the hierarchy physical states are annihilated by A_n and $[A_n, A]$. For isometries, the corresponding Noether charges vanish both classically and quantally.
3. The algebra A_n effectively reduces to a finite-D algebra and A_n would be analogous to normal subgroup, which suggests that this hierarchy relates to a hierarchy of Galois groups associated with composite polynomials and having a decomposition to a product of normal subgroups.
4. These hierarchies could naturally relate to the hierarchies of inclusions of hyperfinite factors of type II_1 and also to hierarchies of Galois groups for extensions of rationals defined by composites $P_n \circ P_{n-1} \circ ...P_1$ of polynomials.

The Galois correspondence raises questions.

1. Could the Dynkin diagrams for A_n be assigned to the McKay graphs of Galois groups acting as Weyl groups?
2. The Galois groups acting as Weyl group could be assigned to finite subgroups of $SU(2)$ acting as the covering group of quaternion automorphisms and of $SL(2, C)$ as covering group of H^3 isometries acting on tessellations of H^3. Also the finite subgroups of $SL(n, C)$ can be considered.

The proposed interpretation for the hierarchies of inclusions of HFFs is that they correspond to hierarchies for the inclusions of Galois groups defined by hierarchies of composite polynomials $P_n \circ ... \circ P_1$ interpreted as number theoretical evolutionary hierarchies.

If the relative Galois groups act as Weyl groups, they would be associated with the inclusions of HFFs naturally and the corresponding affine algebra (perhaps its finite field or p-adic variant) would characterize the inclusion. The proposed interpretation of the inclusion is in terms of measurement resolution defined by the included algebra. This suggests that a finite field version of the affine algebra could be in question.

This picture would suggest that hierarchies of polynomials for which the relative Galois groups act as Weyl groups are very special and could be selected in the number theoretical fight for survival.

One could argue that since number theoretic degrees of freedom relate to cognition, the quantum arithmetics for the irreps of Galois groups could make possible cognitive representations of the ordinary quantum states: roots would be represented by irreps. Irreps as quantum roots would correspond to ordinary roots as roots of the same monic polynomial and the direct sums of irreps would correspond to ordinary algebraic numbers.

About the interpretation of EDDs

An innocent layman can wonder whether the tensor products for 2-D spinor ground states for the discrete subgroups of the covering group of quaternionic automorphisms or of $SL(2, C)$ as covering group of H^3 isometries could give rise to representations contained by ADE type affine algebras characterized by the same EDD. These representations would be only a small part of the representations and perhaps define representation from which all states can be generated.

1. The reflections for the roots represented as irreps of Gal_I by Weyl group represented as Gal should assign to the irreps of G new copies so that the nodes of the entire root diagram would correspond to a set of representations obtained from the ground state. Infinite number of states labelled by conformal weight n is obtained.
2. Adjacency matrix A should characterize the angles between the roots represented as irreps? If the irreps of Gal_I and their Weyl images correspond to roots of a monic polynomial, they can be mapped to roots of an ordinary algebraic extension of rationals and the angles could correspond to angles between the points of extension regarded as vectors.

How the EDD characterizing the tensor products of the irreps of finite subgroups G with 2-D canonical representation V could define an ADE type affine algebra?

1. Roots are replaced with representations of G, which are in the general case direct sums of irreps. The identity representation should correspond to the scaling generator L_0, whose eigenvalues define integer value conformal weights.

   The inner products between the roots appearing in the Cartan matrix would correspond to the symmetric matrices defined by the structure constant n_{2ij} characterizing the tensor product. One might say that the inner products are matrix elements of the operator $\langle \xi_j | V \otimes \xi_i \rangle$ defined by the tensor product action of V. The diagonal elements of the Cartan matrix have value $+2$ and non-diagonal elements are negative integers or vanish.
2. Weyl reflections of roots with respect to roots involve negatives of the non-diagonal elements of Cartan matrix, which are negative so that the coefficient of the added root is positive represented as a direct sum. The negatives of the positive roots would correspond to negative integers and make sense only p-adically or for finite fields.

The expression for the generalized Cartan matrix for McKay graph is known (https://cuttly/QLRqrGt) for the tensor products of representation with dimension d and multiplicities n_{ij}^d and is given by

$$C_{ij}^d = d\delta_{ij} - n_{ij}^d \ .$$

For Dynkin diagrams the Cartan matrix satisfies additional conditions.

Weyl reflection (https://cutt.ly/kLRuXBP) of the root v with respect to root α in the space of roots is defined as

$$s_\alpha v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \ .$$

where $(.,.)$ is the inner product in V, which now corresponds to extension of rationals associated with Gal.

The Weyl chamber is identified as the set of points of V for which the inner products (α, v) are positive. The Weyl group permutes the Weyl chambers.

3. The root system would be obtained from the roots of the quantum Dynkin diagram by Weyl reflections (Galois group as Weyl group) with respect to other roots. The number N of these roots is $n = d_C + 1$,where d_C is the dimension of Cartan algebra of the Dynkin diagram. The number N_I of irreps is the same: $N = N_I$. The Cartan matrix defines metric in the roots so that the reflections are well-defined also in the generalized picture.

4. It would seem that one must introduce an infinite number of copies of the Lie algebra realized in the usual manner (in terms of oscillator operators) with copies labelled by the conformal weight n. The commutators of these copies would be like for an ordinary affine algebra. Only the roots as labels of generators and possibly also the coefficient field would be replaced with their quantum variants.

5. What about the realization of the scaling generator L_0, whose Sugawara representation involves bilinears of the generators and their Hermitian conjugates with negative conformal weight? In the case of finite fields there are no obvious problems. Also the analog of Virasoro algebra can be realized in the case of finite fields. If one restricts consideration to finite quantum integers and their negatives as conformal weights, the map of the roots to algebraic numbers in extension of rationals is well defined.

8.3.4 Could the inclusion hierarchies of extensions of rationals correspond to inclusion hierarchies of hyperfinite factors?

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (https://cutt.ly/lXp6MDB) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving

up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

It is. The number of conjugacy classes $g_i\sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD comes to rescue.

1. In the TGD framework, I am primarily interested in Galois groups, which are finite groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

One would have a deep connection between number theory and HFFs.

8.4 Appendix: Isometries and holonomies of WCW as counterparts of exact and broken gauge symmetries

The detailed interpretation of various candidates for the symmetries of WCW [L70] has remained somewhat obscure. At the level of H, isometries are exact symmetries and analogous to unbroken gauge symmetries assignable to color interactions. Holonomies do not give rise to Noether charges and are analogous to broken gauge symmetries assignable to electroweak interactions. This observation can serve as a principle in attempts to understand WCW symmetries.

The division to isometries and holonomies is expected to take place at the level of WCW and this decomposition would naturally correspond to exact and broken gauge symmetries.

8.4.1 Isometries of WCW

The identification of the isometries of WCW is still on shaky ground.

1. In the H picture, the conjecture has been that symplectic transformations of δM^4_+ act as isometries. The hierarchies of dynamically emerging symmetries could relate to the hierarchies of sub-algebras (SSA_n) of super symplectic algebra SSA [L70] acting as isometries of the "world of classical worlds" (WCW) [K80] [L104].

Each level in the hierarchy of subalgebras SSA_n of SSA corresponds to a transformation in which SSA_n acts as a gauge symmetry and its complement acts as genuine isometries of WCW: gauge symmetry breaking in the complement generates a genuine symmetry, which could correspond to Kac-Moody symmetry. By Noether's theorem, the isometries of WCW would give rise to local integrals of motion: also super-charges are involved. These charges are well-defined but they need not be conserved so that the interpretation as dynamically emerging symmetries must be considered.

The symmetries would naturally correspond to a long range order. The hierarchies of SSA$_n$:s, of relative Galois groups and of inclusions of hyperfinite factors [K105, K36] could relate to each other as $M^8 - H$ duality suggests [L112].

What can one say about the algebras SSA_n and the corresponding affine analogs KM_n (for affine algebras the generalized Cartan matrix is a product of a diagonal matrix with integer entries with a symmetric matrix). If n is prime, one can regard these algebras as local algebras in a finite field $G(p)$. Also extensions $G(p,n)$ of $G(p)$ induced by extensions of rationals can be considered. KM algebras in finite fields define what are called the incomplete Kac-Moody groups. Some of their aspects are discussed in the article "Abstract simplicity of complete Kac-Moody groups over finite fields" [A30]. It is shown that for $p > 3$, affine groups are abstractly simple, that is, have no proper non-trivial closed subgroups. Complete KM groups are obtained as completions of incomplete KM groups and are totally disconnected: this suggests that they define p-adic analogs of Kac-Moody groups. Complete KM groups are known to be simple.

2. There are also different kinds of isometries. Consider first the light-cone boundary $\delta M^4_+ \times CP_2$ as an example of a light-like 3-surface. The isometries of CP_2 are symmetries. ΔM^4_+ is metrically equivalent with sphere S^2. Conformal transformations of S^2, which are made local with light-like coordinate r of δM^4_+, induce a conformal scaling of the metric of S^2 depending on r. It is possible to compensate for this scaling by a local radial scaling of r depending on S^2 coordinates such that the transformation acts as an isometry of δM^4_+.

   These isometries of ΔM^4_+ form an infinite-D group. The transformations of this group differ from those of the symplectic group in that the symplectic group of δM^4_+ is replaced with the isometries of δM^4_+ consisting of r-local conformal transformations of S^2 involving S^2-local radial scaling. There are no localizat of CP_2 isometries. This yields an analog of KM algebra.

   This group induces local spinor rotations defining a realization of KM algebra. Also super-KM algebra defined in terms of conserved super-charges associated with the modified Dirac action is possible. These isometries would be Noether symmetries just like those defined by SSA.

3. What about light-like partonic orbits analogous to $\delta M^4_+ \times CP_2$. Can one assign with them Kac-Moody type algebras acting as isometries?

   The infinite-D group of isometries of the light-cone boundary could generalize. If they leave the partonic 2-surfaces at the ends of the orbit X^3_L, they could be seen as 3-D general coordinate transformations acting as internal isometries of the partonic 3-surface, which cannot be regarded as isometries of a fixed subspace of H. These isometries do not affect the partonic 3-surface as a whole and cannot induce isometries of WCW.

   However, if X^3_L is connected by string world sheets to other partonic orbits, these transformations affect the string world sheets and there is a real physical effect, and one has genuine isometries. Same is true if these transformations do not leave the partonic 2-surfaces at the ends of X^3_L invariant.

8.4.2 Holonomies of WCW

What about holonomies at the level of WCW? The holonomies of H acting on spinors induces a holonomy at the level of WCW: WCW spinors identified as Fock states created by oscillator operators of the second quantized H spinors. This would give a generalized KM-type algebra decomposing to sub-algebras corresponding to spin and electroweak quantum numbers. This algebra would have 3 tensor-factors. p-Adic mass calculations imply that the optimal number of tensor factors in conformal algebra is 5 [K52]. 2 tensor factors are needed.

1. SSA would give 2 tensor factors corresponding to δM^4_+ (effectively S^2) and CP_2. This gives 5 tensor factors which is the optimal number of tensor factors in p-adic mass calculations [K52]. SSA Noether charges are well-defined but not conserved. Could SSA only define a hierarchy of dynamical symmetries. Note however that for isometries of H conservation holds true.

2. Also the isometries of δM^4 and of light-like orbits of partonic 2-surfaces give the needed 2 tensor factors. Also this alternative would give inclusion hierarchies of KM sub-algebras with conformal weights coming as multiples of the full algebra. The corresponding Noether charges are well-defined but can one speak of conservation only in the partonic case? One can even argue that the isometries of $\delta M^4_+ \times CP_2$ define a more plausible candidate for inducing WCW isometries than the symplectic transformations. p-Adic mass calculations conform with this option.

To sum up, WCW symmetries would have a nice geometric interpretation as isometries and holonomies. The details of the interpretation are however still unclear and one must leave the status of SSA open.

Chapter 9

TGD as it is towards end of 2021

9.1 Introduction

The purpose of this article is to give a rough overall view about Topological Geometrodynamics (TGD) as it is now. It must be emphasized that TGD is only a vision, not a theory able to provide precise rules for calculating scattering amplitudes. A collective theoretical and experimental effort would be needed to achieve this.

It is perhaps good to explain what TGD is not and what it is or hoped to be. The article [L89] gives an overview of various aspects of TGD and is warmly recommended.

1. "Geometro-" refers to the idea about the geometrization of physics. The geometrization program of Einstein is extended to gauge fields allowing realization in terms of the geometry of surfaces so that Einsteinian space-time as abstract Riemann geometry is replaced with sub-manifold geometry. The basic motivation is the loss of classical conservation laws in General Relativity Theory (GRT)(see **Fig. 9.1**). Also the interpretation as a generalization of string models by replacing string with 3-D surface is natural.

   Standard model symmetries uniquely fix the choice of 8-D space in which space-time surfaces live to $H = M^4 \times CP_2$ [L2]. Also the notion of twistor is geometrized in terms of surface geometry and the existence of twistor lift fixes the choice of H completely so that TGD is unique [L45, L58](see **Fig. 9.6**). The geometrization applies even to the quantum theory itself and the space of space-time surfaces - "world of classical worlds" (WCW) - becomes the basic object endowed with Kähler geometry (see **Fig. 9.7**). General Coordinate Invariance (GCI) for space-time surfaces has dramatic implications. Given 3-surface fixes the space-time surface almost completely as analog of Bohr orbit (preferred extremal).This implies holography and leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces.

2. Consider next the attribute "Topological". In condensed matter physical topological physics has become a standard topic. Typically one has fields having values in compact spaces, which are topologically non-trivial. In the TGD framework space-time topology itself is non-trivial as also the topology of $H = M^4 \times CP_2$.

   The space-time as 4-surface $X^4 \subset H$ has a non-trivial topology in all scales and this together with the notion of many-sheeted space-time brings in something completely new. Topologically trivial Einsteinian space-time emerges only at the QFT limit in which all information about topology is lost (see **Fig. 9.3**).

   Practically any GCI action has the same universal basic extremals: CP_2 type extremals serving basic building bricks of elementary particles, cosmic strings and their thickenings to flux tubes defining a fractal hierarchy of structure extending from CP_2 scale to cosmic scales, and massless extremals (MEs) define space-time correletes for massless particles. World as a set or particles is replaced with a network having particles as nodes and flux tubes as bonds between them serving as correlates of quantum entanglement.

   "Topological" could refer also to p-adic number fields obeying p-adic local topology differing radically from the real topology (see **Fig. 9.10**).

3. Adelic physics fusing real and various p-adic physics are part of the number theoretic vision, which provides a kind of dual description for the description based on space-time geometry and the geometry of "world of classical" orders. Adelic physics predicts two fractal length scale hierarchies: p-adic length scale hierarchy and the hierarchy of dark length scales labelled by $h_{eff} = nh_0$, where n is the dimension of extension of rational. The interpretation of the latter hierarchy is as phases of ordinary matter behaving like dark matter. Quantum coherence is possible in all scales.

   The concrete realization of the number theoretic vision is based on $M^8 - H$ duality (see **Fig. 9.8**). The physics in the complexification of M^8 is algebraic - field equations as partial differential equations are replaced with algebraic equations associating to a polynomial with rational coefficients a X^4 mapped to H by $M^8 - H$ duality. The dark matter hierarchy corresponds to a hierarchy of algebraic extensions of rationals inducing that for adeles and has interpretation as an evolutionary hierarchy (see **Fig. 9.9**).

   $M^8 - H$ duality provides two complementary visions about physics (see **Fig. 9.2**), and can be seen as a generalization of the q-p duality of wave mechanics, which fails to generalize to quantum field theories (QFTs).

4. In Zero energy ontology (ZEO), the superpositions of space-time surfaces inside causal diamond (CD) having their ends at the opposite light-like boundaries of CD, define quantum states. CDs form a scale hierarchy (see **Fig. 9.12** and **Fig. 9.13**).

   Quantum jumps occur between these and the basic problem of standard quantum measurement theory disappears. Ordinary state function reductions (SFRs) correspond to "big" SFRs (BSFRs) in which the arrow of time changes (see **Fig. 9.14**). This has profound thermodynamic implications and the question about the scale in which the transition from classical to quantum takes place becomes obsolete. BSFRs can occur in all scales but from the point of view of an observer with an opposite arrow of time they look like smooth time evolutions.

   In "small" SFRs (SSFRs) as counterparts of "weak measurements" the arrow of time does not change and the passive boundary of CD and states at it remain unchanged (Zeno effect).

TGD develops by explaining what TGD is and also this work led to considerable progress in several aspects of TGD.

1. The mutual entanglement of fermions (bosons) as elementary particles is always maximal so that only fermionic and bosonic degrees can entangle in QFTs. The replacement of point-like particles with 3-surfaces forces us to reconsider the notion of identical particles from the category theoretical point of view. The number theoretic definition of particle identity seems to be the most natural and implies that the new degrees of freedom make possible geometric entanglement.

   Also the notion particle generalizes: also many-particle states can be regarded as particles with the constraint that the operators creating and annihilating them satisfy commutation/anticommutation relations. This leads to a close analogy with the notion of infinite prime.

2. The understanding of the details of the $M^8 - H$ duality forces us to modify the earlier view. The notion of causal diamond (CD) central to zero energy ontology (ZEO) emerges as a prediction at the level of H. The pre-image of CD at the level of M^8 is a region bounded by two mass shells rather than CD. $M^8 - H$ duality maps the points of cognitive representations as momenta of quarks with fixed mass in M^8 to either boundary of CD in H.

3. Galois confinement at the level of M^8 is understood at the level of momentum space and is found to be necessary. Galois confinement implies that quark momenta in suitable units are algebraic integers but integers for Galois singlet just as in ordinary quantization for a particle in a box replaced by CD. Galois confinement could provide a universal mechanism for the formation of all bound states.

4. There is considerable progress in the understanding of the quantum measurement theory based on ZEO. From the point of view of cognition BSFRs would be like heureka moments and the sequence of SSFRs would correspond to an analysis having as a correlate the decay of 3-surface to smaller 3-surfaces.

9.2 Physics as geometry

The following provides a sketchy representation of TGD based on the vision about physics as geometry which is complementary to the vision of physics as number theory. M^8-H duality relates these two visions. A longer representation can be found in [L89].

9.2.1 Space-time as 4-surface in $H=M^4\times CP_2$

1. The energy problem of GRT means that since space-time is curved, one cannot define Poincare charges as Noether charges (see **Fig. 9.1**). If space-time X^4 is a surface in $H=M^4\times CP_2$, the situation changes. Poincare symmetries are lifted to the level of $M^4\subset H$.

2. Generalization of the notion of particle is in question: point-like particle $\rightarrow$ 3-surface so that TGD can be seen also as a generalization of string model. String $\rightarrow$ 3-surface. String world sheet $\rightarrow X^4$. The notions of the particle and space are unified.

3. Einstein's geometrization program is extended to standard model interactions. CP_2 codes for standard model symmetries and gauge fields. Isometries $\leftrightarrow$ color SU(3). Holonomies of spinor connection $\leftrightarrow$ electroweak U(2) [L2]. Genus-generation correspondence provides a topological explanation of the family replication phenomenon of fermions [K21]: 3 fermion families are predicted.

4. Induction of spinors structure as projection of components of spinor connection from CP_2 to X^4 is central for the geometrization. The projections of Killing vectors of color isometries yield color gauge potentials. Parallel translation at X^4 using spinor connection of H. Also spinor structure is induced and means projection of gamma matrices.

5. Dynamics for X^4 is determined by an action S consisting of Kähler action plus volume term (cosmological constant) following from the twistor lift of TGD [L10, L58].

6. The dynamics for fermions at space-time level is determined by modified Dirac action determined by S being super-symmetrically related to it. Gamma matrices are replaced with modified gamma matrices determined by the S as contractions of canonical momentum currents with gamma matrices. Preferred extremal property follows as a condition of hermiticity for the modified Dirac operator.

   Second quantized H-spinors, whose modes satisfy free massless Dirac equation in H restricted to X^4: this induces second quantization to X^4 and one avoids the usual problems of quantization in a curved background. This picture is consistent with the modified Dirac equation satisfied by the induced spinors in X^4.

   Only quarks are needed if leptons are 3-quark composites in CP_2 scale: this is possible only if one accepts the TGD view about color symmetries. This also provides a new view about matter antimatter asymmetry [L74, L94]. CP violation is forced by the M^4 part of Kähler form forced by the twistor lift.

Basic extremals of classical action

Practically any GCI action allows the same basic extremals (for basic questions related to classical TGD see **Fig. 9.3**).

1. CP_2 type extremals having light-like geodesic as M^4 projection and Euclidian signature of the induced metric serve as building bricks of elementary particles. If the volume term is absent as it might be at infinite volume limit, the geodesics become light-like curves [L107]. Wormhole contacts connecting two Minkowskian space-time sheets can be regarded as a piece

of a deformed CP_2 type extremal. Monopole flux through contact stabilizes the wormhole contact.

2. Massless extremals (MEs)/topological light rays are counterparts for massless modes. They allow superposition of modes with single direction of ligth-like momentum. Ideal laser beam is a convenient analogy here.

3. Cosmic strings $X^2 \times Y^2 \subset M^4 \times CP_2$ and their thickenings to flux tubes are also a central notion.

QFT limit of TGD

The induced gauge fields and gravitational field are expressible in terms of only 4 H- coordinates. Locally the theory is too simple to be physical.

1. Many-sheeted space-time means that X^4 is topologically extremely complex. CP_2 coordinates are many-valued functions of M^4 coordinates or vice versa or both. In contrast to this, the space-time of EYM theory is topologically extremely simple.

2. Einsteinian space-times have 4-D projection to M^4. Small test particle experiences the sum of the classical gauge potentials associated with various space-time sheets. At QFT limit the sheets are replaced with a single region of M^4 made slightly curved and gauge potentials are defined as the sums of gauge potentials from different space-time sheets having common M^4 projection. Topological complexity and local simplicity are replaced with topological simplicity and local complexity. (see **Fig. 9.3**).

9.2.2 World of classical worlds (WCW)

The notion of WCW emerges as one gives up the idea about quantizing by path integral.

The failure of path integral forces WCW geometry

The extreme non-linearity implies that the path integral for surfaces space-time surfaces fails. A possible solution is generalize Einstein's geometrization program to the level of the entire quantum theory.

1. "World of classical worlds" (WCW) can be identified as the space of 3-surfaces endowe with a metric and spinor structure (see **Fig. 9.7**). Hermitian conjugation must have a geometrization. This requires Kähler structure requiring also complex structure. WCW has Kähler form and metric.

2. WCW spinors are Fock states created by fermionic oscillator operators assignable to spinor modes of H basically [L86]. WCW gamma matrices as linear combinations of fermionic (quark) oscillator operators defining analog of vielbein.

   WCW has also spinor connection and curvature in WCW. correspond The quantum states of world correspond formally to *classical* spinor fields in WCW. Gamma matrices of WCW expressinble in terms of fermionic oscillator operators are also purely classical objects.

Implications of General Coordinate Invariance

General Coordinate Invariance (GCI) in 4-D sense forces to assign to 3-surface X^3 a 4-surface $X^4(X^3)$, which is as unique as possible. This gives rise to Bohr orbitology and quantum classical correspondence (QCC), and holography. Also zero energy ontology (ZEO) emerges.

Quantum states quantum superpositions of space-time surfaces as analogs of Bohr orbits. QCC means that the classical theory is an exact part of quantum theory (QCC).

A solution to the basic paradox of quantum measurement theory emerges [L73]: superposition of deterministic time evolutions is replaced with a new one in state function reduction (SFR): SFR does not force any failure of determinism for individual time evolutions.

WCW Kähler geometry from classical action

WCW geometry is determined by a classical action defining Kähler function $K(X^3)$ for a preferred extremal $X^4(X^3)$ defining the preferred extremal/Bohr orbit [K45] (see **Fig. 9.7**).

1. QCC suggests that the definition of Kähler function assigns a more or less unique 4-surface $X^4(X^3)$ to 3-surface X^3. Finite non-uniqueness is however possible [L107].

2. $X^4(X^3)$ is identified as a *preferred* extremal of some general coordinate invariant (GCI) action forcing the Bohr orbit property/holography/ZEO. This means a huge reduction of degrees of freedom.

   Remark:: Already the notion of induced gauge field and metric eliminates fields as primary dynamical variables and GCI leaves locally only 4 H-coordinates as dynamical variables.

3. Twistor lift [L45, L58] of TGD geometrizes the twistor Grassmann approach to QFTs. The 6-D extremal X^6 of 6-D Kähler action as a 6- surface in the product $T(M^4) \times T(CP_2)$ of twistor spaces of M^4 and CP_2 represents the twistor space of X^4.

   The condition that X^6 reduces to an S^2 bundle with X^4 as base space, forces a dimensional reduction of 6-D Kähler action to 4-D Kähler action + volume term, whose value for the preferred extremal defines the Kähler function for $X^4(X^3)$.

4. The volume term corresponds to a p-adic length scale dependent cosmological constant Λ approach zero at long p-adic length scale so that a solution of the cosmological constant problem emerges. Preferred extremal/Bohr orbit property means a simultaneous extremal property for *both* Kähler action and volume term. This forces X^4 to have a generalized complex structure (Hamilton-Jacobi structure) so that field equations trivialize and there is no dependence on coupling parameters. Universality of dynamics follows and the TGD Universe is quantum critical. In particular, Kähler coupling strength is analogous to a critical temperature and is quantized [L101].

5. Soap film analogy is extremely useful [L107]: the analogs of soap film frames are singular surfaces of dimension $D < 4$. At the frame the space-time surface fails to be a simultaneous extremal of both actions separately and Kähler and volume actions couple to each other. The corresponding contributions to conserved isometry currents diverge but sum up to a finite contribution. The frames define the geometric analogs for the vertices of Feynman diagrams.

WCW geometry is unique

WCW geometry is fixed by the existence of Riemann connection and requires maximal symmetries.

1. Dan Freed [A37] found that loop space for a given Lie group allows a unique Kähler geometry: maximal isometries needed in order to have a Riemann connection. Same expected to be true now [K24, K80].

2. Twistor lift of TGD [L45, L58] means that one can replace X^4 with its twistor space $X^6(X^4)$ in the product $T(M^4) \times T(CP_2)$ of the 6-D twistor spaces $T(M^4)$ and $T(CP_2)$. $X^6(X^4)$ is 6-surface with the structure of S^2 bundle.

   Dimensionally reduced 6-D Kähler action gives sum of 4-D Kähler action and volume term. Twistor space must however have a Kähler structure and only the twistor spaces of M^4,E^4, and CP_2 have Kähler structure [A54]. TGD is unique both physically and mathematically!

Isometries of WCW

What can one say about the isometries of WCW? Certainly, they should generalize conformal symmetries of string models.

1. The crucial observation is that the 3-D light-cone boundary δM^4_+ has metric, which is effectively 2-D. Also the light-like 3-surfaces $X^3_L \subset X^4$ at which the Minkowskian signature of the induced metric changes to Euclidian are metrically 2-D. This gives an extended conformal invariance in both cases with complex coordinate z of the transversal cross section and radial light-coordinate r replacing z as coordinate of string world sheet. Dimensions $D = 4$ for X^4 and M^4 are therefore unique.

2. $\delta M^4_+ \times CP_2$ allows the group symplectic transformations of $S^2 \times CP_2$ made local with respect to the light-like radial coordinate r. The proposal is that the symplectic transformations define isometries of WCW [K24].

3. To the light-like partonic orbits one can assign Kac-Moody symmetries assignable to $M^4 \times CP_2$ isometries with additional light-like coordinate. They could correspond to Kac-Moody symmetries of string models assignable to elementary particles.

   The preferred extremal property raises the question whether the symplectic and generalized Kac-Moody symmetries are actually equivalent. The reason is that isometries are the only normal subgroup of symplectic transformations so that the remaining generators would naturally annihilate the physical states and act as gauge transformations. Classically the gauge conditions would state that the Noether charges vanish: this would be one manner to express preferred extremal property.

A possible problem related to the twistor lift

The twistor lift strongly suggests that the Kähler form of M^4 exists. The Kähler gauge potential would be the sum of M^4 and CP_2 contributions. The definition of M^4 Kähler structure is however not straightforward [L82, L83]. The naive guess would be that J represents an imaginary unit as the square root of -1 represented by the metric tensor. This would give the condition $J^2 = -g$ for the tensor square but this leads to problems.

To understand the situation, notice that the analogs of symplectic/Kähler structures in $M^4 \subset H$ have a moduli space, whose points correspond to what I have called Hamilton-Jacobi structures defined by integrable distributions of orthogonal decompositions $M^4 = M^2(x) \times E^2(x)$: $M^2(x)$ is analogous to string world sheet and Y^2 to partonic 2-surface. This means the presence of slicing by string world sheets $X^2(x)$, where x labels a point of Y^2. $X^2(x)$ is orthogonal to Y^2 at x. One can interchange the roles X^2 ad Y^2 in the slicing.

The induced Kähler form has an analogous decomposition. The decomposition is completely analogous to the decomposition of polarizations to non-physical time-like ones and physical space-like ones. This decomposition allows a natural modification of the definition of the symplectic structure so that the problem caused by $J^2 = -g$ conditions is avoided.

Consider first the problem. The $E^2(x)$ part of M^4 Kähler metric produces no problems since the signature of the metric is Euclidean. For $M^2(x)$ part, the Minkowskian signature produces problems. If one assumes that the $M^2(x)$ part of the Kähler form is non-vanishing, it should be imaginary in order to satisfy $J^2(M^2(x)) = -g(M^2(x))$. This implies that Kähler gauge potential is imaginary and this spoils the hermiticity of the modified Dirac equation [K106]. Also the electric contribution to the Kähler energy is negative.

The solution of the problem turned out to be ridiculously simple and I should have noticed it a long time ago.

1. $M^2(x)$ has a hypercomplex structure, which means that the imaginary unit e satisfies $e^2 = 1$ rather than $e^2 = -1$. Hamilton-Jacobi structure allows one to decompose J locally into two parts $J = J(M^2(x)) + J(E^2(x))$ such that $J^2 = g(M^2(x)) - g(E^2(x))$. This gives $J^4 = g(M^4)$. The Kähler energy of the canonically embedded M^4 is non-vanishing and positive whereas Kähler action vanishes by self-duality. Situation is identical to that in Maxwell's electrodynamics.

2. Kähler action for the canonically embedded M^4 vanishes and it is possible to define also Lagrangian 2-surfaces as surfaces for which the induced Kähler form vanishes. These are of special interest since they would guarantee small CP violation: string world sheets could be examples of these surfaces. Note that since the magnetic part of J induces violation of CP,

the violation is vanishing for CP_2 type extremals and cosmic strings and also small for flux tubes.

If the notion of symplectic/canonical transformation generated by Hamiltonian preserving J generalizes, one could generate an infinite number of slicings.

Consider first ordinary symplectic transformations.

1. For the ordinary symplectic transformations, the closedness of the symplectic for J is essential ($dJ = 0$ corresponds to topological half of Maxwell's equations).
2. Second essential element is that symplectic transformation is generated as a flow for some Hamiltonian H: $j_H = i_{dH}J$ or more explicitly: $j^l_H = J^{kl}\partial_l H$. It is essential that one has $i_{j_H}J = -dH$: having a vanishing exterior derivative. In other words, $J_{kl}j^l_H = -\partial_k H$ is a gradient vector field and has therefore a vanishing curl. Together with $dJ = 0$, this guarantees the vanishing of the Lie derivative of J: $d_{j_H}J = d(i_{j_H}J) + i_{j_H}dJ = ddH + dJ(j_H) = 0$ so that J is preserved.

Could one talk about symplectic transformations in M^4?

1. The analogs of symplectic/canonical transformations should map the Hamilton-Jacobi structure to a new one and leave $J(M^2(x))$ and $J(E^2(x))$ invariant. The induced metrics of X^2 and Y^2 need not be preserved since only the diagonal metric $g^k_l(X^2/Y^2)$ appears in the conditions $J^2 = g(X^2) - g(Y^2)$.
2. The symplectic transformation generated by the Hamiltonian H would be a flow defined by the vector field $j_H = i_{dH}J$ and one would have $i_{j_H}J = -d_1H + d_2H$, where d_1 and d_2 are gradients operators in X^2 and Y^2. Usually one would have $J_{kl}j^l = dH$ satisfying $d^2H = 0$.

   The condition $ddH = 0$ satisfied by the ordinary symplectic transformations is replaced with the condition $d(-d_1H + d_2H) = 0$. This can be written as $-d_1^2H + d_2^2H + [d_2, d_1]H = 0$, and is satisfied. Therefore this part is not a problem.
3. Also the orthogonality of $M^2(x)$ and $E^2(x)$ must be preserved. This is a highly non-trivial condition since the metrics are induced and the symplectic transformations change the slicing and the metrics. An arbitrary Hamiltonian flow f, which depends on the coordinates of Y^2 only, maps Y^2 to itself but takes the tangent space $E^2(x)$ to $E^2(f(x))$. Unless the slicing satisfies special conditions, $E^2(f(x))$ is not orthogonal to $M^2(x)$.
4. The orthogonality is expressed as orthogonality of the projectors $P(X^2)$ and $P(Y^2)$: $P(X^2)P(Y^2) = 0$. This condition must be respected by the Hamiltonian flow. The product involves 4 components giving 4 conditions which turn out to be partial differential equations for Hamiltonian. The naive expectation is that there are very few solutions. The Lie-derivative of the product must therefore vanish:

   $$L_{j_H}[P(X^2)P(Y^2)] = L_{j_H}(P(X^2))P(Y^2) + P(X^2))L_{j_H}(P(Y^2)) = 0 \ . \tag{9.2.1}$$

   The projector $P_{mn}(X^2)$ can be expressed as

   $$P^{mn} = g^{\alpha\beta}\partial_\alpha m^k\partial_\beta m^l \ . \tag{9.2.2}$$

   Here $g_{\alpha\beta} = m_{kl}\partial_\alpha m^k\partial_\beta m^l$ is the induced metric of X^2 or Y^2. m_{kl} is Minkowski metric and one can use linear Minkowski coordinates so that m_{kl} is constant.

The Lie derivative of $P^{mn}(X^2) \equiv P$ can be written as

$$L_jP^{mn} = L_j(g^{\alpha\beta})\partial_\alpha m^k\partial_\beta m^l + g^{\alpha\beta})(\partial_r j^k\partial_\alpha m^r\partial_\beta m^l + \partial_r j^l k\partial_\alpha m^r\partial_\beta m^k \ . \quad (9.2.3)$$

The Lie derivative of the induced metric is

$$\begin{aligned} L_jg^{\alpha\beta} &= g^{\alpha\mu}g^{\beta\nu}L_Jg_{\mu\nu} \ , \\ L_jg_{\alpha\beta} &= m_{kl}(\partial_\alpha j^k\partial_\beta m^l + \partial_\alpha m^k\partial_\beta j^l \ . \end{aligned} \quad (9.2.4)$$

Although the existence of symplectic transformations in the general case seems implausible, one can construct special slicings for which symplectic transformations are possible.

1. One can start from a trivial slicing defined by $M^2 \times E^2$ decomposition and perform slicings of M^2 and E^2. The orthogonality is trivially true for all slicings of this kind since $Y^2(y)$ is orthogonal to X^2 not only at y but at every point x. Symplectic transformations of M^2 and Y^2 produce new slicings of this kind. Even symplectic flowqs defined by general Hamiltonians respect the orthogonality.
2. Second example is provided by the slicing of the light-one boundary by light-like 2-surfaces Y_v^2 labelled by the value of light-like radial coordinate v with metrics differing by r^2 factor. The surfaces X^2 would be planes $X^2(y)$ orthogonal to Y^2 at y with light-like coordinates u and v. The orthogonality would be preserved by symplectic transformations.

   The open question is whether these slicings are the only possible slicings allowing symplectic transformations. Although the construction of these slicings looks trivial, they are not trivial physically.

9.2.3 Should unitarity be replaced with the Kähler-like geometry of the fermionic state space?

Physical states correspond to WCW spinor fields and in ZEO. WCW spinors at a given point of WCW correspond to pairs of Fock states assignable to the 3-surfaces at the opposite boundaries of CD defining space-time surface. These pairs of many-fermion states in fermionic degrees of freedom define the TGD counterpart of the S-matrix.

Unitary is a natural notion in non-relativistic wave-mechanics but already in quantum field theory it becomes problematic. In the twistor approach to the scattering amplitudes of massless gauge theories both unitarity and locality are problematic. Whether TGD can give rise to a unitary S-matrix has been a continual head-ache. This leads to a heretic question.

Is unitarity possible at all in TGD framework and should it be replaced with some deeper principle? I have considered these questions several times and in [L91] a rather radical solution was proposed. The implications of this proposal for the construction of scattering amplitudes are discussed in [L92].

Assigning an S-matrix to a unitary time evolution works in non-relativistic theory but fails already in the generic QFT and correlation functions replace S-matrix.

1. Einstein's great vision was to geometrize gravitation by reducing it to the curvature of space-time. Could the same recipe work for quantum theory? Could the replacement of the flat Kähler metric of Hilbert space with a non-flat one allow the identification of the analog of unitary S-matrix as a geometric property of Hilbert space? Kähler metric is required to geometrize hermitian conjugation. It turns out that the Kähler metric of a Hilbert bundle determined by the Kähler metric of its base space could replace the unitary S-matrix.

2. An amazingly simple argument demonstrates that one can construct scattering probabilities from the matrix elements of Kähler metric and assign to the Kähler metric a unitary S-matrix assuming that some additional conditions guaranteeing that the probabilities are real and non-negative are satisfied. If the probabilities correspond to the real part of the complex analogs of probabilities, it is enough to require that they are non-negative: complex analogs of probabilities would define the analog of the Teichmüller matrix.

   Teichmüller space parameterizes the complex structures of Riemann surface: could the allowed WCW Kähler metrics - or rather the associated complex probability matrices - correspond to complex structures for some space? By the strong form of holography (SH), the most natural candidate would be Cartesian product of Teichmüller spaces of partonic 2 surfaces with punctures and string world sheets.

3. Under some additional conditions one can assign to Kähler metric a unitary S-matrix but this does not seem necessary. The experience with loop spaces suggests that for infinite-D Hilbert spaces the existence of non-flat Kähler metric requires a maximal group of isometries. Hence one expects that the counterpart of S-matrix is highly unique.

4. In the TGD framework the "world of classical worlds" (WCW) has Kähler geometry allowing spinor structure. WCW spinors correspond to Fock states for second quantized spinors at space-time surface and induced from second quantized spinors of the embedding space. Scattering amplitudes would correspond to the Kähler metric for the Hilbert space bundle of WCW spinor fields realized in zero energy ontology and satisfying Teichmüller condition guaranteeing non-negative probabilities.

5. Equivalence Principle generalizes to the level of WCW and its spinor bundle. In ZEO one can assign also to the Kähler space of zero energy states spinor structure and this strongly suggests an infinite hierarchy of second quantizations starting from space-time level, continuing at the level of WCW, and continuing further at the level of the space of zero energy states. This would give an interpretation for an old idea about infinite primes as an infinite hierarchy of second quantizations of an arithmetic quantum field theory.

6. There is also an objection. The transition probabilities would be given by $P(A,B) = g^{A,\overline{B}}g_{\overline{B},A}$ and the analogs for unitarity conditions would be satisfied by $g^{A,\overline{B}}g_{\overline{B},C} = \delta^A_C$. The problem is that $P(A,B)$ is not real without further conditions. Can one imagine any physical interpretation for the imaginary part of $Im(P(A,B))$?

In this framework, the twistorial scattering amplitudes as zero energy states define the covariant Kähler metric $g_{A\overline{B}}$, which is non-vanishing between the 3-D state spaces associated with the opposite boundaries of CD. $g^{A\overline{B}}$ could be constructed as the inverse of this metric. The problem with the unitarity would disappear.

This view is developed in detail in [L92] and one ends up with a very concrete and surprisingly simple number theoretic view about scattering amplitudes.

9.2.4 About Dirac equation in TGD framework

Three Dirac equations

In TGD spinors appear at 3 levels:

1. At the level of embedding space $H = M^4 \times CP_2$ the spinor field embedding space $M^4 \times CP_2$spinor fields (quark field) is a superposition of the harmonics of the Dirac operator. In the complexified M^8 having interpretation as complexified octonions, spinors are octonionic spinors. In accordance with the fact that M^8 is analogous to momentum space, the Dirac equation is purely algebraic and its solutions correspond to discrete points analogous to occupied points of Fermi ball.

2. The spinors at the level of 4-surfaces $X^4 \subset H$ are restrictions of the second quantized embedding space spinor field in X^4 so that the problematic second quantization in curved background is avoided. At the level of M^8 the restriction selects the points of M^8 belonging

to 4-surface and carrying quark. The simplest manner to realize Fermi statistics is to assume that there is at most a single quark at a given point.

3. The third realization is at the level of the "world of classical worlds" (WCW) assigned to H consisting of 4-surfaces as preferred extremals of the action. Gamma matrices of WCW are expressible as superpositions of quark oscillator operators so that anti-commutation relations are geometrized. The conditions stating super-symplectic symmetry are a generalization of super-Kac-Moody symmetry and of super-conformal symmetry and give rise to the WCW counterpart of the Dirac equation [K80] [L89].

4. What the realization of WCW at the level of M^8 is, has remained unclear. The notion of WCW geometry does not generalize to his level and should be replaced with an essentially number theoretic notion.

   Adelic physics as a fusion of real and p-adic physics suggests a possible realization. Given extension of rationals induces extensions of various p-adic number fields. These can be glued to a book-like structure having as pages real numbers and the extensions of p-adic number fields.

   The pages would intersect along points with coordinates in the extension of rationals. These points form a cognitive representation. The additional condition that the active points are occupied by quarks guarantees that this makes sense also for octonions, quaternions and 4-surface in M^8. The p-adic sector could consist of discrete and finite cognitive representations continued to the p-adic surface and define the counterpart of WCW at the level of M^8?

The relationship between Dirac operator of H and modified Dirac operator

At the level of $X^4 \subset H$, the proposal is that modified Dirac action for the induced spinor fields defines the dynamics somehow. Modified Dirac equation or operator should be also consistent with the second quantization of induced spinor fields performed at the level of H and inducing the second quantization at the level of X^4.

1. The modified gamma matrices Γ^α are defined by the contractions of H gamma matrices Γ_k and canonical momentum currents $T^{k\alpha}$ associated with the action defining space-time surface. The modified Dirac operator $D = \Gamma^\alpha D_\alpha$, where D_α is X^4 projection of the vector defined by the covariant derivative operators of H ($D_\alpha = \partial_\alpha h^k D_k$). Hermiticity requires $D_\alpha \Gamma^\alpha = 0$ implying that classical field equations are satisfied.

2. Can one assume that the modified Dirac equation is satisfied? Or is it enough to assume that this is not the case so that the modified Dirac operator defines the propagator as its inverse as the QFT picture would suggest?

   In fact, the propagators in H allow to compute N-point functions involving quarks and at the level of H the theory is free and the restriction to the space-time surface brings in the interactions. Therefore the notion of space-time propagator is not absolutely necessary. One can however ask whether some weaker condition could be satisfied and provide new insights.

   One can also ask whether the solutions of the modified Dirac equation correspond to external particles, which correspond to space-time surfaces for which the solution of the modified Dirac equation is consistent with the solution of the Dirac equation in H. Are these kinds of space-time surfaces possible?

3. The intuitive picture is that the solutions of the modified Dirac equation correspond to the external particles of a scattering diagram having an interpretation on mass shell states and are possible only for a very special kind of preferred extremals. Intuitively they should correspond to singular surfaces in M^8 and their mapping to H would involve blow-up due to the non-uniqueness of the normal space along lower than 4-D surface. String like objects and CP_2 type extremals would be basic entities of this kind. Could the modified Dirac equation or its weakened form hold true for these surfaces.

The strong form of equivalence of modified Dirac equation and ordinary Dirac equation would mean the equivalence of the actions of two Dirac operators acting on the second quantized induced spinor field.

1. The modified Dirac operator is given by $\Gamma_k T^{\alpha k}\partial_\alpha h^k D_k$ and its action should be same as H Dirac operator $\Gamma^k D_k$. This would require

$$\Gamma_k T^{\alpha k}\partial_\alpha h^k D_k \Psi = \Gamma^k D_k \Psi \ . \tag{9.2.5}$$

   Not surprisingly, it turns out that this condition is too strong.

2. One can express Γ_k using an overcomplete basis defined by the Killing vector fields j_A^k for H isometries. In the case of M^4 it is enough to use translations by using the identity $\sum_A j_A^k j_A^l = h^{kl}$. This allows to define gamma matrices $\Gamma_A = \Gamma_k j_A^k$ and to write the equation in the form

$$\Gamma_A T^{A\alpha}\partial_\alpha h^k D_k \Psi = \Gamma_A j_A^k D_k \Psi \ . \tag{9.2.6}$$

   Here $T^{A\alpha}$ is the conserved isometry current associated with the Killing vector j_A^k. Is it possible to satisfy the condition

$$T^{A\alpha}\partial_\alpha h^k = j_A^k \tag{9.2.7}$$

   or its suitably weakened form?

   The strong form of the condition cannot be satisfied. The left hand side of the equation is determined by the gradients of H coordinates and parallel to X^4 whereas the right hand side also involves the component normal to X^4. Therefore the condition cannot be satisfied in the general case.

3. By projecting the condition to the tangent space, one obtains a weaker condition stating that the tangential parts of two Dirac operators are proportional to each other with a position dependent proportionality factor $\Lambda(x)$:

$$\begin{aligned} T^{A\alpha} &= \Lambda(x) j_A^\alpha \\ j_A^\alpha &= j_A^k \partial^\alpha h_k = j_A^k h_{kl} g^{\alpha\beta}\partial_\beta h^l \ . \end{aligned} \tag{9.2.8}$$

   The conserved isometry current is proportional to the projection of the Killing vector to the tangent space of X^4. $\Lambda(x)$ is proportionality constant depending on the point of X^4. Isometry current is analogous to a Hamiltonian vector field being parallel to the Killing vector field.

4. If the action were a mere cosmological volume term, the isometry currents would be proportional to j^α so that the conditions would be automatically satisfied. The contribution to $\Lambda(x)$ is proportional to the p-adic length scale dependent cosmological constant.

   Kähler action receives contributions from both M^4 and CP_2. Both add to $T^{A\alpha}$ a term of form $T^{\alpha\beta} j_{A\beta}$ coming from the variation of the Kähler action with respect to $g_{\alpha\beta}$. $T^{\alpha\beta}$ is the energy momentum tensor with a form similar to that for Maxwell action.

   Besides this, M^4 *resp.* CP_2 contribute a term proportional to $J^{\alpha\beta}J_{kl}\partial_\beta h^k j_A^k$ coming from the variation of the Kähler action with respect to $J_{\alpha\beta}$ contributing only to M^4 *resp.* CP_2 isometries. These contributions make the conditions non-trivial. The Kähler contribution to $\Lambda(x)$ need not be constant. Note that the Kähler contributions to the energy momentum tensor vanish if X^4 is (minimal) surface of form $X^2 \times Y^2 \subset M^4 \times CP_2$ so that both X^2 and Y^2 are Lagrangian.

5. The vanishing of the divergence of $T^{A\alpha}$ using the Killing property $D_l j_{Ak} + D_k j_{Al} =$ of j_{Ak} gives

$$j^{A\alpha}\partial_\alpha \Lambda = 0 \ . \tag{9.2.9}$$

   Λ is constant along the flow lines of $j^{A\alpha}$ and is therefore analogous to a Hamiltonian. The constant contribution from the cosmological term to Λ does not contribute to this condition.

6. An attractive hypothesis, consistent with the hydrodynamic interpretation, is that the proposed condition is true for all preferred extremals. The conserved isometry current along the X^4 projection of the flow line is proportional to the projection of Killing vector: this conservation law is analogous to the conservation of energy density $\rho v^2/2 + p$ along the flow line). One can say that isometries as flows in the embedding space are projected to flows along the space-time surface. One could speak of projected or lifted representation.

7. The projection to the normal space does not vanish in the general case. One could however ask whether a weaker condition stating that the second fundamental form $H^k_{\alpha\beta} = D_\alpha h^k$, which is normal to X^4, defines the notion of the normal space in terms of data provided by space-time surface. If X^4 is a geodesic submanifold of H, in particular a product of geodesic submanifolds of M^4 and CP_2, one has $H^k_{\alpha\beta} = 0$.

Gravitational and inertial representations of isometries

The lift/projection of the isometry flows to X^4 strongly suggests a new kind of representation of isometries as analog of the braid representation considered earlier.

1. Projected/lifted representation would clarify the role of the classical conserved charges and currents and generalize hydrodynamical conservation laws along the flow lines of isometries. In particular, quark lines would naturally correspond to time-like flow lines of time translations. In the case of CP_2 type extremals, quark momenta for the lifted representations would be light-like.

2. The conservation conditions along the flow lines are very strong, and one can wonder if they might provide a new formulation of the preferred extremal property. It is quite possible that the conditions apply only to a sub-algebra. Quantum classical correspondence (QCC) suggests Cartan algebra for which the quantum charges can have well-defined eigen values simultaneously. In accordance with QCC, the choice of the quantization axes would affect the space-time surfaces considered and could be interpreted as a higher level quantum measurement.

3. Projected/lifted representation provides a new insight also to the Equivalence Principle (EP) stating that gravitational and inertial masses are identical. At the level of scattering amplitudes involving isometry charges defined at the level of H, the isometries affect the entire space-time surface, and one could see EP as an almost trivial statement. QCC however forces us to consider EP more seriously.

   I have proposed that QCC could be seen as the identification of the eigenvalues of Cartan algebra isometry charges for quantum states with the classical charges associated with the preferred extremals. EP would follow from QCC: gravitational charges would correspond to the representation of the flows defined by isometries as their projections/lifts to X^4 whereas inertial charges would correspond to the representation at the level of H with isometries affecting the entire space-time surfaces.

4. The lifted/projected/gravitational representation of isometries, which seems possible in 4-D situation, is analogous to braid group representation making sense only in 2-D situation. Indeed, for the many-sheeted space-time surfaces assignable to $h_{eff} > h_0$, it can happen that rotation by 2π leads to a new space-time sheet and that the $SO(2)$ subgroup of the rotation group associated with the Cartan algebra is lifted to n-fold covering. Same can can happen

in the case of color rotations. This leads to a fractionation of quantum numbers usually assigned with quantum group representations suggested to correspond to $h_{eff} > h$ [K72].

Also for the quantum groups, Cartan algebra plays a special role. In the case of the Poincare group, the 2-D nature of braid group representations would correspond to the selection $M^2 \times SO(2)$ as a Cartan subgroup implying effective 2-dimensionality in the case rotation group. Gravitational representations could therefore correspond to quantum group representations.

5. The gravitational representation provides also a new insight on $M^8 - H$ duality. The source of worries has been whether Uncertainty Principle (UP) is realized if a given 4-surface in M^8 is mapped to a single space-time surface in M^8. It seems that UP can be realized both in terms of inertial and gravitational representations.

   (a) In the case of the "inertial" representation of H-isometries at the level of H, one must regard $X^4 \subset H$ representing images of particle-like 4-surface in M^8 analog of Bohr orbit (holography) and map it to an analog of plane wave define as superposition of its translates and by the total momentum associated with the either boundary of CD associated with the particle. The same applies to the transforms to other Cartan algebra generators.

   In a cognitive representation based on extension of rationals, the shifts for Cartan algebra would be discrete: the values of the plane wave would be roots of unity belonging to the extension and satisfy periodic boundary conditions at the boundary of larger CD.

   Periodic boundary conditions pose rather strong conditions on the time evolution by scaling between two SSFRs. The scaling must respect the boundary conditions. If the momenta assignable to the plane waves of massive particles are conserved and h_{eff} is conserved, the scaling must multiply CD size by integers. The iterations of integer scalings, in particular $n = 2$ scalings (period doubling), are in a preferred position.

   (b) If one replaces the inertial representation of isometries with the gravitational representation, the quantum states can be realized at the level of a single space-time surface. One would have two representations: gravitational and inertial -subjective and objective, one might say.

   (c) Gravitational representations make also sense for the super-symplectic group acting at the boundary of light-cone as well as for the Kac-Moody type algebra associated with the isometries of H realized the light-like orbits of partonic 2-surfaces.

9.2.5 Different ways to understand the "complete integrability" of TGD

There are several ways to see how TGD could be a completely integrable theory.

Preferred extremal property

Preferred extremal property requires Bohr orbit property and holography and is an extremely powerful condition.

1. Twistor lift of TGD implies that X^4 in H is simultaneous extremal of volume action and Kähler action. Minimal surface property is counterpart for massless field equations and extremality for Kähler action gives interpretation for massless field as Kähler form as part of induced electromagnetic field.

   The simultaneous preferred extremal property strongly suggests that 2-D complex structure generalizes for 4-D space-time surfaces and so called Hamilton-Jacobi structure [L68] meaning a decomposition of M^4 to orthogonal slicings by string world sheets and orthogonal partonic 2-surfaces would realize this structure.

2. Generalized Beltrami property [L95] implies that 3-D Lorentz force and dissipation for Kähler form vanish. The Kähler form is analogous to the classical Maxwell field. Energy momentum tensor has vanishing divergence, which makes it plausible that QFT limit is analogous to Einstein-Maxwell theory.

The condition also implies that the Kähler current defines an integrable flow so that there is global coordinate varying along flow lines. This is a natural classical correlate for quantum coherence. Quantum coherence would be always present but broken only by the finite size of the region of the space-time considered.

Beltrami property plus current conservation implies gradient flow and an interesting question is whether conserved currents define gradient flows: non-trivial space-time topology would allow this at the fundamental level. Beltrami condition is a very natural classical condition in the models of supraphases.

3. The condition that the isometry currents for the Cartan algebra of isometries are proportional to the projections of the corresponding Killing vectors is a strong condition and could also be at least an important aspect of the preferred extremal property.

Supersymplectic symmetry

The third approach is based on the super-symplectic symmetry of WCW. Isometry property would suggest that an infinite number of super-symplectic Noether charges are defined at the boundaries of CD by the action of the theory. They need not be conserved since supersymplectic symmetries cannot be symmetries of the action: if they were, the WCW metric would be trivial.

The gauge conditions for Virasoro algebra and Kac-Moody algebras suggest a generalization. Super-symplectic algebra (SSA) involves only non-negative conformal weights n suggesting extension to a Yangian algebra (this is essential!). Consider the hierarchy of subalgebras SSA_m for which the conformal weights are m-tiples of those of entire algebra. These subalgebras are isomorphic with the entire algebra and form a fractal hierarchy.

Assume that the sub-algebra SSA_m and commutator $[SSA_m, SSA]$ have vanishing classical Noether charges for $m > m_{max}$. These conditions could fix the preferred extremal. One can also assume that the fermionic realizations of these algebras annihilate physical states. The remaining symmetries would be dynamical symmetries.

The generators are Hamiltonians of $\delta M^4_+ \times CP_2$. The symplectic group contains Hamiltonians of the isometries as a normal sub-algebra. Also the Hamiltonians of and one could assume that only the isometry generators correspond to non-trivial classical and quantal Noether charges. Could the actions of SSA and Kac-Moody algebras of isometries be identical if a similar construction applies to Kac-Moody half-algebras associated with the light-like partonic orbits. Super-symplectic symmetry would reduce to a hierarchy of gauge symmetries.

9.3 Physics as number theory

Number theoretic physics involves the combination of real and various p-adic physics to adelic physics [L43, L42], and classical number fields [K91].

9.3.1 p-Adic physics

The motivation for p-adicization came from p-adic mass calculations [K52, K21].

1. p-Adic thermodynamics for mass squared operator M^2 proportional to scaling generator L_0 of Virasoro algebra. Mass squared thermal mass from the mixing of massless states with states with mass of order CP_2 mass.

2. $exp(-E/T) \rightarrow p^{L_0/T_p}$, $T_p = 1/n$. Partition function p^{L_0/T_p}. p-Adic valued mass squared mapped to a real number by canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$. Eigenvalues of L_0 must be integers for the Boltzmann weights to exist. Conformal invariance guarantees this.

3. p-adic length scale $L_p \propto \sqrt{p}$ from Uncertainty Principle ($M \propto 1/\sqrt{p}$). p-Adic length scale hypothesis states that p-adic primes characterizing particles are near to a power of 2: $p \simeq 2^k$. For instance, for an electron one has $p = M^{127} - 1$, Mersenne prime. This is the largest not completely super-astrophysical length scale.

   Also Gaussian Mersenne primes $M_{G,n} = (1+i)^n - 1$ seem to be realized (nuclear length scale, and 4 biological length scales in the biologically important range 10 nm,2.5 μm).

4. p-Adic physics [K62] is interpreted as a correlate for cognition. Motivation comes from the observation that piecewise constant functions depending on a finite number of pinary digits have a vanishing derivative. Therefore they appear as integration constants in p-adic differential equations. This could provide a classical correlate for the non-determinism of imagination.

Unlike the Higgs mechanism, p-adic thermodynamics provides a universal description of massivation involving no other assumptions about dynamics except super-conformal symmetry which guarantees the existence of p-adic Boltzmann weights.

The number theoretic picture leads to a deeper understanding of a long standing objection against p-adic thermodynamics [K52] as a thermodynamics for the scaling generator L_0 of Super Virasoro algebra.

If one requires super-Virasoro symmetry and identifies mass squared with a scaling generator L_0, one can argue that only massless states are possible since L_0 must annihilate these states! All states of the theory would be massless, not only those of fundamental particles as in conformally invariant theories to which twistor approach applies! This looks extremely beautiful mathematically but seems to be in conflict with reality already at single particle level!

The resolution of the objection is that *thermodynamics* is indeed in question.

1. Thermodynamics replaces the state of the entire system with the density matrix for the subsystem and describes approximately the interaction with the environment inducing the entanglement of the particle with it. To be precise, actually a "square root" of p-adic thermodynamics could be in question, with probabilities being replaced with their square roots having also phase factors. The excited states of the entire system indeed are massless [**?**]

2. The entangling interaction gives rise to a superposition of products of single particle massive states with the states of environment and the entire mass squared would remain vanishing. The massless ground state configuration dominates and the probabilities of the thermal excitations are of order $O(1/p)$ and extremely small. For instance, for the electron one has $p = M_{127} = 2^{127} - 1 \sim 10^{38}$.

3. In the p-adic mass calculations [K52, K21], the effective environment for quarks and leptons would in a good approximation consist of a wormhole contact (wormhole contacts for gauge bosons and Higgs and hadrons). The many-quark state many-quark state associated with the wormhole throat (single quark state for quarks and 3-quark-state for leptons [L94].

4. In M^8 picture [L82, L83], tachyonicity is unavoidable since the real part of the mass squared as a root of a polynomial P can be negative. Also tachyonic real but algebraic mass squared values are possible. At the H level, tachyonicity corresponds to the Euclidean signature of the induced metric for a wormhole contact.

   Tachyonicity is also necessary: otherwise one does not obtain massless states. The super-symplectic states of quarks would entangle with the tachyonic states of the wormhole contacts by Galois confinement.

5. The massless ground state for a particle corresponds to a state constructed from a massive single state of a single particle super-symplectic representation (CP_2 mass characterizes the mass scale) obtained by adding tachyons to guarantee masslessness. Galois confinement is satisfied. The tachyonic mass squared is assigned with wormhole contacts with the Euclidean signature of the induced metric, whose throats in turn carry the fermions so that the wormhole contact would form the nearby environment.

   The entangled state is in a good approximation a superposition of pairs of massive single-particle states with the wormhole contact(s). The lowest state remains massless and massive single particle states receive a compensating negative mass squared from the wormhole contact. Thermal mass squared corresponds to a single particle mass squared and does not take into account the contribution of wormhole contacts except for the ground state.

6. There is a further delicate number theoretic element involved [L100, L107]. The choice of $M^4 \subset M^8$ for the system is not unique. Since M^4 momentum is an M^4 projection of a massless M^8 momentum, it is massless by a suitable choice of $M^4 \subset M^8$. This choice must be made for the environment so that both the state of the environment and the single particle ground state are massless. For the excited states, the choice of M^4 must remain the same, which forces the massivation of the single particle excitations and p-adic massivation.

These arguments strongly suggest that pure states, in particular the state of the entire Universe, are massless. Mass would reflect the statistical description of entanglement using the density matrix. The proportionality between p-adic thermal mass squared (mappable to real mass squared by canonical identification) and the entropy for the entanglement of the subsystem-environment pair is therefore natural. This proportionality conforms with the formula for the blackhole entropy, which states that the blackhole entropy is proportional to mass squared. Also p-adic mass calculations inspired the notion of blackhole-elementary particle analogy [K66] but without a deeper understanding of its origin.

One implication is that virtual particles are much more real in the TGD framework than in QFTs since they would be building bricks of physical states. A virtual particle with algebraic value of mass squared would have a discrete mass squared spectrum given by the roots of a rational, possibly monic, polynomial and M^8-H duality suggests an association to an Euclidean wormhole contact as the "inner" world of an elementary particle. Galois confinement, universally responsible for the formation of bound states, analogous to color confinement and possibly explaining it, would make these virtual states invisible [L108, L109].

9.3.2 Adelic physics

Adelic physics fuses real and various p-adic physics to a single structure [L42].

1. One can combine real numbers and p-adic number fields to a product: number fields would be like pages of a book intersecting along rationals acting as the back of the book.

2. Each extension of rational induces extensions of p-adic number fields and extension of the basic adele. Points in the extension of rationals are now common to the pages. The infinite hierarchy of adeles defined by the extensions forms an infinite library.

3. This leads to an evolutionary hierarchy (see **Fig. 9.9**) . The order n of the Galois group as a dimension of extension of rationals is identified as a measure of complexity and of evolutionary level, "IQ". Evolutionary hierarchy is predicted.

4. Also a hierarchy of effective Planck constants interpreted in terms of phases of ordinary matter is predicted. X^4 decomposes to n fundamental regions related by Galois symmetry. Action is n times the action for the fundamental region. Planck constant h is effectively replaced with $h_{eff} = nh$. Quantum coherence scales are typically proportional to h_{eff}. Quantum coherence in arbitrarily long scales is implied. Dark matter at the magnetic body of the system would serve as controller of ordinary matter in the TGD inspired quantum biology [L120].

   $h_{eff} = nh_0$ is a more general hypothesis. Reasons to believe that h/h_0 could be the ratio R^2/L_p^2 for CP$_2$ length scale R deduced from p-adic mass calculations and Planck length L_P [L101]. The CP_2 radius R could actually correspond to L_P and the value of R deduced from the p-adic mass calculations would correspond to a dark CP_2 radius $\sqrt{h/h_0}l_P$.

9.3.3 Adelic physics and quantum measurement theory

Adelic physics [L42] forces us to reconsider the notion of entanglement and what happens in state function reductions (SFRs). Let us leave the question whether the SFR can correspond to SSFR or BSFR or both open for a moment.

1. The natural assumption is that entanglement is a number-theoretically universal concept and therefore makes sense in both real and various p-adic senses. This is guaranteed if the

entanglement coefficients are in an extension E of rationals associated with the polynomial Q defining the space-time surface in M^8 and having rational coefficients.

In the general case, the diagonalized density matrix ρ produced in a state function reduction (SFR) has eigenvalues in an extension E_1 of E. E_1 is defined by the characteristic polynomial P of ρ.

2. Is the selection of one of the eigenstates in SFR possible if E_1 is non-trivial? If not, then one would have a number-theoretic entanglement protection.

3. On the other hand, if the SFR can occur, does it require a phase transition replacing E with its extension by E_1 required by the diagonalization?

Let us consider the option in which E is replaced by an extension coding for the measured entanglement matrix so that something also happens to the space-time surface.

1. Suppose that the observer and measured system correspond to 4-surfaces defined by the polynomials O and S somehow composed to define the composite system and reflecting the asymmetric relationship between O and S. The simplest option is $Q = O \circ S$ but one can also consider as representations of the measurement action deformations of the polynomial $O \times P$ making it irreducible. Composition conforms with the properties of tensor product since the dimension of extension of rationals for the composite is a product of dimensions for factors.

2. The loss of correlations would suggest that a classical correlate for the outcome is a union of uncorrelated surfaces defined by O and S or equivalently by the reducible polynomial defined by the $O \times S$ [L97]. Information would be lost and the dimension for the resulting extension is the sum of dimensions for the composites. O however gains information and quantum classical correspondence (QCC) suggests that the polynomial O is replaced with a new one to realize this.

3. QCC suggests the replacement of the polynomial O the polynomial $P \circ O$, where P is the characteristic polynomial associated with the diagonalization of the density matrix ρ. The final state would be a union of surfaces represented by $P \circ O$ and S: the information about the measured observable would correspond to the increase of complexity of the space-time surface associated with the observer. Information would be transferred from entangled Galois degrees of freedom including also fermionic ones to the geometric degrees of freedom $P \circ O$. The information about the outcome of the measurement would in turn be coded by the Galois groups and fermionic state.

4. This would give a direct quantum classical correspondence between entanglement matrices and polynomials defining space-time surfaces in M^8. The space-time surface of O would store the measurement history as kinds of Akashic records. If the density matrix corresponds to a polynomial P which is a composite of polynomials, the measurement can add several new layers to the Galois hierarchy and gradually increase its height.

   The sequence of SFRs could correspond to a sequence of extensions of extensions of..... This would lead to the space-time analog of chaos as the outcome of iteration if the density matrices associated with entanglement coefficients correspond to a hierarchy of powers P^k [L84, L96].

Does this information transfer take place for both BSFRs and SSFRs? Concerning BSFRs the situation is not quite clear. For SSFRs it would occur naturally and there would be a connection with SSFRs to which I have associated cognitive measurement cascades [?]

1. Consider an extension, which is a sequence of extensions $E_1 \rightarrow ..E_k \rightarrow E_{k+1}.. \rightarrow E_n$ defined by the composite polynomial $P_n \circ .... \circ P_1$. The lowest level corresponds to a simple Galois group having no non-trivial normal subgroups.

2. The state in the group algebra of Galois group $G = G_n$ having G_{n-1} as a normal subgroup can be expressed as an entangled state associated with the factor groups G_n/G_{n-1} and subgroup G_{n-1} and the first cognitive measurement in the cascade would reduce this entanglement.

After that the process could but need not to continue down to G_1. Cognitive measurements considerably generalize the usual view about the pair formed by the observer and measured system and it is not clear whether $O - S$ pair can be always represented in this manner as assumed above: also small deformations of the polynomial $O \times S$ can be considered.

These considerations inspire the proposal the space-time surface assigned to the outcome of cognitive measurement G_k, G_{k-1} corresponds to polynomial the $Q_{k,k-1} \circ P_n$, where $Q_{k,k-1}$ is the characteristic polynomial of the entanglement matrix in question.

9.3.4 Entanglement paradox and new view about particle identity

A brain teaser that the theoretician sooner or later is bound to encounter, relates to the fermionic and bosonic statistics. This problem was also mentioned in the article of Keimer and Moore [D2] discussing quantum materials `https://cutt.ly/bWdTRj0`. The unavoidable conclusion is that both the fermions and bosons of the entire Universe are maximally entangled. Only the reduction of entanglement between bosonic and fermionic states of freedom would be possible in SFRs. In the QFT framework, gauge boson fields are primary fields and the problem in principle disappears if entanglement is between states formed by elementary bosons and fermions.

In the TGD Universe, all elementary particles are composites of fundamental fermions (quarks in the simplest scenario) so that if Fock space the Fock states of fermions and bosons express everything worth expressing, SFRs would not be possible at all!

Remark: In the TGD Universe all elementary particles are composites of fundamental fermions (quarks in the simplest scenario) localized at the points of space-time surface defining a number theoretic discretization that I call cognitive representation. Besides this there are also degrees of freedom associated with the geometry of 3-surfaces representing particles. These degrees of freedom represent new physics. The quantization of quarks takes place at the level of H so that anticommutations hold true over the entire H.

Obviously, something is entangled and this entanglement is reduced. What these entangled degrees of freedom actually are if Fock space cannot provide them?

1. Mathematically entanglement makes sense also in a purely classical sense. Consider functions $\Psi_i(x)$ and $\Psi_j(y)$) and form the superposition $\Psi(x) = \sum_{ij} c_{ij} \Psi_i(x) \Psi_j x)$. This function is completely analogous to an entangled state.

2. Number theoretical physics implies that the Galois group becomes the symmetry group of physics and quantum states are representations of the Galois group [L90, L93]. For an extension of extension of ...., the Galois group has decomposition by normal subgroups to a hierarchy of coset groups.

   The representation of a Galois group can be decomposed to a tensor product of representations of these coset groups. The states in irreps of the Galois group are entangled and the SFR cascade produces a product of the states as a product of representations of the coset groups. Galois entanglement allows us to express the asymmetric relation between observer and observed very naturally. This cognitive SSFR cascade - as I have called it - could correspond to what happens in at least cognitive SFRs.

   If so, then SFR would in TGD have nothing to do with fermions and bosons (consisting of quarks too) since the maximal fermionic entanglement remains. For instance, when one for instance talks about long range entanglement the entanglement that matters would correspond to entanglement between degrees of freedom, which do not allow Fock space description.

In the TGD framework, the replacement of particles with 3-surfaces brings in an infinite number of non-Fock degrees of freedom. Could it make sense to speak about the reduction of entanglement in WCW degrees of freedom? There is no second quantization at WCW level so that one cannot talk about Fock spaces WCW level but purely classical entanglement is possible as observed.

1. In WCW unions of disjoint 3-surfaces correspond to classical many-particle states. One can form single particle wave functions for 3-surfaces with a single component, products of these

single particle wave functions, and also analogs of entangled states as their superposition realized as building bricks of WCW spinor fields.

If one requires that these wave functions are completely symmetric under the exchange of 3-surfaces, maximal entanglement in this sense would be realized also now and SFR would not be possible. But can one require the symmetry? Under what conditions one can regard two 3-surfaces as identical? For point-like particles one has always identical particles but in TGD the situation changes.

2. Here theoretical physics and category theory meet since the question when two mathematical objects can be said to be identical is the basic question of category theory. The mathematical answer is they are isomorphic in some sense. The physical answer is that the two systems are identical if they cannot be distinguished in the measurement resolution used.

9.4 $M^8 - H$ duality

There are several observations motivating $M^8 - H$ duality (see **Fig. 9.8**).

1. There are four classical number fields: reals, complex numbers, quaternions, and octonions with dimensions $1, 2, 4, 8$. The dimension of the embedding space is $D(H) = 8$, the dimension of octonions. Spacetime surface has dimension $D(X^4) = 4$ of quaternions. String world sheet and partonic 2-surface have dimension $D(X^2) = 2$ of: complex numbers. The dimension $D(string) = 1$ of string is that of reals.

2. Isometry group of octonions is a subgroup of automorphism group G_2 of octonions containing $SU(3)$ as a subgroup. $CP_2 = SU(3)/U(2)$ parametrizes quaternionic 4-surfaces containing a fixed complex plane.

Could M^8 and $H = M^4 \times CP_2$ provide alternative dual descriptions of physics (see **Fig. 9.8**)?

1. Actually a complexification $M_c^8 \equiv E_c^8$ by adding an imaginary unit i commuting with octonion units is needed in order to obtain sub-spaces with real number theoretic norm squared. M_c^8 fails to be a field since $1/o$ does not exist if the complex valued octonionic norm squared $\sum o_i^2$ vanishes.

2. The four-surfaces $X^4 \subset M^8$ are identified as "real" parts of 8-D complexified 4-surfaces X_c^4 by requiring that $M^4 \subset M^8$ coordinates are either imaginary or real so that the number theoretic metric defined by octonionic norm is real. Note that the imaginary unit defining the complexification commutes with octonionic imaginary units and number theoretical norm squared is given by $\sum_i z_i^2$ which in the general case is complex.

3. The space H would provide a geometric description, classical physics based on Riemann metric, differential geometric structures and partial differential equations deduced from an action principle. M_c^8 would provide a number theoretic description: no partial differential equations, no Riemannian metric, no connections...

   M_c^8 has only the number theoretic norm squared and bilinear form, which are real only if M_c^8 coordinates are real or imaginary. This would define "physicality". One open question is whether all signatures for the number theoretic metric of X^4 should be allowed? Similar problem is encountered in the twistor Grassmannian approach.

4. The basic objection is that the number of algebraic surfaces is very small and they are extremely simple as compared to extremals of action principle. Second problem is that there are no coupling constants at the level of M^8 defined by action.

   Preferred extremal property realizes quantum criticality with universal dynamics with no dependence on coupling constants. This conforms with the disappearance of the coupling constants from the field equations for preferred extremals in H except at singularities, with the Bohr orbitology, holography and ZEO. $X^4 \subset H$ is analogous to a soap film spanned by frame representing singularities and implying a failure of complete universality.

5. In M^8, the dynamics determined by an action principle is replaced with the condition that the *normal* space of X^4 in M^8 is associative/quaternionic. The distribution of normal spaces is always integrable to a 4-surface.

   One cannot exclude the possibility that the normal space is complex 2-space, this would give a 6-D surface [L82, L83]. Also this kind of surfaces are obtained and even 7-D with a real normal space. They are interpreted as analogs of branes and are in central role in TGD inspired biology.

   Could the twistor space of the space-time surface at the level of H have this kind of 6-surface as M^8 counterpart? Could $M^8 - H$ duality relate these spaces in 16-D M^8_c to the twistor spaces of the space-time surface as 6-surfaces in 12-D $T(M^4) \times T(CP_2)$?

6. Symmetries in M^8 number theoretic: octonionic automorphism group G_2 which is complexified and contains $SO(1,3)$. G_2 contains $SU(3)$ as M^8 counterpart of color $SU(3)$ in H. Contains also $SO(3)$ as automorphisms of quaternionic subspaces. Could this group appear as an (approximate) dynamical gauge group?

   $M^8 = M^4 \times E^4$ as $SO(4)$ as a subgroup. It is not an automorphism group of octonions but leaves the octonion norm squared invariant. Could it be analogous to the holonomy group $U(2)$ of CP_2, which is not an isometry group and indeed is a spontaneously broken symmetry.

   A connection with hadron physics is highly suggestive. $SO(4) = SU(2)_L \times SU(2)_R$ acts as the symmetry group of skyrmions identified as maps from a ball of M^4 to the sphere $S^3 \subset E^4$. Could hadron physics $\leftrightarrow$ quark physics duality correspond to $M^8 - H$ duality. The radius of S^3 is proton mass: this would suggest that M^8 has an interpretation as an analog of momentum space.

7. What is the interpretation of M^8? Massless Dirac equation in M^8 for the octonionic spinors must be algebraic. This would be analogous to the momentum space Dirac equation. Solutions would be discrete points having interpretation as quark momenta! Quarks pick up discrete points of $X^4 \subset M^8$.

   States turn out to be massive in the M^4 sense: this solves the basic problem of 4-D twistor approach (it works for massless states only). Fermi ball is replaced with a region of a mass shell (hyperbolic space H^3).

   M^8 duality would generalize the momentum-position duality of the wave mechanics. QFT does not generalize this duality since momenta and position are not anymore operators.

9.4.1 Associative dynamics in M^8_c

How to realize the associative dynamics in M^8_c [L82, L83]?

1. Number theoretical vision requires hierarchy of extensions of rationals and polynomials with rational coefficients would realize them. Rational coefficients make possible the interpretation as a polynomial with p-adic argument and therefore number theoretical universality.

   One cannot exclude the possibility that also real argument is allowed and that number theoretic universality and adelization applies only for the space-time surfaces defined by polynomials with rational coefficients.

2. Algebraic physics suggests that X^4 is in some sense a root of a M^8_c valued polynomial. One can continue polynomials P with rational coefficients to M^8_c by replacing the real argument with a complexified octonion.

3. The algebraic conditions should imply that the normal space of X^4 is quaternionic/associative. One can decompose octonions to sums $q_1 + I_4 q_2$, or "real" and "imaginary" parts q_i, which are quaternions and I_4 is octonion unit orthogonal to quaternions. The condition is that the "real" part of the octionic polynomial vanishes. Complexified 4-D surface whose projection to a real section (M^8 coordinates imaginary or real so that complexified octonion norm squared is real) is 4-D.

4. $M^8 - H$ duality requires an additional condition. The normal space contains also a complex plane M^2 which is commutative. This guarantees that normal spaces correspond to a point of CP_2. This is necessary in order to define $M^8 - H$ duality mapping X^4 from M^8 to H. M^2 can be replaced with an integrable distribution of M^2s if the assignment of the CP_2 point to tangent space can be made unique. This is the case if the spaces $M^2(x)$ are obtained from $M^2(y)$ by a unique G_2 automorphism $g(x, y)$.

Associativity condition at the level of M^8

Associativity condition for polynomials allows to characterize space-time surfaces in terms of polynomials with rational coefficients and possibly also analytic functions with rational Taylor coefficients at M^8 level. $M^8 - H$ duality would map $X^4 \subset M^8$ to $X^4 \subset H$. In M^8_c the space-time surfaces could be also seen as graphs of local (complex) G_2 gauge transformations.

Remark: Even non-rational coefficients can be considered. In this case polynomials with rational coefficients would define a unique discretion of WCW and allow p-adicization and adelization.

In the generic case the set of points in the extension of rationals defining cognitive representation is discrete and finite. The surprise was that the "roots"can be solved explicitly and that the discrete cognitive representation is dense so that momentum quantization due to the finite volume of CD must be assumed to obtain finite cognitive representation inside CD. Cognitive representation could be defined by the points which correspond to the 8-momenta solving octonionic Dirac equation. This is excellent news concerning practical applications.

The outcome of a detailed examination of the "roots" of the octonionic polynomial having real part $X = Re_Q(P)$ and imaginary part $Y = Im_Q(P)$ in quaternionic sense, yielded a series of positive and negative surprises and demonstrated the failure of the naive arguments based on dimension counting.

1. Although no interesting associative space-time surfaces are possible, every distribution of normal associative planes (co-associativity) is integrable. Note that the distribution of normal spaces must have an integrable distribution of commutative planes in order to guarantee the existence of $M^8 - H$ duality. Generic arguments fail in the presence of symmetries.

2. Another positive surprise was that Minkowski signature is the only possible option. Equivalently, the image of M^4 as real co-associative subspace of O_c (complex valued octonion norm squared is real valued for them) by an element of local $G_{2,c}$ or its subgroup $SU(3, c)$ gives a real co-associative space-time surface.

3. The conjecture based on naive dimensional counting, which was not correct, was that the polynomials P determine these 4-D surfaces as roots of $Re_Q(P)$. The normal spaces of these surfaces possess a fixed 2-D commuting sub-manifold or possibly their distribution allowing the mapping to H by $M^8 - H$ duality as a whole.

   If this conjecture were correct, strong form of holography (SH) would not be needed and would be replaced with extremely powerful number theoretic holography determining space-time surface from its roots and selection of real subspace of O_c characterizing the state of motion of a particle.

4. One of the cold showers during the evolution of the ideas about $M^8 - H$ duality was that the naive expectation that one obtains complex 4-D surfaces as solutions is wrong. The equations for $Re_Q(P) = 0$ ($ImQ_P = 0$) reduce to roots of ordinary real polynomials defined by the odd (even() parts of P and have interpretation as complex values of 8-D mass squared. These surfaces have complex dimension 7. 4 complex dimensions should be eliminated in order to have a complex 4-D surface, whose real parts would give a real 4-surface X^4. The explanation for the unexpected result comes from the symmetries of the octonionic polynomial implying that generic arguments fail.

How does one obtain 4-D space-time surfaces?

Contrary to the naive expections, the solutions of the vanishing conditions for the $Re_Q(P)$ ($Im_Q(P)$) (real (imaginary) part in quaternionic sense) are 7-D complex mass shells $r^2 = r_{n,1}$ as roots of $P_1(r) = 0$ or $r^2 = r_{n,2}$ of $P_2(r) = 0$ rather than 4-D complex surfaces (for a detailed discussion see [K16]) A solution of both conditions requires that P_1 and P_2 have a common root but the solution remains a 7-D complex mass shell! This was one of the many cold showers during the development of the ideas about $M^8 - H$ duality! It seems that the adopted interpretation is somehow badly wrong. Here zero energy ontology (ZEO) and holography come to the rescue.

1. Could the roots of P_1 or P_2 define only complex mass shells of the 4-D complex momentum space identifiable as M^4_c? ZEO inspires the question whether a proper interpretation of mass shells could be as pre-images of boundaries of cd:s (intersections of future and past directed light-cones) as pairs of mass shells with opposite energies. If this is the case, the challenge would be to understand how X^4_c is determined if P does not determine it.

   Here holography, considered already earlier, suggests itself: the complex 3-D mass shells belonging to X^4_c would only define the 3-D boundary conditions for holography and the real mass shells would be mapped to the boundaries of cds. This holography can be restricted to X^4_R. Bohr orbit property at the level of H suggests that the polynomial P defines the 4-surface more or less uniquely.

2. Let us take the holographic interpretation as a starting point. In order to obtain an X^4_c mass shell from a complex 7-D light-cone, 4 complex degrees of freedom must be eliminated. $M^8 - H$ duality requires that X^4_c allows M^4_c coordinates.

   Note that if one has $X^4_c = M^4_c$, the solution is trivial since the normal space is the same for all points and the H image under $M^8 - H$ duality has constant $CP_2 = SU(3)/U(2)$ coordinates. X^4_c should have interpretation as a non-trivial deformation of M^4_c in M^8.

3. By $M^8 - H$ duality, the normal spaces should be labelled by $CP_2 = SU(3)/U(2)$ coordinates. $M^8 - H$ duality suggests that the image $g(p)$ of a momentum $p \in M^4_c$ is determined essentially by a point $s(p)$ of the coset space $SU(3)/U(2)$. This is achieved if M^4_c is deformed by a local $SU(3)$ transformation $p \to g(p)$ in such a way that each image point is invariant under $U(2)$ and the mass value remains the same: $g(p)^2 = p^2$ so that the point represents a root of P_1 or P_2.

   Remark: I have earlier considered the possibility of G_2 and even $G_{2,c}$ local gauge transformation. It however seems that that local $SU(3)$ transformation is the only possibility since G_2 and $G_{2,c}$ would not respect $M^8 - H$ duality. One can also argue that only real $SU(3)$ maps the real and imaginary parts of the normal space in the same manner: this is indeed an essential element of $M^8 - H$ duality.

4. This option defines automatically $M^8 - H$ duality and also defines causal diamonds as images of mass shells $m^2 = r_n$. The real mass shells in H correspond to the real parts of r_n. The local $SU(3)$ transformation g would have interpretation as an analog of a color gauge field. Since the H image depends on g, it does not correspond physically to a local gauge transformation but is more akin to an element of Kac-Moody algebra or Yangian algebra which is in well-defined half-algebra of Kac-Moody with non-negative conformal weights.

The following summarizes the still somewhat puzzling situation as it is now.

1. The most elegant interpretation achieved hitherto is that the polynomial P defines only the mass shells so that mass quantization would reduce to number theory. Amusingly, I started to think about particle physics with a short lived idea that the d'Alembert equation for a scalar field could somehow give the mass spectrum of elementary particles so that the issue comes full circle!

2. Holography assigns to the complex mass shells complex 4-surfaces for which $M^8 - H$ duality is well-defined even if these surfaces would fail to be 4-D co-associative. These surfaces are expected to be highly non-unique unless holography makes them unique. The Bohr orbit property of their images in H indeed suggests this apart from a finite non-determinism [L107].

Bohr orbit property could therefore mean extremely powerful number theoretical duality for which the roots of the polynomial determine the space-time surface almost uniquely. $SU(3)$ as color symmetry emerges at the level of M^8. By M^8-H duality, the mass shells are mapped to the boundaries of CDs in H.

3. Do we really know that X^4_r co-associative and has distribution of 2-D commuting subspaces of normal space making possible M^8-H duality? The intuitive expectation is that the answer is affirmative [A27]. In any case, M^8-H duality is well-defined even without this condition.

4. The special solutions to $P=0$, discovered already earlier, are restricted to the boundary of CD_8 and correspond to the values of energy (rather than mass or mass squared) coming as roots of the real polynomial P. These mass values are mapped by inversion to "very special moments in the life of self" (a misleading term) at the level of H as special values of light-cone proper time rather than linear Minkowski time as in the earlier interpretation [L65]. The new picture is Lorenz invariant.

Octonionic Dirac equation requires co-associativity

The octonionic Dirac equation allows a second perspective on associativity [L83].

1. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

2. Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q\cdot q=m^2=r_n$, where $q\cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p=I_4q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

3. For the co-associative option, the co-associative octonion p representing the momentum is given in terms of quaternion q as $p=I_4q$. One obtains $p\cdot p=q\overline{q}=m^2=r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of p p are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

4. For associative option, one would obtain $qq=m^2$, which cannot be satisfied: q reduces to a complex number $zx+Iy$ and one has analog of equation $z^2=z^2-y^2+2Ixy=m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

This picture combined with zero energy ontology leads also to a view about quantum TGD at the level of M^8. Local $SU(3)$ element g has properties suggesting a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

Hamilton-Jacobi structure and Kähler structure of $M^4\subset H$ and their counterparts in $M^4\subset M^8$

The Kähler structure of $M^4\subset H$, forced by the twistor lift of TGD, has deep physical implications and seems to be necessary. It implies that for Dirac equation in H, modes are eigenstates of only the longitudinal momentum and in the 2 transversal degrees of freedom one has essentially harmonic oscillator states [L104, L100], that is Gaussians determined by the 2 longitudinal momentum components. For real longitudinal momentum the exponents of Gaussians are purely imaginary or purely real.

The longitudinal momentum space $M^2 \subset M^4$ and its orthogonal complement E^2 is in a preferred role in gauge theories, string models, and TGD. The localization of this decomposition leads to the notion of Hamilton-Jacobi (HJ) structure of M^4 and the natural question is how this relates to Kähler structures of M^4. At the level of H spinors fields only the Kähler structure corresponding to constant decomposition $M^2 \oplus E^2$ seems to make sense and this raises the question how the H-J structure and Kähler structure relate. TGD suggests the existence of two geometric structure in M^4: HJ structure and Kähler structure. It has remained unclear whether HJ structure and Kähler structure with covariantly constant self-dual Kähler form are equivalent notions or whether there several H-J structures accompaning the Kähler structure.

In the following I argue that H-J structures correspond to different choices of symplectic coordinates for M^4 and that the properties of $X^4 \subset H$ determined bt $M^- H$ duality make it natural to to choose particular symplectic coordinates for M^4.

Consider first what H-J structure and Kähler structure could mean in H.

1. The H-J structure of $M^4 \subset H$ would correspond to an integrable distribution of 2-D Minkowskian sub-spaces of M^4 defining a distribution of string world sheets $X^2(x)$ and orthogonal distribution of partonic 2-surfaces $Y^2(x)$. Could this decomposition correspond to self-dual covariantly Kähler form in M^4?

   What do we mean with covariant constancy now? Does it mean a separate covariant constancy for the choices of $M^2(x)$ and $Y^2(x)$ or only of their sum, which in Minkowski coordinates could correspond to a constant electric and magnetic fields orthogonal to each other?

2. The non-constant choice of $M^2(x)$ ($E^2(x)$) cannot be covariantly constant. One can write $J(M^4) = J(M^2(x)) \oplus J(E^2(x)$ corresponding to decomposition to electric and magnetic parts. Constancy of $J(M^2(x)$ would require that the gradient of $J(M^2(x)$ is compensated by the gradient of an antisymmetric tensor with square equal to the projector to $M^2(x)$. Same condition holds true for $J(E^2(x))$. The gradient of the antisymmetric tensor would be parallel to itself implying that the tensor is constant.

3. H-J structure can only correspond to a transformation acting on J but leaving $J_{kl}dm^kdm^l$ invariant. One should find analogs of local gauge transformations leaving J invariant. In the case of CP_2, these correspond to symplectic transformations and now one has a generalization of the notion. The M^4 analog of the symplectic group would parameterize various decompositions of $J(M^4)$.

   Physically the symplectic transformations define local choices of 2-D space $E^2(x)$ of transversal polarization directions and longitudinal momentum space M^2 emerging in the construction of extremals of Kähler action.

4. For the simplest Kähler form for $M^4 \subset H$, this decomposition in Minkowski coordinates would be constant: orthogonal constant electric and magnetic fields. This Kähler form extends to its number theoretical analog in M^8. The local $SU(3)$ element g would deform M^4 to $g(M^4)$ and define an element of local CP_2 defining $M^8 - H$ duality. g should correspond to a symplectic transformation of M^4.

Consider next the number theoretic counterparts of H-J- and Kähler structures of $M^4 \subset H$ in $M^4 \subset M^8$.

1. In M^4 coordinates H-J structure would correspond to a constant $M^2 \times E^2$ decomposition. In M^4 coordinates Kähler structure would correspond to constant E and B orthogonal to each other. Symplectic transformations give various representations of this structure as H-J structures.

2. The number theoretic analog of H-J structure makes sense also for $X^4 \subset M^8$ as obtained from the distribution of quaternionic normal spaces containing 2-D commutative sub-space at each point by multiplying then by local unit $I_4(x)$ orthogonal to the quaternionic units $\{1, I_1 = I_2 = I_3\}$ with respect to octonionic inner product. There is a hierarchy of CDs and the choices of these structures would be naturally parameterized by G_2.

This would give rise to a number theoretically defined slicing of $X^4_c \subset M^8_c$ by complexified string world sheets X^2_c and partonic 2-surfaces Y^2_c orthogonal with respect to the octonionic inner product for complexified octonions.

3. In M^8-H duality defined by $g(p) \subset SU(3)$ assigns a point of CP_2 to a given point of M^4. $g(p)$ maps the number theoretic H-J to H-J in $M^4 \subset M^8$. The space-time surface itself - that is $g(p)$ - defines these symplectic coordinates and the local $SU(3)$ element g would naturally define this symplectic transformation.

4. For $X^4 \subset M^8$ g reduces to a constant color rotation satisfying the condition that the image point is $U(2)$ invariant. Unit element is the most natural option. This would mean that g is constant at the mass and energy shells corresponding to the roots of P and the mass shell is a mass shell of M^4 rather than some deformed mass shell associated with images under $g(p)$.

   This alone does not yet guarantee that the 4-D tangent space corresponds to M^4. The additional physically very natural condition on g is that the 4-D momentum space at these mass shells is the same. M^8-H duality maps these mass shells to the boundaries of these cd:s in M^4 (CD= $cd \times CP_2$). This conforms with the identification of zero energy states as pairs of 3-D states at the boundaries of CD.

This generalizes the original intuitive but wrong interpretation of the roots r_n of P as "very special moments in the life of self" [L65].

1. Since the roots correspond to mass squared values, they are mapped to the boundaries of cd with size $L = \hbar_{eff}/m$ by M^8-H duality in M^4 degrees of freedom. During the sequence of SSFRs the passive boundary of CD remains does not shift only changes in size, and states at it remain unaffected. Active boundary is shifted due to scaling of cd.

   The hyperplane at which upper and lower half-cones of CD meet, is shifted to the direction of geometric future. This defines a geometric correlate for the flow of experienced time.

2. A natural proposal is that the moments for SSFRs have as geometric correlates the roots of P defined as intersections of geodesic lines with the direction of 4-momentum p from the tip of CD to its opposite boundary (here one can also consider the possibility that the geodesic lines start from the center of cd). Also energy shells as roots $E = r_n$ of P are predicted. They decompose to a set of mass shells $m_{n.,k}$ with the same $E = r_n$: similar interpretation applies to them.

3. What makes these moments very special is that the mass and energy shells correspond to surfaces in M^4 defining the Lorentz quantum numbers. SSFRs correspond to quantum measurements in this basis and are not possible without this condition. At $X^4 \subset M^8$ the mass squared would remain constant but the local momentum frame would vary. This is analogous to the conservation of momentum squared in general relativistic kinematics of point particle involving however the loss of momentum conservation.

4. These conditions, together with the assumption that g is a rational function with real coefficients, strongly suggest what I have referred to as preferred extremal property, Bohr orbitology, strong form of holography, and number theoretical holography.

In principle, by a suitable choice of M^4 one can make the momentum of the system light-like: the light-like 8-momentum would be parallel to M^4. I have asked whether this could be behind the fact that elementary particles are in a good approximation massless and whether the small mass of elementary particles is due to the presence of states with different mass squares in the zero state allowed by Lorentz invariance.

The recent understanding of the nature of right-handed neutrinos based on M^4 Kähler structure [L100] makes this mechanism un-necessary but poses the question about the mechanism choosing some particular M^4. The conditions that $g(p)$ leaves mass shells and their 4-D tangent spaces invariant provides this kind of mechanism. Holography would be forced by the condition that the 4-D tangent space is same for all mass shels representing inverse images for very special moments of time.

9.4.2 Uncertainty Principle and $M^8 - H$ duality

The detailed realization of $M^8 - H$ duality involves still uncertainties. The quaternionic normal spaces containing fixed 2-space M^2 (or an integrable distribution of M^2) are parametrized by points of CP_2. One can map the normal space to a point of CP_2.

The tough problem has been the precise correspondence between M^4 points in $M^4 \times E^4$ and $M^4 \times CP_2$ and the identification of the sizes of causal diamonds (CDs) in M^8 and H. The identification is naturally linear if M^8 is analog of space-time but if M^8 is interpreted as momentum space, the situation changes. The option discussed in [L82, L83] maps mass hyperboloids to light-cone proper time =constant hyperboloids and it has turned out that this correspondence does not correspond to the classical picture suggesting that a given momentum in M^8 corresponds in H to a geodesic line emanating from the tip of CD.

$M^8 - H$ duality in M^4 degrees of freedom

The following proposal for $M^8 - H$ duality in M^4 degrees of freedom relies on the intuition provided by UP and to the idea that a particle with momentum p^k corresponds to a geodesic line with this direction emanating from the tip of CD.

1. The first constraint comes from the requirement that the identification of the point $p^k \in X^4 \subset M^8$ should classically correspond to a geodesic line $m^k = p^k \tau/m$ ($p^2 = m^2$) in M^8 which in Big Bang analogy should go through the tip of the CD in H. This geodesic line intersects the opposite boundary of CD at a unique point.

   Therefore the mass hyperboloid H^3 is mapped to the 3-D opposite boundary of $cd \subset M^4 \subset H$. This does not fix the size nor position of the CD ($= cd \times CP_2$) in H. If CD does not depend on m, the opposite light-cone boundary of CD would be covered an infinite number of times.

2. The condition that the map is 1-to-1 requires that the size of the CD in H is determined by the mass hyperboloid M^8. Uncertainty Principle (UP) suggests that one should choose the distance T between the tips of the CD associated with m to be $T = \hbar_{eff}/m$.

   The image point m^k of p^k at the boundary of $CD(m, h_{eff})$ is given as the intersection of the geodesic line $m^k = p^k\tau$ from the origin of $CD(m, h_{eff})$ with the opposite boundary of $CD(m, h_{eff})$:

$$m^k = \hbar_{eff} X \frac{p^k}{m^2} \quad , X = \frac{1}{1+p_3/p_0} \quad . \tag{9.4.1}$$

   Here p_3 is the length of 3-momentum.

   The map is non-linear. At the non-relativistic limit ($X \to 1$), one obtains a linear map for a given mass and also a consistency with the naive view about UP. m^k is on the proper time constant mass shell so the analog of the Fermi ball in $H^3 \subset M^8$ is mapped to the light-like boundary of $cd \subset M^4 \subset H$.

3. What about massless particles? The duality map is well defined for an arbitrary size of CD. If one defines the size of the CD as the Compton length $\hbar_{eff}/m$ of the massless particle, the size of the CD is infinite. How to identify the CD? UP suggests a CD with temporal distance $T = 2\hbar_{eff}/p_0$ between its tips so that the geometric definition gives $p^k = \hbar_{eff} p^k/p_0^2$ as the point at the 2-sphere defining the corner of CD. p-Adic thermodynamics [K52]) strongly suggests that also massless particles generate very small p-adic mass, which is however proportional to $1/p$ rather than $1/\sqrt{p}$. The map is well defined also for massless states as a limit and takes massless momenta to the 3-ball at which upper and lower half-cones meet.

4. What about the position of the CD associated with the mass hyperboloid? It should be possible to map all momenta to geodesic lines going through the 3-ball dividing the largest CD involved with T determined by the smallest mass involved to two half-cones. This is because this 3-ball defines the geometric "Now" in TGD inspired theory of consciousness. Therefore all CDs in H should have a common center and have the same geometric "Now".

$M^8 - H$ duality maps the slicing of momentum space with positive/negative energy to a Russian doll-like slicing of $t \geq 0$ by the boundaries of half-cones, where t has origin at the bottom of the double-cone. The height of the $CD(m, h_{eff})$ is given by the Compton length $L(m, h_{eff}) = \hbar_{eff}/m$ of quark. Each value of h_{eff} corresponds its own scaled map and for $h_{gr} = GMm/v_0$, the size of $CD(m, h_{eff}) = GM/v_0$ does not depend on m and is macroscopic for macroscopic systems such as Sun.

5. The points of cognitive representation at quark level must have momenta with components, which are algebraic integers for the extension of rationals considered. A natural momentum unit is $m_{Pl} = \hbar_0/R$, h_0 is the minimal value of $h_{eff} = h_0$ and R is CP_2 radius. Only "active" points of $X^4 \subset M^8$ containing quark are included in the cognitive representation. Active points give rise to active CD:s $CD(m, h_{eff})$ with size $L(m, h_{eff})$.

   It is possible to assign $CD(m, h_{eff})$ also to the composites of quarks with given mass. Galois confinement suggest a general mechanism for their formation: bound states as Galois singlets must have a rational total momentum. This gives a hierarchy of bound states of bound states of ..... realized as a hierarchy of CDs containing several CDs.

6. This picture fits nicely with the general properties of the space-time surfaces as associative "roots" of the octonionic continuation of a real polynomial. A second nice feature is that the notion of CD at the level H is forced by this correspondence. "Why CDs?" at the level of H has indeed been a longstanding puzzle. A further nice feature is that the size of the largest CD would be determined by the smallest momentum involved.

7. Positive and negative energy parts of zero energy states would correspond to opposite boundaries of CDs and at the level of M^8 they would correspond to mass hyperboloids with opposite energies.

8. What could be the meaning of the occupied points of M^8 containing fermion (quark)? Could the image of the mass hyperboloid containing occupied points correspond to sub-CD at the level of H containing corresponding points at its light-like boundary? If so, $M^8 - H$ correspondence would also fix the hierarchy of CDs at the level of H.

It is enough to realize the analogs of plane waves only for the actualized momenta corresponding to quarks of the zero energy state. One can assign to CD as total momentum and passive *resp.* active half-cones give total momenta $P_{tot,P}$ *resp.* $P_{tot,A}$, which at the limit of infinite size for CD should have the same magnitude and opposite sign in ZEO.

The above description of $M^8 - H$ duality maps quarks at points of $X^4 \subset M^8$ to states of induced spinor field localized at the 3-D boundaries of CD but necessarily delocalized into the interior of the space-time surface $X^4 \subset H$. This is analogous to a dispersion of a wave packet. One would obtain a wave picture in the interior.

Does Uncertainty Principle require delocalization in H or in X^4?

One can argue that Uncertainty Principle (UP) requires more than the naive condition $T = \hbar_{eff}/m$ on the size of sub-CD. I have already mentioned two approaches to the problem: they could be called inertial and gravitational representations.

1. The inertial representations assigns to the particle as a space-time surface (holography) an analog of plane wave as a superposition of space-time surfaces: this is natural at the level of WCW. This requires delocalization space-time surfaces and CD in H.

2. The gravitational representation relies on the analog of the braid representation of isometries in terms of the projections of their flows to the space-time surface. This does not require delocalization in H since it occurs in X^4.

Consider first the inertial representation. The intuitive idea that a single point in M^8 corresponds to a discretized plane wave in H in a spatial resolution defined by the total mass at the passive boundary of CD. UP requires that this plane wave should be realized at the level of H and also WCW as a superposition of shifted space-time surfaces defined by the above correspondence.

1. The basic observation leading to TGD is that in the TGD framework a particle as a point is replaced with a particle as a 3-surface, which by holography corresponds to 4-surface.

   Momentum eigenstate corresponds to a plane wave. Now planewave could correspond to a delocalized state of 3-surface - and by holography that of 4-surface - associated with a particle.

   A generalized plane wave would be a quantum superposition of shifted space-time surfaces inside a larger CD with a phase factor determined by the 4-momentum. $M^8 - H$ duality would map the point of M^8 containing an object with momentum p to a generalized plane wave in H. Periodic boundary conditions are natural and would force the quantization of momenta as multiples of momentum defined by the larger CD. Number theoretic vision requires that the superposition is discrete such that the values of the phase factor are roots of unity belonging to the extension of rationals associated with the space-time sheet. If momentum is conserved, the time evolutions for massive particles are scalings of CD between SSFRs are integer scalings. Also iterated integer scalings, say by 2 are possible.

2. This would also provide WCW description. Recent physics relies on the assumption about single background space-time: WCW is effectively replaced with M^4 since 3-surface is replaced with point and CP_2 is forgotten so that one must introduce gauge fields and metric as primary field variables.

As already discussed, the gravitational representation would rely on the lift/projection of the flows defined by the isometry generators to the space-time surface and could be regarded as a "subjective" representation of the symmetries. The gravitational representation would generalize braid group and quantum group representations.

The condition that the "projection" of the Dirac operator in H is equal to the modified Dirac operator, implies a hydrodynamic picture. In particular, the projections of isometry generators are conserved along the lifted flow lines of isometries and are proportional to the projections of Killing vectors. QCC suggests that only Cartan algebra isometries allow this lift so that each choice of quantization axis would also select a space-time surface and would be a higher level quantum measurement.

Exact ZEO emerges only at the limit of CD with infinite size

At the limit when the volume of CD becomes infinite, the sum of the momenta associated with opposite boundaries of CD should automatically vanish and one would obtain ideal zero energy states. The original assumption that ideal zero energy states are possible for finite size of CD, is not strictly true. The situation is the same for quantization in a finite volume.

1. Denote the sum of the total momenta with positive energy associated with passive boundaries of all CDs by $P_{tot,P} =\equiv P_{tot}$. For finite size of CD, $P_{tot,P}$ need not be the same as the total momentum $P_{tot,A}$ associated with the active boundary which can change during the sequence of SSFRs. Denote the difference $P_{tot,P} - P_{tot,A}$ by ΔP.

   This momentum is P_{tot} is large for large CDs, and naturally defines the spatial resolution. Denote by $M^k = nXh_{eff}P^k_{tot}/\cdot P^2_{tot}$, $X = 1/(1 + P_3/P_0)$, the shift defined by P_{tot}. The analogs of plane waves for the sub-CDs should be discretized with this spatial resolution and at the limit of large total mass the discretization improves.

2. The image of X^4 in H for a given mass hyperboloid H^3 should define a geometric analog of a plane wave in WCW for the total momentum $P^k = \sum_i p^k_i$, $p^2_i = m^2$ of H^3, associated with the CD(M) in M^8. It is also possible to include the momenta with different masses since they have images also at the boundaries of all CDs in the Russian doll hierarchy. For $\hbar_{gr}$ there is a common CD for all particle masses with size Λ_{gr}.

   The WCW plane wave would not be a superposition of points but of shifted space-time surfaces. The argument of the plane wave would correspond to the shift of the $X^4 \subset CD(M) \subset H$.

   Maximal spatial resolution is achieved if one shifts the X^4 and corresponding CD(m) in H inside the large CD by nM^k, $M^k = nh_{eff}XP^k_{tot}/\cdot P^2_{tot}$ and forms the WCW spinor field as a superposition of shifted space-time surfaces $X^4(m)$ with $U_n = exp(i\Delta P \cdot nM)$ appearing as plane wave phase factor.

3. At the limit when the size of the largest CD becomes infinite (the mass M defining Λ_{gr} becomes very large), the sum $\sum_n U_n$ obtained as integral over the identical shifted copies of the space-time surfaces is non-vanishing only for $\Delta P = 0$ and one obtains an momentum conserving ideal zero energy state.

These states would be analogs of single particle states as plane waves, with particle replaced with many-quark state inside $CD(m)$. The generalization is obvious: perform the analog of second quantization by forming N-particle states in which one has N $CD(m)$ plane waves.

The revised view about $M^8 - H$ duality and the "very special moments in the life of self"

The polynomial equations allow at M^8 level also highly unique brane-like solutions having the topology of 6-sphere S^6 and intersecting M^4 along $p^0 = E = constant$ hyperplane. These quantizd values of energy E correspond to the roots of the polynomial defining the solution and are algebraic numbers and algebraic integers for monic polynomials of form $P(x) = x^n + p_{n-1}x^{n-1} + ...$

The TGD inspired theory of consciousness motivated the interpretation of these hyperplanes as "very special moments in the life of self": this interpretation [L65] emerged before the realization that M^8 corresponds to momentum space. The images of these planes under $M^8 - H$ duality should however allow this interpretation also in the new picture. Is this possible?

To answer the question one must understand what the image of S^6 under $M^8 - H$ duality is.

1. The image must belong to $M^4 \times CP_2$. The 2-D normal space of the point of S^6 is a complex commutative plane of octonions. Since 4-D normal planes of space-time surface containing complex plane correspond to points of CP_2, the natural proposal is that the image now corresponds to point of CP_1 identified as homologically trivial geodesic sub-manifold S^2_G of CP_2 carrying Kähler magnetic charge.

2. The first thing to notice about the H-image of the 3-D $E = constant$ surface $X^3(E) \subset M^4$ is that it is indeed 3-D rather than 4-D. In M^4 the map has the form $m^k = X\hbar_{eff}/m^2$, $X = 1/(1 + p_3/p_0)$ already discussed.

   The value of $m^2 = E^2 - p_3^2$ decreases as p_3^2 increases so that the values of light-cone proper time $a = t^2 - r^2$ for the image are larger than $a_{min} = \hbar_{eff}/m$. "Fermi-spheres" $S^2_F(p_3)$ are mapped to 2-spheres $S^2(r) \subset M^4 \subset H$ with an increasing radius $r(t) = \sqrt{t^2 - a^2_{min}}$. 2-sphere is born at $t = a_{min}$ and starts to increase in size and the expansion velocity approaches light velocity asymptotically. This expanding sphere would be magnetically charged.

   The sequence a_n of "very special moments in the life of self" in the life of self would mean the birth of this kind of expanding sphere and a_n would correspond to the roots of the polynomial considered identified as quantized energies. The dispersion relation $E = constant$ means that energy does not depend on the momentum: plasmons provide the condensed matter analogy.

3. There are interesting questions to be answered. Do the surfaces $X^3(E)$ intersect the 4-D space-time surface $X^4 \subset H$? At the level of M^8 the intersections of 4-D and 6-D surfaces are 2-D. The proposal is that these 2-surfaces M^8 are mapped to partonic vertices identified as 2-surfaces $X^2 \subset X^4 \subset H$ at which 4-D surfaces representing particles meet. This should happen also for the new identification of $M^8 - H$ duality.

   However, in the generic case the intersections of 3-surfaces and 4-surfaces in H are empty. The recent situation is however not a generic one since the S^6 solutions are non-generic (one would expect only 4-D solutions) and 4-D and 6-D solutions are determined by the same polynomial. Therefore the points to which the 2-spheres contract for $t = a_{min}$ should be mapped to partonic 2-surfaces in H. Single point should correspond to the geodesic sphere S^2_G.

   Does this conform with the view that 4-D CP_2 type extremals in H correspond to "blow-ups" of 1-D line singularities of $X^4 \subset M^8$ for which the quaternionic tangent spaces at singularity are not unique and define 3-D surface as points of CP_2. Now the 2-D normal spaces of S^2_F would span $S^2_G \subset CP_2$ and at the limit of S^2_F contracting to a point, one would have a 2-D singularity having an interpretation as a partonic vertex.

4. Cosmic strings $X^4 = X^2 \times S^2_G \subset M^4 \times CP_2$ carrying monopole charge are basic solutions of field equations. Could these cosmic strings relate to the images of $X^3(E)$? For instance, could $X^3(E_1)$ and $X^3(E_2)$ correspond to the ends of a cosmic string thickening to a monopole flux tube? Thickening would correspond to the growth of M^4 projection $S^2(r(t))$ of the flux tube having $r(t) = \sqrt{t^2 - a^2_{min}}$. The interpretation would be as a pair of magnetic poles connected by a monopole flux tube. Cosmic strings would be highly dynamical entities if this is the case.

An objection against $M^8 - H$ duality

Objections are the best manner to proceed. $M^8 - H$ duality maps the point M^8 at mass shell m to points of CD corresponding to the Compton length $\hbar_{eff}/m$ obtained as intersection of line with momentum p starting at the center point of CD and intersecting either boundary of CD. Each quaternionic normal space contains a commuting subspace (in octonionic sense) such that the distribution of the latter spaces is integrable. These normal spaces are parameterized by CP_2. This implies a complete localization in CP_2 so that the restriction of the induced quark field does not have well-defined color quantum numbers.

How to circumvent this objection? The proposed identification of string-like and particle-like space-time surfaces suggests a solution to the problem. Consider first CP_2 type extremals.

1. Consider first CP_2 type extremals as analogs of particles proposed to correspond to line singularities of algebraic 4-surfaces in M^8 with the property that the normal co-quaternionic space is not unique and the normal spaces at given point of the line are parametrized by a 3-D surface of CP_2 at each point of the light-like curve. Algebraic geometers speak of blow-up singularity. This kind of singularity is analogous to the tip of a cone.

   For polynomials the M^4 projection is a light-like geodesic. Also the octonionic continuations of analytic functions of real argument with rational Taylor coefficients can define space-time surfaces and in this case more general light-like curves are expected to be possible. This gives rise to a 4-D surface of H, which has the same Euclidean metric and Kähler form as CP_2 and only the induced gamma matrices are different.

2. The induced spinor field as restriction of the second quantized spinor field of H decomposes into modes, which are modes of H d'Alembertian. The modes have well-defined color quantum numbers so that one can speak of color quarks. This would mean that one can speak about colored quarks only inside CP_2 type extremals and possibly also inside string-like objects. This would trivialize the mysteries of quark and color confinement.

   Gluons would correspond to pairs of quark and antiquark associated with distinct wormhole throats or even - contacts. The mass squared for a given mode is well-defined but at the level of H only the right-handed neutrino is massless. Other states have mass of order CP_2 mass.

3. One can argue that the average momenta associated with these kinds of states have M^4 projection parallel to the light-like geodesic so that the momentum is light-like. There are several justifications for the claim.

   (a) The gravitational representation of isometries already discussed as lift/projection of the corresponding flows in H to X^4 restricts the action of M^4 isometries to a light-like geodesic and implies that the states are massless in this sense.

   (b) The claim conforms with an earlier intriguing observation that the restriction of a massive quark propagator to a pair of space-time points with light-like M^4 distance is essentially a massless propagator irrespective of the value of the mass.

   (c) With a suitable choice of $M^4 \subset M^8$ the ground state mass can be chosen to vanish. The reason is that the 8-D momentum is light-like and if M^4 contains the momentum, then also the M^4 mass vanishes. This choice can be made only for a single mode in the superposition. p-Adic thermodynamics would describe the contribution of higher modes in the quantum superposition of states to the mass squared having interpretation as thermal mass squared.

(d) One can look at the situation also at the space-time level. If one has a light-like curve or a curve consisting of segments, which are light-like geodesic lines, the situation changes. Since the average velocity for this kind of zigzag (zitterbewegung) curve is below light velocity, the intuitive expectation is that this represents the TGD analog of the Higgs mechanism having interpretation as massivation.

This finding was the original motivation for p-adic thermodynamics. The conditions stating the light-likeness of the projection are nothing but Virasoro conditions. p-Adic thermodynamics involves also the inclusion of supersymplectic symmetries.

$H(M^4)$ is orthogonal to the space-time surface and has an interpretation as a local acceleration of the space-time surface as an extended particle. The CP_2 part of H was the original proposal for the Higgs field considered in my thesis. Indeed, $H(CP_2)$ behaves like a complex doublet in complex coordinates. The physical interpretation is that the minimal surface property forces zitterbewegung with acceleration $H(M^4) = H(CP_2)$, which in turn means that light-like curve looks in the average sense like time-like geodesic for a massive particle.

The problem is that the proposed Higgs field vanishes in the interiors of space-time surfaces. However, the general field equations do not imply minimal surface property and also for preferred extremals it fails at singularities analogous to frames of soap films. At these point one can have non-vanishing $H(CP_2)$. 8-D light-likeness suggests that at these points $H(H)$ is light-like.

What happens to string like-objects corresponding to 2-D singularities such that the normal spaces at a given point correspond to a 2-D surface of CP_2, which in the most general situation can be either complex 2-surface of CP_2 or a minimal Lagrangian 2-manifold? One cannot exclude 1-D singularities associated with surfaces $X^3 \times X^1 \subset M^4 \times CP_2$ for which CP_2 projection is 1-D, presumably a geodesic circle.

(a) The simplest string-like objects come in 2 variants corresponding to CP_2 projection, which is a geodesic sphere, which can be homologically non-trivial or non-trivial. M^4 projection is in the simplest situation 2-D plane M^2.

These two options correspond to the reduction of $SU(3)$ to $U(2)$ or $SO(3)$. The interpretation in terms of spontaneous symmetry breaking is highly suggestive. The representations of $SU(3)$ decompose to those of $U(2)$ or $SO(3)$. Color confinement could weaken to that for $U(2)$ or $SO(3)$ so that the total color quantum numbers I_3 and Y would still vanish but color multiplets would allow these kinds of states.

(b) The simplest symmetry breaking to $U(1)$ could correspond to extremals of form $M^3 \times S^1$ and only $U(1)$ confinement would hold true. In the case of M^4 it does not make sense to speak of color quantum numbers.

9.4.3 Generalizations related to $M^8 - H$ duality

It has become clear that $M^8 - H$ duality generalizes and there is a connection with the twistorialization at the level of H.

M^8-H duality at the level of WCW and p-adic prime as the maximal ramified prime of polynomial

The vacuum functional as an exponent of the Kähler function determines the physics at WCW level. $M^8 - H$ duality suggests that it should have a counterpart at the level of M^8 and appear as a weight function in the summation. Adelic physics requires that weight function is a power of p-adic prime and ramified primes of the extension are the natural candidates in this respect.

1. The discriminant D of the algebraic extension defined by a polynomial P with rational coefficients (`https://en.wikipedia.org/wiki/Discriminant`) is expressible as a square for

the product of the non-vanishing differences $r_i - r_j$ of the roots of P. For a polynomial P with rational coefficients, D is a rational number as one can see for polynomial $P = ax^2 + bx + c$ from its expression $D = b^2 - 4ac$. For monic polynomials of form $x^n + a_{n-1}x^{m-1} + ...$ with integer coefficients, D is an integer. In both cases, one can talk about ramified primes as prime divisors of D.

If the p-adic prime $p$is identified as a ramified prime, D is a good candidate for the weight function since it would be indeed proportional to a power of p and have p-adic norm proportional to negative power of p. Hence the p-adic interpretation of the sum over scattering amplitudes for polynomials P is possible if p corresponds to a ramified prime for the polynomials allowed in the amplitude.

p-Adic thermodynamics [K52] suggest that p-adic valued scattering amplitudes are mapped to real numbers by applying to the Lorentz invariants appearing in the amplitude the canonical identification $\sum x_n p^n \to \sum x_n p^{-n}$ mapping p-adics to reals in a continuous manner

2. For monic polynomials, the roots are powers of a generating root, which means that D is proportional to a power of the generating root, which should give rise to some power of p. When the degree of the monic polynomial increases, the overall power of p increases so that the contributions of higher polynomials approach zero very rapidly in the p-adic topology. For the p-adic prime $p = M_{127} = 2^{127} - 1 \sim 10^{38}$ characterizing electrons, the convergence is extremely rapid.

   Polynomials of lowest degree should give the dominating contribution and the scattering amplitudes should be characterized by the degree of the lowest order polynomial appearing in it. For polynomials with a low degree n the number of particles in the scattering amplitude could be very small since the number n of roots is small. The sum $x_i + p_i$ cannot belong to the same mass shell for timelike p_i so that the minimal number of roots r_n increases with the number of external particles.

3. $M^8 - H$ duality requires that the sum over polynomials corresponds to a WCW integration at H-side. Therefore the exponent of Kähler function at its maximum associated to a given polynomial should be apart from a constant numerical factor equal to the discriminant D in canonical identification.

   The condition that the exponent of Kähler function as a sum of the Kähler action and the volume term for the preferred extremal $X^4 \subset H$ equals to power of D apart from a proportionality factor, should fix the discrete number theoretical and p-adic coupling constant evolutions of Kähler coupling strength and length scale dependent cosmological constant proportional to inverse of a p-adic length scale squared. For Kähler action alone, the evolution is logarithmic in prime p since the function reduces to the logarithm of D.

$M^8 - H$ duality suggests that the exponent $exp(-K)$ of Kähler function has an M^8 counterpart with a purely number theoretic interpretation. The discriminant D of the polynomial P is the natural guess. For monic polynomials D is integer having ramified primes as factors.

There are two options for the correspondence between $exp(-K)$ at its maximum and D assuming that P is monic polynomial.

1. In the real topology, one would naturally have $exp(-K) = 1/D$. For monic polynomials with high degree, D becomes large so that $exp(-K)$ is large.

2. In a p-adic topology defined by p-adic prime p identified as a ramified prime of D, one would have naturally $exp(-K) = I(D)$, where one has $I(x) = \sum x_n p^n = \sum x_n p^{-n}$.

   If p is the largest ramified prime associated with D, this option gives the same result as the real option, which suggests a unique identification of the p-adic prime p for a given polynomial P. P would correspond to a unique p-adic length scale L_p and a given L_p would correspond to all polynomials P for which the largest ramified prime is p.

   This might provide some understanding concerning the p-adic length scale hypothesis stating that p-adic primes tend to be near powers of integer. In particular, understanding about why Mersenne primes are favored might emerge. For instance, Mersennes could correspond to primes for which the number of polynomials having them as the largest ramified prime is especially large. The quantization condition $exp(-K) = D(p)$ could define which p-adic primes are the fittest ones.

The condition that $exp(-K)$ at its maximum equals to D via canonical identification gives a powerful number theoretic quantization condition.

Space-time surfaces as images of associative surfaces in M^8

$M^8 - H$ duality would provide an explicit construction of space-time surfaces as algebraic surfaces with an associative normal space [L82, L83]. M^8 picture codes space-time surface by a real polynomial with rational coefficients. One cannot exclude coefficients in an extension of rationals and also analytic functions with rational or algebraic coefficients can be considered as well as polynomials of infinite degree obtained by repeated iteration giving rise algebraic numbers as extension and continuum or roots as limits of roots.

$M^8 - H$ duality maps these solutions to H and one can consider several forms of this map. The weak form of the duality relies on holography mapping only 3-D or even 2-D data to H and the strongest form maps entire space-time surfaces to H. The twistor lift of TGD allows to identify the space-time surfaces in H as base spaces of 6-D surfaces representing the twistor space of space-time surface as an S^2 bundle in the product of twistor spaces of M^4 and CP_2. These twistor spaces must have Kähler structure and only the twistor spaces of M^4 and CP_2 have it [A54] so that TGD is unique also mathematically.

An interesting question relates to the possibility that also 6-D commutative space-time surfaces could be allowed. The normal space of the space-time surface would be a commutative subspace of M^8_c and therefore 2-D. Commutative space-time would be a 6-D surface X^6 in M^8.

This raises the following question: Could the inverse image of the 6-D twistor-space of 4-D space-time surface X^4 so that X^6 would be M^8 analog of twistor lift? This requires that $X^6 \subset M^8_c$ has the structure of an S^2 bundle and there exists a bundle projection $X^6 \rightarrow X^4$.

The normal space of an associative space-time surface actually contains this kind of commutative normal space! Its existence guarantees that the normal space of X^4 corresponds to a point of CP_2. Could one obtain the M^8_c analog of the twistor space and the bundle bundle projection $X^6 \rightarrow X^4$ just by dropping the condition of associativity. Space-time surface would be a 4-surface obtained by adding the associativity condition.

One can go even further and consider 7-D surfaces of M^8 with real and therefore well-ordered normal space. This would suggest dimensional hierarchy: $7 \rightarrow 6 \rightarrow 4$.

This leads to a possible interpretation of twistor lift of TGD at the level of M^8 and also about generalization of $M^8 - H$ correspondence to the level of twistor lift. Also the generalization of twistor space to a 7-D space is suggestive. The following arguments representa vision about "how it must be" that emerged during the writing of this article and there are a lot of details to be checked.

Commutative 6-surfaces and twistorial generalization of $M^8 - H$ correspondence

One can generalize the notion of complex 4-surface $X^6_c \subset M^8_c$ to that of complex 6-surface $X^6_c \subset M^8$ with a complexified commutative normal space. The 6-surface would correspond to a surface obtained by a local $SU(3)$ element invariant under $U(1) \times U(1) \subset SU(2)$. In complete analogy with 4-D case, these 6-surfaces would contain 5-D mass shells determined by the roots of P. The space $F = SU(3)/U(1) \times U(1)$ of points is nothing but the twistor space of CP_2!

The deformed M^6 defining $X^6 \subset M^8$ regarded as surface in M^8 suggests an interpretation as an analog of 6-D twistor space of M^4. Maybe one could identify the M^6 as the projective space $C^4/C_\times$ obtained from C^4 by dividing with complex scalings? This would give the twistor space $CP_3 = SU(4)/U(3)$ of M^4. This is not obvious since one has (complexified) octonions rather than C^4 or its hypercomplex analog. This would be analogous to using several (4) coordinate charts glued together as in the case of sphere CP_1.

The map $M^6 \rightarrow F$ obtained in this manner would define mapping of the twistor spaces of M^4 and CP_2 to each other. The twistor lift of TGD indeed defines this kind of map. The twistor lift involves the additional assumption that the S^2 fibers of these twistor spaces correspond to each other isometrically. This could correspond to a choice of Hamilton-Jacobi structure defining a local decomposition of $M^6 = M^2 \oplus E^4$ such that M^2 defines the analog of the Riemann sphere for M^6.

It might be also possible to identify the octonionic analog of the projective space $CP_3 = C^4/C_\times$. Could the octonionic M^8 momenta be scaled down by dividing with the momentum projection in the commutative normal space so that one obtains an analog of projective space? Could one use these as coordinates for M^6? The scaled 8-momenta would correspond to the points of the octonionic analog of CP_3. The scaled down 8-D mass squared would have a constant value.

A possible problem is that one must divide either from left or right and results are different in the general case. Could one require that the physical states are invariant under the automorphisms generated $o \to gog^{-1}$, where g is an element of the commutative subalgebra in question?

Physical interpretation of the counterparts of twistors at the level of M^8_c

What about the physical interpretation at the level of M^8_c. The twistor space allows a geometrization of spin so that momentum and spin would combine to a purely geometric entity with 6 components. The active points would correspond to fermions (quarks) with a given momentum and spin.

1. The first thing to notice is that in the twistor Grassmannian approach twistor space provides an elegant description of spin. Partial waves in the fiber S^2 of twistor space representation of spin as a partial wave. All spin values allow a unified treatment.

   The problem is that this requires massless particles. In the TGD framework 4-D masslessness is replaced with its 8-D variant so that this difficulty is circumvented. This kind of description in terms of partial waves is expected to have a counterpart at the level of the twistor space $T(M(^4) \times T(CP_2)$. At level of M^8 the description is expected to be in terms of discrete points of M^8_c.

2. Consider first the real part of $X^6_c \subset M^8_c$. At the level of M^8 the points of X^4 correspond to points. The same must be true also at the level of X^6. Single point in the fiber space S^2 would be selected. The interpretation could be in terms of the selection of the spin quantization axis.

   Spin quantization axis corresponds to 2 diametrically opposite points of S^2. Could the choice of the point also fix the spin direction? There would be two spin directions and in the general case of a massive particle they must correspond to the values $S_z = \pm 1/2$ of fermion spin. For massless particles in the 4-D sense two helicities are possible and higher spins cannot be excluded. The allowance of only spin 1/2 particles conforms with the idea that all elementary particles are constructed from quarks and antiquarks. Fermionic statistics would mean that for fixed momentum one or both of the diametrically opposite points of S^2 defining the same and therefore unique spin quantization axis can be populated by quarks having opposite spins.

3. For the 6-D tangent space of X^6_c or rather, its real projection, an analogous argument applies. The tangent space would be parametrized by a point of $T(CP_2)$ and mapped to this point. The selection of a point in the fiber S^2 of $T(CP_2)$ would correspond to the choice of the quantization axis of electroweak spin and diametrically opposite points would correspond to opposite values of electroweak spin 1/2 and unique quantization axis allows only single point or pair of diametrically opposite points to be populated.

   Spin 1/2 property would hold true for both ordinary and electroweak spins and this conforms with the properties of $M^4 \times CP_2$ spinors.

4. The points of $X^6_c \subset M^8_c$ would represent geometrically the modes of H-spinor fields with fixed momentum. What about the orbital degrees of freedom associated with CP_2?

   M^4 momenta represent orbital degrees of M^4 spinors so that E^4 parts of E^8 momenta should represent the CP_2 momenta. The eigenvalue of CP_2 Laplacian defining mass squared eigenvalue in H should correspond to the mass squared value in E^4 and to the square of the radius of sphere $S^3 \subset E^4$.

This would be a concrete realization for the $SO(4) = SU(2)_L \times SU(2)_R \leftrightarrow SU(3)$ duality between hadronic and quark descriptions of strong interaction physics. Proton as skyrmion would correspond to a map S^3 with radius identified as proton mass. The skyrmion picture would generalize to the level of quarks and also to the level of bound states of quarks allowed by the number theoretical hierarchy with Galois confinement. This also includes bosons as Galois confined many quark states.

5. The bound states with higher spin formed by Galois confinement should have the same quantization axis in order that one can say that the spin in the direction of the quantization axis is well-defined. This freezes the S^2 degrees of freedom for the quarks of the composite.

What does the map of the twistor space $T(M^4)$ to $T(CP_2)$ mean physically? Does spin correspond to color isospin or electroweak spin? Color $U(2)$ corresponds to electroweak $U(2)$ as the holonomy group of CP_2 as symmetric space so that the latter option is possible.

Quarks are doublets with respect to spin and electroweak spin but color triplet contains also isospin singlet. This is not a problem since color is not a spin-like quantum number in TGD but corresponds to color partial waves. This leaves spin-ew spin correspondence realized for quarks. Does the map between spin and electroweak degrees of freedom allow all pairings of spin and electroweak isospin doublets? The map between the spheres S^2 is determined only modulo relative rotation so that this might be the case for spin and color isospin. For composites of quarks obtained as Galois singlets, the relation between spin and ew spin could be more complex.

7-surfaces with real normal space and generalization of the notion of twistor space

The next step is to ask whether it makes sense to consider 7-surfaces with a real normal space allowing well-ordering? This would give a hierarchy of surfaces of M^8 with dimensions 7, 6, and 4. The 7-D space would have bundle projection to 6-D space having bundle projection to 4-D space.

One can also consider the complex 7-D surfaces with a complexified normal space for which the real projection is well-ordered so that the hierarchy of number fields would be realized. These surfaces would be realized by local elements of $SU(3)$ invariant under $U(1) \subset SU(3)$ and would define maps to $SU(3)/U(1)$ defining a generalization of twistor space. Now 6-D complex mass shells would take the role of 3-D complex mass shells and would correspond to the roots of P-

For the 7-D surface also the 7:th component of H- momentum should have some physical interpretation. Fermi statistics at the level of M^8 could be expressed purely geometrically: a single point of X^7 can contain only a single fermion (quark).

What could be the physical interpretation of 7-D surfaces of M^8 with real normal space in the octonionic sense and of their H images?

1. The first guess is that the images in H correspond to 7-D surfaces as generalizations of 6-D twistor space in the product of similar 7-D generalization of twistor spaces of M^4 and CP_2. One would have a bundle projection to the twistor space and to the 4-D space-time.

2. $SU(3)/U(1) \times U(1)$ is the twistor space of CP_2. $SU(3)/SU(2) \times U(1)$ is the twistor space of M^4? Could 7-D $SU(3)/U(1)$ *resp.* $SU(4)/SU(3)$ correspond to a generalization of the twistor spaces of M^4 *resp.* CP_2? What could be the interpretation of the fiber added to the twistor spaces of M^4, CP_2 and X^4? S^3 isomorphic to $SU(2)$ and having $SO(4)$ as isometries is the obvious candidate.

3. The analog of $M^8 - H$ duality in Minkowskian sector in this case could be to use coordinates for M^7 obtained by dividing M^8 coordinates by the real part of the octonion. Is it possible to identify $RP_7 = M^8/R_\times$ with $SU(4)/SU(3)$ or at least relate these spaces in a natural manner. It should be easy to answer these questions with some knowhow in practical topology.

   A possible source of problems or of understanding is the presence of a commuting imaginary unit implying that complexification is involved in Minkowskian degrees of freedom whereas in CP_2 degrees of freedom it has no effect. RP_7 is complexified to CP_7 and the octonionic analog of CP_3 is replaced with its complexification.

What could be the physical interpretation of the extended 7-D twistor space?

1. Twistorialization takes care of spin and electroweak spin and correlates them for quarks. The remaining standard model quantum numbers are Kähler and Kähler magnetic charges for M^4 and CP_2. Could the additional dimension allow a geometrization of these quantum numbers in terms of partial waves in the 3-D fiber? The example with the twistorialization suggests that the M^4 and CP_2 Kähler charges are identical apart from the sign.

2. The first thing to notice is that it is not possible to speak about the choice of quantization axis for $U(1)$ charge. It is however possible to generalize the momentum space picture also to the 7-D branes X^7 of M^8 with real normal space and select only discrete points of cognitive representation carrying quarks. The coordinate of 7-D generalized momentum in the 1-D fiber would correspond to some charge interpreted as a $U(1)$ momentum in the fiber of 7-D generalization of the twistor space.

3. One can start from the level of the 7-D surface with a real normal space. For both M^4 and CP_2, a plausible guess for the identification of 3-D fiber space is as 3-sphere S^3 having Hopf fibration $S^3 \to S^2$ with $U(1)$ as a fiber.

   At H side one would have a wave $exp(iQ\phi/2\pi)$ in $U(1)$ with charge Q and at M^8 side a point of X^7 representing Q as 7:th component of 7-D momentum.

   Note that for X^6 as a counterpart of twistor space the 5:th and 6:th components of the generalized momentum would represent spin quantization axis and sign of quark spin as a point of S^2. Even the length of angular momentum might allow this kind representation.

4. Since both M^4 and CP_2 allow induced Kähler field, a possible identification of Q would be as a Kähler magnetic charge. These charges are not conserved but in ZEO the non-conservation allows a description in terms of different values of the magnetic charge at opposite halfs of the light-cone of M^8 or CD.

   Instanton number representing a change of magnetic charge would not be a charge in strict sense and drops from consideration.

One expects that the action in the 7-D situation is analogous to Chern-Simons action associated with 8-D Kahler action, perhaps identifiable as a complexified 4-D Kähler action.

1. At M^4 side, the 7-D bundle would be $SU(4)/SU(3) \to SU(4)/SU(3) \times U(1)$. At CP_2 side the bundle would be $SU(3)/U(1) \to SU(3)/U(1) \times U(1)$.

2. For the induced bundle as 7-D surface in the $SU(4)/SU(3) \times SU(3)/U(1)$, the two $U(1)$:s are identified. This would correspond to an identification $\phi(M^4) = \phi(CP_2)$ but also a more general correspondence $\phi(M^4) = (n/m)\phi(CP_2)$ can be considered. m/n can be seen as a fractional $U(1)$ winding number or as a pair of winding numbers characterizing a closed curve on torus.

3. At M^8 level, one would have Kähler magnetic charges $Q_K(M^4)$, $Q_K(CP_2)$ represented associated with $U(1)$ waves at twistor space level and as points of X^7 at M^8 level involving quark. The same wave would represent both M^4 and CP_2 waves that would correlate the values of Kähler magnetic charges by $Q_{K,m}(M^4)/Q_{K,m}(CP_2) = m/n$ if both are non-vanishing. The value of the ratio m/n affects the dynamics of the 4-surfaces in M^8 and via twistor lift the space-time surfaces in H.

How could the Grassmannians of standard twistor approach emerge number theoretically?

One can identify the TGD counterparts for various Grassmann manifolds appearing in the standard twistor approach.

Consider first, the various Grassmannians involved with the standard twistor approach (https://cutt.ly/XE3vDKj) can be regarded as flag-manifolds of 4-complex dimensional space T.

1. Projective space is FP_{n-1} the Grasmannian $F_1(F^n)$ formed by the k-D planes of V^n where F corresponds to the field of real, complex or quaternionic numbers, are the simplest spaces of this kind. The F-dimension is $d_F = n-1$. In the complex case, this space can be identified as $U(n)/U(n-1) \times U(1) = CP_{n-1}$.

2. More general flag manifolds carry at each point a flag, which carries a flag which carries ... so that one has a hierarchy of flag dimensions $d_0 = 0 < d_1 < d_2 ... d_k = n$. Defining integers $n_i = d_i - d_{i-1}$, this space can in the complex case be expressed as $U(n)/U(n_1) \times .....U(n_k)$. The real dimension of this space is $d_R = n^2 - \sum_i n_i^2$.

3. For $n = 4$ and $F = C$, one has the following important Grassmannians.

   (a) The twistor space CP_3 is projective is of complex planes in $T = C^4$ and given by $CP_3 = U(4)/U(3) \times U(1)$ and has real dimension $d_R = 6$.

   (b) $M = F_2$ as the space of complex 2-flags corresponds to $U(4)/U(2) \times U(2)$ and has $d_R = 16 - 8 = 8$. This space is identified as a complexified Minkowski space with $D_C = 4$.

   (c) The space $F_{1,2}$ consisting of 2-D complex flags carrying 1-D complex flags has representation $U(4)/U(2) \times U(1) \times U(1)$ and has dimension $D_R = 10$.

   $F_{1,2}$ has natural projection ν to the twistor space CP_3 resulting from the symmetry breaking $U(3) \rightarrow U(2) \times U(1)$ when one assigns to 2-flag a 1-flag defining a preferred direction. $F_{1,2}$ also has a natural projection μ to the complexified and compactified Minkowski space $M = F_2$ resulting in the similar manner and is assignable to the symmetry breaking $U(2) \times U(2) \rightarrow U(1) \times U(1)$ caused by the selection of 1-flag.

   These projections give rise to two correspondences known as Penrose transform. The correspondence $\mu \circ \nu_{-1}$ assigns to a point of twistor space CP_3 a point of complexified Minkowski space. The correspondence $\nu \circ \mu_{-1}$ assigns to the point of complexified Minkowski space a point of twistor space CP_3. These maps are obviously not unique without further conditions.

This picture generalizes to TGD and actually generalizes so that also the real Minkowski space is obtained naturally. Also the complexified Minkowski space has a natural interpretation in terms of extensions of rationals forcing complex algebraic integers as momenta. Galois confinement would guarantee that physical states as bound states have real momenta.

1. The basic space is $Q_c = Q^2$ identifiable as a complexified Minkowski space. The idea is that number theoretically preferred flags correspond to fields R, C, Q with real dimensions 1,2,4. One can interpret Q_c as Q^2 and Q as C^2 corresponding to the decomposition of quaternion to 2 complex numbers. C in turn decomposes to $R \times R$.

2. The interpretation $C^2 = C^4$ gives the above described standard spaces. Note that the complexified and compactified Minkowski space is not same as $Q_c = Q^2$ and it seems that in TGD framework Q_c is more natural and the quark momenta in M_c^4 indeed are complex numbers as algebraic integers of the extension.

Number theoretic hierarchy $R \rightarrow C \rightarrow Q$ brings in some new elements.

1. It is natural to define also the quaternionic projective space $Q_c/Q = Q^2/Q$ `https://cutt.ly/LE3vMOG`, which corresponds to real Minkowski space. By non-commutativity this space has two variants corresponding to left and right division by quaternionic scales factor. A natural condition is that the physical states are invariant under automorphisms $q \rightarrow hqh^{-1}$ and depend only on the class of the group element. For the rotation group this space is characterized by the direction of the rotation axis and by the rotation angle around it and is therefore 2-D.

   This space is projective space QP_1, quaternionic analog of Riemann sphere CP_1 and also the quaternionic analog of twistor space CP_3 as projective space. Therefore the analog of real Minkowski space emerges naturally in this framework. More generally, quaternionic projective spaces Q^n have dimension $d = 4n$ and are representable as coset spaces of symplectic groups defining the analogs of unitary/orthogonal groups for quaternions as $Sp(n+1)/Sp(n) \times Sp(1)$ as one can guess on basis of complex and real cases. M^4_R would therefore correspond to $Sp(2)/Sp(1) \times SP(1)$.

QP_1 is homeomorphic to 4-sphere S^4 appearing in the construction of instanton solutions in E^4 effectively compactified to S^4 by the boundary conditions at infinity. For Minkowski signature it would be replaced by 4-D hyperboloid $H^4 = SO(1,4)/SO(3)$ known also as anti-de Sitter space AdS(4,1) (`https://cutt.ly/RRuXIBS`). An interesting question is whether the self-dual Kähler forms in E^4 could give rise to M^4 Kähler structure and could correspond to this kind of self-dual instantons and therefore what I have called H-J structures.

2. The complex flags can also contain real flags. For the counterparts of twistor spaces this means the replacement of $U(1)$ with a trivial group in the decompositions.

   The twistor space CP_3 would be replaced $U(4)/U(3)$ and has real dimension $d_R = 7$. It has a natural projection to CP_3. The space $F_{1,2}$ is replaced with representation $U(4)/U(2)$ and has dimension $D_R = 12$.

To sum up, the Grassmannians associated with M^4 as 6-D twistor space and its 7-D extension correspond to a complexification by a commutative imaginary unit i - that is "vertical direction". The Grassmannians associated with CP_2 correspond to "horizontal ", octonionic directions and to associative, commutative and well-ordered normal spaces of the space-time surface and its 6-D and 7-D extensions. Geometrization of the basic quantum states/numbers - not only momentum - representing them as points of these spaces is in question.

How could the quark content of the physical state determine the geometry of the space-time surface?

In the standard quantum field theory, fermionic currents serve as sources of the gauge fields. This correlation must have a counterpart in the TGD framework. Somehow the selection of the active points of the cognitive representation containing quarks must determine the 4-surface of M^8 determined by a polynomial P with rational coefficients. $M8-H$ duality would in turn determine the space-time surface.

This requirement gives a motivation for the earlier assumption that the roots of P defining 6-D surfaces fix P. Two kinds of surfaces appear.

1. The special $E = E_n$ roots of P having interpretation as energy have 3-D hyperplanes as M^4 intersections that I have misleadingly called "special moments in the life of self".

   The proposal [L82, L83] was that quarks are associated with the 2-D intersections of 4-D space-time surfaces with these planes. At the level of H, these 2-D intersections were assigned to partonic 2-surfaces serving as vertices of topological Feynman diagrams represented as space-time surfaces. Knowledge of the values of energy E_n defining 3-D complex planes at which the quarks of the quantum state are located in momentum space fixes the minimal polynomial P and therefore also space-time surface.

2. Besides energy hyper-planes there are also complex mass hyperboloids. The general 4-D solution of co-associativity conditions is 4-D (in real sense) intersection of two complex mass shells with mass squared $m^2_{c,odd}$ *resp.* $m^2_{c,even}$ with complex mass squared equal to a root of the odd *resp.* even part of the polynomial P defining the 4-surface [L82]. The real projection of the 4-D intersection is 2-D and might have interpretation as counterpart of a partonic 2-surface.

   This complex surface has complex dimension 4 and 4-D real projection in the sense that the number theoretic quadratic form is real. The 6-surface defined by the root reduces to a 3-D real mass shell if the imaginary part of m^2_c can vanishes: this is possible for real roots only. The 4-D intersection of these complex mass shells provide natural seats for the quark momenta as algebraic integers, which in general are complex. This data can fix the roots of the imaginary part of P as complex mass squared values.

3. Interestingly, also 6-D surfaces having these 4-surfaces as sub-manifolds emerge. A good guess is that these are just the surfaces with commutative normal space and serve as M^8 counterparts of twistor space.

How to understand leptons as bound states of 3 quarks?

A benchmark test for the view about the twistorial aspects of M^8 is the challenge of describing leptons as bound states of 3 quarks assignable to single wormhole contact, single throat, or even single point. The assumption that wormhole contacts correspond to blow-ups of line singularities in M^8 containing quarks favors the strongest option.

1. At the level of H , quarks with different colors (color partial waves in CP_2) could have exactly the same M^4 location inside a single wormhole throat but different CP_2 locations to realize statics. Color can be realized as H partial waves and this would require that the oscillator operators act at the level of M^8 allowing to put several oscillators at a single M^4 point at the level of H.

2. At the level of M^8 the Fermi statistics would state that only a single quark corresponds to a given point. If one works at the level of 4-surface so that only momentum is taken into account, this is not possible. Could the 3 quarks be at different points in the 7-D extension of the twistor space bringing in quark spin and Kähler magnetic charge?

   The total spin of lepton is 1/2 so that two spins are opposite. Kähler magnetic charges of quarks are proposed to be proportional to color hypercharge (2,-1,-1) for quarks to realize Fermi statistics topologically. The points (p,1/2,-1),(p,1/2,-1) and (p,-1/2,2) and the states obtained by permuting Kähler charges would allow arealization of lepton as a 3 quark state with identical momenta.

9.4.4 Hierarchies of extensions for rationals and of inclusions of hyperfinite factors

TGD suggests 3 different views of finite measurement resolution.

1. At the space-time level, finite measurement resolution is realized in terms of cognitive representations at the level of M^8 actualized in terms of fermionic momenta with momentum components identifiable as algebraic integers. Galois group has natural action on the momentum components.

2. The inclusion $N \subset M$ of group algebras of Galois groups is proposed to realize finite measurement resolution for which the number theoretic counterpart is Galois singlet property of N with respect to the Galois group of M relative to N identifiable as the coset group of Galois groups of M and N. If the origin serves as a root of all polynomials considered, the composite $P \circ Q$ inherits the roots of Q.

   The idea generalizes to infinite-D Galois groups [L96, L93]. The HFF in question would be infinite-D group algebra of infinite Galois group for a polynomial R obtained as a composite $R = P_{infty} \circ Q$ of an infinite iterate P_{infty} of polynomial P and of some polynomial Q of finite degree (inverse limit construction). The roots of R at the limit correspond to the attractor basin associated with P_∞, which is bounded by the Julia set so that a connection with fractals emerges.

3. The inclusions $N \subset M$ of hyperfinite factors of type II_1 (HFFs) [K105, K36] is a natural candidate for the representation of finite measurement resolution. N would represent the degrees of freedom below measurement resolution mathematically very similar to gauge degrees of freedom except that gauge algebra would be replaced with the super-symplectic algebra and analogs of Kac Moody algebra with non-negative conformal weights and gauge conditions would apply to sub-algebra with conformal weights larger than the weight h_{max} defining the measurement resolution.

For HFFs, the index $[M : N]$ of the inclusion defines the quantum dimension $d(N \subset M) \leq 1$ as a quantum trace of the projector $P(M \rightarrow N)$ (the identify operator of M has quantum trace equal to one). $d(N \subset M)$ is defined in terms of quantum phase q and serves as a dimension for the analog of factor space M/N representing the system with N regarded as degrees of freedom below the measurement resolution and integrated out in "quantum algebra" M/N. Quantum groups and quantum spaces are closely related notions [K105, K36].

Galois confinement would suggest that $N \subset M$ corresponds to the algebra creating Galois singlets with respect to the Galois group of N relative to M whereas M includes also operators which are not this kind of singlets. In the above example $R = P \circ Q$, the Galois group of P would be represented trivially and the Galois group of Q or its subgroup would act non-trivially. In the case of hadrons, color degrees of freedom perhaps assignable to the Galois group Z^3 in the case of quarks would correspond to the degrees of freedom below the measurement resolution.

The universality of the quantum dimension and its expressibility in terms of quantum phase suggests that the integer m in $q = rxp(i2\pi/m)$ is closely related to the dimension for the extension of rationals $n = h_{eff}/h_0$ and depends therefore only very weakly on the details of the extension. The simplest guess is $m = n$. This conforms with the concrete interpretation of charge fractionation as being due to the many-valuedness of the graphs of space-time surfaces as maps from $M^4 \to CP_2$ or vice versa.

9.4.5 Galois confinement

The notion of Galois confinement emerged in TGD inspired biology [L120, L87, L93, L98]. Galois group for the extension of rationals determined by the polynomial defining the space-time surface $X^4 \subset M^8$ acts as a number theoretical symmetry group and therefore also as a physical symmetry group.

1. The idea that physical states are Galois singlets transforming trivially under the Galois group emerged first in quantum biology. TGD suggests that ordinary genetic code is accompanied by dark realizations at the level of magnetic body (MB) realized in terms of dark proton triplets at flux tubes parallel to DNA strands and as dark photon triplets ideal for communication and control [L87, L98, L97]. Galois confinement is analogous to color confinement and would guarantee that dark codons and even genes, and gene pairs of the DNA double strand behave as quantum coherent units.

2. The idea generalizes also to nuclear physics and suggests an interpretation for the findings claimed by Eric Reiter [L105] in terms of dark N-gamma rays analogous to BECs and forming Galois singlets. They would be emitted by N-nuclei - also Galois singlets - quantum coherently [L105]. Note that the findings of Reiter are not taken seriously because he makes certain unrealistic claims concerning quantum theory.

Galois confinement as a number theoretically universal manner to form bound states?

It seems that Galois confinement might define a notion much more general than thought originally. To understand what is involved, it is best to proceed by making questions.

1. Why not also hadrons could be Galois singlets so that the somewhat mysterious color confinement would reduce to Galois confinement? This would require the reduction of the color group to its discrete subgroup acting as Galois group in cognitive representations. Could also nuclei be regarded as Galois confined states? I have indeed proposed that the protons of dark proton triplets are connected by color bonds [L72, L85, L34].

2. Could all bound states be Galois singlets? The formation of bound states is a poorly understood phenomenon in QFTs. Could number theoretical physics provide a universal mechanism for the formation of bound states. The elegance of this notion is that it makes the notion of bound state number theoretically universal, making sense also in the p-adic sectors of the adele.

3. Which symmetry groups could/should reduce to their discrete counterparts? TGD differs from standard in that Poincare symmetries and color symmetries are isometries of H and their action inside the space-time surface is not well-defined. At the level of M^8 octonionic automorphism group G_2 containing as its subgroup $SU(3)$ and quaternionic automorphism group $SO(3)$ acts in this way. Also super-symplectic transformations of $\delta M^4_\pm \times CP_2$ act at the level of H. In contrast to this, weak gauge transformations acting as holonomies act in the tangent space of H.

One can argue that the symmetries of H and even of WCW should/could have a reduction to a discrete subgroup acting at the level of X^4. The natural guess is that the group in question is Galois group acting on cognitive representation consisting of points (momenta) of M_c^8 with coordinates, which are algebraic integers for the extension.

Momenta as points of M_c^8 would provide the fundamental representation of the Galois group. Galois singlet property would state that the sum of (in general complex) momenta is a rational integer invariant under Galois group. If it is a more general rational number, one would have fractionation of momentum and more generally charge fractionation. Hadrons, nuclei, atoms, molecules, Cooper pairs, etc.. would consist of particles with momenta, whose components are algebraic, possibly complex, integers.

Also other quantum numbers, in particular color, would correspond to representations of the Galois group. In the case of angular moment Galois confinement would allow algebraic half-integer valued angular momenta summing up to the usual half-odd integer valued spin.

4. Why Galois confinement would be needed? For particles in a box of size L the momenta are integer valued as multiples of the basic unit $p_0 = \hbar n \times 2\pi/L$. Group transformations for the Cartan group are typically represented as exponential factors which must be roots of unity for discrete groups. For rational valued momenta this fixes the allowed values of group parameters. In the case of plane waves, momentum quantization is implied by periodic boundary conditions.

   For algebraic integers the conditions satisfied by rational momenta in general fail. Galois confinement for the momenta would however guarantee that they are integer valued and boundary conditions can be satisfied for the bound states.

Explicit conditions for Galois confinement

It is interesting to look more explicitly at the conditions for the Galois confinement.

Single quark states have momenta, which are algebraic integers generated by so called integral basis (`https://cutt.ly/SRuZySX`) spanning algebraic integers as a lattice and analogous to unit vectors of momentum lattice but for single component of momentum as a vector in extension. There is also a theorem stating that one can form the basis of extension as powers of a single root. It is also known that irreducible monic polynomials have algebraic integers as roots.

1. In its minimal form Galois confinement states that only momenta, which are rational integers are allowed by Galois confinement. Note that for irreducible polynomials with rational coefficients one does not obtain any rational roots. Monic polynomials with integer coefficients can allow integer roots. If one assumes that single particle states can have arbitrary algebraic integer as momentum, one obtain also rational integers for momentum values. These states are not at mass - or energy shell associated with the single particle momenta.

2. A stronger condition would be that also the inner products of the momenta involved are real so that one has $Re(p_i) \cdot Im(p_j) = 0$. For $i = j$ this gives a condition is possible only for the real roots for the real polynomials defining the space-time surface.

To see that real roots are necessary, some facts about the realization of the co-associativity condition [L82] are necessary.

1. The expectation is that that the vanishing condition for the real part (in quaternionic sense) of the octonionic polynomial gives a co-associative surface. By the Lorentz symmetry one actually obtains as a solution a 6-D complex mass shell $m_c^2 \equiv m_{Re}^2 - m_{Im}^2 + 2iRe(p) \cdot Im(p) = r_1$, where the real and imaginary masses are defined are $m_{Re}^2 = Re(p)^2$ and $m_{Im}^2 = Im(p)^2$ and r_1 is some root for the odd part of the polynomial P assumed to determining the 4-surface.

2. This surface can be co-associative but would be also co-commutive. Maximally co-associative surface requires quaternionic normal space. The first proposal is that the space-time surface is the intersection of the surface defined by the polynomial and its conjugate with respect to i. This gives 4-D surface as the intersection of the two 6-D surfaces.

Second proposal is that the 6-surface having a structure of S^2 bundle defines as its base space quaternionic 4-surface. This space would correspond to a gauge choices selecting point of S^2 at very point of M^4. To a given polynomial one could assign entire family of 4-surfaces mapped to different space-time surfaces in H. A possible interpretation of gauge group would be as quaternionic automorphisms acting on the 2-sphere.

These proposals are equivalent if the base base is the intersection of the 6-D bundle spaces. One could say that the fibers are conjugates of each other. This might be relevant for ZEO.

Concerning Galois confinement, the basic result is that for complex roots r_1 the conditions $Re(p_i) \cdot Im(p_i) = 0$ cannot be satisfied unless one requires that r_1 is real. Therefore the stronger option makes sense for real roots only.

1. Galois confinement allows the momenta p_i forming the bound state to be in an extension of rationals defined by the polynomial defining the space-time surface. Galois confinement condition states that the total momentum is rational integer when a suitable unit defined by the size of CD is used (periodic boundary conditions).

2. Another natural condition is the vanishing of the inner products between the real part $Re(p)$ and imaginary part $Im(p)$ of p. This guarantees that the number theoretical norm squared for the momentum is real. For time-like p, this means that $Im(p)$ belongs to the 3-D orthogonal complement E^3 of $Re(p)$. For light-like p, $Im(p)$ belongs to 2-D orthogonal complement E^2.

3. Suppose one has several number theoretic momenta p_i such that $\sum p_i = p$ is rational integer and $p_i \propto p$ holds true. Also in this case, the number theoretic inner products must be real. The orthogonality conditions read as

$$Re(p_i) \cdot Im(p_j) = 0 \ . \tag{9.4.2}$$

   For a given pair (i, j), one has several conditions corresponding to algebraically independent imaginary momentum components and it is quite possible that very few solutions exist besides $Im(p_i) = 0$. If $Re(p_i)$ is not a rational integer, the number of conditions still increases.

4. The proposal for Galois confinement is that the real parts of p_i are parallel or even identical: $Re(p_i) \propto Re(\sum p_i) = p$, which is a rational integer. In this case the conditions reduce to $Re(p) \cdot Im(p_i) = 0$ and their number is much smaller.

5. For a given momentum component, the basis $p_{i,k}$ has the dimension n of extension. The basis contains m complex elements e_k and their conjugates $\overline{e}_k$ plus $n - 2m - 1$ real but algebraically trivial elements r_k besides the real unit 1. The sums $E_k = e_k + \overline{e}_k$ are algebraic integers and give m real basis elements. Note that $F_k = e_k - \overline{e}_k$ are purely imaginary algebraic integers.

   r_k and E_i give $n-m-1$ algebraically non-trivial real momenta. The momentum components $p_{i,k}$ formed as linear combinations of r_k, E_i, and 1 are real. This gives $n-m$-dimensional real subspace and momenta formed in this way satisfy the reality conditions for the inner products.

6. One can also construct complex momenta such that $Im(p_i)$ is a linear combination $Im(p_i) = \sum n_{i,k} F_k$. If $Re(p_i)$ are parallel and rational integers and $p_i \propto p$ holds true, the reality conditions reduce to

$$p \cdot Im(p_i) = \sum_k p^i n_{i,k} F_k = 0 \ . \tag{9.4.3}$$

   One can construct a maximal set of complex momenta P_K characterized by matrices n_{ik}^K satisfying these conditions. Also linear combinations of P_K satisfy the reality conditions and one obtains a lattice of momenta.

This looks like nice construction but it seems that mere Galois confinement is more realistic.

9.4.6 M^8-H duality at the level of WCW

WCW emerges in the geometric view of quantum TGD. M^8-H duality should lso work for WCW. What is the number theoretic counterpart of WCW? What is the geometric counterpart of the discretization characteristic to the number theoretic approach?

In the number theoretic vision in which WCW is discretized by replacing space-time surfaces with their number theoretical discretizations determined by the points of $X^4 \subset M^8$ having the octonionic coordinates of M^8 in an extension of rationals and therefore making sense in all p-adic number fields? How could an effective discretization of the real WCW at the geometric H level, making computations easy in contrast to all expectations, take place?

1. The key observation is that any functional or path integral with integrand defined as exponent of action, can be *formally* calculated as an analog of Gaussian integral over the extrema of the action exponential $exp(S)$. The configuration space of fields would be effectively discretized. Unfortunately, this holds true only for the so called integrable quantum field theories and there are very few of them and they have huge symmetries. But could this happen for WCW integration thanks to the maximal symmetries of the WCW metric?

2. For the Kähler function K, its maxima (or maybe extrema) would define a natural effective discretization of the sector of WCW corresponding to a given polynomial P defining an extension of rationals.

   The discretization of the sector defined by P should be equivalent with the number theoretical discretization induced by the number theoretical discretization of space-time surfaces. Various p-adic physics and corresponding discretizations should emerge naturally from the real physics in WCW.

3. The physical interpretation is clear. The TGD Universe is analogous to the spin glass phase [?] The discretized WCW corresponds to the energy landscape of spin glass having an ultrametric topology. Ultrametric topology of WCW means that discretized WCW decomposes to p-adic sectors labelled by polynomials P. The ramified primes of P label various p-adic topologies associated with P.

9.4.7 Some questions and ideas related to M^8-H duality

In the following some questions and ideas, which do not quite fit under the titles of the previous sections, are considered.

A connection with Langlands program

Langlands correspondence [A82, K47, A40, A39], which I have tried to understand several times [K47] [L1, L8, L26] relates in an interesting manner to M^8-H duality and Galois confinement.

1. Global Langlands correspondence (GLC) states that there is connection between representations of continuous groups and Galois groups of extensions of rationals.

2. Local LC states (LLC) states this in the case of p-adics.

There is a nice interpretation for the two LCs in terms of sensory experience and cognition in TGD inspired theory of consciousness.

1. In adelic physics real numbers and p-adic number fields define the adele. Sensory experience corresponds to reals and cognition to p-adics. Cognitive representations are in their discrete intersection and for extensions of rationals belonging to the intersection.

   (a) Sensory world, "real" world corresponds to representation of continuous groups/Galois groups of rationals. GLC.

(b) "p-Adic" worlds correspond to cognition and representations of p-adic variants of continuous groups and Galois groups over p-adics. Local LLC.

(c) One could perhaps talk also about Adelic LC: ALC in the TGD framework. Adelic representations would combine real and p-adic representations for all primes and give as complete information about reality as possible.

TGD provides a geometrization for the identification of Galois groups as discrete subgroups of Lie groups, not only of the isometry (automorphism) groups of H (M^8) but perhaps also as discrete sub-groups of more general Lie groups to which the action of super-symplectic representations could reduce. A naive guess is that these groups correspond to the ADE groups appearing in the McKay correspondence [L36, L76, L77].

The representation of real continuous groups assignable to the real numbers as a piece of adele [L43, L42] would be related to the representations of Galois groups GLC. Also p-adic representations of groups are needed to describe cognition and these p-adic group representations and representations of p-adic Galois groups would be related by LLC.

Could the notion of emergence of space-time have some analog in the TGD Universe?

The idea about the emergence of space-time from entanglement is as such not relevant for TGD. One can however ask whath the emergence of *observed* space-time could mean in TGD. Space-time surface as a continuum exists in TGD but they are not directly observable due to a finite measurement resolution. One can ask what a body with an outer boundary means physically. The space-time regions defined by solid bodies have boundaries. What makes the boundaries of the bodies "hard"?

1. In momentum space Fermi statistics does not allow fermions to get through the boundary of Fermi ball. This is a good guideline.

2. Second feature of a spatial object such as an atom is that it is a bound state quantum mechanically. If it has parts they stay together. In QFT theory the notion of a bound state is however poorly understood.

3. Quantum coherence is a further property considered in the article. Spatial objects correspond to quantum coherent structures. Quantum coherence reduces to entanglement. Quantum coherence length and time determine the size of a quantum object. Somehow one must have stable entanglement in long scales.

Let us see what these guidelines could give in the framework of $M^8 - H$ duality which generalizes the wave particle duality of wave mechanics.

1. In adelic physics space-times can be seen as either surfaces in M^8 or $H = M^4 \times CP_2$. $X^4 \subset M^8$ is analogous to momentum space cognitive representations consist of points of $X^4 \subset M^8$, whose points are algebraic integers in the extension of rationals defined by the polynomial defining the space-time surface and are algebraic integers as roots of monic polynomials of form $x^n + ...$ This defines a unique discretization of the space-time surface. The discretization guarantees number theoretical universality: the cognitive representation makes sense also p-adically and space-time has also p-adic variants.

   Cognitive representations give rise to "cognitive emergence" of the space-time in cognitive sense and since cognitive representations are intersection of reality and p-adicities they must closely related to the "sensory emergence"".

2. $X^4 \subset M^8$ is mapped to H by $M^8 - H$ duality determined by the condition that it momentum is mapped to a geodesic with a direction of momentum and starting from either tip of CD: the image point is its intersection with the opposite light-like boundary of CD and selects a point of space-time surface. The size of CD is $T = h_{eff}/m$ for quark with mass m to satisfy Uncertainty Principle. The map generalizes to bound states of quarks (whatever they are).

Consider the problem of "sensory emergence" in this framework.

1. What makes a point of a cognitive representation "hard"? Quarks are associated with points (not necessarily all) of a cognitive representation: one can say that the point is activated when there is a quark at it. Fermi ball corresponds to a discrete set of activated points at the level of momentum space. These points define activated points also in $X^4 \subset H$ by $M^8 - H$ duality. One could perhaps say that these activated points in M^8 and their H-image containing fermions define the spatial objects as something "hard" and having a boundary. Another fermion knows that there is a space-time point there because it cannot get to this point. The presence of a fermion (quark) would make a space-time point "hard".

2. What about the role of entanglement? The size and duration of the space-time surface (inside a causal diamond CD) defines quantum coherence length and time. Fermionic statistics makes fundamental fermions - to be distinguished from elementary fermions - maximally entangled. One cannot reduce fermionic entanglement in SFR and quantum measurements would be impossible. The entanglement in the WCW degrees of freedom comes to the rescue. This entanglement can be reduced in SFRs since the particles as surfaces are identical under very special - naturally number theoretical - conditions.

   Negentropy Maximization Principle and hierarchy of $h_{eff} = n \times h_0$ phases favor the generation of stable entanglement in the TGD Universe. Also, if the coefficients of the entanglement matrix belong to extension of rationals, entanglement probabilities in general belong to its extension and the density matrix is not diagonalizable without going to a larger extension. This might require "big" SFR increasing the extension: only after this state function reduction to an eigenstate could occur. This leads to a concrete proposal for how the information about the diagonal form of the density matrix expressed by its characteristic polynomial is coded into the geometry of the space-time surface [L93].

3. Bound state formation is third essential element. Momenta are points of the space-time surface $X^4 \subset M^8$ with components which are algebraic integers. Physical momenta are however ordinary integers for a particle in a finite volume defined by causal diamond (CD). This means that one can allow only composites of quarks with rational integer valued momenta which correspond to Galois singlets.

   Galois confinement would be the universal mechanism behind formation of all bound states and also give rise to stable entanglement. One would obtain a hierarchy of bound states corresponding to a hierarchy of polynomials and corresponding Galois groups and extensions of rationals. By $M^8 - H$ duality, bound states of quarks and higher structures formed from them in M^8 would give rise to spatial objects.

9.5 Zero energy ontology (ZEO)

ZEO [K109] forms the cornerstone of the TGD inspired quantum theory extending to a theory of consciousness. ZEO has so far reaching consequences that it would have deserved a separate section. Since it involves in an essential manner the notion of CD, it is natural to include it to the section discussing $M^8 - H$ duality.

9.5.1 The basic view about ZEO and causal diamonds

The following list those ideas and concepts behind ZEO that seem to be rather stable.

1. GCI for the geometry of WCW implies holography, Bohr orbitology and ZEO [L73] [K109].

2. X^3 is more or less equivalent with Bohr orbit/preferred extremal $X^4(X^3)$. Finite failure of determinism is however possible [L107]. Zero energy states are superpositions of $X^4(X^3)$. Quantum jump is consistent with causality of field equations.

3. Causal diamond (CD) defined as intersection of future and past directed light cones ($\times CP_2$) plays the role of quantization volume, and is not arbitrarily chosen. CD determines momentum scale and discretization unit for momentum (see **Fig. 9.12 Fig. 9.13**).

4. The opposite light-like boundaries of CD correspond for fermions dual vacuums (bra and ket) annihilated by fermion annihilation *resp.* creation operators. These vacuums are also time reversals of each other.

   The first guess is that zero energy states in fermionic degrees of freedom correspond to pairs of this kind of states located at the opposite boundaries of CD. This seems to be the correct view in H. At the M^8 level the natural identification is in terms of states localized at points inside light-cones with opposite time directions. The slicing would be by mass shells (hyperboloids) at the level of M^8 and by CDs with same center point at the level of H.

5. Zeno effect can be understood if the states at either cone of CD do not change in "small" state function reductions (SSFRs). SSFRs are analogs of weak measurements. One could call this half-cone call as a passive half-cone. I have earlier used a somewhat misleading term passive boundary.

   The time evolutions between SSFRs induce a delocalization in the moduli space of CDs. Passive boundary/half-cone of CD does not change. The active boundary/half-cone of CD changes in SSFRs and also the states at it change. Sequences of SSFRs replace the CD with a quantum superposition of CDs in the moduli space of CDs. SSFR localizes CD in the moduli space and corresponds to time measurement since the distance between CD tips corresponds to a natural time coordinate - geometric time. The size of the CD is bound to increase in a statistical sense: this corresponds to the arrow of geometric time.

6. There is no reason to assume that the same boundary of CD is always the active boundary. In "big" SFRs (BSFRs) their roles would indeed change so that the arrow of time would change. The outcome of BSFR is a superposition of space-time surfaces leading to the 3-surface in the final state. BSFR looks like deterministic time evolution leading to the final state [L62] as observed by Minev *et al* [L62].

7. h_{eff} hierarchy [K27, K28, K29, K30] implied by the number theoretic vision [L82, L83] makes possible quantum coherence in arbitrarily long length scales at the magnetic bodies (MBs) carrying $h_{eff} > h$ phases of ordinary matter. ZEO forces the quantum world to look classical for an observer with an opposite arrow of time. Therefore the question about the scale in which the quantum world transforms to classical, becomes obsolete.

8. Change of the arrow of time changes also the thermodynamic arrow of time. A lot of evidence for this in biology. Provides also a mechanism of self-organization [L69]: dissipation with reversed arrow of time looks like self-organization [L120].

9.5.2 Open questions related to ZEO

There are many unclear details related to the time evolution in the sequence of SSRs. Before discussing these unclear details let us make the following assumptions.

1. The size of CDs increases at least in a statistical sense in the sequence of CD and the second boundary remains stationary apart from scaling (note that one can also consider the possibility that the entire CD is scaled and temporal shift occurs in both directions).

2. Mental mentals (say after images) are in kind of Karma's cycle: they are born and die roughly periodically.

3. I do not experience directly mental images with the opposite arrow of time.

4. I can have memories only about states of consciousness with the same arrow of time that I have. This explains why I do not have memories about periods of sleep if sleep is interpreted as a time reversed state of some subself of me responsible for self-ness.

One can use three empirical inputs in an attempt to fix the model.

1. After images appear and disappear roughly periodically. Also I fall asleep and wake up with a standard arrow of time roughly periodically.

(a) The first interpretation is that as a sequence of wake up-sleep periods I am a time crystal-like structure consisting of nearly copies of the mental image, such that each mental image - including me as mental images of higher level self - continues Karma's cycle in my geometric past. How "me" is transferred to a new almost copy of my biological body? Does my MB just redirect its attention?

(b) The second interpretation is that me and my mental images somehow drift towards my geometric future, while performing the Karma's cycle so that my mental images follow me in my time travel. This would require that the sub-CDs of mental images drift towards the geometric future.

Also sleep could be a "small" death at some layer of the personal hierarchy of MBs. I do not however wake-up in BSFR at the moment of geometric time defined by the moment of falling asleep but later. So it seems that my CD must drift to the geometric future with the same speed that those of other living beings in the biosphere.

2. There is however an objection. In cosmology the observation of stars older than the Universe would have a nice solution if the stars evolve forth and back in time in our distant geometric past rather than drifting towards the future so that they could age by continuing their Karma's cycle with a constant center of mass value of time. Can these three observations be consistent?

Could the scaling dynamics CD induce the temporal shifting of sub-CDs as 4-D perceptive fields?

Suppose that the sub-CDs within a bigger CD "follow the flow". How the dynamics of the bigger CD could induce this flow?

1. The scalings of bigger CD in unitary evolutions between SSFRs induce the scaling of sub-CS. This would not be shifting but scaling and the distance between given CD and larger CDs would gradually scale up.

   This would remove the objection. The astrophysical objects in distant geometric past would move towards the geometric future but with much smaller velocity as the objects with cosmic scale so that the temporal distance to future observers would increase. These objects would be aging in their personal Karma's cycle, and the paradox would disappear.

2. The flow would be defined by the scalings of a larger CD containing our CDs and those of others at my level. Each CD would define a shared time for its sub-CDs. If the CDs form a hierarchy structure with a common center, this is indeed true of the time evolutions as scalings of CDs. There would be scalings induced by scalings at higher levels and "personal" scalings.

3. It however seems that the common center is too strong an assumption and shifted positions for the sub-CDs and associated hierarchy inside a given CD are indeed possible for the proposed realization of M^8-H duality and actually required by Uncertainty Principle.

A further open question is what happens to the size of CD in the BSFR. Does it remain the same so that the size of the CD would increase indefinitely? Or is the size reduced in the sense that there would be scaling, reducing the size of the CD in which the passive boundary of the CD would be shifted towards the active one. After every BSFR, the self would experience a "childhood".

Are we sure about what really occurs in BSFR?

It has been assumed hitherto that a time reversal occurs in BSFR. The assumption that SSFRs correspond to a sequence of time evolutions identified as scalings, forces to challenge this assumption. Could BSFR involve a time reflection T natural for time translations or inversion $I: T \rightarrow 1/T$ natural for the scalings or their combination TI?

I would change the scalings increasing the size of CD to scalings reducing it. Could any of these options: time reversal T, inversion I, or their combination TI take place in BSFRs whereas arrow would remain as such in SSFRs? T (TI) would mean that the active boundary of CD is frozen and CD starts to increase/decrease in size.

There is considerable evidence for T in BSFRs identified as counterparts of ordinary SFRs but could it be accompanied by I?

1. Mere I in BSFR would mean that CD starts to decrease but the arrow of time is not changed and passive boundary remains passive boundary. What comes to mind is blackhole collapse.

   I have asked whether the decrease in size could take place in BSFR and make it possible for the self to get rid of negative subjective memories from the last moments of life, start from scratch and live a "childhood". Could this somewhat ad hoc looking reduction of size actually take place by a sequence of SSFRs? This brings into mind the big bang and big crunch. Could this period be followed by a BSFR involving inversion giving rise to increase of the size of CD as in the picture considered hitherto?

2. If BSFR involves TI, the CD would shift towards a fixed time direction like a worm, and one would have a fixed arrow of time from the point of view of the outsider although the arrow of time would change for sub-CD. This modified option does not seem to be in conflict with the recent picture, in particular with the findings made in the experiments of Minev *et al* [L62] [L62].

   This kind of shifting must be assumed in the TGD inspired theory of consciousness. For instance, after images as a sequence of time reversed lives of sub-self, do not remain in the geometric past but follow the self in travel through time and appear periodically (when their arrow of time is the same as of self). The same applies to sleep: it could be a period with a reversed arrow of time but the self would shift towards the geometric future during this period: this could be interpreted as a shift of attention towards the geometric future. Also this option makes it possible for the self to have a "childhood"".

3. However, the idea about a single arrow of time does not look attractive. Perhaps the following observation is of relevance. If the arrow of time for sub-CD correlates with that of sub-CD, the change of the arrow of time for CD, would induce its change for sub-CDs and now the sub-CDs would increase in the opposite direction of time rather than decrease.

To sum up, TI or T can be considered as competing options for what happens in BSFR. T should however be able to explain why sub-selves (sub- CDs) drift to the direction of the future. If the time evolutions between SSFRs correspond to scalings rather than time translations, and if the scalings occur also for sub-CDs this can be understood. The dynamics of spin glasses strongly suggests that SSFRs correspond to scalings [L103].

9.5.3 What happens in quantum measurement?

According to the proposed TGD view about particle identity, the systems for which mutual entanglement can be reduced in SFR must be non-identical in the category theoretical sense.

When SFR corresponds to quantum measurement, it involves the asymmetric observer-system $O-S$ relationship. One cannot exclude SFRs without this asymmetry. Some kind of hierarchy is suggestive.

The extensions of rationals realize this kind of $O-S$ hierarchy naturally. The notion of finite measurement resolution strongly suggests discretization, which favors number theoretical realization. The hierarchies of effective Planck constants and p-adic length scale hierarchies reflect this hierarchy. What about the topological situation: can one order topologies to a hierarchy by their complexity and could this correspond to $O-S$ relationship?

The intuitive picture about many-sheeted space-time is as a hierarchical structure consisting of sheets condensed at larger sheets by wormhole contacts, whose throats carry fermion number. Intuitively, the larger sheet serves as an observer. p-Adic primes assignable to the space-time sheet could arrange them hierarchically and one could have entanglement between wavefunctions for the Minkowskian regions of the space-time sheets and the surface with a larger value for p would be in the role of O

Number theoretic view about measurement interaction

Quantum measurement involves also a measurement interaction. There must be an interaction between two different levels O and S of the hierarchy.

One can look at the measurement interaction from a number theoretic point of view.

1. For cognitive measurements the step forming the composite $O \circ S$ of polynomials would represent the measurement interaction. Before measurement interaction systems would be represented by O and S and measurement interaction would form $O \circ S$ and after the measurement the situation would be as proposed.

   Could one think that in BSFR the pair of uncorrelated surface defined by $O \times S$ with degree $n_O + n_S$ (analog for the additivity of classical degrees of freedom) is replaced with $O \circ S$ with degree $n_O \times n_S$ (analog for multiplicativity of degrees of freedom in tensor product) in BSFR? This would mean that the formation of $O \circ S$ is like a formation of an intermediate state in particle reaction or in chemical reaction.

   Could the subsequent SSFR cascade define a cascade of cognitive measurements [L90]. I have proposed that this occurs in all particle reactions. For instance, nuclear reactions involving tunneling would involve formation of dark nuclei with $h_{eff} > h$ in BSFR and a sequence of SSFRs in opposite time direction performing cognitive quantum measurement cascade [L72] and also the TGD based model for "cold fusion" relies on this picture [L34, L85]. After the SSFR cascade, a second BSFR would occur and bring back the original arrow of time and lead to the final state of the nuclear reaction.

   From the point of view of cognition, BSFR would correspond to the heureka moment and the sequences of SSFRs to the cognitive analysis decomposing the space-time surface defined by $O \circ S$ to pieces.

2. One can also consider small perturbations of the polynomials $O \circ S$ as a measurement interaction. For instance, quantum superpositions of space-time surfaces determined by polynomials depending on rational valued parameters are possible. The Galois groups for two polynomials with parameters which are near to each other are the same but for some critical values of the parameters the polynomials separate into products. This would reduce the Galois group effectively to a product of Galois groups. Quantum measurement could be seen as a localization in the parameter space [L93].

Topological point of view about measurement interaction

The measurement interaction can be also considered from the topological point of view.

1. Wormhole contacts are Euclidean regions of $X^4 \subset H$ couples two parallel space-time regions with Minkowskian signature and could give rise to measurement interaction. Wormhole contact carries a monopole flux and there must be a second monopole contact to make flux loop possible. This structure has an interpretation as an elementary particle, for instance a boson. The measurement interaction could correspond to the formation of this structure and splitting by reconnection to flux loops associated with the space-time sheets after the interaction has ceased.

   Remark: Wormhole contacts for $X^4 \subset H$ correspond in $M^8 - H$ duality images of singularities of $X^4 \subset M^8$. The quaternionic normal space at a given point is not unique but has all possible directions, which correspond to all points of CP_2. This is like the monopole singularity of an electric or magnetic field. At the level of CP_2 wormhole contact is the "blow-up" of this singularity.

2. Flux tube pairs connecting two systems serve also as a good candidate for the measurement interaction. U-shaped monopole flux tubes are like tentacles and their reconnection creates a flux tube pair connecting two systems. SFR would correspond geometrically to the splitting of the flux tube pair by inverse re-connection.

Geometric view about SSFR

The considerations of [L92] strongly suggest the following picture about SSFRs.

In the measurement interaction a quantum superposition of functional composites of polynomials P_i defining the space-time surfaces of external states as Galois singlets is formed. A priori all orders for the composites in the superposition are allowed but if one requires that the same SSFR cascade can occur for all of them simultaneously, only single ordering and its cyclic permutations can be allowed.

The SSFR cascade can of course begin with a reduction selection single permutation and its cyclic permutations: localization in S_n/Z_n would take place.

Incoming states at passive boundary of CD correspond to prepared states and outgoing states at active boundary to state function reduced states. The external states could correspond to products of polynomials as number theoretical correlates for the absence of correlations in unentangled states.

Number theoretic existence for the scattering amplitudes [?] require that the p-adic primes characterizing the external states correspond to maximal ramified primes of the corresponding polynomials and therefore also to unique p-adic length scales L_p. In the interaction regions this ramified prime is the largest p-adic (that is ramified) prime for particles participating in the reaction. This correlation between polynomial and p-adic length scale allows a rather concrete geometric vision about what happens in the cascade.

SSFR cascade begins with a reduction of the state to a superposition of single composite with its cyclic variants for positive and negative energy parts separately: this kind of cyclic superpositions appear also in the twistor Grassmann picture [L92] and in string models. In the recent situation this makes possible a well-defined state preparation and SFR cascades at the two sides of CD. In ZEO, the cascade could take place for positive energy states only during SSFR.

A number theoretic SFR cascade would take place and decompose the Galois state group of the composite having decomposition to normal sub-groups to a product of states for the relative Galois groups for the composite.

A given step of the cascade would be a measurement of a density matrix ρ producing information coded by its reduction probabilities as its eigenvalues in turn coded by the characteristic polynomial P_M of the density matrix.

The simplest guess is that the final state polynomial is simply the product $\prod P_{i_-}$ of the polynomials P_{i_-} for the passive boundary of CD and product $\prod P_{i_+}$ for the active boundary.

Question of quantum information theorist

Quantum information theorists could however ask what happens to the information yielded by a given step of the measurement cascade.

1. Could the information about the measured ρ coded by P_M as its algebraic roots be stored to the final state coded by the final state polynomials $P_{i,+}$?

   Could the outcome at the active boundary of CD for which the SSFR cascade is actually not the 4-surface determined by the polynomials $P_{i,+}$ but $P_{M_{i_+}} \circ P_i$, or more generally a quantum superposition of $P_{M_{i_+}} \circ P_i$, and $P_i \circ P_{M_{i_+}}$.

   The "unitary time evolution" preceding the next SSFR would correspond to a functional composite of these polynomials so that the space-time surface would evolve during the SSFR sequence. The basic process would be a formation of functional composite followed by SSFR cascade storing the information about the measured density matrices to the space-time surface.

2. There are strong constraints on this proposal. $P_{M_{i_+}}$ should have rational coefficients in the extension of rationals defined by the composite polynomial, or even polynomial P_i. Monic polynomial property would pose even stronger conditions on entanglement coefficients and the representations of the entire Galois group.

There is also the notion of Galois confinement for physical states. What constraints does this give?

These conditions pose very strong conditions on the allowed entanglement matrix and could make the proposal unrealistic.

9.5.4 About TGD based description of entanglement

The general classification of possible quantum entanglements is an interesting challenge and there are many approaches (`https://cutt.ly/iREIg1u`). One interesting approach relies on the irreducible representations of the unitary group $U(n)$ acting as the isometry group of n-D Hilbert space (`https://cutt.ly/ZREIEAT`). The assumption about irreducibility is however not essential for what follows.

1. A system with n-D state space H_n identified as a sub-system of a larger system with N-D state space H_N can entangle with its $M = N - n$-D complement H_M. Suppose $n \leq M$. Entanglement implies that the n-D state space or its sub-space is embedded isometrically into a subspace of the M-D state space. For a non -trivial subspace one can replace H_n with this subspace H_m in what follows. The diagonal form of the density matrix describes this correspondence explicitly. If the subspace is 1-D one has an unentangled situation.

2. $U(n)$ and its subgroups act as automorphism groups of H_n This inspires the idea that the irreducible representations of $U(n)$ define physically very special entanglements $H_n \subset H_M$. The isometric inclusions $H_n \subset H_M$ are parametrized by a flag-manifold $F_{n,M} = U(M)/U(n) \times U(M-n)$. If one allows second quantization in the sense that the wave functions in the space of entanglements make sense, this flag manifold represents additional degrees of freedom for entanglements $H_n \subset H_M$. If the entanglement does not have maximal dimension, the product of flag manifolds $F_{n,M}$ and $F_{m,n}$ characterizes the space of entanglements.

3. Flag manifold has a geometric interpretation as the space of n-D spaces C^n (flags) embedded in C^M. Interestingly,twistor spaces and more general spaces of twistor Grassmannian approach are flag manifolds and twistor spaces are also related to Minkowski space.

4. I have not been personally enthusiastic about the notion of emergence of 3-space or space-time from entanglement but one can wonder whether flag manifolds related naturally to entanglement could lead to the emergence of Minkowski space. Or perhaps better, whether the notion of entanglement and Minkowski space could be natural aspects of a more general description.

5. One can also have flags inside flags inside leading to more complex flag manifolds $F(n_1, n_2, .., n_k = M) = U(M)/U(m_1) \times ... \times U(m_k)$, $m_k = n_k - n_{k-1}$ assuming $n_0 = 0$. In consciousness theories, the challenge is to understand the quantum correlates of attention. Entanglement is the most obvious candidate in this respect. Attention seems to be something with a directed arrow. This is difficult to understand in terms of the ordinary entanglement. Flag hierarchy would suggest a hierarchical structure of entanglement in which the system entangles with a higher-D system, which entangles with a higher-D system. In this picture the state function reduction would be replaced by a cascade starting from the top.

6. The analog of flags inside flags is what happens in what I call number theoretic measurement cascades for wavefunctions [L90] in the Galois groups which are associated with extension of extensions of..... The already mentioned cognitive measurement cascade corresponds to a hierarchy of normal subgroups of Galois group and one can perhaps say that discret Galois group replaces the unitary group. Each normal subgroup in the hierarchy is the Galois group of the extension of the extension below it. This automatically realizes the hierarchical entanglement as an attentional hierarchy. The cognitive measurement cascade can actually start at any level of the hierarchy of extensions of extensions and if it starts from the top all factors are reduced to a pure state.

   If the polynomials defining the 4-surfaces in M^8 satisfy $P(0) = 0$, the the composite polynomial $P_n \circ P_{n-1} ... \circ P_1$ has the roots of $P_1, ..., P_{n-1}$ as its roots. In this case the inclusion of state spaces are unique so that flag manifolds are not needed.

9.5.5 Negentropy Maximization Principle

Negentropy Maximization Principle (NMP) [L99] is the basic variational principle of TGD based quantum measurement theory giving rise to a theory of consciousness.

1. The adelic entanglement entropy is the sum of the real entanglement entropy and p-adic entropies. The adelic negentropy is its negative.

   The real part of adelic entropy is non-negative but p-adic negentropies can be positive. The sum of p-adic negentropies can be larger than the real entropy for non-trivial extensions of rationals. NMP is expected to take care that this is indeed the case. Second law for the real entropy would still hold true and guarantee NMP.

2. NMP states that SFRs cannot reduce the *overall* entanglement entropy although this can happen to subsystems. In SFRs this local reduction of negentropy would happen. Entanglement is not destroyed in SFRs in general and new entanglement negentropy can be generated.

3. Although real entanglement entropy tends to increase, the positive p-adic negentropies assignable to the cognition would do the same so that net negentropy would increase. This would not mean only entanglement protection, but entanglement generation and cognitive evolution. This picture is consistent with the paradoxical proposal of Jeremy England [?] [L14] that biological evolution involves an increase of entropy.

4. It should be noticed that the increase of real entanglement entropy as such does not imply the second law. The reduction of real entropy transforms it to ensemble entropy since the outcome of the measurement is random. This entropy is entropy of fermions at space-time sheets. The fermionic entanglement would be reduced but transformed to Galois entanglement.

9.6 Appendix

9.6.1 Comparison of TGD with other theories

Table 9.1 compares GRT and TGD and **Table 9.2** compares standard model and TGD.

9.6.2 Glossary and figures

The following glossary explains some basic concepts of TGD and TGD inspired biology.

- ***Space-time as surface***. Space-times can be regarded as 4-D surfaces in an 8-D space $M^4 \times CP_2$ obtained from empty Minkowski space (M^4) by adding four small dimensions (CP_2). The study of field equations characterizing space-time surfaces as "orbits" of 3-surfaces (3-D generalization of strings) forces the conclusion that the topology of space-time is non-trivial in all length scales.

- ***Geometrization of classical fields***. Both weak, electromagnetic, gluonic, and gravitational fields are known once the space-time surface in H as a solution of field equations is known.

  Many-sheeted space-time (see **Fig. 9.4**) consists of space-time sheets with various length scales with smaller sheets being glued to larger ones by ***wormhole contacts*** (see **Fig. 9.5**) identified as the building bricks of elementary particles. The sizes of wormhole contacts vary but are at least of CP_2 size (about 10^4 Planck lengths) and thus extremely small.

  Many-sheeted space-time replaces reductionism with ***fractality*** . The existence of scaled variants of physics of strong and weak interactions in various length scales is implied, and biology is especially interesting in this respect.

- ***Topological field quantization (TFQ)*** . TFQ replaces classical fields with space-time quanta. For instance, magnetic fields decompose into space-time surfaces of finite size representing flux tubes or -sheets. Field configurations are like Bohr orbits carrying

	GRT	**TGD**
Scope of geometrization	classical gravitation	all interactions and quantum theory
Spacetime		
Geometry	abstract 4-geometry	sub-manifold geometry
Topology	trivial in long length scales	many-sheeted space-time
Signature	Minkowskian everywhere	also Euclidian
Fields		
classical	primary dynamical variables	induced from the geometry of H
Quantum fields	primary dynamical variables	modes of WCW spinor fields
Particles	point-like	3-surfaces
Symmetries		
Poincare symmetry	lost	Exact
GCI	true	true - leads to SH and ZEO
	Problem in the identication of coordinates	$H = M^4 \times CP_2$ provides preferred coordinates
Super-symmetry	super-gravitation	super variant of H: super-surfaces
Dynamics		
Equivalence Principle	true	true
Newton's laws and		
notion of force	lost	generalized
Einstein's equations	from GCI and EP	remnant of Poincare invariance at QFT limit of TGD
Bosonic action	EYM action	Kähler action + volume term
Cosmological constant	suggested by dark energy	length scale dependent coefficient of volume term
Fermionic action	Dirac action	Modified Dirac action for induced spinors
Newton's constant	given	predicted
Quantization	fails	Quantum states as modes of WCW spinor field

Table 9.1: Differences and similarities between GRT and TGD

	SM	TGD
Symmetries		
Origin	from empiria	reduction to CP_2 geometry
Color symmetry	gauge symmetry	isometries of CP_2
Color	analogous to spin	analogous to angular momentum
Ew symmetry	gauge symmery	holonomies of CP_2
Symmetry breaking	Higgs mechanism	CP_2 geometry
Spectrum		
Elementary particles	fundamental	consist of fundamental fermions
Bosons	gauge bosons, Higgs	gauge bosons, Higgs, pseudo-scalar
Fundamental fermions	quarks and leptons	quarks: leptons as local 3-quark composites
Dynamics		
Degrees of freedom	gauge fields, Higgs, and fermions	3-D surface geometry and spinors
Classical fields	gauge fields, Higgs	induced spinor connection
	SU(3) Killing vectors of CP_2	
Quantal degrees of freedom	gauge bosons,Higgs,	quantized induced spinor fields
Massivation	Higgs mechanism	p-adic thermodynamics with superconformal symmetry

Table 9.2: Differences and similarities between standard model and TGD

"archetypal" classical field patterns. Radiation fields correspond to topological light rays or massless extremals (MEs), magnetic fields to magnetic flux quanta (flux tubes and sheets) having as primordial representatives "cosmic strings", electric fields correspond to electric flux quanta (e.g. cell membrane), and fundamental particles to CP_2 type vacuum extremals.

- ***Field body*** (FB) and ***magnetic body*** (MB). Any physical system has field identity - FB or MB - in the sense that a given topological field quantum corresponds to a particular source (or several of them - e.g. in the case of the flux tube connecting two systems).

  Maxwellian electrodynamics cannot have this kind of identification since the fields created by different sources superpose. Superposition is replaced with a set theoretic union: only the *effects* of the fields assignable to different sources on test particle superpose. This makes it possible to define the QFT limit of TGD.

- ***p-Adic physics*** [K62] as a physics of cognition and intention and the fusion of p-adic physics with real number based physics are new elements.

- ***Adelic physics*** [L43, L49] is a fusion of real physics of sensory experience and various p-adic physics of cognition.

- ***p-Adic length scale hypothesis*** states that preferred p-adic length scales correspond to primes p near powers of two: $p \simeq 2^k$, k positive integer.

- A ***Dark matter hierarchy*** realized in terms of a hierarchy of values of effective Planck constant $h_{eff} = nh_0$ as integers using $h_0 = h/6$ as a unit. Large value of h_{eff} makes possible macroscopic quantum coherence which is crucial in living matter.

- ***MB as an intentional agent using biological body (BB) as a sensory receptor and motor instrument*** . The personal MB associated with the living body - as opposed to larger MBs assignable with collective levels of consciousness - has a hierarchical onion-like layered structure and several MBs can use the same BB making possible remote mental interactions such as hypnosis [L9].

- ***Cosmic strings Magnetic flux tubes*** belong to the basic extremals of practically any general coordinate invariant action principle. Cosmic strings are surfaces of form $X^2 \times Y^2 \subset M^4 \times CP_2$. X^2 is analogous to string world sheet. Cosmic strings come in two varieties and both seem to have a deep role in TGD.

  Y^2 is either a complex or Lagrangian 2-manifold of CP_2. Complex 2-manifold carries monopole flux. For Lagrangian sub-manifold the Kähler form and magnetic flux and Kähler action vanishes. Both types of cosmic strings are are simultaneous extremals of both Kähler action and volume action: this holds true quite generally for preferred extremals.

  Cosmic strings are unstable against perturbations thickening the 2-D M^4 projection to 3-D or 4-D: this gives rise to monopole (see **Fig. ??**) and non-monopole magnetic flux tubes. Using $M^2 \times Y^2$ coordinates, the thickening corresponds to the deformation for which $E^2 \subset M^4$ coordinates are not constant anymore but depend on Y^2 coordinates.

- ***Magnetic flux tubes and sheets*** serve as "body parts" of MB (analogous to body parts of BB), and one can speak about magnetic motor actions. Besides concrete motion of flux quanta/tubes analogous to ordinary motor activity, basic motor actions include the contraction of magnetic flux tubes by a phase transition possibly reducing Planck constant, and the change in thickness of the magnetic flux tube, thus changing the value of the magnetic field, and in turn the cyclotron frequency. Transversal oscillatory motions of flux tubes and oscillatory variations of the thickness of the flux tubes serve as counterparts for Alfwen waves.

  Reconnections of the U-shaped flux tubes allow two MBs to get in contact based on a pair of flux tubes connecting the systems and temporal variations of magnetic fields inducing motor actions of MBs favor the formation of reconnections.

  In hydrodynamics and magnetohydrodynamics reconnections would be essential for the generation of turbulence by the generation of vortices having monopole flux tube at core and Lagrangian flux tube as its exterior.

  Flux tube connections at the molecular level bring a new element to biochemistry making it possible to understand bio-catalysis. Flux tube connections serve as a space-time correlates for attention in the TGD inspired theory of consciousness.

- ***Cyclotron Bose-Einstein condensates (BECs)*** of various charged particles can accompany MBs. Cyclotron energy $E_c = hZeB/m$ is much below thermal energy at physiological temperatures for magnetic fields possible in living matter. In the transition $h \to h_{eff}$ E_c is scaled up by a fractor $h_{eff}/h = n$. For sufficiently high value of h_{eff} cyclotron energy is above thermal energy $E = h_{eff}\ ZeB/m$. Cyclotron Bose-Einstein condensates at MBs of basic biomolecules and of cell membrane proteins - play a key role in TGD based biology.

- ***Josephson junctions*** exist between two superconductors. In TGD framework, ***generalized Josephson junctions*** accompany membrane proteins such as ion channels and pumps. A voltage between the two super-conductors implies a ***Josephson current***. For a constant voltage the current is oscillating with the ***Josephson frequency*** . The Josephson current emits ***Josephson radiation***. The energies come as multiples of ***Josephson energy***.

  In TGD generalized Josephson radiation consisting of dark photons makes communication of sensory input to MB possible. The signal is coded to the modulation of Josephson frequency depending on the membrane voltage. The cyclotron BEC at MB receives the radiation producing a sequence of resonance peaks.

- ***Negentropy Maximization Principle*** (NMP). NMP [K57] [L99] is the variational principle of consciousness and generalizes SL. NMP states that the negentropy gain in SFR is non-negative and maximal. NMP implies SL for ordinary matter.

- ***Negentropic entanglement*** (NE). NE is possible in adelic physics and NMP does not allow its reduction. NMP implies a connection between NE, the dark matter hierarchy, p-adic physics, and quantum criticality. NE is a prerequisite for an experience defining abstraction as a rule having as instances the state pairs appearing in the entangled state.

- ***Zero energy ontology (ZEO)*** In ZEO physical states are pairs of positive and negative energy parts having opposite net quantum numbers and identifiable as counterparts of initial and final states of a physical event in the ordinary ontology. Positive and negative energy parts of the zero energy state are at the opposite boundaries of a ***causal diamond*** (CD, see **Fig. 9.12**)) defined as a double-pyramid-like intersection of future and past directed light-cones of Minkowski space.

  CD defines the "spot-light of consciousness": the contents of conscious experience associated with a given CD is determined by the space-time sheets in the embedding space region spanned by CD.

- ***SFR*** is an acronym for state function reduction. The measurement interaction is universal and defined by the entanglement of the subsystem considered with the external world [L73] [K109]. What is measured is the density matrix characterizing entanglement and the outcome is an eigenstate of the density matrix with eigenvalue giving the probability of this particular outcome. SFR can in principle occur for any pair of systems.

  SFR in ZEO solves the basic problem of quantum measurement theory since the zero energy state as a superposition of classical deterministic time evolutions (preferred extremals) is replaced with a new one. Individual time evolutions are not made non-deterministic.

  One must however notice that the reduction of entanglement between fermions (quarks in TGD) is not possible since Fermi- and als Bose statistics predicts a maximal entanglement. Entanglement reduction must occur in WCW degrees of freedom and they are present because point-like particles are replaced with 3-surfaces. They can correspond to the number theoretical degrees of freedom assignable to the Galois group - actually its decomposition in terms of its normal subgroups - and to topological degrees of freedom.

- ***SSFR*** is an acronym for "small" SFR as the TGD counterpart of weak measurement of quantum optics and resembles classical measurement since the change of the state is small [L73] [K109]. SSFR is preceded by the TGD counterpart of unitary time evolution replacing the state associated with CD with a quantum superposition of CDs and zero energy states associated with them. SSFR performs a localization of CD and corresponds to time measurement with time identifiable as the temporal distance between the tips of CD. CD is scaled up in size - at least in statistical sense and this gives rise to the arrow of time.

  The unitary process and SSFR represent also the counterpart for Zeno effect in the sense that the passive boundary of CD as also CD is only scaled up but is not shifted. The states remain unchanged apart from the addition of new fermions contained by the added part of the passive boundary. One can say that the size of the CD as analogous to the perceptive field means that more and more of the zero energy state at the passive boundary becomes visible. The active boundary is however both scaled and shifted in SSFR and states at it change. This gives rise to the experience of time flow and SSFRs as moments of subjective time correspond to geometric time as a distance between the tips of CD. The analog of unitary time evolution corresponds to "time" evolution induced by the exponential of the scaling generator L_0. Time translation is thus replaced by scaling. This is the case also in p-adic thermodynamics. The idea of time evolution by scalings has emerged also in condensed matter physics.

- ***BSFR*** is an acronym for "big" SFR, which is the TGD counterpart of ordinary state function reduction with the standard probabilistic rules [L73] [K109]. What is new is that the arrow of time changes since the roles of passive and active boundaries change and CD starts to increase in an opposite time direction.

  This has profound thermodynamic implications. Second law must be generalized and the time corresponds to dissipation with a reversed arrow of time looking like self-organization for an observed with opposite arrow of time [L69]. The interpretation of BSFR is as analog of biological death and the time reversed period is analogous to re-incarnation but with non-standard arrow of time. The findings of Minev *et al* [L62] give support for BSFR at atomic level. Together with h_{eff} hierarchy BSFR predicts that the world looks classical in all scales for an observer with the opposite arrow of time.

9.6.3 Figures

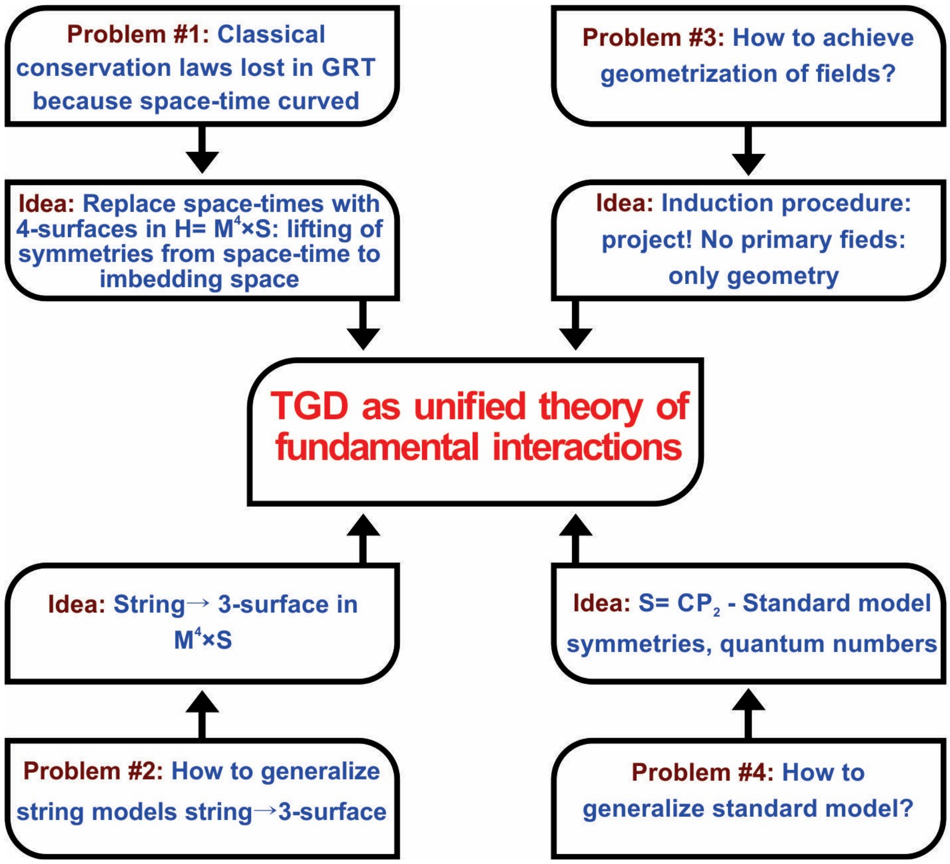

Figure 9.1: The problems leading to TGD as their solution.

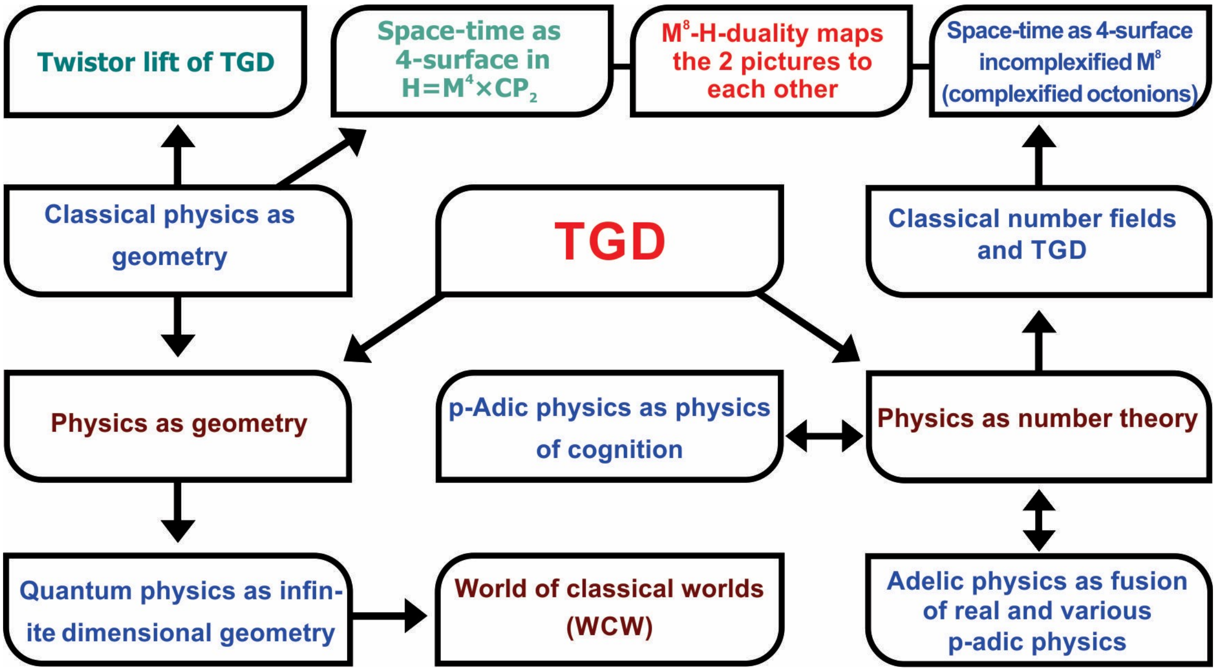

Figure 9.2: TGD is based on two complementary visions: physics as geometry and physics as number theory.

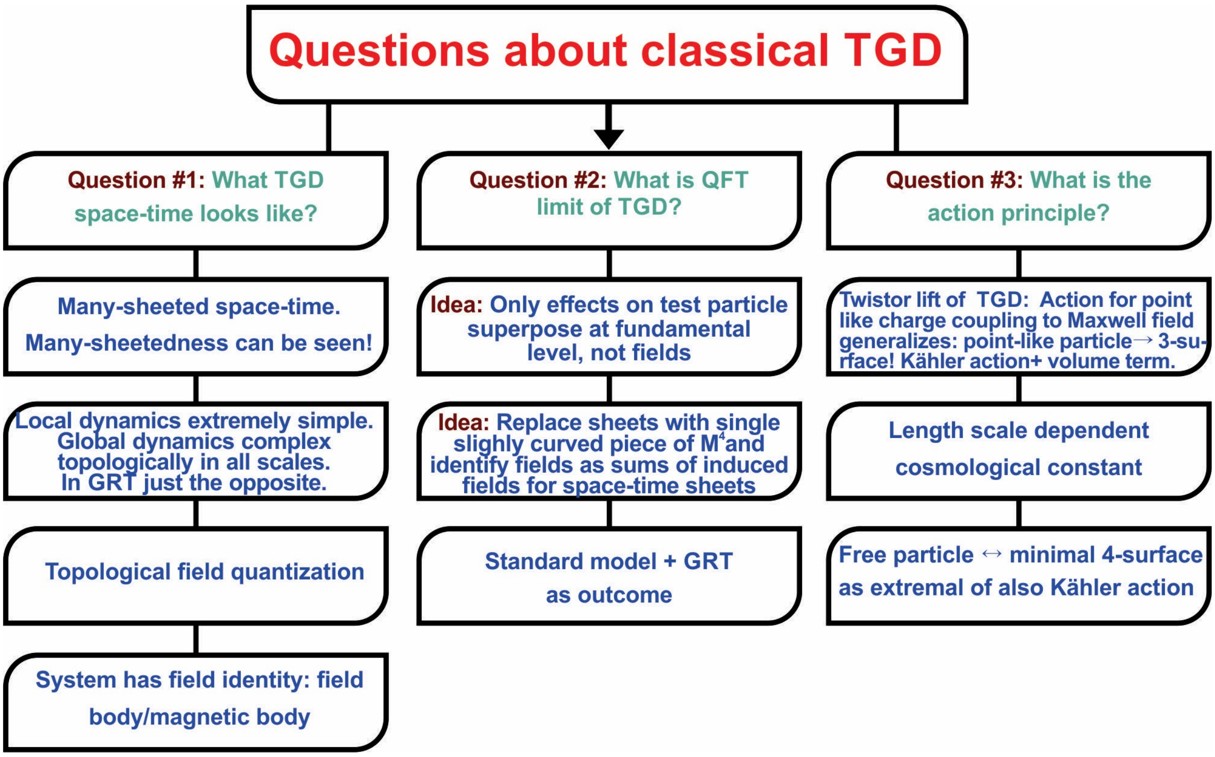

Figure 9.3: Questions about classical TGD.

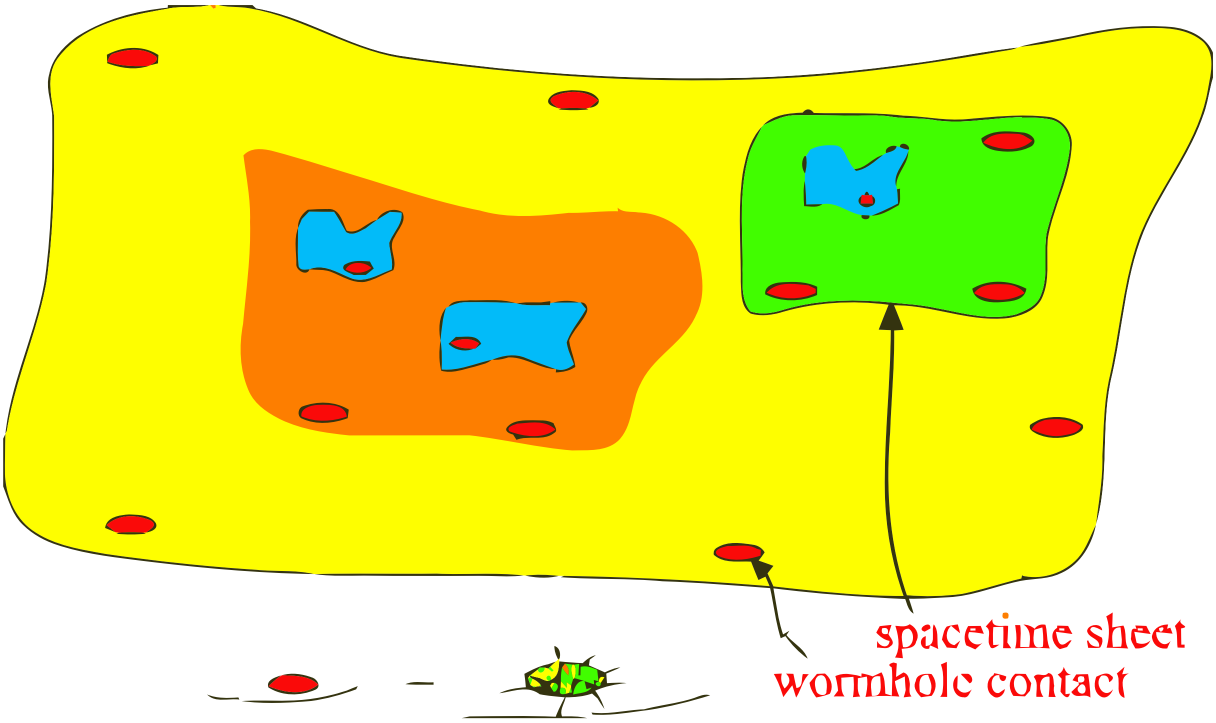

Figure 9.4: Many-sheeted space-time.

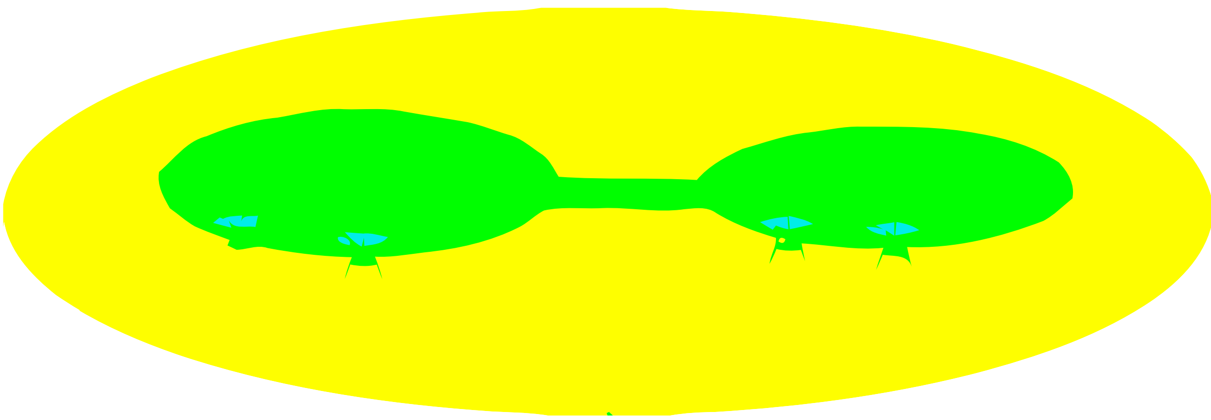

Figure 9.5: Wormhole contacts.

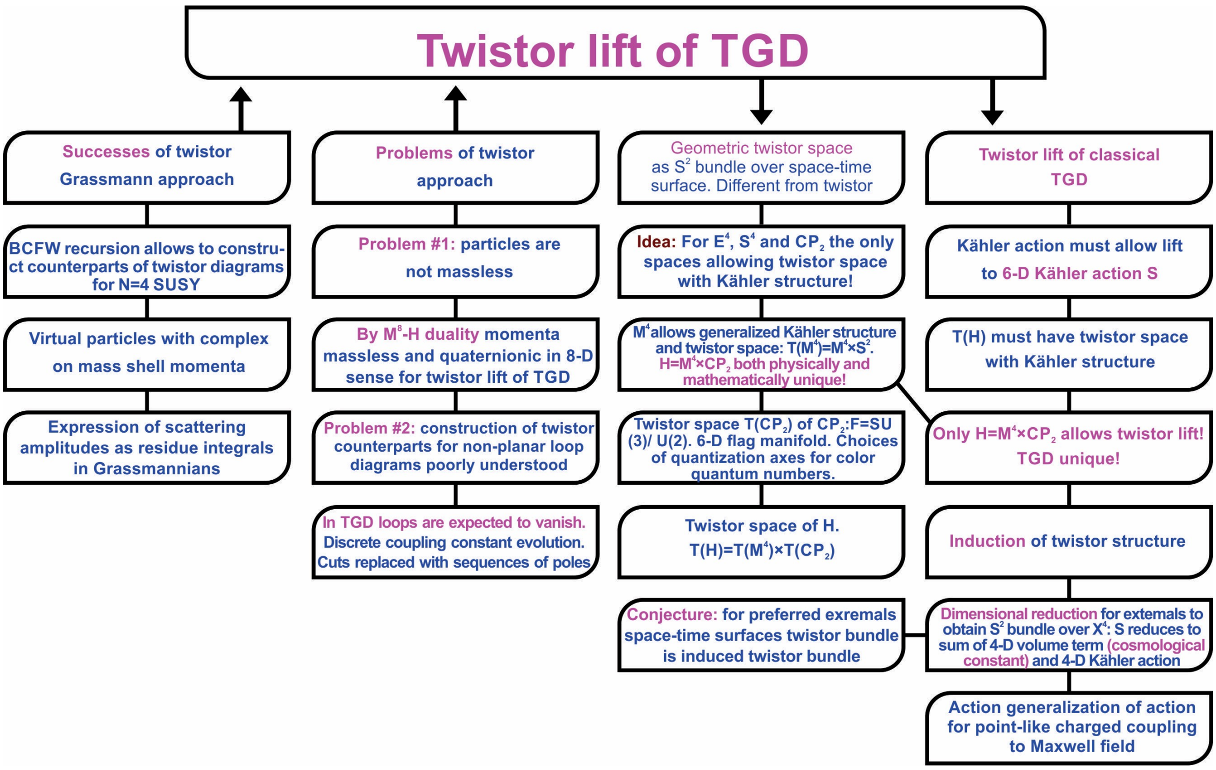

Figure 9.6: Twistor lift

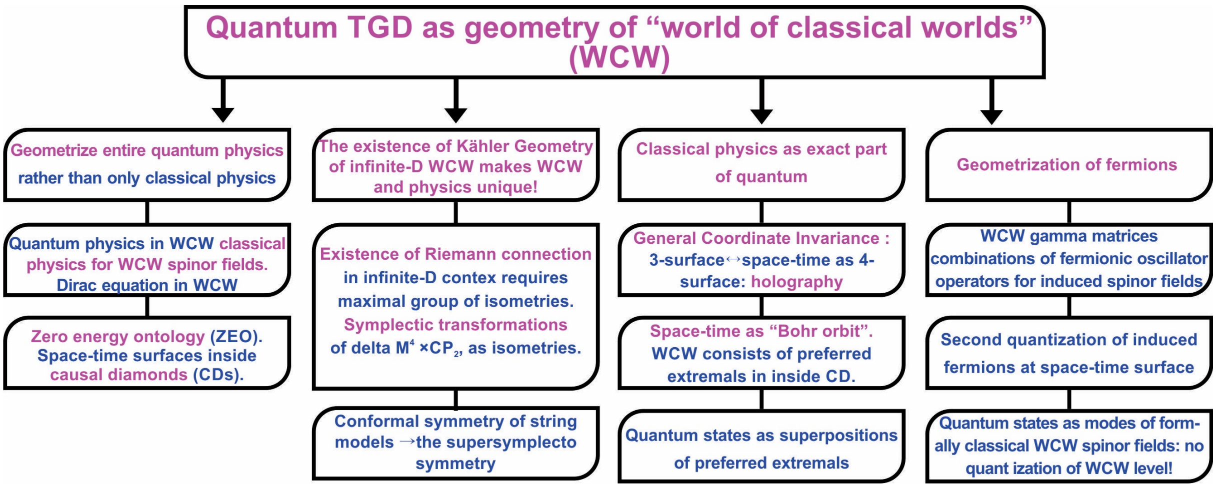

Figure 9.7: Geometrization of quantum physics in terms of WCW

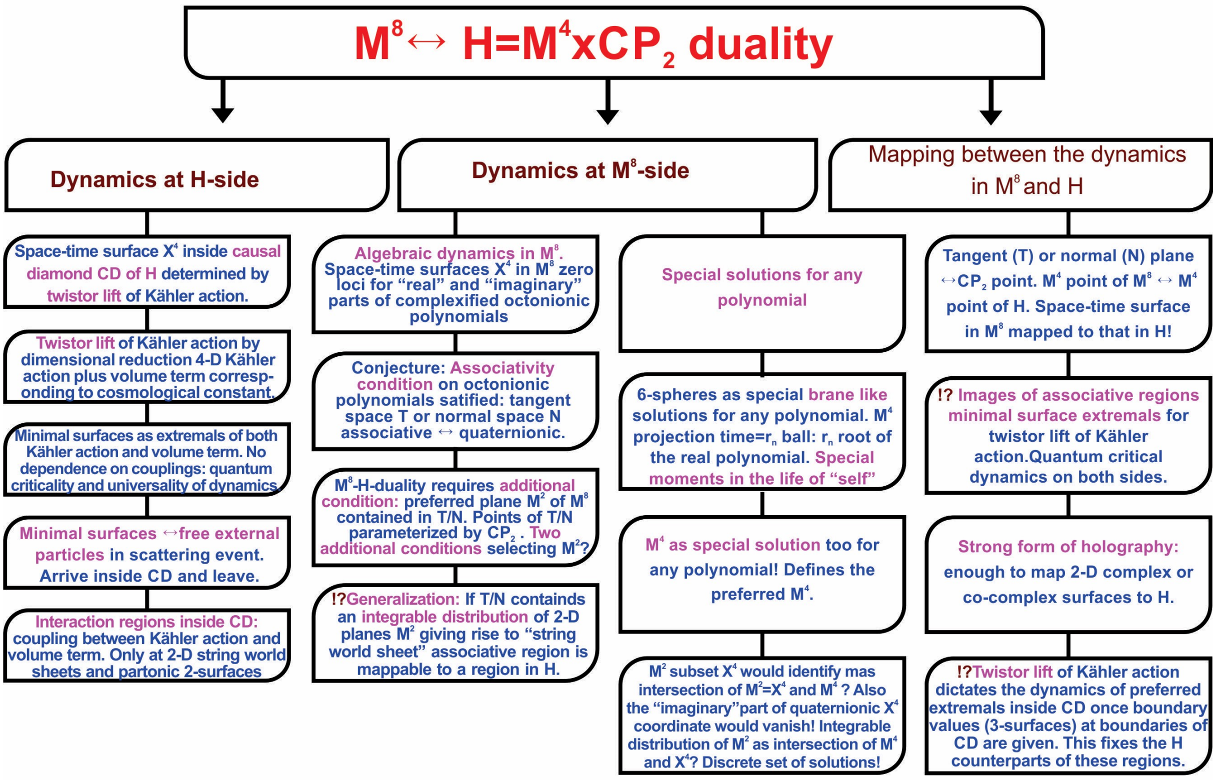

Figure 9.8: $M^8 - H$ duality

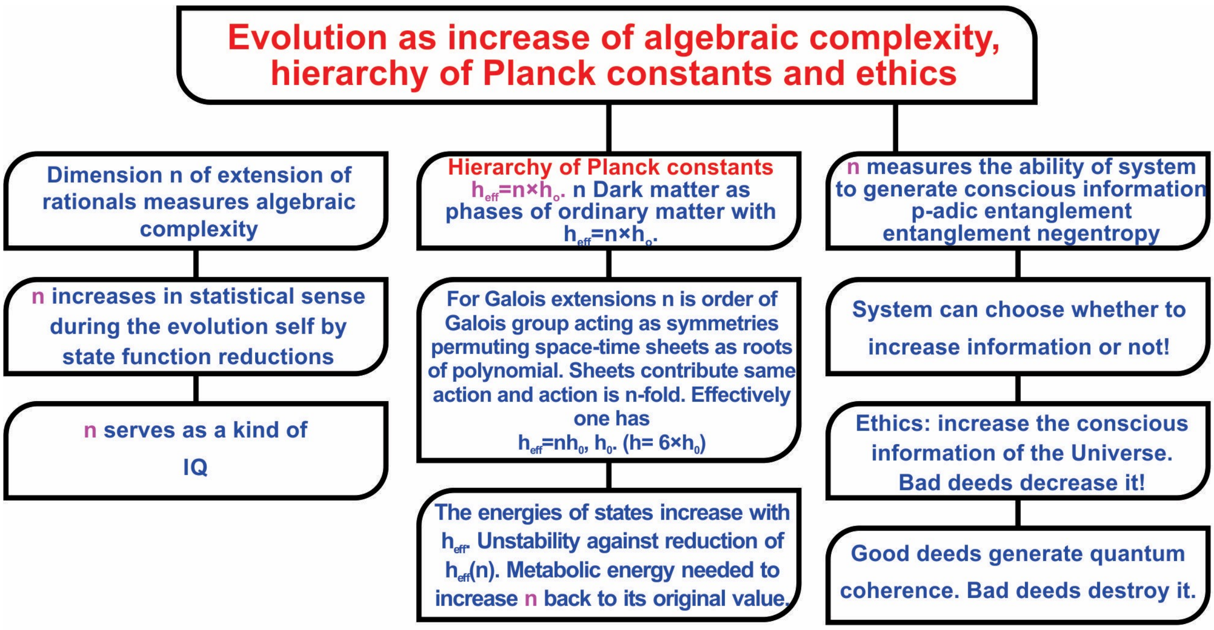

Figure 9.9: Number theoretic view of evolution

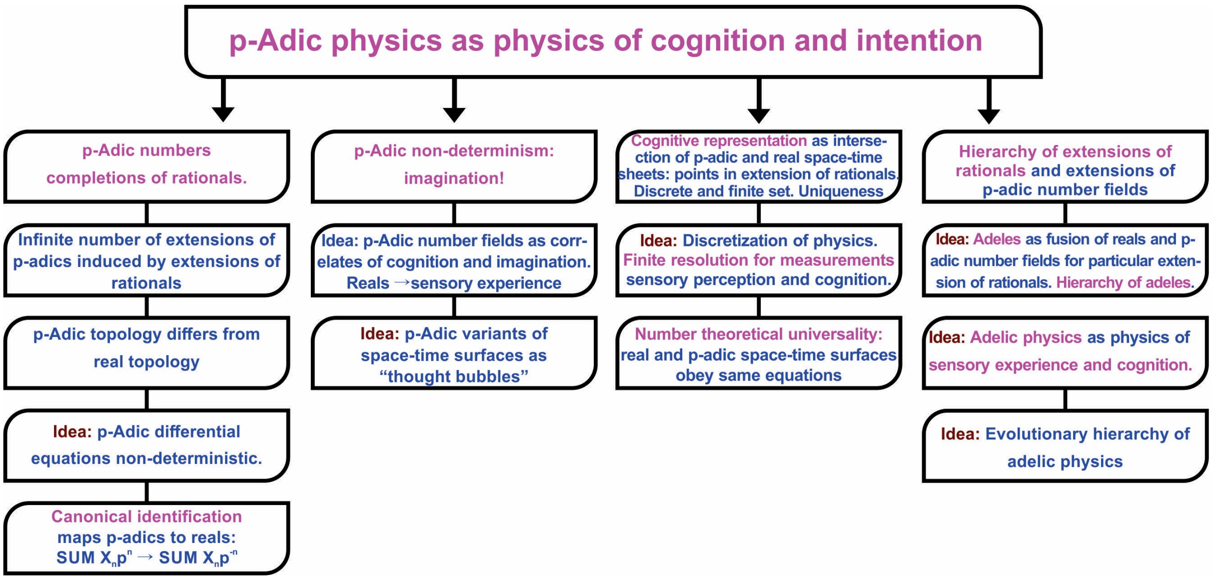

Figure 9.10: p-Adic physics as physics of cognition and imagination.

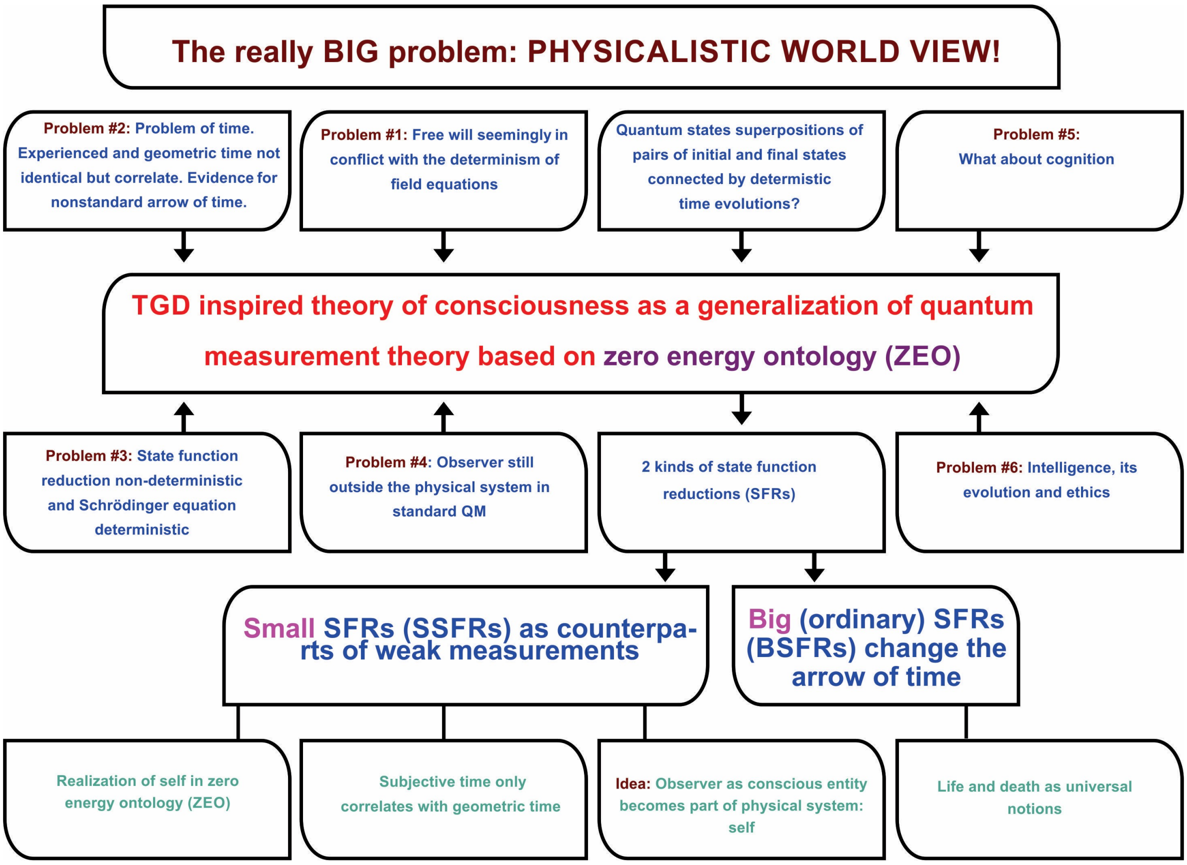

Figure 9.11: Consciousness theory from quantum measurement theory

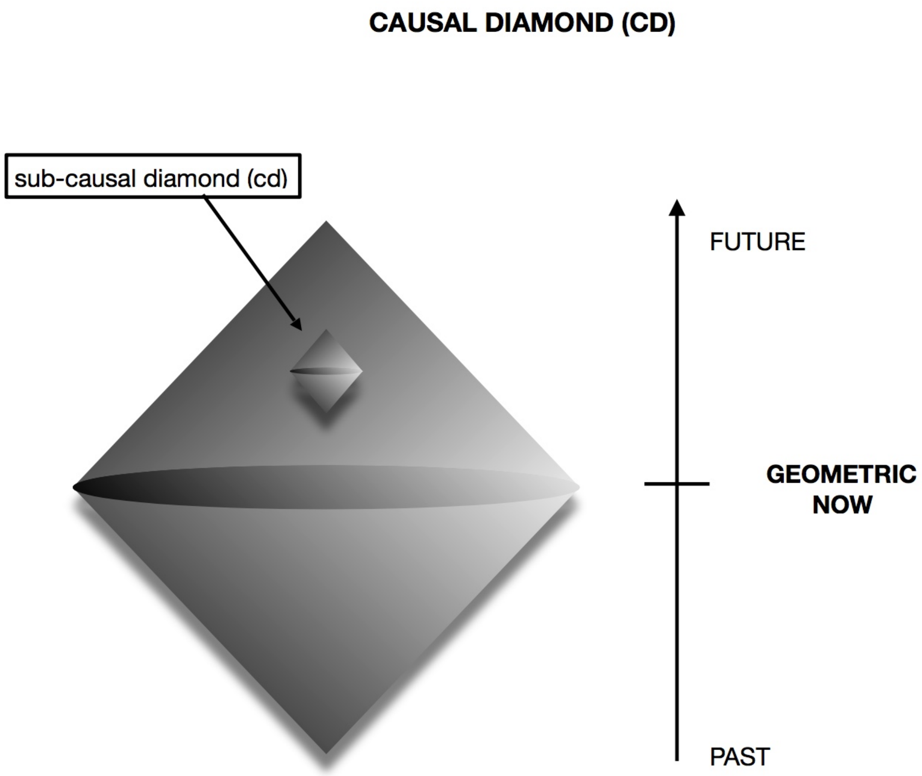

Figure 9.12: Causal diamond

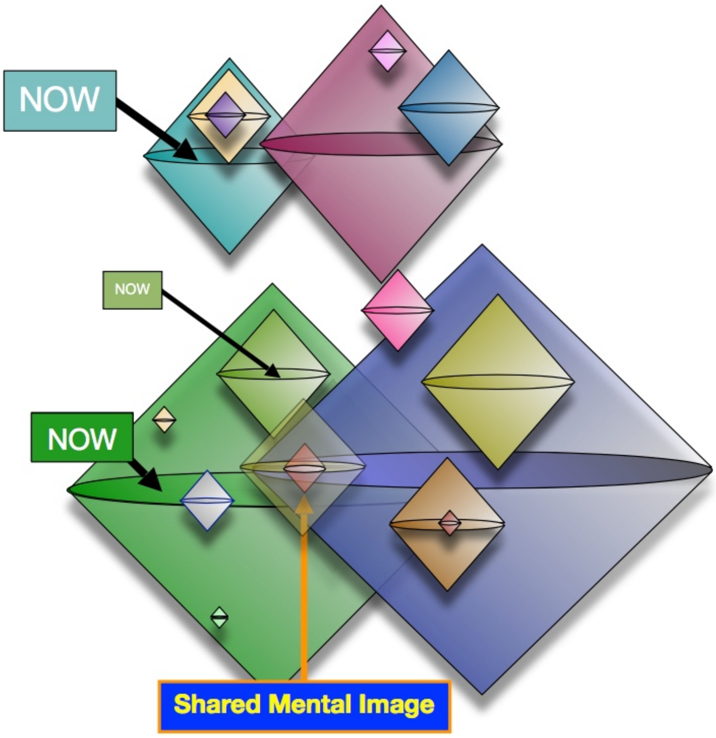

Figure 9.13: CDs define a fractal "conscious atlas"

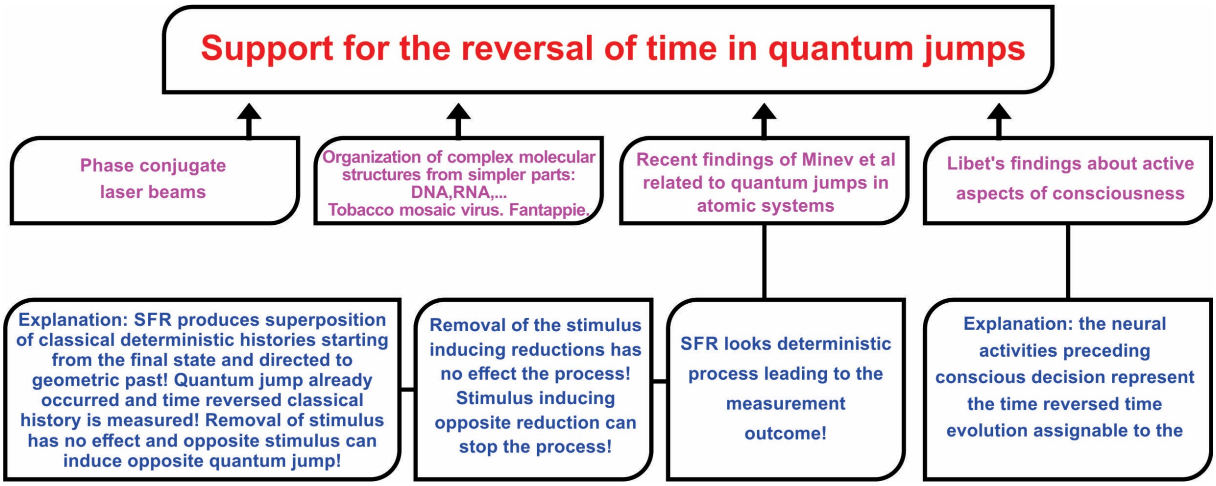

Figure 9.14: Time reversal occurs in BSFR

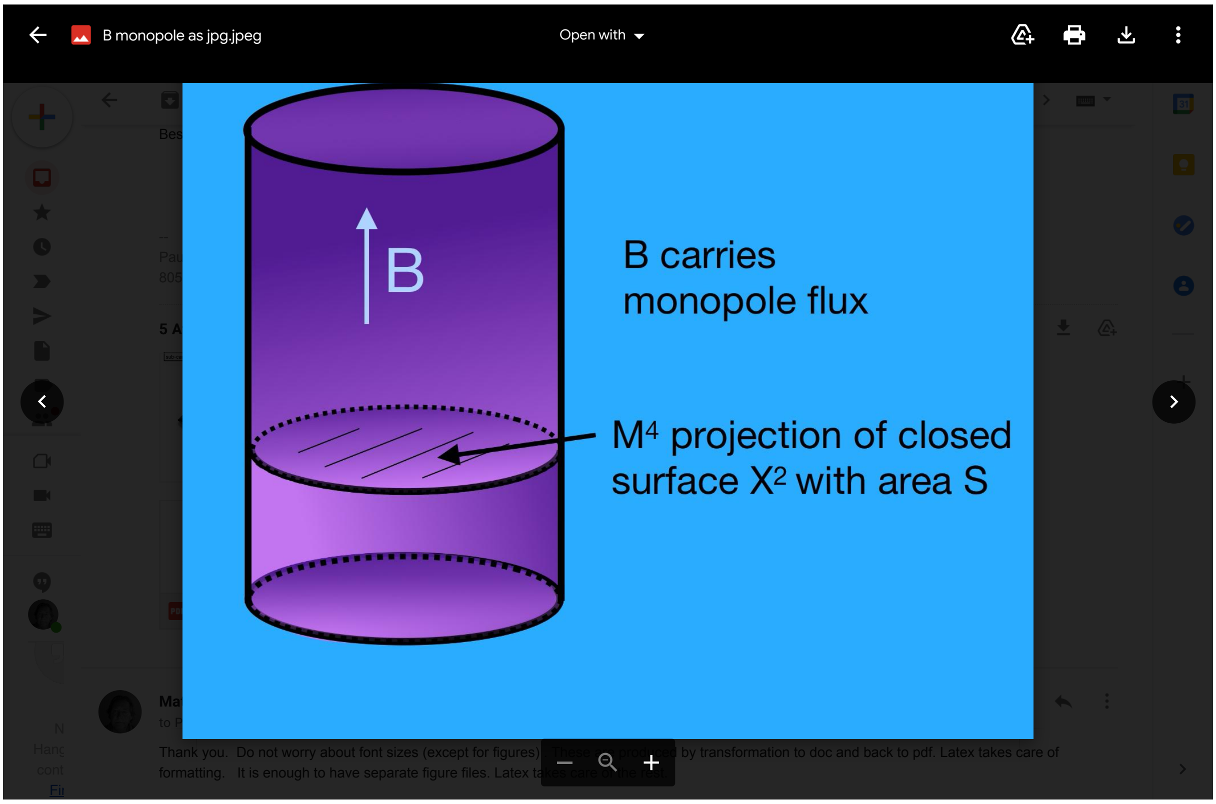

Figure 9.15: The M^4 projection of a closed surface X^2 with area S defining the cross section for monopole flux tube. Flux quantization $e\oint B\cdot dS = eBS = kh$ at single sheet of n-sheeted flux tube gives for cyclotron frequency $f_c = ZeB/2\pi m = khZ/2\pi mS$. The variation of S implies frequency modulation.

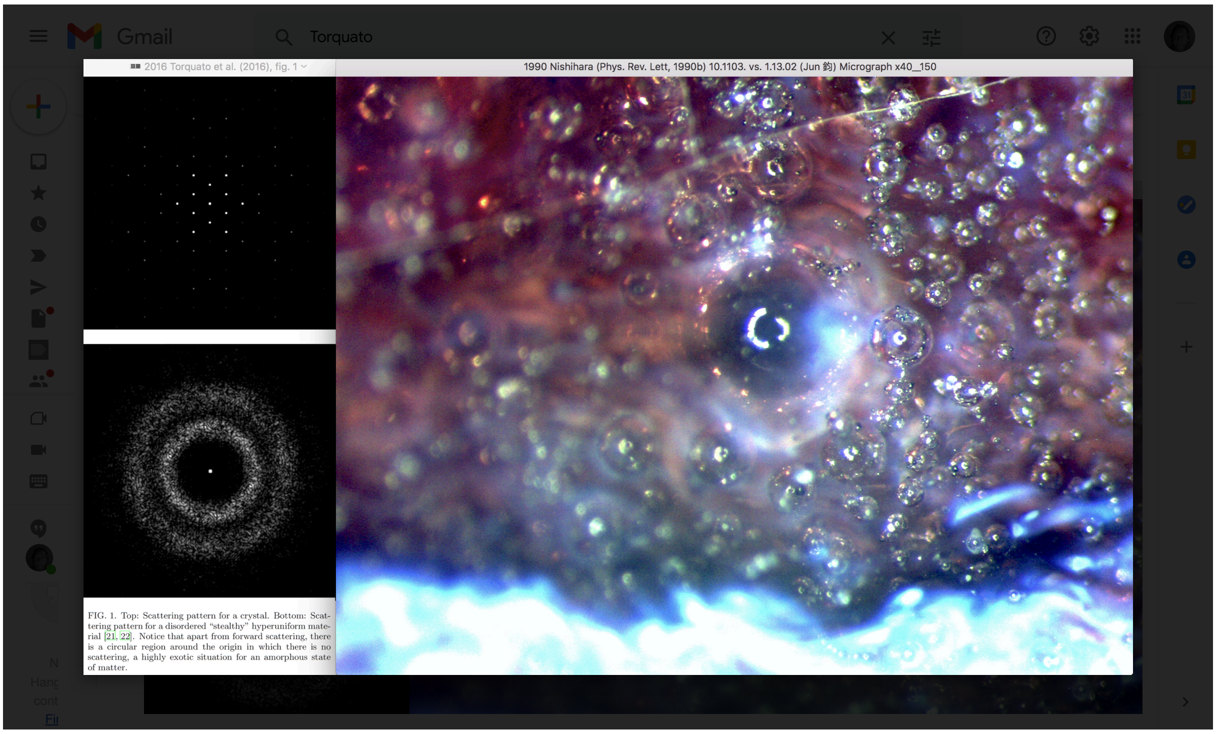

Figure 9.16: The scattering from a hyperuniform amorphous material shows no scattering in small angles apart from the forward peak (`https://cutt.ly/ZWyLgjk`). This is very untypical in amorphous matter and might reflect the diffraction pattern of dark photons at the magnetic body of the system.

Chapter 10

About TGD counterparts of twistor amplitudes

10.1 Introduction

The twistor program was originally introduced by Penrose [B63]. The application of twistors to gauge theories, in particular $\mathcal{N} = 4$ SUSY, led to a dramatic progress in the mathematical understanding of these theories. For beginners like me (still), the article of Elvang and Huang [B32] is an extremely helpful introduction to twistor scattering amplitudes.

I am not a specialist in the field. Therefore the following list of works that have had effect in my attempts to understand how twistors might relate to TGD, must look rather random in the eyes of a professional. It however gives some idea about the timeline of ideas.

- Witten's work (2003) [B29] on perturbative string theory in twistor space.
- The proof of Britto, Cachazo, Feng and Witten (2005) [B20] for tree level recursion relation (BCFW recursion) in Yang-Mills theory.
- The work of Hodges (2005) [B8] about twistor diagram recursion for gauge-theory amplitudes.
- The works of Mason and Skinner (2009) on scattering amplitudes and BCFW recursion in twistor space [B58] and on dual superconformal invariance, momentum twistors and Grassmannians (2009) [B59]. There is also the work of Bullimore, Mason and Skinner (2009) on twistor strings, Grassmannians and leading singularities [B21].
- The work of Drummond, Henn and Plefka (2009) [B27] on Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SUSY.
- The work of Goncharov *et al* (2010) [B42] on classical polylogarithms for amplitudes and Wilson loops.
- Nima Arkani-Hamed and colleagues have made impressive contributions. There is a work by Arkani-Hamed *et al* on S-Matrix in twistor space (2009) [B35, B34]; a work about unification of residues and Grassmannian dualities (2010) [B37]; a proposal for all-loop integrand for scattering amplitudes for planar $\mathcal{N} = 4$ SUSY (2011) [**?**] a work on scattering amplitudes and positive Grassmannian (2012) [B33]; the proposal of amplituhedron (2013) [B15] and work about positive amplitudes in amplituhedron [B14] (2014); a proposal of MHV on-shell amplitudes beyond the planar limit (2014) [B39] ; the notion of associahedron (2017) [B13].

The TGD approach to twistors [L10, L45] [L58, L78] has developed gradually during the last decade. The evolution of ideas began with the attempt to geometrize twistors in the same way as standard model gauge fields are geometrized in TGD. Only quite recently, the number theoretic approach to twistors has started to evolve.

The twistor lift of TGD geometrizes the notion of twistor by replacing the twistor field configurations with 6-D surfaces assigning to space-time surfaces analog of its twistor space obtained by inducing the twistor structure of the product $T(M^4) \times T(CP_2)$ of the twistor spaces of M^4 and CP_2. The construction requires that these twistor spaces have a Kähler structure. M^4 and CP_2 are unique in that only their twistor spaces allow a Kähler structure [A54]. Therefore TGD is mathematically unique: the same conclusion is forced by standar model symmetries and M^8-H duality. This gives strong motivation for an attempt to construct the TGD counterparts of the twistor scattering amplitudes.

The number theoretic view about twistors based on M^8-H duality [L82, L83, L104] has developed during this year (2021) and this article tries to articulate this vision and leads to a proposal for how to construct twistor scattering amplitudes in the TGD framework.

10.1.1 Some background

In the following, the basic facts related to twistors are described. I cannot say anything about the technicalities of the twistorial computations and my basic aim is to clarify myself the contents of the notions involved and understand how the twistors diagrammatics might generalize to the TGD context.

Basic facts about twistors and bi-spinors

It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality (see `http://tinyurl.com/y6bnznyn`).

1. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle \lambda, \mu \rangle &= \epsilon_{ab}\lambda^a \mu^b \ , \\ \left[\tilde{\lambda}, \tilde{\mu}\right] &= \epsilon_{a'b'}\tilde{\lambda}^{a'}\tilde{\mu}^{b'} \ , \\ p \cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu}\right] \ , \ (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ . \end{aligned} \tag{10.1.1}$$

2. Spinor indices are lowered and raised using antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda}\right]} \ , \ \text{negative helicity} \ . \end{aligned} \tag{10.1.2}$$

   In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

3. What makes 4-D twistors unique is the existence of the index raising and lifting operations using antisymmetric ϵ tensors. In higher dimensions they do not exist and this causes difficulties. For octonionic twistors with quaternionic components possibly only in $D=8$ the situation changes.

Also massive momenta and any point of M^4 can be expressed in terms of helicity spinors but momenta different by a light-like momenta on some light-like geodesic give rise to the same twistor.

1. One has $p_{a\dot{b}} = \mu_a \tilde{\lambda}_{\dot{b}}$. The spinors μ and λ are determined only modulo opposite complex scalings. One can say that the twistor line (sphere CP_1) determines a point of M^4. A possible interpretation is that the points of CP_1 correspond to the choices of spin quantization axis for momentum p and the scaling changes its direction.

2. The incidence relation $\mu^a = p^{a\dot{b}}\lambda_{\dot{b}}$ is also true for $p^{a\dot{b}}+k\lambda^a\lambda\dot{b}$, for any k, so that the points of a light-like line in M^4 are mapped to a point of the twistor space and therefore would correspond to the same direction of spin quantization axis. Physically this could be interpreted by saying that this is the case because the points with a light-like separation are not causally independent.

Twistors allow an elegant formulation of the kinematics and the Mandelstam variable $s_{ij} = (p_i - p_j)^2 = m_i^2 + m_j^2 - 2p_i \cdot p_j$ can be expressed in terms of twistors by expressing p as

$$p = |\mu\rangle[\tilde{\lambda}| + |\tilde{\mu}]\langle\lambda|$$

Since the states are massive, the inner product $p_1 \cdot 2$ can be expressed as

$$p_1 \cdot p_2 = \langle\lambda_1\mu_2\rangle[\tilde{\lambda}_1\tilde{\mu}_2] \ ,$$

Since <> and [] are not complex conjugates of each other and can be regarded as independent complex variables. For massless case this is not case that the expression for $p_1 \cdot p_2$ reduces to modulus squared=

The notion of momentum twistor is nicely explained by Claude Durr in the slides of a talk "Momentum twistors, special functions and symbols" (`https://cutt.ly/AY7QYv3`). Momentum twistors are essential in the twistorial construction of the scattering amplitudes.

1. The notion makes sense for planar diagrams for which the momenta can be ordered. For non-planar diagrams this is not the case. Whether the embedding of non-planar diagrams to a surface with some minimal genus could allow the ordering (if two lines which cross in plane, the other line could go along the handle), is not clear to me.

2. One ends up with the momentum twistors Z_i, as opposed to ordinary twistors denoted by W_i, by performing a Fourier transform of a massless twistor amplitude, which is holomorphic in variables $\langle\lambda_i\lambda_j\rangle$ so that the relation of the helicity spinor μ to λ is essentially that of wave vector to a position vector. The helicity spinor pair $Z = (\omega, \lambda)$, where ω is essentially the complex conjugate of λ in massless case is replaced with (ω, μ). This transform makes sense also in the massive case.

   Momentum twistors correspond to what are called dual or area momenta. The ordinary momenta p_i can be expressed as their differences $p_i = x_{i+1} - x_i$ and area momenta in turn as as $x_i = \sum_{1\leq k\leq i} x_k$. The term area momentum comes from the observation that the planar diagrams divide the plane into disjoint regions and the area momenta can be assigned to these regions.

3. At the level of symmetries the possibility of momentum twistors means extension of the algebra of conformal symmetries of M^4 to a Yangian algebra whose generators are labeled by non-negative integers and which are poly-local so that the corresponding charges contain multilocal contributions (note that potential energy is bilocal and somewhat analogous notion). The generators generating conformal symmetries in the space of area momenta correspond to generators of conformal weight $h = 1$ and whereas ordinary conformal generators have conformal weight $h = 0$.

Remark: TGD suggests the interpretation of two kinds of twistors in terms of $M^8 - H$ duality. Area momenta and momentum twistors could correspond to M^8 level and ordinary momenta and twistors to H level. M^8 indeed has interpretation as analog of momentum space and $M^8 - H$ duality as the TGD counterpart of momentum-position duality having no generalization in quantum field theories where momentum and position are not dynamical variables.

MHV amplitudes as basic amplitudes

The following comments about MHV amplitudes sketch only the main points as I see them from my limited TGD perspective. One reason for this, besides my very limited practical experience with these amplitudes, is that it seems that The TGD approach in its recent form does not force their introduction.

The article of Elvang and Huang [B32] provides an excellent summary about the construction of twistor amplitudes explaining the important details (see also the slides by Claude Durr at https://cutt.ly/AY7QYv3). Maximally helicity violating (MHV) amplitudes with $k = 2$ negative helicity gluons are defined as tree amplitudes of say $\mathcal{N} = 4$ SUSY and involve gluons and their superpartners. It is convenient to drop the group theory factor $Tr(T_1T_2 \cdots T_n)$ related to gluons.

NMHV amplitudes have $k > 2$ and can be classified by the number of loops as also $k = 2$ diagrams. NMHV diagrams are constructible in terms of MHV diagrams and the construction is known as BSFW construction which by recursion reduces these diagrams to $k = 2$ diagrams, about which 3-gluon vertices is the simplest example. To my amateurish understanding, it is not yet clear whether also the planar Feynman diagrams allow twistorialization. The basic problem is that the area moment x_i with $p_i = x_{i+1} - x_i$ must be ordered and this is not possible for non-planar diagrams.

The construction gives a recursion formula allowing to express the amplitudes in terms of MHV tree amplitudes. Rather remarkably, all loop amplitudes are proportional to the tree level MHV amplitudes so that the singularity structure of the amplitudes is completely determined by the MHV amplitudes. A holography at the level of momentum space is realized in the sense that the singularities dictate the amplitudes completely.

1. The starting point is the observation that tree amplitude with $k = 0$ or $k = 1$ vanishes. The simplest MHV amplitudes have exactly $k = n - 2$ gluons of same helicity- taken by a convention to be negative - have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{10.1.3}$$

   When the sign of the helicities is changed $\langle .. \rangle$ is replaced with $[..]$.

2. It is essential that the amplitudes are expressible in terms of the antisymmetric bi-linears $\langle \lambda_i, \lambda_j \rangle = \epsilon^{ab} \lambda_{i,a} \lambda_{j,b}$. This implies holomorphy and homogeneity with respect to $\langle \lambda_i, \lambda_j \rangle$ follows for massless field theories by the cancellation of the []:s or ¡¿:s of spinor inner products with [] or ¡¿ appearing in $p_i \cdot p_j$ appearing in the massless propagator.

3. $k = 2$ MHV amplitudes take the role of vertices in the construction of amplitudes with $k > 2$ negative helicity gluons. These amplitudes are connected together by off-shell propagator factors $1/P^2$. MHV diagrams allow to develop expressions for the planar on tree amplitudes and also of loop amplitudes using recursion.

4. The treatment of off-mass shell gluons forces to introduce an arbitrary fixed spinor η such that η is not a complex conjugate of λ. η is not the helicity spinor μ assignable uniquely to a massive particle (now a virtual particle). This assumption makes sense for momentum twistors assignable to internal lines of the MHV diagrams since area momenta are in general off-mass-shell.

Yangian symmetry, Grassmannians, positive Grassmannians, and amplituhedron

The work by Nima Arkani Hamed [B34, B38, B33, B5, B15, B67] and other pioneers has led to a very beautiful vision in which the twistorial scattering amplitudes $A_{k,n}$ for $\mathcal{N} = 4$ SUSY are expressible as residue integrals over Grassmannians $Gr_{k,n}$ of integrands which depend on the twistors characterizing the external only via delta functions forcing the integration to surfaces of $Gr(k,n)$. BCFW diagrams and therefore the Grassmannian integrals as their representations are Yangian invariants.

> The amplitudes are defined as residue integrals over $Gr(k,n)$ and contain data about momenta coded by twistors in the arguments of delta functions. The counterparts of the $< ij >$ or $ij]$ determining the integrand are the Plücker coordinates defined as the k-minors, that is determinants of the $k \times n$ matrices, characterizing the point of $Gr(k,n)$. The included minors are taken in cyclic order and contain subsequent columns [B32] (`https://cutt.ly/yY7QzQg`). One integrates over the k-planes, or equivalently, over $n-k$-planes, of C^n and the integral is residue integral. $Gr(n,k) = U(n)/U(k) \times U(n-k)$ has also an interpretation as a flag-manifold. The residues are located in the positive Grassmannian $Gr^{\geq 0}_{n,k}$. The integral reduces to a mere residue selecting a special k-plane of Grassmannian (note that a gauge fixing eliminating gauge degrees of freedom due to the $Gr(k)$ and $Gr(n)$ symmetries is performed). In the massless case, the delta function constraints state that the n-helicity spinors are orthogonal to $k-D$ and $n-k$-D planes of $GR_{k,n}$ and the conditions imply momentum conservation. In the massive case, the momentum conservation constraint states $\sum p_i = |\mu_i > [\tilde{\lambda}_i| + |\tilde{\mu}]_i < \lambda|_i = 0$. Also now, the interpretation as the inner product of n-helicity spinors is suggestive. A technically important detail is that the quadratic momentum delta function $\delta(\sum_i \lambda_i la\tilde{m}bda_i)$ is forced by a product of linear delta function constraints associated with part of $Gr_(k,n)$ to two parts corresponding to k and $n-k$ gluons with opposite helicities. The gauge invariance of these parts with respect to $Gl(k)$ and $Gl(n-k)$ allows a coordinate choice in $Gr(k,n)$ simplifying the calculation drastically.

This work has led to the notions of positive Grassmannian $Gr^{\geq 0}_{k,n}$ [B32] (`https://arxiv.org/abs/2110.10856`) defined as a sub-space of Grassmannian in which all Plücker coordinates defined by the $k \times k$ minors appearing in the expression of the twistor amplitude are non-negative. Any $n \times (k+m)$, whose minors are positive induces a map from $Gr^{\geq 0}_{k,n}$ whose image is the amplituhedron $\mathcal{A}_{\backslash,\Vert,\updownarrow\Updownarrow}$ (`https://arxiv.org/pdf/1912.06125.pdf` and `https://en.wikipedia.org/wiki/Amplituhedron`) introduced by Arkani-Hamed and Trnka. For $m=4$ the BSFW recurrence relations for the scattering amplitudes can be used to produce collections of $4k$-dimensional cells in $Gr^{\geq 0}_{k,n}$, whose images are conjectured to sub-divide the amplituhedron. $\mathcal{A}_{\backslash,\Vert,\updownarrow\Updownarrow}$ generalizes the positive Grassmannian.

Tree-level amplituhedron can be regarded as a generalization of convex hull of external data and the scattering amplitudes can be extracted from a unique differential form having poles at the boundaries of the amplituhedron.

10.1.2 How to generalize twistor amplitudes in the TGD framework?

Twistor approach works so beautifully in massless case such as $calN = 4$ SUSY because the scattering amplitudes for massless gluons can be written as holomorphic homogeneous functions of arguments constructed from the helicity spinors characterizing the momenta of the external massless particles.

It is always best to start from a problem and the basic problem of the twistor approach is that physical particles are not massless. In the massive QFT, one cannot write a simple twistorial expression of the amplitudes, which would be holomorphic homogenous polynomials in the twistor components and involve only the twistor bilinears $< ij >$ or $[ij]$. The reason is that the external and internal particles are massive. For massive particles, the Mandelstam variables $s_{ij} = (p_i - p_j)^2$ do not factorize as $s_{ij} =< ij > [ij]$.

> The intuitive TGD based proposal has been that since quark spinors are massless in 8-D sense in H, the masslessness in the 8-D sense could somehow solve the problems caused by the massivation in the construction of twistor scattering amplitudes. However, no obvious mechanism has been identified. One step in this direction was however the realization that in H quarks propagate with well-defined M^4 chiralities and only the $D^2(H)$ of Dirac operator annihilates the spinors. M^8 quark momenta are in general complex as algebraic integers. They are identifiable as the counterparts of the area momenta x_i of the momentum twistor

space whereas H momenta can be identified as ordinary momenta. The total momenta of Galois confined states have as components ordinary integers and the momentum spectra in H and M^8 are identical by $M^8 - H$ duality. The mass squared spectrum is quantized as integers for Galois confined states in accordance with supersymplectic invariance implying "stringy" mass spectrum. The natural first guess is that in H the free quarks satisfy the Dirac equation $D(H)\Psi = 0$. There are however excellent reasons to ask whether H spinors satisfy $D(M^4)\Psi = 0$. If so, the M^8 spinors as octonionic spinors would correspond to off-mass shell states with mass squared values given by the roots $m^2 = r_n$ of P, which in general are complex. This conforms with an idea that the super-symplectic conformal weights have an imaginary part and conformal confinement forces total conformal weights to be integers. This would give rise to twistor holomorphy.

The outcome is an extremely simple proposal for the scattering amplitudes.

1. Vertices correspond to trilinears of Galois confined many-quark states as states of super symplectic algebra acting as isometries of the "world of classical worlds" (WCW).

2. Both M^8 and H quarks are on-shell with H momentum p_i and M^8 momenta x_i, x_{i+1}, $p_i = x_{i+1} - x_i$. Dirac operator $x^k\gamma_k$ restricted to a fixed helicity L, R appears as a vertex factor and has an interpretation as a residue of a pole from an on-mass-shell propagator D so that a correspondence with twistorial construction becomes obvious. M^8 quarks are effectively massless but off-shell but the helicity spinors μ and λ are independent unlike for massless particles.

3. The solutions of the octonionic Dirac operator $D(X^4)$ is expressible in terms of helicity spinors of given chirality and this gives two independent holomorphic factors: in the case of massless theories they would be complex conjugates and the other one must cancel by a spinor contraction.

4. The scattering amplitudes would be rational functions in accordance with the number theoretic vision.

5. In the TGD framework the construction of the scattering amplitudes for a single space-time surface is not enough. One must also understand what the WCW integration could mean at the level of scattering amplitudes based on cognitive representations. WCW integration would be naturally replaced by a summation over polynomials such that the corresponding 4-surface correspond at the level of H maxima of the Kähler function. Monic polynomials are highly suggestive.

   A connection with the p-adicization emerges via the identification of the p-adic prime as one of the ramified primes of P. Only (monic) polynomials having a common ramified prime are allowed in the sum. The counterpart of the vacuum functional $exp(-K)$ is naturally identified as the discriminant D of the extension associated with P and p-adic coupling constant evolution emerges from the identification of $exp(-K)$ with D. This leads to the proposal that discriminant equals the exponent of Kähler function. This forces the identification of p-adic prime as ramified prime and fixes coupling constant evolution to a high degree.

10.1.3 Scattering as recombination of quarks to Galois singlets

The view about scattering event is as follows.

1. External particles are Galois singlets consisting of off-mass shell massless quarks with mass squared values coming as roots of the polynomial P characterizing the interaction region. External particles are characterized by polynomials P_i satisfying $P_i(0) = 0$. P is identified as the functional composite of P_i since it inherits the roots (mass squared values) of the incoming particles. The TGD view about cognitive state function reduction [L90] allows only cyclic permutations of P_i in the superposition.

2. The scattering event is essentially a re-combination of incoming Galois singlets to new Galois singlets and quarks propagate freely: hence OZI rule generalizes. Also a connection with the dual resonance models emerges. Finiteness is manifest since the integration of virtual moments is restricted to a summation over a finite number of mass shells.

10.1.4 Comparison with the gauge theory picture

There are several differences between the standard twistor approach applied in gauge theories and the TGD based vision.

1. Vertices involve external H line and two internal N^8 lines. If it indeed does not make sense to speak about internal on-mass-shell quark lines in H, the BCFW construction using MHV amplitudes as building bricks and utilizing now also internal H quark lines, is not needed. One can of course ask, whether the M^8 quark lines could be regarded as analogs of lines connecting different MHV diagrams replaced with Galois singlets. It seems that also Grassmannians, positive Grassmannians, and amplituhedron are unnecessary.

2. The identification of the twistor amplitudes as Yangian invariants is extremely attractive. The proposal has been that the super-symplectic algebra (SSA) and the extended half-Kac Moody algebra of isometries acting as symmetries of WCW extend to Yangians and that the higher charges of Grassmannisans with conformal weight $h > 0$ correspond to multiparticle contributions to conserved charges with potential energy as a very familiar 2-particle example.

   Hence the TGD based construction should produce the scattering amplitudes as Yangian invariants. One cannot of course exclude the possibility that the integration over the "world of classical worlds", which is not considered in this article, could produce analogs BCFW diagrams and their Grassmannian representations.

   Since ordinary particles correspond basically to massless Galois singlests with mass resulting from p-adic thermodynamics, it is very natural to expect that the QFT limit of TGD is a massless QFT. At this limit, the twistor Grassmannian approach would be very natural.

3. Another difference relates to the M^4 conformal invariance of the twistor approach. M^4 conformal invariance is not a symmetry of TGD and the fact that quarks in M^8 are massive in the M^4 sense, reflects this. Massivation forces to extend the twistor holomorphy to both bi-spinors defining the twistor for massive momenta. By the properties of M^8 mass, the masses do not appear explicitly in the amplitudes so effectively the M^8 quarks are massless off-mass shell states. The Yangians would be therefore associated with various super-symplectic algebras rather than with the M^4 conformal group.

4. In the TGD framework, the loop corrections are predicted to vanish and the scattering amplitudes for a given space-time surface would therefore be rational functions in accordance with the number theoretic vision. The absence of logarithmic radiative corrections is not a problem: the coupling constant evolution would be discrete and defined by the hierarchy of extensions of rationals. Also this supports the view that Grassmannians are not needed.

10.1.5 What about unitarity?

Unitary, locality, and the failure to find the twistorial counterparts of non-planar Feynman diagrams are the basic problems of the twistor Grassmannian approach. Also the non-existence of twistor spaces for most Riemannian manifolds is a problem in GRT framework but in TGD the existence of twistor spaces for M^4 and CP_2 solves this problem. In the TGD framework, the replacement of point-like particles with 3-surfaces leads to the loss of locality at the fundamental level. The analogs of non-planar diagrams are eliminated since only cyclic permutations of P_i are allowed.

This leaves only the problem with unitarity. Unitary is essentially a non-relativistic concept and unitary time evolution is a completely ad hoc notion. My feeling is that this problem reflects a lack of some deep principle. In the spirit of Einstein's program for the geometrization of physics, I have proposed in [L91] a geometrization of the state space. Replace the unitary S-matrix with the Kähler metric of Hilbert space. If this metric is non-trivial it is by infinite dimension highly unique. The unitarity conditions are replaced with the conditions $g^{A\overline{B}}g^{\overline{B}C} = \delta^A_C$. The twistorial scattering amplitudes as zero energy states define the Kähler metric $g_{A\overline{B}}$ of quark state space, which is non-vanishing between the 3-D state spaces associated with the opposite boundaries of CD. $g^{A\overline{B}}$ could be constructed as the inverse of this metric.

Scattering probabilities are identified as products of covariant and contravariant matrix elements of the metric and are complex but real and imaginary parts are separately conserved.

The interpretation in terms of Fisher information is possible. Due to the infinite-D character of the state space, the Kähler geometry exists only if it has a maximal group of isometries and is a unique constant curvature geometry. Also the interpretation of this approach in zero energy ontology is discussed.

10.1.6 Objections and critical questions

Objections and critical questions are the best way to make progress by making the picture more precise, and allowing us to see which assumptions might not be final. For instance, twistor holomorphy, M^4 conformal symmetry number theoretically, and many other arguments strongly suggest that free quark spinors do not satisfy $D(H)\Psi = 0$ but $D(M^4)\Psi = 0$ and are therefore massless. The propagation of any massive particle along a light-like geodesic is however effectively massless and CP_2 type extremals have light-like M^4 projection so that one must leave this issue open.

10.1.7 Number theoretical generalizations of scattering amplitudes

Last section discusses the number theoretical generalizations of the scattering amplitudes. For an iterate of fixed P (say large number of gravitons), the roots of the iterate of P defined virtual mass squared values, approach to the Julia set of P. The construction of scattering amplitudes thus leads to chaos theory at the limit of an infinite number of identical particles.

The construction generalizes also to the surfaces defined by real analytic functions and the fermionic variant of Riemann zeta and elliptic functions are discussed as examples.

10.2 TGD related considerations and ideas

The goal is to generalize twistorial construction of scattering amplitudes in the simplest possible manner to the TGD framework. One of the key challenges is the twistorial description of massivation. In this section I summarize briefly the ideas of TGD which seem to be relevant for the construction of the twistor amplitudes.

10.2.1 The basic view about ZEO and causal diamonds

In the following are listed the ideas and concepts behind ZEO [K109] that seem to be rather stable.

1. General Coordinate Invarince (GCI) plays a crucial role in the construction of the Kähler geometry of WCW and implies holography, Bohr orbitology and zero energy ontology (ZEO) [L73, L104] [K109].

2. X^3 is more or less equivalent with Bohr orbit/preferred extremal $X^4(X^3)$. A finite failure of determinism is however possible and is discussed in [L107]. Preferred extremals would be simultaneous extremals of both volume action and Kähler action outside singularities and thus minimal surfaces analogous to soap films spanned by frames. Zero energy states are superpositions of $X^4(X^3)$. Quantum jump is consistent with causality of field equations.

3. Causal diamond (CD=$cd\times CP_2$) defined as intersection of future and past directed light cones (cds) plays the role of quantization volume, and is not arbitrarily chosen. CD determines momentum scale and discretization unit for momentum (see **Fig. 9.12 Fig. 9.13**).

4. The opposite light-like boundaries of CD correspond for fermions dual vacuums (bra and ket) annihilated by fermion annihilation - *resp.* creation operators. These vacuums are also time reversals of each other.

   The first guess is that zero energy states in the fermionic degrees of freedom correspond to pairs of this kind of states located at the opposite boundaries of CD. This seems to be the correct view in H. At the M^8 level the natural identification is in terms of states localized at points inside light-cones with opposite time directions. The slicing would be by mass shells (hyperboloids) at the level of M^8 and by CDs with same center point at the level of H.

5. Zeno effect can be understood if the states at either cone of CD do not change in "small" state function reductions (SSFRs). SSFRs are analogs of weak measurements (`https://cutt.ly/nURW3QE`). One could call this half-cone call as a passive half-cone. I have also talked about passive boundary.

   The time evolutions between SSFRs induce a delocalization in the moduli space of CDs. Passive boundary/half-cone of CD does not change. The active boundary/half-cone of CD changes in SSFRs and also the states at it change. Sequences of SSFRs replace the CD with a quantum superposition of CDs in the moduli space of CDs. SSFR localizes CD in the moduli space and corresponds to time measurement since the distance between CD tips corresponds to a natural time coordinate identifiable as geometric time. The size of the CD is bound to increase in a statistical sense: this corresponds to the arrow of geometric time.

6. There is no reason to assume that the same boundary of CD is always the active boundary. In "big" SFRs (BSFRs) their roles would indeed change so that the arrow of time would change. The outcome of BSFR is a superposition of space-time surfaces leading to the 3-surface in the final state. BSFR looks like deterministic time evolution leading to the final state [L62] as observed by Minev *et al* [L62].

7. h_{eff} hierarchy [K27, K28, K29, K30] implied by the number theoretic vision [L82, L83] makes possible quantum coherence in arbitrarily long length scales at the magnetic bodies (MBs) carrying $h_{eff} > h$ phases of ordinary matter. ZEO forces the quantum world to look classical for an observer with an opposite arrow of time. Therefore the question about the scale in which the quantum world transforms to classical, becomes obsolete.

8. Change of the arrow of time changes also the thermodynamic arrow of time. A lot of evidence for this in biology. Provides also a mechanism of self-organization [L69]: dissipation with reversed arrow of time looks like self-organization [L120].

10.2.2 Galois confinement

The notion of Galois confinement emerged originally in TGD inspired quantum biology [L120, L87, L93, L98]. Galois group for the extension of rationals determined by the polynomial defining the space-time surface $X^4 \subset M^8$ acts as a number theoretical symmetry group and therefore also as a physical symmetry group.

1. The idea that physical states are Galois singlets transforming trivially under the Galois group emerged first in quantum biology. TGD suggests that ordinary genetic code is accompanied by dark realizations at the level of magnetic body (MB) realized in terms of dark proton triplets at flux tubes parallel to DNA strands and as dark photon triplets ideal for communication and control [L87, L98, L97]. Galois confinement is analogous to color confinement and would guarantee that dark codons and even genes, and gene pairs of the DNA double strand behave as quantum coherent units.

2. The idea generalizes also to nuclear physics and suggests an interpretation for the findings claimed by Eric Reiter [L105] in terms of dark N-gamma rays analogous to BECs and forming Galois singlets. They would be emitted by N-nuclei - also Galois singlets - quantum coherently [L105]. Note that the findings of Reiter are not taken seriously because he makes certain unrealistic claims concerning quantum theory.

It seems that Galois confinement might define a notion, which is much more general than thought originally. To understand what is involved, it is best to proceed by making questions.

1. Why not also hadrons could be Galois singlets so that the somewhat mysterious color confinement would reduce to Galois confinement? This would require the reduction of the color group to its discrete subgroup acting as Galois group in cognitive representations. Could also nuclei be regarded as Galois confined states? I have indeed proposed that the protons of dark proton triplets are connected by color bonds [L72, L85, L34].

2. Could all bound states be Galois singlets? The formation of bound states is a poorly understood phenomenon in QFTs. Could number theoretical physics provide a universal mechanism for the formation of bound states? The elegance of this notion is that it makes the notion of bound state number theoretically universal, making sense also in the p-adic sectors of the adele.

3. Which symmetry groups could/should reduce to their discrete counterparts? TGD differs from standard in that Poincare symmetries and color symmetries are isometries of H and their action inside the space-time surface is not well-defined. At the level of M^8 octonionic automorphism group G_2 containing as its subgroup $SU(3)$ and quaternionic automorphism group $SO(3)$ acts in this way. Also super-symplectic transformations of $\delta M^4_\pm \times CP_2$ act at the level of H. In contrast to this, weak gauge transformations acting as holonomies act in the tangent space of H.

   One can argue that the symmetries of H and even of WCW should/could have some kind of reduction to a discrete subgroup acting at the level of X^4. The natural guess is that the group in question is Galois group acting on cognitive representation consisting of points (momenta) of M^8_c with coordinates, which are algebraic integers for the extension.

   Momenta as points of M^8_c would provide the fundamental representation of the Galois group. Galois singlet property would state that the sum of (in general complex) momenta is a rational integer invariant under Galois group. If it is a more general rational number, one would have fractionation of momentum and more generally charge fractionation. Hadrons, nuclei, atoms, molecules, Cooper pairs, etc.. would consist of particles with momenta, whose components are algebraic, possibly complex, integers.

   Also other quantum numbers, in particular color, could correspond to representations of the Galois group. In the case of angular momentum, Galois confinement would allow algebraic fractional angular momenta summing up to the usual half-odd integer valued spin.

4. Why Galois confinement would be needed? For particles in a box of size L, the momenta are integer valued as multiples of the basic unit $p_0 = \hbar n \times 2\pi/L$. Group transformations for the Cartan group are typically represented as exponential phase factors, which must be roots of unity for discrete groups. For rational valued momenta this fixes the allowed values of group parameters. In the case of plane waves, momentum quantization is implied by periodic boundary conditions.

   For algebraic integers, the conditions satisfied by rational momenta in general fail. Galois confinement for the momenta would however guarantee that they are integer valued and boundary conditions can be satisfied for the bound states.

10.2.3 No loops in TGD

There are several arguments suggesting that there is no counterpart for loops of quantum field theories (QFTs) in TGD. Purely rational scattering amplitudes are required by number theoretic vision but the logarithmic corrections from loops would spoil the number theoretic beauty.

Loops however give rise to coupling constant evolution, which is a physical fact. What could be the TGD counterpart of coupling constant evolution?

1. The number theoretic and p-adic coupling constant evolutions, which are discrete rather than continuous, look natural. The effective coupling constant should be renormalized because the allowed momentum exchanges depend on the roots of a polynomial P or at least on their number. If the p-adic prime p corresponds to a ramified prime of extension, the dependence of the effective coupling parameters on the extension of rationals defined by P implies dependence on the prime p characterizing the p-adic length scale. The emerging picture will be described in more detail in the next section.

   In the scatterin amplitudes, a power of coupling g identifiable as Kähler coupling constant g_K appears. Also the factors from Galois singlets appear as well as the states, which correspond to the super-symplectic representations.

It seems that for given external momenta a sum of several terms appear. If the number of momenta is small, a higher dimension of extension gives a larger number of diagrams and this could lead to number theoretic coupling constant evolution. If a given extension of rationals prefers some p-adic primes, mot naturally the ramified primes of the extension, number theoretic coupling constant evolution translates to a p-adic coupling constant evolution.

2. Does the integration over the WCW give Kähler coupling strength and various couplings or is Kähler coupling present at vertices from the beginning? The latter option would look natural. $M^8 - H$ duality strongly suggests that the exponent $exp(-K)$ of Kähler function K defining vacuum functional has a number theoretic counterpart. The unique counterpart would be the discriminant of the polynomial P and suggests that the value of $exp(-K)$ is equal to discriminant for maxima of K, which would naturally correspond to the space-time surface defining the cognitive representation.

10.2.4 Twistor lift of TGD

One could end up with the twistor lift of TGD from problems of the twistor Grassmannian approach originally due to Penrose [B63] and developed to a powerful computational tool in $\mathcal{N} = 4$ SYM [B30, B20, B39, B13]. For a very readable representation see [B32].

Twistor lift of TGD [L30, L80, L81] generalizes the ordinary twistor approach [L59, L60]. The 4-D masslessness implying problems in twistor approach is replaced with 8-D masslessness so that masses can be non-vanishing in 4-D sense. This gives hopes about massive twistorialization.

The basic recipe is simple: replace fields with surfaces. Twistors as field configurations are replaced with 6-D surfaces in the 12-D product $T(M^4) \times T(CP_2)$ of 6-D twistor spaces $T(M^4)$ and $T(CP_2)$ having the structure of S^2 bundle and analogous to twistor space $T(X^4)$. Bundle structure requires dimensional reduction. The induction of twistor structure allows to avoid the problems with the non-existence of twistor structure for arbitrary 4-geometry encountered in GRT.

he pleasant surprise was that the wistor space has the necessary Kähler structure only for M^4 and CP_2 [A54]: this had been discovered already when started to develop TGD! Since the Kähler structure is necessary for the twistor lift of TGD (the action principle is 6-D variant of Kähler action), TGD is unique. One outcome is length scale dependent cosmological constant Λ assignable to any system - even hadron - taking a central role in the theory [L45]. At long length scales Λ approaches zero and this solves the basic problem associated with it. At this limit action reduces to Kähler action, which for a long time was the proposal for the variational principle.

10.2.5 Yangian of supersymplectic algebra

The notion of Yangian for conformal symmetry group of Minkowski space plays a key role in the construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY as Yangian invariants. There are excellent reasons to expect that also in TGD the scattering amplitudes are Yangian invariants.

Yangian symmetry

The notion equivalent to that of Yangian [A77] [B26, B27, B45] was originally introduced by Faddeev and his group in the study of integrable systems. Yangians are Hopf algebras which can be assigned with Lie algebras as the deformations of their universal enveloping algebras.

The elegant but rather cryptic looking definition is in terms of the modification of the relations for generating elements [L10]. Besides ordinary product in the enveloping algebra there is co-product Δ, which maps the elements of the enveloping algebra to its tensor product with itself. One can visualize product and co-product is in terms of particle reactions. Particle annihilation is analogous to annihilation of two particle so single one and co-product is analogous to the decay of particle to two. Δ allows to construct higher generators of the algebra.

Lie-algebra can mean here ordinary finite-dimensional simple Lie algebra, Kac-Moody algebra or Virasoro algebra. In the case of SUSY it means conformal algebra of M^4- or rather its super counterpart. Witten, Nappi and Dolan have described the notion of Yangian for superconformal algebra in very elegant and concrete manner in the article *Yangian Symmetry in D=4 superconformal Yang-Mills theory* [B26]. Also Yangians for gauge groups are discussed.

In the general case Yangian resembles Kac-Moody algebra with the discrete index n being replaced with a continuous one. Discrete index poses conditions on the Lie group and its representation (adjoint representation in the case of $\mathcal{N}=4$ SUSY). One of the conditions conditions is that the tensor product $R\otimes R^*$ for representations involved contains adjoint representation only once. This condition is non-trivial. For $SU(n)$ these conditions are satisfied for any representation. In the case of $SU(2)$ the basic branching rule for the tensor product of representations implies that the condition is satisfied for the product of any representations.

Yangian algebra with a discrete basis is in many respects analogous to Kac-Moody algebra. Now however the generators are labelled by non-negative integers labeling the light-like incoming and outgoing momenta of scattering amplitude whereas in in the case of Kac-Moody algebra also negative values are allowed. Note that only the generators with non-negative conformal weight appear in the construction of states of Kac-Moody and Virasoro representations so that the extension to Yangian makes sense.

The generating elements are labelled by the generators of ordinary conformal transformations acting in M^4 and their duals acting in momentum space. These two sets of elements can be labelled by conformal weights $n=0$ and $n=1$ and and their mutual commutation relations are same as for Kac-Moody algebra. The commutators of $n=1$ generators with themselves are however something different for a non-vanishing deformation parameter h.

Serre's relations characterize the difference and involve the deformation parameter h. Under repeated commutations the generating elements generate infinite-dimensional symmetric algebra, the Yangian. For $h=0$ one obtains just one half of the Virasoro algebra or Kac-Moody algebra. The generators with $n>0$ are $n+1$-local in the sense that they involve $n+1$-forms of local generators assignable to the ordered set of incoming particles of the scattering amplitude. This non-locality generalizes the notion of local symmetry and is claimed to be powerful enough to fix the scattering amplitudes completely.

How to generalize Yangian symmetry in TGD framework?

As far as concrete calculations are considered, there is not much to say. It is however possible to keep discussion at general level and still say something interesting (as I hope!). The key question is whether it could be possible to generalize the proposed Yangian symmetry and geometric picture behind it to TGD framework.

1. The first thing to notice is that the Yangian symmetry of $\mathcal{N}=4$ SUSY in question is quite too limited since it allows only single representation of the gauge group and requires massless particles. One must allow all representations and massive particles so that the representation of symmetry algebra must involve states with different masses, in principle arbitrary spin and arbitrary internal quantum numbers. The candidates are obvious: Kac-Moody algebras [A5] and Virasoro algebras [A14] and their super counterparts. Yangians indeed exist for arbitrary super Lie algebras.

2. In the twistor approach conformal symmetries of M^4 are crucial. The isometries of H do not include scalings and inversions. The massless states of the super-symplectic representation would allow conformal invariance of M^4 as dynamical symmetries.

   There are however several alternatives.

   (a) The spectrum of the Dirac operator $D(H)$ contains only right-handed neutrino ν_R as a massless state and if M^4 Kähler structure is assumed it becomes tachyon.

   (b) The second option is that $D(M^4)$ annilates spinor modes. Dirac propagator would reduce to a delta function in CP_2 degrees of freedom. This option is favored by M^8-H duality and also by the associativity of the octonionic spinors implying that M^8 momenta reduce to M^4 momenta. This is actually achieved by a suitable choice of $M^4\subset M^8$ always.

   (c) If $D(M^4)$ contains no coupling to M^4 Kähler gauge potential $A(M^4)$, on-mass-shell quarks are massless and realize M^4 conformal invariance. The appearance of roots polynomials as mass squared values in quark propagators would realize number theoretic breaking of M^4 conformal invariance at the level scattering amplitudes and allow twistor holomorphy.

If $A(M^4)$ coupling is present, all quarks appear as spin doublets with positive and negative mass squared. M^4 conformal symmetry at the quark level is achieved only at long length scales when the spin term vanishes. The quark propagator in the scattering amplitudes would contain the coupling to $A(M^4)$ so that twistor holomorphy seems to be lost. M^4 gauge potential could explain small CP breaking, and one can imagine that the induced M^4 gauge potential appears only in the modified Dirac equation for the induced spinors.

3. The formal generalization looks surprisingly straightforward at the formal level. In zero energy ontology one replaces point like particles with partonic two-surfaces appearing at the ends of light-like orbits of wormhole throats located to the future and past light-like boundaries of causal diamond ($cd \times CP_2$ or briefly CD). Here CD is defined as the intersection of future and past directed light-cones.

   The polygon with light-like momenta would be naturally replaced with a polygon with more general momenta in zero energy ontology and having partonic surfaces as its vertices. Non-point-likeness forces to replace the finite-dimensional super Lie-algebra with infinite-dimensional Kac-Moody algebras and corresponding super-Virasoro algebras assignable to partonic 2-surfaces.

4. This description replaces disjoint holomorphic surfaces in twistor space with partonic 2-surfaces at the boundaries of $cd \times CP_2$ so that there seems to be a close analogy with Cachazo-Svrcek-Witten picture. These surfaces are connected by either light-like orbits of partonic 2-surface or space-like 3-surfaces at the ends of CD so that one indeed obtains the analog of polygon.

What does this then mean concretely?

1. At least it means that ordinary Super Kac-Moody and Super Virasoro algebras associated with isometries of $M^4 \times CP_2$ annihilating the scattering amplitudes must be extended to a co-algebras with a non-trivial deformation parameter. Kac-Moody group is thus the product of Poincare and color groups.

   This algebra acts as deformations of the light-like 3-surfaces representing the light-like orbits of particles which are extremals of Chern-Simon action with the constraint that weak form of electric-magnetic duality holds true. I know so little about the mathematical side that I cannot tell whether the condition that the product of the representations of Super-Kac-Moody and Super-Virasoro algebras contains adjoint representation only once, holds true in this case. In any case, it would allow all representations of finite-dimensional Lie group in vertices whereas $\mathcal{N} = 4$ SUSY would allow only the adjoint.

2. Besides this ordinary kind of Kac-Moody algebra there is the analog of Super-Kac-Moody algebra associated with the light-cone boundary which is metrically 3-dimensional. The finite-dimensional Lie group is in this case replaced with infinite-dimensional group of symplectomorphisms of $\delta M^4_{+/-}$ made local with respect to the internal coordinates of the partonic 2-surface. This picture also justifies p-adic thermodynamics applied to either symplectic or isometry Super-Virasoro and giving thermal contribution to the vacuum conformal and thus to mass squared.

3. The construction of TGD leads also to other super-conformal algebras and the natural guess is that the Yangians of all these algebras annihilate the scattering amplitudes.

4. Obviously, already the starting point symmetries look formidable but they still act on single partonic surface only. The discrete Yangian associated with this algebra associated with the closed polygon defined by the incoming momenta and the negatives of the outgoing momenta acts in multi-local manner on scattering amplitudes. It might make sense to speak about polygons defined also by other conserved quantum numbers so that one would have generalized light-like curves in the sense that state are massless in 8-D sense.

Could Yangian symmetry provide a new view about conserved quantum numbers?

The Yangian algebra has some properties which suggest a new kind of description for bound states. The Cartan algebra generators of $n = 0$ and $n = 1$ levels of Yangian algebra commute. Since the co-product Δ maps $n = 0$ generators to $n = 1$ generators and these in turn to generators with high value of n, it seems that they commute also with $n \geq 1$ generators. This applies to four-momentum, color isospin and color hyper charge, and also to the Virasoro generator L_0 acting on Kac-Moody algebra of isometries and defining mass squared operator.

Could one identify total four momentum and Cartan algebra quantum numbers as sum of contributions from various levels? If so, the four momentum and mass squared would involve besides the local term assignable to wormhole throats also n-local contributions. The interpretation in terms of n-parton bound states would be extremely attractive. n-local contribution would involve interaction energy. For instance, string like object would correspond to $n = 1$ level and give $n = 2$-local contribution to the momentum. For baryonic valence quarks one would have 3-local contribution corresponding to $n = 2$ level. The Yangian view about quantum numbers could give a rigorous formulation for the idea that massive particles are bound states of massless particles.

How could the Yangian structure of the super-symplectic algebra emerge?

The isometries of WCW should generalize conformal symmetries of string models and supersymplectic transformations of the light-like boundary of CD are a highly natural candidate in this respect.

1. The crucial observation is that the 3-D light-cone boundary δM^4_+ has metric, which is effectively 2-D. Also the light-like 3-surfaces $X^3_L \subset X^4$ at which the Minkowskian signature of the induced metric changes to Euclidian are metrically 2-D. This gives an extended conformal invariance in both cases with complex coordinate z of the transversal cross section and radial light-coordinate r replacing z as coordinate of string world sheet. Dimensions $D = 4$ for X^4 and M^4 are therefore unique.

2. $\delta M^4_+ \times CP_2$ allows the group symplectic transformations of $S^2 \times CP_2$ made local with respect to the light-like radial coordinate r. The proposal is that the symplectic transformations define isometries of WCW [K24].

3. To the light-like partonic orbits one can assign Kac-Moody symmetries assignable to $M^4 \times CP_2$ isometries with additional light-like coordinate. They could correspond to Kac-Moody symmetries of string models assignable to elementary particles.

   The preferred extremal property raises the question whether the symplectic and generalized Kac-Moody symmetries are actually equivalent. The reason is that isometries are the only normal subgroup of symplectic transformations so that the remaining generators would naturally annihilate the physical states and act as gauge transformations. Classically the gauge conditions would state that the Noether charges vanish: this would be one manner to express preferred extremal property.

Consider next the general structure of the super-symplectic algebra (SSA).

1. The SSA and the TGD analogs of Kac-Moody algebras assignable to light-like partonic 3-surfaces have the property that the conformal weights assigned to the light-like coordinate r are non-negative integers. One can say that they are analogs of "half"-Kac-Moody algebras. Same holds true for the Yangian algebras, which suggests that these algebras could extend to Yangian algebras.

2. SCA (and also the Kac-Moody analogs) has fractal hierarchies of sub-algebras isomorphic to the algebra SSA itself at the lowest level. The conformal weights of sub-algebra SSA_n an n-multiplets of those of SSA: one obtains hierarchies of sub-algebras $SSA \supset SCA_{n_1} \supset SSA_{n_2 n_1}, ...$

3. This leads to the proposal that there is a hierarchy of analogs of "gauge symmetry" breakings. For the maximal "gauge symmetry", the entire SSA annihilates the states and classical

Noether charges vanish. For SSA_n, only SSA_n and the commutator $[SSA_n, SSA]$ annihilate the physical states.

One can ask whether these hierarchies could correspond to the hierarchies of extensions for rationals defined by the composition of polynomials defining 4-surfaces in M^8 and by M^8-H duality in H.

Cognitive representations play a key role and correspond to many quarks states.

1. Cognitive representations consist of the points of $X^4 \subset M^8$ with $M^4 \subset M^8$ coordinates belonging to an extension of rationals defined by a polynomial P defining X^4. It has become clear that here only the mass shells corresponding to the roots r_n of P need to be considered and that only algebraic integers defining the components of M^4 momenta need to be considered.

2. Cognitive representations consist of only those points which are "active", i.e. contain quark or antiquark. $M^8 - H$ duality maps the cognitive representations to H. The points of a given mass shell to the light-like boundary of CD. Momentum p as a point of $M^4 \subset M^8$ is mapped to a geodesic line starting from the center of CD and yields the image point as its intersection with the boundary of CD. The momenta at a given mass shell are actually mapped to the boundaries of all CDs forming a Russian doll hierarchy with common center points.

3. The cognitive representation codes for the physical states in quark degrees of freedom and should reflect themselves in the properties of the SSA state construction. The natural condition is that the Hamiltonians of SSA generate transformations leaving invariant the image points of cognitive representation at the boundaries of CD. This requires that the Hamiltonians vanish at the points of the cognitive representation. This is achieved if the Hamiltonians are obtained by multiplying the usual Hamiltonians, which can be chosen to define irreducible representations of $SU(2) \times SU(3)$, by a Hamiltonian H_{cogn}, which vanishes at the points of the cognitive representation.

   The condition that also the super-generators vanish at the points of cognitive representation implies that also the corresponding Hamiltonian vector field j vanishes so that at the points of cognitive representation all Hamiltonians vanish and are extrema. One would have a modification of the hierarchy of SSA_n but the gauge conditions would remain as such. These conditions could be regarded as a realization of quantum criticality.

4. The cognitive representation defined by the multi-quark states in M^8 would modify the SSA in H by multiplying its Hamiltonians with H_{cogn}. The level of WCW the role of the subalgebra SSA_{cogn} defined by cognitive representation would be similar to the algebra of isotropy group $SO(3)$ of particle momentum as a subgroup of $SO(3,1)$.

   This suggests that the induction procedure generating the irreducible representations for finite-dimensional Lie groups generalizes. The representations of $SO(3)$ have as an analog the representations of SSA_{cogn}. From these representations one would obtain by general symplectic transformations states analogous to the Lorentz boosts of a particle at rest. Note that for cognitive representations the Galois group acts non-trivially but one would have Galois singlet. One could have it in geometric sense so that the momenta would simply add up as vectors or in quantum sense as a many-quark state, with quarks at different points of the mass shell or at different mass shells.

How could one understand the generalization of the duality between momenta and area momenta?

1. The duality between ordinary momentum space and area momentum space means that dual conformal transformations act on area momenta x_i as symmetries of the scattering amplitudes. At the level of ordinary momenta this symmetry extends conformal symmetry algebra to a Yangian algebra.

2. Is this possible in the case of M^8-H duality? Does SSA realized at CD boundaries have a counterpart at the $M^4\subset H$ mass shells? The counterparts of SSA transformations in M^8 must map the mass shells to itself and leave the points of the cognitive representation invariant. In the interior of $X^4\subset M^8$ they would induce a deformation of X^4 consistent with the assumption that X^4 is obtained as a local element of $CP_2=SU(3)/U(2)$, i.e. the deformation is induced by $SU(3)$ element $g(x)$ acting as octonionic automorphism such that $U(2)\subset SU(3)$ leaves the image point invariant. This would guarantee M^8-H duality.

   This deformation at the mass shell would induce in $X^4\subset H$ an action having interpretation in terms of a local $SU(3)$ (CP_2) transformation, or possibly an symplectic transformation of CP_2 local with respect to light-cone. At the level of H one has group symplectic transformations of $S^2\times CP_2$ expressible in terms of Hamiltonian in irreps of $SU(3)$.

3. Could the local $SU(3)/U(2)=CP_2$ transformations be representable as symplectic transformations as the duality would suggest? Does this somehow relate to the facts that both CP_2 and its twistor space $SU(3)/U(1)\times U(1)$ have Kähler structure [A54] and therefore also symplectic structure: this in fact makes CP_2 and M^4 completely unique.

4. What about the M^8 counterparts S^2 Hamiltonians. Could they somehow correspond to quaternionic automorphism group $SO(3)$. Could $SO(3)$ correspond to the allowed symplectic (contact) transformations for the mass shell itself whereas $SU(3)$ would act in the interior of $X^4\subset M^8$?

The dual conformal transformations induce bilocal transformations in the ordinary Minkowski space and this leads to the notion of Yangian, which also implies higher multi-local actions. Why would be the physical origin of this multilocality?

1. Quantum group structure is involved and bi-local elements should correspond to tensor products $f_{abc}T^b\otimes T_c$ of Lie-algebra generators. This generalizes to higher multilocal states. Galois confinement is a multilocal phenomenon in M^8. M^8-H duality maps this multilocality to H. The simplest bi-local state is the quark-antiquark pair with total momentum which is an ordinary integer (necessarily non-tachyonic even if the roots r_n had negative real parts). Leptons would be tri-local states of quarks in CP_2 scale.

   The multilocality of the Galois confined many quark states in M^8 strongly suggests that the total charges include, besides the 1-local contributions, there are also multilocal contributions to Noether charges.

2. Galois confinement should force the multilocality of the symmetry generators. In particular, since the total momenta of quarks sum up to an ordinary integer, one cannot perform Lorentz transformations for them independently but one must transform several momenta simultaneously in order to guarantee that the total momentum changes in such a way that Galois confinement condition is satisfied.

   The Galois group acts also on spinors which can have number theoretic analogs of spinor space assignable to algebraic extensions as linear spaces and providing a finite-D number theoretic counterpart for WCW spinors. Therefore the generators of Lorentz transformations must contain bi-local and also n-local terms. Same applies to scalings and conformal transformations and in fact to all other symmetries.

3. In the case of energy, these multilocal contributions could have an interpretation as binding energy or potential energy depending on the distance between the image points of different momenta at the boundary of CD. The question is how these multilocal contributions would emerge in H for the super-symplectic algebra having a representation as classical Noether charges and fermionic Noether charges.

4. The notion of gravitational coupling constant suggests strongly that conserved quantities have besides the local contribution also bilocal contribution for which gravitional Planck constant defines unit of quantization. A possible identification is as a bilocal Yangian contribution.

In $\mathcal{N} = 4$ SUSY, scattering amplitudes are invariants of the Yangian defined by conformal transformations of M^4 and its dual acting in the space of area momenta. Since SSA is proposed to act as isometries of the "world of classical worlds" (WCW), also zero energy states having interpretation as scattering amplitudes should be Yangian invariants.

10.2.6 $M^8 - H$ duality and twistorialization of scattering amplitudes

The precise formulation of twistor amplitudes has remained a challenge although I have considered several proposals in this direction. The progress made in the understanding of the details of $M^8 - H$ duality [L104] motivate the attempts to find more explicit formulation for the scattering amplitudes. The following tries to give a brief overall vision.

1. In its recent form $M^8 - H$ duality predicts the twistor spaces of M^4 and CP_2 and their map to each other having interpretation in terms of 6-D twistor spaces of space-time surfaces as 6-surfaces in the product of the twistor spaces of M^4 and CP_2 replacing space-time surfaces with their twistor spaces in the twistor lift of TGD [L104].

2. Momentum twistors and space-time twistors are related by M^8-duality. M^8 momenta are identified as area momenta different from M^4-momenta in H. The notion of area momentum makes sense only for planar diagrams (it is not clear to me whether the embedding of diagrams genus g topology could allow a definition of area momentum).

3. In the usual twistor Grassmann approach to massless QFTs, the momenta of internal lines are massless and thus on-mass-shell but complex. The simplest option conforming is that both area momenta x_i and H-momenta p_i are on-mass-shell. Area momenta are indeed in general complex as algebraic integers. For a given polynomial P area mass squared spectrum of quarks is fixed as - in general complex - roots of polynomial P.

4. What looks first like a problem is that H momenta have naturally integer valued components (periodic boundary conditions) and mass squared is integer using a suitable unit determined by the p-adic length L_p for the CD. However, at the M^8 side the momenta have components which are algebraic integers in the extension determined by the polynomial P.

   A natural solution of the problem is provided by Galois confinement requiring that momentum components of confined states, which are Galois singlets, are integer valued rather than algebraic integers. This provides a universal mechanism for the formation of bound states. This allows also to have identical spectra for area momenta and ordinary momenta.

   In this picture, the particle would be a Galois singlet formed as a composite of quarks. This notion of a particle is extremely general as compared to the QFT view about elementary particles. The external lines of twistor diagrams carrying H quantum numbers would correspond to states in the representations of super-symplectic algebra (SSA) with Yangian structure.

5. The second quantization for quark fields of H means an enormous simplification. One avoids all problems related to quantization in a curved background. Here an essential role is played by the Kähler structure of M^4 forced by the twistor lift. The generators of supersymplectic algebra and generalized Kac-Moody algebras can be expressed in terms of quark oscillator operators.

6. For given H momenta, the momentum transfers are fixed by $p_i = x_{i-1} - x_i$. The twistor sphere S^2 characterizes the momentum directions. Momentum plus S^2 point s characterized by helicity spinor, defines a point in the twistor space and the geometric interpretation for s is that it characterizes the direction of spin quantization axis.

   The direction of quantization axies is defined only apart from a sign and for spin 1/2 particles the interpretation is as the sign of the spin projection. For massless states the spin axis is parallel to momentum.

7. Galois confinement is crucial. The conditions allow integer valued H momenta only if the area momenta correspond to Galois bound states of quarks. Entire composite of quarks at

the same mass shell propagates as particle with total momentum which has integer components. By duality one can assign to the momentum p_i quantum numbers in supersymplectic representation.

Clearly the notion of a particle as a Galois singlet is very general and corresponds to a multilocal state in both M^8 and H leading also to the notion of Yangian. In H, a particle is a state of a super-symplectic representation. At the level of M^8 it is a Galois confined state. These states correspond to each other.

The basic ideas related to the construction of scattering amplitudes are as follows.

1. M^8-H duality remains as such. M^8-H duality maps. Total area momenta X_i of Galois confined states to points at the boundary of corresponding CD with size determined by the total area momentum by M^8-H duality.
2. Basic vertices for Galois confined states involve many-quark Galois singlet in H with total momentum P_i and 2 many-quark Galois singlets in M^8 involving area momenta X_i and X_{i+1} satisfying $P_i = X_{i+1} - X_i$. The scattering amplitude reduces to quark level and one can say that quark lines connect different mass shells of $X^4 \subset M^8$.
3. 3-vertices are between two M^8 Galois singlets and super-symplectic Galois singlet in H at different M^8 mass shells and lines connecting them carrying momenta calculated at the level of H. Quarks in Galois singlets have collinear rational parts which are analogous to SUSY where monomials of theta parameters assignable to higher spin states are analogous to collinear many-fermion states.

10.3 Are holomorphic twistor amplitudes for massive particles possible in TGD?

Massive particles are believed to make twistorialization impossible. For instance, for a scalar field theory with Yukawa coupling to fermions, the part of scattering amplitude involving vertex with Yukawa coupling plus scalar propagator gives $g < 12 > \times 1/(p_1-p_2)^2$. For massless particles, one has $(p_1-p_2)^2 =< pq > [pq]$ and the expression reduces to $g/ < pq >$. This is essential for the holomorphy in twistor components in turn reflecting conformal invariance.

In MHV construction the MHV amplitudes with 2 negative helicities are used as building bricks of twistorial representations of more complex planar tree amplitudes and loop amplitudes connecting them with off-mass-shell lines involving propagators. The obvious question is whether this construction could be generalized.

The simplest MHV diagrams would be replaced with diagrams assignable to single CD and involving only on-mass-shell area momenta in M^8 and on-mass-shell area momenta in H as external particles. One would take several diagrams of this kind and connect them by a line carrying off-mass-shell M^8 momentum and quantum numbers of a state in SSA representation. In a given vertex involving this kind of virtual H-line, the on-mass-shell fermion momenta would be replaced by two 2 on-mass-shell area momenta and off-mass-shell momentum of the scalar particle would correspond to M^8 momentum.

The intuitive idea is that somehow 8-D massless at the level of H solves the problem but it is not at all clear whether it is possible to obtain twistor holomorphy somehow. One hint comes from the fact that twistors associated with massive particles involve two independent helicity spinors μ and λ? Could one have holomorphy with respect to both? A further hint comes from the observation that at the level of H tachyonic right-handed neutrino makes possible the construction of massless states. A further hint comes from Galois confinement: could the external particles be Galois confined states and could the propagating particles be quarks in M^8 having complex masses coming as roots of the polynomial P?

10.3.1 Is it possible to have twistor holomorphy for massive scalar and fermions?

Consider first the simple example of massive fermions and a massive scalar field. Assume that fermions are on-mass-shell with masses m_1 and m_2 and scalar off-mass-shell with mass m.

1. Assume Dirac spinors expressible in terms of left and right handed components. For massive scalar particle, the propagator factor reads as $(p_1 - p_2)^2 - m^2 = m_1^2 + m_2^2 - m^2 - 2(p_1 \cdot p_2)$.

2. The completeness relation for spinor modes reads in massive case as $p^k\gamma_k + m = O(p)$, $O(p) = |p\rangle\,[p| + |p[\,\langle p|$

   One can express $O(p)$ as $p^k\gamma_k = O(p) - m$. One obtains for Dirac spinor with left and right handed parts

   $$2p_1 \cdot p_2 = \frac{1}{4}Tr[(O(p_1) - m)(O(p_2) - m)] = -m^2 - \frac{1}{4}Tr[O(p_1)O(p_2)] \ .$$

   For

   $$m_1^2 + m_2^2 = 2m^2 \ ,$$

   the propagator factor reduces to $1/(Tr(O(p)O(q)) =< pq > [pq]$ as if the particles were massless. The part of the amplitude considered would reduce to $g < pq >$.

3. Could the masses for the generalized twistor diagram satisfy a generalization of the condition $m_1^2 + m_2^2 = 2m^2$ guaranteeing the holomorphy with respect to $< .. >$ or $[..]$? The prediction for spinors would be an effective prediction of massless QFT. Note that this result is also true when the masses are identical. This in turn might relate to SUSY. The additivity of mass squared values might in turn relate to 2-D conformal invariance in which mass squared operator is scaling generator and mass squared values are conformal weights. 2-D conformal invariance would generalize to its 4-D counterpart.

   Could this picture generalize to TGD in such a way that external on mass states correspond to states constructed in H area momenta are off-mass-shell? It is easy to see that this generalization does not work as such.

10.3.2 Scattering amplitudes in a picture based on $M^8 - H$ duality

The basic assumptions are inspired by $M^8 - H$ duality, ZEO, and geometric view about helicity spinors.

The first guess is that area momenta x_i are assignable to M^8 quarks and are at complex mass shells $m^2 = r_n$. x_i algebraic integers in the extension determined by a polynomial P. Galois confinement implies that the quark momenta associated with mass shells belong to quark composites forming Galois singlets and have a total momentum, which is integer valued with respect to the p-adic mass scale assignable to the mass shell. Also mass squared values would be integers. For general Galois singlets the momenta are assignable to several mass shells $m^2 = r_n$ and thus multi-local entities in M^8, which suggests possible origin of the Yangian symmetry. The mass shells are mapped to the boundaries of corresponding CD in H by $M^8 - H$ duality mapping p-adic mass scale m to its inverse defining p-adic length scale $L = \hbar_{eff}/m$ implying multi-locality in H. CDs form a Russian doll-like structure.Assume that the incoming momenta p_i are H assignable to supersymplectic representations constructed from spinor harmonics in H for a second quantized quark field. $M^8 - H$ duality suggests that the momentum and mass squared spectra are identical at M^8 and H sides. This conforms with Galois confinement at M^8 side. Particles would be Galois confined multi-quark states. Assume that twistors and momentum twistors have a geometric interpretation so that helicity spinors do not represent fermions but points in the CP_1 fiber of CP_3 as a bundle and the states with given spin correspond to wave functions in CP_2 having also half-integer spins. Twistor amplitudes would be constructed as contractions of these wave functions with the scattering amplitudes that the basic scattering amplitude would be independent of spin. In this framework, the many-quark states constructed by elements of Clifford algebra would be analogous to components of a super-field. By Galois confinement, the rational parts of quark momenta would be collinear, which conforms with the basic idea of SUSY that n-monomials of theta parameters are analogous to states of p collinear fermions. The spin of a given state would correspond to a product of spin 1/2 sherical harmonics in the space defined by the helicity spinor. A huge generalization of the notion of particle would be in question. Particle would correspond to an arbitrary Galois singlet assignable to single CD. This would conform with the WCW picture in which physical states of the Universe correspond to WCW spinor fields identified as zero energy states. Vertices would correspond to the states of Yangian supersymplectic representation identifiable as mode of WCW spinor field and representing general fermionic state analogous to a component of super field but without Majorana condition. In the standard model, all couplings except the coupling of Higgs to itself and to fermions respect helicity conservation. Assume that this is true also in TGD so that one can decompose quark spinors to left and right handed parts and that they can be described by spin wave functions in the fiber of twistor space corresponding to the momentum of the quark. Note however that the helicity twistors would be purely geometric quantities rather than representing spinor basis of a fermion. At the level of the twistor space of H, spin states would be described by partial waves at the twistor sphere. At the level of M^8 twistor space, a completely geometric description as a point of twistor space characterizing momentum and spin quantization axis and the sign of the spin 1/2 projection is possible. Helicity spinors μ and $\tilde{\lambda}$ would characterize the direction of the spin quantization axis as a point twistor sphere S^2. This conforms with the fact that for massive particles the direction of helicity spinor is not unique since the spin μ is determined only apart from a spinor proportional to λ. For massless particles the direction of the quantization axis is unique. Since only quarks with spin 1/2 are fundamental fermions, the twistor sphere with a fixed radius is enough. This interpretation is similar to the interpretation of the twistor sphere of $SU(3)/U(1) \times U(1)$ as a characterizer of the color quantization axes. For many-quark states a common quantization axis would force the spins to be parallel or antiparallel. The sum of spins associated with different momenta as different points of twistor space would be the sum of these spins.

The special twistorial role of quarks as spin 1/2 particles supports the idea that the construction of scattering amplitudes should be reduced to quark level although the physical states are Galois singlets. The situation would be very similar to that in QCD, where the challenge is to understand how the scattering amplitudes between hadrons are constructible in terms of scattering amplitudes for quarks and gluons. The basic problem in QCD is that a mechanism for the formation of bound states is missing: in TGD it is provided by Galois confinement.

The basic assumption is therefore that the quarks in M^8 are on-mass-shell states with $m^2 = r_n$. If Galois singlets were regarded as fundamental objects, one would encounter problems with the description of spin degrees of freedom. Situation is essentially the same as in hadron physics.

One can speak about Galois singlet states as a generalization of super-field but without Majorana conditions with oscillator operator monomials replacing the components of superfield: Galois singlets having quark momenta with parallel rational components would in this sense propagate linearly. Each quark Dirac operator $p^k\gamma_k$ is added to the vertex and is expressible in terms of a pair of holomorphic quantities $< .. >$ and [..] which are independent for massive quarks.

10.3.3 Twistor amplitudes using only mass shell M^8 momenta as internal lines

The simplest proposal for the twistor amplitudes assignable to single 4-surface assumes that the physical particles correspond to Galois singlets with integer valued momentum components p_i and integer valued mass squared spectrum. The components of quark momenta in M^8 would be algebraic integers.

$M^8 - H$ duality requires that physical states in M^8 and H correspond to each other and have the same mass and momentum spectrum. A stronger form of $M^8 - H$ duality would force the identification of the quark momenta in M^8 and H. Quark momenta would be virtual momenta. If the coupling to M^4 Kähler potential is not present, the twistor holomorphy is achieved if spinor modes satisfy $D(M^4)\Psi = 0$.

What could be the basic assumptions?

The following summarize the assumptions, which look plausible.

1. All quark states in both H and M^8 are on-mass-shell states with momenta which are algebraic integers in the extensions determined by polynomial P determining the quark mass shells $m^2 = r_n$ as its roots. Momenta for Galois singlets could also be rationals but periodic boundary conditions allow only integers.

   The physical states are Galois singlets with integer valued momenta in a given p-adic length scale. Mass squared values are integers and one obtains a stringy mass squared spectrum. By $M^8 - H$ duality the spectra at M^8 and H sides are identical.

2. The analog of the idea that the scattering amplitudes are poles of residue integral in momentum space is adopted. This means that in M^8 the purely algebraic 4-D quark Dirac operators $D(M^4)$, rather than propagators as in Feynman diagrams, act on the vertex defined by the trilinear of 3 Galois singlets (particles do not propagate in momentum space as they do in x-space!). The Galois singlets have an interpretation as representations of super-symplectic algebra.

   The Galois singlet with total momentum $P_i = \sum p_{i,k}$ corresponds to H-state and the two other Galois singlets corresponds to states with area momenta X_i, X_{i+1} having similar decompositions $X_k =\sim x_{i,r}$ in terms of in general complex algebraic integer valued area momenta x_i. The complex on-mass-shell area momenta are analogous to the complex on-mass- shell light-like virtual momenta in the twistor Grassmann approach.

3. The total momentum of the vertex is conserved and gives a constraint on the quark momenta associated with the 3 states. In each vertex one has sum over all possible quark momenta consistent with the Galois singlet property and the structure of the state. Momentum conservation at vertex does not make sense at quark level since fermion number conservation would fail unless one introduces fundamental bosons.

   Momentum conservation constraints $P_i = X_{i+1} - X_i$, which completely fixes the momentum exchanges as $2X_i \cdot X_j = P_i^2 - X_{i+1}^2 - X_i^2 - 2(X_i - X_j)^2$. Momentum conservation implies in ZEO that one can see scattering diagrams as polygons having momenta at mass shells at the half-light-cones of M^8.

4. An essential constraint is that the rational parts of the area momenta x_i are parallel to each other. This gives rise to an analogy with supersymmetry in which one could regard the higher components of the super field as parallelly propagating Majorana fermions.

5. The propagator lines correspond in M^8 to vertex factors with the analog of $D = x_i^k \gamma_k$ acting on Galois singlet i. This would mean that one has a residue of the Feynman propagator. By adding a multiplicative factor m^2, one could equally well use Feynman propagator $1/D = D/m^2$, where $m^2 = r_n$ is quark mass squared. The number of diagrams is limited by the number of roots and only the number of Galois singlets poses a limit to the summation if one considers only amplitudes for a single surface X^4.

   In principle all pairs of Galois singlets in M^8 with a non-vanishing trilinear overlap with a given Galois singlet in H are allowed in the vertex. Note that same Galois singlets can contain quarks assignable to different quark mass shells $m^2 = r_n$.

6. The details of the algebraic extension are not visible in the properties of Galois singlets as analogs of hadrons. The details of algebraic extension are however visible in the details of quark propagators and give rise to a number theoretic coupling constant evolution as will be found. Also the increase of the dimension of extension with the degree of P implies that the number of contributing diagrams increases.

   In principle, also roots r_n with negative rational parts are possible and one cannot exclude tachyonic states. From tachyonic states one can form non-tachyonic ones by requiring that the 3-momenta sum up to zero.

7. The big difference with respect to standard massive QFTs is that although the states are massive, they propagate with well-defined helicities. There is therefore a doubling of helicity spinors appearing as L-R degeneration. The division to positive and negative helicities corresponds to the presence of quarks and antiquarks.

8. It seems that quarks and antiquarks can correspond to the same CD and to the same diagram of the proposed kind. For a single space-time surface BCFW construction does not make sense since it would require an off-mass-shell H particle. One must however notice that the quark propagators bring in mind the $1/P^2$ lines connecting BCFW sub-diagrams and Galois singlets bring in mind the MHV diagrams.

   Can one construct Galois singlets from both quarks and antiquarks? It would seem that in this case the scattering amplitudes involve products of holomorphic and antiholomorphic monomials of the twistor variables. This option looks intuitively more plausible.

A possible solution of the mass problem

The basic problem of the twistor approach is that physical particles are not massless. The intuitive TGD based proposal has been that since quark spinors are massless in H, the masslessness in the 8-D sense could somehow solve the problems caused by the massivation in the construction of twistor scattering amplitudes.

1. The first key observation stimulated by the recent findings about right-handed neutrino candidate [L100] was that although neutrinos are massive, their right-handed component has not been observed. This leads to a proposal that in H quarks should propagate with well-defined chiralities so that only the square of Dirac equation $D^2(H)\Psi = 0$ is satisfied.

2. At the level of M^8 the octonionic M^4 quark spinor reducing to a quaternionic spinor corresponds to H spinors. A spinor with a given chirality can be identified as a helicity spinor λ_{dota} and is annihilated by the operator $p^{a\dot{b}} = \mu^a\lambda^{\dot{a}}$. This makes sense by the fact that in the TGD Universe quarks are the only fundamental particles implying that all other particles, including elementary particles, emerge as their many particle states as Galois singlets.

   The M^8 counterpart of the 8-D massless condition in H is the restriction of the quark momenta to mass shells $m^2 = r_n$ determined as roots of P. The M^8 counterpart of Dirac equation in H is octonionic Dirac equation, which is algebraic. The solution is a helicity spinor $\tilde{\lambda}$ associated with the massive momentum p.

What about tachyons?

Polynomials P allow also roots r_n, which are negative and correspond to tachyonic mass shells. Should one restrict the roots inside the future light-cone? Should one require that the mass squared values of the masses of Galois singlets are non-negative integers? In principle, one can have integer valued momenta with tachyonic mass squared. The sum of this kind of momenta however gives always a non-tachyonic state if the energies are of the same sign as they are for a given half-light-cone.

1. M^4 Kähler structure implies that covariantly constant right- handed neutrino in CP_2 is a tachyon [L100]. This gives rise to the highly desired tachyon required by p-adic mass calculations [K52, K21]: with it the scale of mass spectrum would be huge and given by CP_2 mass. Tachyonic property is not consistent with the unitarity and ν_R cannot appear as a free particle.

2. Situation remains the same if the right-handed neutrino spinor mode is a good approximation for a Galois and color singlet of 3 quarks assignable to the same wormhole throat in H. ν_R as Galois singlet with tachyonic mass can be understood if tachyonic mass squared values are allowed for quarks.

   Could all quark masses could be tachyonic? Could this explain quark confinement? By generalizing slightly, also complex mass squared values for quarks could be seen as tachyonic so that Galois confinement would be essentially quark confinement.

3. A long-standing question has been whether ν_R could generate $N = 2$ SUSY. It seems that the tachyon property does not allow the analog of ordinary SUSY. States without ν_R would have huge masses of order CP_2 mass. One can also say that $calN = 2$ SUSY is broken in CP_2 scale.

Is the proposed picture consistent with coupling constant evolution?

Can one understand the discrete number theoretic coupling constant evolution in the proposed framework? As the number of roots of P increases, the number of scattering diagrams with N external particles with fixed momenta p_i increases since the number of Galois confined states characterized by mass shells $m_i^2 = n_i$ increases.

The number of diagrams contributing to the scattering increases and it becomes possible to speak about number theoretical coupling constant evolution. Otherwise the dependence on polynomials P is rather weak and brings in mind logarithmic coupling constant evolution replaced in TGD by discrete p-adic length scale evolution.

How does this relate to the p-adic coupling constant evolution and p-adic length scale hypothesis $p \simeq 2^k$, k some selected integer? For instance, could the p-adic primes preferred by a given extension correspond to the ramified primes of the extension dividing the product $\prod_i(r_i - r_j)$?

1. The dimensionless roots of $P(x)$ are of the form $r_n = R_n/M_p$, where R_n is the dimensional root of $P(M_px)$. M_p would define the p-adic mass scale and the p-adic length scale of the corresponding CD. This would suggest that p-adic coupling constant evolution is not related to number theoretic coupling constant evolution.

2. On the other hand, the scattering amplitudes depend on the p-adic scale of the momenta. The reduction of scattering amplitudes to homogeneous functions of the factors $p_i \cdot p_j$ appearing in propagator denominators implies very simple dependence on momenta and the characteristic logarithmic dependence is absent. Does this mean that there should be a correlation between the p-adic length scale and algebraic extension? Why should a given extension prefer some p-adic primes, say ramified primes?

3. What about the vertices between Galois singlets, which involve a trilinear of an on-mass-shell state in H and two M^8 off-mass-shell states? How does the p-adic mass scale manifest itself in the properties of these Galois singlets? The conditions for Galois singlet property are scale invariant and the scale invariance is only broken by the condition that mass squared values are roots of polynomial P.

   $M^8 - H$ duality suggests the identification of the discriminant D of the polynomial as an exponent $exp(-K)$ of Kähler function defining vacuum functional and the identification of p-adic prime as a ramified prime dividing D. The real mass squared value would be determined by the canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ for ramified prime and depend on P.

4. p-Adic physics depends on the value of p-adic prime p. Could this bring in the p-adic coupling constant evolution and preferred p-adic primes number theoretically? The dimension of extensions of p-adics induced by a given extension of rationals depends on p since some roots exist as ordinary p-adic numbers. If p-adic physics as physics of cognition is essential also for real physics as p-adic mass calculations [K52, K21] suggest, it could force the natural selection of preferred p-adic primes and p-adic length scale evolution.

5. Only the identification of the preferred p-adic primes as ramified primes of extension comes into mind. What could make them so special? The p-adic variant of the polynomial P has a double root in order $O(p) = 0$ for a ramified prime. Double root is the mathematical counterpart of criticality and quantum criticality indeed is the basic dynamical principle of TGD. Could something which is of order $O(p^0)$ become order $O(p)$ for a ramified prime? The roots of P correspond to mass squared values: one would have $m_1^2 - m_2^2 = r_1 - r_2 = O(p)$ p-adically.

   For instance, could it be a generic mass squared scale defined by the difference $m_1^2 - m_2^2$ reduces from $M^2(CP_2)$ to $M(CP_2)^2/p$ for ramified primes or p-adic mass scale $M_p = M^2(CP)/p$ reduces to secondary p$_adicmassscale$M$_{p,2} = M^2(CP)/p^2$. Could the interpretation be in terms of emergence of a massless excitation as counterpart of quantum criticality. Kind of number theoretic analog of Goldstone boson.

   There is some support for this idea. In the living matter, the 10 Hz biorhythm is fundamental. It corresponds to the secondary p-adic length scale of the electron characterized by Mersenne prime $M_{127} = 2^{127} - 1$ [K52]. 10 Hz biorhythm could correspond to a kind of Goldstone boson. This argument still leaves open the question why ramified primes near powers of 2 (or of a small integer such as 3 [?, ?]) should be so special?

6. One can even speculate with the possibility that a kind of natural selection takes place already at this level. A high number of zero energy states could be possible for Galois singlet states associated with very special polynomials. In the functional composition $P_1 \circ P_2$ of polynomials conservation of roots takes place if the condition $P_i(0) = 0$ is satisfied. This could make possible evolutionary hierarchies in which conserved roots would be analogous to conserved genes.

An open challenge is to formulate a precise criterion fixing what diagrams are allowed. The intuitive picture is that the lines of the diagrams connecting mass shells $m_i^2 = n_i$ diagrams define convex polygons.

10.3.4 How can one include the WCW degrees of freedom?

The above consideration has been restricted to a single cognitive representation defined by a polynomial P. Already the inclusion of color degrees of freedom requires color partial waves in H and the superposition over space-time surfaces related by color rotation and therefore WCW spinor fields.

"Objective" and "subjective" representations of physics

The usual understanding of Uncertainty Principle (UP) requires that one has a WCW spinor field providing for instance the analogs of the plane waves in the center of mass degrees of freedom for 3-surface. This representation at the level of WCW might be called "objective" representation since one looks at the system from the H or WCW perspective. The localization of particles to the space-time surface violates UP in this "objective" sense.

Discrete cognitive representations define in ZEO what might be called a "subjective" representation of the Poincare and color group since one looks at the system from the perspective of a single space-time surface.

1. The "subjective" representations of isometries would be realized as flows inside X^4 rather than in H. The flows would be defined by the projections of Killing vectors on the space-time surface [L104].

2. The "subjective" representation is actually highly analogous to quantum group representation. For instance, for many-sheeted space-time surface, rotation by 2π would not bring the particle to a different space-time sheet and one would obtain charge fractionalization closely related to the hierarchy of many-sheeted structure corresponding to $h_{eff}/h_0 = n$ hierarchy were n is the dimension of the extension of rationals determined by the polynomial P. This representation could be restricted to Cartan algebra and does not require a 2-D system since the Cartan algebra effectively replaces the 2-D system.

3. The notion of "subjective" representation allows to generalize the gravitational and inertial mass to all conserved charges. Inertial charges would relate to the action in H and gravitational charges to the quantum group charges for flows restricted to $X^4 \subset H$. $M^8 - H$ duality indeed maps the momenta at mass shells associated with $X^4 \subset M^8$ to positions at the boundaries of CD and the action of Lorentz symmetries keeps the image points at the boundaries of CD.

Is WCW needed at the level of M^8?

The inclusion of WCW degrees of freedom is necessary for several reasons. WCW provides the "objective" perspective extending the "subjective" perspective provided by scattering amplitudes at a single space-time surface. Also the understanding of classical physics as an exact correlate of quantum physics requires WCW.

WCW has been introduced at the level of H and the question whether the notion of WCW makes sense also at the level of M^8, has remained open for a long time.

It is now clear that the polynomials P alone determine only the mass shells as their roots [L104]. Could the adelization and p-adization alone serve as the counterpart of WCW for M^8?

On the other hand, the interiors of 4-surfaces in M^8 involve the local CP_2 element and at the mass shells one has a local $S^2 = SO(3)/SO(2)$ element. Hence WCW might be realized at both sides as $M^8 - H$ duality suggests. An interesting conjecture is that by $M^8 - H$ duality, the two WCWs are one and the same thing. Therefore it would seem that adelization does not provide the counterpart of WCW in M^8.

Summation over polynomials as M^8 analog for the WCW integration

What could be the "cognitive" M^8 analog of WCW and integration over WCW?

1. The preferred extremal property of space-time surface $X^4 \subset H$ means that it is defined by its intersections with the boundary of CD. $M^8 - H$ duality requires that this is the case also in M^8. This would mean that the polynomial P determines, not only the 3-D mass shells of selected M^4 as its roots contained in $X^4 \subset M^8_c$, but also the 4-surface as an $SU(3)/U(2)$ local deformation of M^4 containing them and mapped to H by $M^8 - H$ duality.

2. In the full theory, one has integration over WCW spinor fields. Number theoretical approach means number theoretically unique discretization using cognitive representation rather than its "active" points (containing quark) defining a representation of the Galois group.

   The natural proposal is that WCW integration reduces to a summation over some subset of polynomials and amplitudes associated with the corresponding cognitive representations for which the area momenta for quarks are algebraic integers. External momenta would be ordinary integers for a given p-adic prime p. Therefore the summation over polynomials of varying degree makes sense for amplitudes with fixed external momenta if one uses extension of rationals containing all extensions defined by the polynomials.

3. The rational coefficients of polynomials would serve as WCW coordinates for the polynomials. The assumption that they are rational, however, creates a problem since the summation over rationals defining the coefficients understood as real numbers does not define an analog of integration measure.

One can imagine two number theoretical solutions of the problem: both are inspired by p-adic thermodynamics [K62, K42].

1. One manner to overcome the problem would be a restriction of the coefficients of P to integers. This is natural if the polynomials are monic polynomials of the form $x^n + an - 1x^{n-1} + ...$ This would mean a loss of scaling invariance since $P(kx)$ is not a monic polynomial. The good news is that this might select preferred p-adic primes and explain even the p-adic length scale hypothesis.

2. For a monic polynomial of degree n, the summation would reduce to a summation over $n-1$ integers. The roots would be powers of a single generating root r_0 giving rise to a basis for algebraic integers, and one would have fractility since the quark mass shells correspond to the powers for the modulus of the generating root. The moduli for the differences of roots would be proportional to the power of the modulus of the root and it would be natural to assign p-adic prime to the root with the smallest modulus. This option is highly attractive both physically and mathematically.

3. One expects a rapid p-adic convergence in the sense that polynomials with coefficients, which differ by a large power of p give to scattering amplitudes p-adically very similar contributions. The sum over these contributions should converge rapidly.

   It would seem that the exponent of Kähler function must enter into the picture and give rise to something resembling p-adic thermodynamics with the Boltzmann weight $exp(-E/T)$ being replaced with p-adic number p^{S/T_p}, where the p-adic temperature T_p is inverse integer and S is integer valued. p-Adic number p^{S/T_p} would correspond to the exponent $exp(-K)$ of Kähler function for the H imasage of the surface associated with P. Canonical identification would map p^{S/T_p} to its p-adic norm p^{-S/T_p} identified with $exp(-K)$.

4. The values of S/T_p correspond to the maxima of the Kähler function K for preferred extremals. These exponents exist p-adically only if the value of Kähler coupling strength α_K as an analog of inverse of a critical temperature satisfies strong number theoretic conditions reducing the exponent to an integer power of p (unless one assumes that also the roots of p can appear in the extension considered). These conditions would give rise to a p-adic coupling constant evolution for α_K and also to a coupling constant evolution as a function of algebraic extension.

5. One expects that these conditions can be satisfied only in a very restricted subset of preferred extremals so that one should assume a localization of WCW spinor field to a subset of maxima of the Kähler function. TGD is analogous to a complex square root of thermodynamics and this kind of localization takes place quite generally (spontaneous magnetization) in thermodynamics and also in quantum field theories (Higgs mechanism).

   For spin glass discussed from the TGD point of view in [L103], this kind of localization occurs also and in the ultrametric topology of the spin glass energy landscape emerges naturally. p-Adic topologies represent basic examples about ultrametric topologies. The TGD inspired proposal indeed is that p-adic thermodynamics [K62, K52] allows the formulation of spin glass thermodynamics free of ad hoc assumptions.

   TGD is Universe is indeed highly analogous to a spin glass in long scales, where the action approaches Kähler action having a huge vacuum degeneracy involving classical non-determinism as the length scale dependent cosmological constant Λ predicted by the twistor lift [L45, L58] approaches zero. An attractive proposal is that this kind of localization has a purely number theoretic origin making p-adic thermodynamics for a suitably chosen value of α_K possible [L103].

6. Also the summation over amplitudes associated with different polynomials of various degrees is in principle possible and could correspond to the summation appearing in perturbation theory and to the summation appearing in p-adic thermodynamics.

One cannot exclude a more general option in which there is a summation over all polynomials with rational coefficients analogous to the summation over the valleys of the energy landscape for spin glass phase.

1. For general rational polynomials, one would have a scaling invariance $P(x) \to P(kx)$. There would be a summation over scaled roots of P and rationally scaled mass shells. For monic polynomials the scaling invariance is lost and this seems the only realistic possibility.

2. One might hope that the summations over rationals assigned to the coefficients of P with fixed degree reduce to a p-adic integration and that a p-adic integration measure for this integral exists and reduces essentially to summation over p-adic integers with a given norm p^k plus to a summation over the norms p^k at the limit when the norm approaches infinity (`https://cutt.ly/UUbit6f`). Here the problem is that there is no natural lower bound on the p-adic norm of the coefficients as for monic polynomials and the integral need not converge.

The restriction to monic polynomials looks highly attractive. Another possible restriction is that polynomials are proportional to x so that the roots of P are also the roots of the functional composite $P \circ Q$. This restriction might be also an outcome of a number theoretical evolution.

M^8 analog of vacuum functional

The vacuum functional as an exponent of the Kähler function determines the physics at WCW level. $M^8 - H$ duality suggests that it should have a counterpart at the level of M^8 and appear as a weight function in the summation. Adelic physics requires that weight function is a power of p-adic prime and ramified primes of the extension are the natural candidates in this respect.

1. The discriminant D of the algebraic extension defined by a polynomial P with rational coefficients (`https://en.wikipedia.org/wiki/Discriminant`) is expressible as a square for

the product of the non-vanishing differences $r_i - r_j$ of the roots of P. For a polynomial P with rational coefficients, D is a rational number as one can see for polynomial $P = ax^2 + bx + c$ from its expression $D = b^2 - 4ac$. For monic polynomials of form $x^n + a_{n-1}x^{m-1} + ...$ with integer coefficients, D is an integer. In both cases, one can talk about ramified primes as prime divisors of D.

If the p-adic prime $p$is identified as a ramified prime, D is a good candidate for the weight function since it would be indeed proportional to a power of p and have p-adic norm proportional to negative power of p. Hence the p-adic interpretation of the sum over scattering amplitudes for polynomials P is possible if p corresponds to a ramified prime for the polynomials allowed in the amplitude.

p-Adic thermodynamics [K52] suggest that p-adic valued scattering amplitudes are mapped to real numbers by applying to the Lorentz invariants appearing in the amplitude the canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ mapping p-adics to reals in a continuous manner

2. For monic polynomials, the roots are powers of a generating root, which means that D is proportional to a power of the generating root, which should give rise to some power of p. When the degree of the monic polynomial increases, the overall power of p increases so that the contributions of higher polynomials approach zero very rapidly in the p-adic topology. For the p-adic prime $p = M_{127} = 2^{127} - 1 \sim 10^{38}$ characterizing electrons, the convergence is extremely rapid.

   Polynomials of lowest degree should give the dominating contribution and the scattering amplitudes should be characterized by the degree of the lowest order polynomial appearing in it. For polynomials with a low degree n the number of particles in the scattering amplitude could be very small since the number n of roots is small. The sum $x_i + p_i$ cannot belong to the same mass shell for timelike p_i so that the minimal number of roots r_n increases with the number of external particles.

3. $M^8 - H$ duality requires that the sum over polynomials corresponds to a WCW integration at H-side. Therefore the exponent of Kähler function at its maximum associated to a given polynomial should be apart from a constant numerical factor equal to the discriminant D in canonical identification.

   The condition that the exponent of Kähler function as a sum of the Kähler action and the volume term for the preferred extremal $X^4 \subset H$ equals to power of D apart from a proportionality factor, should fix the discrete number theoretical and p-adic coupling constant evolutions of Kähler coupling strength and length scale dependent cosmological constant proportional to inverse of a p-adic length scale squared. For Kähler action alone, the evolution is logarithmic in prime p since the function reduces to the logarithm of D.

$M^8 - H$ duality suggests that the exponent $exp(-K)$ of Kähler function has an M^8 counterpart with a purely number theoretic interpretation. The discriminant D of the polynomial P is the natural guess. For monic polynomials D is integer having ramified primes as factors.

There are two options for the correspondence between $exp(-K)$ at its maximum and D assuming that P is monic polynomial.

1. In the real topology, one would naturally have $exp(-K) = 1/D$. For monic polynomials with high degree, D becomes large so that $exp(-K)$ is large.

2. In a p-adic topology defined by p-adic prime p identified as a ramified prime of D, one would have naturally $exp(-K) = I(D)$, where one has $I(x) = \sum x_n p^n = \sum x_n p^{-n}$.

   If p is the largest ramified prime associated with D, this option gives the same result as the real option, which suggests a unique identification of the p-adic prime p for a given polynomial P. P would correspond to a unique p-adic length scale L_p and a given L_p would correspond to all polynomials P for which the largest ramified prime is p.

   This might provide some understanding concerning the p-adic length scale hypothesis stating that p-adic primes tend to be near powers of integer. In particular, understanding about why Mersenne primes are favored might emerge. For instance, Mersennes could correspond to primes for which the number of polynomials having them as the largest ramified prime is especially large. The quantization condition $exp(-K) = D(p)$ could define which p-adic primes are the fittest ones.

The condition that $exp(-K)$ at its maximum equals to D via canonical identification gives a powerful number theoretic quantization condition. Is this condition realized for preferred extremals as extremals of both Kähler action and volume term, or should one regard these conditions as additional conditions?

1. P fixes only the mass-shells as its roots r_n. The real parts of these roots belong to the same M^4. M^8-H duality is realized by assuming that the mass shells are connected by a 4-surface X^4, which is a deformation of M^4 by a local $SU(3)$ element $g(x)$ such that the subgroup $U(2)$ leaves the points of deformation invariant: this condition gives rise to an explicit form of M^8-H duality.

   P itself poses no conditions on the local CP_2 element. Could the condition $exp(-K) = I(D)$ for the image of $X^4 \subset M^8$ in H fix the $g(x)$ and thus $X^4 \subset H$?

2. The twistor lift should determine the surface $X^4 \subset H$. The counterpart of twistor lift is defined also at the level of M^8. It maps 6-D surface connecting 5-D mass shells of M^8 as roots of P identified as a local $SU(3)$ deformation of M^6 remaining invariant under $U(1)\times U(1)$ at each point. Hence a point of CP_2 twistor space is assigned to M^6 identified locally as a point of M^4 twistor space.

   One can assign to the twistor space of X^4 as 6-surface $X^6 \subset T(M^4)\times T(CP_2)$ 6-D Kähler action reducing to 4-D Kähler action plus volume term by a dimensional reduction required by the bundle property. One can define the twistorial variant of WCW with the Kähler function K_6 defined by the 6-D Kähler action for X^6. The vacuum functional $exp(-K_6)$ would be the same as for WCW.

   Since S^2 degrees are non-dynamical, the two WCWs are more or less one and the same thing apart from delicacies of non-trivial windings numbers for the maps from the fiber S^2 of $T(X^4)$ to the fibers of $T(M^4)$ and $T(CP_2)$.

3. The $U(2)$ *resp.* $U(1)\times U(1)$ invariant points of the deformation of M^4 *resp.* M^6 would define X^4 *resp.* its twistor space $T(X^4)$. The condition that the image of the deformed M^6 is a preferred extremal of 6-D Kähler action, should determine $g(x)$. $I(D) = exp(-K)$ fixes the 6-D Kähler action action.

4. The formulation of the variational problem in H as a variational problem in $M^4 \subset M^8$ might provide some insight. The 6-D Kähler action for $X^6 \subset H$ naturally assigns an action to the deformed $M^6 \subset M^8$. At the level of M^8, the quantization condition $exp(-K) = I(D)$ plus the boundary conditions defined by the roots of P would select $X^6 \subset M^8$ as a preferred extremal of 6-D Kähler action. This condition could also induce a natural selection of p-adic primes explaining p-adic length scale hypothesis.

The evolution of α_K and of cosmological constant from number theory?

I have considered earlier the evolution of cosmological constant [L10, L45, L58] but it is interesting to look at it in a more detail from the number theoretic perspective.

1. There are three parameters involved: Kähler coupling strength α_K and the winding numbers n_1 and n_2 for the maps of the twistor sphere $T(X^4)$ of $X^4 \subset H$ to the twistor spheres $S^2(M^4)$ and $S^2(CP_2)$ associated with the twistor spaces $T(M^4)$ and $T(CP_2)$: these maps essentially identify the latter twistor spheres.

2. The 6-D Kähler action for $X^6 = T(X^4) \subset T(M^4)\times T(CP_2)$ is proportional to Kähler coupling strength and the scale factor $1/R^2$, which is equal to CP_2 radius squared. The recent interpretation is that CP_2 radius corresponds to the Planck length L_{Pl} scaled up by h_{eff}/h_0. So that for $h_{eff} = h_0$, the CP_2 radius would reduce to Planck length apart from a numerical constant.

3. Dimensional reduction is necessary in order that X^6 has the structure of the induced twistor bundle with $X^4 \subset H$ as a base-space. This requires maps of the twistor sphere S^2 of the twistor space $T(X^4)$ of $X^4 \subset H$ to the twistor spheres $S^2(M^4)$ and $S^2(CP_2)$: this map identifies these twistor spheres locally.

4. Dimensional reduction gives rise to the usual 4-D Kähler action and a volume term with a cosmological constant Λ determined by the Kähler action for the S^2 part of 6-D Kähler action. The induced Kähler form in S^2 is the sum of the contributions from $S^2(M^4)$ and $S^2(CP_2)$.

   Unless the winding numbers of the maps differ from unit, the induced Kähler form is zero or twice the Kähler form of $S^2(CP_2)$ depending on the relative sign of the Kähler forms, whose normalization is fixed by the condition that the magnetic flux is quantized to unity. The form of the maps in spherical coordinates (θ, ϕ) for $S^2(X^4)$ is given by $\theta(M^4) = \theta(CP_2) = \theta$ and $\phi(M^4) = n_1\phi$ and $\phi(CP_2) = n_2\phi$.

5. If the winding numbers n_i are different and of opposite sign (assuming the same sign for Kähler forms), the induced Kähler form is given by $J = (n_2 - n_1)J(S^2(CP_2))$, where n_i are positive.

   The induced line element is $ds^2 = d\theta^2 + sin^2(\theta)(n_1^2 + n_2^2)\phi^2$. The determinant $\sqrt{g}$ of the induced metric of S^2 is $\sqrt{g} = \sqrt{n_1^2 + n_2^2}\sqrt{g(CP_2)}$. The contravariant induced Kähler form is given by

$$J^{\theta\phi} = \frac{g^{\theta\theta}g^{\phi\phi}}{J}_{\theta\phi} = (n_1 - n_2)/n_1^2 + n_2^2 J^{\theta\phi}(CP_2) \ . \qquad (10.3.1)$$

   The Kähler action for S^2 is given by

$$J^{\theta\phi}J_{\theta\phi}\sqrt{g} = \frac{n_1 - n_2}{\sqrt{n_1^2 + n_2^2}} J^{\theta\phi}(CP_2)J_{\theta\phi}(J(CP_2))\sqrt{g(CP_2)} \ .$$

   For small values of $n_1 - n_2$ and large values of $n_1 \sim n_2$ the contribution to action behaves like $\Delta n/n_1$ and can become arbitrarily small. This would predict that cosmological constant approaches to zero in long p-adic length scales.

   This poses a condition on the integers n_i depending on the p-adic prime p identified as a ramified prime: $\Delta n/n_1$ should behave like the inverse of the p-adic length scale scale. The p-adic length scale evolution of both α_K and integers n_i should follow from the condition that the total action equals to the discriminant D (also a polynomial of discriminant can in principle be considered but this seems artificial). The best one can hope is that $M^8 - H$ duality completely fixes both coupling constant evolutions.

6. For the cosmological constant Λ in cosmological scales, the dark energy density is parameterized as $\rho_{vac} = 1/L^4$, $L \sim L_{neuron}$, where $L_{neuron} \simeq 10^{-4}$ m corresponds to the size scale of neuron.

   This rough estimate follows from the identification $\Lambda/8\pi G = 1/L^4$ giving $L(8\pi G/\Lambda)^{1/4}$. Λ itself would correspond to an inverse of p-adic length square, which is of order of the horizon size (naturally the size of cosmological CD).

Do Grassmannians emerge at the QFT limit of TGD?

There is no obvious use for Grassmannians and related concepts in the construction of twistor amplitudes for a space-time surface associated with a given polynomial P.

However, a given scattering amplitude is a sum of contributions associated with monic polynomials P with an increasing number of roots such that a given p-adic prime p appears as their ramified prime. The discriminant D is assumed to play the role of the vacuum functional $exp(-K)$. This picture is highly analogous to a perturbation theory in a given p-adic length scale.

This suggests that QFT with massless particles is a reasonable approximation of TGD at the QFT limit and that the basic twistorial structures could appear at this limit.

Apart from masses given by p-adic thermodynamics [K52, K21], elementary particles, to be distinguished from fundamental quarks, correspond to massless states so that massless QFT is a good guess for the QFT limit.

The emergence of the massless states requires M^4 Kähler structure forced by the twistor lift [L100]. This breaks the Lorentz symmetry to that of $M^2 \times E2$ and the transversal degrees of freedom correspond to harmonic oscillator type degrees of freedom just as in string model and are characterized by two conformal weights. This spontaneous breaking of Lorentz symmetry characterizes massless particles and hadronic quarks. It makes possible the required tachyonic ν_R making it possible to construct massless ground states in p-adic mass calculations.

1. In M^8, the mass shells in general correspond to complex roots. It is possible to have tachyonic Galois confined states. Covariantly constant right-handed neutrino ν_R would be such a state and needed to construct massless Galois confined physical states in H.

2. In H, only the ν_R constructed from quarks is tachyonic in the approximation that H-spinor mode with Kähler charge $Q_K = 3$ describes leptons as 3-quark Galois singlets. $M^8 - H$ duality suggests that there are no other tachyonic quark states and that all Galois confined states are non-tachyonic so that the momenta belong to the interior of the light-cone in M^8.

3. If the amplitudes in the massless sector are indeed Yangian invariants, Grassmannians would emerge naturally at the QFT limit.

The following series of questions is an attempt to crystallize my ignorance.

1. Could a QFT based on twistorial amplitudes for massless Galois confined external particles in H provide a QFT limit of TGD?

2. Could the sum over amplitudes for different polynomials having a given p-adic prime p as a ramified prime correspond to structure resembling that produced in BCFW recursion?

3. Or could MHV structure emerge at the level of a single polynomial P: this is the case if the quark propagators connecting Galois singlets in the amplitudes be regarded as analogs of the propagators $1/P^2$ connecting parts of MHV amplitudes?

4. How the coupling constant evolution emerges at the QFT limit. Number theoretic approach does not allow logarithmic contributions coming from loops but it would not be surprising if the discrete p-adic coupling constant evolution would allow a logarithmic coupling evolution as a reasonable approximation.

   This is also suggested by the fact that the expression of α_K in terms of discriminant D involves logarithm of the p-adic length scale (, that is p). If $exp(-K)$ equals to the image $I(D)$ under canonical identification, one has $\alpha_K = S/log(I(D))$, where $S = K\alpha_K$ is the total action without the proportionality factor $1/\alpha_K$. For ramified primes α_K is proportional to $1/log(p)$.

10.3.5 What about the twistorialization in CP_2 degrees of freedom?

The proposed picture does not use CP_2 twistor space at all. One should understand why this is the case.

The treatment of color degrees of freedom involves several new aspects. First of all, color is not a spin-like quantum number in the TGD framework.

1. One can identify colored states as color partial waves in WCW degrees of freedom associated with the center of mass degrees of freedom of 3-surface. H spinor modes can be indeed regarded as color partial waves in H.

It would seem that one cannot speak of color for a single space-time surface. This is indeed true for an "objective" view about the isometries of H. One can however define the "subjective" representations of the isometries by replacing them with flows defined by the projections of Killing vectors to the space-time surfaces [L104].

For cognitive representations the "subjective" representations could in some situations be reduced to those for the discrete Galois group. One can wonder whether color confinement could reduce to Galois confinement.

2. "Subjective" representations are analogous to quantum group representations [L104]. Objective-subjective dichotomy coud also generalize the inertial-gravitational dichotomy. Note that one can also assign Noether charges to the projected flows. This applies also to supersymplectic symmetries.

The treatment of CP_2 degrees of freedom for the twistor amplitudes remains a challenge and in the following I can only try to clarify my thoughts.

1. Twistor lift strongly suggests that M^8-H duality defines a map of the twistor spaces of H and M^8 to each other. The M^8 counterparts of 6-D twistor space as a surface $X^6 \subset T(M^4)\times T(CP_2)$ would be 6-D surface with a commutative normal space defined by a deformation of complexified Minkowski space M^6 by a local $SU(3)$ element, which is left-invariant under $U(1)\subset U(1)$. This would give a 6-surface Y^6 as a counterpart of the 6-surface. It would seem that M^6 should correspond to the twistor $T(M^4)$, perhaps via the identification with a projective space of M^8 by 2-D projective scalings (perhaps by hypercomplex numbers).

2. This map would preserve S^2 bundle structure so that the twistor spheres of $T(M^4)$ and $T(CP_2)$ would be mapped to each other. This looks strange at first but conforms with the general picture.

   At the level of $T(H)$ twistor wave functions at the twistor spheres S^2 of $T(M^4)$ and $T(CP_2)$, which have been identified, describe spin and color or electroweak quantum numbers (the holonomy group of the spinor connection of CP_2 defining weak gauge group can be identified as $U(2)\subset SU(3)$). This implies a correlation for spin and electroweak spin doublets defined quarks apart from the sign factors.

   In the algebraic picture a single point of M^8 does not define only the momentum of quark momentum: rather quark momentum and spin corresponds to a single point of $X^6\subset M^8$. Fermi statistics boil down to the condition that each point of X^6 can contain only a single quark. Also now directions of the quantization axis characterize the sign of spin and electroweak spin.

3. Spin-isospin correspondence makes sense only because quarks are both spin and weak isospin doublets. The fact that spin value $\pm 1/2$ corresponds to the two directions of the quantization axis allows all possible pairings of spin and electroweak (or color) isospin.

   This map between $T(M^4)$ and $T(CP_2)$ can be understood at M^8 level and generalizes the mapping of M^4 to CP_2 for a space-time surface with 4-D M^4 projection. There are 4-suraces X^4 for which the dimensions of the projections M^4 or CP_2 projection are not maximal. These 4-surfaces correspond to singularities in which normal space at the points of the singularity is not unique [L107].

   It is enough that the twistor spheres of $T(M^4)$ and $T(CP_2)$ are mapped to each other by locally 1-to-1 projection to the twistor sphere of $T(X^4)$: the base space of the twistor space X^6 need not have 4-D projection to M^4 or CP_2.

4. CP_2 twistors can be regarded as functions of M^4 twistors for a given space-time surface with 4-D M^4 projection. The implications for the construction of scattering amplitudes remain to be understood.

How color degrees of freedom are described at M^8 level? There are two equivalent ways to understand the emergence of CP_2 in M^8-H duality.

1. The normal spaces of $X^4 \subset M^8$ define an integrable distribution. Normal space of X^4 is regarded as a CP_2 point characterizing the deformation of fixed M^4 [L104, L82, L83] so that one obtains $M^8 - H$ duality.

   This distribution contains an integrable distribution of commutative 2-surfaces in turn defining a 6-D surface X^6, which is a good candidate for the counterpart of twistor space. The assignment of the normal space defines a point of the twistor space $SU(3)/U(1) \times U(1)$.

2. Second view [L104, L82, L83], which emerged only quite recently from the detailed study of the surfaces determined by polynomials P, is that the element of local $SU(3)$ naturally defines a deformation of X^4, which is invariant under left or right action by $U(2) \subset SU(3)$ so that local element of $CP_(2)$ is in question. This means that color $SU(3)$ corresponds to a subgroup of the automorphism group G_2 of octonions. P as such does not etermines the local CP_2 element. What determines P, will be discussed later.

   The counterpart for the distribution of commutative normal spaces of X^6 is a deformation of M^6, or its variant with some signature of metric, defined by a local element of $SU(3)$ such that the imagine point remains invariant by $U(1) \times U(1) \subset SU(3)$ so that it assigns a point of the twistor space $SU(3)/U(1)U(1)$ to each point of X^6.

3. The equivalence of these views is not rigorously proven. Note that the polynomial P itself defines only 3-D complex mass shells as its roots and the 4-surface connecting them is determined from the condition that $M^8 - H$ duality makes sense.

There is an objection against CP_2 type extremal as a blow-up of 1-D singularity of $X^4 \subset M^8$. Is it really possible to describe CP_2 type extremal as 1-D singularity of $X^{\subset} M^8$ using the $U(2)$ invariant map $M^4 \rightarrow CP_2$?

1. The line singularity can be identified as an 1-D intersection of 2 Minkowskian space-time sheets as roots of P. At H level, this leads to a generation of wormhole contact with an Euclidean signature of metric, CP_2 type extremal, connecting the space-time sheets. The Minkowskian space-time becomes Euclidean at the wormhole throats.

2. At each point of 1-D curve L the singularity should be 3-D surface in CP_2. This requires that the normal space is non-unique and the normal spaces at a point x of L form a 3-D surface in CP_2. If one however thinks about how this could be achieved, one ends up with a problem. One can think that the images of an arbitrarily small sphere S^2 around the point of L is a sphere of CP_2. At the limit one would obtain 2-D rather than 3-D surface of CP_2.

3. The $U(2)$ invariant local $SU(3)$ transformation as a deformation of M^4 defining a local CP_2 transformation is not quite enough to describe the situation. The solution is to consider its inverse as a map from CP_2 to M^4 having a singularity at which a 4-D region of CP_2 is mapped to a line of M^4.

10.4 What about unitarity?

Unitarity is a poorly understood problem of the twistor approach and also of TGD.

10.4.1 What do we mean with time evolution?

The first questions relate to the identification of the TGD counterpart of S-matrix.

1. Zero energy states correspond to superpositions of pairs of ordinary 3-D states assignable to the opposite boundaries of CD. The simplest assumption corresponds to the idea about state preparation is that the states are unentangled. Unitarity would mean that the 3-D zero energy states at the active boundary of CD are orthogonal if the 3-D states at the passive boundary of CD are orthogonal. The scattering amplitudes considered in this article would naturally correspond to zero energy states. Is there any reason for zero energy states to satisfy this kind of orthogonality?

2. The time evolutions between "small" state function reductions (SSFRs) are assumed to increase the size of CD in a statistical sense at least and affect the states at the active boundary of CD but leave the "visible" part of the state at the passive boundary unaffected. These time evolutions are proposed to correspond to the scalings of CD rather than time translations. In this case unitarity would look a reasonable property.

   The sequence of (ordinary) "big" SFRs (BSFRs) could allow approximate description as being associated with unitary time evolutions with time translations rather than scalings and followed by BSFR changing the arrow of time. The characteristic features of these time evolutions would be polynomial and exponential decay and the relaxation of spin glass would be a key example about time evolution by SSFRs [L103].

10.4.2 What really occurs in BSFR?

It has been assumed hitherto that a time reversal occurs in BSFR. The assumption that SSFRs correspond to a sequence of time evolutions identified as scalings, forces to challenge this assumption. Could BSFR involve a time reflection T natural for time translations or inversion $I : T \rightarrow 1/T$ natural for the scalings or their combination TI?

I would change the scalings increasing the size of CD to scalings reducing it. Could any of these options: time reversal T, inversion I, or their combination TI take place in BSFRs whereas arrow would remain as such in SSFRs? T (TI) would mean that the active boundary of CD is frozen and CD starts to increase/decrease in size.

There is considerable evidence for T in BSFRs identified as counterparts of ordinary SFRs but could it be accompanied by I?

1. Mere I in BSFR would mean that CD starts to decrease but the arrow of time is not changed and passive boundary remains passive boundary. What comes to mind is blackhole collapse.

   I have asked whether the decrease in size could take place in BSFR and make it possible for the self to get rid of negative subjective memories from the last moments of life, start from scratch and live a "childhood". Could this somewhat ad hoc looking reduction of size actually take place by a sequence of SSFRs? This brings into mind the big bang and big crunch. Could this period be followed by a BSFR involving inversion giving rise to increase of the size of CD as in the picture considered hitherto?

2. If BSFR involves TI, the CD would shift towards a fixed time direction like a worm, and one would have a fixed arrow of time from the point of view of the outsider although the arrow of time would change for sub-CD. This modified option might be consistent with the recent picture, in particular with the findings made in the experiments of Minev *et al* [L62] [L62].

   This kind of shifting must be assumed in the TGD inspired theory of consciousness. For instance, after images as a sequence of time reversed lives of sub-self, do not remain in the geometric past but follow the self in travel through time and appear periodically (when their arrow of time is the same as of self). The same applies to sleep: it could be a period with a reversed arrow of time but the self would shift towards the geometric future during this period: this could be interpreted as a shift of attention towards the geometric future. Also this option makes it possible for the self to have a "childhood"".

3. However, the idea about a single arrow of time does not look attractive. Perhaps the following observation is of relevance. If the arrow of time for sub-CD correlates with that of sub-CD, the change of the arrow of time for CD, would induce its change for sub-CDs and now the sub-CDs would increase in the opposite direction of time rather than decrease.

10.4.3 Should unitarity be replaced with the Kähler-like geometry of the fermionic state space?

After these preliminaries we can state the key question. Is unitarity possible at all and should it be replaced with some deeper principle? I have considered these questions several times and in [L91] a rather radical solution was proposed.

Assigning an S-matrix to a unitary time evolution works in non-relativistic theory but fails already in the generic QFT and correlation functions replace S-matrix.

1. Einstein's great vision was to geometrize gravitation by reducing it to the curvature of space-time. Could the same recipe work for quantum theory? Could the replacement of the flat Kähler metric of Hilbert space with a non-flat one allow the identification of the analog of unitary S-matrix as a geometric property of Hilbert space? Kähler metric is required to geometrize hermitian conjugation. It turns out that the Kähler metric of a Hilbert bundle determined by the Kähler metric of its base space could replace the unitary S-matrix.

2. An amazingly simple argument demonstrates that one can construct scattering probabilities from the matrix elements of Kähler metric and assign to the Kähler metric a unitary S-matrix assuming that some additional conditions guaranteeing that the probabilities are real and non-negative are satisfied. If the probabilities correspond to the real part of the complex analogs of probabilities, it is enough to require that they are non-negative: complex analogs of probabilities would define the analog of the Teichmüller matrix.

   Teichmüller space parameterizes the complex structures of Riemann surface: could the allowed WCW Kähler metrics - or rather the associated complex probability matrices - correspond to complex structures for some space? By the strong form of holography (SH), the most natural candidate would be Cartesian product of Teichmüller spaces of partonic 2 surfaces with punctures and string world sheets.

3. Under some additional conditions one can assign to Kähler metric a unitary S-matrix but this does not seem necessary. The experience with loop spaces suggests that for infinite-D Hilbert spaces the existence of non-flat Kähler metric requires a maximal group of isometries. Hence one expects that the counterpart of S-matrix is highly unique.

4. In the TGD framework the "world of classical worlds" (WCW) has Kähler geometry allowing spinor structure. WCW spinors correspond to Fock states for second quantized spinors at space-time surface and induced from second quantized spinors of the embedding space. Scattering amplitudes would correspond to the Kähler metric for the Hilbert space bundle of WCW spinor fields realized in zero energy ontology and satisfying Teichmüller condition guaranteeing non-negative probabilities.

5. Equivalence Principle generalizes to the level of WCW and its spinor bundle. In ZEO one can assign also to the Kähler space of zero energy states spinor structure and this strongly suggests an infinite hierarchy of second quantizations starting from space-time level, continuing at the level of WCW, and continuing further at the level of the space of zero energy states. This would give an interpretation for an old idea about infinite primes as an infinite hierarchy of second quantizations of an arithmetic quantum field theory.

6. There is also an objection. The transition probabilities would be given by $P(A,B) = g^{A,\overline{B}}g_{\overline{B},A}$ and the analogs for unitarity conditions would be satisfied by $g^{A,\overline{B}}g_{\overline{B},C} = \delta^A_C$. The problem is that $P(A,B)$ is not real without further conditions. Can one imagine any physical interpretation for the imaginary part of $Im(P(A,B))$?

In this framework, the twistorial scattering amplitudes as zero energy states define the covariant Kähler metric $g_{A\overline{B}}$, which is non-vanishing between the 3-D state spaces associated with the opposite boundaries of CD. $g^{A\overline{B}}$ could be constructed as the inverse of this metric. The problem with the unitarity would disappear.

Explicit expressions for scattering probabilities

The proposed identification of scattering probabilities as $P(A \rightarrow B) = g^{A\overline{B}}g_{\overline{B}A}$ in terms of components of the Kähler metric of the fermionic state space.

Contravariant component $g^{A\overline{B}}$ of the metric is obtained as a geometric series $\sum_{n\geq 0} T^n$ from from the deviation $T_{A\overline{B}} = g_{A\overline{B}} - \delta_{A\overline{B}}$ of the covariant metric $g_{A\overline{B}}$ from $\delta A\overline{B}$.

g this is not a diagonal matrix. It is convenient to introduce the notation Z^A, $A \in \{1,...,n\}$ $Z^{\overline{A}} = Z^{n+k}$, $k = n+1,...,2n$. So that the $g_{\overline{B}C}$ corresponds to $g_{k+n,l} = \delta_{k,l} + T_{k,l}$. and one has $g^{A\overline{B}}$ to $g^{k,l+n} = \delta_{k,l} + T^1_{k,l}$.

The condition $g^{A\overline{B}} g_{\overline{B}C} = \delta^A_C$ gives

$$g^{k,l+n} g_{l+n,m} = \delta^k_m \quad . \tag{10.4.1}$$

giving

$$\sum_l (\delta_{k,l} + T^1_{k,l})(\delta_{l,m} + T_{l,m}) = \delta_{k,m} + (T^1 + T + T^1 T)_{km} = \delta_{k,m} \quad . \; . \tag{10.4.2}$$

which resembles the corresponding condition guaranteeing unitarity. The condition gives

$$T_1 = -\frac{T}{1+T} = -\sum_{n>1} ((-1)^n T^n. \quad . \tag{10.4.3}$$

The expression for $P(A \to B)$ reads as

$$\begin{array}{l} P(A \to B) = g^{A\overline{B}} g_{\overline{B}A} \\ = [1 - \frac{T}{1+T} + T^\dagger - (\frac{T}{1+T})_{AB} T^\dagger]_{AB} \quad . \end{array} \tag{10.4.4}$$

It is instructive to compare the situation with unitary S-matrix $S = 1 + T$. Unitarity condition $SS^\dagger = 1$ gives

$$T^\dagger = -\frac{T}{1+T} \quad ,$$

and

$$P(A \to B) = \delta_{AB} + T_{AB} + T^\dagger_{AB} + T^\dagger_{AB} T_{AB} = [\delta_{AB} - (\frac{T}{1+T})_{AB} + T_{AB} - (\frac{T}{1+T})_{AB} T_{AB} \quad .$$

The formula is the same as in the case of Kähler metric.

10.4.4 Critical questions

One can pose several critical questions helping to further develop the proposed number theoretic picture.

Is mere recombinatorics enough as fundamental dynamics?

Fundamental dynamics as mere re-combination of free quarks to Galois singlets is attractive in its simplicity but might be an over-simplification. Can quarks really continue with the same momenta in each SSFR and even BSFR?

1. For a given polynomial P, there are several Galois singlets with the same incoming integer-valued total momentum p_i. Also quantum superpositions of different Galois singlets with the same incoming momenta p_i but fixed quark and antiquark numbers are in principle possible. One must also remember Galois singlet property in spin degrees of freedom.

2. WCW integration corresponds to a summation over polynomials P with a common ramified prime (RP) defining the p-adic prime. For each P of the Galois singlets have different decomposition to quark momenta.

   One can even consider the possibility that only the total quark number as the difference of quark and antiquark numbers is fixed so that polynomials P in the superposition could correspond to different numbers of quark-antiquark pairs.

3. One can also consider a generalization of Galois confinement by replacing classical Galois singlet property with a Galois-singlet wave function in the product of quark momentum spaces allowing classical Galois non-singlets in the superposition.

   Hydrogen atom serves as an illustration: electron at origin would correspond to classical ground state and s-wave correspond to a state invariant under rotations such that the position of electron is not anymore invariant under rotations. The proposal for transition amplitudes remains as such otherwise.

Note however that the basic dynamics at the level of a single polynomial would be recombinatorics for all these options.

General number theoretic picture of scattering

Only the interaction region has been considered hitherto. One must however understand how the interaction region is determined by the 4-surfaces and polynomials associated with incoming Galois singlets. Also the details of the map of p-adic scatting amplitude to a real one must be understood.

The intuitive picture about scattering is as follows.

1. The incoming particle "i" is characterized by p-adic prime p_i, which is RP for the corresponding 4-surface in M^8. Also the "adelic" option that all RPs characterize the particle, is considered below.

2. The interaction region corresponds to a polynomial P. The integration over WCW corresponds to a sum over several P:s with at least one common RP allowing to map the superposition of amplitudes to real amplitude by canonical identification I: $\sum x_n p^n \rightarrow \sum x_n p^{-n}$.

   If one gives up the assumption about a shared RP, the real amplitude is obtained by applying I to the amplitudes in the superposition such that RP varies. Mathematically, this is an ugly option.

3. If there are several shared RPs, in the superposition over polynomials P, one can consider an adelic picture. The amplitude would be mapped by I to a product of the real amplitudes associated with the shared RP:s. This brings in mind the adelic theorem stating that rational number is expressible as a product of the inverses of its p-adic norms. The map I indeed generalizes the p-adic norm as a map of p-adics to reals. Could one say that the real scattering amplitude is a product of canonical images of the p-adic amplitudes for the shared RP:s? Witten has proposed this kind of adelic representation of real string vacuum amplitude.

   Whether the adelization of the scattering amplitudes in this manner makes sense physically is far from clear. In p-adic thermodynamics one must choose a single p-adic prime p as RP. Sum over ramified primes for mass squared values would give CP_2 mass scale if there are small p-adic primes present.

The incoming polynomials P_i should determine a unique polynomial P assignable to the interaction regions as CD to which particles arrive. How?

1. The natural requirement would be that P possess the RPs associated with P_i:s. This can be realized if the condition $P_i = 0$ is satisfied and P is a functional composite of polynomials P_i. All permutations π of $1, ..., n$ are allowed: $P = P_{i_1} \circ P_{i_2} \circ ....P_{i_n}$ with $(i_1, ...i_n) = (\pi(1), ..., \pi(n))$. P possesses the roots of P_i.

   Different permutations π could correspond to different permutations of the incoming particles in the proposal for scattering amplitudes so that the formation of area momenta $x_{i+1} = \sum_{k=1}^{i} p_k$ in various orders would corresponds to different orders of functional compositions.

2. Number theoretically, interaction would mean composition of polynomials. I have proposed that so-called cognitive measurements as a model for analysis could be assigned with this kind of interaction [L90, L93]. The preferred extremal property realized as a simultaneous extremal property for both Kähler action and volume action suggests that the classical non-determinism due to singularities as analogs of frames for soap films serves as a classical correlate for quantum non-determinism [L107].

3. If each incoming state "i" corresponds to a superposition of P_i:s with some common RPs, only the RP:s shared by all compositions P from these would appear in the adelic image. If all polynomials P_i are unique (no integration over WCW for incoming particles), the canonical image of the amplitude could be the product over images associated with common RPs.

   The simplest option is that a complete localization in WCW occurs for each external state, perhaps as a result of cognitive state preparation and reduction, so that P has the RP:s of P_is as RP:s and adelization could be maximal.

Do the notions of virtual state, singularity and resonance have counterparts?

Is the proposal physically acceptable? Does the approach allow to formulate the notions of virtual state, singularity and resonance, which are central for the standard approach?

1. The notion of virtual state plays a key role in the standard approach. On-mass-shell internal lines correspond to singularities of S-matrix and in a twistor approach for $\mathcal{N} = 4$ SUSY, they seem to be enough to generate the full scattering amplitudes.

   If only off-mass-shell scattering amplitudes between on- mass-shell states are allowed, one can argue that only the singularities are allowed, which is not enough.

2. Resonance should correspond to the factorization of S-matrix at resonance, when the intermediate virtual state reduces to an on-mass-shell state. Can the approach based on Kähler metric allow this kind of factorization if the building brick of the scattering amplitudes as the deviation of the covariant Kähler metric from the unit matrix $\delta_{A\overline{B}}$ is the basic building bricks and defined between on mass shell states?

   Note that in the dual resonance model, the scattering amplitude is some over contribution of resonances and I have proposed that a proper generalization of this picture could make sense in the TGD framework.

The basic question concerns the number theoretical identification of on-mass-shell and off-mass-shell states.

1. Galois singlets with integer valued momentum components are the natural identification for on-mass-shell states. Galois non-singlet would be off-mass-shell state naturally having complex quark masses and momentum components as algebraic integers.

   Virtual states could be arbitrary states without any restriction to the components of quark momentum except that they are in the extension of rationals and the condition coming from momentum conservation, which forces intermediate states to be Galois singlets or products of them.

   Therefore momentum conservation allows virtual states as on mass shell states, that is intermediate states, which are Galois singlets but consist of Galois non-singlets identified as off-mass-shell lines. The construction of bound states formed from Galois non-singlets would indeed take place in this way.

2. The expansion of the contravariant part of the scattering matrix $T_1 = T/(1+T)$ appearing in the probability

$$\begin{array}{l} P(A \to B) = g^{A\overline{B}} g_{\overline{B}A} \\ = [1 - \frac{T}{1+T} + T^\dagger - (\frac{T}{1+T})_{AB} T^\dagger]_{AB} \quad . \end{array}$$

   would give a series of analogs of diagrams in which Galois singlets of intermediate states are deformed to non-singlets states.

3. Singularities and resonances would correspond to the reduction of an intermediate state to a product of Galois singlets.

What about the planarity condition in TGD?

The simplest proposal inspired by the experience with the twistor amplitudes is that only planar polygon diagrams are possible since otherwise the area momenta are not well-defined. In the TGD framework, there is no obvious reason for not allowing diagrams involving permutations of external momenta with positive energies *resp.* negative energies since the area momenta $x_{i+1} = \sum_{k=1}^{i} p_k$ are well-defined irrespective of the order. The only manner to uniquely order the Galois singlets as incoming states is with respect to their mass squared values given by integers.

Generalized OZI rule

In TGD, only quarks are fundamental particles and all elementary particles and actually all physical states in the fermionic sector are composites of them. This implies that quark and antiquark numbers are separately conserved in the scattering diagrams and the particle reaction only means the-arrangement of the quarks to a new set of Galois singlets.

At the level of quarks, the scattering would be completely trivial, which looks strange. One would obtain a product of quark propagators connecting the points at mass shells with opposite energies plus entanglement coefficients arranging them at positive and negative energy light-cones to groups which are Galois singlets.

This is completely analogous to the OZI role. In QCD it is of course violated by generation of gluons decaying to quark pairs. In TGD, gauge bosons are also quark pairs so that there is no problem of principle.

There is an objection against this picture.

1. If particle reactions are mere recombinations of Galois singlets with Galois singlets, the quark and antiquark numbers N_q and $N_{\overline{q}}$ of quark and antiquark numbers are separately conserved (as also their difference $N_q - N_{\overline{q}}$). This forbids many reactions, for instance those in which a gauge boson is emitted unless one assumes that many quark states are superpositions of states with a varying total quark number N. This would mean that the extremely simple re-combinatorics picture is lost.

2. Crossing symmetry, which is a symmetry of standard QFTs, suggests a solution to the problem. Crossing symmetry would mean that one can transfer quarks between initial and final states by changing the sign of the quark four-momentum so that momentum conservation is not violated. Crossing means analytic continuation of the scattering amplitude by replacing incoming (outgoing) momentum p with outgoing (incoming) momentum $-p$. The scattering amplitudes for reactions for which the quark number is conserved can be constructed using mere recombinatorics, and the remaining amplitudes would be obtained by crossing.

3. Crossing must respect the Galois singlet property. For instance, the crossing of a single quark destroys Galois singlet. Unless one allows destruction and recombination of Galois singlets, the crossing can apply only to Galois singlets. These rules bring to mind the vanishing of twistor amplitudes when one gluon has negative helicity and the remaining gluons have positive helicity.

10.4.5 Western and Eastern ontologies of physics

This picture forces us to ask whether something deeper might lurk behind the usual ideas about particle physics in which scattering rates encode the information. Could the imaginary part of $P(A, B)$ have a well-defined physical meaning in some more general framework?

1. In ZEO, single classical time evolution and zero energy state as a pair of initial and final states becomes the basic entity. One can even ask whether it might make sense to speak about probability density for different zero energy states as time evolutions, events.

   Could the "western" view about existing reality evolving in time be replaced with an ontology in which events in both classical sense (zero energy states) and quantum transitions would be what really exists.

   In the "eastern" view, the relevant probabilities would not be for transitions $A \to B$ for a given state A but for the occurrence of these transitions $A \to B$ in given state, whatever its definition might be, and one would measure the relative rates for occurrence for the various transitions $A \to B$.

The ensemble would not consist of entities A but transitions $A \to B$. In biology and neuroscience, the states are indeed replaced with behaviors. IKn computer science the program, rather than the state of the computer, is the basic notion.

2. In order to develop this picture at the level of scattering amplitudes, one could start from the QFT description for the n-point correlation functions used to construct S-matrix. One adds to the exponent of action a term, which is a combination of small current terms assignable to external particles and calculates functional Taylor series with respect to the small parameters. The Taylor coefficients are identified as n-point functions.

   In QFTs this is regarded as a mere calculational trick and the "state" defined by the exponential as an analog of that in statistical physics is defined by the exponential of action when the values of the parameters vanish.

   One can of course ask what it would mean if these parameters do not vanish. In perturbation theory one actually has this situation. These deformed states look formally like coherent states. Could the physical states at a deeper level correspond to these analogs of coherent states as analogs of thermo-dynamical states?

3. TGD can be formally regarded as a complex square root of thermodynamics, which suggests a generalization of the formulation of quantum theory as algebraic QFT promoted for instance by Connes [A32], and this is what this new interpretation would mean also physically.

4. In the TGD framework, one would add to the exponent of $exp(-K)$ a superposition of oscillator operator monomials of quark oscillator operators creating positive and negative energy parts of the zero energy states with complex coefficients Z_i as parameters and essentially defining coordinates for the Hilbert space. Z_i would be analogous to the complex numbers defining coherent states.

   The exponential can be expanded and fermionic vacuum expectation forces conservation of quark number and the combination of the positive and negative energy parts to give a non-vanishing result. At the limit of infinitely large CD conservation of 4-momentum is obtained.

5. The ordinary transition amplitudes are obtained by performing the limit $Z_i \to 0$, and calculating Taylor coefficients as transition amplitudes. The analog of $G_{A,\overline{B}}$ would be obtained for the analogs 2-point functions having as arguments the parts of zero energy states and $P(A,B) = Re(G_{A,\overline{B}} G_{\overline{B},A})$ would give transition probabilities. For Kähler geometry the analog of probability conservation and unitarity would hold true.

6. That these amplitudes are obtained as second derivatives with respect to the fermionic Hilbert space complex coordinates Z_i and $\overline{Z}_j$ conforms with the interpretation of the exponential containing the additional terms as a generalization of an exponential of Kähler function associated with the fermionic degrees of freedom. Kähler metric indeed corresponds to $\partial_{Z_I}\partial_{\overline{Z}_J}K$, where K is the Kähler function.

7. Could the expressions of higher n-point functions in fermionic degrees of freedom boil down to the curvature tensor and its covariant derivatives so that quantum theory would be geometrized? If one has a constant curvature space, as strongly suggested by the mere existence of infinite-D Kähler metric, then only $G_{A,\overline{B}}$ would be needed so that it is enough to measure only the scattering probabilities (rates at infinite-volume limit for CD).

   Could the parameters Z_i be non-vanishing and define a square root of a thermodynamic state as an analog of a coherent state? If a constant curvature metric is in question, the scattering rates for non-vanishing Z_i could be expressed in terms of those for $Z_i = 0$. Could different phases of quantum theory correlate with the value ranges of the parameters Z_i?

10.4.6 Connection with the notion of Fisher information

The notion of Fisher information (`https://cutt.ly/GUPvF37`) relates in an interesting manner to the proposed Kähler geometrization of quantum theory.

1. Fisher information matrix F is associated with a probability density function $f(X,Z)$ for random variables X_i depending on the parameters Z_i (Z_i are denoted by θ_i in the Wikipedia article at `https://cutt.ly/GUPvF37`). Matrix F gives information about the $f(X,\theta)$, which must be deduced from the measurements of X. The matrix element F_{ij} is essentially integral over X for the quantity $\langle \partial_{\theta_i}\partial_{\theta_j} log(f)\rangle$, where $\langle ..\rangle$ denotes the expectation obtained by integrating over X. F_{ij} determines a statistical metric and for complex parameters Z_i one obtains a Kähler metric.

2. In TGD, X would correspond to WCW coordinates and f would be analogous to the vacuum functional $exp(-K)$ but containing also a parameter dependent part defined by the combination of positive and negative energy parts of the fermionic zero energy states. The complex coefficients Z_i *resp.* $\overline{Z}_i$ of monomials of creation *resp.* annihilation operators would define the parameters. Fermionic Kähler metric would have an interpretation as Fisher information, which can be also complex valued.

3. Also the higher derivatives with respect to coefficients of zero energy states would provide information about the vacuum functional. One would have n-point functions for zero energy states possibly reducing to covariant derivatives of the analog curvature tensor. If the space of fermionic zero energy states is analog of a constant curvature space, the scattering amplitudes at the limit $Z_i = 0$ would give all the needed information needed to calculate the scattering amplitudes for $Z_i \neq 0$. $P(A,B)$ would be complex as components of the Fisher information matrix.

4. Basically, the information provided by the scattering amplitudes would be about the generalization of the vacuum functional of WCW including also the fermionic part. Scattering amplitudes would give information Kähler function of the WCW metric and about parameters Z_i.

   The scattering amplitudes indeed correlate strongly with the properties of space-time surfaces determined by polynomials. The p-adic prime p, crucial for the real scattering amplitudes as canonical images of p-adic amplitudes, corresponds to a ramified prime for P and this means localization of the vacuum functional to polynomials having a ramified prime equal to p. The number of Galois singlets in the scattering amplitude means lower bound for the degree of P.

10.4.7 About the relationship of Kähler approach to the standard picture

The replacement of the notion of unitary S-matrix with Kähler metric of fermionic state space generalizes the notion of unitarity. The rows of the matrix defined by the contravariant metric are orthogonal to the columns of the covariant metric in the inner product $(T\circ U)_{A\overline{B}} = T_{A\overline{C}}\eta^{\overline{C}D}U_{D\overline{B}}$, where $\eta^{\overline{C}D}$ is flat contravariant Kähler metric of state space. Although the probabilities are complex, their real parts sum up to 1 and the sum of the imaginary parts vanishes.

The counterpart of the optical theorem in TGD framework

The Optical Theorem generalizes. In the standard form of the optical theorem $i(T-T^{\dagger})_{mm} = 2Im(T) = TT^{\dagger}_{m,m}$ states that the imaginary part of the forward scattering amplitude is proportional to the total scattering rate. Both quantities are physical observables.

In the TGD framework the corresponding statement

$$T^{A\overline{B}}\eta_{\overline{B}C} + \eta^{A\overline{B}}T_{\overline{B}C} + T^{A\overline{B}}T_{\overline{B}C} = 0 \quad . \tag{10.4.5}$$

Note that here one has $G = \eta + T$, where G and T are hermitian matrices. The correspondence with the standard situation would require the definition $G = \eta + iU$. The replacement $T \rightarrow T = iU$, where U is antihermitian matrix, gives

One has

$$i[U^{A\overline{B}}\eta_{\overline{B}C}+\eta^{A\overline{B}}I_{\overline{B}C}]=U^{A\overline{B}}U_{\overline{B}C} \ . \tag{10.4.6}$$

This statement does not reduce to single condition but gives two separate conditions.

1. The first condition is analogous to Optical Theorem:

$$Im[\eta^{A\overline{B}}U_{C\overline{B}}+U^{A\overline{B}}\eta_{\overline{B}C}]=-Re[U^{A\overline{B}}U_{\overline{B}C}]=Re[U^{A\overline{B}}U_{C\overline{B}}] \ . \tag{10.4.7}$$

2. Second condition is new and reflects the fact that the probabilities are complex. It is necessary to guarantee that the sum of the probabilities reduces to the sum of their real parts.

$$Re[\eta^{A\overline{B}}U_{C\overline{B}}+U^{A\overline{B}}\eta_{\overline{B}C}]=-Im[U^{A\overline{B}}U_{C\overline{B}}] \ . \tag{10.4.8}$$

   The challenge would be to find a physical meaning for the imaginary parts of scattering probabilities. This is discussed in [L91]. The idea is that the imaginary parts could make themselves visible in a Markov process involving a power of the complex probability matrix.

In the applications of the optical theorem, the analytic properties of the scattering matrix T make it possible to construct the amplitude as a function of mass shell momenta using its discontinuity at the real axis. Indeed, $2Im(T)$ for the forward scattering amplitude can be identified as the discontinuity $Disc(T)$. In the recent case, this identification would suggests the generalization

$$Disc[T^{A\overline{B}}\eta_{\overline{B}C}]=T^{A\overline{B}}\eta_{\overline{B}C}+\eta^{A\overline{B}}T_{C\overline{B}} \ . \tag{10.4.9}$$

Therefore covariant and contravariant Kähler metric could be limits of the same analytic function from different sides of the real axis. One assigns the hermitian conjugate of S-matrix to the time reflection. Are covariant and contravariant forms of Kähler metric related by time reversal? Does this mean that T symmetry is violated for a non-flat Kähler metric.

The emergence of QFT type scattering amplitudes at long length scale limit

The basic objection against the proposal for the scattering amplitudes is that they are non-vanishing only at mass shells with $m^2=n$. A detailed analysis of this objection improves the understanding about how the QFT limit of TGD emerges.

1. The restriction to the mass shells replaces cuts of QFT approach with a discrete set of masses. The TGD counterpart of unitarity and optical theorem holds true at the discrete mass shells.
2. The p-adic mass scale for the reaction region is determined by the largest ramified prime RP for the functional composite of polynomials characterizing the Galois singlets participating in the reaction. For large values of ramified prime RP for the reaction region, the p-adic mass scale increases and therefore the momentum resolution improves.
3. For large enough RP below measurement resolution, one cannot distinguish the discrete sequence of poles from a continuum, and it is a good approximation to replace the discrete set of mass shells with a cut. The physical analogy for the discrete set of masses along the real axis is as a set of discrete charges forming a linear structure. When their density becomes high enough, the description as a line charge is appropriate and in 2-D electrostatistics this replaces the discrete set of poles with a cut.

This picture suggests that the QFT type description emerges at the limit when RP becomes very large. This kind of limit is discussed in the article considering the question whether a notion of a polynomial of infinite degree as an iterate of a polynomial makes sense [L96]. It was found that the number of the roots is expected to become dense in some region of the real line so that effectively the QFT limit is approached.

1. If the polynomial characterizing the scattering region corresponds to a composite of polynomials participating in the reaction, its degree increases with the number of external particles. At the limit of an infinite number of incoming particles, the polynomial approaches a polynomial of infinite degree. This limit also means an approach to a chaos as a functional iteration of the polynomial defining the space-time surface [L84]. In the recent picture, the iteration would correspond to an addition of particles of a given type characterized by a fixed polynomial. Could the characteristic features for the approach of chaos by iteration, say period doubling, be visible in scattering in some situations. Could p-adic length scale hypothesis stating that p-adic primes near powers of two are favored, relate to this.

2. For a large number of identical external particles, the functional composite defining RG involves iteration of polynomials associated with particles of a particular kind, if their number is very large. For instance, the radiation of IR photons and IR gravitons in the reaction increases the degree of RP by adding to P very high iterates of a photonic or gravitonic polynomial.

   Gravitons could have a large value of ramified prime as the approximately infinite range of gravitational interaction and the notion of gravitational Planck constant [L50, L106] originally proposed by Nottale [E1] suggest. If this is the case, graviton corresponds to a polynomial of very high degree, which increases the p-adic length scale of the reaction region and improves the momentum resolution. If the number of gravitons is large, this large RP appears at very many steps of the SFR cascade.

A connection with dual resonance models

There is an intriguing connection with the dual resonances models discussed already in [L58].

1. The basic idea behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. The 4-particle amplitude of Veneziano was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged.

2. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have a representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

3. The resonances have zero width and the imaginary part of the amplitude has a discontinuity only at the resonance poles, which is not consistent with unitarity so that one must force unitarity by hand by an iterative procedure. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of the twistor Grassmann approach.

It is interesting to compare this picture with the twistor Grassman approach and TGD picture.

1. In the TGD framework, one just picks up the residue of what would be analogous to stringy scattering amplitude at mass shells. In the dual resonance models, one keeps the entire amplitude and encounters problems with the unitarity outside the poles. In the twistor Grassmann approach, one assumes that the amplitudes are completely determined by the singularities whereas in TGD they *are* the residues at singularities. At the limit of an infinite-fold iterate the amplitudes approach analogs of QFT amplitudes.

2. In the dual resonance model, the sums over s- and t-channel resonances are the same. This guarantees crossing symmetry. An open question is whether this can be the case also in the TGD framework. If this is the case, the continuum limit of the scattering amplitudes should have a close resemblance with string model scattering amplitudes as the $M^4 \times CP_2$ picture having magnetic flux tubes in a crucial role indeed suggests.

3. In dual resonance models, only the cyclic permutations of the external particles are allowed. As found, the same applies in TGD if the scattering event is a cognitive measurement [L90], only the cyclic permutations of the factors of a fixed functional composite are allowed. Non-cyclic permutations would produce the counterparts of non-planar diagrams and the cascade of cognitive state function reductions could not make sense for all polynomials in the superposition simultaneously. Remarkably, in the twistor Grassmann approach just the non-planar diagrams are the problem.

10.5 Some useful objections

The details of the proposed construction of the scattering amplitudes starting from twistors are still unclear and the best way to proceed is to invent objections and critical questions.

10.5.1 How the quark momenta in M^8 and H relate to each other?

The relationship between quark momenta in M^8 and H is not clear. There are four options to consider corresponding to the Dirac propagators in H and M^4 with or without coupling to $A(M^4)$. I assign to these options attributea $D(H, A)$, $D(H)$, $D(M^4, A)$ and $D(M^4)$. For all options something seems to go wrong.

Consider fits the list of criteria that the correct option should satisfy.

1. $M^8 - H$ duality suggests the same momentum and mass spectrum for quarks in M^8 and H.

   (a) However, the mass spectrum of color partial waves for quark spinors for $D(H)$ and $D(H, A)$ is very simple and characterized by 2 integers labeling triality $t = 1$ representations of $SU(3)$ [L2]. Neither $D(H)$ or $D(H, A)$ allows a mass spectrum as algebraic roots of polynomials and seems to be excluded.

   (b) If $M^8 - H$ duality holds true in a strong sense so that these spectra are identical, the only possible conclusion seems to be that the propagator in both M^8 and H is just the M^4 Dirac propagator $D(M^4)$ and that the roots of the polynomial P give the spectrum of off-mass-shell masses. Also tachyonic mass squared values are allowed as roots of P. The real on-shell masses would be associated with Galois singlets.

2. Twistor holomorphy and associativity leave only the $D(M^4)$ option. The couplings to $A(M^4)$ and presence of $D(CP_2)$ spoil these properties. $D(M^4)$ option has very nice features. The integration over the momentum space reduces to a finite summation over virtual mass shells defined by the roots of P and one avoids divergences. This tightens the connection with QFTs. For $D(M^4()$ this nice property is lost. Massless quarks are also consistent with the QCD picture about quarks.

3. The predictions of p-adic mass calculations [K52, K21] were sensitive to the negative ground state conformal weight h_{vac} depending on the electroweak isospin and gave rise to electroweak symmetry breaking. h_{vac} could be generated by conformal generators with weights h coming as algebraic integers determined by P. This favors $D(H)$ and $D(H, A)$. $D(H, A)$ predicts tachyonic ν_R, which was necessary for the calculation. Only $D(H, A)$ survives.

4. For some years ago, I found that the space-time propagators for points of H connected by a light-like geodesic behave like massless propagators irrespective of mass. CP_2 type extermals have a light-like geodesic as an M^4 projection. This would suggest that quarks associated with CP_2 type extremals effectively propagate as massless particles even if one assumes that they correspond to modes of the full H Dirac operator. This allows us to consider $D(H)$ as an alternative. For this option most quarks in the interior of the space-time surface would be extremely massive and practically absent.

5. Suppose that one takes seriously the idea that the situation can be described also by using massless M^8 momenta. This implies that for some choices of $M^4 \subset M^8$ the momentum is parallel to M^4 and therefore massless in 4-D sense. Only the quarks associated with the same M^4 can interact. Hence M^4 can be always chosen so that the on mass-shell 4-momenta are light-like. $D(H,A)$ option could be correct but $D(M^4)$ option would appear as an effective option obtained by a suitable choice of $M^4 \subset M^8$.

6. The consideration of problems related to right-handed neutrino [L100] led to the question whether the quark spinor modes in H are annihilated only by the H d'Alembertian $D^2(H,A(M^4))$ but not by the H Dirac operator [L100]. The assumption that on mass shell H-spinors are annihilated by $D(M^4,A)$ leads to the same outcome.

   D^2 options allow different M^4 chiralities to propagate separately and solves problems related to the notion of right-handed neutrino ν_R (assumed to be 3-antiquark state and modellable using leptonic spinors in H. This also conforms with the right and left-handed character of the standard model couplings. However, the mixing of M^4 chiralities serves as a signature for the massivation and is lost.

   If leptons are allowed as fundamental fermions, $D(H)$ allows ν_R as a spinor mode, which is covariantly constant in CP_2. If leptons are not allowed, one can argue that ν_R as a 3-quark state can be modeled as a mode of H spinor with Kähler coupling yielding correct leptonic charges.

   The M^4 Kähler structure favored by the twistor lift of TGD [L58] implies that ν_R with negative mass squared appears as a mode of $D(H)$. This mode allows the construction of tachyonic ground states. This is lost for $D(M^4)$ with coupling to $A(M^4)$.

   For $D(M^4,A)$, one obtains for all spinor modes states with both positive and negative mass squared from the $J_{kl}\Sigma^{kl}$ term. Physical on-mass- shell states with negative mass squared cannot be allowed. These would however allow to construct tachyonic ground states needed in the p-adic mass calculations. Now the problem is that $D(M^4,A)$ as propagator spoils twistor holomorphy.

7. Since the color group acts as symmetries, one can assume that spinor modes correspond to color partial waves as eigen states of CP_2 spinor d'Alembertian $D^2(CP_2)$. This predicts that different M^4 chiralities propagate independently. $D(M^4)$ and $D(M^4,A)$ options make the same prediction. For the $D(H)$ and $D(H,A)$ option one obtains a mixing of M^4 chiralities having interpretation in terms of massivation.

   For all options the correlation between color and electroweak quantum numbers is "wrong". This is however not a problem for off-mass-shell fundamental quarks since the physical states are obtained as SSA representations.

To sum up, $D(H,A)$ is strongly favored by the p-adic thermodynamics, by the possibility to build the physical quarks using SSA, by the fact that propagators over-light-like distances do not depend on mass, and also by the freedom to choose $M^4 \subset M^8$ in such a way that on mass shell spinor mode is massless. $D(M^4)$ is strongly favoured by M^8-H duality (associativity) and by twistor analyticity. Both options seem to be both right and wrong. This suggests that something is wrong with the interpretation of the notion of the Dirac propagator.

1. From the view point of H, M^8 quarks are off-mass-shell whereas from the M^8 point of view they are on-mass-shell. Suppose that off-mass shell quarks in the sense of $D(H,A)$ differ from on-mass-shell quarks only in that they have M^4 momentum $p_{off}=p_{on}+\Delta p$ differing by Δp from the on-mass shell momentum p_{on} with integer components and satisfying mass shell condition for $D(H)$. In CP_2 these states are on-mass-shell. Suppose that p_{off} is on M^8 mass shell determined as a root of P.

   With these assumptions, one can write Dirac operator as $D(H,A,off) = D(H,A,on) + \Delta p^k gamma_k$, whose action to incoming Galois singlets reduces to $D(H,A,off) = \Delta p^k gamma_k = D(M^4)$. This is just the free massless propagator.

2. The propagating entities would be basically solutions of $D(H,A)$ with an off-mass-shell M^4-momentum with Δp having mass. In particular, they are superpositions of components with

left- and right-handed M^4 chiralities having opposite CP_2 chiralities and the mixing of M^4 chiralities can be seen as a signature of massivation. On the other hand, $D(M^4)$ does not depend on M^4 chirality. Maybe this option could avoid all objections!

10.5.2 Can one allow "wrong" correlation between color and electroweak quantum numbers for fundamental quarks?

For CP_2 harmonics, the correlation between color and electroweak quantum numbers is wrong [K52]. Therefore the physical quarks cannot correspond to the solutions of $D^2(H)\Psi = 0$. The same applies also to the solutions of $D(M^4)\Psi = 0$ if one assumes that they belong to irreducible representations of the color group as eigenstates of $D(CP_2)$.

How to construct quark states, which are physical in the sense that they are massless and color-electroweak correlation is correct?

1. The reduction of quark masses to zero requires a tachyonic ground state in p-adic mass calculations [K52]. The assumption that physical states are constructed using quarks, which are on-mass-shell in the M^8 sense but off-mass-shell in the H sense.

   Colored operators with non-vanishing conformal weight are required to make all quark states massless color triplets. This is possible only if the ground state is tachyonic, which gives strong support for M^4 Kähler structure.

2. This is achieved by the identification of physical quarks as states of super-symplectic representations. Also the generalized Kac-Moody algebra assignable to the light-like partonic orbits or both of these representations can be considered. These representations could correspond to inertial and gravitational representations realized at "objective" embedding space level and "subjective" space-time level.

   Supersymplectic generators are characterized by a conformal weight h completely analogous to mass squared. The conformal weights naturally correspond to algebraic integers associated with P. The mass squared values for the Galois singlets are ordinary integers.

3. It is plausible that also massless color triplet states of quarks can be constructed as color singlets. From these one can construct hadrons and leptons as color singlets for a larger extension of rationals. This conforms with the earlier picture about conformal confinement. These physical quarks constructed as states of super-symplectic representation, as opposed to modes of the H spinor field, would correspond to the quarks of QCD.

   One can argue that Galois confinement allows to construct physical quarks as color triplets for some polynomial Q and also color singlets bound states of these with extended Galois group for a higher polynomial $P \circ Q$ and with larger Galois group as representation of group $Gal(P)/Gal(Q)$ allowing representations of a discrete subgroup of color group.

10.5.3 Can one allow complex quark masses?

One objection relates to unitarity. Complex energies and mass squared values are not allowed in the standard picture based on unitary time evolution.

1. Here several new concepts lend a hand. Galois confinement could solve the problems if one considers only Galois singlets as physical particles. ZEO replaces quantum states with entangled pairs of positive and negative energy states at the boundaries of CD and entanglement coefficients define transition amplitudes.

   The notion of the unitary time evolution is replaced with the Kähler metric in quark degrees of freedom and its components correspond to transition amplitudes. The analog of the time evolution operator assignable to SSFRs corresponds naturally to a scaling rather than time translation and mass squared operator corresponds to an infinitesimal scaling.

2. The complex eigenvalues of mass squared as roots of P be allowed when unitarity at quark level is not required to achieve probability conservation. For complex mass squared values, the entanglement coefficients for quarks would be proportional to mass squared exponents

$exp(im^2\lambda)$, λ the scaling parameter analogous to the duration of time evolution. For Galois singlets these exponentials would sum up to imaginary ones so that probability conservation would hold true.

10.5.4 Are M^8 spinors as octonionic spinors equivalent with H-spinors?

At the level of M^8 octonionic spinors are natural. M^8-H duality requires that they are equivalent with H-spinors. The most natural identification of octionic spinors is as bi-spinors, which have octonionic components. Associativity is satisfied if the components are complexified quaternionic so that they have the same number of components as quark spinors in H. The H spinors can be induced to $X^4 \subset M^8$ by using M^8-H duality. Therefore the M^8 and H pictures fuse together.

The quaternionicity condition for the octonionic spinors is essential. Octonionic spinor can be expressed as a complexified octonion, which can be identified as momentum p. It is not an on-mass shell spinor. The momenta allowed in scattering amplitudes belong to mass shells defined by the polynomial P. That octonionic spinor has only quaternionic components conforms with the quaternionicity of $X^4 \subset M^8$ eliminating the remaining momentum components and also with the use of $D(M^4)$.

10.5.5 Two objections against p-adic thermodynamics and their resolution

Unlike the Higgs mechanism, p-adic thermodynamics provides a universal description of massivation involving no other assumptions about dynamics except super-conformal symmetry, which guarantees the existence of p-adic Boltzmann weights.

There are two basic objections against p-adic thermodynamics. The mass calculations require the presence of states with negative conformal weights giving rise to tachyons. Furthermore, by conformal invariance L_0 should annihilate physical states so that all states should have vanishing mass squared! In this article a resolution of these objections, based on the very definition of thermodynamics and on number theoretic vision predicting quark states with discretized tachyonic mass, which are counterparts for virtual states in QFTs, is discussed.

Physical states for the entire Universe would be indeed massless but for subsystems such as elementary particles the thermal expectation of the mass squared is non-vanishing. This conforms with the formula of blackhole entropy stating that it is proportional to the mass square of the blackhole and vanishes for vanishing mass: this would indeed correspond to a pure state.

p-Adic thermodynamics

Number theoretic physics involves the combination of real and various p-adic physics to adelic physics [L43, L42], and classical number fields [K91]. p-Adic mass calculations is a rather successful application of p-adic thermodynamics for the mass squared operator identified as conformal scaling generator L_0. p-Adic thermodynamics can be also understood as a constraint on a real thermodynamics for the mass squared from the condition that it can be also regarded as a p-adic thermodynamics.

The motivation for p-adicization came from p-adic mass calculations [K52, K21].

1. p-Adic thermodynamics for mass squared operator M^2 proportional to scaling generator L_0 of Virasoro algebra. Mass squared thermal mass from the mixing of massless states with states with mass of order CP_2 mass.

2. $exp(-E/T) \rightarrow p^{L_0/T_p}$, $T_p = 1/n$. Partition function p^{L_0/T_p}. p-Adic valued mass squared mapped to a real number by canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$. Eigenvalues of L_0 must be integers for the Boltzmann weights to exist. Conformal invariance guarantees this.

3. p-adic length scale $L_p \propto \sqrt{p}$ from Uncertainty Principle ($M \propto 1/\sqrt{p}$). p-Adic length scale hypothesis states that p-adic primes characterizing particles are near to a power of 2: $p \simeq 2^k$. For instance, for an electron one has $p = M^{127} - 1$, Mersenne prime. This is the largest not completely super-astrophysical length scale.

Also Gaussian Mersenne primes $M_{G,n} = (1+i)^n - 1$ seem to be realized (nuclear length scale, and 4 biological length scales in the biologically important range 10 nm,2.5 μm).

4. p-Adic physics [K62] is interpreted as a correlate for cognition. Motivation comes from the observation that piecewise constant functions depending on a finite number of pinary digits have a vanishing derivative. Therefore they appear as integration constants in p-adic differential equations. This could provide a classical correlate for the non-determinism of imagination.

Objections and their resolution

The number theoretic picture leads to a deeper understanding of a long standing objection against p-adic thermodynamics [K52] as a thermodynamics for the scaling generator L_0 of Super Virasoro algebra.

If one requires super-Virasoro symmetry and identifies mass squared with a scaling generator L_0, one can argue that only massless states are possible since L_0 must annihilate these states! All states of the theory would be massless, not only those of fundamental particles as in conformally invariant theories to which twistor approach applies! This looks extremely beautiful mathematically but seems to be in conflict with reality already at single particle level!

The resolution of the objection is that *thermodynamics* is indeed in question.

1. Thermodynamics replaces the state of the entire system with the density matrix for the subsystem and describes approximately the interaction with the environment inducing the entanglement of the particle with it. To be precise, actually a "square root" of p-adic thermodynamics could be in question, with probabilities being replaced with their square roots having also phase factors. The excited states of the entire system indeed are massless [L113].

2. The entangling interaction gives rise to a superposition of products of single particle massive states with the states of environment and the entire mass squared would remain vanishing. The massless ground state configuration dominates and the probabilities of the thermal excitations are of order $O(1/p)$ and extremely small. For instance, for the electron one has $p = M_{127} = 2^{127} - 1 \sim 10^{38}$.

3. In the p-adic mass calculations [K52, K21], the effective environment for quarks and leptons would in a good approximation consist of a wormhole contact (wormhole contacts for gauge bosons and Higgs and hadrons). The many-quark state many-quark state associated with the wormhole throat (single quark state for quarks and 3-quark-state for leptons [L94].

4. In M^8 picture [L82, L83], tachyonicity is unavoidable since the real part of the mass squared as a root of a polynomial P can be negative. Also tachyonic real but algebraic mass squared values are possible. At the H level, tachyonicity corresponds to the Euclidean signature of the induced metric for a wormhole contact.

   Tachyonicity is also necessary: otherwise one does not obtain massless states. The super-symplectic states of quarks would entangle with the tachyonic states of the wormhole contacts by Galois confinement.

5. The massless ground state for a particle corresponds to a state constructed from a massive single state of a single particle super-symplectic representation (CP_2 mass characterizes the mass scale) obtained by adding tachyons to guarantee masslessness. Galois confinement is satisfied. The tachyonic mass squared is assigned with wormhole contacts with the Euclidean signature of the induced metric, whose throats in turn carry the fermions so that the wormhole contact would form the nearby environment.

   The entangled state is in a good approximation a superposition of pairs of massive single-particle states with the wormhole contact(s). The lowest state remains massless and massive single particle states receive a compensating negative mass squared from the wormhole contact. Thermal mass squared corresponds to a single particle mass squared and does not take into account the contribution of wormhole contacts except for the ground state.

6. There is a further delicate number theoretic element involved [L100, L107]. The choice of $M^4 \subset M^8$ for the system is not unique. Since M^4 momentum is an M^4 projection of a massless M^8 momentum, it is massless by a suitable choice of $M^4 \subset M^8$. This choice must be made for the environment so that both the state of the environment and the single particle ground state are massless. For the excited states, the choice of M^4 must remain the same, which forces the massivation of the single particle excitations and p-adic massivation.

All physical states are massless!

These arguments strongly suggest that pure states, in particular the state of the entire Universe, are massless. Mass would reflect the statistical description of entanglement using a density matrix. The proportionality between p-adic thermal mass squared (mappable to real mass squared by canonical identification) and the entropy for the entanglement of the subsystem-environment pair is therefore natural.

This proportionality conforms with the formula for the blackhole entropy, which states that the blackhole entropy is proportional to mass squared. Also p-adic mass calculations inspired the notion of blackhole-elementary particle analogy [K66] but without a deeper understanding of its origin.

One implication is that virtual particles are much more real in the TGD framework than in QFTs since they would be building bricks of physical states. A virtual particle with algebraic value of mass squared would have a discrete mass squared spectrum given by the roots of a rational, possibly monic, polynomial and M^8-H duality suggests an association to an Euclidean wormhole contact as the "inner" world of an elementary particle. Galois confinement, universally responsible for the formation of bound states, analogous to color confinement and possibly explaining it, would make these virtual states invisible [L108, L109].

Relationship with Higgs mechanism

Polynomials P have two kinds of solutions depending on whether their roots determine either mass or energy shells. For the energy option a space-time region corresponds by M^8-H duality to a solution spectrum in which the roots correspond to energies rather than mass squared values and light-cone proper time is replaced with linear Minkoski time [L82, L83]. The physical interpretation of the energy shell option has remained unclear.

The energy shell option gives rise to a p-adic variant of the ordinary thermodynamics and requires integer quantization of energy. This option is natural for massless states since scalings leave the mass shell invariant in this case. Scaling invariance and conformal invariance are not violated.

One can wonder what the role of these massless virtual quark states in TQC could be. A good guess is that the two options correspond to phases with broken *resp.* unbroken conformal symmetry. In gauge theories they correspond to phases with broken and unbroken gauge symmetries. The breaking of gauge symmetry indeed induces breaking of conformal symmetry and this breaking is more fundamental.

1. Particle massivation corresponds in gauge theories to symmetry breaking caused by the generation of the Higgs vacuum expectation value. Gauge symmetry breaking induces a breaking of conformal symmetry and particle massivation. In the TGD framework, the generation of entanglement between members of state pairs such that members having opposite values of mass squared determined as roots of polynomial P in the most general case, leads to a breaking of conformal symmetry for each tensor factor and the description in terms of p-adic thermodynamics gives thermal mass squared.

2. What about the situation when energy, instead of mass squared, comes as a root of P. Also now one can construct physical states from massless virtual quarks with energies coming as algebraic integers. Total energies would be ordinary integers. This gives massless entangled states, if the rational integer parts of 4-momenta are parallel. This brings in mind a standard twistor approach with parallel light-like momenta for on-mass shell states. Now however the virtual states can have transversal momentum components which are algebraic numbers (possibly complex) but sum up to zero.

Quantum entangled states would be superpositions over state pairs with parallel massless momenta. Massless extremals (topological light rays) are natural classical space-time correlates for them. This phase would correspond to the phase with unbroken conformal symmetry.

3. One can also assign a symmetry breaking to the thermodynamic massivation. For the energy option, the entire Galois group appears as symmetry of the mass shell whereas for the mass squared option only the isotropy group does so. Therefore there is a symmetry breaking of the full Galois symmetry to the symmetry defined by the isotropy group. In a loose sense, the real valued argument of P serves as a counterpart of the Higgs field.

   If the symmetry breaking in the model of electroweak interaction corresponds to this kind of symmetry breaking, the isotropy group, which presumably involves also a discrete subgroup of quaternionic automorphisms as an analog of the Galois group. Quaternionic group could act as a discrete subgroup of $SU(2) \subset SU(2)_L \times U(1)$. The hierarchy of discrete subgroups associated with the hierarchy of Jones inclusions assigned with measurement resolution suggests itself. It has the isometry groups of Platonic solids as the groups with genuinely 3-D action. $U(1)$ factor could correspond to Z_n as the isotropy group of the Galois group. In the QCD picture about strong interactions there is no gauge symmetry breaking so that a description based on the energy option is natural. Hadronic picture would correspond to mass squared option and symmetry breaking to the isotropy group of the root.

To sum up, in the maximally symmetric scenario, conformal symmetry breaking would be only apparent, and due to the necessity to restrict to non-tachyonic subsystems using p-adic thermodynamics. Gauge symmetry breaking would be replaced with the replacement of the Galois group with the isotropy group of the root representing mass squared value. The argument of the polynomial defining space-time region would be the analog of the Higgs field.

10.5.6 Some further comments about the notion of mass

In the sequel some further comments related to the notion of mass are represented.

$M^8 - H$ duality and tachyonic momenta

Tachyonic momenta are mapped to space-like geodesics in H or possibly to the geodesics of X^4 [L82, L83, L104]. This description could allow to describe pair creation as turning of fermion backwards in time [L109]. Tachyonic momenta correspond to points of de Sitter space and are therefore outside CD and would go outside the space-time surface, which is inside CD. Could one avoid this?

1. Since the points of the twistor spaces $T(M^4)$ and $T(CP_2)$ are in 1-1 correspondence, one can use either $T(M^4)$ or $T(CP_2)$ so that the projection to M^4 or CP_2 would serve as the base space of $T(X^4)$. One could use CP_2 coordinates or M^4 coordinates as space-time coordinates if the dimension of the projection is 4 to either of these spaces. In the generic case, both dimensions are 4 but one must be very cautious with genericity arguments which fail at the level of M^8.

2. There are exceptional situations in which genericity fails at the level of H. String-like objects of the form $X^2 \times Y^2 \subset M^4 \subset CP_2$ is one example of this. In this case, X^6 would not define 1-1 correspondence between $T(M^4)$ or $T(CP_2)$.

   Could one use partial projections to M^2 and S^2 in this case? Could $T(X^4)$ be divided locally into a Cartesian product of 3-D M^4 part projecting to $M^2 \subset M^4$ and of 3-D CP_2 part projected to $Y^2 \subset CP_2$.

3. One can also consider the possibility of defining the twistor space $T(M^2 \times S^2)$. Its fiber at a given point would consist of light-like geodesics of $M^2 \times S^2$. The fiber consists of direction vectors of light-like geodesics. S^2 projection would correspond to a geodesic circle $S^1 \subset S^2$ going through a given point of S^2 and its points are parametrized by azimuthal angle Φ. Hyperbolic tangent $tanh(\eta)$ with range $[-1, 1]$ would characterize the direction of a time like geodesic in M^2. At the limit of $\eta \rightarrow \pm\infty$ the S^2 contribution to the S^2 tangent vector to

length squared of the tangent vector vanishes so that all angles in the range $(0, 2\pi)$ correspond to the same point. Therefore the fiber space has a topology of S^2.

There are also other special situations such as $M^1 \times S^3$, $M^3 \times S^1$ for which one must introduce specific twistor space and which can be treated in the same way.

During the writing of this article I realized that the twistor space of H defined geometrically as a bundle, which has as H as base space and fiber as the space of light-like geodesic starting from a given point of H need not be equal to $T(M^4) \times T(CP_2)$, where $T(CP_2)$ is identified as $SU(3)/U(1) \times U(1)$ characterizing the choices of color quantization axes.

1. The definition of $T(CP_2)$ as the space of light-like geodesics from a given point of CP_2 is not possible. One could also define the fiber space of $T(CP_2)$ geometrically as the space of geodesics emating from origin at $r = 0$ in the Eguchi-Hanson coordinates [L4] and connecting it to the homologically non-trivial geodesic sphere S^2_G $r = \infty$. This relation is symmetric.

   In fact, all geodesics from $r = 0$ end up to S^2. This is due to the compactness and symmetries of CP_2. In the same way, the geodesics from the North Pole of S^2 end up to the South Pole. If only the endpoint of the geodesic of CP_2 matters, one can always regard it as a point S^2_G.

   The two homologically non-trivial geodesic spheres associated with distinct points of CP_2 always intersect at a single point, which means that their twistor fibers contain a common geodesic line of this kind. Also the twistor spheres of $T(M^4)$ associated with distinct points of M^4 with a light-like distance intersect at a common point identifiable as a light-like geodesic connecting them.

2. Geometrically, a light-like geodesic of H is defined by a 3-D momentum vector in M^4 and 3-D color momentum along CP_2 geodesic. The scale of the 8-D tangent vector does not matter and the 8-D light-likeness condition holds true. This leaves 4 parameters so that $T(H)$ identified in this way is 12-dimensional.

   The M^4 momenta correspond to a mass shell H^3. Only the momentum direction matters so that also in the M^4 sector the fiber reduces to S^2 . If this argument is correct, the space of light-like geodesics at point of H has the topology of $S^2 \times S^2$ and $T(H)$ would reduce to $T(M^4) \times T(CP_2)$ as indeed looks natural.

Conformal confinement at the level of H

The proposal of [L117], inspired by p-adic thermodynamics, is that all states are massless in the sense that the sum of mass squared values vanishes. Conformal weight, as essentially mass squared value, is naturally additive and conformal confinement as a realization of conformal invariance would mean that the sum of mass squared values vanishes. Since complex mass squared values with a negative real part are allowed as roots of polynomials, the condition is highly non-trivial.

$M^8 - H$ duality [L82, L83] would make it natural to assign tachyonic masses with CP_2 type extremals and with the Euclidean regions of the space-time surface. Time-like masses would be assigned with time-like space-time regions. In [L115] it was found that, contrary to the beliefs held hitherto, it is possible to satisfy boundary conditions for the action action consisting of the Kähler action, volume term and Chern-Simons term, at boundaries (genuine or between Minkowskian and Euclidean space-time regions) if they are light-like surfaces satisfying also $detg_4 = 0$. Masslessness, at least in the classical sense, would be naturally associated with light-like boundaries (genuine or between Minkowskian and Euclidean regions).

About the analogs of Fermi torus and Fermi surface in H^3

Fermi torus (cube with opposite faces identified) emerges as a coset space of E^3/T^3, which defines a lattice in the group E^3. Here T^3 is a discrete translation group T^3 corresponding to periodic boundary conditions in a lattice.

In a realistic situation, Fermi torus is replaced with a much more complex object having Fermi surface as boundary with non-trivial topology. Could one find an elegant description of the situation?

1. Hyperbolic manifolds as analogies for Fermi torus?

The hyperbolic manifold assignable to a tessellation of H^3 defines a natural relativistic generalization of Fermi torus and Fermi surface as its boundary. To understand why this is the case, consider first the notion of cognitive representation.

1. Momenta for the cognitive representations [L116] define a unique discretization of 4-surface in M^4 and, by M^8-H duality, for the space-time surfaces in H and are realized at mass shells $H^3 \subset M^4 \subset M^8$ defined as roots of polynomials P. Momentum components are assumed to be algebraic integers in the extension of rationals defined by P and are in general complex.

   If the Minkowskian norm instead of its continuation to a Hermitian norm is used, the mass squared is in general complex. One could also use Hermitian inner product but Minkowskian complex bilinear form is the only number-theoretically acceptable possibility. Tachyonicity would mean in this case that the real part of mass squared, invariant under $SO(1,3)$ and even its complexification $SO_c(1,3)$, is negative.

2. The active points of the cognitive representation contain fermion. Complexification of H^3 occurs if one allows algebraic integers. Galois confinement [L116, L112] states that physical states correspond to points of H^3 with integer valued momentum components in the scale defined by CD.

   Cognitive representations are in general finite inside regions of 4-surface of M^8 but at H^3 they explode and involve all algebraic numbers consistent with H^3 and belonging to the extension of rationals defined by P. If the components of momenta are algebraic integers, Galois confinement allows only states with momenta with integer components favored by periodic boundary conditions.

Could hyperbolic manifolds as coset spaces $SO(1,3)/\Gamma$, where Γ is an infinite discrete subgroup $SO(1,3)$, which acts completely discontinuously from left or right, replace the Fermi torus? Discrete translations in E^3 would thus be replaced with an infinite discrete subgroup Γ. For a given P, the matrix coefficients for the elements of the matrix belonging to Γ would belong to an extension of rationals defined by P.

1. The division of $SO(1,3)$ by a discrete subgroup Γ gives rise to a hyperbolic manifold with a finite volume. Hyperbolic space is an infinite covering of the hyperbolic manifold as a fundamental region of tessellation. There is an infinite number of the counterparts of Fermi torus [L98]. The invariance respect to Γ would define the counterpart for the periodic boundary conditions.

   Note that one can start from $SO(1,3)/\Gamma$ and divide by $SO(3)$ since Γ and $SO(3)$ act from right and left and therefore commute so that hyperbolic manifold is $SO(3)\setminus SO(1,3)/\Gamma$.

2. There is a deep connection between the topology and geometry of the Fermi manifold as a hyperbolic manifold. Hyperbolic volume is a topological invariant, which would become a basic concept of relativistic topological physics (`https://cutt.ly/RVsdNl3`).

   The hyperbolic volume of the knot complement serves as a knot invariant for knots in S^3. Could this have physical interpretation in the TGD framework, where knots and links, assignable to flux tubes and strings at the level of H, are central. Could one regard the effective hyperbolic manifold in H^3 as a representation of a knot complement in S^3?

   Could these fundamental regions be physically preferred 3-surfaces at H^3 determining the holography and M^8-H duality in terms of associativity [L82, L83]. Boundary conditions at the boundary of the unit cell of the tessellation should give rise to effective identifications just as in the case of Fermi torus obtained from the cube in this way.

2 .De Sitter manifolds as tachyonic analogs of Fermi torus do not exist

Can one define the analogy of Fermi torus for the real 4-momenta having negative, tachyonic mass squared? Mass shells with negative mass squared correspond to De-Sitter space $SO(1,3)/SO(1,2)$ having a Minkowskian signature. It does not have analogies of the tessellations of H^3 defined by discrete subgroups of $SO(1,3)$.

The reason is that there are no closed de-Sitter manifolds of finite size since no infinite group of isometries acts discontinuously on de Sitter space: therefore these is no group replacing the Γ in H^3/Γ. (https://cutt.ly/XVsdLwY).

3.Do complexified hyperbolic manifolds as analogs of Fermi torus exist?

The momenta for virtual fermions defined by the roots defining mass squared values can also be complex. Tachyon property and complexity of mass squared values are not of course not the same thing.

1. Complexification of H^3 would be involved and it is not clear what this could mean. For instance, does the notion of complexified hyperbolic manifold with complex mass squared make sense.

2. $SO(1,3)$ and its infinite discrete groups Γ act in the complexification. Do they also act discontinuously? p^2 remains invariant if $SO(1,3)$ acts in the same way on the real and imaginary parts of the momentum leaves invariant both imaginary and complex mass squared as well as the inner product between the real and imaginary parts of the momenta. So that the orbit is 5-dimensional. Same is true for the infinite discrete subgroup Γ so that the construction of the coset space could make sense. If Γ remains the same, the additional 2 dimensions can make the volume of the coset space infinite. Indeed, the constancy of $p_1 \cdot p_2$ eliminates one of the two infinitely large dimensions and leaves one.

   Could one allow a complexification of $SO(1,3)$, $SO(3)$ and $SO(1,3)_c/SO(3)_c$? Complexified $SO(1,3)$ and corresponding subgroups Γ satisfy $OO^T = 1$. Γ_c would be much larger and contain the real Γ as a subgroup. Could this give rise to a complexified hyperbolic manifold H_c^3 with a finite volume?

3. A good guess is that the real part of the complexified bilinear form $p \cdot p$ determines what tachyonicity means. Since it is given by $Re(p)^2 - Im(p)^2$ and is invariant under $SO_c(1,3)$ as also $Re(p) \cdot Im(p)$, one can define the notions of time-likeness, light-likeness, and space-likeness using the sign of $Re(p)^2 - Im(p^2)$ as a criterion. Note that $Re(p)^2$ and $Im(p)^2$ are separately invariant under $SO(1,3)$.

   The physicist's naive guess is that the complexified analogs of infinite discrete and discontinuous groups and complexified hyperbolic manifolds as analogs of Fermi torus exist for $Re(P^2) - Im(p^2) > 0$ but not for $Re(P^2) - Im(p^2) < 0$ so that complexified dS manifolds do not exist.

4. The bilinear form in H_c^3 would be complex valued and would not define a real valued Riemannian metric. As a manifold, complexified hyperbolic manifold is the same as the complex hyperbolic manifold with a hermitian metric (see https://cutt.ly/qVsdS7Y and https://cutt.ly/kVsd3Q2) but has different symmetries. The symmetry group of the complexified bilinear form of H_c^3 is $SO_c(1,3)$ and the symmetry group of the Hermitian metric is $U(1,3)$ containing $SO(1,3)$ as a real subgroup. The infinite discrete subgroups Γ for $U(1,3)$ contain those for $SO(1,3)$. Since one has complex mass squared, one cannot replace the bilinear form with hermitian one. The complex H^3 is not a constant curvature space with curvature -1 whereas H_c^3 could be such in a complexified sense.

10.5.7 Is pair creation really understood in the twistorial picture?

Twistorialization leads to a beautiful picture about scattering amplitudes at the level of M^8 [L108, L109]. In the simplest picture, scattering would be just a re-organization of Galois singlets to new Galois singlets. Fundamental fermions would move as free particles.

The components of the 4-momentum of virtual fundamental fermion with mass m would be algebraic integers and therefore complex. The real projection of 4-momentum would be mapped by $M^8 - H$ duality to a geodesic of M^4 starting from either vertex of the causal diamond (CD) . Uncertainty Principle at classical level requires inversion so that one has $a = \hbar_{eff}/m$, where ab denotes light-cone proper time assignable to either half-cone of CD and m is the mass assignable to the point of the mass shell $H^3 \subset M^4 \subset M^8$.

The geodesic would intersect the $a = \hbar_{eff}/m$ 3-surface and also other mass shells and the opposite light-cone boundaries of CDs involved. The mass shells and CDs containing them would have a common center but Uncertainty Principle at quantum level requires that for each CD and its contents there is an analog of plane wave in CD cm degrees of freedom.

One can however criticize this framework. Does it really allow us to understand pair creation at the level of the space-time surfaces $X^4 \subset H$?

1. All elementary particles consist of fundamental fermions in the proposed picture. Conservation of fermion number allows pair creation occurring for instance in the emission of a boson as fermion-antifermion pair in $f \to f + b$ vertex.

2. The problem is that if only non-space-like geodesics of H are allowed, both fermion and antifermion numbers are conserved separately so that pair creation does not look possible. Pair creation is both the central idea and source of divergence problems in QFTs.

3. One can identify also a second problem: what are the anticommutation relations for the fermionic oscillator operators labelled by tachyonic and complex valued momenta? Is it possible to analytically continue the anticommutators to complexified $M^4 \subset H$ and $M^4 \subset M^8$? Only the first problem will be considered in the following.

Is it possible to understand pair creation in the proposed picture based on twistor scattering amplitudes or should one somehow bring the bff 3-vertex or actually $ff\overline{f}\overline{f}$ vertex to the theory at the level of quark lines? This vertex leads to a non-renormalizable theory and is out of consideration.

One can first try to identify the key ingredients of the possible solution of the problem.

1. Crossing symmetry is fundamental in QFTs and also in TGD. For non-trivial scattering amplitudes, crossing moves particles between initial and final states. How should one define the crossing at the space-time level in the TGD framework? What could the transfer of the end of a geodesic line at the boundary of CDs to the opposite boundary mean geometrically?

2. At the level of H, particles have CP_2 type extremals -wormhole contacts - as building bricks. They have an Euclidean signature (of the induced metric) and connect two space-time sheets with a Minkowskian signature.

   The opposite throats of the wormhole contacts correspond to the boundaries between Euclidean and Minkowskian regions and their orbits are light-like. Their light-like boundaries, orbits of partonic 2-surfaces, are assumed to carry fundamental fermions. Partonic orbits allow light-like geodesics as possible representation of massless fundamental fermions.

   Elementary particles consist of at least two wormhole contacts. This is necessary because the wormhole contacts behave like Kähler magnetic charges and one must have closed magnetic field lines. At both space-time sheets, the particle could look like a monopole pair.

3. The generalization of the particle concept allows a geometric realization of vertices. At a given space-time sheet a diagram involving a topological 3-vertex would correspond to 3 light-like partonic orbits meeting at the partonic 2-surface located in the interior of X^4. Could the topological 3-vertex be enough to avoid the introduction of the 4-fermion vertex?

Could one modify the definition of the particle line as a geodesic of H to a geodesic of the space-time surface X^4 so that the classical interactions at the space-time surface would make it possible to describe pair creation without introducing a 4-fermion vertex? Could the creation of a fermion pair mean that a virtual fundamental fermion moving along a space-like geodesics of a wormhole throat turns backwards in time at the partonic 3-vertex. If this is the case, it would correspond to a tachyon. Indeed, in M^8 picture tachyons are building bricks of physical particles identified as Galois singlets.

1. If fundamental fermion lines are geodesics at the light-like partonic orbits, they can be light-like but are space-like if there is motion in transversal degrees of freedom.

2. Consider a geodesic carrying a fundamental fermion, starting from a partonic 2-surface at either light-like boundary of CD. As a free fermion, it would propagate to the opposite boundary of CD along the wormhole throat.

   What happens if the fermion goes through a topological 3-vertex? Could it turn backwards in time at the vertex by transforming first to a space-like geodesic inside the wormhole contact leading to the opposite throat and return back to the original boundary of CD? It could return along the opposite throat or the throat of a second wormhole contact emerging from the 3-vertex. Could this kind of process be regarded as a bifurcation so that it would correspond to a classical non-determinism as a correlate of quantum non-determinism?

3. It is not clear whether one can assign a conserved space-like M^4 momentum to the geodesics at the partonic orbits. It is not possible to assign to the partonic 2-orbit a 3-momentum, which would be well-defined in the Noether sense but the component of momentum in the light-like direction would be well-defined and non-vanishing.

   By Lorentz invariance, the definition of conserved mass squared as an eigenvalue of d'Alembertian could be possible. For light-like 3-surfaces the d'Alembertian reduces to the d'Alembertian for the Euclidean partonic 2-surface having only non-positive eigenvalues. If this process is possible and conserves M^4 mass squared, the geodesic must be space-like and therefore tachyonic.

   The non-conservation of M^4 momentum at single particle level (but not classically at n-particle level) would be due to the interaction with the classical fields.

4. In the M^8 picture, tachyons are unavoidable since there is no reason why the roots of the polynomials with integer coefficients could not correspond to negative and even complex mass squared values. Could the tachyonic real parts of mass squared values at M^8 level, correspond to tachyonic geodesics along wormhole throats possibly returning backwards along the another wormhole throat?

How does this picture relate to p-adic thermodynamics [L117] as a description of particle massivations?

1. The description of massivation in terms of p-adic thermodynamics [L117] suggests that at the fundamental level massive particles involve non-observable tachyonic contribution to the mass squared assignable to the wormhole contact, which cancels the non-tachyonic contribution.

   All articles, and for the most general option all quantum states could be massless in this sense, and the massivation would be due the restriction of the consideration to the non-tachyonic part of the mass squared assignable to the Minkowskian regions of X^4.

2. p-Adic thermodynamics would describe the tachyonic part of the state as "environment" in terms of the density matrix dictated to a high degree by conformal invariance, which this description would break. A generalization of the blackhole entropy applying to any system emerges and the interpretation for the fact that blackhole entropy is proportional to mass squared. Also gauge bosons and Higgs as fermion-antifermion pairs would involve the tachyonic contribution and would be massless in the fundamental description.

3. This could solve a possible and old problem related to massless spin 1 bosons. If they consist of a collinear fermion and antifermion, which are massless, they have a vanishing helicity and would be scalars, because the fermion and antifermion with parallel momenta have opposite helicities. If the fermion and antifermion are antiparallel, the boson has correct helicity but is massive.

   Massivation could solve the problem and p-adic thermodynamics indeed predicts that even photons have a very small thermal mass. Massless gauge bosons (and particles in general) would be possible in the sense that the positive mass squared is compensated by equally small tachyonic contribution.

4. It should be noted however that the roots of the polynomials in M^8 can also correspond to energies of massless states. This phase would be analogous to the Higgs=0 phase. In this

phase, Galois symmetries would not be broken: for the massive phase Galois group permutes different mass shells (and different $a = constant$ hyperboloids) and one must restrict Galois symmetries to the isotropy group of a given root. In the massless phase ,Galois symmetries permute different massless momenta and no symmetry breaking takes place.

10.6 Antipodal duality and TGD

I learned of a new particle physics duality from the popular article "Particle Physicists Puzzle Over a New Duality" published in Quanta Magazine (`https://cutt.ly/jZOaDhd`). The article describes the findings of Dixon et al reported in the article "Folding Amplitudes into Form Factors: An Antipodal Duality" [B40] (`https://cutt.ly/EZOsfGl`) This work relies on the calculations of Goncharov et al published in the article "Classical Polylogarithms for Amplitudes and Wilson Loops" [B48] (`https://cutt.ly/sZOsuu6`).

The calculations of Goncharov et al lead to an explicit formula for the loop contributions to the 6-gluon scattering amplitude in $\mathcal{N} = 4$ SUSY. The new duality is called antipodal duality and relates 6-gluon amplitude for the forward scattering to a 3-gluon form factor of stress tensor analogous to a quantum field describing a scalar particle. This amplitude can be identified as a contribution to the scattering amplitude $h + g \rightarrow g + g$. The result is somewhat mysterious since in the standard model strong and electroweak interactions are completely separate.

10.6.1 Findings of Dixon et al

Consider first the findings of Dixon et al [B40].

1. One considers [B48] twistor amplitudes in $\mathcal{N} = \triangle$ SUSY. Only the maximally helicity violating amplitudes (MHV) are considered and one restricts the consideration to planar diagrams (to my best understanding, non-planar diagrams are still poorly understood). The contribution of the loop corrections is studied and the number of loops is rather high in the computations checking the claimed result.

   6-gluon forward scattering amplitude and 3-gluon form factor of stress energy tensor regarded as a quantum field are discussed. Conformal invariance fixes the Lorentz invariants appearing in the 6-gluon forward amplitude and in the 3-gluon form factor of stress tensor to be 3 conformally invariant cross ratios formed from the 3 gluon momenta.

   The claimed antipodal duality is found to hold true for each number of loops separately at the limit when one of conformal invariants approaches zero: the interpretation is that momentum exchange between 2 gluons vanishes at this limit. For 6-gluon forward amplitudes, this limit corresponds to in the 3-D space of conformal invariants to the edges of a tetrahedron.

2. $3g \rightarrow 3g$ scattering amplitude is studied at the limit when the scattering is in forward direction. One has effectively 3 gluons but not 3-gluon scattering since there is no momentum conservation constraining the total momentum of 3 gluons except effectively for the forward scattering of the stress tensor.

   As far as total quantum numbers are considered, the stress tensor can give rise to a quantum field behaving like Higgs as far as QCD is considered. The surprising finding is that the so-called antipodal duality applied to the 6-gluon amplitude gives a 3-gluon form factor of the stress tensor, which is scalar having no spin and vanishing color quantum numbers.

3. The antipodal transformation is carried for the 6-gluon amplitude in forward direction so that only 3 gluon momenta are involved. One starts from the 6-gluon amplitude constructed using the standard rules, which require that the amplitude involves only cyclic permutations of the gluons (elements of S_6 of the gluons.

   One considers permutation group $S_3 \subset S_6$ acting in the same way on the first 3 first and 3 remaining gluons, and constructs an S_3 singlet as a sum of the amplitudes obtained by applying S_3 transformations. S_3 operations are not allowed in the twistor diagrammatics since only planar amplitudes are considered usually (the construction of twistor counterparts of non-planar amplitudes is not well-understood).

4. One also constructs the 3-gluon form factor of stress energy tensor by using the twistor rules and considers the so-called soft limit at which the sum of the 3 gluon momenta vanishes so that the effectite particle assignable to the stress tensor scatters in the forward direction. It comes as a surprise that this amplitude is related to the amplitude obtained from the forward 6-gluon amplitude by the antipodal transformation.

5. The duality also involves a simple transformation of the 3 conformal invariants formed from the gluon momenta involved to the 3-gluon form factor of the energy momentum tensor. The antipodal duality holds true at the edges of the 2-D tetrahedron surface defined by the image of the 3-gluon form factor in the space of 3 conformal invariants characterizing the 6-gluon forward amplitude.

   The term antipodal derives from the fact that the 6-gluon amplitude can be expressed as a "word" formed from 6 "letters" and the above described transformation reverses the order of the letters.

6. It is conjectured that this result generalizes to large values of n so that antipodal images of $2n$-gluon scattering amplitude in forward direction could correspond to $n-$-gluon form factor for stress tensor energy and this in turn would be associated with scattering of Higgs and n gluons.

10.6.2 Questions

Since the stress tensor is a scalar, it is not totally surprising that a term proportional to this amplitude contributes to the scattering amplitude $h+g\rightarrow g+g$, where h denotes Higgs particle. What looks somewhat mysterious is that Higgs is an electro-weakly interacting particle and has no direct color interactions. The description of the scattering in the standard model involves electroweak interactions and involves at least one decay of a gluon to a quark pair in turn interacting with the Higgs.

This inspires several questions.

1. Can one consider more general subgroups $S_m \subset S_{2n}$ and by forming S_m singlets construct amplitudes with a physical interpretation?

2. Can one imagine a deep duality between color and electroweak interactions such that $\mathcal{N}=4$ SUSY would reflect this duality? Could one even think that the strong and electroweak interactions are in some sense dual?

In TGD color interactions and electroweak interactions are related to the isometries and holonomies of $CP2$ and there indeed exists quite a number of pieces of evidence for this kind of duality. However, the possibility that electroweak or color interactions alone could provide a full description of scattering amplitudes looks unrealistic: both electroweak and color quantum numbers are needed. The number-theoretical view of TGD [L104, L42, L108, L109] could however come into rescue.

10.6.3 In what sense could electroweak and color interactions be dual?

Some kind of duality of electroweak and color interactions is suggested by the antipode duality having an interpretation in terms of Hopf algebras (`https://en.wikipedia.org/wiki/Hopf_algebra`): antipode generalizes the notion of inverse for an element of algebra.

TGD contains several mysterious looking and not-well understood features suggesting some kind of duality between electroweak and color interactions. What could make this duality possible in the TGD framework, would be the presence of Galois symmetry, which would allow us to describe electroweak or color particle multiplets number-theoretically using representations of the Galois group.

1. The electric-magnetic duality or Montonen-Olive duality (`https://en.wikipedia.org/wiki/Montonen\OT1\textendashOlive_duality`) is inspired by the homology of CP_2 in TGD [?]. The generalization of this duality in gauge theories relates the perturbative description of

gauge interactions for gauge group G to a non-perturbative description in terms of magnetic monopoles associated with the dual gauge group G_L. Langlands duality [A40, A39], discussed from the TGD perspective in [K47, L26], relates the representations of Galois groups and those of Lie groups, and involves Lie group and its Langlands dual. Therefore gauge groups, magnetic monopoles and the corresponding dual gauge group, and number theory seem to be mathematically related, and TGD suggests a physical realization of this view.

2. The dual groups G and G_L should be very similar but electroweak gauge group $U(2)$ and color group $SU(3)$, albeit naturally related as holonomy and isometry groups of CP_2, do not satisfy this condition. Here the Galois group could come into rescue and provide the missing quantum numbers.

3. Depending on the situation, Galois confinement could relate to color confinement or electroweak confinement. In the context of electric-magnetic duality [K45, K8, K59], I have discussed electroweak confinement and as a possible dual description for the electroweak massivation, involving summation of electroweak $SU(2)$ quantum numbers to zero in the scale of monopole flux tubes assignable to elementary particles. The screening of weak isospin would take place by a pair of neutrino and right-handed neutrino in the Compton scale of weak boson or fermion: $h_{eff} > h$ allows longer scales.

4. Also magnetic charge or flux assignable to the flux tubes could make possible a topological description of color hypercharge topologically whereas color isospin could might have description in terms of weak iosospin. I considered this idea already in my thesis. As a matter of fact, already before the discovery of CP_2 around 1980, I proposed that magnetic (homology-) charges 2,-1,-1 for cP_2 could correspond to em charges 2/3,-1/3,-1/3 of quarks and that quark confinement could be a topological phenomenon. Maybe these almost forgotten ideas might find a place in TGD after all.

Consider now the possible duality between electroweak and color interactions.

H level

At the level of H spinors do not couple classically to gluons and color is not spin-like quantum number.

1. The proposal is that the zero energy states are singlets either with respect to the Galois group or the isotropy group of a given root. Z_3 as a subgroup or possibly normal subgroup of the Galois group would act on the space of fermion momenta for which components are algebraic integers belonging to the extension of rationals defined by P.

2. Color confinement could correspond to Galois confinement. Alternatively, the confinement of color isospin could correspond to Galois confinement whereas the confinement of color hypercharge would have a description in terms of the already mentioned monopole confinement. Both number theoretic and topological color would be invisible.

Could antipodal duality be understood number-theoretically?

1. The antipodal duality produces an S_3 singlet from a twistor amplitude. Could color singlets correspond to Z_3 Galois-singlets and electroweak singlets above Compton scale to Z_2 singlets.

2. Could Z_2 be realized as an exchange of two gluons ordered cyclically in the amplitude? Could one think that S^6 acts as a Galois group or its isotropy group?

   The stress tensor as a Higgs like state is not a doublet. Could one obtain Higgs as a Z_2 doublet by allowing the entire orbit of S_3 but requiring only that Z_3 singlet property holds true?

3. Could all isotropy groups or even all subgroups of S^3 be allowed. Could S_n quite generally have a representation as a Galois group? This picture applies also to $2n$-gluon amplitudes but also more general conditions for Galois singlet property can be imagined.

M^8 level

The roles of color and electroweak quantum numbers are changed in $M^8 - H$ duality [L82, L83].

1. At the level of M^8, complexified octonionic 2-spinors [L77, L82, L83] decompose to the representations of the subgroup $SU(3) \subset G_2$ of octonionic automorphisms as $1 + \overline{1} + 3 + \overline{3}$. One obtains leptons and quarks with spin but electroweak quantum numbers do not appear as spin-like quantum numbers. This would suggests that one should assume both lepton and quark spinors at the level of H although the idea about leptons as 3-quark composites in CP_2 scale is attractive [L94].

   One can however construct octonionic spinor fields $M^4 \times E^4$ with the spinor partial waves belonging to the representations of $SO(4) = SU(2) \times SU(2)$ decomposing to representation of $U(2)$ with quantum numbers having interpretation as orbital angular momentum like electroweak quantum numbers.

2. At the level of 4-surfaces of M^8, weak isospin doublet could correspond to Galois doublet associated with a Z_2 factor of the Galois group.

Twistor space level

Also at the level of twistor spaces, the roles of electroweak and color numbers are changed in $M^8 - H$ duality.

1. At the level of H, $M^4 \times CP_2$ is replaced by the product of the twistor spaces $T(M^4)$ and $T(CP_2) = SU(3)/U(1)\times U(1)$. Since spinors are not involved anymore, electroweak quantum numbers disappear. Number theoretic description should apply. Here Galois subgroup Z_2 could help.

   This suggests that $U(2) \subset SU(3)$ must be interpreted in terms of electroweak quantum numbers. There indeed exists a natural embedding of the holonomy group of CP_2 to its isometry group. At the level of space-time, surface color hyper-charge and isopin could correspond to electroweak hyper-charge and isospin. This works if, for given electroweak quantum numbers, the choice of the quantization axes of color quantum numbers depends on the state so that the regions of space-time surface assignable to a fermion depends on its color quantum numbers in H. This would give a correlation between space-time geometry and quantum numbers.

2. At the level of M^8 the twistor space $T(E^4)$ contains information about weak quantum numbers but no information of color quantum numbers since octonionic spinors are given up. Z_6 as a subgroup of the Galois group could help now.

Also the induced twistor structure at the level of space-time surface in H and at the level of 4-surface in M^8 gives strong consistency conditions.

1. The induced twistor structure for the surface $T(X^4) \subset T(H)$ has S^2 bundle structure characterizing twistor space. This structure is obtained by dimensional reduction to $X^6 = X^4 \times S^2$ locally such that S^2 corresponds to the twistor sphere of both $T(M^4)$ and $T(CP_2)$.

2. For cognitive representations as unique number theoretic discretizations of the space-time surface, the twistor spheres S^2 of $T(M^4)$ *resp.* $T(CP_2)$ must correspond to each other. The point of S^2 represents the direction of the quantization axis and the value $\pm 1/2$ of spin *resp.* color isospin or appropriately normalized color hypercharge respectively.

   For quark triplets this kind of correlation can make sense between spin and color hypercharge only and only at the level of the space-time surface. Since the quantization directions of color isospin are not fixed, only the correlation between representations, rather states, is required and makes sense for quarks. This suggests that color isospin at the space-time level must correspond to Galois quantum number.

3. What about leptons? For leptons color hypercharge vanishes. However, both leptonic and quark-like induced spinors have anomalous hypercharge proportional to electromagnetic charge so that also leptonic spinors would form doublets with respect to anomalous color [L52].

The induced twistor structure for 4-surfaces in M^8 does not correspond to dimensional reduction but one expects an analogous correlation between spin and electroweak quantum numbers induced by the mapping of the twistor spheres S^2 to each other.

1. This correlation spin H-spinors correspond to tensor products of spin and electroweak doublets and all elementary particles are constructed from these.

2. Something seems to be however missing: also M^4 spinors should have a $U(1)$ charge to make the picture completely symmetric. The spinor lift strongly suggests that also M^4 has the analog of Kähler structure [L100] and this would give rise to $U(1)$ charge for M^4 spinors [L45] [K8]. This coupling could give rise to small CP breaking effects at the level of fundamental spinors [L100].

The experimental picture about strong and electroweak interactions suggests that the description of standard model interactions as either color interactions or electroweak interactions combined with a number theoretic/topological description of the missing quantum numbers is enough.

1. In hadron physics, only electroweak quantum numbers are visible. Color could be described using number-theory and topology and also these descriptions might be dual. In the QCD picture at high energies only color quantum numbers are visible and electroweak quantum numbers could be described number-theoretically. For a given particle, electroweak confinement would work above its Compton scale of weak scale.

2. In the old fashioned hadron physics conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC) relate hadron physics and electroweak physics in a way which is not fully understood since also quark confinement is still poorly understood. PCAC reflects the massivation of hadrons and can be also seen as caused by the massivation of quarks and leptons and makes successful predictions. In the TGD framework PCAC is applied to the model of so-called lepto-hadrons [K97].

   One can say that hadronic description uses $SO(4) = SU(2)_L \times U(2)_R$ or rather, $U_{ew}(2)$ as a symmetry group whereas QCD uses $SU(3)$ in accordance with the duality between color and electroweak interactions. This conforms with the $M^8 - H$ duality.

3. What about CP_2 type extremals (wormhole contacts), which have Euclidean metric. Could electroweak spin be described as the spin of an octo-spinor and could M^4 spin be described number-theoretically.

What about leptons? For leptons color hypercharge vanishes. However, both leptonic and quark-like induced spinors have anomalous hypercharge proportional to electromagnetic charge so that also leptonic spinors would form doublets with respect to anomalous color.

10.7 How could Julia sets and zeta functions relate to Galois confinement?

In this section the limit of large particle number of identical particles for the scattering is considered. It is found that the mass spectrum belongs to the Julia set of an infinitely iterated polynomial defining the many-particle state. Also a generalization replacing polynomials with real analytic functions is discussed and it is found that zeta functions and elliptic functions are especially interesting concerning conformal confinement as analog of Galois confinement.

10.7.1 The mass spectrum for an iterate of polynomial and chaos theory

Suppose that the number theoretic interaction in the scattering corresponds to a functional composition of the polynomials characterizing the external particles. If the number of the external particles is large, the composite can involve a rather high iterate of a single polynomial. This motivates the study of the scattering of identical particles described by the same polynomial P at the limit of a large particle number. These particles could correspond to elementary particles, in particular IR photons and gravitons. This situation leads to an iteration of a complex polynomial.

If the polynomials satisfy $P(0) = 0$ requiring $P(x) = xP_1(x)$, the roots of P are inherited. In this case fixed points correspond to the points with $P(x) = 1$. Assume also that the coefficients are rational. Monic polynomials are an especially interesting option.

For a k:th iterate of P, the mass squared spectrum is obtained as a union of spectra obtained as images of the spectrum of P under iterates P^{-r}, $r \leq k$, for the inverse of P, which is an n-valued algebraic function if P has degree n. This set is a subset of Fatou set (`https://cutt.ly/hOgq6Yy`)and for polynomials a subset of filled Julia set.

At the limit of large k, the limiting contributions to the spectrum approach a subset of Julia set defined as a P-invariant set which for polynomials is the boundary of the set for which the iteration divergences. The iteration of all roots except $x = 0$ (massless particles) leads to the Julia set asymptotically.

All inverse iterates of the roots of P are algebraic numbers. The Julia set itself is expected to contain transcendental complex numbers. It is not clear whether the inverse iterates at the limit are algebraic numbers or transcendentals. For instance, one can ask whether they could consist of n-cycles for various values of n consisting of algebraic points and forming a dense subset of the Julia set. The fact that the number of roots is infinite at this limit, suggests that a dense subset is in question.

The basic properties of Julia set deserve to be listed.

1. At the real axis , the fixed points satisfying $P(x) = x$ with $|dP/dx| > 1$ are repellers and belong to the Julia set. In the complex plane, the definition of points of the Julia set is $|P(w) - P(z)| \geq |w - z|$ for point w near to z.

2. Julia set is the complement of the Fatou set consisting of domains. Each Fatou domain contains at least one critical point with $dP/dz = 0$. At the real axis, this means that P has maximum or minimum. The iteration of P inside Fatou domain leads to a fixed point inside the Fatou set and inverse iteration to its boundary. The boundaries of Fatou domains combine to form the Julia set. In the case of polynomials, Fatou domains are labeled by the $n - 1$ solutions of $dP/dz = P_1 + zdP_1/dz = 0$.

3. Julia set is a closure of infinitely many periodic repelling orbits. The limit of inverse iteration leads towards these orbits. These points are fixed points for powers P^n of P.

4. For rational functions Julia set is the boundary of a set consisting of points whose iteration diverges to infinity. For polynomials Julia set is the boundary of the so-called filled Julia set consisting of points for which the iterate remains finite.

Chaos theory also studies the dependence of Julia set on the parameters of the polynomials. Mandelbrot fractal is associated to the polynomial $Q(z) = a + z^2$ for which origin is an stable critical point and corresponds to the boundary of the region in a-plane containing origin such that outside the boundary the iteration leads to infinity and in the interior to origin.

The critical points of P with $dP/dz = 0$ for $z = z_{cr}$ located inside Fatou domains are analogous to point $z = 0$ for $Q(z)$ associated with Fatou domains and quadratic polynomial $a + b(z - z_{cr})^2$, $b > 0$, would serve as an approximation. The variation of a is determined by the variation of the coefficients of P required to leave z_{cr} invariant.

Feigenbaum studied iteration of a polynomial $a - x^2$ for which origin is unstable critical point and found that the variation of a leads to a period doubling sequence in which a sequence of 2^n-cycles is generated (`https://cutt.ly/pOgwuqj`). Origin would correspond to an unstable critical point $dP(z)/dz = 0$ belonging to a Julia set.

The physical implications of this picture are highly interesting.

1. For a large number of interacting quarks, the mass squared spectrum of quarks as roots of the iterate of P in the interaction region would approach the Julia set as infinite inverse iterates of the roots of P. This conforms with the idea that the complexity increases with the particle number.

   Galois confinement forces the mass squared spectrum to be integer valued when one uses as a unit the p-adic mass scale defined by the larger ramified prime for the iterate. The complexity manifests itself only as the increase of the microscopic states in interaction regions.

2. Julia set contains a dense set consisting of repulsive n-cycles, which are fixed points of P and the natural expectation is that the mass spectrum decomposes into n-multiplets. Whether all values of n are allowed, is not clear to me. The limit of a large quark number would also mean an approach to (quantum) criticality.

To sum up, it would seem that chaos (or rather complexity-) theory could be an essential part of the fundamental physics of many-quark systems rather than a mere source of pleasures of mathematical aesthetics.

10.7.2 A possible generalization of number theoretic approach to analytic functions

M^8-H duality also allows the possibility that space-time surfaces in M^8 are defined as roots of real analytic functions. This option will be considered in this subsection.

Are polynomials 4-surfaces only an approximation

One of the open problems of the number-theoretic vision is whether the space-time surfaces associated with rational or even monic polynomials are an approximation or not.

1. One could argue that the cognitive representations are only a universal discretization obtained by approximating the 4-surface in M^8 by a polynomial. This discretization relies on an extension of rationals and more general than rational discretizations, which however appear via Galois confinement for the momenta of Galois singlets.

   One objection against space-time surfaces as being determined by polynomials in M^8 was that the resulting 4-surfaces in M^8 would bre algebraic surfaces. There seems to be no hope about Fourier analysis. The problem disappeared with the realization that polynomials determine only the 3-surfaces as mass-shells of M^4 and that M^8-H duality is realized by an explicit formula subject to $I(D)=exp{-K}$ condition.

2. Galois confinement provides a universal mechanism for the formation of bound states. Could evolution be a development of real states for which cognitive representations in terms of quarks become increasingly precise.

   That the quarks defining the active points of the representation are at 3-D mass shells would correspond to holography at the level of M^8. At the level of H they would be at the boundaries of CD. This would explain why we experience the world as 3-dimensional.

Also the 4-surfaces containing quark mass shells defined in terms of roots of arbitrary real analytic functions are possible.

1. Analytic functions could be defined in terms of Taylor or Laurent series. In fact, any representation can be considered. Also now one can consider representation involving only integers, rationals, algebraic numbers, and even reals as parameters playing a role of Taylor coefficients.

2. Does the notion of algebraic integers generalize? The roots of the holomorphic functions defining the meromorphic functions as their ratios define an extension of rationals, which is in the general transcendental. It is plausible that the notion of algebraic integers generalizes and one can assume that quarks have momentum components, which are transcendental integers. One can also define the transcendental analog of Galois confinement.

3. One can form functional composites to construct scattering amplitudes and this would make possible particle reactions between particles characterized by analytic functions. Iteration of analytic functions and approach to chaos would emerge as the functions involved appear very many times as one expects in case of IR photons and gravitons.

What about p-adicization requiring the definition discriminant D and identification of the ramified primes and maximal ramified prime? Under what conditions do these notions generalize?

1. One can start from rational functions. In the case of rational functions R, one can generalize the notion of discriminant and define it as a ratio $D = D_1/D_2$ of discriminants D_1 and D_2 for the polynomials appearing as a numerator and denominator of R. The value of D is finite irrespective of the values of the degrees of polynomials.

2. Analytic functions define function fields. Could a generalization of discriminant exist. If the analytic function is holomorphic, it has no poles so that D could be defined as the product of squares of root differences.

   If the roots appear as complex conjugate pairs, D is real. This is guaranteed if one has $f(\overline{z}) = \overline{f(z)}$. The real analyticity of f guarantees this and is necessary in the case of polynomials. A stronger condition would be that the parameters such as Taylor coefficients are rational.

   If the roots are rationals, the discriminant is a rational number and one can identify ramified primes and p-adic prime if the number of zeros is finite.

3. Meromorphic functions are ratios of two holomorphic functions. If the numbers of zeros are finite, the ratio of the discriminants associated with the numerator and denominator is finite and rational under the same assumptions as for holomorphic functions.

4. $M^8 - H$ duality leads to the proposal that the discriminant interpreted as a p-adic number for p-adic prime defined by the largest ramified prime, is equal to the exponent of $exp(-K)$ of Kähler function for the space-time surface in H.

   If one can assign ramified primes to D, it is possible to interpret D as a p-adic number having a finite real counterpart in canonical identification. For instance, if the roots of zeta are rationals, this could make sense.

Questions related to the emergence of mathematical consciousness

These considerations inspire further questions about the emergence of mathematical consciousness.

1. Could some mathematical entities such as analytic functions have a direct realization in terms of space-time surfaces? Could cognitive processes be identified as a formation of functional composites of analytic functions? They would be analogs of particle reactions in which the incoming particles consist of quarks, which are associated with mass-shells defined by the roots of analytic function.

   These composites would decay to products of polynomials in cognitive measurements involving a cascade of SSFRs reducing the entanglement between a relative Galois group and corresponding normal group acting as Galois group of rationals [L90].

2. Could the basic restriction to cognition come from the Galois confinement: momenta of composite states must be integers using p-adic mass scale as a unit.

   Or could one think that the normal sub-group hierarchies formed by Galois groups actually give rise to hierarchies of states, which are Galois confined for an extension of the Galois group.

   Could these higher levels relate to the emergence of consciousness about algebraic numbers. Could one extend computationalism allow also extensions of rationals and algebraic integers as discussed in [L88].

   Galois confinement for an extension of rationals would be analogous to the replacement of a description in terms of hadrons with that in terms of quarks and mean increase of cognitive resolution. Also Galois confinement could be generalized to its quantum version. One could have many quark states for which wave function in the space of total momenta is Galois singlet whereas total momenta are algebraic integers. S-wave states of a hydrogen atom define an obvious analog.

3. During the last centuries the evolution of mathematical consciousness has made huge steps due to the discovery of various mathematical concepts. Essentially a transformation of rational arithmetics with real analysis and calculus has taken place since the times of Newton. Could these evolutionary explosions correspond to the emergence of space-time surfaces defined by analytic functions or is it that only a conscious awareness about their existence has emerged?

Space-time surfaces defined by zeta functions and elliptic functions

Several physical interpretations of Riemann zeta have been proposed. Zeta has been associated with chaotic systems, and the interpretation of the imaginary parts of the roots of zeta as energies has been considered. Also an interpretation as a formal analog of a partition function has been considered. The interpretation as a scattering amplitude was considered by Grant Remmen [B47] (`https://cutt.ly/TID1kjU`).

1.Conformal confinement as Galois confinement for polynomials?

TGD suggests a totally different kind of approach in the attempts to understand Riemann Zeta. The basic notion is conformal confinement [K61].

1. The proposal is that the zeros of zeta correspond to complex conformal weights $s_n = 1/2+iy_n$. Physical states should be conformally confined meaning that the total conformal weight as the sum of conformal weights for a composite particle is real so that the state would have integer value conformal weight n, which is indeed natural. Also the trivial roots of zeta with $s = -2n$, $n > 0$, could be considered.

2. In $M^8 - H$ duality, the 4-surfaces $X^4 \subset M^8$ correspond to roots of polynomials P. M^8 has an interpretation as an analog of momentum space. The 4-surface involves mass shells $m^2 = r_n$, where r_n is the root of the polynomial P, algebraic complex number in general.

   The 4-surface goes through these 3-D mass-shells having $M^4 \subset M^8$ as a common real projection. The 4-surface is fixed from the condition that it defines $M^8 - H$ duality mapping it to $M^4 \times CP_2$. One can think X^4 as a deformation of M^4 by a local $SU(3)$ element such that the image points are $U(2)$ invariant and therefore define a point of CP_2. $SU(3)$ has an interpretation as octonionic automorphism.

3. Galois confinement states that physical states as many-quark states with quark momenta as algebraic integers in the extension defined by the polynomial have integer valued momentum components in the scale defined by the causal diamond also fixed by the p-adic prime identified as the largest ramified prime associated with the discriminant D of P.

   Mass squared in the stringy picture corresponds to conformal weight so that the mass squared values for quarks are analogous to conformal weights and the total conformal weight is integer by Galois confinement.

2.Conformal confinement for zeta functions

At least formally, TGD also allows a generalization of real polynomials to analytic functions. For a generic analytic function it is not possible to find superpositions of roots that would be integers and this could select Riemann Zeta and possible other analytic functions are those with infinite number of roots since they might allow a large number of bound states and be therefore winners in the number theoretic selection.

Riemann zeta is a highly interesting analytic function in this respect.

1. Actually an infinite hierarchy of zeta functions, one for any extension of rationals and conjectured to have zeros at the critical line, can be considered. Could one regard these zetas

as analogous to polynomials with an infinite degree so that the allowed mass squared values for quarks would correspond to the roots of zeta?

2. Conformal confinement [K61] requires integer valued momentum components and total conformal weights as mass squared values. The fact that the roots of zetas appear as complex conjugates allows for a very large number of states with real conformal weights. This is however not enough. The fact that the roots are of the form $z_n = 1/2 + iy_n$ or $z = -2n$ implies that the conformal weights of Galois/conformal singlets are integer-valued and the spectrum is the same as in conformal field theories.

3. Riemann zeta has only a single pole at $s = 1$. Discriminant would be the product $\prod_{m\neq n}(y_m - y_n^2)\prod_{m\neq n} 4(m-n)^2 \prod_{m,n}(4m^2 + y_n^2)$ since the pole gives $D = 1$. D would be infinite.

4. Fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ is analogous to the partition function for fermionic statistics and looks more appropriate in the case of quarks. In this case, the zeros are z_n *resp.* $z_n/2$ and the ratio of determinants would reduce to an infinite power of 2. The ramified prime would be the smallest possible: $p = 2$!

   $D = D_1/D_2$ would be infinite power of 2 and 2-adically zero so that $exp(-K)$ should vanish and Kähler function would diverge. 3-adically it would be infinite power of -1. If one can say that the number of roots is even, one has $D = 1$ 3-adically. Kähler function would be equal to zero, which is in principle possible.

   For Mersenne primes $M_n = 2^n - 1$, 2^n would be equal to $1 + M_n = 1 \ \ modM_n$ and one would obtain an infinite power $1 + M_n$, which is equal to 1 mod M_n. Could this relate to the special role of Mersenne primes?

5. What about Riemann Hypothesis? By $\zeta(\overline{s}) = \overline{zeta}(s)$, the zeros of zeta appear in complex conjugate pairs. By functional equation, also s and $1 - s$ are zeros. Suppose that there is a zero $s_+ = s_0 + iy_n$ with $s_0 \neq 1/2$ in the interval $(0, 1)$. This is accompanied by zeros $\overline{s}_+$, $1 - s_+$, $s_- = 1 - \overline{s}_+$. The sum of these four zeros is equal to $s = 2$. Therefore Galois singlet property does not allow us to say anything about the Riemann hypothesis.

3.Conformal confinement for elliptic functions

Elliptic functions (`https://cutt.ly/dINxAeQ`) provide examples of analytic functions with infinite number of roots forming a doubly periodic lattice and are therefore candidates for analogs of polynomials with infinite degree.

1. Weierstrass $\mathcal{P}(z)$-function $\mathcal{P}(z) = \sum_\lambda 1/(z - \lambda)^2$, where the summation is over the lattice defined by a complex modular parameter τ, is the fundamental elliptic function. The basic objection is that $\mathcal{P}(z)$ is not real analytic. Despite this it is interesting to look at its properties so that conformal weights do not appear in complex conjugate pairs. Therefore it is not clear whether conformal confinement is possible. One can also ask whether the notion of integer could be replaced with that of "modular" integers $m + n\tau$.

2. Elliptic functions are doubly periodic and characterized by the ratio τ of complex periods ω_1 and ω_2. One can assume the convention $\omega_1 = 1$ giving $\omega_2 = \tau$. The roots of the elliptic function for an infinite lattice and complex rational roots are of obvious interest concerning the generalization of Galois/conformal confinement.

3. The fundamental set of zeros is associated with a cell of this lattice. The finite number of zeros (with zero with multiplicity m counted as m zeros) in the cell is the same as the number poles and characterizes partially the elliptic function besides τ.

4. Weierstrass $\mathcal{P}$-function and its derivative $d\mathcal{P}/\lceil\ddagger$ are the building blocks of elliptic functions. A general elliptic function is a rational function of $\mathcal{P}$ and $d\mathcal{P}/\lceil\ddagger$. In even elliptic functions only the even funktion $\mathcal{P}$ appears.

5. The roots of Weierstrass $\mathcal{P}$-function $\mathcal{P}(z) = \sum_\lambda 1/(z - \lambda)^2$ appear in pairs $\pm z$ whereas the double poles at at the points of the modular lattice: see the article "The zeros of the

Weierstrass $\mathcal{P}$-function and hypergeometric series" of Duke and Imamoglu [A88] (https://cutt.ly/uIZSK4T).

The roots are given by Eichler-Zagier formula $z_{\pm}(m,n) = 1/2+m+n\tau \pm z_1$, where z_1 contains an imaginary transcendental part $log(5+2\sqrt{6})/2\pi)$ plus second part, which depends on τ (see formula 6) of https://cutt.ly/uIZSK4T).

6. Conformally confined states with conformal weights $h = 1 + (m_1+m_2) + (n_1+n_2)\tau$ can be realized as pairs with conformal weights $(z_+(m_1,n_1), z_-(m_2,n_2)$. The condition $n_1 = -n_2$ guarantees integer-valued conformal weights and conformal confinement for a general value of τ.

7. A possible problem is that the total conformal weights can be also negative, which means tachyonicity. This is not a problem also in the case of Riemann zeta if trivial zeros are included.

   As a matter of fact, already at the level of M^8, M^4 Kähler structure implies that right-handed neutrino ν_R is a tachyon [L100]. However, ν_R provides the tachyon needed to construct massless super-symplectic ground states and also allows us to understand why neutrinos can be massive although right-handed neutrinos are not detected. The point is that only the square of Dirac equation in H is satisfied so that different M^4 chiralities can propagate independently.

   In $M^8 - H$ duality, non-tachyonicity makes it possible to map the momenta at mass shell to the boundary of CD in H. Hence the natural condition would be that the total conformal weight of a physical state is non-negative.

What about the notion of discriminant and ramified prime? One can assign to the algebraic extensions primes as prime ideals for algebraic integers and this suggests that the generalization of p-adicity and p-adic prime is possible. If this is the case also for transcendental extensions, it would be possible to define transcendental p-adicity.

One can however ask whether the discriminant is rational under some conditions. D could also allow factorization to the primes of the transcendental extension.

1. Elliptic functions are meromorphic and have the same number of poles and zeros in the basic cell so that there are some hopes that the ratio of discriminants is finite and even rational or integer for a suitable choice of the modular parameter τ as the ratio of the periods and the other parameters. Discriminant D as the ratio D_1/D_2 of the discriminants defined by the products of differences of roots and poles could be finite although they diverge separately.

2. For the Weierstrass $\mathcal{P}$-function, the zeros appear as pairs $\pm z_0$ and also as complex conjugate pairs. Complex pairs are required by real analyticity essential for the number theoretical vision. It might be possible to define the notion of ramified prime under some assumptions.

   For $z_+(m,n)$ or $z_-(m,n)$, the defining D_1 in D_1/D_2 would reduce to a product $\prod_{m,n}\Delta_{m,n})^2(\Delta_{m,n}+2z_1)(\Delta_{m,n}-2z_1)$, $\Delta_{m,n} = \Delta m + \Delta n\tau$, which is a complex integer valued if τ has integer components. D_1 would be a product of Gaussian integers.

3. The number of poles and zeros for the basic cell is the same so that D_2 as a product of the pole differences would have an identical general form. For large values of m,n, the factors in the product approach $\Delta_{m,n}$ for both zeros and poles so that the corresponding factors combine to a factor approaching unity.

   The double poles of $\mathcal{P}(z) = \sum_\lambda 1/(z-\lambda)^2$ are at points of the lattice. One has $D_2 = \prod_{m,n}\Delta_{m,n})^4$. This gives

$$D = \frac{D_1}{D_2} = \prod_{m,n}(1+\frac{2z_0}{\Delta_{m,n}})(1-\frac{2z_0}{\Delta_{m,n}}) = \prod_{m,n}(1-4(\frac{2z_0}{\Delta_{m,n}})^2) \ .$$

   Therefore D is finite and in general complex and transcendental so that the notion of ramified prime does not make sense as an ordinary prime. z_0 contains a transcendental constant term plus a term depending on τ (https://cutt.ly/uIZSK4T). Whether values of τ for which D is rational, might exist, is not clear.

In the number theoretic vision, the construction of many-particle states corresponds to the formation of functional composites of polynomials P. If the condition $P(0) = 0$ is satisfied, the $n - fold$ composite inherits the roots of $n - 1$-fold composites and the roots are like conserved genes. If one multiplies zeta functions and elliptic functions by z, one obtains similar families and the formation of composites gives rise to iteration sequences and approach to chaos [L84].

Riemann zeta, quantum criticality, and conformal confinement

There are strong indications Riemann zeta (`https://cutt.ly/iVTV1kqs`) has a deep role in physics, in particular in the physics of critical systems. TGD Universe is quantum critical. What quantum criticality would mean at the space-time level is discussed in [L115]. This raises the question whether Riemann zeta could have a deep role in TGD.

First some background relating to the number theoretic view of TGD.

1. In TGD, space-time regions are characterized by polynomials P with rational coefficients [L82, L83]. Galois confinement defines a universal mechanism for the formation of bound states. Momenta for virtual fermions have components, which are algebraic integers in an extension of rationals defined by a polynomial P characterizing space-time region. For the physical many fermion states, the total momentum as the sum of fermion momenta has components, which are integers using the unit defined by the size of the causal diamond (CD).

   This defines a universal number theoretical mechanism for the formation of bound states. The condition is very strong but for rational coefficients it can be satisfied since the sum of all roots is always a rational number as the coefficient of the first order term.

2. Galois confinement implies that the sum of the mass squared values, which are in general complex algebraic numbers in E, is also an integer. Since the mass squared values correspond to conformal weights as also in string models, one has conformal confinement: states are conformal singlets. This condition replaces the masslessness condition of gauge theories [L117].

Riemann zeta is not a polynomial but has infinite number of root. How could one end up with Riemann zeta in TGD? One can also consider the replacement of the rational polynomials with analytic functions with rational coefficients or even more general functions [L109].

1. For real analytic functions roots come as pairs but building many-fermion states for which the sum of roots would be a real integer, is very difficult and in general impossible.

2. Riemann zeta and the hierarchy of its generalizations to extensions of rationals (Dedekind zeta functions) is however a complete exception! If the roots are at the critical line as the generalization of Riemann hypothesis assumes, the sum of the root and its conjugate is equal to 1 and it is easy to construct many fermion states as 2N fermion states, such that they have integer value conformal weight.

   One can wonder whether one could see Riemann zeta as an analog of a polynomial such that the roots as zeros are algebraic numbers. This is however not necessary. Could zeta and its analogies allow it to build a very large number of Galois singlets and they would form a hierarchy corresponding to extensions of rationals. Could they represent a kind of second abstraction level after rational polynomials?

Part II

CATEGORY THEORY AND QUANTUM TGD

Chapter 11

Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

11.1 Introduction

Goro Kato has proposed an ontology of consciousness relying on category theory [A47, A68]. Physicist friendly summary of the basic concepts of category theory can be found in [A57]) whereas the books [A24, A51] provide more mathematically oriented representations. Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. To mention only one example, C. J. Isham [A57] has proposed that topos theory could provide a new approach to quantum gravity in which space-time points would be replaced by regions of space-time and that category theory could geometrize and dynamicize even logic by replacing the standard Boolean logic with a dynamical logic dictated by the structure of the fundamental category purely geometrically [A78].

Although I am an innocent novice in this field and know nothing about the horrible technicalities of the field, I have a strong gut feeling that category theory might provide the desired systematic approach to quantum TGD proper, the general theory of consciousness, and the theory of cognitive representations [K65].

11.1.1 Category Theory As A Purely Technical Tool

Category theory could help to disentangle the enormous technical complexities of the quantum TGD and to organize the existing bundle of ideas into a coherent conceptual framework. The construction of the geometry of the configuration space ("world of classical worlds") [K45, K24]. of classical configuration space spinor fields [K106]. and of S-matrix [K22] using a generalization of the quantum holography principle are especially natural applications. Category theory might also help in formulating the new TGD inspired view about number system as a structure obtained by "gluing together" real and p-adic number fields and TGD as a quantum theory based on this generalized notion of number [K90, K91, K89].

11.1.2 Category Theory Based Formulation Of The Ontology Of TGD Universe

It is interesting to find whether also the ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences [**?**] could be expressed elegantly using the language of the category theory.

There are indeed natural and non-trivial categories involved with many-sheeted space-time and the geometry of the configuration space ("the world of classical worlds"); with configuration space spinor fields; and with the notions of quantum jump, self and self hierarchy. Functors between these categories could express more precisely the quantum classical correspondences and self-referentiality of quantum states allowing them to express information about quantum jump sequence.

1. Self hierarchy has a structure of category and corresponds functorially to the hierarchical structure of the many-sheeted space-time.

2. Quantum jump sequence has a structure of category and corresponds functorially to the category formed by a sequence of maximally deterministic regions of space-time sheet. Even the quantum jump could have space-time correlates made possible by the generalization of the Boolean logic to what might be space-time correlate of quantum logic and allowing to identify space-time correlate for the notion of quantum superposition.

3. The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

4. In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.

5. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K91]. The duality would allow to construct new preferred extremals of Kähler action.

11.1.3 Other Applications

One can imagine also other applications.

1. Categories posses inherent logic [A78] based on the notion of sieves relying on the notion of presheaf which generalizes Boolean logic based on inclusion. In TGD framework inclusion is naturally replaced by topological condensation and this leads to a two-valued logic realizing space-time correlate of quantum logic based on the notions of quantum sieve and quantum topos.

   This suggests the possibility to geometrize the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through logic in which the law of excluded middle is not true. Interestingly, the p-adic logic of cognition is naturally 2-valued whereas the real number based logic of sensory experience allows excluded middle (is the person at the door in or out, in and out, or neither in nor out?). The quantum logic naturally associated with spinors (in the "world of classical worlds") is consistent with the logic based on quantum sieves.

2. Simple Boolean logic of right and wrong does not seem to be ideal for understanding moral rules. Same applies to the beauty-ugly logic of aesthetic experience. The logic based on quantum sieves would perhaps provide a more flexible framework.

3. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience. Here the new elements associated with the ontology of space-time due to the generalization of number concept would be central. Category theory could be also helpful in the modelling of conscious communications, in particular the telepathic communications based on sharing of mental images involving the same mechanism which makes possible space-time correlates of quantum logic and quantum superposition.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L11]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L12].

11.2 What Categories Are?

In the following the basic notions of category theory are introduced and the notion of presheaf and category induced logic are discussed.

11.2.1 Basic Concepts

Categories [A24, A51, A57] are roughly collections of objects A, B, C... and morphisms $f(A \rightarrow B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Topological/linear spaces form a category with continuous/linear maps acting as morphisms. Also algebraic structures of a given type form a category: morphisms are now homomorphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can can be very general: for instance, partial ordering $a \leq b$ can define morphism $f(A \rightarrow B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Group representation is example of this kind of a functor: now group action in group is mapped to a linear action at the level of the representations. Commuting square is an easy visual manner to understand the basic properties of a functor, see **Fig. 11.1**.

The product $C = AB$ for objects of categories is defined by the requirement that there are projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \rightarrow A)$ and $g(D \rightarrow B)$ there exist morphism $h(D \rightarrow C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically (see **Fig. 11.1**) this corresponds to a square diagram in which pairs A, B and C, D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC.

Examples of product categories are Cartesian products of topological and linear spaces, of differentiable manifolds, groups, etc. Also tensor products of linear spaces satisfies these axioms. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

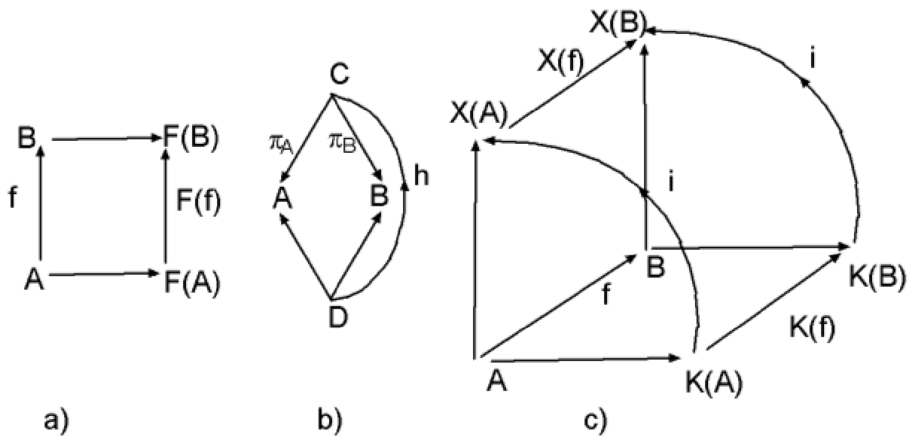

Figure 11.1: Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".)

11.2.2 Presheaf As A Generalization For The Notion Of Set

Presheafs can be regarded as a generalization for the notion of set. Presheaf is a functor X that assigns to any object of a category **C** an object in the category **Set** (category of sets) and maps morphisms to morphisms (maps between sets for **C**). In order to have a category of presheafs, also morphisms between presheafs are needed. These morphisms are called natural transformations $N : X(A) \rightarrow Y(A)$ between the images $X(A)$ and $Y(A)$ of object A of **C**. They are assumed to obey the commutativity property $N(B)X(f) = Y(f)N(A)$ which is best visualized as a commutative square diagram. Set theoretic inclusion $i : X(A) \subset Y(A)$ is obviously a natural transformation.

An easy manner to understand and remember this definition is commuting diagram consisting of two pages of book with arrows of natural transformation connecting the corners of the pages: see **Fig. ??**.

As noticed, presheafs are generalizations of sets and a generalization for the notion of subset to a sub-object of presheaf is needed and this leads to the notion of topos [A78, A57]. In the classical set theory a subset of given sets X can be characterized by a mapping from set X to the set $\Omega = \{true, false\}$ of Boolean statements. Ω itself belongs to the category **C**. This idea generalizes to sub-objects whose objects are collections of sets: Ω is only replaced with its Cartesian power. It can be shown that in the case of presheafs associated with category **C** the sub-object classifier Ω can be replaced with a more general algebra, so called Heyting algebra [A78, A57] possessing the same basic operations as Boolean algebra (and, or, implication arrow, and negation) but is not in general equivalent with any Boolean algebra. What is important is that this generalized logic is inherent to the category **C** so that many-valued logic ceases to be an ad hoc construct in category theory.

In the theory of presheafs sub-object classifier Ω, which belongs to **Set**, is defined as a particular presheaf. Ω is defined by the structure of category **C** itself so that one has a geometrization of the notion of logic implied by the properties of category. The notion of sieve is essential here. A sieve for an object A of category **C** is defined as a collection of arrows $f(A \rightarrow ...)$ with the property that if $f(A \rightarrow B)$ is an arrow in sieve and if $g(B \rightarrow C)$ is any arrow then $gf(A \rightarrow C)$ belongs to sieve.

In the case that morphism corresponds to a set theoretic inclusion the sieve is just either empty set or the set of all sets of category containing set A so that there are only two sieves corresponding to Boolean logic. In the case of a poset (partially ordered set) sieves are sets for which all elements are larger than some element.

11.2.3 Generalized Logic Defined By Category

The presheaf $\Omega : \mathbf{C} \rightarrow \mathbf{Set}$ defining sub-object classifier and a generalization of Boolean logic is defined as the map assigning to a given object A the set of all sieves on A. The generalization of maps $X \rightarrow \Omega$ defining subsets is based on the notion of sub-object K. K is sub-object of presheaf X in the category of presheaves if there exist natural transformation $i : K \rightarrow X$ such that for each A one has $K(A) \subset X(A)$ (so that sub-object property is reduced to subset property).

The generalization of the map $X \rightarrow \Omega$ defining subset is achieved as follows. Let K be a sub-object of X. Then there is an associated characteristic arrow $\chi^K : X \rightarrow \Omega$ generalizing the characteristic Boolean valued map defining subset, whose components $\chi^K_A : X(A) \rightarrow \Omega(A)$ in **C** is defined as

$$\chi^K_A(x) = \{f(A \rightarrow B) | X(f)(x) \in K(B)\} \ .$$

By using the diagrammatic representation of **Fig. 11.1** for the natural transformation i defining sub-object, it is not difficult to see that by the basic properties of the presheaf K $\chi^K_A(x)$ is a sieve. When morphisms f are inclusions in category Set, only two sheaves corresponding to all sets containing X and empty sheaf result. Thus binary valued maps are replaced with sieve-valued maps and sieves take the role of possible truth values. What is also new that truths and logic are in principle context dependent since each object A of **C** serves as a context and defines its own collection of sieves.

The generalization for the notion of point of set X exists also and corresponds to a selection of single element γ_A in the set $X(A)$ for each A object of **C**. This selection must be consistent with the action of morphisms $f(A \rightarrow B)$ in the sense that the matching condition $X(f)(\gamma_A) = \gamma_B$ is satisfied. It can happen that category of presheaves has no points at all since the matching condition need not be satisfied globally.

It turns out that TGD based notion of subsystem leads naturally to what might be called quantal versions of topos, presheaves, sieves and logic.

11.3 More Precise Characterization Of The Basic Categories And Possible Applications

In the following the categories associated with self and quantum jump are discussed in more precise manner and applications to communications and cognition are considered.

11.3.1 Intuitive Picture About The Category Formed By The Geometric Correlates Of Selves

Space-time surface $X^4(X^3)$ decomposes into regions obeying either real or p-adic topology and each region of this kind corresponds to an unentangled subsystem or self lasting at least one quantum jump. By the localization in the zero modes these decompositions are equivalent for all 3-surfaces X^3 in the quantum superposition defined by the prepared WCW spinor fields resulting in quantum jumps. There is a hierarchy of selves since selves can contain sub-selves. The entire space-time surface $X^4(X^3)$ represents the highest level of the self hierarchy.

This structure defines in a natural manner a category. Objects are all possible sub-selves contained in the self hierarchy: sub-self is set consisting of lower level sub-selves, which in turn have a further decomposition to sub-selves, etc... The naïve expectation is that geometrically sub-self belongs to a self as a subset and this defines an inclusion map acting as a natural morphism in this category. This expectation is not quite correct. More natural morphisms are the arrows telling that self as a set of sub-selves contains sub-self as an element. These arrows define a structure analogous to a composite of hierarchy trees.

To be more precise, for a single space-time surface $X^4(X^3)$ this hierarchy corresponds to a subjective time slice of the self hierarchy defined by a single quantum jump. The sequence of hierarchies associated with a sequence of quantum jumps is a natural geometric correlate for the self hierarchy. This means that the objects are now sequences of submoments of consciousness. Sequences are not arbitrary. Self must survive its lifetime although sub-selves at various levels can disappear and reappear (generation and disappearance of mental images). Geometrically this means typically a phase transition transforming real or p_1-adic to p_2-adic space-time region with same topology as the environment. Also sub-selves can fuse to single sub-self. The constraints on self sequences must be such that it takes these processes into account. Note that these constraints emerge naturally from the fact that quantum jumps sequences define the sequences of surfaces $X^4(X^3)$.

By the rich anatomy of the quantum jump there is large number of quantum jumps leading from a given initial quantum history to a given final quantum history. One could envisage quantum jump also as a discrete path in the space of WCW spinor fields leading from the initial state to the final state. In particular, for given self there is an infinite number of closed elementary paths leading from the initial quantum history back to the initial quantum history and these paths in principle give all possible conscious information about a given quantum history/idea: kind of self morphisms are in question (analogous to, say, group automorhisms). Information about point of space is obtained only by moving around and coming back to the point, that is by studying the surroundings of the point. Self in turn can be seen as a composite of elementary paths defined by the quantum jumps. Selves can define arbitrarily complex composite closed paths giving information about a given quantum history.

11.3.2 Categories Related To Self And Quantum Jump

The categories defined by moments of consciousness and the notion of self

Since quantum jump involves state reduction and the sequence of self measurement reducing all entanglement except bound state entanglement, it defines a hierarchy of unentangled subsystems allowing interpretation as objects of a category. Arrows correspond to subsystem-system relationship and the two subsystems resulting in self measurement to the system. What subsystem corresponds mathematically is however not at all trivial and the naïve description as a tensor factor does not work. Rather, a definition relying on the notion of p-adic length scale cutoff identified as a fundamental aspect of nature and consciousness is needed.

It is not clear what the statement that self corresponds to a subsystem which remains unentangled in subsequent quantum jump means concretely since subsystem can certainly change in some limits. What is clear that bound state entanglement between selves means a loss of consciousness. Category theory suggests that there should exist a functor between categories defined by two subsequent moments of consciousness. This functor maps submoments of consciousness to submoments of consciousness and arrows to arrows. Two subsequent submoments of consciousness belong to same sub-self is the functor maps the first one to the latter one. Thus category theory would play essential role in the precise definition of the notion of self.

The sequences of moments of consciousness form a larger category containing sub-selves as sequences of unentangled subsystems mapped to each other by functor arrows functoring subsequent quantum jumps to each other.

What might then be the ultimate characterizer of the self-identity? The theory of infinite primes suggests that space-time surface decomposes into regions labelled by finite p-adic primes. These primes must label also real regions rather than only p-adic ones. A p-adic space-time region characterized by prime p can transform to a real one or vice versa in quantum jump if the sizes of real and p-adic regions are characterized by the p-adic length scale L_p (or n-ary p-adic length scale $L_p(n)$. One can also consider the possibility that real region is accompanied by a p-adic region characterized by a definite prime p and providing a cognitive self-representation of the real region.

If this view is correct, the p-adic prime characterizing a given real or p-adic space-time sheet could be one characterizer of the self-identity. Self identity is lost in bound state entanglement with another space-time sheet (at least when a space-time sheet with smaller value of the p-adic prime joins by flux tube to a one with a higher value of the p-adic prime). Self identity is also lost if a space-time sheet characterized by a given p-adic prime disappears in quantum jump.

The category associated with quantum jump sequences

There are several similarities between the ontologies and epistemologies of TGD and of category theory. Conscious experience is always determined by the discrete paths in the space of configuration space spinor fields defined by a quantum jump connecting two quantum histories (states) and is never determined by single quantum history as such (quantum states are unconscious). Also category theory is about relations between objects, not about objects directly: self-morphisms give information about the object of category (in case of group composite paths would correspond to products of group automorphisms). Analogously closed paths determined by quantum jump sequences give information about single quantum history. The point is however that it is impossible to have direct knowledge about the quantum histories: they are not conscious.

One can indeed define a natural category, call it **QSelf**, applying to this situation. The objects of the category **QSelf** are initial quantum histories of quantum jumps and correspond to prepared quantum states. The discrete path defining quantum jump can be regarded as an elementary morphism. Selves are composites of elementary morphisms of the initial quantum history defined by quantum jumps: one can characterize the morphisms by the number of the elementary morphisms in the product. Trivial self contains no quantum jumps and corresponds to the identity morphism, null path. Thus the collection of all possible sequences of quantum jumps, that is collections of selves allows a description in terms of category theory although the category in question is not a subcategory of the category **Set**.

Category **QSelf** does not possess terminal and initial elements (for terminal (initial) element T there is exactly one arrow $A \to T$ ($T \to A$) for every A: now there are always many paths between quantum histories involved).

11.3.3 Communications In TGD Framework

Goro Kato identifies communications between conscious entities as natural maps between them whereas in TGD natural maps bind submoments of consciousness to selves. In TGD framework quantum measurement and the sharing of mental images are the basic candidates for communications. The problem is that the identification of communications as sharing of mental images is not consistent with the naïve view about subsystem as a tensor factor. Many-sheeted space-time however forces length scale dependent notion of subsystem at space-time level and this saves the situation.

What communications are?

Communication is essentially generation of desired mental images/sub-selves in receiver. Communication between selves need not be directly conscious: in this case communication would generate mental images at some lower level of self hierarchy of receiver: for instance generate large number of sub-sub-selves of similar type. This is like communications between organizations. Communication can be also vertical: self can generate somehow sub-self in some sub-sub....sub-self or sub-sub...sub-self can generate sub-self of self somehow. This is communication from boss to the lower levels organization or vice versa.

These communications should have direct topological counterparts. For instance, the communication between selves could correspond to an exchange of mental image represented as a space-time region of different topology inside sender self space-time sheet. The sender self would simply throw this space-time region to a receiver self like a ball. This mechanism applies also to vertical communications since the ball could be also thrown from a boss to sub...sub-self at some lower level of hierarchy and vice versa.

The sequence of space-time surfaces provides a direct topological counterpart for communication as throwing balls representing sub-selves. Quantum jump sequence contains space-time surfaces in which the regions corresponding to receiver and sender selves are connected by a flux tube (perhaps massless extremal) representing classically the communication: during the communication the receiver and sender would form single self. The cartoon vision about rays connecting the eyes of communication persons would make sense quite concretely.

More refined means of communication would generate sub-selves of desired type directly at the end of receiver. In this case it is not so obvious how the sequence $X^(X^3)$ of space-time surfaces could represent communication. Of course, one can question whether communication is really what happens in this kind of situation. For instance, sender can affect the environment of receiver to be such that receiver gets irritated (computer virus is good manner to achieve this!) but one can wonder whether this is real communication.

Communication as quantum measurement?

Quantum measurement generates one-one map between the states of the entangled systems resulting in quantum measurement. Both state function reduction and self measurement give rise to this kind of map. This map could perhaps be interpreted as quantum communication between unentangled subsystems resulting in quantum measurement. For the state reduction process the space-time correlates are the values of zero modes. For state preparation the space-time correlates should correspond to classical spinor field modes correlating for the two subsystems generated in self measurement.

Communication as sharing of mental images

It has become clear that the sharing of mental images induced by quantum entanglement of sub-selves of two separate selves represents genuine conscious communication which is analogous telepathy and provides general mechanism of remote mental interactions making possible even molecular recognition mechanisms.

1. The sharing of mental images is not possible unless one assumes that self hierarchy is defined by using the notion of length scale resolution defined by p-adic length scale. The notion of scale of resolution is indeed fundamental for all quantum field theories (renormalization group invariance) for all quantum field theories and without it the practical modelling of physics would not be possible. The notion reflects directly the length scale resolution of conscious experience. For a given sub-self of self the resolution is given by the p-adic length scale associated with the sub-self space-time sheet.

2. Length scale resolution emerges naturally from the fact that sub-self space-time sheets having Minkowskian signature of metric are separated from the one representing self by wormhole contacts with Euclidian signature of metric. The signature of the induced metric changes from Minkowskian signature to Euclidian signature at "elementary particle horizons" surrounding the throats of the wormhole contacts and having degenerate induced metric. Elementary

particle horizons are thus metrically two-dimensional light like surfaces analogous to the boundary of the light cone and allow conformal invariance. Elementary particle horizons act as causal horizons. Topologically condensed space-time sheets are analogous to black hole interiors and due to the lack of the causal connectedness the standard description of sub-selves as tensor factors of the state space corresponding to self is not appropriate.

Hence systems correspond, not to the space-time sheets plus entire hierarchy of space-time sheets condensed to it, but rather, to space-time sheets with holes resulting when the space-time sheets representing subsystems are spliced off along the elementary particle horizons around wormhole contacts. This does not mean that all information about subsystem is lost: subsystem space-time sheet is only replaced by the elementary particle horizon. In analogy with the description of the black hole, some parameters (mass, charges, ...) characterizing the classical fields created by the sub-self space-time sheet characterize sub-self.

One can say that the state space of the system contains "holes". There is a hierarchy of state spaces labelled by p-adic primes defining length scale resolutions. This picture resolves a longstanding puzzle relating to the interpretation of the fact that particle is characterize by both classical and quantum charges. Particle cannot couple simultaneously to both and this is achieved if quantum charge is associated with the lowest level description of the particle as CP_2 extremal and classical charges to its description at higher levels of hierarchy.

3. The immediate implication indeed is that it is possible to have a situation in which two selves are unentangled although their sub-selves (mental images) are entangled. This corresponds to the fusion and sharing of mental images. The sharing of the mental images means that union of disjoint hierarchy trees with levels labelled by p-adic primes p is replaced by a union of hierarchy threes with horizontal lines connecting subsystems at the same level of hierarchy. Thus the classical correspondence defines a category of presheaves with both vertical arrows replaced by sub-self-self relationship, horizontal arrows representing sharing of mental images, and natural maps representing binding of submoments of consciousness to selves.

Comparison with Goro Kato's approach

It is of interest to compare Goro Kato's approach with TGD approach. The following correspondence suggests itself.

1. In TGD each quantum jumps defines a category analogous to the Goro Kato's category of open sets of some topological space but set theoretic inclusion replaced by topological condensation. The category defined by a moment of consciousness is dynamical whereas the category of open sets is non-dynamical.

2. The assignment of a 3-surface acting as a causal determinant to each unentangled subsystem defined by a moment of consciousness defines a unique "quantum presheaf" which is the counterpart of the presheaf in Goro Kato's theory. The conscious entity of Kato's theory corresponds to the classical correlate for a moment of consciousness.

3. Natural maps between the causal determinants correspond to the space-time correlates for the functor arrows defining the threads connecting submoments of consciousness to selves. In Goro Kato's theory natural maps are interpreted as communications between conscious entities. The sharing of mental images by quantum entanglement between subsystems of unentangled systems defines horizontal bi-directional arrows between subsystems associated with same moment of consciousness and is counterpart of communication in TGD framework. It replaces the union of disjoint hierarchy trees associated with various unentangled subsystems with hierarchy trees having horizontal connections defining the bi-directional arrows. The sharing of mental images is not possible if subsystem is identified as a tensor factor and thus without taking into account length scale resolution.

11.3.4 Cognizing About Cognition

There are close connections with basic facts about cognition.

1. Categorization means classification and abstraction of common features in the class formed by the objects of a category. Already quantum jump defines category with hierarchical structure and can be regarded as consciously experienced analysis in which totally entangled entire universe $U\Psi_i$ decomposes to a product of maximally unentangled subsystems. The sub-selves of self are like elements of set and are experienced as separate objects whereas sub-sub-selves of sub-self self experiences as an average: they belong to a class or category formed by the sub-self. This kind of averaging occurs also for the contributions of quantum jumps to conscious experience of self.

2. The notions of category theory might be useful in an attempt to construct a theory of cognitive structures since cognition is indeed to high degree classification and abstraction process. The sub-selves of a real self indeed have p-adic space-time sheets as geometric correlates and thus correspond to cognitive sub-selves, thoughts. A meditative experience of empty mind means in case of real self the total absence of thoughts.

3. Predicate logic provides a formalization of the natural language and relies heavily on the notion of n-ary relation. Binary relations $R(a, b)$ corresponds formally to the subset of the product set $A \times B$. For instance, statements like "A does something to B" can be expressed as a binary relation, particular kind of arrow and morphism ($A \leq B$ is a standard example). For sub-selves this relation would correspond to a dynamical evolution at space-time level modelling the interaction between A and B. The dynamical path defined by a sequence of quantum jumps is able to describe this kind of relationships too at level of conscious experience. For instance, "A touches B" would involve the temporary fusion of sub-selves A and B to sub-self C.

11.4 Logic And Category Theory

Category theory allows naturally more general than Boolean logics inherent to the notion of topos associated with any category. Basic question is whether the ordinary notion of topos algebra based on set theoretic inclusion or the notion of quantum topos based on topological condensation is physically appropriate. Starting from the quasi-Boolean algebra of open sets one ends up to the conclusion that quantum logic is more natural. Also WCW spinor fields lead naturally to the notion of quantum logic.

11.4.1 Is The Logic Of Conscious Experience Based On Set Theoretic Inclusion Or Topological Condensation?

The algebra of open sets with intersections and unions and complement defined as the interior of the complement defines a modification of Boolean algebra having the peculiar feature that the points at the boundary of the closure of open set cannot be said to belong to neither interior of open set or of its complement. There are two options concerning the interpretation.

1. 3-valued logic could be in question. It is however not possible to understand this three-valuedness if one defines the quasi-Boolean algebra of open sets as Heyting algebra. The resulting logic is two-valued and the points at boundaries of the closure do not correspond neither to the statement or its negation. In p-adic context the situation changes since p-adic open sets are also closed so that the logic is strictly Boolean. That our ordinary cognitive mind is Boolean provides a further good reason for why cognition is p-adic.

2. These points at the boundary of the closure belong to both interior and exterior in which case a two-valued "quantum logic" allowing superposition of opposite truth values is in question. The situation is indeed exactly the same as in the case of space-time sheet having wormhole contacts to several space-time sheets.

The quantum logic brings in mind Zen consciousness [J5] (which I became fascinated of while reading Hofstadter's book "Gödel, Escher, Bach" [A36]) and one can wonder whether selves having real space-time sheets as geometric correlates and able to live simultaneously in many parallel worlds correspond to Zen consciousness and Zen logic. Zen logic would be also logic of sensory experience whereas cognition would obey strictly Boolean logic.

The causal determinants associated with space-time sheets correspond to light like 3-surfaces which could elementary particle horizons or space-time boundaries and possibly also 3-surfaces separating two maximal deterministic regions of a space-time sheet. These surfaces act as 3-dimensional quantum holograms and have the strange Zen property that they are neither space-like nor time-like so that they represent both the state and the process. In the TGD based model for topological quantum computation (TQC) light-like boundaries code for the computation so that TQC program code would be equivalent with the running program [K5].

11.4.2 Do WCW Spinor Fields Define Quantum Logic And Quantum Topos

I have proposed already earlier that WCW spinor fields define what might be called quantum logic. One can wonder whether WCW spinor s could also naturally define what might be called quantum topos since the category underlying topos defines the logic appropriate to the topos. This question remains unanswered in the following: I just describe the line of though generalizing ordinary Boolean logic.

Finite-dimensional spinors define quantum logic

Spinors at a point of an $2N$-dimensional space span 2^N-dimensional space and spinor basis is in one-one correspondence with Boolean algebra with N different truth values (N bits). 2N=2-dimensional case is simple: Spin up spinor= true and spin-dow spinor=false. The spinors for $2N$-dimensional space are obtained as an N-fold tensor product of 2-dimensional spinors (spin up, spin down): just like in the case of Cartesian power of Ω.

Boolean spinors in a given basis are eigen states for a set N mutually commuting sigma matrices providing a representation for the tangent space group acting as rotations. Boolean spinors define N Boolean statements in the set Ω^N so that one can in a natural manner assign a set with a Boolean spinor. In the real case this group is $SO(2N)$ and reduces to $SU(N)$ for Kähler manifolds. For pseudo-euclidian metric some non-compact variant of the tangent space group is involved. The selections of N mutually commuting generators are labelled by the flag-manifold $SO(2N)/SO(2)^N$ in real context and by the flag-manifold $U(N)/U(1)^N$ in the complex case. The selection of these generators defines a collection of N 2-dimensional linear subspaces of the tangent space.

Spinors are in general complex superpositions of spinor basis which can be taken as the product spinors. The quantum measurement of N spins representing the Cartan algebra of $SO(2N)$ ($SU(N)$) leads to a state representing a definite Boolean statement. This suggests that quantum jumps as moments of consciousness quite generally make universe classical, not only in geometric but also in logical sense. This is indeed what the state preparation process for WCW spinor field seems to do.

Quantum logic for finite-dimensional spinor fields

One can generalize the idea of the spinor logic also to the case of spinor fields. For a given choice of the local spinor basis (which is unique only modular local gauge rotation) spinor field assigns to each point of finite-dimensional space a quantum superposition of Boolean statements decomposing into product of N statements.

Also now one can ask whether it is possible to find a gauge in which each point corresponds to definite Boolean statement and is thus an eigen state of a maximal number of mutually commuting rotation generators Σ_{ij}. This is not trivial if one requires that Dirac equation is satisfied. In the case of flat space this is certainly true and constant spinors multiplied by functions which solve d'Alembert equation provide a global basis.

The solutions of Dirac equation in a curved finite-dimensional space do not usually possess a definite spin direction globally since spinor curvature means the presence of magnetic spin-flipping interaction and since there need not exist a global gauge transformation leading to an eigen state of the local Cartan algebra everywhere. What might happen is that the local gauge transformation becomes singular at some point: for instance, the direction of spin would be radial around given point and become ill defined at the point. This is much like the singularities for vector fields on sphere. The spinor field having this kind of singularity should vanish at singularity but the local gauge rotation rotating spin in same direction everywhere is necessarily ill-defined at the singularity.

In fact, this can be expressed using the language of category theory. The category in question corresponds to a presheaf which assigns to the points of the base space the fiber space of the spinor bundle. The presence of singularity means that there are no global section for this presheaf, that is a continuous choice of a non-vanishing spinor at each point of the base space. The so called Kochen-Specker theorem discussed in [A57] is closely related to a completely analogous phenomenon involving non-existence of global sections and thus non-existence of a global truth value.

Thus in case of curved spaces is not necessarily possible to have spinor field basis representing globally Boolean statements and only the notion of locally Boolean logic makes sense. Indeed, one can select the basis to be eigen state of maximal set of mutually commuting rotation generators in single point of the compact space. Any such choice does.

Quantum logic and quantum topos defined by the prepared WCW spinor fields

The prepared WCW spinor fields occurring as initial and final states of quantum jumps are the natural candidates for defining quantum logic. The outcomes of the quantum jumps resulting in the state preparation process are maximally unentangled states and are as close to Boolean states as possible.

WCW spinors correspond to fermionic Fock states created by infinite number of fermionic (leptonic and quarklike) creation and annihilation operators. The spin degeneracy is replaced by the double-fold degeneracy associated with a given fermion mode: given state either contains fermion or not and these two states represent true and false now. If WCW were flat, the Fock state basis with definite fermion and anti-fermion numbers in each mode would be in one-one correspondence with Boolean algebra.

Situation is however not so simple. Finite-dimensional curved space is replaced with the fiber degrees of freedom of WCW in which the metric is non-vanishing. The precise analogy with the finite-dimensional case suggests that if the curvature form of the WCW spinor connection is nontrivial, it is impossible to diagonalize even the prepared maximally unentangled WCW spinor fields Ψ_i in the entire fiber of WCW (quantum fluctuating degrees of freedom) for given values of the zero modes. Local singularities at which the spin quantum numbers of the diagonalized but vanishing WCW spinor field become ill-defined are possible also now.

In the infinite-dimensional context the presence of the fermion-anti-fermion pairs in the state means that it does not represent a definite Boolean statement unless one defines a more general basis of WCW spinor s for which pairs are present in the states of the state basis: this generalization is indeed possible. The sigma matrices of the WCW appearing in the spinor connection term of the Dirac operator of WCW indeed create fermion-fermion pairs. What is decisive, is not the absence of fermion-anti-fermion pairs, but the possibility that the spinor field basis cannot be reduced to eigen states of the local Cartan algebra in fiber degrees of freedom globally.

Also for bound states of fermions (say leptons and quarks) it is impossible to reduce the state to a definite Boolean statement even locally. This would suggest that fermionic logic does not reduce to a completely Boolean logic even in the case of the prepared states.

Thus WCW spinor fields could have interpretation in terms of non-Boolean quantum logic possessing Boolean logics only as sub-logics and define what might be called quantum topos. Instead of Ω^N-valued maps the values for the maps are complex valued quantum superpositions of truth values in Ω^N.

An objection against the notion of quantum logic is that Boolean algebra operationsandOR do not preserve fermion number so that quantum jump sequences leading from the product state defined by operands to the state representing the result of operation are therefore not possible. One manner to circumvent the objection is to consider the sub-algebra spanned by fermion and anti-fermion pairs for given mode so that fermion number conservation is not a problem. The objection can be also circumvented for pairs of space-time sheets with opposite time orientations and thus opposite signs of energies for particles. One can construct the algebra in question as pairs of many fermion states consisting of positive energy fermion and negative energy anti-fermion so that all states have vanishing fermion number and logical operations become possible. Pairs of MEs with opposite time orientations are excellent candidates for carries of these fermion-anti-fermion pairs.

Quantum classical correspondence and quantum logic

The intuitive idea is that the global Boolean statements correspond to sections of Z^2 bundle. Möbius band is a prototype example here. The failure of a global statement would reduce to the non-existence of global section so that true would transforms to false as one goes around full 2π rotation.

One can ask whether fermionic quantum realization of Boolean logic could have space-time counterpart in terms of Z_2 fiber bundle structure. This would give some hopes of having some connection between category theoretical and fermionic realizations of logic. The following argument stimulated by email discussion with Diego Lucio Rapoport suggests that this might be the case.

1. The hierarchy of Planck constants realized using the notion of generalized embedding space involves only groups $Z_{n_a} \times Z_{n_b}$, $n_a, n_b \neq 2$ if one takes Jones inclusions as starting point. There is however no obvious reason for excluding the values $n_a = 2$ and $n_b = 2$ and the question concerns physical interpretation. Even if one allows only $n_i \geq 3$ one can ask for the physical interpretation for the factorization $Z_{2n} = Z_2 \times Z_n$. Could it perhaps relate to a space-time correlates for Boolean two-valuedness?

2. An important implication of fiber bundle structure is that the partonic 2-surfaces have $Z_{n_a} \times Z_{n_b} = Z_{n_a n_b}$ as a group of conformal symmetries. I have proposed that n_a or n_b is even for fermions so that Z_2 acts as a conformal symmetry of the partonic 2-surface. Both n_a and n_b would be odd for truly elementary bosons. Note that this hypothesis makes sense also for $n_i \geq 3$.

3. Z_2 conformal symmetry for fermions would imply that all partonic 2-surfaces associated with fermions are hyper-elliptic. As a consequence elementary particle vacuum functionals defined in modular degrees of freedom would vanish for fermions for genus $g > 2$ so that only three fermion families would be possible in accordance with experimental facts. Since gauge bosons and Higgs correspond to pairs of partonic 2-surfaces (the throats of the wormhole contact) one has 9 gauge boson states labelled by the pairs (g_1, g_2) which can be grouped to SU(3) singlet and octet. Singlet corresponds to ordinary gauge bosons.

   super-symplectic bosons are truly elementary bosons in the sense that they do not consist of fermion-anti-fermion pairs. For them both n_a and n_b should be odd if the correspondence is taken seriously and all genera would be possible. The super-conformal partners of these bosons have the quantum numbers of right handed neutrino. Since both spin directions are possible, one can ask whether Boolean Z_2 must be present also now. This need not be the case, ν_R generates only super-symmetries and does not define a family of fermionic oscillator operators. The electro-weak spin of ν_R is frozen and it does not couple at all to electro-weak intersections. Perhaps (only) odd values of n_i are possible in this case.

4. If fermionic Boolean logic has a space-time correlate, one can wonder whether the fermionic Z_2 conformal symmetry might correspond to a space-time correlate for the Boolean true-false dichotomy. If the partonic 2-surface contains points which are fixed points of Z_2 symmetry, there exists no everywhere non-vanishing sections. Furthermore, induced spinor fields should vanish at the fixed points of Z_2 symmetry since they correspond to singular orbifold points so that one could not actually have a situation in which true and false are true simultaneously. Global sections could however fail to exist since CP_2 spinor bundle is non-trivial.

11.4.3 Category Theory And The Modelling Of Aesthetic And Ethical Judgements

Consciousness theory should allow to model model the logics of ethics and aesthetics. Evolution (representable as p-adic evolution in TGD framework) is regarded as something positive and is a good candidate for defining universal ethics in TGD framework. Good deeds are such that they support this evolution occurring in statistical sense in any case. Moral provides a practical model for what good deeds are and moral right-wrong statements are analogous to logical statements. Often however the two-valued right-wrong logic seems to be too simplistic in case of moral statements. Same applies to aesthetic judgements. A possible application of the generalized logics defined by the inherent structure of categories relates to the understanding of the dilemmas associated with the moral and aesthetic rules.

As already found, quantum versions of sieves provide a formal generalization of Boolean truth values as a characteristic of a given category. Generalized moral rules could perhaps be seen as sieve valued statements about deeds. Deeds are either right or wrong in what might be called Boolean moral code. One can also consider Zen moral in which some deeds can be said to be right and wrong simultaneously. Some deeds could also be such that there simply exists no globally consistent moral rule: this would correspond to the non-existence of what is called global section assigning to each object of the category consisting of the pairs formed by a moral agents and given deed) a sieve simultaneously.

11.5 Platonism, Constructivism, And Quantum Platonism

During years I have been trying to understand how Category Theory and Set Theory relate to quantum TGD inspired view about fundamentals of mathematics and the outcome section is added to this chapter several years after its first writing. I hope that reader does not experience too unpleasant discontinuity. I managed to clarify my thoughts about what these theories are by reading the article Structuralism, Category Theory and Philosophy of Mathematics by Richard Stefanik [A79]. Blog discussions and email correspondence with Sampo Vesterinen have been very stimulating and inspired the attempt to represent TGD based vision about the unification of mathematics, physics, and consciousness theory in a more systematic manner.

Before continuing I want to summarize the basic ideas behind TGD vision. One cannot understand mathematics without understanding mathematical consciousness. Mathematical consciousness and its evolution must have direct quantum physical correlates and by quantum classical correspondence these correlates must appear also at space-time level. Quantum physics must allow to realize number as a conscious experience analogous to a sensory quale. In TGD based ontology there is no need to postulate physical world behind the quantum states as mathematical entities (theory is the reality). Hence number cannot be any physical object, but can be identified as a quantum state or its label and its number theoretical anatomy is revealed by the conscious experiences induced by the number theoretic variants of particle reactions. Mathematical systems and their axiomatics are dynamical evolving systems and physics is number theoretically universal selecting rationals and their extensions in a special role as numbers, which can can be regarded elements of several number fields simultaneously.

11.5.1 Platonism And Structuralism

There are basically two philosophies of mathematics.

1. Platonism assumes that mathematical objects and structures have independent existence. Natural numbers would be the most fundamental objects of this kind. For instance, each natural number has its own number-theoretical anatomy decomposing into a product of prime numbers defining the elementary particles of Platonia. For quantum physicist this vision is attractive, and even more so if one accepts that elementary particles are labelled by primes (as I do)! The problematic aspects of this vision relate to the physical realization of the Platonia. Neither Minkowski space-time nor its curved variants understood in the sense of set theory have no room for Platonia and physical laws (as we know them) do not seem to allow the realization of all imaginable internally consistent mathematical structures.

2. Structuralist believes that the properties of natural numbers result from their relations to other natural numbers so that it is not possible to speak about number theoretical anatomy in the Platonic sense. Numbers as such are structureless and their relationships to other numbers provide them with their apparent structure. According to [A79] structuralism is

however not enough for the purposes of number theory: in combinatorics it is much more natural to use intensional definition for integers by providing them with inherent properties such as decomposition into primes. I am not competent to take any strong attitudes on this statement but my physicist's intuition tells that numbers have number theoretic anatomy and that this anatomy can be only revealed by the morphisms or something more general which must have physical counterparts. I would like to regard numbers are analogous to bound states of elementary particles. Just as the decays of bound states reveal their inner structure, the generalizations of morphisms would reveal to the mathematician the inherent number theoretic anatomy of integers.

11.5.2 Structuralism

Set theory and category theory represent two basic variants of structuralism and before continuing I want to clarify to myself the basic ideas of structuralism: the reader can skip this section if it looks too boring.

Set theory

Structuralism has many variants. In set theory [A13] the elements of set are treated as structureless points and sets with the same cardinality are equivalent. In number theory additional structure must be introduced. In the case of natural numbers one introduces the notion of successor and induction axiom and defines the basic arithmetic operations using these. Set theoretic realization is not unique. For instance, one can start from empty set Φ identified as 0, identify 1 as $\{\Phi\}$, 2 as $\{0,1\}$ and so on. One can also identify 0 as Φ, 1 as $\{0\}$, 2 as $\{\{0\}\}$, ... For both physicist and consciousness theorist these formal definitions look rather weird.

The non-uniqueness of the identification of natural numbers as a set could be seen as a problem. The structuralist's approach is based on an extensional definition meaning that two objects are regarded as identical if one cannot find any property distinguishing them: object is a representative for the equivalence class of similar objects. This brings in mind gauge fixing to the mind of physicists.

Category theory

Category theory [A2] represents a second form of structuralism. Category theorist does not worry about the ontological problems and dreams that all properties of objects could be reduced to the arrows and formally one could identify even objects as identity morphisms (looks like a trick to me). The great idea is that functors between categories respecting the structure defined by morphisms provide information about categories. Second basic concept is natural transformation which maps functors to functors in a structure preserving manner. Also functors define a category so that one can construct endless hierarchy of categories. This approach has enormous unifying power since functors and natural maps systemize the process of generalization. There is no doubt that category theory forms a huge piece of mathematics but I find difficult to believe that arrows can catch all of it.

The notion of category can be extended to that of n-category. In the blog post "First edge of the cube" (see `http://tinyurl.com/yydjavv8`) I have proposed a geometric realization of this hierarchy in which one defines 1-morphisms by parallel translations, 2-morphisms by parallel translations of parallel translations, and so on. In infinite-dimensional space this hierarchy would be infinite. Abstractions about abstractions about.., thoughts about thoughts about, statements about statements about..., is the basic idea behind this interpretation. Also the hierarchy of logics of various orders corresponds to this hierarchy. This encourages to see category theoretic thinking as being analogous to higher level self reflection which must be distinguished from the direct sensory experience.

In the case of natural numbers category theoretician would identify successor function as the arrow binding natural numbers to an infinitely long string with 0 as its end. If this approach would work, the properties of numbers would reflect the properties of the successor function.

11.5.3 The View About Mathematics Inspired By TGD And TGD Inspired Theory Of Consciousness

TGD based view might be called quantum Platonism. It is inspired by the requirement that both quantum states and quantum jumps between them are able to represent number theory and that all quantum notions have also space-time correlates so that Platonia should in some sense exist also at the level of space-time. Here I provide a brief summary of this view as it is now.

Physics is fixed from the uniqueness of infinite-D existence and number theoretic universality

1. The basic philosophy of quantum TGD relies on the geometrization of physics in terms of infinite-dimensional Kähler geometry of WCW , whose uniqueness is forced by the mere mathematical existence. Space-time dimension and embedding space $H = M^4 \times CP_2$ are fixed among other things by this condition and allow interpretation in terms of classical number fields. Physical states correspond to WCW spinor fields with WCW spinor s having interpretation as Fock states. Rather remarkably, WCW Clifford algebra defines standard representation of so called hyper finite factor of II_1, perhaps the most fascinating von Neumann algebra.

2. Number theoretic universality states that all number fields are in a democratic position. This vision can be realized by requiring generalization of notions of embedding space by gluing together real and p-adic variants of embedding space along common algebraic numbers. All algebraic extensions of p-adic numbers are allowed. Real and p-adic space-time sheets intersect along common algebraics. The identification of the p-adic space-time sheets as correlates of cognition and intentionality explains why cognitive representations at space-time level are always discrete. Only space-time points belonging to an algebraic extension of rationals associated contribute to the data defining S-matrix. These points define what I call number theoretic braids. The interpretation in of algebraic discreteness terms of a physical realization of axiom of choice is highly suggestive. The axiom of choice would be dynamical and evolving quantum jump by quantum jump as the algebraic complexity of quantum states increases.

Holy trinity of existence

In TGD framework one would have 3-levelled ontology numbers should have representations at all these levels [L5].

1. Subjective existence as a sequence of quantum jumps giving conscious sensory representations for numbers and various geometric structures would be the first level.

2. Quantum states would correspond to Platonia of mathematical ideas and mathematician- or if one is unwilling to use this practical illusion- conscious experiences about mathematic ideas, would be in quantum jumps. The quantum jumps between quantum states respecting the symmetries characterizing the mathematical structure would provide conscious information about the mathematical ideas not directly accessible to conscious experience. Mathematician would live in Plato's cave. There is no need to assume any independent physical reality behind quantum states as mathematical entities since quantum jumps between these states give rise to conscious experience. Theory-reality dualism disappears since the theory is reality or more poetically: painting is the landscape.

3. The third level of ontology would be represented by classical physics at the space-time level essential for quantum measurement theory. By quantum classical correspondence space-time physics would be like a written language providing symbolic representations for both quantum states and changes of them (by the failure of complete classical determinism of the fundamental variational principle). This would involve both real and p-adic space-time sheets corresponding to sensory and cognitive representations of mathematical concepts. This representation makes possible the feedback analogous to formulas written by mathematician

crucial for the ability of becoming conscious about what one was conscious of and the dynamical character of this process allows to explain the self-referentiality of consciousness without paradox.

This ontology releases a deep Platonistic sigh of relief. Since there are no physical objects, there is no need to reduce mathematical notions to objects of the physical world. There are only quantum states identified as mathematical entities labelled naturally by integer valued quantum numbers; conscious experiences, which must represent sensations giving information about the number theoretical anatomy of a given quantum number; and space-time surfaces providing space-time correlates for quantum physics and therefore also for number theory and mathematical structures in general.

Factorization of integers as a direct sensory perception?

Both physicist and consciousness theorist would argue that the set theoretic construction of natural numbers could not be farther away from how we experience integers. Personally I feel that neither structuralist's approach nor Platonism as it is understood usually are enough. Mathematics is a conscious activity and this suggests that quantum theory of consciousness must be included if one wants to build more satisfactory view about fundamentals of mathematics.

Oliver Sack's book *The man who mistook his wife for a hat* [J4] (see also [K82]) contains fascinating stories about those aspects of brain and consciousness which are more or less mysterious from the view point of neuroscience. Sacks tells in his book also a story about twins who were classified as idiots but had amazing number theoretical abilities. I feel that this story reveals something very important about the real character of mathematical consciousness.

The twins had absolutely no idea about mathematical concepts such as the notion of primeness but they could factorize huge numbers and tell whether they are primes. Their eyes rolled wildly during the process and suddenly their face started to glow of happiness and they reported a discovery of a factor. One could not avoid the feeling that they quite concretely saw the factorization process. The failure to detect the factorization served for them as the definition of primeness. For them the factorization was not a process based on some rules but a direct sensory perception.

The simplest explanation for the abilities of twins would in terms of a model of integers represented as string like structures consisting of identical basic units. This string can decay to strings. If string containing n units decaying into $m > 1$ identical pieces is not perceived, the conclusion is that a prime is in question. It could also be that decay to units smaller than 2 was forbidden in this dynamics. The necessary connection between written representations of numbers and representative strings is easy to build as associations.

This kind theory might help to understand marvellous feats of mathematicians like Ramanujan who represents a diametrical opposite of Groethendienck as a mathematician (when Groethendienck was asked to give an example about prime, he mentioned 57 which became known as Groethendienck prime!).

The lesson would be that one very fundamental representation of integers would be, not as objects, but conscious experiences. Primeness would be like the quale of redness. This of course does not exclude also other representations.

Experience of integers in TGD inspired quantum theory of consciousness

In quantum physics integers appear very naturally as quantum numbers. In quantal axiomatization or interpretation of mathematics same should hold true.

1. In TGD inspired theory of consciousness [L5] quantum jump is identified as a moment of consciousness. There is actually an entire fractal hierarchy of quantum jumps consisting of quantum jumps and this correlates directly with the corresponding hierarchy of physical states and dark matter hierarchy. This means that the experience of integer should be reducible to a certain kind of quantum jump. The possible changes of state in the quantum jump would characterize the sensory representation of integer.

2. The quantum state as such does not give conscious information about the number theoretic anatomy of the integer labelling it: the change of the quantum state is required. The above

geometric model translated to quantum case would suggest that integer represents a multiplicatively conserved quantum number. Decays of this this state into states labelled by integers n_i such that one has $n = \prod_i n_i$ would provide the fundamental conscious representation for the number theoretic anatomy of the integer. At the level of sensory perception based the space-time correlates a string-like bound state of basic particles representing n=1.

3. This picture is consistent with the Platonist view about integers represented as structured objects, now labels of quantum states. It would also conform with the view of category theorist in the sense that the arrows of category theorist replaced with quantum jumps are necessary to gain conscious information about the structure of the integer.

Infinite primes and arithmetic consciousness

Infinite primes [K89] were the first mathematical fruit of TGD inspired theory of consciousness and the inspiration for writing this posting came from the observation that the infinite primes at the lowest level of hierarchy provide a representation of algebraic numbers as Fock states of a super-symmetric arithmetic QFT so that it becomes possible to realize quantum jumps revealing the number theoretic anatomy of integers, rationals, and perhaps even that of algebraic numbers.

1. Infinite primes have a representation as Fock states of super-symmetric arithmetic QFT and at the lowest level of hierarchy they provide representations for primes, integers, rationals and algebraic numbers in the sense that at the lowest level of hierarchy of second quantizations the simplest infinite primes are naturally mapped to rationals whereas more complex infinite primes having interpretation as bound states can be mapped to algebraic numbers. Conscious experience of number can be assigned to the quantum jumps between these quantum states revealing information about the number theoretic anatomy of the number represented. It would be wrong to say that rationals only label these states: rather, these states represent rationals and since primes label the particles of these states.

2. More concretely, the conservation of number theoretic energy defined by the logarithm of the rational assignable with the Fock state implies that the allowed decays of the state to a product of infinite integers are such that the rational can decompose only into a product of rationals. These decays could provide for the above discussed fundamental realization of multiplicative aspects of arithmetic consciousness. Also additive aspects are represented since the exponents k in the powers p^k appearing in the decomposition are conserved so that only the partitions $k = \sum_i k_i$ are representable. Thus both product decompositions and partitions, the basic operations of number theorist, are represented.

3. The higher levels of the hierarchy represent a hierarchy of abstractions about abstractions bringing strongly in mind the hierarchy of n-categories and various similar constructions including n: th order logic. It also seems that the n+1: th level of hierarchy provides a quantum representation for the n: th level. Ordinary primes, integers, rationals, and algebraic numbers would be the lowest level, -the initial object- of the hierarchy representing nothing at low level. Higher levels could be reduced to them by the analog of category theoretic reductionism in the sense that there is arrow between n: th and n+1: th level representing the second quantization at this level. On can also say that these levels represent higher reflective level of mathematical consciousness and the fundamental sensory perception corresponds the lowest level.

4. Infinite primes have also space-time correlates. The decomposition of particle into partons can be interpreted as a infinite prime and this gives geometric representations of infinite primes and also rationals. The finite primes appearing in the decomposition of infinite prime correspond to bosonic or fermionic partonic 2-surfaces. Many-sheeted space-time provides a representation for the hierarchy of second quantizations: one physical prediction is that many particle bound state associated with space-time sheet behaves exactly like a boson or fermion. Nuclear string model is one concrete application of this idea: it replaces nucleon reductionism with reductionism occurs first to strings consisting of $A \leq 4$ nuclei and which in turn are strings consisting of nucleons. A further more speculative representation of infinite rationals as space-time surfaces is based on their mapping to rational functions.

Number theoretic Brahman=Atman identity

The notion of infinite primes leads to the notion of algebraic holography in which space-time points possess infinitely rich number-theoretic anatomy. This anatomy would be due to the existence of infinite number of real units defined as ratios of infinite integers which reduce to unit in the real sense and various p-adic senses. This anatomy is not visible in real physics but can contribute directly to mathematical consciousness [K89].

The anatomies of single space-time point could represent the entire world of classical worlds and quantum states of universe: the number theoretic anatomy is of course not visible in the structure of these these states. Therefore the basic building brick of mathematics - point- would become the Platonia able to represent all of the mathematics consistent with the laws of quantum physics. Space-time points would evolve, becoming more and more complex quantum jump by quantum jump. WCW and quantum states would be represented by the anatomies of space-time points. Some space-time points are more "civilized" than others so that space-time decomposes into "civilizations" at different levels of mathematical evolution.

Paths between space-time points represent processes analogous to parallel translations affecting the structure of the point and one can also define n-parallel translations up to $n = 4$ at level of space-time and $n = 8$ at level of embedding space. At level of world of classical worlds whose points are representable as number theoretical anatomies arbitrary high values of n can be realized.

It is fair to say that the number theoretical anatomy of the space-time point makes it possible self-reference loop to close so that structured points are able to represent the physics of associated with with the structures constructed from structureless points. Hence one can speak about algebraic holography or number theoretic Brahman=Atman identity.

Finite measurement resolution, Jones inclusions, and number theoretic braids

In the history of physics and mathematics the realization of various limitations have been the royal road to a deeper understanding (Uncertainty Principle, Gödel's theorem). The precision of quantum measurement, sensory perception, and cognition are always finite. In standard quantum measurement theory this limitation is not taken into account but forms a corner stone of TGD based vision about quantum physics and of mathematics too as I want to argue in the following.

The finite resolutions has representation both at classical and quantum level.

1. At the level of quantum states finite resolution is represented in terms of Jones inclusions N subset M of hyper-finite factors of type II_1 (HFFs) [K35]. N represents measurement resolution in the sense that the states related by the action of N cannot be distinguished in the measurement considered. Complex rays are replaced by N rays. This brings in non-commutativity via quantum groups [K11]. Non-commutativity in TGD Universe would be therefore due to a finite measurement resolution rather than something exotic emerging in the Planck length scale. Same applies to p-adic physics: p-adic space-time sheets have literally infinite size in real topology!

2. At the space-time level discretization implied by the number theoretic universality could be seen as being due to the finite resolution with common algebraic points of real and p-adic variant of the partonic 3-surface chosen as representatives for regions of the surface. The solutions of Kähler-Dirac equation are characterized by the prime in question so that the preferred prime makes itself visible at the level of quantum dynamics and characterizes the p-adic length scale fixing the values of coupling constants. Discretization could be also understood as effective non-commutativity of embedding space points due to the finite resolution implying that second quantized spinor fields anti-commute only at a discrete set of points rather than along stringy curve.

In this framework it is easy to imagine physical representations of number theoretical and other mathematical structures.

1. Every compact group corresponds to a hierarchy of Jones inclusions corresponding to various representations for the quantum variants of the group labelled by roots of unity. I would be surprised if non-compact groups would not allow similar representation since HFF can be

regarded as infinite tensor power of n-dimensional complex matrix algebra for any value of n. Somewhat paradoxically, the finite measurement resolution would make possible to represent Lie group theory physically [K35].

2. There is a strong temptation to identify the Galois groups of algebraic numbers as the infinite permutation group S_∞ consisting of permutations of finite number of objects, whose projective representations give rise to an infinite braid group B_∞. The group algebras of these groups are HFFs besides the representation provided by the spinors of the world of classical worlds having physical identification as fermionic Fock states. Therefore physical states would provide a direct representation also for the more abstract features of number theory [K47].

3. Number theoretical braids crucial for the construction of S-matrix provide naturally representations for the Galois groups G associated with the algebraic extensions of rationals as diagonal embeddings $G \times G \times ....$ to the completion of S_∞ representable also as the action on the completion of spinors in the world of classical worlds so that the core of number theory would be represented physically [K47]. At the space-time level number theoretic braid having G as symmetries would represent the G. These representations are analogous to global gauge transformations. The elements of S_∞ are analogous to local gauge transformations having a natural identification as a universal number theoretical gauge symmetry group leaving physical states invariant.

Hierarchy of Planck constants and the generalization of embedding space

Jones inclusions inspire a further generalization of the notion of embedding space obtained by gluing together copies of the embedding space H regarded as coverings $H \rightarrow H/G_a \times G_b$. In the simplest scenario $G_a \times G_b$ leaves invariant the choice of quantization axis and thus this hierarchy provides embedding space correlate for the choice of quantization axes inducing these correlates also at space-time level and at the level of world of classical worlds [K35].

Dark matter hierarchy is identified in terms of different sectors of H glued together along common points of base spaces and thus forming a book like structure. For the simplest option elementary particles proper correspond to maximally quantum critical systems in the intersection of all pages. The field bodies of elementary particles are in the interiors of the pages of this "book".

One can assign to Jones inclusions quantum phase $q = exp(i2\pi/n)$ and the groups Z_n acts as exact symmetries both at level of M^4 and CP_2. In the case of M^4 this means that space-time sheets have exact Z_n rotational symmetry. This suggests that the algebraic numbers q^m could have geometric representation at the level of sensory perception as Z_n symmetric objects. We need not be conscious of this representation in the ordinary wake-up consciousness dominated by sensory perception of ordinary matter with $q = 1$. This would make possible the idea about transcendentals like π, which do not appear in any finite-dimensional extension of even p-adic numbers (p-adic numbers allow finite-dimensional extension by since e^p is ordinary p-adic number). Quantum jumps in which state suffers an action of the generating element of Z_n could also provide a sensory realization of these groups and numbers $exp(i2\pi/n)$.

Planck constant is identified as the ratio n_a/n_b of integers associated with M^4 and CP_2 degrees of freedom so that a representation of rationals emerge again. The so called ruler and compass rationals whose definition involves only a repeated square root operation applied on rationals are cognitively the simplest ones and should appear first in the evolution of mathematical consciousness. The successful [K31] quantum model for EEG is only one of the applications providing support for their preferred role. Other applications are to Bohr quantization of planetary orbits interpreted as being induced by the presence of macroscopically quantum coherent dark matter [K85].

11.5.4 Farey Sequences, Riemann Hypothesis, Tangles, And TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *"Platonia as the best possible world in the* *sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

Farey sequences

Some basic facts about Farey sequences [A3] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \tag{11.5.1}$$

   Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p+1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p+1)$ to the sequence.

   The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the embedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [K23] would naturally relate to the Farey sequence.

Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \ .$$

In other words, $d_{n,N}$ is the difference between the n: th term of the N: th Farey sequence, and the n: th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\sum_{n=1,...,|F_N|} |d_{n,N}| = O(N^r) \ \text{for any} \ r > 1/2 \ ,$$
$$\sum_{n=1,...,|F_N|} d_{n,N}^2 = O(N^r) \ \text{for any} \ r > 1 \ . \tag{11.5.2}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [K23, K21].

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [K23].

Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$: th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [A69] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2, Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The

so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles $[0]$, $[\infty]$, $\pm[1]$, $\pm 1/[1]$, $\pm[2]$, $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M+N-2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M+N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N>2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N >> 2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N>2$ case.

1. Could the commutativity of tangle product allow to characterize the $N>2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N>2$. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N: th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a,b]^T \rightarrow a/b$ from a rational 2-spinor $[a,b]^T$ to which SL(2(N-1), Z) acts. Equivalence means that the columns $[a,b]^T$ and $[c,d]^T$ combine to form element of SL(2, Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N-1$ $2(N-1)$-component projective column-spinors $[a_i^1, a_i^2, .., a_i^{2(N-1)}]^T$, $i=1,...N-1$ so that only the ratios $a_i^k/a_i^{2(N-1)} \leq 1$ matter? Could equivalence for them mean that the $N-1$ spinors combine to form $N-1+N-1$ columns of $SL(2(N-1),Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc=\pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g, Z) so that $N-1$ would be analogous to g and $1 \leq N \geq 3$- braids would correspond to $g \leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2,Q)$ labelled by N (the generator $\tau \rightarrow \tau+2$ of modular group is replaced with $\tau \rightarrow \tau + 2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N-1),Q)$.

Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N: th root of unity instead of phase U satisfying $U^2 = 1$?

How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N: th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

11.6 Quantum Quandaries

John Baez's [A60] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

11.6.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

11.6.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups [K11] too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

11.6.3 Tqft As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob → Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

11.6.4 The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A34] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In

TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of $2\rightarrow 2$ reaction open string is pinched to a point at vertex. $1\rightarrow 2$ vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. There is U-matrix acting in zero energy states. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K61].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics.

11.7 How To Represent Algebraic Numbers As Geometric Objects?

Physics blogs are also interesting because they allow to get some grasp about very different styles of thinking of a mathematician and physicist. For mathematician it is very important that the result is obtained by a strict use of axioms and deduction rules. Physicist is a cognitive opportunist: it does not matter how the result is obtained by moving along axiomatically allowed paths or not, and the new result is often more like a discovery of a new axiom and physicist is ever-grateful for Gödel for giving justification for what sometimes admittedly degenerates to a creative hand-waving. For physicist ideas form a kind of bio-sphere and the fate of the individual idea depends on its ability to survive, which is determined by its ability to become generalized, its consistency with other ideas, and ability to interact with other ideas to produce new ideas.

11.7.1 Can One Define Complex Numbers As Cardinalities Of Sets?

During few days before writing this we have had in Kea's blog a little bit of discussion inspired by the problem related to the categorification of basic number theoretical structures. I have learned that sum and product are natural operations for the objects of category. For instance, one can define sum as in terms of union of sets or direct sum of vector spaces and product as Cartesian product of sets and tensor product of vector spaces: rigs [A19] are example of categories for which natural numbers define sum and product.

Subtraction and division are however problematic operations. Negative numbers and inverses of integers do not have a realization as a number of elements for any set or as dimension of vector space. The naïve physicist inside me asks immediately: why not go from statics to dynamics and take operations (arrows with direction) as objects: couldn't this allow to define subtraction and division? Is the problem that the axiomatization of group theory requires something which purest categorification does not give? Or aren't the numbers representable in terms of operations of finite groups not enough? In any case cyclic groups would allow to realize roots of unity as operations (Z_2 would give -1).

One could also wonder why the algebraic numbers might not somehow result via the representations of permutation group of infinite number of elements containing all finite groups and thus Galois groups of algebraic extensions as subgroups? Why not take the elements of this group as objects of the basic category and continue by building group algebra and hyper-finite factors of type II_1 isomorphic to spinors of world of classical worlds, and so on.

After having written the first half of the section, I learned that something similar to the transition from statics to dynamics is actually carried out but by manner which is by many orders of magnitudes more refined than the proposal above and that I had never been able to imagine. The article *Objects of categories as complex numbers* of Marcelo Fiore and Tom Leinster [A19] describes a fascinating idea summarized also by John Baez [A17] about how one can assign to the objects of a category complex numbers as roots of a polynomial $Z = P(Z)$ defining an isomorphism of object. Z is the element of a category called rig, which differs from ring in that integers are replaced with natural numbers. One can replace Z with a complex number $|Z|$ defined as a root of polynomial. $|Z|$ is interpreted formally as the cardinality of the object. It is essential to have natural numbers and thus only product and sum are defined. This means a restriction: for instance, only complex algebraic numbers associated with polynomials having natural numbers as coefficients are obtained. Something is still missing.

Note that this correspondence assumes the existence of complex numbers and one cannot say that complex numbers are categorified. Maybe basic number fields must be left outside categorification. One can however require that all of them have a concrete set theoretic representation rather than only formal interpretation as cardinality so that one still encounters the problem how to represent algebraic complex number as a concrete cardinality of a set.

11.7.2 In What Sense A Set Can Have Cardinality -1?

The discussion in Kea's blog led me to ask what the situation is in the case of p-adic numbers. Could it be possible to represent the negative and inverse of p-adic integer, and in fact any p-adic number, as a geometric object? In other words, does a set with -1 or $1/n$ or even $\sqrt{-1}$ elements exist? If this were in some sense true for all p-adic number fields, then all this wisdom combined together might provide something analogous to the adelic representation for the norm of a rational number as product of its p-adic norms. As will be found, alternative interpretations of complex algebraic numbers as p-adic numbers representing cardinalities of p-adic fractals emerge. The fractal defines the manner how one must do an infinite sum to get an infinite real number but finite p-adic number.

Of course, this representation might not help to define p-adics or reals categorically but might help to understand how p-adic cognitive representations defined as subsets for rational intersections of real and p-adic space-time sheets could represent p-adic number as the number of points of p-adic fractal having infinite number of points in real sense but finite in the p-adic sense. This would also give a fundamental cognitive role for p-adic fractals as cognitive representations of numbers.

How to construct a set with -1 elements?

The basic observation is that p-adic -1 has the representation

$$-1 = (p-1)/(1-p) = (p-1)(1+p+p^2+p^3....)$$

As a real number this number is infinite or -1 but as a p-adic number the series converges and has p-adic norm equal to 1. One can also map this number to a real number by canonical identification taking the powers of p to their inverses: one obtains p in this particular case. As a matter fact, any rational with p-adic norm equal to 1 has similar power series representation.

The idea would be to represent a given p-adic number as the infinite number of points (in real sense) of a p-adic fractal such that p-adic topology is natural for this fractal. This kind of fractals can be constructed in a simple manner: from this more below. This construction allows to represent any p-adic number as a fractal and code the arithmetic operations to geometric operations for these fractals.

These representations - interpreted as cognitive representations defined by intersections of real and p-adic space-time sheets - are in practice approximate if real space-time sheets are assumed to have a finite size: this is due to the finite p-adic cutoff implied by this assumption and the meaning a finite resolution. One can however say that the p-adic space-time itself could by its necessarily infinite size represent the *idea* of given p-adic number faithfully.

This representation applies also to the p-adic counterparts of algebraic numbers in case that they exist. For instance, roughly one half of p-adic numbers have square root as ordinary p-adic number and quite generally algebraic operations on p-adic numbers can give rise to p-adic numbers so that also these could have set theoretic representation. For $p\ mod\ 4 = 1$ also $\sqrt{(-1)}$ exists: for instance, for $p = 5$: $2^2 = 4 = -1\ mod\ 5$ guarantees this so that also imaginary unit and complex numbers would have a fractal representation. Also many transcendentals possess this kind of representation. For instance $exp(xp)$ exists as a p-adic number if x has p-adic norm not larger than 1: also $log(1 + xp)$ does so.

Hence a quite impressive repertoire of p-adic counterparts of real numbers would have representation as a p-adic fractal for some values of p. Adelic vision would suggest that combining these representations one might be able to represent quite a many real numbers. In the case of π I do not find any obvious p-adic representation (for instance $sin(\pi/6) = 1/2$ does not help since the p-adic variant of the Taylor expansion of $\pi/6 = arcsin(1/2)$ does not converge p-adically for any value of p). It might be that there are very many transcendentals not allowing fractal representation for any value of p.

Conditions on the fractal representations of p-adic numbers

Consider now the construction of the fractal representations in terms of rational intersections of real real and p-adic space-time sheets. The question is what conditions are natural for this representation if it corresponds to a cognitive representation is realized in the rational intersection of real and p-adic space-time sheets obeying same algebraic equations.

1. Pinary cutoff is the analog of the decimal cutoff but is obtained by dropping away high positive rather than negative powers of p to get a finite real number: example of pinary cutoff is $-1 = (p-1)(1+p+p^2+...) \rightarrow (p-1)(1+p+p^2)$. This cutoff must reduce to a fractal cutoff meaning a finite resolution due to a finite size for the real space-time sheet. In the real sense the p-adic fractal cutoff means not forgetting details below some scale but cutting out all above some length scale. Physical analog would be forgetting all frequencies below some cutoff frequency in Fourier expansion.

   The motivation comes from the fact that TGD inspired consciousness assigns to a given biological body there is associated a field body or magnetic body containing dark matter with large $\hbar$ and quantum controlling the behavior of biological body and so strongly identifying with it so as to belief that this all ends up to a biological death. This field body has an onion like fractal structure and a size of at least order of light-life. Of course, also larger onion layers could be present and would represent those levels of cognitive consciousness not depending on the sensory input on biological body: some altered states of consciousness could relate to these levels. In any case, the larger the magnetic body, the better the numerical skills of the p-adic mathematician.

2. Lowest pinary digits of $x = x_0 + x_1p + x_2p^2 + ...$, $x_n \leq p$ must have the most reliable representation since they are the most significant ones. The representation must be also highly redundant to guarantee reliability. This requires repetitions and periodicity. This is guaranteed if the representation is hologram like with segments of length p^n with digit x_n represented again and again in all segments of length p^m, $m > n$.

3. The TGD based physical constraint is that the representation must be realizable in terms of induced classical fields assignable to the field body hierarchy of an intelligent system interested in artistic expression of p-adic numbers using its own field body as instrument. As a matter, sensory and cognitive representations are realized at field body in TGD Universe and EEG is in a fundamental role in building this representation. By p-adic fractality fractal wavelets are the most natural candidate. The fundamental wavelet should represent the p different pinary digits and its scaled up variants would correspond to various powers of p so that the representation would reduce to a Fourier expansion of a classical field.

Concrete representation

Consider now a concrete candidate for a representation satisfying these constraints.

1. Consider a p-adic number

   $$y = p^{n_0}x, \quad x = \sum x_n p^n \quad , \quad n \geq n_0 = 0 \quad .$$

   If one has a representation for a p-adic unit x the representation of is by a purely geometric fractal scaling of the representation by p^n. Hence one can restrict the consideration to p-adic units.

2. To construct the representation take a real line starting from origin and divide it into segments with lengths $1, p, p^2, ...$ In TGD framework this scalings come actually as powers of $p^{1/2}$ but this is just a technical detail.

3. It is natural to realize the representation in terms of periodic field patterns. One can use wavelets with fractal spectrum $p^n\lambda_0$ of "wavelet lengths", where λ_0 is the fundamental wavelength. Fundamental wavelet should have p different patterns correspond to the p values of pinary digit as its structures. Periodicity guarantees the hologram like character enabling to pick n: th digit by studying the field pattern in scale p^n anywhere inside the field body.

4. Periodicity guarantees also that the intersections of p-adic and real space-time sheets can represent the values of pinary digits. For instance, wavelets could be such that in a given p-adic scale the number of rational points in the intersection of the real and p-adic space-time sheet equals to x_n. This would give in the limit of an infinite pinary expansion a set theoretic realization of any p-adic number in which each pinary digit x_n corresponds to infinite copies of a set with x_n elements and fractal cutoff due to the finite size of real space-time sheet would bring in a finite precision. Note however that p-adic space-time sheet necessarily has an infinite size and it is only real world realization of the representation which has finite accuracy.

5. A concrete realization for this object would be as an infinite tree with $x_n+1 \leq p$ branches in each node at level n (x_n+1 is needed in order to avoid the splitting tree at $x_n = 0$). In 2-adic case -1 would be represented by an infinite pinary tree. Negative powers of p correspond to the of the tree extending to a finite depth in ground.

11.7.3 Generalization Of The Notion Of Rig By Replacing Naturals With P-Adic Integers

Previous considerations do not relate directly to category theoretical problem of assigning complex numbers to objects. It however turns out that p-adic approach allows to generalize the proposal of [A19] by replacing natural numbers with p-adic integers in the definition of rig so that any algebraic complex number can define cardinality of an object of category allowing multiplication and sum and that these complex numbers can be replaced with p-adic numbers if they make sense as such so that previous arguments provide a concrete geometric representation of the cardinality. The road to the realization this simple generalization required a visit to the John Baez's Weekly Finds (Week 102) [A17].

The outcome was the realization that the notion of rig used to categorify the subset of algebraic numbers obtained as roots of polynomials with *natural number* valued coefficients generalizes trivially by replacing natural numbers by *p-adic integers*. As a consequence one obtains beautiful p-adicization of the generating function F(x) of structure as a function which converges p-adically for any rational $x = q$ for which it has prime p as a positive power divisor.

Effectively this generalization means the replacement of natural numbers as coefficients of the polynomial defining the rig with all rationals, also negative, and *all* complex algebraic numbers find a category theoretical representation as "cardinalities". These cardinalities have a dual interpretation as p-adic integers which in general correspond to infinite real numbers but are mappable to real numbers by canonical identification and have a geometric representation as fractals.

Mapping of objects to complex numbers and the notion of rig

The idea of rig approach is to categorify the notion of cardinality in such a way that one obtains a *subset* of algebraic complex numbers as cardinalities in the category-theoretical sense. One can assign to an object a polynomial with coefficients, which are *natural numbers* and the condition $Z = P(Z)$ says that $P(Z)$ acts as an isomorphism of the object. One can interpret the equation also in terms of complex numbers. Hence the object is mapped to a complex number Z defining a root of the polynomial interpreted as an ordinary polynomial: it does not matter which root is chosen. The complex number Z is interpreted as the "cardinality" of the object but I do not really understand the motivation for this. The deep further result is that also more general polynomial equations $R(|Z|) = Q(|Z|)$ satisfied by the generalized cardinality Z imply $R(Z) = Q(Z)$ as isomorphism.

I try to reproduce what looks the most essential in the explanation of John Baez and relate it to my own ideas but take this as my talk to myself and visit This Week's Finds [A17], one of the many classics of Baez, to learn of this fascinating idea.

1. Baez considers first the ways of putting a given structure to n-element set. The set of these structures is denoted by F_n and the number of them by $|F_n|$. The generating function $|F|(x) = \sum_n |F_n| x^n$ packs all this information to a single function.

   For instance, if the structure is binary tree, this function is given by $T(x) = \sum_n C_{n-1} x^n$, where C_{n-1} are Catalan numbers and n¿0 holds true. One can show that T satisfies the formula

   $$T = X + T^2 \ ,$$

   since any binary tree is either trivial or decomposes to a product of binary trees, where two trees emanate from the root. One can solve this second order polynomial equation and the power expansion gives the generating function.

2. The great insight is that one can also work directly with structures. For instance, by starting from the isomorphism $T = 1 + T^2$ applying to an object with cardinality 1 and substituting T^2 with $(1+T^2)^2$ repeatedly, one can deduce the amazing formula $T^7(1) = T(1)$ mentioned by Kea, and this identity can be interpreted as an isomorphism of binary trees.

3. This result can be generalized using the notion of rig category [A19]. In rig category one can add and multiply but negatives are not defined as in the case of ring. The lack of subtraction and division is still the problem and as I suggested in previous posting p-adic integers might resolve the problem.

   Whenever Z is object of a rig category, one can equip it with an isomorphism $Z = P(Z)$ where $P(Z)$ is polynomial with *natural numbers* as coefficients and one can assign to object "cardinality" as any root of the equation $Z = P(Z)$. Note that set with n elements corresponds to $P(|Z|) = n$. Thus subset of algebraic complex numbers receive formal identification as cardinalities of sets. Furthermore, if the cardinality satisfies another equation $Q(|Z|) = R(|Z|)$ such that neither polynomial is constant, then one can construct an isomorphism $Q(Z) = R(Z)$. Isomorphisms correspond to equations!

4. This is indeed nice that there is something which is not so beautiful as it could be: why should we restrict ourselves to *natural numbers* as coefficients of $P(Z)$? Could it be possible to replace them with integers to obtain *all complex algebraic numbers* as cardinalities? Could it be possible to replace natural numbers by p-adic integers?

p-Adic rigs and Golden Object as p-adic fractal

The notions of generating function and rig generalize to the p-adic context.

1. The generating function $F(x)$ defining isomorphism Z in the rig formulation converges p-adically for any p-adic number containing p as a factor so that the idea that all structures have p-adic counterparts is natural. In the real context the generating function typically diverges and must be defined by analytic continuation. Hence one might even argue that p-adic numbers are more natural in the description of structures assignable to finite sets than reals.

2. For rig one considers only polynomials $P(Z)$ (Z corresponds to the generating function F) with coefficients which are natural numbers. Any p-adic integer can be however interpreted as a non-negative integer: natural number if it is finite and "super-natural" number if it is infinite. Hence can generalize the notion of rig by replacing natural numbers by p-adic integers. The rig formalism would thus generalize to arbitrary polynomials with integer valued coefficients so that all complex algebraic numbers could appear as cardinalities of category theoretical objects. Even rational coefficients are allowed. This is highly natural number theoretically.

3. For instance, in the case of binary trees the solutions to the isomorphism condition $T = p+T^2$ giving $T = [1 \pm (1-4p)^{1/2}]/2$ and T would be complex number $[p \pm (1-4p)^{1/2}]/2$. $T(p)$ can be interpreted also as a p-adic number by performing power expansion of square root in case that the p-adic square root exists: this super-natural number can be mapped to a real number by the canonical identification and one obtains also the set theoretic representations of the category theoretical object $T(p)$ as a p-adic fractal. This interpretation of cardinality is much more natural than the purely formal interpretation as a complex number. This argument applies completely generally. The case $x = 1$ discussed by Baez gives $T = [1 \pm (-3)^{1/2}]/2$ allows p-adic representation if $-3 == p-3$ is square mod p. This is the case for $p = 7$ for instance.

4. John Baez [A17] poses also the question about the category theoretic realization of "Golden Object", his big dream. In this case one would have $Z = G = -1 + G^2 = P(Z)$. The polynomial on the right hand side does not conform with the notion of rig since -1 is not a natural number. If one allows p-adic rigs, $x = -1$ can be interpreted as a p-adic integer $(p-1)(1+p+...)$, positive and infinite and "super-natural", actually largest possible p-adic integer in a well defined sense.

   A further condition is that Golden Mean converges as a p-adic number: this requires that $\sqrt{5}$ must exist as a p-adic number: $(5 = 1+4)^{1/2}$ certainly converges as power series for $p = 2$ so that Golden Object exists 2-adically. By using [A12] of Euler, one finds that 5 is square mod p only if p is square mod 5. To decide whether given p is Golden it is enough to look whether p mod 5 is 1 or 4. For instance, $p = 11, 19, 29, 31$ (=M_5) are Golden. Mersennes M_k, $k = 3, 7, 127$ and Fermat primes are not Golden. One representation of Golden Object as p-adic fractal is the p-adic series expansion of $[1/2 \pm 5^{1/2}]/2$ representable geometrically as a binary tree such that there are $0 \leq x_n + 1 \leq p$ branches at each node at height n if n: th p-adic coefficient is x_n. The "cognitive" p-adic representation in terms of wavelet spectrum of classical fields is discussed in the previous posting.

5. It would be interesting to know how quantum dimensions of quantum groups assignable to Jones inclusions [K105, K35, K11] relate to the generalized cardinalities. The root of unity property of quantum phase ($q^{n+1} = q$) suggests $Q = Q^{n+1} = P(Q)$ as the relevant isomorphism. For Jones inclusions the cardinality $q = exp(i2\pi/n)$ would not be however equal to quantum dimension $D(n) = 4cos^2(\pi/n)$.

Is there a connection with infinite integers?

Infinite primes [K89] correspond to Fock states of a super-symmetric arithmetic quantum field theory and there is entire infinite hierarchy of them corresponding to repeated second quantization. Also infinite primes and rationals make sense. Besides free Fock states spectrum contains at each level also what might be identified as bound states. All these states can be mapped to polynomials. Since the roots of polynomials represent complex algebraic numbers and as they seem to characterize objects of categories, there are reasons to expect that infinite rationals might allow also interpretation in terms of say rig categories or their generalization. Also the possibility to identify space-time coordinate as isomorphism of a category might be highly interesting concerning the interpretation of quantum classical correspondence.

11.8 Gerbes And TGD

The notion of gerbes has gained much attention during last years in theoretical physics and there is an abundant gerbe-related literature in hep-th archives. Personally I learned about gerbes from the excellent article of Jouko Mickelson [A64] (Jouko was my opponent in PhD dissertation for more than two decades ago: so the time flows!).

I have already applied the notion of bundle gerbe in TGD framework in the construction of the Dirac determinant which I have proposed to define the Kähler function for the WCW (see [K106]). The insights provided by the general results about bundle gerbes discussed in [A64] led, not only to a justification for the hypothesis that Dirac determinant exists for the Kähler-Dirac action, but also to an elegant solution of the conceptual problems related to the construction of Dirac determinant in the presence of chiral symmetry. Furthermore, on basis of the special properties of the Kähler-Dirac operator there are good reasons to hope that the determinant exists even without zeta function regularization. The construction also leads to the conclusion that the space-time sheets serving as causal determinants must be geodesic sub-manifolds (presumably light like boundary components or "elementary particle horizons"). Quantum gravitational holography is realized since the exponent of Kähler function is expressible as a Dirac determinant determined by the local data at causal determinants and there would be no need to find absolute minima of Kähler action explicitly.

In the sequel the emergence of 2-gerbes at the space-time level in TGD framework is discussed and shown to lead to a geometric interpretation of the somewhat mysterious cocycle conditions for a wide class of gerbes generated via the $\wedge d$ products of connections associated with 0-gerbes. The resulting conjecture is that gerbes form a graded-commutative Grassmman algebra like structure generated by -1- and 0-gerbes. 2-gerbes provide also a beautiful topological characterization of space-time sheets as structures carrying Chern-Simons charges at boundary components and the 2-gerbe variant of Bohm-Aharonov effect occurs for perhaps the most interesting asymptotic solutions of field equations especially relevant for anyonics systems, quantum Hall effect, and living matter [K5].

11.8.1 What Gerbes Roughly Are?

Very roughly and differential geometrically, gerbes can be regarded as a generalization of connection. Instead of connection 1-form (0-gerbe) one considers a connection $n+1$-form defining n-gerbe. The curvature of n-gerbe is closed $n+2$-form and its integral defines an analog of magnetic charge. The notion of holonomy generalizes: instead of integrating n-gerbe connection over curve one integrates its connection form over n+1-dimensional closed surface and can transform it to the analog of magnetic flux.

There are some puzzling features associated with gerbes. Ordinary $U(1)$-bundles are defined in terms of open sets U_α with gauge transformations $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$ defined in $U_\alpha \cap U_\beta$ relating the connection forms in the patch U_β to that in patch U_α. The 3-cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad (11.8.1)$$

makes it possible to glue the patches to a bundle structure.

In the case of 1-gerbes the transition functions are replaced with the transition functions $g_{\alpha\beta\gamma} = g^{-1}_{\gamma\beta\alpha}$ defined in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and 3-cocycle must be replaced with 4-cocycle:

$$g_{\alpha\beta\gamma} g_{\beta\gamma\delta} g_{\gamma\delta\alpha} g_{\delta\alpha\beta} = 1 \ . \quad (11.8.2)$$

The generalizations of these conditions to n-gerbes is obvious.

In the case of 2-intersections one can build a bundle structure naturally but in the case of 3-intersections this is not possible. Hence the geometric interpretation of the higher gerbes is far from obvious. One possible interpretation of non-trivial 1-gerbe is as an obstruction for lifting projective bundles with fiber space CP_n to vector bundles with fiber space C^{n+1} [A64]. This involves the lifting of the holomorphic transition functions g_α defined in the projective linear group $PGL(n+1, C)$ to $GL(n+1, C)$. When the 3-cocycle condition for the lifted transition functions $\overline{g}_{\alpha\beta}$ fails it can be replaced with 4-cocycle and one obtains 1-gerbe.

11.8.2 How Do 2-Gerbes Emerge In TGD?

Gerbes seem to be interesting also from the point of view of TGD, and TGD approach allows a geometric interpretation of the cocycle conditions for a rather wide class of gerbes.

Recall that the Kähler form J of CP_2 defines a non-trivial magnetically charged and self-dual $U(1)$-connection A. The Chern-Simons form $\omega = A \wedge J = A \wedge dA$ having CP_2 Abelian instanton density $J \wedge J$ as its curvature form and can thus be regarded as a 3-connection form of a 2-gerbe. This 2-gerbe is induced by 0-gerbe.

The coordinate patches U_α are same as for $U(1)$ connection. In the transition between patches A and ω transform as

$$\begin{aligned} A &\rightarrow A + d\phi \ , \\ \omega &\rightarrow \omega + dA_2 \ , \\ A_2 &= \phi \wedge J \ . \end{aligned} \tag{11.8.3}$$

The transformation formula is induced by the transformation formula for $U(1)$ bundle. Somewhat mysteriously, there is no need to define anything in the intersections of U_α in the recent case.

The connection form of the 2-gerbe can be regarded as a second $\wedge d$ power of Kähler connection:

$$A_3 \equiv A \wedge dA \ . \tag{11.8.4}$$

The generalization of this observation allows to develop a different view about n-gerbes generated as $\wedge d$ products of 0-gerbes.

The hierarchy of gerbes generated by 0-gerbes

Consider a collection of $U(1)$ connections $A^{i)}$. They generate entire hierarchy of gerbe-connections via the $\wedge d$ product

$$A_3 = A^{1)} \wedge dA^{2)} \tag{11.8.5}$$

defining 2-gerbe having a closed curvature 4-form

$$F_4 = dA^{1)} \wedge dA^{2)} \ . \tag{11.8.6}$$

$\wedge d$ product is commutative apart from a gauge transformation and the curvature forms of $A^{1)} \wedge dA^{2)}$ and $A^{2)} \wedge dA^{1)}$ are the same.

Quite generally, the connections A_m of $m-1$ gerbe and A_n of $n-1$-gerbe define $m+n+1$ connection form and the closed curvature form of $m+n$-gerbe as

$$\begin{aligned} A_{m+n+1} &= A_m^{1)} \wedge dA_n^{2)} \ , \\ F_{m+n+2} &= dA_m^{1)} \wedge dA_n^{2)} \ . \end{aligned} \tag{11.8.7}$$

The sequence of gerbes extends up to $n = D-2$, where D is the dimension of the underlying manifold. These gerbes are not the most general ones since one starts from 0-gerbes. One can of course start from $n > 0$-gerbes too.

The generalization of the $\wedge d$ product to the non-Abelian situation is not obvious. The problems stem from the that the Lie-algebra valued connection forms $A^{1)}$ and $A^{2)}$ appearing in the covariant version $D = d + A$ do not commute.

11.8.3 How To Understand The Replacement Of 3-Cycles With N-Cycles?

If n-gerbes are generated from 0-gerbes it is possible to understand how the intersections of the open sets emerge. Consider the product of 0-gerbes as the simplest possible case. The crucial observation is that the coverings U_α for $A^{1)}$ and V_β for $A^{2)}$ need not be same (for CP_2 this was the case). One can form a new covering consisting of sets $U_\alpha \cap V_{\alpha_1}$. Just by increasing the index range one can replace V with U and one has covering by $U_\alpha \cap U_{\alpha_1} \equiv U_{\alpha\alpha_1}$.

The transition functions are defined in the intersections $U_{\alpha\alpha_1} \cap U_{\beta\beta_1} \equiv U_{\alpha\alpha_1\beta\beta_1}$ and cocycle conditions must be formulated using instead of intersections $U_{\alpha\beta\gamma}$ the intersections $U_{\alpha\alpha_1\beta\beta_1\gamma\gamma_1}$.

Hence the transition functions can be written as $g_{\alpha\alpha_1\beta\beta_1}$ and the 3-cocycle are replaced with 5-cocycle conditions since the minimal co-cycle corresponds to a sequence of 6 steps instead of 4:

$$U_{\alpha\alpha_1\beta\beta_1} \rightarrow U_{\alpha_1\beta\beta_1\gamma} \rightarrow U_{\beta\beta_1\gamma\gamma_1} \rightarrow U_{\beta_1\gamma\gamma_1\alpha} \rightarrow U_{\gamma\gamma_1\alpha\alpha_1} \quad .$$

The emergence of higher co-cycles is thus forced by the modification of the bundle covering necessary when gerbe is formed as a product of lower gerbes. The conjecture is that any even gerbe is expressible as a product of 0-gerbes.

An interesting application of the product structure is at the level of WCW ("world of classical worlds"). The Kähler form of WCW defines a connection 1-form and this generates infinite hierarchy of connection $2n+1$-forms associated with $2n$-gerbes.

11.8.4 Gerbes As Graded-Commutative Algebra: Can One Express All Gerbes As Products Of -1 And 0-Gerbes?

If one starts from, say 1-gerbes, the previous argument providing a geometric understanding of gerbes is not applicable as such. One might however hope that it is possible to represent the connection 2-form of any 1-gerbe as a $\wedge d$ product of a connection 0-form ϕ of "-1" -gerbe and connection 1-form A of 0-gerbe:

$$A_2 = \phi dA \equiv A \wedge d\phi \quad ,$$

with different coverings for ϕ and A. The interpretation as an obstruction for the modification of the underlying bundle structure is consistent with this interpretation.

The notion of -1-gerbe is not well-defined unless one can define the notion of -1 form precisely. The simplest possibility that 0-form transforms trivially in the change of patch is not consistent. One could identify contravariant n-tensors as $-n$-forms and d for them as divergence and d^2 as the antisymmetrized double divergence giving zero. ϕ would change in a gauge transformation by a divergence of a vector field. The integral of a divergence over closed M vanishes identically so that if the integral of ϕ over M is non-vanishing it corresponds to a non-trivial 0-connection. This interpretation of course requires the introduction of metric.

The requirement that the minimal intersections of the patches for 1-gerbes are of form $U_{\alpha\beta\gamma}$ would be achieved if the intersections patches can be restricted to the intersections $U_{\alpha\beta\gamma}$ defined by $U_\alpha \cap V_\gamma$ and $U_\beta \cap V_\gamma$ (instead of $U_\beta \cap V_\delta$), where the patches V_γ would be most naturally associated with -1-gerbe. It is not clear why one could make this restriction. The general conjecture is that any gerbe decomposes into a multiple $\wedge d$ product of -1 and 0-gerbes just like integers decompose into primes. The $\wedge d$ product of two odd gerbes is anti-commutative so that there is also an analogy with the decomposition of the physical state into fermions and bosons, and gerbes for a graded-commutative super-algebra generalizing the Grassmann algebra of manifold to a Grassmann algebra of gerbe structures for manifold.

11.8.5 The Physical Interpretation Of 2-Gerbes In TGD Framework

2-gerbes could provide some insight to how to characterize the topological structure of the many-sheeted space-time.

1. The cohomology group H^4 is obviously crucial in characterizing 2-gerbe. In TGD framework many-sheetedness means that different space-time sheets with induced metric having Minkowski signature are separated by elementary particle horizons which are light like 3-surfaces at which the induced metric becomes degenerate. Also the time orientation of the space-time sheet can change at these surfaces since the determinant of the induced metric vanishes.

   This justifies the term elementary particle horizon and also the idea that one should treat different space-time sheets as generating independent direct summands in the homology group of the space-time surface: as if the space-time sheets not connected by join along boundaries bonds were disjoint. Thus the homology group H^4 and 2-gerbes defining instanton numbers would become important topological characteristics of the many-sheeted space-time.

2. The asymptotic behavior of the general solutions of field equations can be classified by the dimension D of the CP_2 projection of the space-time sheet. For $D = 4$ the instanton density defining the curvature form of 2-gerbe is non-vanishing and instanton number defines a topological charge. Also the values of the Chern-Simons invariants associated with the boundary components of the space-time sheet define topological quantum numbers characterizing the space-time sheet and their sum equals to the instanton charge. CP_2 type extremals represent a basic example of this kind of situation. From the physical view point $D = 4$ asymptotic solutions correspond to what might be regarded chaotic phase for the flow lines of the Kähler magnetic field. Kähler current vanishes so that empty space Maxwell's equations are satisfied.

3. For $D = 3$ situation is more subtle when boundaries are present so that the higher-dimensional analog of Aharonov-Bohm effect becomes possible. In this case instanton density vanishes but the Chern-Simons invariants associated with the boundary components can be non-vanishing. Their sum obviously vanishes. The space-time sheet can be said to be a neutral C-S multipole. Separate space-time sheets can become connected by flux tubes in a quantum jump replacing a space-time surface with a new one. This means that the cohomology group H^4 as well as instanton charges and C-S charges of the system change.

Concerning the asymptotic dynamics of the Kähler magnetic field, $D = 3$ phase corresponds to an extremely complex but highly organized phase serving as an excellent candidate for the modelling of living matter. Both the TGD based description of anyons and quantum Hall effect and the model for topological quantum computation based on the braiding of magnetic flux tubes rely heavily on the properties $D = 3$ phase [K5].

The non-vanishing of the C-S form implies that the flow lines of the Kähler magnetic are highly entangled and have as an analog mixing hydrodynamical flow. In particular, one cannot define non-trivial order parameters, say phase factors, which would be constant along the lines. The interpretation in terms of broken super-conductivity suggests itself. Kähler current can be non-vanishing so that there is no counterpart for this phase at the level of Maxwell's equations.

11.9 Appendix: Category Theory And Construction Of S-Matrix

The construction of WCW geometry, spinor structure and of S-matrix involve difficult technical and conceptual problems and category theory might be of help here. As already found, the application of category theory to the construction of WCW geometry allows to understand how the arrow of psychological time emerges.

The construction of the S-matrix involves several difficult conceptual and technical problems in which category theory might help. The incoming states of the theory are what might be called free states and are constructed as products of the WCW spinor fields. One can effectively regard them as being defined in the Cartesian power of WCW divided by an appropriate permutation group. Interacting states in turn are defined in the WCW .

Cartesian power of WCW of 3-surfaces is however in geometrical sense more or less identical with WCW since the disjoint union of N 3-surfaces is itself a 3-surface in WCW . Actually it differs from WCW itself only in that the 3-surfaces of many particle state can intersect each other and if one allows this, one has paradoxical self-referential identification $CH = \overline{CH^2}/S_2 = ... = \overline{CH^N}/S_N ...$, where over-line signifies that intersecting 3-surfaces have been dropped from the product.

Note that arbitrarily small deformation can remove the intersections between 3-surfaces and four-dimensional general coordinate invariance allows always to use non-intersecting representatives. In case of the spinor structure of the Cartesian power this identification means that the tensor powers SCH^N of the WCW spinor structure are in some sense identical with the spinor structure SCH of the WCW . Certainly the oscillator operators of the tensor factors must be assumed to be mutually anti-commuting.

The identities $CH = \overline{CH^2}/S_2 = ..$ and corresponding identities $SCH = SCH^2 = ...$ for the space SCH of WCW spinor fields might imply very deep constraints on S-matrix. What comes into mind are counterparts for the Schwinger-Dyson equations of perturbative quantum field theory providing defining equations for the n-point functions of the theory [A58]. The isomorphism between SCH^2 and SCH is actually what is needed to calculate the S-matrix elements. Category theory might help to understand at a general level what these self-referential and somewhat paradoxical looking identities really imply and perhaps even develop TGD counterparts of Schwinger-Dyson equations.

There is also the issue of bound states. The interacting states contain also bound states not belonging to the space of free states and category theory might help also here. It would seem that the state space must be constructed by taking into account also the bound states as additional "free" states in the decomposition of states to product states.

A category naturally involved with the construction of the S-matrix (or U-matrix) is the space of preferred extremals of the Kähler action which might be called interacting category. The symplectic transformations acting as isometries of the configuration space geometry act naturally as the morphisms of this category. The group $Diff^4$ of general coordinate transformations in turn acts as gauge symmetries.

S-matrix relates free and interacting states and is induced by the classical long range interactions induced by the criticality of the preferred extremals in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes S-matrix elements are essentially Glebch-Gordan coefficients relating the states in the tensor power of the interacting super-symplectic representation with the interacting super-symplectic representation itself. More concretely, N-particle free states can be seen as WCW spinor fields in CH^N obtained as tensor products of ordinary WCW spinor fields. Free states correspond classically to the unions of space-time surfaces associated with the 3-surfaces representing incoming particles whereas interacting states correspond classically to the space-time surfaces associated with the unions of the 3-surfaces defining incoming states. These two states define what might be called free and interacting categories with canonical transformations acting as morphisms.

The classical interaction is represented by a functor $S : \overline{CH^N}/S_N \rightarrow CH$ mapping the classical free many particle states, that is objects of the product category defined by $\overline{CH^N}/S_N$ to the interacting category CH. This functor assigns to the union $\cup_i X^4(X_i^3)$ of the absolute minima $X^4(X_i^3)$ of Kähler action associated with the incoming, free states X_i^3 the preferred extreal $X^4(\cup X_i^3)$ associated with the union of 3-surfaces representing the outgoing interacting state. At quantum level this functor maps the state space SCH^N associated with $\cup_i X^4(X_i^3)$ to SCH in a unitary manner. An important constraint on S-matrix is that it acts effectively as a flow in zero modes correlating the quantum numbers in fiber degrees of freedom in one-to-one manner with the values of zero modes so that quantum jump $U\Psi_i \rightarrow \Psi_0...$ gives rise to a quantum measurement.

Chapter 12

Category Theory and Quantum TGD

12.1 Introduction

TGD predicts several hierarchical structures involving a lot of new physics. These structures look frustratingly complex and category theoretical thinking might help to build a bird's eye view about the situation. I have already earlier considered the question how category theory might be applied in TGD [K22, K19]. Besides the far from complete understanding of the basic mathematical structure of TGD also my own limited understanding of category theoretical ideas have been a serious limitation. During last years considerable progress in the understanding of quantum TGD proper has taken place and the recent formulation of TGD is in terms of light-like 3-surfaces, zero energy ontology and number theoretic braids [K103, ?]. There exist also rather detailed formulations for the fusion of p-adic and real physics and for the dark matter hierarchy. This motivates a fresh look to how category theory might help to understand quantum TGD.

The fusion rules for the symplectic variant of conformal field theory, whose existence is strongly suggested by quantum TGD, allow rather precise description using the basic notions of category theory and one can identify a series of finite-dimensional nilpotent algebras as discretized versions of field algebras defined by the fusion rules. These primitive fusion algebras can be used to construct more complex algebras by replacing any algebra element by a primitive fusion algebra. Trees with arbitrary numbers of branches in any node characterize the resulting collection of fusion algebras forming an operad. One can say that an exact solution of symplectic scalar field theory is obtained.

Conformal fields and symplectic scalar field can be combined to form symplecto-formal fields. The combination of symplectic operad and Feynman graph operad leads to a construction of Feynman diagrams in terms of n-point functions of conformal field theory. M-matrix elements with a finite measurement resolution are expressed in terms of a hierarchy of symplecto-conformal n-point functions such that the improvement of measurement resolution corresponds to an algebra homomorphism mapping conformal fields in given resolution to composite conformal fields in improved resolution. This expresses the idea that composites behave as independent conformal fields. Also other applications are briefly discussed.

Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K91]. The duality would allow to construct new preferred extremals of Kähler action.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L11]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L12].

12.2 S-Matrix As A Functor

John Baez's [A66] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

12.2.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

12.2.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

12.2.3 TSFT As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

12.2.4 The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A34] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of 2→2 reaction open string is pinched to a point at vertex. 1→2 vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

The new element would be quantum measurements performed separately for observables assignable to positive and negative energy states. These measurements would be characterized in terms of Jones inclusions. The state function reduction for the negative energy states could be interpreted as a detection of a particle reaction.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K61].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics. Note that also the presence of factor of type I coming from embedding space degrees of freedom forces thermal S-matrix.

Time-like entanglement coefficients as a square root of density matrix?

All quantum states do not correspond to thermal states and one can wonder what might be the most general identification of the quantum state in zero energy ontology. Density matrix formalism defines a very general formulation of quantum theory. Since the quantum states in zero energy ontology are analogous to operators, the idea that time-like entanglement coefficients in some sense define a square root of density matrix is rather natural. This would give the defining conditions

$$\begin{aligned} \rho^+ &= SS^\dagger \ , \rho^- = S^\dagger S \ , \\ Tr(\rho^\pm) &= 1 \ . \end{aligned} \tag{12.2.1}$$

$\rho^\pm$ would define density matrix for positive/negative energy states. In the case HFFs of type II_1 one obtains unitary S-matrix and also the analogs of pure quantum states are possible for factors of type I. The numbers $p^+_{m,n} = |S^2_{m,n}|/\rho^+_{m,m}$ and $p^-_{m,n} = |S^2_{n,m}|/\rho^-_{m,m}$ give the counterparts of the usual scattering probabilities.

A physically well-motivated hypothesis would be that S has expression $S = \sqrt{\rho}S_0$ such that S_0 is a universal unitary S-matrix, and $\sqrt{\rho}$ is square root of a state dependent density matrix. Note that in general S is not diagonalizable in the algebraic extension involved so that it is not possible to reduce the scattering to a mere phase change by a suitable choice of state basis.

What makes this kind of hypothesis aesthetically attractive is the unification of two fundamental matrices of quantum theory to single one. This unification is completely analogous to the combination of modulus squared and phase of complex number to a single complex number: complex valued Schrödinger amplitude is replaced with operator valued one.

S-matrix as a functor and the groupoid structure formed by S-matrices

In zero energy ontology S-matrix can be seen as a functor from the category of Feynman cobordisms to the category of operators. S-matrix can be identified as a "square root" of the positive energy density matrix $S = \rho_+^{1/2}S_0$, where S_0 is a unitary matrix and ρ_+ is the density matrix for positive energy part of the zero energy state. Obviously one has $SS^\dagger = \rho_+$. $S^\dagger S = \rho_-$ gives the density matrix for negative energy part of zero energy state. Clearly, S-matrix can be seen as matrix valued generalization of Schrödinger amplitude. Note that the "indices" of the S-matrices correspond to WCW spinor s (fermions and their bound states giving rise to gauge bosons and gravitons) and to WCW degrees of freedom. For hyper-finite factor of II_1 it is not strictly speaking possible to speak about indices since the matrix elements are traces of the S-matrix multiplied by projection operators to infinite-dimensional subspaces from right and left.

The functor property of S-matrices implies that they form a multiplicative structure analogous but not identical to groupoid [A4]. Recall that groupoid has associative product and there exist always right and left inverses and identity in the sense that ff^{-1} and $f^{-1}f$ are always defined but not identical and one has $fgg^{-1} = f$ and $f^{-1}fg = g$.

The reason for the groupoid like property is that S-matrix is a map between state spaces associated with initial and final sets of partonic surfaces and these state spaces are different so that inverse must be replaced with right and left inverse. The defining conditions for groupoid are replaced with more general ones. Also now associativity holds but the role of inverse is taken by hermitian conjugate. Thus one has the conditions $fgg^\dagger = f\rho_{g,+}$ and $f^\dagger fg = \rho_{f,-}g$, and the conditions $ff^\dagger = \rho_+$ and $f^\dagger f = \rho_-$ are satisfied. Here $\rho_\pm$ is density matrix associated with positive/negative energy parts of zero energy state. If the inverses of the density matrices exist, groupoid axioms hold true since $f_L^{-1} = f^\dagger\rho_{f,+}^{-1}$ satisfies $ff_L^{-1} = Id_+$ and $f_R^{-1} = \rho_{f,-}^{-1}f^\dagger$ satisfies $f_R^{-1}f = Id_-$.

There are good reasons to believe that also tensor product of its appropriate generalization to the analog of co-product makes sense with non-triviality characterizing the interaction between the systems of the tensor product. If so, the S-matrices would form very beautiful mathematical structure bringing in mind the corresponding structures for 2-tangles and N-tangles. Knowing how incredibly powerful the group like structures have been in physics one has good reasons to hope that groupoid like structure might help to deduce a lot of information about the quantum dynamics of TGD.

A word about nomenclature is in order. S has strong associations to unitarity and it might be appropriate to replace S with some other letter. The interpretation of S-matrix as a generalized Schrödinger amplitude would suggest Ψ-matrix. Since the interaction with Kea's M-theory blog at (see `http://tinyurl.com/yb3lsbjq` (M denotes Monad or Motif in this context) was led ot the realization of the connection with density matrix, also M-matrix might be considered. S-matrix as a functor from the category of Feynman cobordisms in turn suggests C or F. Or could just Matrix denoted by M in formulas be enough? Certainly it would inspire feeling of awe!

12.3 Further Ideas

The work of John Baez and students has inspired also the following ideas about the role of category theory in TGD.

12.3.1 Operads, Number Theoretical Braids, And Inclusions Of HFFs

The description of braids leads naturally to category theory and quantum groups when the braiding operation, which can be regarded as a functor, is not a mere permutation. Discreteness is a natural notion in the category theoretical context. To me the most natural manner to interpret discreteness is - not something emerging in Planck scale- but as a correlate for a finite measurement resolution and quantum measurement theory with finite measurement resolution leads naturally to number theoretical braids as fundamental discrete structures so that category theoretic approach becomes well-motivated. Discreteness is also implied by the number theoretic approach to quantum TGD from number theoretic associativity condition [L6] central also for category theoretical thinking as well as from the realization of number theoretical universality by the fusion of real and p-adic physics to single coherent whole.

Operads are formally single object multi-categories [A9, A76]. This object consist of an infinite sequence of sets of n-ary operations. These operations can be composed and the compositions are associative (operations themselves need not be associative) in the sense that the is natural isomorphism (symmetries) mapping differently bracketed compositions to each other. The coherence laws for operads formulate the effect of permutations and bracketing (association) as functors acting as natural isomorphisms. A simple manner to visualize the composition is as an addition of $n_1, ...n_k$ leaves to the leaves $1, ..., k$ of k-leaved tree.

An interesting example of operad is the braid operad formulating the combinatorics for a hierarchy of braids formed from braids by grouping subsets of braids having $n_1, ...n_k$ strands and defining the strands of a k-braid. In TGD framework this grouping can be identified in terms of the formation bound states of particles topologically condensed at larger space-time sheet and coherence laws allow to deduce information about scattering amplitudes. In conformal theories braided categories indeed allow to understand duality of stringy amplitudes in terms of associativity condition.

Planar operads [A28] define an especially interesting class of operads. The reason is that the inclusions of HFFs give rise to a special kind of planar operad [A10]. The object of this multi-category [A8] consists of planar k-tangles. Planar operads are accompanied by planar algebras. It will be found that planar operads allow a generalization which could provide a description for the combinatorics of the generalized Feynman diagrams and also rigorous formulation for how the arrow of time emerges in TGD framework and related heuristic ideas challenging the standard views.

12.3.2 Generalized Feynman Diagram As Category?

John Baez has proposed a category theoretical formulation of quantum field theory as a functor from the category of n-cobordisms to the category of Hilbert spaces [A66, A26]. The attempt to generalize this formulation looks well motivated in TGD framework because TGD can be regarded as almost topological quantum field theory in a well defined sense and braids appear as fundamental structures. It however seems that formulation as a functor from nCob to Hilb is not general enough.

In zero energy ontology events of ordinary ontology become quantum states with positive and negative energy parts of quantum states localizable to the upper and lower light-like boundaries of causal diamond (CD).

1. Generalized Feynman diagrams associated with a given CD involve quantum superposition of light-like 3-surfaces corresponding to given generalized Feynman diagram. These superpositions could be seen as categories with 3-D light-like surfaces containing braids as arrows and 2-D vertices as objects. Zero energy states would represent quantum superposition of categories (different topologies of generalized Feynman diagram) and M-matrix defined as Connes tensor product would define a functor from this category to the Hilbert space of zero energy states for given CD (tensor product defines quite generally a functor).

2. What is new from the point of view of physics that the sequences of generalized lines would define compositions of arrows and morphisms having identification in terms of braids which replicate in vertices. The possible interpretation of the replication is in terms of copying of information in classical sense so that even elementary particles would be information carrying and processing structures. This structure would be more general than the proposal of John

Baez that S-matrix corresponds to a function from the category of n-dimensional cobordisms to the category Hilb.

3. p-Adic length scale hypothesis follows if the temporal distance between the tips of CD measured as light-cone proper time comes as an octave of CP_2 time scale: $T = 2^n T_0$. This assumption implies that the p-adic length scale resolution interpreted in terms of a hierarchy of increasing measurement resolutions comes as octaves of time scale. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

This preliminary picture is of course not far complete since it applies only to single CD. There are several questions. Can one allow CDs within CDs and is every vertex of generalized Feynman diagram surrounded by this kind of CD. Can one form unions of CDs freely?

1. Since light-like 3-surfaces in 8-D embedding space have no intersections in the generic position, one could argue that the overlap must be allowed and makes possible the interaction of between zero energy states belonging to different CDs. This interaction would be something new and present also for sub-CDs of a given CD.

2. The simplest guess is that the unrestricted union of CDs defines the counterpart of tensor product at geometric level and that extended M-matrix is a functor from this category to the tensor product of zero energy state spaces. For non-overlapping CDs ordinary tensor product could be in question and for overlapping CDs tensor product would be non-trivial. One could interpret this M-matrix as an arrow between M-matrices of zero energy states at different CDs: the analog of natural transformation mapping two functors to each other. This hierarchy could be continued ad infinitum and would correspond to the hierarchy of n-categories.

This rough heuristics represents of course only one possibility among many since the notion of category is extremely general and the only limits are posed by the imagination of the mathematician. Also the view about zero energy states is still rather primitive.

12.4 Planar Operads, The Notion Of Finite Measurement Resolution, And Arrow Of Geometric Time

In the sequel the idea that planar operads or their appropriate generalization might allow to formulate generalized Feynman diagrammatics in zero energy ontology will be considered. Also a description of measurement resolution and arrow of geometric time in terms of operads is discussed.

12.4.1 Zeroth Order Heuristics About Zero Energy States

Consider now the existing heuristic picture about the zero energy states and coupling constant evolution provided by CDs.

1. The tentative description for the increase of the measurement resolution in terms CDs is that one inserts to the upper and/or lower light-like boundary of CD smaller CDs by gluing them along light-like radial ray from the tip of CD. It is also possible that the vertices of generalized Feynman diagrams belong inside smaller CD: s and it turns out that these CD: s must be allowed.

2. The considerations related to the arrow of geometric time suggest that there is asymmetry between upper and lower boundaries of CD. The minimum requirement is that the measurement resolution is better at upper light-like boundary.

3. In zero energy ontology communications to the direction of geometric past are possible and phase conjugate laser photons represent one example of this.

4. Second law of thermodynamics must be generalized in such a way that it holds with respect to subjective time identified as sequence of quantum jumps. The arrow of geometric time can however vary so that apparent breaking of second law is possible in shorter time scales at least. One must however understand why second law holds true in so good an approximation.

5. One must understand also why the contents of sensory experience is concentrated around a narrow time interval whereas the time scale of memories and anticipation are much longer. The proposed mechanism is that the resolution of conscious experience is higher at the upper boundary of CD. Since zero energy states correspond to light-like 3-surfaces, this could be a result of self-organization rather than a fundamental physical law.

   (a) CDs define the perceptive field for self. Selves are curious about the space-time sheets outside their perceptive field in the geometric future of the embedding space and perform quantum jumps tending to shift the superposition of the space-time sheets to the direction of geometric past (past defined as the direction of shift!). This creates the illusion that there is a time=snapshot front of consciousness moving to geometric future in fixed background space-time as an analog of train illusion.

   (b) The fact that news come from the upper boundary of CD implies that self concentrates its attention to this region and improves the resolutions of sensory experience and quantum measurement here. The sub-CD: s generated in this manner correspond to mental images with contents about this region. As a consequence, the contents of conscious experience, in particular sensory experience, tend to be about the region near the upper boundary.

   (c) This mechanism in principle allows the arrow of the geometric time to vary and depend on p-adic length scale and the level of dark matter hierarchy. The occurrence of phase transitions forcing the arrow of geometric time to be same everywhere are however plausible for the reason that the lower and upper boundaries of given CD must possess the same arrow of geometric time.

   (d) If this is the mechanism behind the arrow of time, planar operads can provide a description of the arrow of time but not its explanation.

This picture is certainly not general enough, can be wrong at the level of details, and at best relates to the whole like single particle wave mechanics to quantum field theory.

12.4.2 Planar Operads

The geometric definition of planar operads [A11, A9, A10, A28] without using the category theoretical jargon goes as follows.

1. There is an external disk and some internal disks and a collection of disjoint lines connecting disk boundaries.

2. To each disk one attaches a non-negative integer k, called the color of disk. The disk with color k has k points at each boundary with the labeling $1, 2, ...k$ running clockwise and starting from a distinguished marked point, decorated by "*". A more restrictive definition is that disk colors are correspond to even numbers so that there are $k = 2n$ points lines leaving the disk boundary boundary. The planar tangles with $k = 2n$ correspond to inclusions of HFFs.

3. Each curve is either closed (no common points with disk boundaries) or joins a marked point to another marked point. Each marked point is the end point of exactly one curve.

4. The picture is planar meaning that the curves cannot intersect and diks cannot overlap.

5. Disks differing by isotopies preserving *'s are equivalent.

Given a planar k-tangle-one of whose internal disks has color k_i- and a k_i-tangle S, one can define the tangle $T \circ_i S$ by isotoping S so that its boundary, together with the marked points and the *'s co-incides with that of D_i and after that erase the boundary of D_i. The collection of planar tangle together with the composition defined in this manner- is called the colored operad of planar tangles.

One can consider also generalizations of planar operads.

1. The composition law is not affected if the lines of operads branch outside the disks. Branching could be allowed even at the boundaries of the disks although this does not correspond to a generic situation. One might call these operads branched operads.
2. The composition law could be generalized to allow additional lines connecting the points at the boundary of the added disk so that each composition would bring in something genuinely new. Zero energy insertion could correspond to this kind of insertions.
3. TGD picture suggests also the replacement of lines with braids. In category theoretical terms this means that besides association one allows also permutations of the points at the boundaries of the disks.

The question is whether planar operads or their appropriate generalizations could allow a characterization of the generalized Feynman diagrams representing the combinatorics of zero energy states in zero energy ontology and whether also the emergence of arrow of time could be described (but probably not explained) in this framework.

12.4.3 Planar Operads And Zero Energy States

Are planar operads sufficiently powerful to code the vision about the geometric correlates for the increase of the measurement resolution and coupling constant evolution formulated in terms of CDs? Or perhaps more realistically, could one improve this formulation by assuming that zero energy states correspond to wave functions in the space of planar tangles or of appropriate modifications of them? It seems that the answer to the first question is almost affirmative.

1. Disks are analogous to the white regions of a map whose details are not visible in the measurement resolution used. Disks correspond to causal diamonds (CDs) in zero energy ontology. Physically the white regions relate to the vertices of the generalized Feynman diagrams and possibly also to the initial and final states (strictly speaking, the initial and final states correspond to the legs of generalized Feynman diagrams rather than their ends).
2. The composition of tangles means addition of previously unknown details to a given white region of the map and thus to an increase of the measurement resolution. This conforms with the interpretation of inclusions of HFFs as a characterization of finite measurement resolution and raises the hope that planar operads or their appropriate generalization could provide the proper language to describe coupling constant evolution and their perhaps even generalized Feynman diagrams.
3. For planar operad there is an asymmetry between the outer disk and inner disks. One might hope that this asymmetry could explain or at least allow to describe the arrow of time. This is not the case. If the disks correspond to causal diamonds (CDs) carrying positive *resp.* negative energy part of zero energy state at upper *resp.* lower light-cone boundary, the TGD counterpart of the planar tangle is CD containing smaller CD: s inside it. The smaller CD: s contain negative energy particles at their upper boundary and positive energy particles at their lower boundary. In the ideal resolution vertices represented 2-dimensional partonic at which light-like 3-surfaces meet become visible. There is no inherent asymmetry between positive and negative energies and no inherent arrow of geometric time at the fundamental level. It is however possible to model the arrow of time by the distribution of sub-CD: s. By previous arguments self-organization of selves can lead to zero energy states for which the measurement resolution is better near the upper boundary of the CD.
4. If the lines carry fermion or anti-fermion number, the number of lines entering to a given CD must be even as in the case of planar operads as the following argument shows.

(a) In TGD framework elementary fermions correspond to single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat.

(b) Elementary bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets of opposite time orientation and modellable as a piece of CP_2 type extremal. Each boson therefore corresponds to 2 lines within CP_2 radius.

(c) As a consequence the total number of lines associated with given CD is even and the generalized Feynman diagrams can correspond to a planar algebra associated with an inclusion of HFFs.

5. This picture does not yet describe zero energy insertions.

   (a) The addition of zero energy insertions corresponds intuitively to the allowance of new lines inside the smaller CD: s not coming from the exterior. The addition of lines connecting points at the boundary of disk is possible without losing the basic geometric composition of operads. In particular one does not lose the possibility to color the added tangle using two colors (colors correspond to two groups G and H which characterize an inclusion of HFFs [A28]).

   (b) There is however a problem. One cannot remove the boundaries of sub-CD after the composition of CDs since this would give lines beginning from and ending to the interior of disk and they are invisible only in the original resolution. Physically this is of course what one wants but the inclusion of planar tangles is expected to fail in its original form, and one must generalize the composition of tangles to that of CD: s so that the boundaries of sub-CD: s are not thrown away in the process.

   (c) It is easy to see that zero energy insertions are inconsistent with the composition of planar tangles. In the inclusion defining the composition of tangles both sub-tangle and tangle induce a color to a given segment of the inner disk. If these colors are identical, one can forget the presence of the boundary of the added tangle. When zero energy insertions are allowed, situation changes as is easy to see by adding a line connecting points in a segment of given color at the boundary of the included tangle. There exists no consistent coloring of the resulting structure by using only two colors. Coloring is however possible using four colors, which by four-color theorem is the minimum number of colors needed for a coloring of planar map: this however requires that the color can change as one moves through the boundary of the included disk - this is in accordance with the physical picture.

   (d) Physical intuition suggests that zero energy insertion as an improvement of measurement resolution maps to an improved color resolution and that the composition of tangles generalizes by requiring that the included disk is colored by using new nuances of the original colors. The role of groups in the definition of inclusions of HFFs is consistent with idea that G and H describe color resolution in the sense that the colors obtained by their action cannot be resolved. If so, the improved resolution means that G and H are replaced by their subgroups $G_1 \subset G$ and $H_1 \subset H$. Since the elements of a subgroup have interpretation as elements of group, there are good hopes that by representing the inclusion of tangles as inclusion of groups, one can generalize the composition of tangles.

6. Also CD: s glued along light-like ray to the upper and lower boundaries of CD are possible in principle and -according the original proposal- correspond to zero energy insertions according. These CD: s might be associated with the phase transitions changing the value of $\hbar$ leading to different pages of the book like structure defined by the generalized embedding space.

7. p-Adic length scale hypothesis is realized if the hierarchy of CDs corresponds to a hierarchy of temporal distances between tips of CDs given as $a = T_n = 2^{-n}T_0$ using light-cone proper time.

8. How this description relates to braiding? Each line corresponds to an orbit of a partonic boundary component and in principle one must allow internal states containing arbitrarily high fermion and anti-fermion numbers. Thus the lines decompose into braids and one must allow also braids of braids hierarchy so that each line corresponds to a braid operad in improved resolution.

12.4.4 Relationship To Ordinary Feynman Diagrammatics

The proposed description is not equivalent with the description based on ordinary Feynman diagrams.

1. In standard physics framework the resolution scale at the level of vertices of Feynman diagrams is something which one is forced to pose in practical calculations but cannot pose at will as opposed to the measurement resolution. Light-like 3-surfaces can be however regarded only locally orbits of partonic 2-surfaces since generalized conformal invariance is true only in 3-D patches of the light-like 3-surface. This means that light-like 3-surfaces are in principle the fundamental objects so that zero energy states can be regarded only locally as a time evolutions. Therefore measurement resolution can be applied also to the distances between vertices of generalized Feynman diagrams and calculational resolution corresponds to physical resolution. Also the resolution can be better towards upper boundary of CD so that the arrow of geometric time can be understood. This is a definite prediction which can in principle kill the proposed scenario.

2. A further counter argument is that generalized Feynman diagrams are identified as light-like 3-surfaces for which Kähler function defined by a preferred extremal of Kähler action is maximum. Therefore one cannot pose any ad hoc rules on the positions of the vertices. One can of course insist that maximum of Kähler function with the constraint posed by $T_n = 2^n T_0$ (or $T_p = p^n T_0$) hierarchy is in question.

It would be too optimistic to believe that the details of the proposal are correct. However, if the proposal is on correct track, zero energy states could be seen as wave functions in the operad of generalized tangles (zero energy insertions and braiding) as far as combinatorics is involved and the coherence rules for these operads would give strong constraints on the zero energy state and fix the general structure of coupling constant evolution.

12.5 Category Theory And Symplectic QFT

Besides the counterpart of the ordinary Kac-Moody invariance quantum TGD possesses so called super-symplectic conformal invariance. This symmetry leads to the proposal that a symplectic variant of conformal field theory should exist. The n-point functions of this theory defined in S^2 should be expressible in terms of symplectic areas of triangles assignable to a set of n-points and satisfy the duality rules of conformal field theories guaranteeing associativity. The crucial prediction is that symplectic n-point functions vanish whenever two arguments co-incide. This provides a mechanism guaranteeing the finiteness of quantum TGD implied by very general arguments relying on non-locality of the theory at the level of 3-D surfaces.

The classical picture suggests that the generators of the fusion algebra formed by fields at different point of S^2 have this point as a continuous index. Finite quantum measurement resolution and category theoretic thinking in turn suggest that only the points of S^2 corresponding the strands of number theoretic braids are involved. It turns out that the category theoretic option works and leads to an explicit hierarchy of fusion algebras forming a good candidate for so called little disk operad whereas the first option has difficulties.

12.5.1 Fusion Rules

Symplectic fusion rules are non-local and express the product of fields at two points s_k an s_l of S^2 as an integral over fields at point s_r, where integral can be taken over entire S^2 or possibly also over a 1-D curve which is symplectic invariant in some sense. Also discretized version of fusion rules makes sense and is expected serve as a correlate for finite measurement resolution.

By using the fusion rules one can reduce n-point functions to convolutions of 3-point functions involving a sequence of triangles such that two subsequent triangles have one vertex in common. For instance, 4-point function reduces to an expression in which one integrates over the positions of the common vertex of two triangles whose other vertices have fixed. For n-point functions one has n-3 freely varying intermediate points in the representation in terms of 3-point functions.

The application of fusion rules assigns to a line segment connecting the two points s_k and s_l a triangle spanned by s_k, s_l and s_r. This triangle should be symplectic invariant in some sense and its symplectic area A_{klm} would define the basic variable in terms of which the fusion rule could be expressed as $C_{klm} = f(A_{klm})$, where f is fixed by some constraints. Note that A_{klm} has also interpretations as solid angle and magnetic flux.

12.5.2 What Conditions Could Fix The Symplectic Triangles?

The basic question is how to identify the symplectic triangles. The basic criterion is certainly the symplectic invariance: if one has found N-D symplectic algebra, symplectic transformations of S^2 must provide a new one. This is guaranteed if the areas of the symplectic triangles remain invariant under symplectic transformations. The questions are how to realize this condition and whether it might be replaced with a weaker one. There are two approaches to the problem.

Physics inspired approach

In the first approach inspired by classical physics symplectic invariance for the edges is interpreted in the sense that they correspond to the orbits of a charged particle in a magnetic field defined by the Kähler form. Symplectic transformation induces only a $U(1)$ gauge transformation and leaves the orbit of the charged particle invariant if the vertices are not affected since symplectic transformations are not allowed to act on the orbit directly in this approach. The general functional form of the structure constants C_{klm} as a function $f(A_{klm})$ of the symplectic area should guarantee fusion rules.

If the action of the symplectic transformations does not affect the areas of the symplectic triangles, the construction is invariant under general symplectic transformations. In the case of uncharged particle this is not the case since the edges are pieces of geodesics: in this case however fusion algebra however trivializes so that one cannot conclude anything. In the case of charged particle one might hope that the area remains invariant under general symplectic transformations whose action is induced from the action on vertices. The equations of motion for a charged particle involve a Kähler metric determined by the symplectic structure and one might hope that this is enough to achieve this miracle. If this is not the case - as it might well be - one might hope that although the areas of the triangles are not preserved, the triangles are mapped to each other in such a way that the fusion algebra rules remain intact with a proper choice of the function $f(A_{klm})$. One could also consider the possibility that the function $f(A_{klm})$ is dictated from the condition that the it remains invariant under symplectic transformations. It however turns that this approach does not work as such.

Category theoretical approach

The second realization is guided by the basic idea of category theoretic thinking: the properties of an object are determined its relationships to other objects. Rather than postulating that the symplectic triangle is something which depends solely on the three points involved via some geometric notion like that of geodesic line of orbit of charged particle in magnetic field, one assumes that the symplectic triangle reflects the properties of the fusion algebra, that is the relations of the symplectic triangle to other symplectic triangles. Thus one must assign to each triplet (s_1, s_2, s_3) of points of S^2 a triangle just from the requirement that braided associativity holds true for the fusion algebra.

All symplectic transformations leaving the N points fixed and thus generated by Hamiltonians vanishing at these points would give new gauge equivalent realizations of the fusion algebra and deform the edges of the symplectic triangles without affecting their area. One could even say that symplectic triangulation defines a new kind geometric structure in S^2. The quantum fluctuating degrees of freedom are parameterized by the symplectic group of $S^2 \times CP_2$ in TGD so that symplectic the geometric representation of the triangulation changes but its inherent properties remain invariant.

The elegant feature of category theoretical approach is that one can in principle construct the fusion algebra without any reference to its geometric realization just from the braided associativity and nilpotency conditions and after that search for the geometric realizations. Fusion algebra has also a hierarchy of discrete variants in which the integral over intermediate points in fusion is replaced by a sum over a fixed discrete set of points and this variant is what finite measurement resolution implies. In this case it is relatively easy to see if the geometric realization of a given abstract fusion algebra is possible.

The notion of number theoretical braid

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [K57].

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of CDs and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining X^2 make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [K57]. The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ...$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions anti-fermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW , the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of M-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [K57] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic pointsandthen perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or anti-fermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

Symplectic triangulations and braids

The identification of the edges of the symplectic triangulation as the end points of the braid is favored by conceptual economy. The nodes of the symplectic triangulation would naturally correspond to the points in the intersection of the braid with the light-like boundaries of CD carrying fermion or anti-fermion number. The number of these points could be arbitrarily large in the generic case but in the intersection of real and p-adic worlds these points correspond to subset of algebraic points belonging to the algebraic extension of rationals associated with the definition of partonic 2-surfaces so that the sum of fermion and anti-fermion numbers would be bounded above. The presence of fermions in the nodes would be the physical prerequisite for measuring the phase factors defined by the magnetic fluxes. This could be understood in terms of gauge invariance forcing to assign to a pair of points of triangulation the non-integrable phase factor defined by the Kähler gauge potential.

The remaining problem is how uniquely the edges of the triangulation can be determined.

1. The allowance of all possible choices for edges would bring in an infinite number of degrees of freedom. These curves would be analogous to freely vibrating strings. This option is not attractive. One should be able to pose conditions on edges and whatever the manner to specify the edges might be, it must make sense also in the intersection of real and p-adic worlds. In this case the total phase factor must be a root of unity in the algebraic extension of rationals involved and this poses quantization rules analogous to those for magnetic flux. The strongest condition is that the edges are such that the non-integrable phase factor is a root of unity for each edge. It will be found that similar quantization is implied also by the associativity conditions and this justifies the interpretation of phase factors defining the fusion algebra in terms of the Kähler magnetic fluxes. This would pose strong constraints on the choice of edges but would not fix completely the phase factors, and it seems that one must allow all possible triangulations consistent with this condition and the associativity conditions so that physical state is a quantum superposition over all possible symplectic triangulations characterized by the fusion algebras.

2. In the real context one would have an infinite hierarchy of symplectic triangulations and fusion algebras satisfying the associativity conditions with the number of edges equal to the total number N of fermions and anti-fermions. Encouragingly, this hierarchy corresponds also to a hierarchy of $\mathcal{N} = N$ SUSY algebras [?] (large values of $\mathcal{N}$ are not a catastrophe in TGD framework since the physical content of SUSY symmetry is not the same as that in the standard approach). In the intersection of real and p-adic worlds the value of $\mathcal{N}$ would be bounded by the total number of algebraic points. Hence the notion of finite measurement resolution, cutoff in $\mathcal{N}$ and bound on the total fermion number would make physics very simple in the intersection of real and p-adic worlds.

Two kinds of symplectic triangulations are possible since one can use the symplectic forms associated with CP_2 and $r_M = constant$ sphere S^2 of light-cone boundary. For a given collection of nodes the choices of edges could be different for these two kinds of triangulations. Physical state would be proportional to the product of the phase factors assigned to these triangulations.

12.5.3 Associativity Conditions And Braiding

The generalized fusion rules follow from the associativity condition for n-point functions modulo phase factor if one requires that the factor assignable to n-point function has interpretation as n-point function. Without this condition associativity would be trivially satisfied by using a product of various bracketing structures for the n fields appearing in the n-point function. In conformal field theories the phase factor defining the associator is expressible in terms of the phase factor associated with permutations represented as braidings and the same is expected to be true also now.

1. Already in the case of 4-point function there are three different choices corresponding to the 4 possibilities to connect the fixed points s_k and the varying point s_r by lines. The options are (1-2, 3-4), (1-3, 2-4), and (1-4, 2-3) and graphically they correspond to s-, t-, and u-channels in string diagrams satisfying also this kind of fusion rules. The basic condition would be that same amplitude results irrespective of the choice made. The duality conditions guarantee associativity in the formation of the n-point amplitudes without any further assumptions. The reason is that the writing explicitly the expression for a particular bracketing of n-point function always leads to some bracketing of one particular 4-point function and if duality conditions hold true, the associativity holds true in general. To be precise, in quantum theory associativity must hold true only in projective sense, that is only modulo a phase factor.

2. This framework encourages category theoretic approach. Besides different bracketing there are different permutations of the vertices of the triangle. These permutations can induce a phase factor to the amplitude so that braid group representations are enough. If one has representation for the basic braiding operation as a quantum phase $q = exp(i2\pi/N)$, the phase factors relating different bracketings reduce to a product of these phase factors since $(AB)C$ is obtained from $A(BC)$ by a cyclic permutation involving to permutations represented as a braiding. Yang-Baxter equations express the reduction of associator to braidings. In the general category theoretical setting associators and braidings correspond to natural isomorphisms leaving category theoretical structure invariant.

3. By combining the duality rules with the condition that 4-point amplitude vanishes, when any two points co-incide, one obtains from $s_k = s_l$ and $s_m = s_n$ the condition stating that the sum (or integral in possibly existing continuum version) of $U^2(A_{klm})|f|^2(x_{kmr})$ over the third point s_r vanishes. This requires that the phase factor U is non-trivial so that Q must be non-vanishing if one accepts the identification of the phase factor as Bohm-Aharonov phase.

4. Braiding operation gives naturally rise to a quantum phase. A good guess is that braiding operation maps triangle to its complement since only in this manner orientation is preserved so that area is A_{klm} is mapped to $A_{klm} - 4\pi$. If the f is proportional to the exponent $exp(-A_{klm}Q)$, braiding operation induces a complex phase factor $q = exp(-i4\pi Q)$.

5. For half-integer values of Q the algebra is commutative. For $Q = M/N$, where M and N have no common factors, only braided commutativity holds true for $N \geq 3$ just as for quantum groups characterizing also Jones inclusions of HFFs. For $N = 4$ anti-commutativity and associativity hold true. Charge fractionization would correspond to non-trivial braiding and presumably to non-standard values of Planck constant and coverings of M^4 or CP_2 depending on whether S^2 corresponds to a sphere of light-cone boundary or homologically trivial geodesic sphere of CP_2.

12.5.4 Finite-Dimensional Version Of The Fusion Algebra

Algebraic discretization due to a finite measurement resolution is an essential part of quantum TGD. In this kind of situation the symplectic fields would be defined in a discrete set of N points of S^2: natural candidates are subsets of points of p-adic variants of S^2. Rational variant of S^2 has as its points points for which trigonometric functions of θ and ϕ have rational values and there exists an entire hierarchy of algebraic extensions. The interpretation for the resulting breaking of the rotational symmetry would be a geometric correlate for the choice of quantization axes in quantum measurement and the book like structure of the embedding space would be direct correlate for this symmetry breaking. This approach gives strong support for the category theory inspired philosophy in which the symplectic triangles are dictated by fusion rules.

General observations about the finite-dimensional fusion algebra

1. In this kind of situation one has an algebraic structure with a finite number of field values with integration over intermediate points in fusion rules replaced with a sum. The most natural option is that the sum is over all points involved. Associativity conditions reduce in this case to conditions for a finite set of structure constants vanishing when two indices are identical. The number $M(N)$ of non-vanishing structure constants is obtained from the recursion formula $M(N) = (N-1)M(N-1)+(N-2)M(N-2)+...+3M(3) = NM(N-1)$, $M(3) = 1$ given $M(4) = 4$, $M(5) = 20$, $M(6) = 120$, ... With a proper choice of the set of points associativity might be achieved. The structure constants are necessarily complex so that also the complex conjugate of the algebra makes sense.

2. These algebras resemble nilpotent algebras ($x^n = 0$ for some n) and Grassmann algebras ($x^2 = 0$ always) in the sense that also the products of the generating elements satisfy $x^2 = 0$ as one can find by using duality conditions on the square of a product $x = yz$ of two generating elements. Also the products of more than N generating elements necessary vanish by braided commutativity so that nilpotency holds true. The interpretation in terms of measurement resolution is that partonic states and vertices can involve at most N fermions

in this measurement resolution. Elements anti-commute for $q=-1$ and commute for $q=1$ and the possibility to express the product of two generating elements as a sum of generating elements distinguishes these algebras from Grassman algebras. For $q=-1$ these algebras resemble Lie-algebras with the difference that associativity holds true in this particular case.

3. I have not been able to find whether this kind of hierarchy of algebras corresponds to some well-known algebraic structure with commutativity and associativity possibly replaced with their braided counterparts. Certainly these algebras would be category theoretical generalization of ordinary algebras for which commutativity and associativity hold true in strict sense.

4. One could forget the representation of structure constants in terms of triangles and think these algebras as abstract algebras. The defining equations are $x_i^2=0$ for generators plus braided commutativity and associativity. Probably there exists solutions to these conditions. One can also hope that one can construct braided algebras from commutative and associative algebras allowing matrix representations. Note that the solution the conditions allow scalings of form $C_{klm}\rightarrow \lambda_k\lambda_l\lambda_m C_{klm}$ as symmetries.

Formulation and explicit solution of duality conditions in terms of inner product

Duality conditions can be formulated in terms of an inner product in the function space associated with N points and this allows to find explicit solutions to the conditions.

1. The idea is to interpret the structure constants C_{klm} as wave functions C_{kl} in a discrete space consisting of N points with the standard inner product

$$\langle C_{kl},C_{mn}\rangle = \sum_r C_{klr}\overline{C}_{mnr} \quad . \tag{12.5.1}$$

2. The associativity conditions for a trivial braiding can be written in terms of the inner product as

$$\langle C_{kl},\overline{C}_{mn}\rangle = \langle C_{km},\overline{C}_{ln}\rangle = \langle C_{kn},\overline{C}_{ml}\rangle \quad . \tag{12.5.2}$$

3. Irrespective of whether the braiding is trivial or not, one obtains for $k=m$ the orthogonality conditions

$$\langle C_{kl},\overline{C}_{kn}\rangle = 0 \quad . \tag{12.5.3}$$

For each k one has basis of $N-1$ wave functions labeled by $l\neq k$, and the conditions state that the elements of basis and conjugate basis are orthogonal so that conjugate basis is the dual of the basis. The condition that complex conjugation maps basis to a dual basis is very special and is expected to determine the structure constants highly uniquely.

4. One can also find explicit solutions to the conditions. The most obvious trial is based on orthogonality of function basis of circle providing representation for Z_{N-2} and is following:

$$C_{klm} = E_{klm}\times exp(i\phi_k+\phi_l+\phi_m) \ , \quad \phi_m = \frac{n(m)2\pi}{N-2} \quad . \tag{12.5.4}$$

Here E_{klm} is non-vanishing only if the indices have different values. The ansatz reduces the conditions to the form

$$\sum_r E_{klr}E_{mnr}exp(i2\phi_r) = \sum_r E_{kmr}E_{lnr}exp(i2\phi_r) = \sum_r E_{knr}E_{mlr}exp(i2\phi_r) \quad . \tag{12.5.5}$$

In the case of braiding one can allow overall phase factors. Orthogonality conditions reduce to

$$\sum_r E_{klr}E_{knr}exp(i2\phi_r) = 0 \quad . \tag{12.5.6}$$

If the integers $n(m)$, $m \neq k, l$ span the range $(0, N-3)$ ortogonality conditions are satisfied if one has $E_{klr} = 1$ when the indices are different. This guarantees also duality conditions since the inner products involving k, l, m, n reduce to the same expression

$$\sum_{r \neq k,l,m,n} exp(i2\phi_r) \quad . \tag{12.5.7}$$

5. For a more general choice of phases the coefficients E_{klm} must have values differing from unity and it is not clear whether the duality conditions can be satisfied in this case.

Do fusion algebras form little disk operad?

The improvement of measurement resolution means that one adds further points to an existing set of points defining a discrete fusion algebra so that a small disk surrounding a point is replaced with a little disk containing several points. Hence the hierarchy of fusion algebras might be regarded as a realization of a little disk operad [A7] and there would be a hierarchy of homomorphisms of fusion algebras induced by the fusion. The inclusion homomorphism should map the algebra elements of the added points to the algebra element at the center of the little disk.

A more precise prescription goes as follows.

1. The replacement of a point with a collection of points in the little disk around it replaces the original algebra element ϕ_{k_0} by a number of new algebra elements ϕ_K besides already existing elements ϕ_k and brings in new structure constants C_{KLM}, C_{KLk} for $k \neq k_0$, and C_{Klm}.

2. The notion of improved measurement resolution allows to conclude

$$C_{KLk} = 0 \quad , \quad k \neq k_0 \quad , \quad C_{Klm} = C_{k_0lm} \quad . \tag{12.5.8}$$

3. In the homomorphism of new algebra to the original one the new algebra elements and their products should be mapped as follows:

$$\begin{array}{l} \phi_K \rightarrow \phi_{k_0} \quad , \\ \phi_K\phi_L \rightarrow \phi_{k_0}^2 = 0 \quad , \quad \phi_K\phi_l \rightarrow \phi_{k_0}\phi_l \quad . \end{array} \tag{12.5.9}$$

Expressing the products in terms of structure constants gives the conditions

$$\sum_M C_{KLM} = 0 \quad , \quad \sum_r C_{Klr} = \sum_r C_{k_0lr} = 0 \quad . \tag{12.5.10}$$

The general ansatz for the structure constants based on roots of unity guarantees that the conditions hold true.

4. Note that the resulting algebra is more general than that given by the basic ansatz since the improvement of the measurement resolution at a given point can correspond to different value of N as that for the original algebra given by the basic ansatz. Therefore the original ansatz gives only the basic building bricks of more general fusion algebras. By repeated local improvements of the measurement resolution one obtains an infinite hierarchy of algebras labeled by trees in which each improvement of measurement resolution means the splitting of the branch with arbitrary number N of branches. The number of improvements of the measurement resolution defining the height of the tree is one invariant of these algebras. The fusion algebra operad has a fractal structure since each point can be replaced by any fusion algebra.

How to construct geometric representation of the discrete fusion algebra?

Assuming that solutions to the fusion conditions are found, one could try to find whether they allow geometric representations. Here the category theoretical philosophy shows its power.

1. Geometric representations for C_{klm} would result as functions $f(A_{klm})$ of the symplectic area for the symplectic triangles assignable to a set of N points of S^2.

2. If the symplectic triangles can be chosen freely apart from the area constraint as the category theoretic philosophy implies, it should be relatively easy to check whether the fusion conditions can be satisfied. The phases of C_{klm} dictate the areas A_{klm} rather uniquely if one uses Bohm-Aharonov ansatz for a fixed the value of Q. The selection of the points s_k would be rather free for phases near unity since the area of the symplectic triangle associated with a given triplet of points can be made arbitrarily small. Only for the phases far from unity the points s_k cannot be too close to each other unless Q is very large. The freedom to chose the points rather freely conforms with the general view about the finite measurement resolution as the origin of discretization.

3. The remaining conditions are on the moduli $|f(A_{klm})|$. In the discrete situation it is rather easy to satisfy the conditions just by fixing the values of f for the particular triangles involved: $|f(A_{klm})| = |C_{klm}|$. For the exact solution to the fusion conditions $|f(A_{klm})| = 1$ holds true.

4. Constraints on the functional form of $|f(A_{klm})|$ for a fixed value of Q can be deduced from the correlation between the modulus and phase of C_{klm} without any reference to geometric representations. For the exact solution of fusion conditions there is no correlation.

5. If the phase of C_{klm} has A_{klm} as its argument, the decomposition of the phase factor to a sum of phase factors means that the A_{klm} is sum of contributions labeled by the vertices. Also the symplectic area defined as a magnetic flux over the triangle is expressible as sum of the quantities $\int A_\mu dx^\mu$ associated with the edges of the triangle. These fluxes should correspond to the fluxes assigned to the vertices deduced from the phase factors of $\Psi(s_k)$. The fact that vertices are ordered suggest that the phase of $\Psi(s_j)$ fixes the value of $\int A_\mu dx^\mu$ for an edge of the triangle starting from s_k and ending to the next vertex in the ordering. One must find edges giving a closed triangle and this should be possible. The option for which edges correspond to geodesics or to solutions of equations of motion for a charged particle in magnetic field is not flexible enough to achieve this purpose.

6. The quantization of the phase angles as multiples of $2\pi/(N-2)$ in the case of N-dimensional fusion algebra has a beautiful geometric correlate as a quantization of symplecto-magnetic fluxes identifiable as symplectic areas of triangles defining solid angles as multiples of $2\pi/(N-2)$. The generalization of the fusion algebra to p-adic case exists if one allows algebraic extensions containing the phase factors involved. This requires the allowance of phase factors $exp(i2\pi/p)$, p a prime dividing $N-2$. Only the exponents $exp(i\int A_\mu dx^\mu) = exp(in2\pi/(N-2))$ exist p-adically. The p-adic counterpart of the curve defining the edge of triangle exists if the curve can be defined purely algebraically (say as a solution of polynomial equations with rational coefficients) so that p-adic variant of the curve satisfies same equations.

Does a generalization to the continuous case exist?

The idea that a continuous fusion algebra could result as a limit of its discrete version does not seem plausible. The reason is that the spatial variation of the phase of the structure constants increases as the spatial resolution increases so that the phases $exp(i\phi(s)$ cannot be continuous at continuum limit. Also the condition $E_{klm} = 1$ for $k \neq l \neq m$ satisfied by the explicit solutions to fusion rules fails to have direct generalization to continuum case.

To see whether the continuous variant of fusion algebra can exist, one can consider an approximate generalization of the explicit construction for the discrete version of the fusion algebra by the effective replacement of points s_k with small disks which are not allowed to intersect. This would mean that the counterpart $E(s_k, s_l, s_m)$ vanishes whenever the distance between two arguments is below a cutoff a small radius d. Puncturing corresponds physically to the cutoff implied by the finite measurement resolution.

1. The ansatz for C_{klm} is obtained by a direct generalization of the finite-dimensional ansatz:

$$C_{klm} = \kappa_{s_k,s_l,s_m}\Psi(s_k)\Psi(s_l)\Psi(s_m) \ . \tag{12.5.11}$$

   where κ_{s_k,s_l,s_m} vanishes whenever the distance of any two arguments is below the cutoff distance and is otherwise equal to 1.

2. Orthogonality conditions read as

$$\Psi(s_k)\Psi(s_l)\int \kappa_{s_k,s_l,s_r}\kappa_{s_k,s_n,s_r}\Psi^2(s_m)d\mu(s_r) = \Psi(s_k)\Psi(s_l)\int_{S^2(s_k,sl,s_n)}\Psi^2(s_r)d\mu(s_r) = 0 \tag{12.5.12}$$

   The resulting condition reads as

$$\int_{S^2(s_k,sl,s_n)}\Psi^2(s_r)d\mu(s_r) = 0 \tag{12.5.13}$$

   This condition holds true for any pair s_k, s_l and this might lead to difficulties.

3. The general duality conditions are formally satisfied since the expression for all fusion products reduces to

$$\begin{aligned} & \Psi(s_k)\Psi(s_l)\Psi(s_m)\Psi(s_n)X \ , \\ X &= \int_{S^2}\kappa_{s_k,s_l,s_m,s_n}\Psi(s_r)d\mu(s_r) \\ &= \int_{S^2(s_k,s_l,s_m,s_n)}\Psi(s_m)d\mu(s_r) \\ &= -\int_{D^2(s_i)}\Psi^2(s_r)d\mu(s_r) \ , \quad i = k,l,s,m \ . \end{aligned} \tag{12.5.14}$$

   These conditions state that the integral of Ψ^2 any disk of fixed radius d is same: this result follows also from the orthogonality condition. This condition might be difficult to satisfy exactly and the notion of finite measurement resolution might be needed. For instance, it might be necessary to restrict the consideration to a discrete lattice of points which would lead back to a discretized version of algebra. Thus it seems that the continuum generalization of the proposed solution to fusion rules does not work.

12.6 Could Operads Allow The Formulation Of The Generalized Feynman Rules?

The previous discussion of symplectic fusion rules leaves open many questions.

1. How to combine symplectic and conformal fields to what might be called symplecto-conformal fields?

2. The previous discussion applies only in super-symplectic degrees of freedom and the question is how to generalize the discussion to super Kac-Moody degrees of freedom. One must of course also try to identify more precisely what Kac-Moody degrees of freedom are!

3. How four-momentum and its conservation in the limits of measurement resolution enters this picture? Could the phase factors assocaited with the symplectic triangulation carry information about four-momentum?

4. At least two operads related to measurement resolution seem to be present: the operads formed by the symplecto-conformal fields and by generalized Feynman diagrams. For generalized Feynman diagrams causal diamond (CD) is the basic object whereas disks of S^2 are the basic objects in the case of symplecto-conformal QFT with a finite measurement resolution. Could these two different views about finite measurement resolution be more or less equivalent and could one understand this equivalence at the level of details.

5. Is it possible to formulate generalized Feynman diagrammatics and improved measurement resolution algebraically?

12.6.1 How To Combine Conformal Fields With Symplectic Fields?

The conformal fields of conformal field theory should be somehow combined with symplectic scalar field to form what might be called symplecto-conformal fields.

1. The simplest thing to do is to multiply ordinary conformal fields by a symplectic scalar field so that the fields would be restricted to a discrete set of points for a given realization of N-dimensional fusion algebra. The products of these symplecto-conformal fields at different points would define a finite-dimensional algebra and the products of these fields at same point could be assumed to vanish.

2. There is a continuum of geometric realizations of the symplectic fusion algebra since the edges of symplectic triangles can be selected rather freely. The integrations over the coordinates z_k (most naturally the complex coordinate of S^2 transforming linearly under rotations around quantization axes of angular momentum) restricted to the circle appearing in the definition of simplest stringy amplitudes would thus correspond to the integration over various geometric realizations of a given N-dimensional symplectic algebra.

Fusion algebra realizes the notion of finite measurement resolution. One implication is that all n-point functions vanish for $n > N$. Second implication could be that the points appearing in the geometric realizations of N-dimensional symplectic fusion algebra have some minimal distance. This would imply a cutoff to the multiple integrals over complex coordinates z_k varying along circle giving the analogs of stringy amplitudes. This cutoff is not absolutely necessary since the integrals defining stringy amplitudes are well-defined despite the singular behavior of n-point functions. One can also ask whether it is wise to introduce a cutoff that is not necessary and whether fusion algebra provides only a justification for the $1 + i\epsilon$ prescription to avoid poles used to obtain finite integrals.

The fixed values for the quantities $\int A_\mu dx^\mu$ along the edges of the symplectic triangles could indeed pose a lower limit on the distance between the vertices of symplectic triangles. Whether this occurs depends on what one precisely means with symplectic triangle.

1. The conformally invariant condition that the angles between the edges at vertices are smaller than π for triangle and larger than π for its conjugate is not enough to exclude loopy edges

and one would obtain ordinary stringy amplitudes multiplied by the symplectic phase factors. The outcome would be an integral over arguments $z_1, z_2, ..z_n$ for standard stringy n-point amplitude multiplied by a symplectic phase factor which is piecewise constant in the integration domain.

2. The condition that the points at different edges of the symplectic triangle can be connected by a geodesic segment belonging to the interior of the triangle is much stronger and would induce a length scale cutoff since loops cannot be used to create large enough value of $\int A_\mu dx^\mu$ for a given side of triangle. Symplectic invariance would be obtained for small enough symplectic transformations. How to realize this cutoff at the level of calculations is not clear. One could argue that this problem need not have any nice solution and since finite measurement resolution requires only finite calculational resolution, the approximation allowing loopy edges is acceptable.

3. The restriction of the edges of the symplectic triangle within a tubular neighborhood of a geodesic -more more generally an orbit of charged particle - with thickness determined by the length scale resolution in S^2 would also introduce the length scale cutoff with symplectic invariance within measurement resolution.

Symplecto-conformal should form an operad. This means that the improvement of measurement resolution should correspond also to an algebra homomorphism in which super-symplectic symplecto-conformal fields in the original resolution are mapped by algebra homomorphism into fields which contain sum over products of conformal fields at different points: for the symplectic parts of field the products reduces always to a sum over the values of field. For instance, if the field at point s is mapped to an average of fields at points s_k, nilpotency condition $x^2 = 0$ is satisfied.

12.6.2 Symplecto-Conformal Fields In Super-Kac-Moody Sector

The picture described above applies only in super-symplectic degrees of freedom. The vertices of generalized Feynman diagrams are absent from the description and CP_2 Kähler form induced to space-time surface which is absolutely essential part of quantum TGD is nowhere visible in the treatment.

How should one bring in Super Kac-Moody (SKM) algebra? The condition that the basic building bricks are same for the treatment of these degrees of freedom is a valuable guideline.

What does SKM algebra mean?

The first thing to consider is what SKM could mean. The recent view is that symplectic algebra corresponds to symplectic transformations for the boundary of causal diamond CD which looks locally like $\delta M^4_\pm \times CP_2$. For this super-algebra fermionic generators would be contractions of covariantly constant right-handed neutrino with the second quantized induced spinor field to which the contraction$j^k_A \Gamma_k$ of symplectic vector field with gamma matrices acts. For SKM algebra corresponding generators would be similar contractions of other spinor modes but involving only Killing vectors fields that is symplectic isometries.

The recent view about quantum criticality strongly suggests that the conformal symmetries act as almost gauge symmetries producing from a given preferred extremal new ones with same action and conserved charges. "Almost" means that sub-algebra of conformal algebra annihilates the physical states. The subalgebras in question form a fractal hierarchy and are isomorphic with the conformal algebra itself. They contain generators for which the conformal weight is multiple of integer n characterizing also the value of Planck constant given by $h_{eff} = n \times h$. n defines the number of conformal equivalence classes of space-time surfaces connecting fixed 3-surfaces at the boundaries of CD (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg` or **Fig. ??** in the appendix of this book).

Since Kähler action reduces for the general ansatz for the preferred extremals to 3-D Chern-Simons terms, the action of the conformal symmetries reduces also to the 3-D space-like surfaces where it is trivial by definition and to non-trivial action to the light-like 3-surfaces at which the signature of the induced metric changes: I have used to call this surface partonic orbits.

It must be however observed that one can consider also the possibility that SKM algebra corresponds to the isometries of $\delta M^4 \pm \times CP_2$ continued to the space-time surface by field equations. These isometries are conformal transformations of S^2 ($\delta M^4_\pm = S^2 \times R_+$) with conformal scaling compensated by the local scaling of the light-like radial coordinate r_M to guarantee that the metric reducing to that for S^2 apart from conformal scaling factor R^2_M remains invariant. If this is the case the SKM contains also other than symplectic isometries.

Attempt to formulate symplectic triangulation for SKM algebra

The analog of symplectic triangulation for SKM algebra obviously requires that SKM algebra corresponds to symplectic isometries rather than including all $\delta M^4_\pm = S^2 \times R_+$ isometries in one-one correspondence with conformal transformations of S^2.

1. In the transition from super-symplectic to SKM degrees of freedom the light-cone boundary is naturally replaced with the light-like 3-surface X^3 representing the light-like random orbit of parton and serving as the basic dynamical object of quantum TGD. The sphere S^2 of light-cone boundary is in turn replaced with a partonic 2-surface X^2. This suggests how to proceed.

2. In the case of SKM algebra the symplectic fusion algebra is represented geometrically as points of partonic 2-surface X^2 by replacing the symplectic form of S^2 with the induced CP_2 symplectic form at the partonic 2-surface and defining $U(1)$ gauge field. This gives similar hierarchy of symplecto-conformal fields as in the super-symplectic case. This also realizes the crucial aspects of the classical dynamics defined by Kähler action. In particular, for vacuum 2-surfaces symplectic fusion algebra trivializes since Kähler magnetic fluxes vanish identically and 2-surfaces near vacua require a large value of N for the dimension of the fusion algebra since the available Kähler magnetic fluxes are small.

3. In super-symplectic case the projection along light-like ray allows to map the points at the light-cone boundaries of CD to points of same sphere S^2. In the case of light-like 3-surfaces light-like geodesics representing braid strands allow to map the points of the partonic two-surfaces at the future and past light-cone boundaries to the partonic 2-surface representing the vertex. The earlier proposal was that the ends of strands meet at the partonic 2-surface so that braids would replicate at vertices. The properties of symplectic fields would however force identical vanishing of the vertices if this were the case. There is actually no reason to assume this condition and with this assumption vertices involving total number N of incoming and outgoing strands correspond to symplecto-conformal N-point function as is indeed natural. Also now Kähler magnetic flux induces cutoff distance.

4. SKM braids reside at light-like 3-surfaces representing lines of generalized Feynman diagrams. If super-symplectic braids are needed at all, they must be assigned to the two light-like boundaries of CD meeting each other at the sphere S^2 at which future and past directed light-cones meet.

12.6.3 The Treatment Of Four-Momentum

Four-momentum enjoys a special role in super-symplectic and SKM representations in that it does not correspond to a quantum number assignable to the generators of these algebras. It would be nice if the somewhat mysterious phase factors associated with the representation of the symplectic algebra could code for the four-momentum - or rather the analogs of plane waves representing eigenstates of four-momentum at the points associated with the geometric representation of the symplectic fusion algebra.

Also the vision about TGD as almost topological QFT suggests that the symplectic degrees of freedom added to the conformal degrees of freedom defining alone a mere topological QFt somehow code for the physical degrees of freedom should and also four-momentum. If so, the symplectic triangulation might somehow code for four-momentum.

The representation of longitudinal momentum in terms of phase factors

The following argument suggests that S^2 and X^2 triangulations cannot naturally represent four-momentum and that one needs extension into 3-D light-like triangulation to achieve this.

1. The basic question is whether four-momentum could be coded in terms of non-integrable phase factors appearing in the representations of the symplectic fusion algebras.
2. In the symplectic case S^2 triangulation suggests itself as a representation of angular momentum only: it would be kind of spin network. In the SKM case X^2 would suggest representation of color hyper charge and isospin in terms of phases since CP_2 symmetries act non-trivially in Chern-Simons action. Does this mean that symplectic and SKM triangulations must be extended so that they are 3-D and defined for space-like 3-surface and the light-like orbit of partonic 2-surface. This would give additional phase factors assignable to presumably light-like edges. Ligh-like momentum would be natural and the recent twistorial formulation of quantum TGD indeed assigns massless momenta to fermion lines.

Suppose that one has 3-D light-like triangulation eith at δCD or at light-like orbits of partonic 2-surface. Consider first coding of four-momentum assuming only Kähler gauge potential of CP_2 possibly having M^4 part which is pure gauge.

1. Four different phase factors are needed if all components of four-momentum are to be coded. Both number theoretical vision about quantum TGD and the realization of the hierarchy of Planck constants assign to each point of space-time surface the same plane $M^2 \subset M^4$ having as the plane of non-physical polarizations. This condition allows to assign to a given light-like partonic 3-surface unique extremal of Kähler action defining the Kähler function as the value of Kähler action.

   Also p-adic mass calculations support the view that the physical states correspond to eigen states for the components of longitudinal momentum only (also the parton model for hadrons assumes this). This encourages to think that only M^2 part of four-momentum is coded by the phase factors. Transversal momentum squared would be a well defined quantum number and determined from mass shell conditions for the representations of super-symplectic (or equivalently SKM) conformal algebra much like in string model.
2. The phase factors associated with the 3-D symplectic fusion algebra in $S^2 \times R_+$ mean a deviation from conformal n-point functions, and the innocent question is whether these phase factors could be identified as plane-wave phase factors in S^2 could be associated with the transversal part of the four-momentum so that the n-point functions would be strictly analogous with stringy amplitudes. Alternative, and perhaps more natural, interpretation is in terms of spin and angular momentum.
3. Suppose one allows a gauge transformation of Kähler gauge potential inducing a pure gauge M^4 component to the Kähler gauge potential expressible as scalar function of M^4 coordianates. This kind of term might allow to achieve the vanishing of $j^\alpha A_\alpha$ term of at least its integral over space-time surface in Kähler action implying reduction of Kähler action to Chern-Simons terms if weak form of electric magnetic duality holds true. The scalar function can be interpreted as integral of a position dependent momentum along curve defined by $S^2 \times R_+$ triangulation and gives hopes of coding four-momentum in terms of Kähler gauge potential.

   In fact, the identification of the phase factors $exp(i \int A_\mu dx^\mu/\hbar)$ along a path as phase factors $exp(ip_{L,k}\Delta m^k)$ defined by the ends of the path and associated with the longitudinal part of four-momentum would correspond to an integral form of covariant constancy condition $\frac{dx^\mu}{ds}(\partial_\mu - iA_\mu)\Psi = 0$ along the edge of the symplectic triangle of more general path.
4. For the SKM triangulation associated with the light-like orbit X_l^3 of partonic 2-surface analogous phase factor would come from the integral along the (most naturally) light-like curve defining braid strand associated with the point in question. A geometric representation for the two projections of the four-momentum would thus result in SKM degrees of freedom and apart from the non-uniqueness related to the multiples of a 2π the components of M^2

momentum could be deduced from the phase factors. If one is satisfied with the projection of momentum in M^2, this is enough.

5. Neither of these phase factors is able to code all components of four-momentum. One might however hope that together they could give enough information to deduce the four-momentum if it is assumed to correspond to the rest system.

6. The phase factors assignable to the symplectic triangles in S^2 and X^2 have nothing to do with momentum. Because the space-like phase factor $exp(iS_z\Delta\phi/\hbar)$ associated with the edge of the symplectic triangle is completely analogous to that for momentum, one can argue that the symplectic triangulation could define a kind of spin network utilized in discretized approaches to quantum gravity. The interpretation raises the question about the interpretation of the quantum numbers assignable to the Lorentz invariant phase factors defined by the CP_2 Kähler gauge potential.

The quantum numbers associated with phase factors for CP_2 parts of Kähler gauge potentials

Suppose that it is possible to assign two independent and different phase factors to the same geometric representation, in other words have two independent symplectic fields with the same geometric representation. The product of two symplectic fields indeed makes sense and satisfies the defining conditions. One can define prime symplectic algebras and decompose symplectic algebras to prime factors. Since one can allow permutations of elements in the products it becomes possible to detect the presence of product structure experimentally by detecting different combinations for products of phases caused by permutations realized as different combinations of quantum numbers assigned with the factors. The geometric representation for the product of n symplectic fields would correspond to the assignment of n edges to any pair of points. The question concerns the interpretation of the phase factors assignable to the CP_2 parts of Kähler gauge potentials of S^2 and CP_2 Kähler form.

1. The natural interpretation for the two additional phase factors would be in terms of color quantum numbers. Color hyper charge and isospin are mathematically completely analogous to the components of four-momentum so that a possible identification of the phase factors is as a representation of these quantum numbers. The representation of plane waves as phase factors $exp(ip_k\Delta m^k/\hbar)$ generalizes to the representation $exp(iQ_A\Delta\Phi^A/\hbar)$, where Φ_A are the angle variables conjugate to the Hamiltonians representing color hyper charge and isospin. This representation depends on end points only so that the crucial symplectic invariance with respect to the symplectic transformations respecting the end points of the edge is not lost ($U(1)$ gauge transformation is induced by the scalar j^kA_k, where j^k is the symplectic vector field in question).

2. One must be cautious with the interpretation of the phase factors as a representation for color hyper charge and isospin since a breaking of color gauge symmetry would result since the phase factors associated with different values of color isospin and hypercharge would be different and could not correspond to same edge of symplectic triangle. This is questionable since color group itself represents symplectic transformations. The construction of CP_2 as a coset space $SU(3)/U(2)$ identifies $U(2)$ as the holonomy group of spinor connection having interpretation as electro-weak group. Therefore also the interpretation of the phase factors in terms of em charge and weak charge can be considered. In TGD framework electro-weak gauge potential indeed suffer a non-trivial gauge transformation under color rotations so that the correlation between electro-weak quantum numbers and non-integrable phase factors in Cartan algebra of the color group could make sense. Electro-weak symmetry breaking would have a geometric correlate in the sense that different values of weak isospin cannot correspond to paths with same values of phase angles $\Delta\Phi^A$ between end points.

3. If the phase factors associated with the M^4 and CP_2 are assumed to be identical, the existence of geometric representation is guaranteed. This however gives constraints between rest mass, spin, and color (or electro-weak) quantum numbers.

Some general comments

Some further comments about phase factors are in order.

1. By number theoretical universality the plane wave factors associated with four-momentum must have values coming as roots of unity (just as for a particle in box consisting of discrete lattice of points). At light-like boundary the quantization conditions reduce to the condition that the value of light-like coordinate is rational of form m/N, if N: th roots of unity are allowed.

2. In accordance with the finite measurement resolution of four-momentum, four-momentum conservation is replaced by a weaker condition stating that the products of phase factors representing incoming and outgoing four-momenta are identical. This means that positive and negative energy states at opposite boundaries of CD would correspond to complex conjugate representations of the fusion algebra. In particular, the product of phase factors in the decomposition of the conformal field to a product of conformal fields should correspond to the original field value. This would give constraints on the trees physically possible in the operad formed by the fusion algebras. Quite generally, the phases expressible as products of phases $exp(in\pi/p)$, where $p \leq N$ is prime must be allowed in a given resolution and this suggests that the hierarchy of p-adic primes is involved. At the limit of very large N exact momentum conservation should emerge.

3. Super-conformal invariance gives rise to mass shell conditions relating longitudinal and transversal momentum squared. The massivation of massless particles by Higgs mechanism and p-adic thermodynamics pose additional constraints to these phase factors.

12.6.4 What Does The Improvement Of Measurement Resolution Really Mean?

To proceed one must give a more precise meaning for the notion of measurement resolution. Two different views about the improvement of measurement resolution emerge. The first one relies on the replacement of braid strands with braids applies in SKM degrees of freedom and the homomorphism maps symplectic fields into their products. The homomorphism based on the averaging of symplectic fields over added points consistent with the extension of fusion algebra described in previous section is very natural in super-symplectic degrees of freedom. The directions of these two algebra homomorphisms are different. The question is whether both can be involved with both super-symplectic and SKM case. Since the end points of SKM braid strands correspond to both super-symplectic and SKM degrees of freedom, it seems that division of labor is the only reasonable option.

1. Quantum classical correspondence requires that measurement resolution has a purely geometric meaning. A purely geometric manner to interpret the increase of the measurement resolution is as a replacement of a braid strand with a braid in the improved resolution. If one assigns the phase factor assigned with the fusion algebra element with four-momentum, the conservation of the phase factor in the associated homomorphism is a natural constraint. The mapping of a fusion algebra element (strand) to a product of fusion algebra elements (braid) allows to realize this condition. Similar mapping of field value to a product of field values should hold true for conformal parts of the fields. There exists a large number equivalent geometric representations for a given symplectic field value so that one obtains automatically an averaging in conformal degrees of freedom. This interpretation for the improvement of measurement resolution looks especially natural for SKM degrees of freedom for which braids emerge naturally.

2. One can also consider the replacement of symplecto-conformal field with an average over the points becoming visible in the improved resolution. In super-symplectic degrees of freedom this looks especially natural since the assignment of a braid with light-cone boundary is not so natural as with light-like 3-surface. This map does not conserve the phase factor but this could be interpreted as reflecting the fact that the values of the light-like radial coordinate

are different for points involved. The proposed extension of the symplectic algebra proposed in the previous section conforms with this interpretation.

3. In the super-symplectic case the improvement of measurement resolution means improvement of angular resolution at sphere S^2. In SKM sector it means improved resolution for the position at partonic 2-surface. This generalizes also to the 3-D symplectic triangulations. For SKM algebra the increase of the measurement resolution related to the braiding takes place inside light-like 3-surface. This operation corresponds naturally to an addition of sub-CD inside which braid strands are replaced with braids. This is like looking with a microscope a particular part of line of generalized Feynman graph inside CD and corresponds to a genuine physical process inside parton. In super-symplectic case the replacement of a braid strand with braid (at light-cone boundary) is induced by the replacement of the projection of a point of a partonic 2-surface to S^2 with a a collection of points coming from several partonic 2-surfaces. This replaces the point s of S^2 associated with CD with a set of points s_k of S^2 associated with sub-CD. Note that the solid angle spanned by these points can be rather larger so that zoom-up is in question.

4. The improved measurement resolution means that a point of S^2 (X^2) at boundary of CD is replaced with a point set of S^2 (X^2) assignable to sub-CD. The task is to map the point set to a small disk around the point. Light-like geodesics along light-like X^3 defines this map naturally in both cases. In super-symplectic case this map means scaling down of the solid angle spanned by the points of S^2 associated with sub-CD.

12.6.5 How Do The Operads Formed By Generalized Feynman Diagrams And Symplecto-Conformal Fields Relate?

The discussion above leads to following overall view about the situation. The basic operation for both symplectic and Feynman graph operads corresponds to an improvement of measurement resolution. In the case of planar disk operad this means to a replacement of a white region of a map with smaller white regions. In the case of Feynman graph operad this means better space-time resolution leading to a replacement of generalized Feynman graph with a new one containing new sub-CD bringing new vertices into daylight. For braid operad the basic operation means looking a braid strand with a microscope so that it can resolve into a braid: braid becomes a braid of braids. The latter two views are equivalent if sub-CD contains the braid of braids.

The disks D^2 of the planar disk operad has natural counterparts in both super-symplectic and SKM sector.

1. For the geometric representations of the symplectic algebra the image points vary in continuous regions of S^2 (X^2) since the symplectic area of the symplectic triangle is a highly flexible constraint. Posing the condition that any point at the edges of symplectic triangle can be connected to any another edge excludes symplectic triangles with loopy sides so that constraint becomes non-trivial. In fact, since two different elements of the symplectic algebra cannot correspond to the same point for a given geometric representation, each element must correspond to a connected region of S^2 (X^2). This allows a huge number of representations related by the symplectic transformations S^2 in super-symplectic case and by the symplectic transformations of CP_2 in SKM case. In the case of planar disk operad different representations are related by isotopies of plane.

   This decomposition to disjoint regions naturally correspond to the decomposition of the disk to disjoint regions in the case of planar disk operad and Feynman graph operad (allowing zero energy insertions). Perhaps one might say that N-dimensional elementary symplectic algebra defines an N-coloring of S^2 (S^2) which is however not the same thing as the 2-coloring possible for the planar operad. TGD based view about Higgs mechanism leads to a decomposition of partonic 2-surface X^2 (its light-like orbit X^3) into conformal patches. Since also these decompositions correspond to effective discretizations of X^2 (X^3), these two decompositions would naturally correspond to each other.

2. In SKM sector disk D^2 of the planar disk operad is replaced with the partonic 2-surface X^2 and since measurement resolution is a local notion, the topology of X^2 does not matter. The

improvement of measurement resolution corresponds to the replacement of braid strand with braid and homomorphism is to the direction of improved spatial resolution.

3. In super-symplectic case D^2 is replaced with the sphere S^2 of light-cone boundary. The improvement of measurement resolution corresponds to introducing points near the original point and the homomorphism maps field to its average. For the operad of generalized Feynman diagrams CD defined by future and past directed light-cones is the basic object. Given CD can be indeed mapped to sphere S^2 in a natural manner. The light-like boundaries of CDs are metrically spheres S^2. The points of light-cone boundaries can be projected to any sphere at light-cone boundary. Since the symplectic area of the sphere corresponds to solid angle, the choice of the representative for S^2 does not matter. The sphere defined by the intersection of future and past light-cones of CD however provides a natural identification of points associated with positive and negative energy parts of the state as points of the same sphere. The points of S^2 appearing in n-point function are replaced by point sets in a small disks around the n points.

4. In both super-symplectic and SKM sectors light-like geodesic along X^3 mediate the analog of the map gluing smaller disk to a hole of a disk in the case of planar disk operad defining the decomposition of planar tangles. In super-symplectic sector the set of points at the sphere corresponding to a sub-CD is mapped by SKM braid to the larger CD and for a typical braid corresponds to a larger angular span at sub-CD. This corresponds to the gluing of D^2 along its boundaries to a hole in D^2 in disk operad. A scaling transformation allowed by the conformal invariance is in question. This scaling can have a non-trivial effect if the conformal fields have anomalous scaling dimensions.

5. Homomorphisms between the algebraic structures assignable to the basic structures of the operad (say tangles in the case of planar tangle operad) are an essential part of the power of the operad. These homomorphisms associated with super-symplectic and SKM sector code for two views about improvement of measurement resolution and might lead to a highly unique construction of M-matrix elements.

The operad picture gives good hopes of understanding how M-matrices corresponding to a hierarchy of measurement resolutions can be constructed using only discrete data.

1. In this process the n-point function defining M-matrix element is replaced with a superposition of n-point functions for which the number of points is larger: $n \rightarrow \sum_{k=1,...,m} n_k$. The numbers n_k vary in the superposition. The points are also obtained by downwards scaling from those of smaller S^2. Similar scaling accompanies the composition of tangles in the case of planar disk operad. Algebra homomorphism property gives constraints on the compositeness and should govern to a high degree how the improved measurement resolution affects the amplitude. In the lowest order approximation the M-matrix element is just an n-point function for conformal fields of positive and negative energy parts of the state at this sphere and one would obtain ordinary stringy amplitude in this approximation.

2. Zero energy ontology means also that each addition in principle brings in a new zero energy insertion as the resolution is improved. Zero energy insertions describe actual physical processes in shorter scales in principle affecting the outcome of the experiment in longer time scales. Since zero energy states can interact with positive (negative) energy particles, zero energy insertions are not completely analogous to vacuum bubbles and cannot be neglected. In an idealized experiment these zero energy states can be assumed to be absent. The homomorphism property must hold true also in the presence of the zero energy insertions. Note that the Feynman graph operad reduces to planar disk operad in absence of zero energy insertions.

12.7 Possible Other Applications Of Category Theory

It is not difficult to imagine also other applications of category theory in TGD framework.

12.7.1 Categorification And Finite Measurement Resolution

I read a very stimulating article by John Baez with title "Categorification" (see http://tinyurl.com/ych6a8oa) [A59] about the basic ideas behind a process called categorification. The process starts from sets consisting of elements. In the following I describe the basic ideas and propose how categorification could be applied to realize the notion of finite measurement resolution in TGD framework.

What categorification is?

In categorification sets are replaced with categories and elements of sets are replaced with objects. Equations between elements are replaced with isomorphisms between objects: the right and left hand sides of equations are not the same thing but only related by an isomorphism so that they are not tautologies anymore. Functions between sets are replaced with functors between categories taking objects to objects and morphisms to morphisms and respecting the composition of morphisms. Equations between functions are replaced with natural isomorphisms between functors, which must satisfy certain coherence laws representable in terms of commuting diagrams expressing conditions such as commutativity and associativity.

The isomorphism between objects represents equation between elements of set replaces identity. What about isomorphisms themselves? Should also these be defined only up to an isomorphism of isomorphism? And what about functors? Should one continue this replacement ad infinitum to obtain a hierarchy of what might be called n-categories, for which the process stops after n: th level. This rather fuzzy buisiness is what mathematicians like John Baez are actually doing.

Why categorification?

There are good motivations for the categofication. Consider the fact that natural numbers. Mathematically oriented person would think number "3" in terms of an abstract set theoretic axiomatization of natural numbers. One could also identify numbers as a series of digits. In the real life the representations of three-ness are more concrete involving many kinds of associations. For a child "3" could correspond to three fingers. For a mystic it could correspond to holy trinity. For a Christian "faith, hope, love". All these representations are isomorphic representation of threeness but as real life objects three sheeps and three cows are not identical.

We have however performed what might be called decategorification: that is forgitten that the isomorphic objects are not equal. Decatecorification was of course a stroke of mathematical genius with enormous practical implications: our information society represents all kinds of things in terms of numbers and simulates successfully the real world using only bit sequences. The dark side is that treating people as mere numbers can lead to a rather cold society.

Equally brilliant stroke of mathematical genius is the realization that isomorphic objects are not equal. Decategorization means a loss of information. Categorification brings back this information by bringing in consistency conditions known as coherence laws and finding these laws is the hard part of categorization meaning discovery of new mathematics. For instance, for braid groups commutativity modulo isomorphisms defines a highly non-trivial coherence law leading to an extremely powerful notion of quantum group having among other things applications in topological quantum computation.

The so called associahedrons (see http://tinyurl.com/ng2fqro) [B13] emerging in n-category theory could replace space-time and space as fundamental objects. Associahedrons are polygons used to represent geometrically associativity or its weaker form modulo isomorphism for the products of n objects bracketed in all possible ways. The polygon defines a hierarchy containing sub-polygons as its edges containing.... Associativity states the isomorphy of these polygons. Associahedrons and related geometric representations of category theoretical arrow complexes in terms or simplexes allow a beautiful geometric realization of the coherence laws. One could perhaps say that categories as discrete structures are not enough: only by introducing the continuum allowing geometric representations of the coherence laws things become simple.

No-one would have proposed categorification unless it were demanded by practical needs of mathematics. In many mathematical applications it is obvious that isomorphism does not mean identity. For instance, in homotopy theory all paths deformable to each other in continuous manner are homotopy equivalent but not identical. Isomorphism is now homotopy. These paths can be connected and form a groupoid. The outcome of the groupoid operation is determined up to homotopy. The deformations of closed path starting from a given point modulo homotopies form homotopy group and one can interpret the elements of homotopy group as copies of the point which are isomorphic. The replacement of the space with its universal covering makes this distinction explicit. One can form homotopies of homotopies and continue this process ad infinitum and obtain in this manner homotopy groups as characterizes of the topology of the space.

Cateforification as a way to describe finite measurement resolution?

In quantum physics gauge equivalence represents a standard example about equivalence modulo isomorphisms which are now gauge transformations. There is a practical strategy to treat the situation: perform a gauge choice by picking up one representative amongst infinitely many isomorphic objects. At the level of natural numbers a very convenient gauge fixing would correspond the representation of natural number as a sequence of decimal digits rather than image of three cows.

In TGD framework a excellent motivation for categorification is the need to find an elegant mathematical realization for the notion of finite measurement resolution. Finite measurement resolutions (or cognitive resolutions) at various levels of information transfer hierarchy imply accumulation of uncertainties. Consider as a concrete example uncertainty in the determination of basic parameters of a mathematical model. This uncertainty is reflected to final outcome as via a long sequence of mathematical maps and additional uncertainties are produced by the approximations at each step of this process.

How could onbe describe the finite measurement resolution elegantly in TGD Universe? Categorification suggests a natural method. The points equivalent with measurement resolution are isomorphic with each other. A natural guess inspired by gauge theories is that one should perform a gauge choice as an analog of decategorification. This allows also to avoid continuum of objects connected by arrows not n spirit with the discreteness of category theoretical approach.

1. At space-time level gauge choice means discretization of partonic 2-surfaces replacing them with a discrete set points serving as representatives of equivalence classes of points equivalent under finite measurement resolution. An especially interesting choice of points is as rational points or algebraic numbers and emerges naturally in p-adicization process. One can also introduce what I have called symplectic triangulation of partonic 2-surfaces with the nodes of the triangulation representing the discretization and carrying quantum numbers of various kinds.

2. At the level of "world classical worlds" (WCW) this means the replacement of the sub-group if the symplectic group of $\delta M^4 \times CP_2$ -call it G - permuting the points of the symplectic triangulation with its discrete subgroup obtained as a factor group G/H, where H is the normal subgroup of G leaving the points of the symplectic triangulation fixed. One can also consider subgroups of the permutation group for the points of the triangulation. One can also consider flows with these properties to get braided variant of G/H. It would seem that one cannot regard the points of triangulation as isomorphic in the category theoretical sense. This because, one can have quantum superpositions of states located at these points and the factor group acts as the analog of isometry group. One can also have many-particle states with quantum numbers at several points. The possibility to assign quantum numbers to a given point becomes the physical counterpart for the axiom of choice.

   The finite measurement resolution leads to a replacement of the infinite-dimensional world of classical worlds with a discrete structure. Therefore operation like integration over entire "world of classical worlds" is replaced with a discrete sum.

3. What suggests itself strongly is a hierarchy of n-categories as a proper description for the finite measurement resolution. The increase of measurement resolution means increase for the number of braid points. One has also braids of braids of braids structure implied by the possibility to map infinite primes, integers, and rationals to rational functions of several variables and the conjecture possibility to represent the hierarchy of Galois groups involved as symplectic flows. If so the hierarchy of n-categories would correspond to the hierarchy

of infinite primes having also interpretation in terms of repeated second quantization of an arithmetic SUSY such that many particle states of previous level become single particle states of the next level.

The finite measurement resolution has also a representation in terms of inclusions of hyperfinite factors of type II_1 defined by the Clifford algebra generated by the gamma matrices of WCW [K105]

1. The included algebra represents finite measurement resolution in the sense that its action generates states which are cannot be distinguished from each other within measurement resolution used. The natural conjecture is that this indistuinguishability corresponds to a gauge invariance for some gauge group and that TGD Universe is analogous to Turing machine in that almost any gauge group can be represented in terms of finite measurement resolution.

2. Second natural conjecture inspired by the fact that symplectic groups have enormous representabive power is that these gauge symmetries allow representation as subgroups of the symplectic group of $\delta M^4 \times CP_2$. A nice article about universality of symplectic groups is the article "The symplectification of science" (see `http://tinyurl.com/y8us9sgw`) by Mark. J. Gotay [A15].

3. An interesting question is whether there exists a finite-dimensional space, whose symplectomorphisms would allow a representation of any gauge group (or of all possible Galois groups as factor groups) and whether $\delta M^4 \times CP_2$ could be a space of this kind with the smallest possible dimension.

12.7.2 Inclusions Of HFFs And Planar Tangles

Finite index inclusions of HFFs are characterized by non-branched planar algebras for which only an even number of lines can emanate from a given disk. This makes possible a consistent coloring of the k-tangle by black and white by painting the regions separated by a curve using opposite colors. For more general algebras, also for possibly existing branched tangle algebras, the minimum number of colors is four by four-color theorem. For the description of zero energy states the 2-color assumption is not needed so that the necessity to have general branched planar algebras is internally consistent. The idea about the inclusion of positive energy state space into the space of negative energy states might be consistent with branched planar algebras and the requirement of four colors since this inclusion involves also conjugation and is thus not direct.

In [A11] if was proposed that planar operads are associated with conformal field theories at sphere possessing defect lines separating regions with different color. In TGD framework and for branched planar algebras these defect lines would correspond to light-like 3-surfaces. For fermions one has single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat. Bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets modellable as a piece of CP_2 type extremal. Each boson thus corresponds to 2 lines within CP_2 radius so that in purely bosonic case the planar algebra can correspond to that associated with an inclusion of HFFs.

12.7.3 2-Plectic Structures And TGD

Chris Rogers and Alex Hoffnung have demonstrated [A81] that the notion of symplectic structure generalizes to n-plectic structure and in $n = 2$ case leads to a categorification of Lie algebra to 2-Lie-algebra. In this case the generalization replaces the closed symplectic 2-form with a closed 3-form ω and assigns to a subset of one-forms defining generalized Hamiltonians vector fields leaving the 3-form invariant.

There are two equivalent definitions of the Poisson bracket in the sense that these Poisson brackets differ only by a gradient, which does not affect the vector field assignable to the Hamiltonian one-form. The first bracket is simply the Lie-derivate of Hamiltonian one form G with respect to vector field assigned to F. Second bracket is contraction of Hamiltonian one-forms with the three-form ω. For the first variant Jacobi identities hold true but Poisson bracket is antisymmetric only modulo gradient. For the second variant Jacobi identities hold true only modulo gradient but Poisson bracket is antisymmetric. This modulo property is in accordance with category theoretic thinking in which commutativity, associativity, antisymmetry, ... hold true only up to isomorphism.

For 3-dimensional manifolds n=2-plectic structure has the very nice property that *all* one-forms give rise to Hamiltonian vector field. In this case any 3-form is automatically closed so that a large variety of 2-plectic structures exists. In TGD framework the natural choice for the 3-form ω is as Chern-Simons 3-form defined by the projection of the Kähler gauge potential to the light-like 3-surface. Despite the fact the induced metric is degenerate, one can deduce the Hamiltonian vector field associated with the one-form using the general defining conditions

$$i_{v_F}\omega \quad = \quad dF \tag{12.7.1}$$

since the vanishing of the metric determinant appearing in the formal definition cancels out in the expression of the Hamiltonian vector field.

The explicit formula is obtained by writing ω as

$$\begin{aligned} \omega = K\epsilon_{\alpha\beta\gamma} \times \epsilon^{\mu\nu\delta} A_\mu J_{\nu\delta}\sqrt{g} \quad &= \epsilon_{\alpha\beta\gamma} \times C - S \quad , \\ C - S = KE^{\alpha\beta\gamma} A_\alpha J_{\beta\gamma} \quad . \end{aligned} \tag{12.7.2}$$

Here $E^{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$ holds true numerically and metric determinant, which vanishes for light-like 3-surfaces, has disappeared.

The Hamiltonian vector field is the curl of F divided by the Chern-Simons action density $C - S$:

$$v_F^\alpha = \frac{1}{2} \times \frac{\epsilon^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)\sqrt{g}}{C - S\sqrt{g}} = \frac{1}{2} \times \frac{E^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)}{C - S} \quad . \tag{12.7.3}$$

The Hamiltonian vector field multiplied by the dual of 3-form multiplied by the metric determinant has a vanishing divergence and is analogous to a vector field generating volume preserving flow. and the value of Chern Simons 3-form defines the analog of the metric determinant for light-like 3-surfaces. The generalized Poisson bracket for Hamiltonian 1-forms defined in terms of the action of Hamiltonian vector field on Hamiltonian as $J_1^\beta D_\beta F_2\alpha - J_2^\beta D_\beta H_2\alpha$ is Hamiltonian 1-form. Here J_i denotes the Hamiltonian vector field associated with F_i. The bracked unique apart from gradient. The corresponding vector field is the commutator of the Hamiltonian vector fields.

The objection is that gauge invariance is broken since the expression for the vector field assigned to the Hamiltonian one-form depends on gauge. In TGD framework there is no need to worry since Kähler gauge potential has unique natural expression and the $U(1)$ gauge transformations of Kähler gauge potential induced by symplectic transformations of CP_2 are not genuine gauge transformations but dynamical symmetries since the induced metric changes and space-time surface is deformed. Another important point is that Kähler gauge potential for a given CD has M^4 part which is "pure gauge" constant Lorentz invariant vector and proportional to the inverse of gravitational constant G. Its ratio to CP_2 radius squared is determined from electron mass by p-adic mass calculations and mathematically by quantum criticality fixing also the value of Kähler coupling strength.

12.7.4 TGD Variant For The Category Ncob

John Baez has suggested that quantum field theories could be formulated as functors from the category of n-cobordisms to the category of Hilbert spaces [A66, A26]. In TGD framework light-like 3-surfaces containing the number theoretical braids define the analogs of 3-cobordisms and surface property brings in new structure. The motion of topological condensed 3-surfaces along 4-D space-time sheets brings in non-trivial topology analogous to braiding and not present in category nCob.

Intuitively it seems possible to speak about one-dimensional orbits of wormhole throats and -contacts (fermions and bosons) in background space-time (homological dimension). In this case linking or knotting are not possible since knotting is co-dimension 2 phenomenon and only objects whose homological dimensions sum up to $D-1$ can get linked in dimension D. String like objects could topologically condense along wormhole contact which is string like object. The orbits of closed string like objects are homologically co-dimension 2 objects and could get knotted if one does not allow space-time sheets describing un-knotting. The simplest examples are ordinary knots which are not allowed to evolve by forming self intersections. The orbits of point like wormhole contact and closed string like wormhole contact can get linked: a point particle moving through a closed string is basic dynamical example. There is no good reason preventing unknotting and unlinking in absolute sense.

12.7.5 Number Theoretical Universality And Category Theory

Category theory might be also a useful tool to formulate rigorously the idea of number theoretical universality and ideas about cognition. What comes into mind first are functors real to p-adic physics and vice versa. They would be obtained by composition of functors from real to rational physics and back to p-adic physics or vice versa. The functors from real to p-adic physics would provide cognitive representations and the reverse functors would correspond to the realization of intentional action. The functor mapping real 3-surface to p-adic 3-surfaces would be simple: interpret the equations of 3-surface in terms of rational functions with coefficients in some algebraic extension of rationals as equations in arbitrary number field. Whether this description applies or is needed for 4-D space-time surface is not clear.

At the Hilbert space level the realization of these functors would be quantum jump in which quantum state localized to p-adic sector tunnels to real sector or vice versa. In zero energy ontology this process is allowed by conservation laws even in the case that one cannot assign classical conserved quantities to p-adic states (their definition as integrals of conserved currents does not make sense since definite integral is not a well-defined concept in p-adic physics). The interpretation would be in terms of generalized M-matrix applying to cognition and intentionality. This M-matrix would have values in the field of rationals or some algebraic extension of rationals. Again a generalization of Connes tensor product is suggestive.

12.7.6 Category Theory And Fermionic Parts Of Zero Energy States As Logical Deductions

Category theory has natural applications to quantum and classical logic and theory of computation [A26]. In TGD framework these applications are very closely related to quantum TGD itself since it is possible to identify the positive and negative energy pieces of fermionic part of the zero energy state as a pair of Boolean statements connected by a logical deduction, or rather- quantum superposition of them. An alternative interpretation is as rules for the behavior of the Universe coded by the quantum state of Universe itself. A further interpretation is as structures analogous to quantum computation programs with internal lines of Feynman diagram would represent communication and vertices computational steps and replication of classical information coded by number theoretical braids.

12.7.7 Category Theory And Hierarchy Of Planck Constants

Category theory might help to characterize more precisely the proposed geometric realization of the hierarchy of Planck constants explaining dark matter as phases with non-standard value of Planck constant. The situation is topologically very similar to that encountered for generalized Feynman diagrams. Singular coverings and factor spaces of M^4 and CP_2 are glued together along 2-D manifolds playing the role of object and space-time sheets at different vertices could be interpreted as arrows going through this object.

Chapter 13

Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD?

13.1 Introduction

The dynamics of TGD is extremely simple locally: space-times are surfaces of 8-D embedding space so that only four field-like dynamical variables are present and preferred extremals satisfy strong form of holography (SH) meaning that almost 2-D data determine them. TGD Universe looks however also extremely complex. There is a hierarchy of space-times sheets, hierarchy of p-adic length scales, hierarchy of dark matters labelled by the values of Planck constant $h_{eff}/h = n$, hierarchy of extensions of rationals defining hierarchy of adeles in adelic physics view about TGD, hierarchy of infinite primes (and rationals), and also the hierarchy of conscious entities (quantum measurement theory in zero energy ontology can be seen as theory of consciousness [L46]).

During years it has become gradually clear that category theory could be the mathematical language of quantum TGD [K19, K18, K11]. Only category theory gives hopes about unifying various hierarchies making TGD Universe to look so horribly complex. Hierarchy formed by categories, categories of categories, ... could be the mathematics needed to keep book about this complexity and provide also otherwise unexpected constraints.

The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks [L23] identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in the standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.

2. The recent view about generalized Feynman diagrams [L22, L24, L45] is inspired by two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to minimal diagrams, which are tree diagrams with braiding. This picture fits very nicely with the picture provided by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction defining basic gate in topological quantum computation [K5] but vertices expressed in terms of twistors depend on momenta of fermions.

3. Braiding transformations for fermionic lines identified as boundaries of string world sheets can take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time

regions with Minkowskian and Euclidian signature of induced metric respectively. Braiding transformation is essentially a permutation for two braid strands mapping tensor product $A \otimes B$ to $B \otimes A$. R-matrix satisfying Yang-Baxter equation [B56] characterizes this operation algebraically.

4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding. I have2-braiding in [K46]: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. I do not actually know whether my intuitive believe that 2-braiding reduces to reconnections is correct. Reconnection induces an exchange of braid strands defined by boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries string world sheets. This requires a generalization of quantum algebras to include also algebraic representation for reconnection: this representation could reduce to a representation in terms of an analog of R-matrix.

Yangians [B26] seem to be especially natural quantum algebras from TGD point of view [L10, L45]. Quantum algebras are bi-algebras having co-product Δ, which in well-defined sense is the inverse of the product. This makes the algebra multi-local: this feature is very attractive as far as understanding of bound states is considered. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Super-Kac-Moody algebras (SKMAs) involved and even with super-symplectic algebra (SSA) [K24, K106, K80], which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some sub-algebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question. There is also extended SKMA associated with the light-like orbits of partonic 2-surfaces and it seems natural to identify appropriate sub-algebras of these two algebras as duals in Yangian sense.

There is also partonic super-Kac-Moody algebra (PSKMA) associated with partonic 2-surfaces extending ordinary SKMA. On old conjecture is that SSA and PSKMA are physically dual in the same sense as the conformal algebra and its dual in twistor Grassmannian approach and that this generalizes equivalence principle (EP) to all conserved charges.

The plan of the article is following.

1. The basic notions and ideas about tensor networks as categories and about Yangians as multi-local symmetries and fundamental description of interactions are described.

2. The questions related to the Yangianization in TGD framework are considered. Yangianization of four-momentum and mass squared operator are discussed as examples.

3. The next section is devoted to category theory as tool of TGD: braided categories and fusion categories are briefly described and the notion of category with reconnection is considered.

4. The last section tries to represent the "great vision" in more detail.

13.2 Basic vision

The existing vision about TGD is summarized first and followed by a proposal about tensor networks as categories and Yangians as a multi-local generalization of symmetries with partonic surfaces replacing point like particles.

13.2.1 Very concise summary about basic notions and ideas of TGD

Let us briefly summarize the basic notions and ideas of TGD.

1. Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$, which is fixed uniquely by the condition that the factors of $H = M^4 \times S$ allow twistor space with Kähler structure [A54]. The

twistor spaces of dynamically allowed space-time surfaces are assumed to be representable as 6-D surfaces in twistor space $T(H) = T(M^4) \times T(CP_2)$ getting their twistor structure by induction from that of $T(H)$. $T(M^4)$ is identified as its purely geometric variant $T(M^4) = M^4 \times CP_1$. At the level of momentum space the usual identification is more appropriate. It is also assumed that these space-time surfaces are obtained as extremals of 6-D Kähler action [L10, L24, L45]. At space-time level this gives rise to dimensionally reduced Kähler action equal to the sum of volume term and 4-D Kähler action. Either the entire action or volume term would correspond to vacuum energy parameterized by cosmological constant in standard cosmology. Planck length corresponds to the radius of twistor sphere of M^4.

2. Strong form of holography (SH) implied by strong form of general coordinate invariance (SGCI) stating that light-like 3-surfaces defined by parton orbits and 3-D space-like ends of space-time surface at boundaries of CD separately code 3-D holography. SH states that 2-D data at string world sheets plus condition fixing the points of space-time surface with H-coordinates in extension of rationals fix the real space-time surface.

   (a) SH strongly suggests that the preferred extremals of the dimensionally reduced action satisfy gauge conditions (vanishing Noether charges) for a subalgebra H of super-symplectic algebras (SSA) isomorphic to it and its commutator $[H, SSA]$ with SSA: this effectively reduces SSA to a finite-dimensional Kac-Moody algebra.

   (b) Similar dimensional reduction would take place in fermionic degrees of freedom, where super-conformal symmetry fixes 4-D Dirac action, when bosonic action is known [K106, K80]. This involves the new notion of modified gamma matrices determined in terms of canonical momentum currents associated with the action.

   Quantum classical correspondence (QCC) states that classical Cartan charges for SSA are equal to the eigenvalues of corresponding fermionic charges. This gives a correlation between space-time dynamics and quantum numbers of positive (negative) parts of zero energy states.

   (c) SH implies that fermions are effectively localized at string world sheets: in other words, the induced spinor fields Ψ_{int} in space-time interior are determined their values Ψ_{string} at string world sheets. There are two options: Ψ_{int} is either continuation of Ψ_{string} or Ψ_{string} serves as the source of Ψ_{int} [L31].

3. At space-time level the dynamics is extremely simple locally since by general coordinate invariance (GCI) only 4 field-like variables are dynamical, and one has also SH by SGCI. Topologically the situation is rather complex: one has many-sheeted space-time having hierarchical structure. The GRT limit of TGD [K99] is obtained in long length scales by mapping the many-sheeted structure to a slightly curved piece of M^4 by demanding that the deformation of M^4 metric is sum of the deformation of he induced metrics of space-time surface from M^4 metric. Similar description implies to gauge potentials in terms of induced gauge potentials. The many-sheetedness is visible as anomalies of GRT and plays central role in quantum biology [K75].

4. Zero energy ontology (ZEO) means that one consider space-time surfaces inside causal diamonds (CDs defined as intersections of future and past directed light-cones with points replaced with CP_2) forming a scale hierarchy. Zero energy states are tensor products of positive and negative energy parts at opposite boundaries of CD. Zero energy property means that the total conserved quantum numbers are opposite at the opposite boundaries of CD so that one has consistency with ordinary positive energy ontology. Zero energy states are analogous to physical events in the usual ontology but is much more flexible since given zero energy energy states is in principle creatable from vacuum.

5. The "world of classical worlds" (WCW) [K45, K24, K80] generalizes the superspace of Wheeler. WCW decomposes to sub-WCWs assignable to CDs forming a scale hierarchy. Note that 3-surface in ZEO corresponds to a pair of disjoint collections 3-surfaces at opposite boundaries of CD- initial and final state in standard ontology. Super-symplectic symmetries

(SCA) act as isometries of WCW. Zero energy states correspond to WCW spinor fields and the gamma matrices of WCW are expressible as linear combinations of fermionic oscillator operators for induced spinor fields. Besides SCA there is partonic super-Kac-Moody algebra (PSCA) acting on light-like orbits of partonic 2-surfaces and these algebras are suggested to be dual physically (generalized EP).

6. One ends up with an extension of real physics to adelic physics [L41]. p-Adic physics for various primes are introduced as physical correlates of cognition and imagination: the original motivation come from p-adic mass calculations [K52]. p-Adic non-determinism (pseudo constants) [K63, K90] strongly suggests that one can always assign to 2-D holographic data a p-adic variant of space-time surface as a preferred extremal. In real case this need not be the case so that the space-time surface realized as preferred extremal is imaginable but not necessarily realizable.

   p-Adic physics and real physics are fused to adelic physics: space-time surface isa book-like structure with pages labelled by real number field and p-adic number fields in an extension induced by some extension of rationals. Planck constants $h_{eff} = n \times h$ corresponds to the dimension of the extension dividing the order of its Galois group and favored p-adic primes correspond to ramified primes for favored extensions. Evolution corresponds to increasing complexity of extension of rationals and favored extensions are the survivors in fight for number theoretic survival.

7. Twistor lift of TGD leads to a proposal for the construction of scattering amplitudes assuming Yangian symmetry assignable to Kac-Moody algebras for embedding space isometries, with electroweak gauge group, and for finite-D Lie dynamically generated Lie group selected by conditions on SSA algebra. 2+2 fermion vertex analogous to braiding interaction serves as the basic vertex in the formulation of [L45].

13.2.2 Tensor networks as categories

The challenge has been the identification of relevant categories and physical realization of them. One can imagine endless number of identifications but the identification of absolutely convincing candidate has been difficult. Quite recently an astonishingly simple proposal emerged.

1. The notion of tensor network [B44] has emerged in condensed matter physics to describe strongly entangled systems and complexity associated with them. Holography is in an essential role in this framework. In TGD framework tensor network is realized physically at the level of the topology and geometry of many-sheeted space-time [L23]. Nodes would correspond to objects and links between them to morphisms. This structure would be realized as partonic 2-surfaces - objects - connected by fermionic strings - morphisms - assignable to magnetic flux tubes. Morphisms would be realized as Hilbert space isometries defined by entanglement. Physical state would be category or set of them!

   Functors are morphisms of categories mapping objects to objects and morphisms to morphisms and respecting the composition of morphisms so that the structure of the category is preserved. For instance, in zero energy ontology (ZEO) S-matrix for given space-time surface could be a unitary functor assigning to an initial category final category: they would be represented as quantum states at the opposite boundaries of causal diamond (CD). Also quantum states could be categories of categories of in accordance with various hierarchies.

2. Skeptic could argue as follows. The passive part of zero energy states for which active part evolves by unitary time evolutions following by state function reductions inducing time localization in moduli space of CDs, could be category. But isn't the active path more naturally a quantum superposition of categories? Should one replace time evolution as a functor with its quantum counterpart, which generates a quantum superposition of categories? If so, then state function reduction to opposite boundary of CD would mean localization in the set of categories! This is quite an abstraction from simple localization in 3-space in wave mechanics.

3. Categories form categories with functors between categories acting as morphisms. In principle one obtains an infinite hierarchy of categories identifiable as quantum states. This would fit

nicely with various hierarchies associated with TGD, most of which are induced by the hierarchy of extensions of rationals.

4. The language of categories fits like glove also to TGD inspired theory of consciousness. The fermionic strings and associated magnetic flux tubes would serve as correlates of attention. The associated morphism would define the direction of attention and also define sensory maps as morphisms. Conscious intelligence relies crucially on analogies and functors realize mathematically the notion of analogy. Categorification means basically classification and this is what cognition does all the time.

13.2.3 Yangian as a generalization of symmetries to multilocal symmetries

Mere networks of arrows are not enough. One needs also symmetry algebra associated with them giving flesh around the bones.

1. Various quantum algebras, in particular Yangians are naturally related to physically interesting categories. The article of Jimbo [B56], one of the pioneers of quantum algebras, gives a nice summary of Yang-Baxter equation central in the construction of quantum algebras. R-matrix performs is an endomorphism permuting two tensor factors in quantal matter.

2. One of the nice features of Yangian is that it gives hopes for a proper description of bound states problematic in quantum field theories (one can argue that QCD cannot really describe hadrons and already QED has problems with Bethe-Salpeter equation for hydrogen atom). The idea would be simple. Yangian would provide many-particle generalization of single particle symmetry algebra and give formulas for conserved charges of many-particle states containing also interaction terms. Interactions would reduce to kinematics. This - as I think - is a new idea.

   The iteration of the co-product Δ would map single particle symmetry operator by homomorphism to operator acting in N-parton state space and one would obtain a hierarchy of algebra generators labelled by N and Yangian inariance would dictate the interaction terms completely (as it indeed does in $\mathcal{N}=4$ SUSY in twistor Grassmannian approach [B27]).

3. There is however a delicacy involved. There is a mysterious looking doubling of the symmetry generators. One has besides ordinary local generators T_0^A generators T_1^A: in twistor Grassmann approach the latter correspond to dual conformal symmetries. For T_0^A the co-product is trivial: $\Delta(J_0^A) = J_0^A \otimes 1 + 1 \otimes J_0^A$, just like in non-interacting theory. This is true for all iterates of Δ.

   For J_1^A one has $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \otimes J_1^A + f^A_{BC} J_0^B \otimes J_0^C$. One has two representations and the duality suggests that the eigenvalues J_0^A and J_1^A are same (note that in Witten's approach [B26] $J_1^A = 0$ holds true so that it does not apply as such to TGD). The differences $T_0^A - T_1^A$ would give a precise meaning for "interaction charges" if the duality holds true, and more generally, to the perturbation theory formed by a pair of free and interacting theory. This picture raises hopes about first principle description of bound states: interactions described in wave mechanics in terms of phenomenological interaction Hamiltonians and interaction potentials would be reduced to kinematics.

   For instance, for four-momentum $\Delta(P_1^k)$ would contain besides free particle term $P_0^k \otimes 1 + 1 \otimes P_0^k$ also the interaction term involving generators of - say - conformal group.

4. What about the physical interpretation of the doubling? The most natural interpretation would be in terms of SSA and the extended super-conformal algebra assignable to the light-like orbits of partonic 2-surfaces. An attractive interpretation is in terms of a generalization of Equivalence Principle (EP) stating that inertial and gravitational charges are identical for the physical states.

5. The tensor summands of Kac-Moody algebra would have different coupling constants k_i perhaps assignable to the 4 fundamental interactions and to the dynamical gauge group emerging from the SCA would give further coupling constant. This would give 5 tensor

factors strongly suggested by p-adic mass calculations - p-adic masses depend only on the number of tensor factors [K52].

13.3 Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

13.3.1 Yang-Baxter equation (YBE)

Yang-Baxter equation (YBE) has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A77]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [B26] (see `http://tinyurl.com/mcvvcqp`) and in conformal field theories and models for anyons. Yangian symmetry appears also in twistor Grassmann approach to scattering amplitudes [B27, B36] and thus involves YBE. At the same time new invariants for links were discovered and new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [B56] (see `http://tinyurl.com/l4z6zyr`, where one can also find a list of references). YBE was first discovered by McGuire (1964) and 3 years later by Yang in quantum mechanical many-body problem involving delta function potential $\sum_{i<j}\delta(x_i-x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as building brick 2-particle scattering matrix - R-matrix. YBE emerged for R-matrix as a consistency condition for factorization. Baxter discovered 1972 solution of the eight vertex model in terms of YBE. Zamolodchikov pointed ot that the algebraic mechanism behind factorization of 2-D QFTs is same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. Interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of `http://tinyurl.com/kyw6xu6`). In 2-D quantum field theories and in models for basic gate in topological quantum computation (for early TGD vision see [K5] were also R-matrix is discussed in more detail) the R-matrix is unitary. One can interpret R-matrix as endomorphism mapping $V_1\otimes V_2$ to $V_2\otimes V_1$ representing permutation of the particles.

YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v)=R_{12}(v)R_{13}(u+v)R_{23}(u) \quad (13.3.1)$$

having interpretation as associativity condition for quantum algebras.

At the limit $u,v\rightarrow\infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braid operations replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on basis of the topological interpretation. Scaling and automorphism induced by linear transformations of

V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V \otimes V$ the condition states that $R(0)$ is proportional to permutation matrix P for the factors.

General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued meromorphic functions to complex plane and define with poles forming an Abelian group. R-matrices can be classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have pole at origin and elliptic solutions have a lattice of poles. In [B56] (see `http://tinyurl.com/l4z6zyr`) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [B56] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.

3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anything to do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

   obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be so interesting as Yangian: in this case co-product Δ does not seem to have a natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$.

   For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.

5. The article of Jimbo discusses also fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

13.3.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \rightarrow A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \rightarrow C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta$. Also ϵ satisfies associativity condition $(\epsilon\otimes 1)\circ\Delta = (1\otimes\epsilon)\circ\Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [B56] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n+1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n+1\geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretation improve the understanding of classical correspondence (QCC).

Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian in $\mathcal{N}=4$ SUSYs [B26], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

It must be however emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC}J_0^B\times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A\otimes 1 + 1\times J_1^A + f^A_{BC}J_0^B\times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n=0$ and $n=1$. The first half of these relations discussed in very clear manner in [B26] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A,J^B\right] = f^{AB}_C J^C \quad , \quad \left[J^A,J^{(1)B}\right] = f^{AB}_C J^{(1)C} \quad . \tag{13.3.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right] + \left[J^{(1)B},\left[J^{(1)C},J^A\right]\right] + \left[J^{(1)C},\left[J^{(1)A},J^B\right]\right] \\
&= \frac{1}{24} f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J_D,J_E,J_F\} \quad , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^C,J^{(1)D}\right]\right] + \left[\left[J^{(1)C},J^{(1)D}\right],\left[J^A,J^{(1)B}\right]\right] \\
&= \frac{1}{24}(f^{AGL}f^{BEM}f^{CD}_K \\
&+ f^{CGL}f^{DEM}f^{AB}_K)f^{KFN}f_{LMN}\{J_G,J_E,J_F\} \quad .
\end{aligned} \tag{13.3.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

The right hand sides have often as a coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A=J^A}$ the first Serre relation reduces to Jacobi identity and second to antisymmetry of Lie bracket. The right hand sided involved completely symmetrized trilinears $\{J_D, J_E, J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} = f^A_{BC} \sum_{i<j} J_i^B J_j^C \quad . \tag{13.3.4}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N}=4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned}
\Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \quad , \\
\Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C
\end{aligned} \tag{13.3.5}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B26].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n = m$ is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f_{AB}^{C'} \sum_{i<j} J_i^A J_j^B \\ &= g_{CC'} f_{AB}^{C'} g^{AA'} g^{BB'} \sum_{i<j} J_{A'}^i J_{B'}^j \ . \end{aligned} \tag{13.3.6}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

13.4 Yangianization in TGD framework

Yangianization of quantum TGD is quite challenging. Super-conformal algebras are much larger than in say $\mathcal{N} = 4$ SUSY and even in superstring models and reconnection and 2-braiding are new topological elements.

13.4.1 Geometrization of super algebras in TGD framework

Super-conformal algebras allow a geometrization in TGD framework and this should be of considerable help in the Yangianization.

1. The basic generators of various Super-algebras follow from modified Dirac action as Noether charges and their super counterparts obtained by replacing fermion field Ψ (its conjugate $\overline{\Psi}$) by a mode u_m ($\overline{u}_n$) of the induced spinor field [K106, K80]. The anti-commutators of these Noetherian super charges labelled by n define WCW gamma matrices. The replacement of both Ψ and $\overline{\Psi}$ with modes u_m and $\overline{u}_n$ gives a collection of conserved c-number currents and charges labelled by (n, m). These c-number charges define the anti-commutation relations for the induced spinor fields so that quantization reduces to dynamics thanks to the notion of modified gamma matrices forced by super-conformal symmetry.

2. The natural generalization of Sugawara formula to the level of Yangian of SKMA starts from the Dirac operator for WCW defined like ordinary Dirac operator in terms of the contractions of WCW gamma matrices with the isometry generators (SCA) replacing the Super Virasoro generators G_r and WCW d'Alembert operator defined as its square replacing Virasoro generators L_n. Anti-commutators of WCW gamma matrices defined by super charges for super-symplectic generators define WCW Kähler metric [K106] for which action for preferred extremal would define Kähler function for WCW metric [K45].

3. Quarks and leptons give rise to a doubling of WCW metric if associated with same space-time sheet that is with the same sector of WCW. The duplication of the super algebra generators - in particular WCW gamma matrices - does not seem to make sense. Do quarks and leptons therefore correspond to different sectors of WCW and live at different space-time surfaces? But what could distinguish between 3-surfaces associated with quarks and leptons?

   Could quarks be associated with homologically non-trivial partonic 2-surfaces with CP_2 homology charges 2,-1,-1 proportional to color hypercharges $2/3, -1/3, -1/3$ and leptons with partonic 2-surfaces with vanishing homology charges coming as multiples of 3? Vanishing of color hypercharge for color-confined states would topologize to a vanishing of total homology charge. Could spin/isospin half property of fundamental fermions topologize to 2-sheeted structure of the space-time surface representing elementary particle consisting of elementary fermions?

SSA acting as isometries of WCW is not the only super-conformal algebra involved.

1. Partonic 2-surfaces are ends of light-like 3-surfaces- partonic orbits - and give rise to a generalization of SKMA of isometries of H so that they act as local isometries preserving the light-likeness property of the orbits. At the ends of the partonic 2-surface SKMA is associated with complex coordinate of partonic 2-surface. What is the role of this algebra, which is also extended SKMA (already christened PSCA) but with light-like coordinate parameterizing the SKMA generators?

   Is it an additional symmetry combining with string world sheet symmetries to a symmetry involving complex coordinate and complex or hypercomplex coordinate? Or is it dual to the string world sheet symmetry? How do these symmetries relate to SSA? Does SGCI implying SH leave only SKMAs associated with isometries, holonomies of CP_2 (electroweak interactions) and dynamical SKMA remaining as remnant of SCA.

2. I have earlier proposed that Equivalence Principle (EP) as identity of inertial and gravitational charges could reduce to the duality between these SSA assignable to strings and the partonic super-conformal algebra. This picture conforms with the expected form of the generators associated with these algebras. The dual generating elements T_0^A *resp.* T_1^A associated with generic Yangian could naturally correspond to isomorphic sub-algebras of super-conformal algebra associated with orbits of partonic 2-surfaces *resp.* super-symplectic algebra assignable to string world sheets.

13.4.2 Questions

There are many open questions to be answered.

Q1: What Yangianization could mean in TGD framework? The answer is not obvious and one can consider two options.

1. Assuming that SH leads to an effective reduction of super-symplectic algebra to finite-D Kac-Moody algebra, assign to partonic 2-surfaces direct sum of Kac-Moody type algebras $L(g) = g(z, z^{-1})$ assigned with complex coordinate z of partonic 2-surface. One could perform Yangianization for this algebra meaning that these symmetries become multi-local with locus identified as partonic 2-surface.

   In Drinfeld's approach this would mean Yangianization of $L(g)$ rather than g and would involve double loop algebra $L(L(g))$ and its positive and negative energy parts. In Minkowskian space-time regions the generators would be functions of complex coordinate z and hypercomplex coordinate u associated with string world sheet: in Euclidian space-time regions one would have 2 complex coordinates z and w. This would conform with holography. I do not know whether mathematicians have considered this generalization and whether it is possible. In the following this is assumed.

2. Physical states at partonic 2-surfaces consist of pointlike fermions and one can ask whether this actually means that one can consider just the Lie algebra g so that in Drinfeld's approach one would have just string world sheets and $Y(g)$. Already this option requires the

algebraization of reconnection mechanism as a new element. Whether this simpler approach make sense for fermions and by QQC for quantum TGD, is not clear.

Q2: Can one really follow the practice of Grassmannian twistor approach and say that T_1^A and TA^0 are dual?

One has $[T_0^A, T_1^B] = f_C^{AB} T_1^C$. Witten's definition $T_1^A = f_{BC}^A T^B \otimes T^C \equiv T_1^A = f_{BC}^A T^B T^C$ with T_1^A identified as total charges for lattice, identifies T_1^A as 2-particle generators of Yangian. One the other hand, in TGD T_0^A would correspond to partonic super-conformal algebra and T_1^A to bi-local super-symplectic algebra and the general definition to be used regards also T_1^A as single particle generators in Yangian sense and defines the generators at 2-particle level as $\Delta(T_0^A) = T_0^A \otimes 1 + 1 \otimes T_0^A$ and $\Delta(T_1^A) = T_1^A \otimes 1 + 1 \otimes T_1^A + f_{BC}^A T_0^B \otimes T_0^C$.

For the Witten's definition one cannot demand that T_0^A and T_1^A have same eigenvalues for the physical states. For the more general definition of Δ to be followed in the sequel it seems to be possible require that T_0^A and T_1^A obey the same commutation relations for appropriate sub-algebras at least, and that it is possible to diagonalize Cartan algebras simultaneously and even require same total Cartan charges. This issue is not however well-understood.

Q3: What algebras are Yangianized in TGD framework?

The Yangians of SKMAs associated with isometries of $M^4 \times CP_2$ and with the holonomy group $SU(2) \times U(1)$ of CP_2 appear as symmetries. M^4 should give SKMA in transversal degrees of freedom for fermionic string. CP_2 isometries would give SKMA associated with SU(3). $SU(2) \times U(1)$ would be assignable to electroweak symmetries. This gives 4 tensor factors.

Five of them are required by p-adic mass calculations [K52], whose outcome depends only on the number of tensor factors in Virasoro algebra. The estimates for the number of tensor factors has been a chronic head ache: in particular, do M^4 SKMA correspond to single tensor factor or two tensor factors assignable to 2 transversal degrees of freedom.

Supersymplectic algebra (SSA) is assumed to define maximal possible isometry group of WCW guaranteeing the existence of Kähler metric with a well-defined Riemann connection. The Yangian of SSA could be the ultimate symmetry group, which could realize the dream about the reduction of all interactions to mere kinematics. If SSA effectively reduces to a finite-D SKMA for fermionic strings, one would have 5 tensor factors.

Q4: What does SSA mean?

1. SSA is associated with light-cone boundary $\delta M_\pm^4$ with one light-like direction. The generators (to be distinguished from generating elements) are products of Hamiltonians of symplectic transformations of CP_2 assignable to representations of color SU(3) and Hamiltonians for the symplectic transformations of light-cone boundary, which reduce to Hamiltonians for symplectic transformations of sphere S^2 depending parametrically on the light-like radial coordinate r. This algebra is generalized to analog of Kac-Moody algebra defined by finite-dimensional Lie algebra.

2. The radial dependence of Hamiltonians of form r^h. The naïve guess that conformal weights are integers for the bosonic generators of SSA is not correct. One must allow complex conformal weights of form $h = 1/2 + iy$: $1/2$ comes from the scaling invariant inner product for functions at $\delta M_\pm^4$ defined by integration measure dr/r [K24, K80].

3. An attractive guess [L17] is that there is an infinite number of generating elements with radial conformal weights given by zeros of zeta. Conformal confinement must holds true meaning that the total conformal weights are real and thus half-odd integers. The operators creating physical states form a sub-algebra assignable by SH and QCC to fermionic string world sheets connecting partonic 2-surfaces.

4. SH inspires the assumption that preferred extremal property requires that sub-algebra H of SSA isomorphic to itself (conformal weights are integer multiples of SSA) and its commutator SH with SH annihilate physical states and classical Noether charges vanish. This could reduce the symmetry algebra to SKMA for a finite-dimensional Lie group. SSA could be replaced also with the sub-algebra creating physical states having half-odd integer valued radial conformal weights.

Similar conditions could make sense for the generalization of super-conformal KM algebra associated with light-like partonic orbits.

Q5: What is the precise meaning of SH in the fermionic sector?

Are string world sheets with their ends behaving like pointlike particles enough or are also partonic 2-surface needed. For the latter option a generalization of conformal field theory (CFT) would be needed assigning complex coordinate with partonic 2-surfaces and hyper-complex or complex coordinates with string world sheets. Elementary particle vacuum functionals depend on conformal moduli of partonic 2-surface [K21], which supports the latter option.

There could be however duality between partonic 2-surfaces and string world sheets so that either of them could be enough [L45]. There is also uncertainty about the relationship between induced spinor fields at string world sheets and space-time interior. Are 4-D induced spinor fields obtained by process analogous to analytic continuation in 2-complex dimensional space-time or do 2-D induced spinor fields serve as sources for 4-D induced spinor fields?

Quantum algebras are characterized by parameters such as complex parameter q characterizing R-matrices for quantum groups. Adelic physics [L41] demands number theoretical universality and in particular demands that the parameters - say q - of quantum algebraic structures involved are products $q = e^{m/n}xU$, where U is root of unity (note that e^p exists as ordinary p-adic number for Q_p) and x is real number in the extension. This guarantees that the induced extensions of p-adic numbers are finite-dimensional (the hypothesis is that the correlates of cognition are finite-D extensions of p-adic number fields) [K80].

In the recent view about twistorial scattering amplitudes [L45] the fundamental fermionic vertices are $2 \rightarrow 2$ vertices. There is no fermionic contact interaction in the sense of QFT but the fermions coming to the topological vertex defined by partonic 2-surface at which 3 partonic orbits meet (analogy for the 3-vertex for Feynman diagram) are re-distributed between partonic two surfaces. Also in integrable 2-D QFTs in M^2 the vertices are $2 \rightarrow 2$ vertices characterized by R-matrix. The twistorial vertex is however not topological.

13.4.3 Yangianization of four-momentum

The QFT picture about bound states is unsatisfactory. The basic question to be answered is whether one should approach the problem in terms of Lorentz invariant mass squared natural in conformal field theories or in terms of Poincare algebra. It is quite possible that the fundamental formulation allowing to understand binding energies is in terms of SCA and PSCA.

Twistor lift of TGD [L45] however suggests that Poincare and even finite-D conformal transformations associated with M^2 could play important role. These longitudinal degrees of freedom are non-dynamical in string dynamics. Maybe there is kind of sharing of labor between these degrees of freedom. In the following we consider two purely pedagogical examples about Yangianization of four-momentum in M^4 and in 8-D context regarding four-momentum as quaternionic 8-momentum in M^8.

Yangianization of four-momentum in conformal algebra of M^4

Consider as an example what the Yangianization for four-momentum P^k could mean. This is a pedagogical example.

1. The first thing to notice is that the commutation relations between P_0^k and P_1^k are inherited from those between P_0^k and force P_1^k and P_0^k to commute. This holds true quite generally for Cartan algebra so that if the correspondence between T_0^A and T_1^A respects Cartan algebra property then Cartan algebras of T_0^A and T_1^A can be simultaneously diagonalized for the physical states. The Serre relations of Eq. 13.3.3 are identically satisfied for Cartan algebra and its image. This is consistent with the assumption that Cartan algebra is mapped to Cartan algebra but does not prove it.

2. The formula $f^A_{BC}T_0^A \otimes T_0^C$ for the interaction term appearing in the expresion of Δ should be non-trivial also when T^A corresponds to four-momentum. Already the Poincare algebra gives this kind of term built from Lorentz generators and translation generators.

The extension of Poincare algebra extended to contain dilatation operator D can be considered as also M^4 conformal algebra with generators of special conformal transformations M^A included (see http://tinyurl.com/nxlmfug). One has doubling of all algebra generators. The interpretation as gravitational and inertial momenta is one possibility, and EP suggests that the two momenta have same values. In twistor Grassmannian approach the conformal algebras are regarded as dual and suggests the same. Hence one would have $P_0^k = P_1^k$ at the level of eigenvalues.

3. For conformal group the proposed co-product for P_i^k would read as

$$\begin{aligned}\Delta(P_0^k) &= P_0^k \otimes 1 + 1 \otimes P_0^k \ , \\ \Delta(P_1^k) &= P_1^k \otimes 1 + 1 \otimes P_1^k + K f^k_{Al}(L_0^A \otimes P_0^l - P_0^k \otimes L_0^A) + K f^k_{Al}(M_0^A \otimes P_0^l - P_0^l \otimes M_0^A) \\ &+ K(D_0 \times P_0^k - P_0^k \times D_0) \ .\end{aligned} \tag{13.4.1}$$

This condition could be combined with the condition for mass squared operator. For $K = 0$ one would have additivity of mass squared requiring that P_1 and P_2 are parallel and light-like. For $K \neq 0$ it might be possible to have a simultaneous solution to the both conditions with massive total momentum.

The Δ-iterates of P_0^k contain no interaction terms. For P_1 one has interaction term. This holds true for all symmetry generators. Assume $P_0 = P_1$: does this mean that the interacting theory associated with P_1 is dual to free theory? The difference $\Delta P_0^k - \Delta(P_1^k)$ defines the analog interaction Hamilton, which would therefore be not due to a somewhat arbitrary decomposition of four-momentum to free and interaction parts. It should be possible to possible to measure this difference and its counterpart for other quantum numbers. One can only make questions about the interpretation for this duality applying to all quantum numbers.

1. In Drinfeld's construction the negative and positive energy parts of loop algebra would be related by the duality. In ZEO it might be possible to relate them to positive and negative energy parts of zero energy states at the opposite boundaries of CD.

2. If n is interpreted as number of partonic surfaces and the generators are interpreted as in Witten's construction then the duality could be seen as a geometric duality in plane mapping edges and vertices (partonic 2-surfaces ordered in sequence and string between them) to each other. In super-conformal algebra of twistor Grassmannian approach the generators T_0^A and T_1^A are associated with vertices and edges of the polygon defining the scattering diagram and this suggests that T_0^A corresponds to partonic 2-surfaces and T_1^A to the strings world sheets.

3. Could the duality be a generalization of for Equivalence Principle identifying inertial and gravitational quantum numbers? This interpretation is encouraged by the presence of SSA action on space-like 3-surfaces at the ends of CDs and extended super-conformal algebra associated with the light-like orbits of partons: SGCI would suggest that these algebras or at least their appropriate sub-algebra are dual. This interpretation conforms also with the above geometric interpretation and twistor Grassmannian interpretation.

Consider for simplicity the situation in which only scaling generator D is present in the extension.

1. Suppose that one has eigenstate of total momentum $\Delta(P_0^k)$ resp. $\Delta(P_1^k)$ with eigenvalue p_0^{tot} resp. p_1^{tot} and that

$$p_0^{tot} = p_1^{tot} \tag{13.4.2}$$

holds true.

2. Since D_0 and P_0^k do not commute, the action of D_0 must be realized as differential operator $D_0 = ip_0^k d/dp_0^k$ so that one has following eigenvalue equations

$$\begin{aligned} \Delta(P_0^k)\Psi &= (p_{0,1}^k + p_{0,2}^k)\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_1^k)\Psi &= (p_{1,1}^k + p_{1,2}^k)\Psi + K(ip_{0,1}^k \otimes p_{0,2}^r \frac{d}{dp_{0,2}^r} - ip_{0,1}^r \frac{d}{dp_{0,1}^r} \otimes p_{0,2}^k)\Psi = p_1^{tot}\Psi \end{aligned} \quad (13.4.3)$$

Ψ must be a superposition of states $|p_{0,1}, p_{0,2}\rangle$. One has non-trivial interaction. Analogous interaction terms mixing states with different momenta emerge from the terms involving Lorentz generators and special conformal generators.

Four-momenta as quaternionic 8-momenta in octonionic 8-space

In octonionic approach to twistorial scattering amplitudes particles can be regarded as massless in 8-D sense [L45]. The light-like octonionic momenta are actually quaternionic and one would obtain massive states in 4-D sense. Different 4-D masses would correspond to discrete set of quaternionic momenta for 8-D massless particle. Could the above conditions generalize to this case?

1. Suppose that the symmetries reduce to Poincare symmetry and to a number theoretic color symmetry acting as automorphisms of octonions. In this case the four-momentum for a given $M^4 \subset M^8$ decomposes to a sum of to a direct sum of M^2 invariant under SU(3) and E^2 invariant under $SU(2) \times U(1) \subset SU(3) \subset G_2$. ΔP_1 would be non-trivial for the transversal momentum and of form

$$\begin{aligned} \Delta(P_0^{L,k})\Psi &= (p_{0,1}^{L,k} + p_{0,2}^{L,k})\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_0^{T,k})\Psi &= (P_0^{T,k} \otimes 1 + 1 \otimes P_0^{T,k})\Psi \ , \\ \Delta(P_1^{L,k})\Psi &= (p_{1,1}^{L,k} + p_{1,2}^{L,k})\Psi = P_1^{L,tot}\Psi \ , \\ \Delta(P_1^{T,k})\Psi &= (P_1^{T,k} \otimes 1 + 1 \otimes P_1^{T,k} + Kf_{Al}^k(ip_{0,1}^l \otimes t_{0,2}^A - i(ip_{0,2}^l \otimes t_{0,2}^A)\Psi \ . \end{aligned} \quad (13.4.4)$$

Here P_0^L resp. P_0^T represents longitudinal resp. transversal momentum and T_0^b denotes $SU(2) \subset SU(3)$ generator representable as differential operator acting on complexified momentum and $p_0^T = p_0^{T,x} + ip_0^{T,y}$ and its conjugate.

2. In transversal degrees of freedom the assumption about momentum eigenstates would be probably too strong. String model suggests Gaussian in transversal oscillator degrees of freedom. Hadronic physics suggests an eigenstate of transversal momentum squared. TGD based number theoretic considerations suggest that the transversal state is characterized by color quantum numbers.

   Hence the conditions

$$p_0^{L,tot} = p_1^{L,tot} \ , \ (p_0^{T,tot})^2 = (p_1^{T,tot})^2 \quad (13.4.5)$$

   are natural. It would be nice if the momenta p_{01} and p_{02} could be chosen to be on mass shell and satisfy stringy formula for mass squared where transverse momentum squared would correspond to stringy contribution.

One can also add to $\Delta(P)$ the terms coming from conformal group of M^4 or its subgroup. Since octonionic momentum is light-like M^2 momentum for a suitable choice of M^2, one must consider the possibility that the conformal group is that of $M^2 \subset M^4$. Twistorialization supports this view [L45]. The action of conformal generations would be on longitudinal momentum only.

One can wonder how gauge interactions and gravitational interaction do fit to this picture. Is the extension to super-conformal algebra and supersymplectic algebra the only manner to obtain gauge interactions and gravitation into the picture?

13.4.4 Yangianization for mass squared operator

It would be nice to have universal mass formulas as a generalization of mass squared formula for string models in terms of the conformal scaling generator $L_0 = zd/dz$. This operator should have besides single particle contributions also many particle contributions in bound states analogous to interaction Hamiltonian and interaction potential. Yangian as an algebra containing multi-local generators is a natural candidate in this respect.

One can consider Yangianization of Super Virasoro algebra (SVA). The Yangianization of various Super Kac-Moody algebras (SKMA) seems however more elegant if it induces the Yangianization of SVA. Consider first direct Yangianization of SVA. The commutation relations for SVA will be used in the sequel. They can be found in Wikipedia (see `http://tinyurl.com/klsgquz`) so that I do not bother to write them here. It must be emphasized that there might be delicate mathematical constraints on algebras which allow Yangianization as the article of Witten [B26] shows. The considerations here rely on physical intuition with unavoidable grain of wishful thinking.

What about the Yangian variant of mass squared operator m^2in terms of the conformal scaling generator $L_0 = zd/dz$? Consider first the definition of various Super algebras in TGD framework.

1. In standard approach the basic condition at single particle level $L_0\Psi = h_{vac}\Psi$ giving the eigenvalues of m^2. Massless in generalize sense requires $h_{vac} = 0$. One would have $m^2_{op} = L_0^{vib} + h_{vac}Id$, where "vib" refers to vibrational degrees of freedom of Kac-Moody algebra (KMA). Sugawara construction [A52] allows to express the left-hand side of this formula in terms of Kac-Moody generators - one has sum over squares $T^a_n T^{-n}_a$. One can say that mass squared is Casimir operator vibrational degrees of freedom for KMA

2. In absence of interactions - and always for $L_{0,0}$ - mass squared formula gives $m_1^2 + m_2^2 = L_0^{vib,1} + L_0^{vib,2}$ for vanishing vacuum weights. It is important to notice that this does *not* imply the additivity of mass squared since one does not have $(p_1 + p_2)^2 = m_1^2 + m_2^2$, which can hold true only for massless and parallel four-momenta. I have considered the possible additivity of mass mass squared for mesons [K64] but it of course fails for systems like hydrogen atom.

One can look what Yangianization of Super Virasoro algebra could mean.

1. One would have doubling of the generators of SKMA and SVA: one possible explanation is in terms of generalized EP. The difference $\Delta(T_0^A) - \Delta(T_1^A)$ would define the analog of interaction Hamiltonian of the duality holds true.

   One has $L_0 = G_0^2/2$. Quite generally, one has $\{G_r, G_{-r}\} = 2L_0$ apart from the central extension term. Generalization Yangian to Super Algebra suggests that one has

$$\begin{aligned}
\Delta(L_{0,0}) &= L_{0,0} \otimes 1 + 1 \otimes L_{0,0} \ , \\
\Delta(L_{1,0}) &= L_{1,0} \otimes 1 + 1 \otimes L_{1,0} + K \sum_n G_{0,r} \otimes G_{0,-r}
\end{aligned} \tag{13.4.6}$$

   Both operators give the value of h_{vac} expected to vanish when acting on physical states and the eigenvalues of the interaction mass squared $K \sum_n G_2 \otimes G_{-r}/2$ would represent the difference $m^2_{0,1} + m^2_{0,2} - m^2_{2,1} - m^2_{2,2}$. By Lorentz invariance the interaction energy is expected to be proportional to the inner product $P_1 \cdot P_2$ and the interpretation in terms of gravitational interaction energy is attractive. The size scale of K would be determined by $l_P^2/R^2 \simeq 2^{-12}$, where l_P is Planck length and R is CP_2 radius gravitational constant [L24, L45].

2. The action of $k \sum_n G_{0,n} \otimes G_{0,-n}/2$ on state $|p_1, p_2\rangle$ is analogous to the action of a tensor product of Dirac operators on tensor product of spinors. Since Dirac operator changes chirality, this suggests that the states are superpositions of eigenstates of chirality of form

$$\Psi = G_{0,0}\Psi_1 \otimes \Psi_2 + \epsilon \times \Psi_1 \otimes G_{0,0}\Psi_2 \ , \ \ \epsilon = \pm 1 \ .$$

$L_{0,0}\Psi_i = 0$ and $\Delta(L_{0,0})\Psi = 0$ holds true. $\Delta(G_{0,0})$ and $\Delta(G_{1,0})$ are given by

$$\begin{aligned} \Delta(G_{0,0}) &= G_{0,0}\otimes 1 - \epsilon\times 1\otimes G_{0,0} \ , \\ \Delta(G_1,0) &= G_{1,0}\otimes 1 - \epsilon\times 1\otimes G_{1,0} - \frac{3K}{2}\sum_r r(L_{0,r}\otimes G_{0,-r} - (G_{0,-r}\otimes L_{0,r}) \ , \end{aligned} \tag{13.4.7}$$

and should annihilate Ψ. This is true if $L_{1,r}$ and $L_{0,r}$ annihilate the states.

3. Perhaps the correct approach reduces to the Yangianization of SKMAs (including the dynamically generated SKM two which SSA effectively reduces by gauge conditions) provided that it induces Yangianization of SVA. Momentum components would be associated with KM generators for M^4 excitations of strings such that only transversal excitations are dynamical.

   For fermionic and bosonic generators of SKMA one would have

$$\begin{aligned} \Delta(F_0^a) &= F_0^a\otimes 1 + 1\times F_0^a \ , \\ (F_1^a) &= F_1^a\otimes 1 + 1\times F_1^a + Kf_a^{Ab}(T_0^A\otimes F_0^b - F_0^b\otimes T_0^A) \ , \\ \Delta(T_0^A) &= T_0^A\otimes 1 + 1\otimes T_0^A \ , \\ \Delta(T_1^A) &= T_1^A\otimes 1 + 1\otimes T_1^A + f_{BC}^A(T_0^B\otimes T_0^C \ . \end{aligned} \tag{13.4.8}$$

   Yangianization of SKMA would introduce interaction terms.

13.5 Category theory as a basic tool of TGD

I have already earlier developed ideas about the role of category theory in TGD [K19, K18, K11]. The hierarchy formed by categories, categories of categories, ... could allow to keep book about the complexity due to various hierarchies. WCW geometry with its huge symmetries combined with adelic physics; quantum states identified in ZEO as WCW spinor fields having topological interpretation as braided fusion categories with reconnection; the local symmetry algebras of quantum TGD extended to Yangians realizing elegantly the construction of interacting many-particle states in terms of iterated Δ operation assigning fundamental interactions to tensor summands of SKMAs: these could be the pillars of the basic vision.

13.5.1 Fusion categories

While refreshing my rather primitive physicist's understanding of categories, I found an excellent representation of fusion categories and braided categories [B7] introduced in topological condensed matter physics. The idea about product and co-product as fundamental vertices is not new in TGD [K11, L10, L45] but the physicist's view described in the article provided new insights.

Consider first fusion categories.

1. In TGD framework scattering diagrams generalize Feynman diagrams in the sense that in 3-vertices the 2-D ends for orbits of 3 partonic 2-surfaces are glued together like the ends of lines in 3-vertex of Feynman diagram. One can say that particles fuse or decay. 3-vertex would be fundamental vertex since higher vertices are unstable against splitting to 3-vertices. Braiding and reconnection would bring in additional topological vertices. Note that reconnection represents basic vertex in closed string theory and appears also in open string theory.

   Also fusions and splittings of 3-surfaces analogous to stringy trouser vertex appear as topological vertices but they do not represent particle decays but give rise to two paths along, which particles travel simultaneously: they appear in the TGD based description of double slit experiment. This is a profound departure from string models.

The key idea is that scattering diagrams are analogous to algebraic computations: the simplest computation corresponds to tree diagram apart from possible braiding and reconnections to be discussed below giving rise to purely topological dynamics. One has a generalization of the duality of the hadronic string model: one does not sum over all diagrams but takes only one of them, most naturally the simplest one. This is highly reminiscent to what happens for twistor Grassmann amplitudes.

One can eliminate all loops by moves and modify the tree diagram by moving lines along lines [?] Scattering diagrams would reduce to tree diagrams having in given vertex either product μ or its time reversal Δ plus propagator factors connecting them. The scattering amplitudes associated with tree diagrams related by these moves were earlier assumed to be identical. With better understanding of fusion categories I realized that the amplitudes corresponding to equivalent computations need not be numerically identical but only unitarily related and in this sense physically equivalent in ZEO.

2. Fusion categories indeed realize algebraically in very simple form the idea that all scattering diagrams reduce to tree diagrams with 3-vertices as basic vertices. Fusion categories [B7] (the illustrations `http://tinyurl.com/l2jsrzc` are very helpful) involve typically tensor product $a\otimes b$ of irreducible representations a and b of an algebraic structure decomposed to irreducible representations c. This product is counterpart for the 3-parton vertex generalizing Feynmanian 3-vertex.

   The article gives a graphical representation for various notions involved and these help enormously to concretize the notions. Fusion coefficients in $a\otimes b = N_{ab}^c c$ must satisfy consistency conditions coming from commutativity and associativity forcing the matrices $(N_a)_{bc} = N_{ab}^c$ to commute. One can diagonalize N_a simultaneously and their largest eigenvalues d_a are so called quantum dimensions. Fusion category contains also identity object and its presence leads to the identification of gauge invariants defining also topological invariants.

   The fusion product $a\otimes b$ has decomposition $V_{ab}^{c\alpha}|c,\alpha\rangle$ for each c. Co-product is an analog of the decay of particle to two particles and product and co-product are inverses of each other in a well-defined sense expressed as an algebraic identities. This gives rise to completeness relations from the condition stating that states associated with various c form a complete basis for states for $a\otimes b$ and orthogonality relations for the states of associated with various c coefficients. Square roots of quantum dimensions d_a appear as normalization factors in the equations.

   Diagrammatically the completeness relation means that scattering $ab\rightarrow c\rightarrow cd$ is trivial. This cannot be the case and the completeness relation must be more general. One would expect unitary S-matrix instead of identity matrix. The orthogonality relation says that loop diagram for $c\rightarrow ab\rightarrow c$ gives identity so that one can eliminate loops.

   Further conditions come from the fact that the decay of particle to 3 particles can occur in two ways, which must give the same outcome apart from a unitary transformation denoted by matrix F (see Eq. (106) of `http://tinyurl.com/l2jsrzc`). Similar consistency conditions for decay to 4 particles give so called pentagon equation as a consistency condition (see Eq. (107) and Fig. 9 of `http://tinyurl.com/l2jsrzc`). These equations are all that is needed to get an internally consistent category.

In TGD framework the fusion algebra would be based on Super Yangian with super Variant of Lie-algebra commutator as product and Yangian co-product of form already discussed and determining the basic interaction vertices in amplitudes. Perhaps the scattering amplitude for a given space-time surface transforming two categories at boundaries of CD to each other could be seen as a diagrammatic representation of category defined by zero energy state.

13.5.2 Braided categories

Braided categories [B7] (see `http://tinyurl.com/l2jsrzc`) are fusion categories with braiding relevant in condensed matter physics and also in TGD.

1. Braiding operation means exchange of braid strands defining particle world-lines at 3-D light-like orbits of partonic 2-surfaces (wormhole throats) defining the boundaries between Minkowskian and Euclidian regions of space-time surface. Braid operation is naturally realized in TGD for fermion lines at orbits of partonic 2-surfaces since braiding occurs in codimension 2.

2. For quantum algebras braiding operation is algebraically realized as R-matrix satisfying YBE (see `http://tinyurl.com/l4z6zyr`). R-matrix is a representation for permutation of two objects represented quantally. Group theoretically the braid group for n-braid system is covering group of the ordinary permutation group.

   In 2-D QFTs braiding operation defines the fundamental $2 \rightarrow 2$ scattering defining R-matrix as a building brick of S-matrix. This scattering matrix is trivial in the sense that the scattering involves only a phase lag but no exchange of quantum numbers: particles just pass by each other in the 2-particle scattering. This kind of S-matrix characterizes also topological quantum field theories used to deduce knot invariants as its quantum trace [A42, A16, A48]. I have considered knots from TGD point of view in [K46] [L7].

3. For braided fusion categories one obtains additional conditions known as hexagon conditions since there are two ways to end up from $1 \rightarrow 3$ fusion diagram involving two 3-vertices and 2 braidings to an equivalent diagram using sliding of lines along lines and braiding operation (see Fig. 10 of `http://tinyurl.com/l2jsrzc`).

13.5.3 Categories with reconnections

Fusion and braiding are not enough to satisfy the needs of TGD.

1. In TGD one does not have just objects - point like particles, whose world lines define braid strands in time direction. One has also the morphisms represented by the strings between the particles. Partonic 2-surfaces are connected by strings and these strings have topological interaction: they can reconnect or just go through each other. Reconnection is in key role in TGD inspired theory of consciousness and quantum biology [K75].

   Reconnection is an additional topological reaction besides braiding and one must assign to it a generalization of R-matrix. Reconnection and going through each other are just the basic operations used to unknot ordinary knots in the construction of knot invariants in topological quantum field theories. Now topological time evolution would be a generalization of this process connecting the knotted and linked structures at boundaries of CD and allowing both knotting and un-knotting.

2. Although 2-knots and braids are difficult to construct and visualize, it seems rather obvious (to me at least) that the reconnections correspond in 4-D space-time surface to basic operations giving rise to 2-knots [A33] - a generalization of ordinary knot that is 1-knot. 2-knots could be seen as a cobordism between 1-knots and this suggests a construction of 2-knot invariants as generalization of that for 1-knots [K46]. 2-knot would be the process transforming 1-knot by re-connections and "going through" the second 1-knot. The trace of the topological unitary S-matrix associated with it would give a knot invariant. If this view is correct, a generalization of TQFT for ordinary braids to include reconnection could give a TQFT for 2-braids with invariants as invariants of knot-cobordism. It must be however emphasized that the identification of 2-braids as knot-cobordisms is only an intuitive guess.

3. From the point of view of braid strands at the ends of strings, reconnection means exchange of braid strands. Composite particles consisting of strands would exchange their building bricks - the analogy with a chemical reaction is obvious and various reactions could be interpreted as knot cobordisms. Since exchange is involved also now, one expects that the generalization of R-matrix to algebraically describe this process should obey the analog of YBE stating that the two braided versions of permutation $abc \rightarrow cba$ are identical.

   If the strings are oriented, one could have YBEs separately for left and right ends such that braid operation would correspond to the exchange of braid between braid pairs. The topological interaction for strings AB and CD could correspond to a) trivial operation "going through" (AB + CD → AB+CD) visible in in the topological intersection matrix characterizing the union of string world sheets, exchanges of either left (AB+CD→ CB+AD) or right ends (AB+CD→ AD+CB), or exchange or right and left ends (AB+CD→ CD+AB) representable as composition of braid operation for string ends and exchange of right or left ends and giving rise to braiding operation for pairs AB and CD.

The following braiding operations would be involved.

(a) Internal braiding operation $A \otimes B \rightarrow B \otimes A$ for string like object.

(b) Braiding operation $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes D) \otimes (A \otimes B)$ for two string like objects.

(c) Reconnection as braiding operation: $(A \otimes B) \otimes (C \otimes D) \rightarrow (A \otimes D) \otimes (C \otimes B)$ and $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes B) \otimes (A \otimes D)$.

I have not found by web search whether this generalization of YBE exists in mathematics literature or whether it indeed reduces to ordinary braiding for the exchanged braids for different options emerging in reconnection. One can ask whether the fusion procedure for R-matrices as an analog for the formation of tensor products already briefly discussed could allow to construct the R-matrix for the reconnection of 2 strings with braids as boundaries.

4. The intersections of braid strands are stable against small perturbations unless one modifies the space-time surface itself (in TGD 2-braids are 2-surfaces inside 4-surfaces). Also the intersections of world lines in M^2 integrable theories are stable. Hence it would be natural to assign analog of R-matrix also to the intersections.

5. Light-like 3-D partonic orbits can contain several fermion lines identifiable as boundaries of string world sheets so that reconnections could induce also more complex reactions in which partonic 2-surfaces exchange fermions. Quite generally one would have braid of braids able to braid and also exchange their constituent braids. This would give rise to a hierarchy of braids within braids and presumably to a hierarchy of categories. This might provide a first principle topological description of both hadronic, nuclear, and (bio-)chemical reactions. For instance, the mysterious looking ability of bio-molecules to find each other in dense molecular soup could rely on magnetic flux tubes (and associated strings) connecting them [K75].

6. Reconnection requires a generalization of various quantum algebras, in particular Yangian, which seems to be especially relevant to TGD since it generalizes local symmetries to multi-local symmetries with locus identifiable as partonic 2-surface in TGD. Since braid strands are replaced with pairs of them, one might expect that the generalization of R-matrix involves two parameters instead of one.

13.6 Trying to imagine the great vision about categorification of TGD

The following tries to summarize the ideas described. This is mostly free play with the ideas in order to see what objects and arrows might be relevant physically and whether category theory might be of help in understanding poorly understood issues related to various hierarchies of TGD.

13.6.1 Different kind of categories

Category theory could be much more than mere book keeping device in TGD. Morphisms and functors could allow to see deep structural similarities between different levels of TGD remaining otherwise hidden.

Geometric and number theoretic categories

There are three geometric levels involved: space-time, CDs at embedding space level, sectors of WCW assignable with CDs their subsectors characterized by a point for moduli space of CDs with second boundary fixed.

There are also number theoretic categories.

1. Adelic physics would define a hierarchy of categories defined by extensions of rationals and identifiable as an evolutionary hierarchy in TGD inspired theory of consciousness. Inclusion of extensions parameterized by Galois group and ramified primes defining preferred p-adic primes would define a functor. The parameters of quantum algebras should be number theoretically universal and belong to the extension of rationals defining the adele in question. Powers or roots of e, roots of unity, and algebraic numbers would appear as building bricks. The larger the p-adic prime p the higher the dimension of extension containing e and possibly also some of its roots, the better the accuracy of the cognitive representation.

2. These inclusions should relate closely to the inclusions of hyperfinite factors of type II_1 assignable to finite measurement resolution [K105]. The measurement resolution at space-time level would characterize the cognitive representation defined in terms of points with embedding space coordinates in the extension of rationals defining the adele. The larger the extension, the larger the cognitive representation and the higher the accuracy of the representation.

   Should the points of cognitive representation be assigned

   (a) only with partonic 2-surfaces (each point of representation is accompanied by fermion)

   (b) or also with the interior of space-time surface (it is not natural to assign fermion to the point unless the point belongs to string world sheet, even in this case this is questionable)?

   Many-fermion states define naturally a tensor product of quantum Boolean algebras at the opposite boundaries of CD in ZEO and the interpretation of time evolution as morphism of quantum Boolean algebras is natural. If cognition is always Boolean then the first option is more plausible.

3. The hierarchy of Planck constants $h_{eff}/h = n$ with $n \leq ord(G)$ naturally the number of sheets and dividing the order $ord(G)$ of the Galois group G of the extension would relate closely to the hierarchy of extensions. n would be dimension of the covering of space-time surface defined by the action of Galois group to space-time sheet. Ramified primes for extensions are in special position for given extension. The conjecture is that p-adic primes near powers of two or more generally of small primes ramified primes for extensions, which are winners in number theoretic fight for survival [L41].

4. The hierarchy of infinite primes [K89] might characterize many-sheeted space-time and leads to a generalization of number concept with infinitely complex number theoretic anatomy provided by infinite rationals, which correspond to real and p-adic units. The inclusion of lower level primes to the higher level primes would define morphism now. One can assign hierarchy of infinite primes with primes of any extension of rationals.

Consciousness and categories

Categories are especially natural from the point of view of cognition. Classification is the basic cognitive function and category is nothing but classification by defining objects as equivalence classes. Morphisms and functors serve as correlates for analogies and would provide the tool of understanding the power of analogies in conscious intelligence. Also attention could involve morphism and its direction would correlate with the direction of attention. Perhaps isomorphism corresponds to the state of consciousness in which the distinction between observer and observed is reported by meditators to cease. Cognitive representations would be provided by adelic physics at both space-time level, embedding space level, and WCW level (the preferred coordinates for WCW would be in extension of rationals defining the adele).

One would have a hierarchy of increasingly complex cognitive representations with inclusions as arrows and their sub-WCWs labelled by moduli of CDs and arrow of geometric time telling which boundary is affected in the sequence of state function reductions defining self as generalized Zeno effect [L46].

13.6.2 Geometric categories

Geometric categories appear at WCW level, embedding space level, and space-time level.

WCW level

The hierarchies formed by the categories defined by the hierarchies of adeles, space-time sheets and hierarchy of CDs would be mapped also to the level of WCW. The preferred coordinates of WCW points would be in extension of rationals defining the adele and one would form inclusion hierarchy. The extension at the level of WCW would induce that at the level of embedding space and space-time surface. Sub-CDs would correspond to sub-WCWs and the moduli space for given CD would correspond to moduli space for corresponding sub-WCWs. The different arrows of embedding space time would correspond to sub-WCW and its time reflection. By the breaking of CP,T, and P the space-time surfaces within time reversed sub-WCWs would not be mere CP, T and P mirror images of each other [L44, L33].

Embedding space level

ZEO emerges naturally at embedding space level and CDs are key notion at this level. Consider next the categories that might be natural in ZEO [K61].

1. Hierarchy of CDs could allow interpretation as hierarchy of categories. Overlapping CDs would define an analog of covering of manifold by open sets: one might speak of atlas with CDs defining conscious maps. Chart maps would be morphisms between different CDs assignable to common pieces of space-time surfaces. These morphisms would be also realized at the level of conscious experience. The sub-CD associated with CD would correspond to mental image defined by sub-self as image of the morphism.

2. Quantum state of single space-time sheet at boundary of CD would define a geometric and topological representation for categories. States at partonic 2-surfaces would be the objects connected by fermionic strings and the associated flux tubes would serve as space-time correlates of attention in TGD inspired theory of consciousness. The arrows represented by fermionic strings would correspond to some morphisms, at least thre Hilbert space isometries defined by entanglement with coefficients in an extension of rationals. Unitary entanglement gives rise to a density matrix proportional to unitary matrix and maximal entanglement in both real and p-adic sense. Much more general entanglement gives rise to maximal entanglement in p-adic sense for some primes.

3. Zero energy states the states at passive boundary would be naturally identifiable as categories. At active boundary quantum superpositions of categories could be in question. Maybe one should talk about quantum categories defined by the superposition of space-time sheets with category assigned with an equivalence class of space-time sheets satisfying the conditions for preferred extremal.

4. One can imagine a hierarchy of zero energy states corresponding to the hierarchy of space-time sheets. One can build zero energy states also by adding zero energy states associated with smaller sub-CDs near the boundaries of CD to get an infinite hierarchy of zero energy states. The interpretation as a hierarchy of reflective levels of consciousness would be natural.

5. Zero energy states would correspond to generalized Feynman diagrams interpreted as unitary functors between initial and final state categories. Scattering diagram would be seen as algebraic computation in a fusion category defined by Yangian. All diagrams would be

reducible to braided tree diagrams with braidings and reconnections. The time evolution between boundaries could be seen as a topological evolution a of tensor net [L23].

Category theory would provide cognitive representations as morphisms. Morphisms would become the key element of physics completely discarded in the existing billiard ball view about Universe: Universe would be like Universal computer mimicking itself at all hierarchy levels. This extends dramatically the standard view about cognition where brain is seen as an isolated seat of cognition.

Space-time level

Many-sheeted space-time is the most obvious application for categorification.

1. Smaller space-time sheets condensed at large space-time surface regarded as categories become objects at the level of larger space-time sheet. Functors between the categories defined by smaller space-time sheets define morphisms between them. Also now fermion lines and flux tubes connecting the condensed space-time sheets to each other via wormhole contacts with flux going along another space-time sheet could define functors. Closed loops involving larger space-time sheets and smaller space-time sheets are needed if monopole flux in question. The loop could visitat smaller space-time sheets.

2. Interactions would reduce to product and co-product. Interaction term in Δ for generalized Yangian would characterize fundamental interactions with dynamically generated SKMAs assignable to SSA as additional interactions. The coupling parameters with Δ assigned to a direct sum of SKMAs would define coupling constants of fundamental interactions. Iteration of the co-product Δ would give rise to a hierarchy of many-particle states. The fact that morphism is in question would map the structure of single particle states to that of many-particle states.

SH would involve a functor mapping the category of string world sheets (and partonic 2-surfaces) to that of space-time surfaces having same points with coordinates in extension of rationals. In p-adic sectors this morphism presumably exists for all p-adic primes thanks to p-adic pseudo-constants. In real sector this need not be the case: all imaginations are not realizable.

The morphisms would be mediated by either continuation of strings world sheets (and partonic 2-surfaces) to space-time interiors (morphism would be analogous to a continuation of holomorphic functions of two complex coordinates from 2-D data at surfaces, where the functions are real). Possible quaternion analyticity [L10] encourages to consider even continuation of 1-D data to 4-D surfaces and twistor lift gives some support for this idea.

In the fermionic sector one must continue induced spinor fields at string world sheets to those at space-time surfaces. The 2-D induced spinor fields could also serve as sources for 4-D spinor fields.

Chapter 14

Are higher structures needed in the categorification of TGD?

14.1 Introduction

I encountered a very interesting work by Urs Schreiber related to so called higher structures and realized that these structures are part of the mathematical language for formulating quantum TGD in terms of Yangians and quantum algebras in a more general way.

14.1.1 Higher structures and categorification of physics

What theoretical physicist Urs Screiber calls "higher structures" are closely related to the categorification program of physics. Baez, David Corfield and Urs Schreiber founded a group blog n-Category Cafe about higher category theory and its applications. John Baez is a mathematical physicists well-known from is pre-blog "This Week's Finds" (see `http://tinyurl.com/yddcabfl`) explaining notions of mathematical physics.

Higher structures or n-structures involve "higher" variants of various mathematical structures such as groups, algebras, homotopy theory, and also category theory (see `http://tinyurl.com/ydz9mbtp`. One can assign a higher structure to practically anything. Typically one loosens some conditions on the structure such as commutativity or associativity: a good example is the product for octonionic units which is associative only apart from sign factors [K91]. Braid groups and fusion algebras [L35], which seem to play crucial role in TGD can be seen as higher structures.

The key idea is simple: replace "=" with homotopy understood in much more general sense than in topology and identified as the procedure proving $A = B$! Physicist would call this operationalism. I would like a more concrete interpretation: "=" is replaced with "=" in a given measurement resolution. Even homotopies can be defined only modulo homotopies of homotopies - that is within measurement resolution - and one obtains a hierarchy of homotopies and at the highest level coherence conditions state that one has "=" almost in the good old sense. This kind of hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adels in adelic TGD, hierarchy of infinite primes, self hierarchy, etc...

14.1.2 Evolution of Schreiber's ideas

One of Schreiber's articles in Physics Forum articles has title "*Why higher category theory in physics?*" (see `http://tinyurl.com/ydcylrun`) telling his personal history concerning the notion of higher category theory. Supersymmetric quantum mechanics and string theory/M-theory are strongly involved with his story.

Wheeler's superspace and its deformations as starting point

Schreiber started with super variant of Wheeler's super-space. Intriguingly, also the "world of classical worlds" (WCW) of TGD [K45, K24, K80] emerged as a counterpart of superspace of Wheeler in which the generalization of super-symmetries is geometrized in terms of spinor structure of WCW expressible in terms of fermionic oscillator operators so that there is something common at least.

Screiber consider deformation theory of this structure. Deformations appear also in the construction of various quantum structures such as quantum groups and Yangians. Both quantum groups characterized by quantum phase, which is root of unity, and Yangians ideal for reduction of many-particle states and their interactions to kinematics seem to be the most important from the TGD point of view [L35].

These deformations are often called "quantizations" but this nomenclature is to my opinion misleading. In TGD framework the basic starting point is "*Do not quantize*" meaning the reduction of the entire quantum theory to classical physics at the level of WCW: modes of a formally classical WCW spinor fields correspond to the states of the Universe.

This does not however prevent the appearance of the deformations of basic structures also in TGD framework and they might be the needed mathematical tool to describe the notions of finite measurement resolution and cognitive resolution appearing in the adelic version of TGD. I proposed more than decade ago that inclusions of hyperfinite factors of II_1 (HFFs) [K105, K36] might provide a natural description of finite measurement resolution: the action of included factor would generate states equivalent under the measurement resolution used.

The description of non-point-like objects in terms of higher structures

Schreiber ends up with the notion of higher gauge field by considering the space of closed loops in 4-D target space [B65]. At the level of target space the loop space connection (1-form in loop space) corresponds to 2-form at the level of target space. At space-time level 1- form A defines gauge potentials in ordinary gauge theory and non-abelian 2-form B as its generalization with corresponding higher gauge field identified as 3-form $F = dB$.

The idea is that the values of 2-form B are defined for a string world sheet connecting two string configuration just like the values of 1-form are defined for a world-line connecting two positions of a point-like particle. The new element is that the ordinary curvature form does not anymore satisfy the usual Bianchi identities stating that magnetic monopole currents are vanishing (see `http://tinyurl.com/ya3ur2ad`).

It however turns out that one has $B = DA = F$ (D denotes covariant derivative) so that B is flat by the usual Bianchi-identities implying $dB = 0$ so that higher gauge field vanishes. B also turns out to be Abelian. In the Abelian case the value of 2-form would be magnetic flux depending only on the boundary of string world sheet. By $dB = 0$ gauge fields in loop space would vanish and only topology of field configurations would make itself manifest as for locally trivial gauge potentials in topological quantum field theories (TQFT): a generalization of Aharonov-Bohm effect would be in question. Schreiber calls this "*fake flatness condition*". This could be seen as an unsatisfactory outcome since dynamics would reduce to topological dynamics.

The assumption that loop space gauge fields reduce to those in target space could be argued to be non-realistic in TGD framework . For instance, high mass excitations of theories of extended structures like strings would be lost. In the case of loop spaces there is also problem with general coordinate invariance (GCI): one would like to have 2-D GCI assignable to string world sheets. In TGD the realization that one must have 4-D GCI for 3-D fundamental objects was a breakthrough, which occurred around 1990 about 12 years after the discovery of the basic idea of TGD and led to the discovery of WCW Kähler geometry and to "Do not quantize".

Understanding "fake flatness" condition

Schreiber tells how he encountered the article of John Baez titled "*Higher Yang-Mills Theory*" [B51] (see `http://tinyurl.com/yagkqsut`) based on the notion of 2-category and was surprised to find that also now the "*fake flatness condition*" emerged.

Schreiber concludes that the "*fake flatness condition*" results from "a kind of choice of coordinate composition": non-Abelian higher gauge field would reduce to Abelian gauge field over a background of ordinary non-Abelian gauge fields. Schreiber describes several string theory related examples involving branes and introduces connection with modern mathematics. Since branes in the stringy sense are not relevant to TGD and I do not know much about them, I will not discuss these here.

However, dimensional hierarchies formed by fermions located to points at partonic 2-surfaces, their world lines at 3-D light-like orbits of partons, strings and string world sheets as their orbits, and space-time surfaces as 4-D orbits of 3-surfaces definitely define a TGD analog for the brane hierarchy of string models. It is not yet completely clear whether strong form of holography (SH) implies that string world sheets and strings provide dual descriptions of 4-D physics or whether one could regard all levels of this hierarchy independent to some degree at least [L31].

Since the motion of measurement resolution is fundamental in TGD [K105, K36], it is interesting to see whether n-structures could emerge naturally also in TGD framework. There is also second aspect involved: various hierarchies appearing in TGD have basically the structure of abstraction hierarchy of statements about statements and higher structures seem to define just this kind of hierarchies. Of course, human mind - at least my mind - is in grave difficulties already with few lowest levels but here category theory and its computerization might come into a rescue.

14.1.3 What higher structures are?

Schreiber describes in very elegant and comprehensible way the notion of higher structures (see `http://tinyurl.com/ydfspcld`). This description is a real gem for a physicists frustrated to the impenetrable formula jungle of the usual mathematical prose. Just the basic ideas and the reader can start to think using his/her own brains. The basic ideas ideas are very simple and general. Even if one were not enthusiastic about the notion of higher gauge field, the notion of higher structure is extremely attractive concerning the mathematical realization of the notion of finite measurement resolution.

1. The idea is to reconsider the meaning of "=". Usually it is understood as equivalence: $A = B$ if A and B belong to same equivalence class defined by equivalence relation. The idea is to replace "=" with its operational definition, with the proof of equivalence. This could be seen as operationalism of physics applied to mathematics. Schreiber calls this proof homotopy identified as a generalization of a map $f_t\colon S \rightarrow X$ depending on parameter $t \in [0, 1]$ transforming two objects of a topological space X to each other in continuous way: $f_0(S)$ is the initial object and $f_1(S)$ is the final object. Now homotopy would be much more general.

2. One can also improve the precision of "=" meaning that equivalence classes decompose to smaller ones and equivalent homotopies decompose to subclasses of equivalent homotopies related by homotopies. One might say that "=" is deconstructed to more precise "=". Physicist would see this as a partial opening of a black box by improving the measurement resolution. This gives rise to n-variants of various algebraic structures.

3. This hierarchy would have a finite number of levels. At highest level the accuracy would be maximal and "=" would have almost its usual meaning. This idea is formulated in terms of coherence conditions. Braiding involving R-matrix represents one example: permutations are replaced by braidings and permutation group is lifted to braid group but associativity still holds true for Yang-Baxter equation (YBE). Second example is 2-group for which associativity holds true only modulo homotopy so that $(x \circ y) \circ z$ is related to $x \circ (y \circ z)$ by homotopy $a_{x,y,z}$ depending on x, y, z and called an associator. For 2-group the composite homotopy $((w \circ x) \circ y) \circ z \rightarrow w \circ (x \circ (y \circ z))$ is however unique albeit non-trivial.

   This gives rise to the so called pentagon identity encountered also in the theory of quantum groups and Yangians. The outcome is that all homotopies associated with re-bracketings of an algebraic expression are identical. One can define in similar way n-group and formally even infinity-group.

14.1.4 Possible applications of higher structures to TGD

Before listing some of the applications of higher structures imaginable in TGD framework, let us summarize the basic principles.

1. Physics as WCW geometry [K96, K45, K24, K80] having super-symplectic algebra (SSA) and partonic super-conformal algebra (PSCA) as fundamental symmetries involving a generalization of ordinary conformal invariance to that for light-like 3-surfaces defined by the boundary of CD and by the light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian.

2. Physics as generalized number theory [K67] [L41] leading to the notion of adelic physics with a hierarchy of adeles defined by the extensions of rationals.

3. In adelic physics finite resolutions for sensory and cognitive representations (see the glossary of Appendix) could would characterize "=". Hierarchies of resolutions meaning hierarchies of n-structures rather than single n-structure would give inclusion hierarchies for HFFs, SSA, and PSCA, and extensions of rationals characterized by Galois groups with order identifiable as $h_{eff}/h = n$ and ramified primes of extension defining candidates for preferred p-adic primes.

   Finite measurement resolution defined by SSA and its isomorphic sub-algebra acting as pure gauge algebra would reduce SSA to finite-dimensional SKMA. WCW could become effectively a coset space of Kac-Moody group or of even Lie group associated with it. Same would take place for PSCA. This would give rise to n-structures. Quantum groups and Yangians would indeed represent examples of n-structures.

   In TGD the "conformal weight" of Yangian however corresponds to the number of partonic surfaces - parton number - whereas for quantum groups and Kac-Moody algebras it is analogous to harmonic oscillator quantum number n, which however has also interpretation as boson number. Maybe this co-incidence involves something much deeper and relates to quantum classical correspondence (QCC) remaining rather mysterious in quantum field theories (QFTs).

4. An even more radical reduction of degrees of freedom can be imagined. Cognitive representations could replace space-time surfaces with discrete structures and points of WCW could have cognitive representations as disretized WCW coordinates.

5. Categorification requires morphisms and homomorphisms mapping group to sub-group having normal sub-group defining the resolution as kernel would define "resolution morphisms". This normal sub-group principle would apply quite generally. One expects that the representations of the groups involved are those for quantum groups with quantum phase q equal to a root of unity.

Some examples helps to make this more concrete.

Scattering amplitudes as computations

The deterministic time devolution connecting two field patterns could define analog of homotopy in generalized sense. In TGD framework space-time surface (preferred extremals) having 3-D space-like surfaces at the opposite boundaries of causal diamond (CD) could therefore define analog of homotopy.

1. Preferred extremal defines a topological scattering diagram in which 3-vertices of Feynman diagram are replaced with partonic 2-surfaces at which the ends of light-like orbits of partonic 2-surfaces meet and fermions moving along lines defined by string world sheets scatter classically, and are redistributed between partonic orbits [L10, L24, L45]. Also braidings and reconnections of strings are possible. It is important to notice that one does not sum over these topological diagrams. They are more like possible classical backgrounds.

   The conjecture is that scattering diagrams are analogous to algebraic computations so that one can find the shortest computation represented by a tree diagram. Homotopy in the roughest sense could mean identification of topological scattering diagrams connecting two states at boundaries of CD and differing by addition of topological loops. The functional integral in WCW is proposed to trivialize in the sense that loop corrections vanish as a manifestation of quantum criticality of Kähler coupling strength and one obtains an exponent of Kähler function which however cancels in scattering amplitudes if only single maximum of Kähler function contributes.

2. In the optimal situation one could eliminate all loops of these diagrams and also move line ends along the lines of diagrams to get tree diagrams as representations of scattering diagrams. Similar conditions hold for fusion algebras. This might however hold true only in the minimal resolution. In an improved measurement resolution the diagrams could become more complex. For instance, one might obtain genuine topological loops.

3. The diagrams and state spaces with different measurement resolutions could be related by Hilbert space *isometries* but would not be unitarily equivalent: Hilbert space isometries are also defined by entanglement in tensor nets [L23]. This would give an n-levelled hierarchy of higher structures (rather than single n-structure!) and at the highest level with best resolution one would have coherence rules. Generalized fusion algebras would partially realize this vision. In improved measurement resolution the diagrams would not be identical anymore and equivalence class would decompose to smaller equivalence classes. This brings in mind renormalization group equations with cutoff.

4. Intuitively the improvement of the accuracy corresponds to addition of sub-CDs of CDs and smaller space-time sheets glued to the existing space-time sheets.

Zero energy ontology (ZEO)

In ZEO [K61] "=" could mean the equivalence of two zero energy states indistinguishable in given measurement resolution. Could one say that the 3-surfaces at the ends of space-time surface are equivalent in the sense that they are connected by preferred extremal and have thus same total Noether charges, or that entangled many-fermion states at the boundaries of CD correspond to quantal logical equivalences (fermionic oscillator algebra defines a quantum Boolean algebra)?

In the case of zero energy states "=" could tolerate a modification of zero energy state by zero energy state in smaller scale analogous to a quantum fluctuation in quantum field theories (QFTs). One could add to a zero energy state for given CD zero energy states associated with smaller CDs within it.

In TGD inspired theory of consciousness [L46] sub-CDs are correlates for the perceptive fields of conscious entities and the states associated with sub-CDs would correspond to sub-selves of self defining its mental images. Also this could give rise to hierarchies of n-structures with n characterizing the number of CDs with varying sizes. An interesting proposal is the distance between the tips of CD is integer multiple of CP_2 for number theoretic reasons. Primes and primes near powers of 2 are suggested by p-adic length scale hypothesis [K52, K58, K59] [L41].

"World of classical worlds" (WCW)

At the level of "world of classical worlds" (WCW) "=" could have both classical meaning and meaning in terms of quantum state defining the measurement resolution. At the level of WCW geometry n-levelled hierarchies formed by the isomorphic sub-algebras of SSA and PSCA are excellent candidates for n-structures. The sub-SCA or sub-PSCA would define the measurement resolution. The smaller the sub-SSA or sub-PSCA, the better the resolution.

This could correspond to a hierarchy of inclusions of HFFs [K105, K36] to which one can assign ADE SKMA by McKay correspondence or its generalization allowing also other Lie groups suggested by the hierarchy of extensions of rationals with Galois groups that are groups of Lie type. The conjecture generalizing McKay correspondence is that the Galois group Gal is representable as a subgroup of G in the case that it is of Lie type.

An attractive idea is that WCW is effectively reduced to a finite-dimensional coset space of the Kac-Moody group defined by the gauge conditions. Number theoretic universality requires that these parameters belong to the extension of rationals considered so that the Kac-Moody group G is discretized and also homotopies are discretized. SH raises the hope that it is enough to consider string world sheets with parameters (WCW coordinates) in the extension of rationals.

One can define quite concretely the action of elements of homotopy groups of Kac-Moody Lie groups G on space-time surfaces as induced action changing the parameters characterizing the space-time surface. $n+1$-dimensional homotopy would be 1-dimensional homotopy of n-dimensional homotopy. Also the spheres defining homotopies could be discretized so that the coordinates of its points would belong to the extension of rationals.

These kind of homotopy sequences could define analogs of Berry phases (see `http://tinyurl.com/yd4agwnt`) in Kac-Moody group. Could gauge theory for Kac-Moody group give an approximate description of the dynamical degrees of freedom besides the standard model degrees of freedom? This need not be a good idea. It is better to base the considerations of the physical picture provided by TGD. I have however discussed the TGD analog of the *fake flatness condition* in the Appendix.

Adelic physics

Also number theoretical meaning is possible for "=". It is good to start with an objection against adelic physics. The original belief was that adelic physics forces preferred coordinates. Indeed, the property of belonging to an extension of rationals does not conform with general coordinate invariance (GCI). Coordinate choice however matters cognitively as any mathematical physicist knows! One can therefore introduce preferred coordinates at the embedding space level as cognitively optimal coordinates: they are dictated to a high degree by the isometries of H. One can use a sub-set of these coordinates also for space-time surfaces, string world sheets, and partonic 2-surfaces.

1. Space-time surfaces can be regarded as multi-sheeted Galois coverings of a representative sheet [L41]. Minimal resolution means that quantum state is Galois singlet. Improving resolution means requiring that singlet property holds true only for normal sub-group H of Galois group Gal and states belong to the representations of Gal/H. Maximal resolution would mean that states are representations of the entire Gal. The hierarchy of normal sub-groups of Gal would define a resolution hierarchy and perhaps an analog of n-structure. $h_{eff}/h = n$ hypothesis suggests hierarchies of Galois groups with dimensions n_i dividing n_{i+1}. The number of extensions in the hierarchy would characterize n-structure.

2. The increase of the complexity for the extension of rationals would bring new points in the *cognitive representations* defined by the points of the space-time surface with embedding space coordinates in the extension of rationals used (see the glossary in Appendix). Also the size of the Gal would increase and higher-D representations would become possible. The value of $h_{eff}/h = n$ identifiable as dimension of Gal would increase. The cognitive representation would become more precise and the topology of the space-time surface would become more complex.

3. In adelic TGD "=" could have meaning at the level of cognitive representations. One could go really radical and ask whether discrete cognitive representations replacing space-time surfaces with the set of points with H-coordinates in an extension of rationals (see the glossary in Appendix) defining the adele should provide the fundamental data and that all group representations involved should be realized as representations of Gal. This might apply in cognitive sector.

   This would also replace space-time surfaces as points of WCW with their cognitive representations defining their WCW coordinates! All finite groups can appear as Galois groups for some number field. Whether this is case when one restricts the consideration to the extensions of rationals, is not known. Most finite groups are groups of Lie type and thus representable as rational points of some Lie group. Note that rational point can also mean rational point in extension of rationals as ratio of corresponding algebraic integers identifiable as roots of monic polynomials $P_n(x) = x^n + ...$ having rational coefficients.

4. By SH space-time surface would in information theoretic sense effectively reduce to string world sheets and even discrete set of points with H-coordinates in extension of rationals. These points could even belong to the partonic 2-surface at the ends of strings at ends of CD carrying fermions and the partonic 2-surfaces defining topological vertices. If only this data is available, the WCW coordinates of space-time surface would reduce to these points of $H = M^4 \times CP_2$ and to the direction angles of strings emerging from these points

and connecting them to the corresponding points at other partonic 2-surfaces besides Gal identifiable as sub-group of Lie group G of some Kac-Moody group! Not all pairs $Gal - G$ are possible.

5. Could these data be enough to describe mathematically what one knows about space-time surface as point of WCW and the physics? One could indeed deduce $h_{eff}/h = n$ as the order of Gal and preferred p-adic primes as ramified primes of extension. The Galois representations acting on the covering defining space-time surface or string world sheets should be identifiable as representations of physical states. There is even number theoretical vision about coupling constant evolution relying on zeros of Riemann zeta [L17],

6. This sounds fine but one must notice that there is also the global information about the conformal moduli of partonic 2-surfaces and the elementary particle vacuum functionals defined in this moduli space [K21] explain family replication phenomenon. There is also information about moduli of CDs. Also the excitations of SKMA representations with higher conformal weights are present and play a crucial role in p-adic thermodynamics predicting particle masses [K52]. It is far from clear whether the approach involving only cognitive representation is able to describe them.

To help the reader I have included a vocabulary at the end of the article and include here a list of the abbreviations used in the text.

General abbreviations: Quantum field theory (QFT); Topological quantum field theory (TQFT); Hyper-finite factor of type II_1 (HFF); General coordinate invariance (GCI); Equivalence Principle (EP).

TGD related abbreviations: Topological Geometrodynamics (TGD); General Relativity Theory (GRT); Zero energy ontology (ZEO); Strong form of holography (SH); Strong form of general coordinate invariance (SGCI); Quantum classical correspondence (QCC); Negentropy Maximization Principle (NMP); Negentropic entanglement (NE); Causal diamond (CD); Super-symplectic algebra (SSA); Partonic superconformal algebra (PSCA); Super Virasoro algebra (SVA); Kac-Moody algebra (KMA); Super-Kac-Moody algebra (SKMA);

14.2 TGD very briefly

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K96] and physics as generalized number theory [K67]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

14.2.1 World of classical worlds (WCW)

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K96] and physics as generalized number theory [K67]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

Construction of WCW geometry briefly

In the following the vision about physics in terms of classical physics of spinor fields of WCW is briefly summarized.

1. The idea is to geometrize not only the classical physics in terms of geometry of space-time surfaces but also quantum physics in terms of WCW [K80]. Quantum states of the Universe would be modes of classical spinor fields in WCW and there would be no quantization. One must construct Kähler metric and Kähler form of WCW: in complex coordinates they differ by a multiplicative imaginary unit. Kähler geometry makes possible to geometrize hermitian conjugation fundamental for quantum theory.

2. One manner to build WCW metric this is via the construction of gamma matrices of WCW in terms of second quantized oscillator operators for fermions described by induced spinor fields at space-time surfaces. By strong form of holography this would reduce to the construction

of second quantized induced spinor fields at string world sheets. The anti-commutators of of WCW gamma matrices expressible in terms of oscillator operators would define WCW metric with maximal isometry group (SCA) [K106, K80].

3. Second manner to achieve the geometrization is to construct Kähler metric and Kähler form directly [K45, K24, K80]. The idea is to induce WCW geometry from the Kähler form J of the embedding space $H = M^4 \times CP_2$. The mere existence of the Riemann connection forces a maximal group of isometries. In fact, already in the case of loop space the Kähler geometry is essentially unique.

   The original construction used only the Kähler form of CP_2. The twistor lift of TGD [L45] forces to endow also M^4 with the Minkowskian analog of Kähler form involving complex and hypercomplex part and the sum of the two Kähler forms can be used to define what might be called flux Hamiltonians. They would define the isometries of WCW as symplectic transformations. What was surprising and also somewhat frustrating was that what I called almost 2-dimensionality of 3-surfaces emerges from the condition of general coordinate invariance and absence of dimensional parameters apart from the size scale of CP_2.

   In the recent formulation this corresponds to SH: 2-D string world sheets and 2-D partonic 2-surfaces would contain data allowing to construct space-time surfaces as preferred extremals. In adelic physics also the specification of points of space-time surface belonging to extension of rationals defining the adele would be needed. There are several options to consider but the general idea is clear.

   SH is analogous to a construction of analytic function of 2-complex from its real values at 2-D surface and the analogy at the level of twistor lift is holomorphy as generalization of holomorphy of solutions gauge fields in the twistor approach of Penrose. Also quaternionic analyticity [L10] is suggestive and might mean even stronger form of holography in which 1-D data allow to construct space-time surfaces as preferred extremals and quantum states.

   I have proposed formulas for the Kähler form of WCW in terms of flux Hamiltonians but the construction as anti-commutators of gamma matrices is the more convincing definition. Fermions and second quantize induced spinor fields could be an absolutely essential part of WCW geometry.

4. WCW allows as infinitesimal isometries huge super-symplectic algebra (SSA) [K45, K24] acting on space-like 3-surfaces at the ends of space-time surfaces inside causal diamond (CD) and also generalization of Kac-Moody and conformal symmetries acting on the 3-D light-like orbits of partonic 2-surfaces (partonic super-conformal algebra (PSCA)). These symmetry algebras have a fractal structure containing a hierarchy of sub-algebras isomorphic to the full algebra. Even ordinary conformal algebra with non-negative conformal weights has similar fractal structure as also Yangian. In fact, quantum algebras are formulated in terms of these half algebras.

   The proposal is that sub-algebra of SSA (with non-negative conformal weights) and isomorphic to entire SSA and its commutator with the full algebra annihilate the physical states. What remains seems to be finite-D Kac-Moody algebra as an effective "coset" algebra obtained. Note that the resulting normal sub-group is actually quantum group.

   There is direct analogy with the decomposition of a group Gal to a product of sub-group and normal sub-group H. If the normal sub-group H acts trivially on the representation the representation of Gal reduces to that of the group Gal/H. Now one works at Lie algebra level: Gal is replaced with SSA and H with its sub-algebra with conformal weights multiples of those for SSA.

Super-symplectic conformal weights, zeros of Riemann zeta, and quantum phases?

In [L17] I have considered the possibility that the generators of super-symplectic algebra could correspond to zeros $h = 1/2 + iy$ of zeta. The hypothesis has several variants.

1. The simplest variant is that the non-trivial zeros of zeta are labelling the generators of SSA associated with Hamiltonians proportional to the functions $f(r_M)$ of the light-like radial

coordinate of light-cone boundary as $f(r_M) = (r_M/_0)^h \equiv exp(hu)$, $u = log(r_M/r_0)$, $h = -1/2 + iy$. For infinitely large size of CD the plane waves are orthogonal but for finite-sized CD orthogonality is lost. Orthogonality requires periodic boundary conditions and these are simultanwously possible only for a finite number of zeros of zeta.

2. One could modify the hypothesis by allowing superpositions of zeros of zeta but with a subtraction of half integer to make the real part of ih equal to 1/2 so that one obtains an analog of plane-wave when using $u = log(r_M/r_0)$ as a radial coordinate. Equivalently, one can take dr_M/r_M out as integration measure and assume $h = iy$ plus the condition that the Riemannian plane waves are orthogonal and satisfy periodic boundary conditions for the allowed zeros $z = 1/2 + iy$.

3. Periodic boundary conditions can be satisfied for given zero of zeta if the condition $r_{max}/r_{min} = p^n$ holds true and the additional conjecture that given non-trivial zeros of zeta correspond to prime $p(y)$ and p^{iy} is a root of unity. Given basis of $f(r_M)$ would correspond to n-ary p-adic length scales and also the size scales of CDs would correspond to powers of p-adic primes. This conjecture is rather attractive physically and I have not been able to prove it wrong.

   One can associate to given zero $z = 1/2+iy$ single and only single prime $p(y)$ by demanding that $p^{iy} = exp(i2\pi q)$, $q = m/n$ rational, implying $log(p)y = 2\pi q$. If there were two primes p_1 and p_2 of this kind, one one ends up with contradiction $p_1^m = p_2^n$ for some integers m and n.

   One could however associate several zeros $y_i(p)$ to the same prime p as discussed in [L17]. If $N = \prod_i n_i$ is the smallest common denominator of q_i allowed conformal weights would be superpositions $ih = iN \sum n_i y_i(p)$ and conformal weights would form higher dimensional lattice rather than 1-D lattice as usually. If only single prime $p(y)$ can be associated to given y, then the original hypothesis identifying $h = 1/2+iy$ as conformal weight would be natural.

4. The understanding of the p-adic length scale hypothesis is far from complete and one can ask whether preferred p-adic primes near powers of 2 and possibly also other small primes could be primes for which there are several roots $y_i(p)$.

14.2.2 Strong form of holography (SH)

There are several reasons why string world sheets and partonic 2-surfaces should code for physics. One reason for SH comes from $M^8 - H$ correspondence [K104]. Second motivation comes from the condition that spinor modes at string world sheets are eigenstates of em charge [K106]. The third reason could come the requirement that the notion of commutative quantum sub-manifold [A20] is equivalent with its number theoretic variant.

SH and $M^8 - H$ correspondence

The strongest form of $M^8 - H$ correspondence [K91, K104, L45] assumes that the 4-surfaces $X^4 \subset M^8$ have fixed $M^2 \subset M^4 \subset M^8$ as part of tangent space. A weaker form states that these 2-D subspaces M^2 define an integrable distribution and therefore 2-D surface in M^4. This condition guarantees that the quaternionic (associative) tangent space of X^4 is parameterized by a point of CP_2 so that the map of X^4 to a 4-surface in $M^4 \times CP_2$ is possible. One can consider also co-associative space-time surfaces having associative normal spaces. m Note that $M^8 - H$ [K91, K104] correspondence respects commutativity and quaternionic property by definition since it maps space-time surfaces having quaternionic tangent space having fixed M^2 as sub-set of tangent space.

What could be the relationship between SH and $M^8 - H$ correspondence? Number theoretic vision suggests rather obvious conjectures.

1. Could the tangent spaces of string world sheets in H be commutative in the sense of complexified octonions and therefore be hyper-complex in Minkowskian regions. By $M^8 - H$ duality the commutative sub-manifolds would correspond to those of octonionic M^8 and finding of these could be the first challenge. The co-commutative manifolds in quaternionic X^4 would have commutative normal spaces. Could they correspond to partonic 2-surfaces?

2. There is however a delicacy involved. Could world sheets and partonic 2-surfaces correspond to hyper-complex and co-hyper-complex sub-manifolds of space-time surface X^4 identifiable as quaternionic surface in octonionic M^8 mappable to similar surfaces in H. Or could their M^4 (CP_2) projections define hypercomplex (co-hypercomplex) 2-manifolds?

3. Could co-commutativity condition for a foliation by partonic 2-surfaces select preferred string world sheets as normal spaces integrable to 2-surfaces identifiable as string world sheets? Note that induced gauge field on 2-surface is always Abelian so that QFT and number theory based views about commutativity co-incide.

   Preferred choices for these 2-surfaces would serve as natural representatives for the equivalence classes of string world sheets and partonic 2-surfaces with fermions at the boundaries of string world sheets serving as markers for the representatives? The end points of the string orbits would belong to extension of rationals or even correspond to singular points at which the different sheets co-incide and have rational coordinates: this possibility was considered in [L48].

Real curves correspond to the lowest level of the dimensional hierarchy of continuous surfaces. Could string world lines along light-like partonic orbits correspond to real sub-manifolds of octonionic M^8 mapped to $M^4 \times CP_2$ by $M^8 - H$ correspondence and carrying fermion number?

What about the set of points with coordinates in the extension of rationals? Do all these points carry fermion number? If so they must correspond to the edges of the boundaries of string world sheets at partonic 2-surfaces at the boundaries of CD or edges at the partonic 2-surfaces defining generalized vertices to which sub-CDs could be assigned.

Well-definedness of em charge forces 2-D fundamental objects

The proposal has been that the representative string world sheets should have vanishing induced W fields so that induced spinors could have well-defined em and Z^0 charges and partonic 2-surfaces would correspond to the ends of 3-D boundaries between Euclidian and Minkowskian space-time regions [K106, K80].

As a matter of fact, the projections of electroweak gauge fields to 2-D surfaces are always Abelian and by using a suitable $SU(2)_L \times U(1)$ rotation one can always find a gauge in which the induced W fields and even Z^0 field vanish. The highly non-trivial conclusion is that string world sheets as fundamental dynamical objects coding 4-D physics by SH would guarantee well-definedness of em charge as fermionic quantum number. Also the projections of all classical color gauge fields, whose components are proportional to $H^A J$, where H^A is color Hamiltonian and J is Kähler form of CP_2, are Abelian and in suitable gauge correspond to hypercharge and isospin.

One can imagine a foliation of space-time surfaces by string world sheets and partonic 2-surfaces. Could there be a $U(1)$ gauge invariance allowing to chose partonic 2-surfaces and string world sheets arbitrarily? If so, the assignment of the partonic 2-surfaces to the light-like boundaries between Minkowskian and Euclidian space-time regions would be only one - albeit very convenient - choice. I have proposed that this choice is equivalent with the choice of complex coordinates of WCW. The change of complex coordinates would introduce a $U(1)$ transformation of Kähler function of WCW adding to it a real part of holomorphic function and of Kähler gauge potential leaving the Kähler form and Kähler metric of WCW invariant.

String world sheets as sub-manifolds of quantum spaces for which commuting sub-set of coordinates are diagonalized?

The third notion of commutativity relates to the notion of non-commutative geometry. Unfortunately, I do not know much about non-commutative geometry.

1. Should one follow Connes [A20] and replace string world sheets with non-commutative geometries with quantum dimension identifiable as fractal dimension. I must admit that I have felt aversion towards non-commutative geometries. For linear structures such as spinors the quantum Clifford algebra looks natural as a "coset space" obtained by taking the orbits of included factor as elements of quantum Clifford algebra. The application of this idea to string world sheets does not look attractive to me.

2. The basic reason for my aversion is that non-commutative quantum coordinates lead to problems with general coordinate invariance (GCI). There is however a possible loophole here. One can approach the situation from two angles: number theoretically and from the point view of non-commutative space. Commutativity could mean two things: number theoretic commutativity and commutativity of quantum coordinates for H seen as observables. Could these two meanings be equivalent as quantum classical correspondence (QCC) encourages to think?

   Could the discreteness for cognitive representations correspond to a discretization of the eigenvalue spectrum of the coordinates as quantum operators? The choice of the coefficient number field for Hilbert space as extension of rationals would automatically imply this and resolve the problems related to continuous spectra.

   Quantum variant of string world sheet could correspond to a quantization using a sub-set of embedding space coordinates as quantum commutative coordinates as coordinates for string world sheet. H-coordinates for string world sheet would correspond to eigenvalues of commuting quantum coordinates.

The above three views about SH suggests that Abelianity at the fundamental level is unavoidable because basic observable objects are 2-dimensional. This would correspond $A = J = -B = 0$ for non-Abelian gauge fields reducing to Abelian ones in Schreiber's approach. Also Schreiber finds that with suitable choice of coordinates this holds true always. In TGD this choice would correspond to gauge choice in which all induced gauge fields are Abelian (see Appendix).

Ordinary twistorialization maps points of M^4 to bi-spinors allowing quantum variants. Could twistorialization of M^4 and CP_2 allow something analogous?

14.3 The notion of finite measurement resolution

Finite measurement resolution [K105, K36] is central in TGD. It has several interpretations and the challenge is to unify the mutually consistent views.

14.3.1 Inclusions of HFFs, finite measurement resolution and quantum dimensions

Concerning measurement resolution the first proposal was that the inclusions of HFFs chacterize it.

1. The key idea is simple. Yangians and/or quantum algebras associated with the dynamical SKMAs defined by pairs of SSA and its isomorphic sub-algebra acting as pure gauge transformations are characterized by quantum phases [L35] characterizing also inclusions of HFFs [K105, K36]. Quantum parameter would characterize the measurement resolution.

   The Lie group characterizing SKMA would be replaced by its quantum counterpart. Quantum groups involve quantum parameter $q \in C$ involved also with n-structures. This parameter - in particular its phase- should belong to the extension of rationals considered. Notions like braiding making sense for 2-D structures are crucial. Remarkably, the representation theory for quantum groups with q different from a root of unity does not differ from that for ordinary groups. For the roots of unity the situation is different.

2. The levels in the hierarchy of inclusions for HFFs [K105] are labelled by integer $n \in [3, \infty)$ or equivalenly by quantum phases $q = exp(i\pi/n)$ and quantum dimension is given by $d_q = 4cos^2(\pi/n)$. $n = 3$ gives $d = 2$ that is ideal SH with minimal measurement resolution. For instance, in extension of rationals only phases, which are powers of $exp(i\pi/3)$ are represented p-adically so that angle measurement is very imprecise. The hierarchy would correspond to an increasing measurement resolution and at the level $n \to \infty$ one would have $d_q \to 4$. Could the interpretation be that one sees space-time as 4-dimensional? This strongly suggests that the hierarchy of Lie groups characterizing SKMAs are characterized by the same quantum phase as inclusions of HFFs.

How does quantal dimension show itself at space-time level?

1. Could SH reduce the 4-surfaces to effectively fractal objects with quantum dimension d_q? Could one speak of quantum variant of SH perhaps describe finite measurement resolution. In adelic picture this limit could correspond to an extension of rational consists of algebraic numbers extended by all rational powers of e. How much does this limit deviate from real numbers?

2. McKay correspondence (see `http://tinyurl.com/z48d92t`) states that the hierarchy of finite sub-groups of $SU(2)$ corresponds to the hierarchy ADE Kac-Moody algebras in the following sense. The so called McKay graph codes for the information about the multiplicities of the tensor products of given representation of finite group (spin 1/2 doublet) - obviously one can assign McKay graph to any Galois group. McKay correspondence says that the McKay graph for the so called canonical representation of finite sub-group of $SU(2)$ co-incides with the Dynkin diagram for ADE type Kac-Moody algebra.

3. A physically attractive idea is that these algebras correspond to a hierarchy of reduced SSAs and PSCAs defined by the gauge conditions of SSA and PSCA. The breaking of maximal effective gauge symmetry characterizing measurement resolution to isomorphic sub-algebra would bring in additional degrees of freedom increasing the quantum dimension of string world sheets from the minimal value $d_q = 2$.

   My naïve physical intuition suggests that McKay correspondence generalizes to a much wider class of Galois groups identifiable as finite groups of Lie type identifiable as sub-groups of Lie groups (for the periodic table of finite groups see (see `http://tinyurl.com/y75r68hp`)). In general, the irreducible representation (irrep) of group is reducible representation of subgroup. The rule could be that the representations of the quantum Lie groups *allowed* as ground states of SKMA representations are *irreducible* also as representations of Galois group in case that it is Lie-type subgroup.

What about the concrete geometric interpretation of d_q? Two interpretations, which do not exclude each other, suggest themselves.

1. A very naïve idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. The idea about fractal string world sheets does not however conform with the fact that preferred extremals must be rather smooth.

   String world sheets could be however locally smooth if they define an analog of discretization for the space-time surface. At the limit $d_q \to 4$ string world sheets would fill space-time surface. Analogously, strings (string orbits) would fill the space-like 3-surfaces at the boundaries of CD (the light-like 3-surfaces connecting the partonic 2-surfaces at boundaries of CD). The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution at the level of physics. For anyonic systems [K72] one indeed would have have large number of fermions at 2-D surfaces.

2. An alternative idea is that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. A precise mathematical representation of ignorance can be extremely successful trick as ordinary thermodynamics and also p-adic thermodynamics for particle masses [K52] demonstrate!

14.3.2 Three options for the identification of quantum dimension

The quantum dimension would increase as the measurement accuracy increases but what quantum dimension of string world sheets could mean at space-time level? Identification of quantum dimension as fractal dimension could be the answer but how could one concretely define this notion? Could one find an elegant formulation for the fractality at space-time level.

Option I

One could argue that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. One would give up the attempts to represent quantum superposition of space-time surfaces with single classical surface. This option would use only the discrete cognitive representations (see the glossary in Appendix).

1. This would mean a radical simplification and could make sense for cognitive representations. String world sheet would be replaced by this discrete cognitive representation and one should be able to deduce its quantum dimension. *Gal* acts on this representation.

2. Could one imagine q-variants of the representations of *Gal* defining also representations of the Lie group defining KMA? If one can imbed *Gal* to Lie-group as discrete sub-group then the q-representation of the Lie-group would define a q-representation of discrete group and one might be able to talk about q-Galois groups.

3. On the other hand, the condition that these representations restricted to representations of Galois group remain irreducible poses similar condition. Are these two criteria equivalent? Could this allow to identify the value of root of unity associated with given Galois group and corresponding Lie group defining SKMA in case that it contains representations that remain irreps of Galois group? If so, the notion of quantum group would follow from adelic physics in a natural manner.

   This would allow to assign quantum dimension to the discretized string world sheet without clumsy fractal constructions at space-time level involving a lot of redundant information. The really nice thing would be that one would use only the information defining the cognitive representations and the fact that one does not know about the rest. Just as in thermodynamics, things would become extremely simple!

4. One might argue that giving just discrete points at partonic 2-surfaces gives very little information. If one however assumes that also the functions characterizing space-time surfaces as points of sub-WCW involved are constructed from rational polynomials with roots in the extension of rationals used, the situation improves dramatically.

Option II

A very naïve idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. Smooth strings would fill the space-like 3-surfaces at the boundaries of CD and light-like 3-surface connecting the partonic 2-surfaces at boundaries of CD. The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution. For anyonic systems one indeed would have have large number of fermions at 2-D surfaces.

This option would be based on fractal dimension of some kind. Most naturally the fractal dimension would be that of space-time surface discretized using string world sheets and possibly also partonic 2-surface instead of points. It is however difficult to imagine a practical realization for fractal dimension in this sense.

1. Assume reference string world sheets in the minimal resolution defined by an extension of rationals with total area S_0. Study the total area S associated with string world sheets as function of the extension of rationals.

2. As the size of the extension grows, new points of extension emerge at partonic 2-surfaces and therefore also new string world sheets and the total area of string worlds sheets increases. Twistor lift suggests that one can take the area S_1 defined by Planck length squared and the area S_2 of CP_2 geodesic sphere as units. Suppose that one has $S/S_0 = (S_1/S_2)^d$, where d depends on the extension and equals to $d = 0$ for rationals, holds true. Could $d+2$ define the fractal dimension equal to d_q for Jones inclusions in the range $[2, 4)$? If the proposed notion of quantum Galois group makes sense this could be the case.

One must admit that the hopes of proving this picture works in practice are rather meager. Too much redundant information is involved.

Option III

One can also imagine an approach quantum dimension identifying quantum dimension as fractal dimension for space-time surface. If SH makes sense, one can consider the possibility that this dimension determined by the geometry of space-time surface as Riemann manifold has fractal dimension equal to the fractal dimension of string world sheets as sub-manifold.

1. The spectral dimension of classical geometry is discussed in `http://tinyurl.com/yadcmjd6`). One considers heat equation describing essentially random walk in a given metric and constructs so called heat kernel as a solution of the heat equation. The Laplacian depends on metric only - now the induced metric. The trace of heat kernel characterizes the probability to return to the original position. The derivative of the logarithm of the heat trace with respect to the logarithm of fictive time coordinate gives time dependent spectral dimension, which for short times approaches to topological dimension and for flat space equals to it always. For long times the dimension is smaller than the topological dimension due to curvature effects and SH raises the hope that this dimension corresponds to the fractal dimension of string world sheets identified as quantum dimension.

2. This approach can be criticized for the introduction of fictive time coordinate. Furthermore, Laplacian would be replaced with d'Alembertian in Minkowskian regions so that one cannot speak about diffusion anymore. Could one replace the heat equation with 4-D spinor d'Alembertian or modified Dirac operator so that also the induced gauge fields would appear in the equation? Artificial time coordinate would be replaced with some time coordinate for M^4 - light-cone proper time is the most natural choice. The probability would be defined as modulus squared for the fermionic propagator integrated over space-time surface.

   The problem is that this approach is rather formal and might be of little practical value.

14.3.3 n-structures and adelic physics

TGD involves several concepts, which could relate to n-structures. The notion of finite measurement resolution realized in terms of HFFs is the oldest notion [K105, K36]. Adelic physics suggests that the measurement resolution could be realized in terms of a hierarchy of extensions of rationals [L41]. The parameters characterizing space-time surfaces and by SH the string world sheets would belong to the extension. Also the points of space-time surface in the extension would be data coding for the preferred extremals. The reconnection points and intersection points would belong to the extension [L35]. n-structures relate closely to the notion of non-commutative space and strings world sheets could be such. Also the role of classical number fields - in particular $M^8 - H$ correspondence suggest the same. The challenge is to develop a coherent view about all these structures.

1. There should be also a connection with the adelic view. In this picture string world sheets and points of space-time surface with coordinates in the extension of rationals defining the adele code for the data for preferred extremals and quantum states. What these points are - could they correspond to points of partonic 2-surfaces carrying fermions or could the correspond also to the points in the interior of space-time surface is not clear. The larger the extension of rationals, the larger the number of these points, and the better the resolution and the larger the deviation of SH from ideal. The hierarchy of Galois groups of extension of rationals should relate closely to the inclusion hierarchies.

2. Galois extension with given Galois group Gal allows hierarchy of intermediate extensions defining inclusion sequence for Galois groups. Besides inclusion homomorphisms there exists homomorphisms from Galois group Gal with order $h_{eff}/h = n$ to its sub-groups $H \subset Gal$ with order $h_{eff}/h = m < n$ dividing n. If it exists the sub-group mapped to identity element is normal sub-group H for which right and left cosets gH and Hg are identical. These homomorphisms to sub-groups identify the sheets of Galois covering of the space-time surface transformed to each other by H and thus define different number theoretical resolutions: measurement resolution would have precise geometric meaning. This would mean looking states with $h_{eff}/h = n$ in poorer resolution defined by $h_{eff}/h = m < n$.

These arrows would define "resolution morphisms" in category theoretic description. Also the analogy with the homotopies of n-structures is obvious. There would be a finite number of normal sub-groups with order dividing n for given higher structure. Quantum phase equal to root of unity ($q = exp(i2\pi/k)$) could appear in these representations and distinguish them from ordinary group representations.

14.3.4 Could normal sub-groups of symplectic group and of Galois groups correspond to each other?

Measurement resolution realized in terms of various inclusion is the key principle of quantum TGD. There is an analogy between the hierarchies of Galois groups, of fractal sub-algebras of SSA, and of inclusions of HFFs. The inclusion hierarchies of isomorphic sub-algebras of SSA and of Galois groups for sequences of extensions of extensions should define hierarchies for measurement resolution. Also the inclusion hierarchies of HFFs are proposed to define hierarcies of measurement resolutions. How closely are these hierarchies related and could the notion of measurement resolution allow to gain new insights about these hierarchies and even about the mathematics needed to realize them?

1. As noticed, SSA and its isomorphic sub-algebras are in a relation analogous to the between normal sub-group H of group Gal (analog of isomorphic sub-algebra) and the group G/H. One can assign to given Galois extension a hierarchy of intermediate extensions such that one proceeds from given number field (say rationals) to its extension step by step. The Galois groups H for given extension is normal sub-group of the Galois group of its extension. Hence Gal/H is a group. The physical interpretation is following. Finite measurement resolution defined by the condition that H acts trivially on the representations of Gal implies that they are representations of Gal/H. Thus Gal/H is completely analogous to the Kac-Moody type algebra conjecture to result from the analogous pair for SSA.

2. How does this relate to McKay correspondence stating that inclusions of HFFs correspond to finite discrete sub-groups of $SU(2)$ acting as isometries of regular n-polygons and Platonic solids correspond to Dynkin diagrams of ADE type SKMAs determined by ADE Lie group G. Could one identify the discrete groups as Galois groups represented geometrically as sub-groups of $SU(2)$ and perhaps also those of corresponding Lie group? Could the representations of Galois group correspond to a sub-set of representations of G defining ground states of Kac-Moody representations. This might be possible. The sub-groups of $SU(2)$ can however correspond only to a very small fraction of Galois groups.

Can one imagine a generalization of ADE correspondence? What would be required that the representations of Galois groups relate in some natural manner to the representations as Kac-Moody groups.

Some basic facts about Galois groups and finite groups

Some basic facts about Galois groups mus be listed before continuing. Any finite group can appear as a Galois group for an extension of some number field. It is known whether this is true for rationals (see http://tinyurl.com/hus4zso).

Simple groups appear as building bricks of finite groups and are rather well understood. One can even speak about periodic table for simple finite groups (see http://tinyurl.com/y75r68hp). Finite groups can be regarded as a sub-group of permutation group S_n for some n. They can be classified to cyclic, alternating , and Lie type groups. Note that alternating group A_n is the subgroup of permutation group S_n that consists of even permutations. There are also 26 sporadic groups and Tits group.

Most simple finite groups are groups of Lie type that is rational sub-groups of Lie groups. Rational means ordinary rational numbers or their extension. The groups of Lie type (see http://tinyurl.com/k4hrqr6) can be characterized by the analogs of Dynkin diagrams characterizing Lie algebras. For finite groups of Lie type the McKay correspondence could generalize.

Representations of Lie groups defining Kac-Moody ground states as irreps of Galois group?

The goal is to generalize the McKay correspondence. Consider extension of rationals with Galois group Gal. The ground staes of KMA representations are irreps of the Lie group G defining KMA. Could the allow ground states for given Gal be irreps of also Gal?

This constraint would determine which group representations are possible as ground states of SKMA representations for a given Gal. The better the resolution the larger the dimensions of the allowed representations would be for given G. This would apply both to the representations of the SKMA associated with dynamical symmetries and maybe also those associated with the standard model symmetries. The idea would be quantum classical correspondence (QCC) space-time sheets as coverings would realize the ground states of SKMA representations assignable to the various SKMAs.

This option could also generalize the McKay correspondence since one can assign to finite groups of Lie type an analog of Dynkin diagram (see `http://tinyurl.com/k4hrqr6`). For Galois groups, which are discrete finite groups of $SU(2)$ the hypothesis would state that the Kac-Moody algebra has same Dynkin diagram as the finite group in question.

To get some perspective one can ask what kind of algebraic extensions one can assign to ADE groups appearing in the McKay correspondence? One can get some idea about this by studying the geometry of Platonic solids (see `http://tinyurl.com/p4rwc76`). Also the geometry of Dynkin diagrams telling about the geometry of root system gives some idea about the extension involved.

1. Platonic solids have p vertices and q faces. One has $\{p,q\} \in \{\{3,3\},\{4,3\},\{3,4\},\{5,3\},\{3,5\}\}$. Tetrahedron is self-dual (see `http://tinyurl.com/qdl4sss` object whereas cube and octahedron and also dodecahedron and icosahedron are duals of each other. From the table of `http://tinyurl.com/p4rwc76` one finds that the cosines and sines for the angles between the vectors for the vertices of tetrahedron, cube, and octahedron are rational numbers. For icosahedron and dodecahedron the coordinates of vertices and the angle between these vectors involve Golden Mean $\phi = (1+\sqrt{5})/2$ so that algebraic extension must involve $\sqrt{5}$ at least.

   The dihedral angle θ between the faces of Platonic solid $\{p,q\}$ is given by $sin(\theta/2) = cos(\pi/q)/sin(\pi/p)$. For tetrahedron, cube and octahedron $sin(\theta)$ and $cos(\theta)$ involve $\sqrt{3}$. For icosahedron dihedral angle is $tan(\theta/2) = \phi$. For instance, the geometry of tetrahedron involves both $\sqrt{2}$ and $\sqrt{3}$. For dodecahedron more complex algebraic numbers are involved.

2. The rotation matrices for for the triangles of tetrahedron and icosahedron involve $cos(2\pi/3)$ and $sin(2\pi/3)$ associated with the quantum phase $q = exp(i2\pi/3)$ associated with it. The rotation matrices performing rotation for a pentagonal face of dodecahedron involves $cos(2\pi/5)$ and $sin(2\pi/5)$ and thus $q = exp(i2\pi/5)$ characterizing the extension. Both $q = exp(i2\pi/3)$ and $q = exp(i2\pi/5)$ are thus involved with icosahedral and dodecahedral rotation matrices. The rotation matrices for cube and for octahedron have rational matrix elements.

3. The Dynkin diagrams characterize both the finite discrete groups of $SU(2)$ and those of ADE groups. The Dynkin diagrams of Lie groups reflecting the structure of corresponding Weyl groups involve only the angles $\pi/2, 2\pi/3, \pi - \pi/6, 2\pi - \pi/6$ between the roots. They would naturally relate to quadratic extensions.

   For ADE Lie groups the diagram tells that the roots associated with the adjoint representation are either orthogonal or have mutual angle of $2\pi/3$ and have same length so that length ratios are equal to 1. One has $sin(2\pi/3) = \sqrt{3}/2$. This suggests that $\sqrt{3}$ belongs to the algebraic extension associated with ADE group always. For the non-simply laced Lie groups of type B, C, F, G the ratios of some root lengths can be $\sqrt{2}$ or $\sqrt{3}$.

For ADE groups assignable to n-polygons ($n > 5$) Galois group must involve the cyclic extension defined by $exp(i2\pi/n)$. The simplest option is that the extension corresponds to the roots of the polynomial $x^n = 1$.

14.3.5 A possible connection with number theoretic Langlands correspondence

I have discussed number theoretic version of Langlands correspondence in [K47, L26] trying to understand it using physical intuition provided by TGD (the only possible approach in my case). Concerning my unashamed intrusion to the territory of real mathematicians I have only one excuse: the number theoretic vision forces me to do this.

Number theoretic Langlands correspondence relates finite-dimensional representations of Galois groups and so called automorphic representations of reductive algebraic groups defined also for adeles, which are analogous to representations of Poincare group by fields. This is kind of relationship can exist follows from the fact that Galois group has natural action in algebraic reductive group defined by the extension in question.

The "Resiprocity conjecture" of Langlands states that so called Artin L-functions assignable to finite-dimensional representations of Galois group Gal are equal to L-functions arising from so called automorphic cuspidal representations of the algebraic reductive group G. One would have correspondence between finite number of representations of Galois group and finite number of cuspidal representations of G.

This is not far from what I am naïvely conjecturing on physical grounds: finite-D representations of Galois group are reductions of certain representations of G or of its subgroup defining the analog of spin for the automorphic forms in G (analogous to classical fields in Minkowski space). These representations could be seen as induced representations familiar for particle physicists dealing with Poincare invariance. McKay correspondence encourages the conjecture that the allowed spin representations are irreducible also with respect to Gal. For a childishly naïve physicist knowing nothing about the complexities of the real mathematics this looks like an attractive starting point hypothesis.

In TGD framework Galois group could provide a geometric representation of "spin" (maybe even spin 1/2 property) as transformations permuting the sheets of the space-time surface identifiable as Galois covering. This geometrization of number theory in terms of cognitive representations analogous to the use of algebraic groups in Galois correspondence might provide a totally new geometric insights to Langlands correpondence. One could also think that Galois group represented in this manner could combine with the dynamical Kac-Moody group emerging from SSA to form its Langlands dual.

Skeptic physicist taking mathematics as high school arithmetics might argue that algebraic counterparts of reductive Lie groups are rather academic entities. In adelic physics the situation however changes completely. Evolution corresponds to a hierarchy of extensions of rationals reflected directly in the physics of dark matter in TGD sense: that is as phases of ordinary matter with $h_{eff}/h = n$ identifiable as divisor of the order of Galois group for an extension of rationals. Algebraic groups and their representations get physical meaning and also the huge generalization of their representation to adelic representations makes sense if TGD view about consciousness and cognition is accepted.

In attempts to understand what Langlands conjecture says one should understand first the rough meaning of many concepts. Consider first the Artin L-functions appearing at the number theoretic side. Consider first the Artin L-functions appearing at the number theoretic side.

1. L-functions (see `http://tinyurl.com/y8dc4zv9`) are meromorphic functions on complex plane that can be assigned to number fields and are analogs of Riemann zeta function factorizing into products of contributions labelled by primes of the number field. The definition of L-function involves Direchlet characters: character is very general invariant of group representation defined as trace of the representation matrix invariant under conjugation of argument.

2. In particular, there are Artin L-functions (see `http://tinyurl.com/y7thhodk`) assignable to the representations of *non-Abelian* Galois groups. One considers finite extension L/K of fields with Galois group G. The factors of Artin L-function are labelled by primes p of K. There are two cases: p is un-ramified or ramified depending on whether the number of primes of L to which p decomposes is maximal or not. The number of ramified primes is finite and in TGD framework they are excellent candidates for physical preferred p-adic primes for given extension of rationals.

These factors labelled by p analogous to the factors of Riemann zeta are identified as characteristic polynomials for a representation matrix associated with any element in a preferred conjugacy class of G. This preferred conjugacy class is known as Frobenius element $Frob(p)$ for a given prime ideal p , whose action on given algebraic integer in O_L is represented as its p:th power. For un-ramified p the characteristic polynomial is explicitly given as determinant $det[I - t\rho(Frob(p))]^{-1}$, where one has $t = N(p)^{-s}$ and $N(p)$ is the field norm of p in the extension L (see http://tinyurl.com/o4saw2l).

In the ramified case one must restrict the representation space to a sub-space invariant under inertia subgroup, which by definition leaves invariant integers of O_L/p that is the lowest part of integers in expansion of powers of p.

At the other side of the conjecture appear representations of algebraic counterparts of reductive Lie groups and their L-functions and the two number theoretic and automorphic L-functions would be identical.

1. Automorphic form F generalizes the notion of plane wave invariant under discrete subgroup of the group of translations and satisfying Laplace equation defining Casimir operator for translation group. Automorphic representations can be seen as analogs for the modes of classical fields with given mass having spin characterized by a representation of subgroup of Lie group G ($SO(3)$ in case of Poincare group).

   Automorphic functions as field modes are eigen modes of some Casimir operators assignable to G. Algebraic groups would in TGD framework relate to adeles defined by the hierarchy of extensions of rationals (also roots of e can be considered in extensions). Galois groups have natural action in algebraic groups.

2. Automorphic form (see http://tinyurl.com/create.php) is a complex vector valued function F from topological group to some vector space V. F is an eigen function of certain Casimir operators of G. In the simplest situation these function are invariant under a discrete subgroup $\Gamma \subset G$ identifiable as the analog of the subgroup defining spin in the case of induced representations.

   In general situation the automorphic form F transforms by a factor j of automorphy under Γ. The factor can also act in a finite-dimensional representation of group Γ, which would suggest that it reduces to a subgroup of Γ obtained by dividing with a normal subgroup. j satisfies 1-cocycle condition $j(g_1, g_2g_3) = j(g_1g_2, g_3)$ in group cohomology guaranteeing associativity (see http://tinyurl.com/on7ffy9). Cuspidality relates to the conditions on the growth of F at infinity.

3. Elliptic functions in complex plane characterized by two complex periods are meromorphic functions of this kind. A less trivial situation corresponds to non-compact group $G = SL(2, R)$ and $\Gamma \subset SL(2, Q)$.

There are more groups involved: Langlands group L_F and Langlands dual group ${}^LG$. A more technical formulation says that the automorphic representations of a reductive Lie group G correspond to homomorphisms from so called Langlands group L_F (see http://tinyurl.com/ycnhkvm2) at the number theoretic side to L-group ${}^LG$ or Langlands dual of algebraic G at group theory side (see http://tinyurl.com/ycnk9ga5). It is important to notice that ${}^LG$ is a complex Lie group. Note also that homomorphism is a representation of Langlands group L_F in L-group ${}^LG$. In TGD this would be analogous to a homomorphism of Galois group defining it as subgroup of the group G defining Kac-Moody algebra.

1. Langlands group L_F of number field is a speculative notion conjectured to be a extension of the Weil group of extension, which in turn is a modification of the absolute Galois group. Unfortunately, I was not able to really understand the Wikipedia definition of Weil group (http://tinyurl.com/hk74sw7). If E/F is finite extension as it is now, the Weil group would be $W_{E/F} = W_F/W_E^c$, W_E^c refers to the commutator subgroup W_E defining a normal subgroup, and the factor group is expected to be finite. This is not Galois group but should be closely related to it.

Only finite-D representations of Langlands group are allowed, which suggests that the representations are always trivial for some normal subgroup of L_F For Archimedean local fields L_F is Weil group, non-Archimedean local fields L_F is the product of Weil group of L and of SU(2). The first guess is that SU(2) relates to quaternions. For global fields the existence of L_F is still conjectural.

2. I also failed to understand the formal Wikipedia definition of the L-group ${}^LG$ appearing at the group theory side. For a reductive Lie group one can construct its root datum $(X^*, \Delta, X_*, \Delta^c)$, where X^* is the lattice of characters of a maximal torus, X_* its dual, Δ the roots, and Δ^c the co-roots. Dual root datum is obtained by switching X^* and X_* and Δ and Δ^c. The root datum for G and ${}^LG$ are related by this switch.

   For a reductive G the Dynkin diagram of ${}^LG$ is obtained from that of G by exchanging the components of type B_n with components of type C_n. For simple groups one has $B_n \leftrightarrow C_n$. Note that for ADE groups the root data are same for G and its dual and it is the Kac-Moody counterparts of ADE groups, which appear in McKay correspondence. Could this mean that only these are allowed physically?

3. Consider now a reductive group over some field with a separable closure K (say k for rationals and K for algebraic numbers). Over K G as root datum with an action of Galois group of K/k. The full group ${}^LG$ is the semi-direct product ${}^LG^0 \rtimes Gal(K/k)$ of connected component as Galois group and Galois group. $Gal(K/k)$ is infinite (absolute group for rationals). This looks hopelessly complicated but it turns it that one can use the Galois group of a finite extension over which G is split. This is what gives the action of Galois group of extension (l/k) in ${}^LG$ having now finitely many components. The Galois group permutes the components. The action is easy to understand as automorphism on Gal elements of G.

Could TGD picture provide additional insights to Langlands duality or vice versa?

1. In TGD framework the action of Gal on algebraic group G is analogous to the action of Gal on cognitive representation at space-time level permuting the sheets of the Galois covering, whose number in the general case is the order of Gal identifiable as $h_{eff}/h = n$. The connected component ${}^LG0$ would correspond to one sheet of the covering.

2. What I do not understand is whether ${}^LG = G$ condition is actually forced by physical contraints for the dynamical Kac-Moody algebra and whether it relates to the notion of measurement resolution and inclusions of HFFs.

3. The electric-magnetic duality in gauge theories suggests that gauge group action of G on electric charges corresponds in the dual phase to the action of ${}^LG$ on magnetic charges. In self-dual situation one would have $G =^L G$. Intriguingly, CP_2 geometry is self-dual (Kähler form is self-dual so that electric and magnetic fluxes are identical) but induced K̈ahler form is self-dual only at the orbits of partonic 2-surfaces if weak form of electric-magnetic duality holds true. Does this condition leads to ${}^LG = G$ for dynamical gauge groups? Or is it possible to distinguish between the two dynamical descriptions so that Langlands duality would correspond to electric-magnetic duality. Could this duality correspond to the proposed duality of two variants of SH: namely, the electric description provided by string world sheets and magnetic description provided by partonic 2-surfaces carrying monopole fluxes?

14.3.6 A formulation of adelic TGD in terms of cognitive representations?

The vision about p-adic physics as cognitive representations of real physics [L41] encourages to consider an amazingly simple formulation of TGD diametrically opposite to but perhaps consistent with the vision based on the notion of WCW and WCW spinor fields. Finiteness of cognitive and measurement resolutions would not be enemies of the theoretician but could make possible to deduce highly non-trivial predictions from the theory by getting rid of all irrelevant information and using only the most significant bits. Number theoretic physics need not of course cover the entire quantum physics and could be analogous to topological quantum field theories: even this might provide huge amounts of precise information about the quantum physics of TGD Universe.

Could the discrete variant of WCW geometry make sense?

The first thing that one can imagine is number theoretic discretization of WCW by assuming that WCW coordinates belong to an extension of rationals. Integration would reduce to a summation but the problem is that there are too many points in the extension so that sums do not make sense in real sense. In the case of space-time surfaces the problems are solved by the fact that space-time surfaces have dimension lower than the embedding space and the number of points with coordinates in the extension is in typical case finite: exceptions are surfaces such as canonically imbedded M^4 or CP_2. This option does not work at the level of WCW.

Cognitive representations however carry information about the points with coordinates in the extension of rationals defining the adele and possibly about the directions of strings emanating from these points. The effective WCW is kind of coset space with most of degrees of freedom not visible in the cognitive representation. Cognitive representations would specify the points in the extension of rationals for space-time surface, string world sheets, or even for their intersection with partonic surfaces at the ends of CD carrying fermion number plus those at the ends of sub-CDs forming a hierarchy.

Could one use the points of cognitive representation as coordinates for this effective WCW so that everything including WCW integration would reduce to well-defined summations? This would solve the problem of too many points in sub-WCW associated with the extension. Could one formulate everything that one can know at given level of cognitive hierarchy defined by extensions?

This idea was already suggested by the interpretation of p-adic mass calculations.

1. p-Adic mass calculations would correspond to cognitive representation of real physics [K21, K52]. For large p-adic primes p-adic thermodynamics converges extremely rapidly as powers $p^{-n/2}$ and the results from two lowest orders are practically exact.

2. What is however required is a justification for the map of p-adic mass squared values to real numbers by canonical identification. Quite generally this map makes sense for group invariants - say Lorentz invariants defined by inner products of momenta. As a matter of fact, the construction of quantum algebras and Yangians demands p-adic topology for the antipode to exist mathematically so that this approach could be forced by mathematical consistency [B6].

Could scattering amplitudes be constructed in terms of cognitive representations?

The crazy looking idea that cognitive representations defined by common points of real and p-adic variants of space-time surfaces or even partonic 2-surfaces is at least worth of showing to be wrong. If the idea works, cognitive representations could code what can be known about classical and even quantum dynamics and reduce physics to number theory. Also WCW would be discretized with points of discretized space-time surface defining WCW coordinates. Functional integral over WCW would reduce to a converging sum over cognitive representations.

It is interesting to look what this could mean if scattering amplitudes correspond in some sense to algebraic computations in bi-algebra besides product also co-product as its time reversal and interpreted as 3-vertex physically.

1. For the simplest option fermions would reside at the intersection points of partonic 2-surfaces and string world sheets. One possibility considered earlier is that at these points the Galois coverings are singular meaning that all sheets co-incide. This might be too strong condition and might be replacable by a weaker condition that Galois group at these points reduces to its sub-group and normal subgroup leaves amplitudes invariant. A reduction of measurement resolution would be in question.

2. If the basic computational operation involves a fusion of representations of Galois group, fusion algebra could describe the situation [L35]. The Galois groups assignable to the incoming lines of 3-vertex must correspond to Galois groups, which define groups of 3-levelled hierarchy of extension of rationals allowing inclusion homomorphism. Therefore the values of Planck constant would be of from $h_{eff}/h \in \{n_1, n_1n_2, n_1n_2n_3\}$. The tensor product decomposition would tell the outcome of tensor product. One can consider also 2-vertices corresponding to a phase transition $n_1 \leftrightarrow n_1n_2$ changing the value of h_{eff}/h.

McKay graphs (see http://tinyurl.com/z48d92t) for Galois groups describe the decomposition of the tensor products of representations of Galois groups. In general the tensor products for corresponding KMAs restricted to Galois group are not irreducible. What could this mean? Are they allowed to occur? Are there general results allowing to conclude how do the analogs of McKay graphs for the tensor products of the irreps of the group defining Kac-Moody group relate to the McKay graphs for its finite discrete sub-groups?

Possible problems relate to the description of momenta and higher excitations of SKMAs. In topological QFTs one loses information about metric properties such as mass but what happens in number theoretic QFT? Could the Galois approach expanded to include also discrete variants of quaternions and octonions assigna ble to extensions of rationals allow also the number theoretic description of also momenta?

1. Octonions and quaternions have $G(2)$ and $SO(3)$ as automorphisms groups (analogs of Galois groups). The octonionic automorphisms respecting chosen imaginary consist of $SU(3)$ rotations. These groups would be replaced with their dicrete variants with matrix elements in an extension of rationals.

   The automorphism group Gal for the extension of rationals and automorphism group $Aut \in \{G_2, SU(3), SO(3)\}$ for octonions/for octonions with fixed unit/for quaternions form a semi-direct product $Gal \rtimes Aut$ with multiplication rule $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_2g_1(g_b))$, where $g_1(g_b)$ represents the element of Aut obtained by performing Gal automorphism g_1 for g_b. For rational elements g_b one has $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_ag_b)$ so that Gal Aut_Q commute. An interesting possibility is that the automorphisms of $Aut \in \{SU(3), SO(3)\}$ can be interpreted in terms of standard model symmetries whereas Gal would relate to the dynamical symmetries.

   In M^8 picture one has naturally wave functions in the space of quaternionic light-like 8-momenta and it is natural to decompose quaternionic momenta to longitudinal M^2 piece and transversal E^2 piece. The physical interpretation of this condition has been discussed thoroughly in [L45]. One has thus more than mere analog of TQFT.

2. If fermions propagate along the lines of the TGD analogs twistor graphs, one must have an analog of propagator. Twistor approach [L45] implies that the propagator is replaced with the inverse of the fermion propagator for quaternionic 8-momentum as a residue with sigma matrices representing the quaternionic units. This is non-vanishing only if the fermion chirality is "wrong". This has co-homological interpretation: for external lines the inverse of the propagator would annihilate the state (co-closedness) unlike for internal lines.

3. Triality holds true for the octonionic vector representation assignable to momenta and octonionic spinors and their conjugates. All these should be quaternionic, in other words belong to some complexified quaternionic $M^4 \subset M^8$. The components of these spinors should belong to an extension of rational used with imaginary unit commuting with octonionic imaginary units.

4. The condition that the amplitudes belong to an extension of rationals could be extremely powerful when combined with category theoretic view implying the Hilbert space isometries allowing to relate amplitudes at different levels of the hierarchy. This conditions should be true also for the twistors in terms which momenta can be expressed. Also the space $SU(3)/U(1) \times U(1)$ of CP_2 twistors would be replaced with a sub-space with points in an extension of rationals.

14.4 Could McKay correspondence generalize in TGD framework?

McKay correspondence is rather mysterious looking correspondence appearing in several fields. This correspondence is extremely interesting from point of view of adelic TGD [L42] [L41].

1. McKay graphs code for the fusion algebra of irreducible representations (irreps) of finite groups (see http://tinyurl.com/z48d92t). For finite subgroups of $G \subset SU(2)$ McKay graphs are extended Dynkin diagrams for affine (Kac-Moody) algebras of ADE type coding the structure of the root diagram for these algebras. The correspondence looks mysterious since Dynkin diagrams have quite different geometric interpretation.

2. McKay graphs for finite subgroups of $G \subset SU(2)$ characterize also the fusion rules of minimal conformal field theories (CFTs) having Kac-Moody algebra (KMA) of $SU(2)$ as symmetries (see http://tinyurl.com/y7doftpe). Fusion rules characterize the decomposition of the tensor products of primary fields in CFT. For minimal CFTs the primary fields belonging to the irreps of $SU(2)$ are in 1-1 correspondence with irreps of G, and the fusion rules for primary fields are same as for the irreps of G. The irreps of $SU(2)$ are also irreps of G.

   Could the ADE type affine algebra appear as dynamical symmetry algebra too? Could the primary fields for ADE defining extended ADE Cartan algebra be constructed as G-invariants formed from the irreps of G and be exponentiated using the standard free field construction using the roots of the ADE KMA a give ADE KMA acting as dynamical symmetries?

3. McKay graphs for $G \subset SU(2)$ characterize also the double point singularities of algebraic surfaces of real dimension 4 in C^3 (or CP^3 , one variant of twistor space!) with real dimension 6 (see http://tinyurl.com/ydz93hle). The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This partially explains the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of spheres is the number of irreps for G is not obvious to me.

   The double point singularities of $C^2 \subset C^3$ allow thus ADE classification. The number of added points corresponds to the dimension of Cartan algebra for ADE type affine algebra, whose Dynkin diagram codes for the finite subgroup $G \subset SU(2)$ leaving the algebraic surface looking locally like C^2 invariant and acting as isotropy group of the singularity.

These results are highly inspiring concerning adelic TGD.

1. The appearance of Dynkin diagrams in the classification of minimal CFTs inspires the conjecture that in adelic physics Galois groups Gal or semi-direct products $G \triangleleft Gal$ of Gal with a discrete subgroup G of automorphism group $SO(3)$ (having $SU(2)$ as double covering!) classifies TGD generalizations of minimal CFTs. Also discrete subgroups of octonionic automorphism group can be considered. The fusion algebra of irreps of Gal would define also the fusion algebra for KMA for the counterparts of minimal fields. This would provide deep insights to the general structure of adelic physics.

2. One cannot avoid the question whether the extended ADE diagram could code for a dynamical symmetry of a minimal CFT or its modification? If the Gal singlets formed from the primary fields of minimal model define primary fields in Cartan algebra of ADE type KMA, then standard free field construction would give the charged KMA generators. In TGD framework this conjecture generalizes.

3. A further conjecture is that the singularities of space-time surface imbedded as 4-surface in its 6-D twistor bundle with twistor sphere as fiber could be classified by McKay graph of Gal. The singular intersection of the Euclidian and Minkowskian regions of space-time surface is especially interesting: the twistor spheres at the common points defining light-like partonic orbits need not be same but have intersections with intersection matrix given by McKay graph for Gal. The basic information about adelic CFT would be coded by the general character of singularities for the twistor bundle.

4. In TGD also singularities in which the group Gal is reduced to its subgroup Gal/H, where H is normal group are possible and would correspond to phase transition reducing the value of Planck constant. What happens in these phase transitions to single particle states would be dictated by the decomposition of representations of Gal to those of Gal/H and transition matrix elements could be evaluated.

One can find from web excellent articles about the topics to be discussed in this article.

1. The article "*Cartan matrices, finite groups of quaternions, and Kleinian singularities*" of John McKay [A63] (see `http://tinyurl.com/ydygjgge`) summarizes McKay correspondence.

2. Miles Reid has written an article " *The Du Val singularities A_n, D_n, E_6, E_7, E_8*" [A73] (see `http://tinyurl.com/ydz93hle`). Also the article " *Chapters on algebraic surfaces*" [A74](see `http://tinyurl.com/yaty9rzy`) of Reid should be helpful. There is also an article "*Resolution of Singularities in Algebraic Varieties*" [A41] (see `http://tinyurl.com/yb7cuwkf`) of Emma Whitten about resolution of singularities.

3. Andrea Cappelli and Jean-Benard Zuber have written an article "*A-D-E Classification of Conformal Field Theories*" [B23] about ADE classification of minimal CFT models (see `http://tinyurl.com/y7doftpe`).

4. McKay correspondence appears also in M-theory, and the thesis "*On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective*" [B49] (see `http://tinyurl.com/ycmyjukn`) of Yang-Hui He might be helful for the reader. In this work the possible generalization of McKay correspondence so that it would apply form finite subgroups of $SU(n)$ is discussed. $SU(3)$ acting as subgroup of automorphism group G_2 of octonions is especially interesting in this respect. The idea is rather obvious: the fusion diagram for the theory in question would be the McKay graph for the finite group in question.

14.4.1 McKay graphs in mathematics and physics

McKay graphs for subgroups of $SU(2)$ reducing to Dynkin diagrams for affine Lie algebras of ADE type appear in several ways in mathematics and physics.

McKay graphs

McKay graphs [A63] (see `http://tinyurl.com/ydygjgge`) code for the fusion algebra of irrpes of finite groups G (for Wikipedia article see `http://tinyurl.com/z48d92t`). One considers the tensor products of irreps with the canonical representation (doublet representation for the finite sub-groups of $SU(2)$), call it V. The irreps V_i correspond to nodes and their number is equal to the number of irreps G.

Two nodes i and j are no connected if the decomposition of $V \otimes V_i$ to irreps does not contain V_j. There is arrow pointing from $i \rightarrow j$ in this case. The number $n_{ij} > 0$ or number of arrows tells how many times j is contained in $V \otimes V_j$. For $n_{ij} = n_{ji}$ there is no arrow.

One can characterize the fusion rules by matrix $A = d\delta_{ij} - n_{ij}$, where d is the dimension of the canonical representation. The eigenvalues of this matrix turn out to be given by $d - \xi_V(g)$, where $\xi_V(g)$ is the character of the canonical representation, which depends on the conjugacy class of g only. The number of eigenvalues is therefore equal to the number $n(class, G)$ of conjugacy classes. The components of eigenvectors in turn are given by the values $\chi_i(g)$ of characters of irreps.

MacKay graphs and Dynkin diagrams

The nodes of the Dynkin diagram (see `http://tinyurl.com/hpm5y9s`) are positive simple root vectors identified as vectors formed by the eigenvalues of the Cartan sub-algebra generators under adjoint action on Lie algebra. In the case of affine Lie algebra the Cartan algebra contains besides the Cartan algebra of the Lie group also scaling generator $L_0 = td/dt$ and the number of nodes increases by one.

The number of positive simple roots equals to the dimension of the root space. The number n_{ij} codes now for the angle between positive simple roots. The number of edges connecting root vectors is $n = 0, 1, 2, 3$ depending on whether the angle between root vectors is $\pi/2, 2\pi/3, 3\pi/4$, or $5\pi/6$. The ratios of lengths of connected roots can have values $\sqrt{n}$, $n \in \{1, 2, 3\}$, and the number n of edges corresponds to this ratio. The arrow is directed to the shorter root if present. For simply laced Lie groups (ADE groups) the roots have unit length so that only single undirected edge can connect the roots. Weyl group acts as symmetries of the root diagram as reflections in hyperplanes orthogonal to the roots.

The Dynkin diagrams of affine algebras are obtained by adding to the Cartan algebra a generator which corresponds to the scaling generator $L_0 = td/dt$ of affine algebra assumed to act via adjoint action to the Lie algebra. Depending on the position of the added node one obtains also twisted versions of the KMA.

For the finite subgroups of $SU(2)$ the McKay graphs reduce to Dynkin diagrams of affine Lie algebras of ADE type [A63] (see `http://tinyurl.com/ydygjgge`) so that one has either $n_{ij} = 0$ or $n_{ij} = 1$ for $i \neq j$. There are no self-loops ($n_{ii} \neq 0$). The result looks mysterious since the two diagrams describe quite different things. One can also raise the question whether ADE type affine algebra might somehow emerge in minimal CFT involving $SU(2)$ KMA for which ADE classification emerges.

In TGD framework the interpretation of finite groups $G \subset SU(2)$ in terms of quaternions is an attractive possibility since rotation group $SO(3)$ acts as automorphisms of quaternions and has $SU(2)$ as its covering group.

ADE diagrams and subfactors

ADE classification emerges also naturally for the inclusions of hyper-finite factors of type II_1 [K105, K36]. Subfactors with index smaller than four have so called principal graphs characterizing the sequence of inclusions equal to one of the A, D or E Coxeter-Dynkin diagrams: see the article "*In and around the origin of quantum groups*" of Vaughan Jones [A87] (see `http://tinyurl.com/ycbbbvpq`). As a matter of fact, only the D_{2n} and E_6 and E_8 do occur. It is also possible to construct $M : N = 4$ sub-factor such that the principle graph is that for any subgroup $G \subset SU(2)$. This suggests that the subfactors $M : N = 4cos^2(\pi/n) < 4$ correspond to quantum groups. The basic objects can be seen as quantum spinors so that again the appearance of subgroups of $SU(2)$ looks natural. One can still wonder whether ADE KMAs might be involved.

ADE classification for minimal CFTs

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CFTs on torus [B23] are characterized by modular invariant partition functions, which can be expressed in terms of characters of the scaling generator L_0 of Virasoro algebra (VA) given by

$$Z(\tau) = Tr(X) \ , \quad X = exp\{i2\pi \left[\tau(L_0 - c/24) - \overline{\tau}(\overline{L}_0 - c/24)\right]\} \ . \tag{14.4.1}$$

Modular invariance requires that $Z(\tau)$ is invariant under modular transformations leaving the conformal equivalence class of torus invariant. Modular group equals to $SL(2, Z)$ has as generators the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The partition function can be expressed as

$$Z(\tau) = \sum N_{j\overline{j}}\chi_j(q)\chi_{\overline{j}}(\overline{q}) \ , \quad q = exp(i2\pi\tau) \ , \quad \overline{q} = exp(-i2\pi\overline{\tau}) \ . \tag{14.4.2}$$

Here χ_j corresponds to the trace of $L_0 - c/24$ for a representation of KMA inducing the VA representation. Modular invariance of the partition function requires $SNS^\dagger = N$ and $TNT^\dagger = N$.

The ADE classification for minimal conformal models summarized in [B23] (see `http://tinyurl.com/y7doftpe`) involves $SU(2)$ affine algebra with central extension parameter k. The central extension parameter for the VA is $c < 1$. The fusion algebra for primary fields in representations of $SU(2)$ KMA characterizes the CFT to a high degree.

The fusion rules characterized the decomposition of the tensor product of representation D_i with representation D_j as $i \otimes j = N_{ij}^k D_k$. Due to the properties of the tensor product the matrices $\mathcal{N}_i = N_{ij}^k$ form and associative and commutative algebra and one can diagonalize these matrices simultaneously. This algebra is known as Verlinde algebra and its elements can be expressed in terms of unitary modular matrix S_{ij} representing the transformation of characters in the modular transformation $\tau \to -1/\tau$.

The generator of the Verlinde algebra is fusion algebra for the 2-D representation of $SU(2)$ generating the fusion algebra (this corresponds to the fact that tensor powers of this representations give rise to all representations of $SU(2)$). It turns out that for minimal models with a finite number of primary fields (KMA representations) the fusion algebra of KMA reduces to that for a finite subgroup of $SU(2)$ and thus corresponds to ADE KMA. The natural interpretation is that the condition that the number of primary fields is finite is realized if the primary fields correspond also to the irreps of finite subgroup of SU(2).

Could the ADE type KMA actually correspond to a genuine dynamical symmetry of minimal CFT? For this conjecture makes sense, the roots of ADE type KMA should be in 1-1 correspondence with the irreps of $G \subset SU(2)$ assignable to primary fields. How could this be possible? In the free field construction of ADE type KMA generators one constructs charged KMA generators from free fields in Cartan algebra by exponentiating the quantities $\alpha \cdot \phi$, where α is the root and ϕ is a primary field corresponding to the element of Cartan algebra of KMA. Could $SU(2)$ invariants formed from the primary fields defined by each G- (equivalently $SU(2)$-) multiplet give rise to $SU(2)$ neutral multiplet of primary fields of ADE type Cartan algebra and could their exponentiation give rise to ADE type KMA acting as dynamical symmetries of a minimal CFT?

The resolution of singularities of algebraic surfaces and extended Dynkin diagrams of ADE type

The classification of singularities of algebraic surfaces leads also to extended Dynkin diagrams of ADE type.

1. Classification of singularities

In algebraic geometry the classification of singularities of algebraic varieties [A41] is a central task. The singularities of curves in plane represent simplest singularities (see `http://tinyurl.com/y8ub2c4s`). The resolution of singularities of complex curves in C^3 is less trivial task.

The resolution of singularity (`http://tinyurl.com/y8veht3p`) is a central concept and means elimination of singularity by modifying it locally. There is extremely general theorem by Hiroka stating that the resolution of singularities of algebraic varieties is always possible for fields with characteristic zero (reals and p-adic number fields included) using a sequence of birational transformations. For finite groups the situation is unclear for dimensions $d > 3$.

The articles of Reid [A73] and Whitten [A41] describe the resolution for algebraic surfaces (2-D surfaces with real dimension equal to four). The article of Reid describes how the resolutions of double-point singularities of $m = d_c = 2$-D surfaces in $n = d_c = 3$-D C^3 or CP_3 (d_c refers to complex dimension) are classified by ADE type extended Dynkin diagrams. Subgroups $G \subset SU(2)$ appear naturally because the surface has dimension $d_c = 2$. This is the simplest non-trivial situation since for Riemann surface with $(m, n) = (1, 2)$ the group would be discrete subgroup of $U(1)$.

2. Singularity and Jacobians

What does one mean with singularity and its resolution? Reid [A73] (see `http://tinyurl.com/ydz93hle`) discusses several examples. The first example is the singularity of the surface $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$.

1. One can look the situation from the point of view of embedding of the 2-surface to C^3: one considers map from tangent space of the surface to the embedding space C^3. The Jacobian of the embedding map $(x_2, x_3) \to (x_1, x_2, x_3) = \pm\sqrt{x_2x_3}, x_2, x_3)$ becomes ill-defined at origin since the partial derivatives $\partial x_1/\partial x_2 = (\sqrt{x_3/x_2})/2$ and $\partial x_1/\partial x_3 = (\sqrt{x_2/x_3})/2$ have all possible limiting values at singularity. The resolution of singularity must as a coordinate transformation singular at the origin should make the Jacobian well-defined. Obviously this must mean addition of points corresponding to the directions of various lines of the surface through origin.

2. A more elegant dual approach replaces parametric representation with representation in terms of conditions requiring function to be constant on the surface. Now the Jacobian of a map from C^3 to the 1-D normal space of the singularity having polynomial $P(x_1, x_2, x_3)$ as coordinate is considered. Singularity corresponds to the situation when the rank of the Jacobian defined by partial derivatives is less than maximal so that one has $\partial P/\partial x_i = 0$. The resolution of singularity means that the rank becomes maximal. Quite generally, for

co-dimension m algebraic surface the vanishing of polynomials P_i, $i = 1, ..., m$ defines the surface. At the singularity the reduction of the rank for the matrix $\partial P_i/\partial x_n$ from its maximal value takes place.

3. Blowing up of singularity

Codimension one algebraic surface is defined by the condition $P(x_1, x_2, ..., x_n) = 0$, where $P(x_1, ..., x_n)$ is polynomial. For higher codimensions one needs more polynomials and the situation is not so neat anymore since so called complete intersection property need not hold anymore. Reid [A73] gives an easy-to- understand introduction to the blowing up of double-point singularities. Also the article "*Resolution of Singularities in Algebraic Varieties*" of Emma Whitten [A41] (see `http://tinyurl.com/yb7cuwkf`) is very helpful.

1. Coordinates are chosen such that the singularity is at the origin $(x, y, z) = (0, 0, 0)$ of complex coordinates. The polynomial has vanishing linear terms at singularity and the first non-vanishing term is second power of some coordinate, say x_1, so that one has $x_1 = \pm\sqrt{P_1(x_1, x_2, x_3}$, where x_1 in P_1 appears in powers higher than 2. At the singularity the two roots co-incide. One can of course have also more complex singularities such as triple-points.

2. The simplest example $P(x_1, x_2, x_3) = x_1^2 - x_2 x_3 = 0$ has been already mentioned. This singularity has the structure of double cone since one as $x_1 = \pm\sqrt{x_2 x_3}$. At $(0, 0, 0)$ the vertices of the two cones meet.

3. One can look this particular situation from the perspective of projective geometry. Homogenous polynomials define a surface invariant under scalings of coordinates so that modulo scalings the surface can be regarded also as complex curve in CP_2. The conical surface can be indeed seen as a union of lines $(x_1 = k^2 x_3, x_2 = k x_3)$, where k is complex number. The ratio $x_1 : x_2 : x_3$ for the coordinates at given line is determined by $x_1 : x_2 = k$ and $x_2 : x_3 = k$ so that the surface can be parameterized by k and the coordinate along given line.

   In this perspective the singularity decomposes to the directions of the lines going through it and the situation becomes non-singular. The replacement of the original view with this gives a geometric view idea about the resolution of singularity: the 2-surface is replaced by a bundle lines of surfaces going through the singularity and singularity is replaced with a union of directions for these lines.

Quite generally, in the resolution of singularity, origin is replaced by a set of points (x_1, x_2, x_3) with a well-defined ratio $(x_1 : x_2 : x_3)$. This interpretation applies also to more general singularities. One can say that origin is replaced with a projective sub-manifold of 2-D projective space CP_2 (very familiar to me)! This procedure is known as blowing up. Strictly speaking, one only replaces origin with the directions of lines in C^3.

Remark: In TGD the wormhole contacts connecting space-time sheets of many-sheeted space-time could be seen as outcomes of blowing up procedure.

Blowing up replaces the singular point with projective space CP_1 for which points with same value of $(x_1 : x_2 : x_3)$ are identified. Blowing up can be also seen as a process analogous to seeing the singularity such as self-intersection of curve as an illusion: the curve is actually a projection of a curve in higher dimensional space to which it is lifted so that the intersection disappears [A41] (see `http://tinyurl.com/yb7cuwkf`). Physicist can of course protest by saying that in space-time physics is is not allowed to introduce additional dimensions in this manner!

There is an analytic description for what happens at the singular point in blowing up process [A41] (see `http://tinyurl.com/yb7cuwkf`).

1. In blowing up one lifts the surface in higher-dimensional space $C^3 \times CP_2$ (C^3 can be replaced by any affine space). The blowing up of the singularity would be the set of lines $\overline{q}$ of the surface S going through the singularity that is the set $B = \{(q, \overline{q}) | q \in S\}$. This set can be seen as a subset of $C^3 \times CP_2$ and one can represent it explicitly by using projective coordinates (y_1, y_2, y_3) for CP_2. Consider points of C^3 and CP_2 with coordinates $z = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The coordinate vectors must be parallel x is to be at line y. This requires that all 2×2 sub-determinants of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad (14.4.3)$$

vanish: that is $x_iy_j - x_jy_i = 0$ for all pairs $i < j$. This description generalizes to the higher-dimensional case. The added CP_1s defined what is called exceptional divisor in the blown up surface. Recall that divisors (see `http://tinyurl.com/yc7x3ohx`) are by definition formal combinations of points of algebraic surface with integer coefficients. The principal divisors defined by functions are sums over their zeros and poles with integer weight equal to the order of zero (negative for pole).

The above example considers a surface $x_1^2 - x_2x_3 = 0$ which allows interpretation as a projective surface. The method however works also for more general case since the idea about replacing point with directions is applied only at origin.

2. One can consider a more practical resolution of singularity by performing a bi-rational coordinate transformation becoming singular at the singular point. This can improve the singularity by blowing it up or make it worse by inducing blowing down. The idea is to perform a sequence of this kind of coordinate changes inducing blowing ups so that final outcome is free of singularities.

   Since one considers polynomial equations both blowing up and its reversal must map polynomials to polynomials. Hence a bi-rational transformation b acting as a surjection from the modified surface to the original one must be in question (for bi-rational geometry see `http://tinyurl.com/yadoo3ot`). At the singularity b is many-to-one y so that at this point inverse image is multivalued and gives rise to the blowing up.

   The equation $P(x_1, x_2, x_3) = 0$ combined with the equations $x_iy_j - x_jy_i = 0$ by putting $y_3 = 1$ (the coordinates are projective) leads to a parametric representation of S using y_1 and y_2 as coordinates instead of x_1 and x_2. Origin is replaced with CP_1. This representation is actually much more general. Whitten [A41] gives a systematic description of resolution of singularities using this representation. For instance, cusp singularity $P(x_1, x_2) = x_1^2 - x_2^3 = 0$ is discussed as a special case.

3. Topologically the blow up process corresponds to the gluing of CP_2 to the algebraic surface $A : A \to A\#CP_2$ and clearly makes it more complex. One can say that gluing occurs along sphere CP_1 and since the process involves several steps several spheres are involved with the resolution of singularities.

4. ADE classification for resolutions of double point singularities of algebraic surfaces

ADE classification emerges for co-dimension one double point singularities of complex surfaces in C^3 known as Du Val singularities. The surface itself can be seen locally as C^2. These surfaces are 4-D in real sense can have self-intersections with real dimension 2. In the singular point the dimension of the intersection is reduced and the dimension of tangent space is reduced (the rank of Jacobian is not maximal). The vertices of cone and cusp are good examples of singularities.

The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This helps to understand the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of projective lines equals to the number of irrepss of G appearing as nodes in McKay graph looks to me rather mysterious. Reid's article [A73] gives the characterization of groups G and canonical forms of the polynomials defining the singular surfaces.

The reason why Du Val singularities are so interesting from TGD point of view is that complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces as preferred extremals have real dimension 4 and assumed possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure [K99].

14.4.2 Do McKay graphs of Galois groups give overall view about classical and quantum dynamics of quantum TGD?

McKay graphs for Galois groups are interesting from TGD view point for several reasons. Galois groups are conjectured to be the number theoretical symmetries for the hierarchy of extensions of rationals defining hierarchy of adelic physics [L42] [L41] and the notion of CFT is expected to generalize in TGD framework so that ADE classification for minimal CFTs might generalize to a classification of minimal number theoretic CFTS by Galois groups.

1. Vision

The arguments leading to the vision are roughly following.

1. Adelic physics postulates a hierarchy of quantum physics with adeles at given level associated with extension of rationals characterized partially by Galois group and ramified primes of extension. The dimension of the extensions dividing the order of Galois group is excellent candidate for defining the value of Planck constant $h_{eff}/h = n$ and ramified primes could correspond to preferred p-adic primes. The discrete sets of points of space-time surface for which embedding space coordinates are in the extension define what I have interpreted as cognitive representations and can be said to be in the intersection of all number fields involved forming kind of book like structure with pages intersecting at the points with coordinates in extension.

   Galois groups would define a hierarchy of theories and the natural first guess is that Galois groups take the role of subgroups of $SU(2)$ in CFTs with $SU(2)$ KMA as symmetry. Could the MacKay graphs defining the fusion algebra of Galois group define the fusion algebra of corresponding minimal number theoretic QFTs in analogy with minimal conformal models? This would fix the primary fields of theories assignable to given level of adele hierarchy to be minimal representations of Gal perhaps having also interpretation as representations of KMAs or their generalization to TGD framework.

2. The analogies between TGD and the theory of Du Val singularities is intriguing. Complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces have real dimension 4 and possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure.

   The twistor bundle of space-time surface has 2-sphere CP_1 as a fiber and space-time surface as base [L24, L45]. Space-time surfaces can be realized as sections in their own 6-D twistor bundle obtained by inducing twistor structure from the product $T(M^4) \times T(CP_2)$ of twistor bundles of M^4 and CP_2. Section is fixed only modulo gauge choice, which could correspond to the choice of the Kähler form defining twistor structure from quaternionic units represented as points of S^2. Even if this choice is made, $U(1)$ gauge transformations remain and could correspond to gauge transformations of WCW changing its Kähler gauge potential by gradient and adding to Kähler function a real part of holomorphic function of WCW coordinates.

   If the embedding of 4-D space-time surface as section can become singular in given gauge, it will have self-intersections with dimension 2 possibly assignable to partonic 2-surfaces and maybe also string world sheets playing a key role in strong form of holography (SH). Could SH mean that information about classical and quantum theory is coded by singularities of the embedding of space-time surface to twistor bundle. This would be highly analogous to what happens in the case of complex functions and also in twistor Grassmann theory whether the amplitudes are determined by the data at singularities.

3. Where would the intersections take place? Space-time regions with Minkowskian and Euclidian signature of metric have light-like orbits of partonic 2-surfaces as intersections. These surfaces are singular in the sense that the metric determinant vanishes and tangent space of space-time surface becomes effectively 3-D: this would correspond to the reduction of tangent space dimension of algebraic surface at singularity. It is attractive to think that the lifts of Minkowskian and Euclidian space-time sheets have twistor spheres, which only intersect and have intersection matrix represented by McKay graph of Gal.

What about string world sheets? Does it make sense to regard them as intersections of 4-D surfaces? This does not look plausible idea but there are also other characterizations of string world sheets. One can also ask about the interpretation of the boundaries of string world sheets, in particular the points at the partonic 2-surfaces. How could they relate to singularities? The points of cognitive representation at partonic 2-surfaces carrying fermion number should belong to cognitive representation with embedding space coordinates belonging to an extension of rationals.

4. In Du Val theory the resolution of singularity means that one adds additional points to a double singularity: the added points form projective sphere CP_1. The blowing up process is like lifting self-intersecting curve to a non-singular curve by embedding it into 3-D space so that the original curve is its projection. Could singularity disappear as one looks at 6-D objects instead of 4-D object? Could the blowing up correspond in TGD to a transition to a new gauge in which the self intersection disappears or is shifted on new place? The intersections of 4-surfaces in 6-space analogous to roots of polynomial are topologically stable suggesting that they can be only shifted by a new choice of gauge.

   Self-intersection be a genuine singularity if the spheres CP_1 defining the fibers of the twistor bundles of branches of the space-time surface do not co-incide in the 2-D intersection. In the generic case they would only intersect in the intersection. Could the McKay diagram of Galois group characterize the intersection matrix?

5. The big vision could be following. Galois groups characterize the singularities at given level of the adelic hierarchy and code for the multiplets of primary fields and for the analogs of their fusion rules for TGD counterparts of minimal CFTs. Note that singularities themselves identified as partonic 2-surfaces and possibly also light partonic orbits and possibly even string world sheets are not restricted in any manner.

This idea need not be so far-fetched as it might look at first.

1. One considers twistor lift and self-intersections indeed occur also in twistor theory. When the M^4 projections of two spheres of twistor space CP_3 (to which the geometric twistor space $T(M^4) = M^4 \times S^2$ has a projection) have light-like separation, they intersect. In twistor diagrams the intersection corresponds to an emission of massless particle.

2. The physical expectation is that this kind of intersections could occur also for the twistor bundle associated with the space-time surface. Most naturally, they could occur along the light-like boundary of causal diamond (CD) for points with light-like separation. They could also occur along the partonic orbits which are light-like 3-surfaces defining the boundaries between Minkowskian and Euclidian space-time regions. The twistor spheres at the ends of light-like curve could intersect.

Why the number of intersecting twistor spheres should reduce to the number $n(irred, Gal)$ of irreducible representations (irreps) of Gal, which equals to $n(Gal)$ in Abelian case but is otherwise smaller? This question could be seen as a serious objection.

1. Does it make sense to think that although there are $n(Gal)$ in the local fiber of twistor bundle, the part of Galois fiber associated with the twistor fiber CP_1 has only $n(irrep, Gal)$ CP_1:s and even that the spheres could correspond to irreps of Gal. I cannot invent any obvious objection against this. What would happen that Could this mean realization of quantum classical correspondence at space-time level.

2. There are $n(irrep, G)$ irreps and $\sum_i n_i^2 = n(G)$. n_i^2 points at corresponding sheet labelled by irrep. The number of twistor spheres collapsing to single one would be n_i for n_i-D irrep so that instead of states of representations the twistor spheres would correspond to irrep. One would have analogy with the fractionization of quantum numbers. The points assignable to n_i-D representations would become effectively $1/n_i$-fractionized. At the level of base space this would not happen.

Phase transitions reducing h_{eff}/h

In TGD framework one can imagine also other kinds of singularities. The reduction of Gal to its subgroup Gal/H, where H is normal subgroup defining Galois group for the Gal as extension of Gal/H is one such singularity meaning that the H orbits of space-time sheets become trivial.

1. The action of Gal could reduce locally to a normal subgroup H so that Gal would be replaced with Gal/H. In TGD framework this would correspond to a phase transition reducing the value of Planck constant $h_{eff}/h = n(Gal)$ labelling dark matter phases to $h_{eff}/h = n(Gal/H) = n(Gal)/n(H)$. The reduction to Gal/H would occur automatically for the points of cognitive representation belonging to a lower dimensional extension having Gal/H as Galois group. The singularity would occur for the cognitive points of both space-time surface and twistor sphere and would be analogous to $n(H)$-point singularity.

2. A singularity of the discrete bundle defined by Galois group would be in question and is assumed to induce similar singularity of $n(Gal)$ -sheeted space-time surface and its twistor lift. Although the singularity would occur for the ends of strings it would induce reduction of the extension of rationals to Gal/H, which should also mean that string world sheets have representation with WCW coordinates in smaller extension of rationals.

3. This would be visible as a reduction in the spectrum of primary fields of number theoretic variant of minimal model. I have considered the possibility that the points at partonic 2-surfaces carrying fermions located at the ends of string world sheets could correspond to singularities of this kind. Could string world sheets could correspond to this kind of bundle singularities? This singularity would not have anything to do with the above described self-interactions of the twistor spheres associated with the Minkowskian and Euclidian regions meeting at light-like orbits of partonic 2-surfaces.

4. This provides a systematic procedure for constructing amplitudes for the phase transitions reducing $h_{eff}/h = n(Gal)$ to $h_{eff}/h = n(Gal/H)$. The representations of Gal would be simply decomposed to the representations of $Gal(G/H)$ in the vertex describing the phase transition. In the simplest 2-particle vertex the representation of Gal remains irreducible as representation of Gal/H. Transition amplitudes are given by overlap integrals of representation functions of group algebra representations of Gal restricted to Gal/H with those of Gal/H.

   The description of transitions in which particles with different Galois groups arrive in same diagram would look like follows. The Galois groups must form an increasing sequence $... \subset Gal_i = Gal_{i+1}/H_{i+1} \subset ....$ The representations of the largest Galois group would be decomposed to the representations of smallest Galois group so that the scattering amplitudes could be constructed using the fusion algebra of the smallest Galois group. The decomposition to should be associative and commutative and could be carried in many ways giving the same outcome at the final step.

Also quaternionic and octonionic automorphisms might be important

What about the role of subgroups of $SU(2)$? What roles they could have? Could also they classify singularities in TGD framework?

1. SU(2) is indeed realize as multiplication of quaternions. $M^8 - H$ correspondence suggests that space-time surfaces in M^8 can be regarded as associative or co-associative (normal space-is associative. Associative translates to quaternionic. Associativity makes sense also at the level of H although it is not necessary. This would mean that the tangent space of space-time surface has quaternionic structure and the multiplication by quaternions is makes sense.

2. The Galois group of quaternions is $SO(3)$ and has discrete subgroups having discrete subgroups of $SU(2)$ as covering groups. Quaternions have action on the spinors from which twistors are formed as pairs of spinors. Could quaternionic automorphisms be lifted to a an $SU(2)$ action on these spinors by quaternion multiplication? Could one imagine that the representations formed as tensor powers of these representations give finite irreps of discrete

subgroups of $SU(2)$ defining ground states of $SU(2)$ KMA a representations and define the primary fields of minimal models in this manner?

3. Galois groups for extensions of rationals have automorphic action on $SO(3)$ and its algebraic subgroups replacing matrix elements with their automorphs: for subgroups represented by rational matrices the action is trivial. One would have analogs of representations of Lorentz group $SL(2,C)$ induced from spin representations of finite subgroups $G \subset SU(2)$ by Lorentz transformations realizing the representation in Lobatchevski space. Lorentz group would be replaced by Gal and the Lobatchevski spaces as orbit with the representation of Gal in its group algebra. An interesting question is whether the hierarchy of discrete subgroups of $SU(2)$ in McKay correspondence relates to quaternionicity.

G_2 acts as octonionic automorphisms and SU(3) appears as its subgroup leaving on octonionic imaginary unit invariant. Could these semi-direct products of Gal with these automorphism groups have some role in adelic physics?

About TGD variant of ADE classification for minimal models

I already considered the ADE classification of minimal models. The first question is whether the finite subgroups $G \subset SU(2)$ are replaced in TGD context with Galois groups or with their semi-direct products $G \triangleleft Gal$. Second question concerns the interpretation of the Dynkin diagram of affine ADE type Lie algebra. Does it correspond to a real dynamical symmetries.

1. Could the MacKay correspondence and ADE classification generalize? Could fusion algebras of minimal models for KMA associated with general compact Lie group G be classified by the fusion algebras of the finite subgroups of G. This generalization seems to be discussed in [B49] (see `http://tinyurl.com/ycmyjukn`).

2. Could the fusion algebra of Galois group Gal give rise to a generalization of the minimal model associated with a KMA of Lie group $G \supset Gal$. The fusion algebra of Gal would be identical with that for the primary fields of KMA for G. Galois groups could be also grouped to classes consisting of Galois groups appearing as a subgroup of a given Lie group G.

3. In TGD one has a fractal hierarchy of isomorphic supersymplectic algebras (SSAs) (the conformal weights of sub-algebra are integer multiples of those of algebra) with gauge conditions stating that given sub-algebra of SSA and its commutator with the entire algebra annihilates the physical states. The remnant of the full SSA symmetry algebra would be naturally KMA.

   The pair formed by full SSA and sub-SSA would correspond to pair formed by group G and normal subgroup H and the dynamical KMA would correspond to the factor group G/H. This conjecture generalizes: one can replace G with Galois group and $SU(2)$ KMA with a KMA continuing Gal as subgroup. One the other hand, one has also hierarchies of extensions of rationals such that $i+1$:th extension of rationals is extension of i:th extension. G_i is a normal subgroup of G_{i+1} so that the group $Gal_{i+1,i} = Gal_{i+1}/Gal_i$ acts as the relative Galois group for $i+1$:th extension as extensions of i:th extension.

   This suggest the conjecture that the Galois groups Gal_i for extension hierarchies correspond to the inclusion hierarchies $SSA_i \supset SSA_{i+1}$ of fractal sub-algebras of SSA such that the gauge conditions for SSA_i define a hierarchy KMA_i of dynamical KMAs acting as dynamical symmetries of the theory. The fusion algebra of KMA_i theory would be characterized by Galois group Gal_i.

4. I have considered the possibility that the McKay graphs for finite subgroups $G \subset SU(2)$ indeed code for root diagrams of ADE type KMAs acting as dynamical symmetries to be distinguished from $SU(2)$ KMA symmetry and from fundamental KMA symmetries assignable to the isometries and holonomies of $M^4 \times CP_2$.

   One can of course ask whether also the fundamental symmetries could have a representation in terms of Gal or its semi-direct product $G \triangleleft Gal$ with a finite sub-group automorphism group $SO(3)$ of quaternions lifting to finite subgroup $G \subset SU(2)$ acting on spinors. This is not necessary since Gal can form semidirect products with the algebraic subgroups of Lie groups of fundamental symmetries (Langlands program relies on this). In the generic case the algebraic subgroups spanned by given extension of rationals are infinite. When the finite subgroup $G \subset SU(2)$ is closed under Gal automorphism, the situation changes, and these extensions are expected to be in a special role physically.

The number theoretic generalization of the idea that affine ADE group acts as symmetries would be roughly like following. The nodes of the McKay graph of $G \triangleleft Gal$ label its irreps, which should be in 1-1 correspondence with the Cartan algebra of the KMA. The KMA counterparts of the local bilinear Gal invariants associated with Gal irreps would give currents of dynamical KMA having unit conformal weight. The convolution of primary fields with respect to conformal weight would be completely analogous to that occurring in the expression of energy momentum tensor as local bilinears of KMA currents.

If the free field construction using the local invariants as Cartan algebra defined by the irreps of $G \triangleleft Gal$ works, it gives rise to charged primary fields for the dynamical KMA labelled by roots of the corresponding Lie algebra. For trivial Gal one would have ADE group acting as dynamical symmetries of minimal model associated with $G \subset SU(2)$.

5. Number theoretic Langlands conjecture [L32] [L26] generalizes this to the semidirect product $G_0 \triangleleft Gal$ algebraic subgroup G_0 of the original KMA Lie group (p-adicization allows also powers of roots of e in extension). One can imagine a hierarchy of KMA type algebras KMA_n obtained by repeating the procedure for the $G_1 \triangleleft Gal$, where G_1 is discrete subgroup of the new KMA Lie group.

6. In CFTs are also other ways to extend VA or SVA (Super-Virasoro algebra) to a larger algebra by discovering new dynamical symmetries. The hope is that symmetries would allow to solve the CFT in question. The general constraint is that the conformal weights of symmetry generators are integer or half-integer valued. For the energy momentum tensor defining VA the conformal weight is $h = 2$ whereas the conformal weights of primary fields for minimal models are rational numbers.

   The simplest extension is SVA involving super generators with $h = 3/2$. Extension of (S)VA by (S)KMA so that (S)VA acts by semidirect product on (S)KMA means adding (S)KMA generators with with $h = 1$ (and 1/2). The generators of W_n-algebras (see `http://tinyurl.com/y93f6eoo`) have either integer or half integer conformal weights and the algebraic operations are defined as ordered products (an associative operation). These extensions are different from the proposed number theoretic extension for which the restriction to a discrete subgroup of KMA Lie group is essential.

14.5 Appendix

I have left the TGD counterpart of *fake flatness condition* in Appendix. Also a little TGD glossary is included.

14.5.1 What could be the counterpart of the fake flatness in TGD framework?

Schreiber considers the n-variant of gauge field concept with gauge potential A and gauge field $F = DA$ replaced with a hierarchy of gauge potential like entities $A^{k)}$, $k = 1,..,n$ with $DA^{n)} = 0$ and ends up in $n = 2$ case to what he calls *fake flatness condition* $DA^{1)} = A^{2)}$. This raises a chain of questions.

Could higher gauge fields of Schreiber and Baez [B65, B51] provide a proper description of the situation in finite measurement resolution? Could induction procedure make sense for them? Should one define the projections of the classical fields by replacing ordinary H-coordinates with their quantum counterparts? Could these reduce to c-numbers for number-theoretically commutative 2-surfaces with commutative tangent space? What about second fundamental form orthogonal to the string world sheet? Must its trace vanish so that one would have minimal 2-surface?

The proposal of Schreiber is a generalization of a massless gauge theory. My gut feeling is that the non-commutative counterpart of space-time surface is not promising in TGD framework. My feelings are however mixed.

1. The effective reduction of SSA and PSCA to quantal variants of Kac-Moody algebras gives rise to a theory much more complex than gauge theory. On the other hand, the reduction to Galois groups by finiteness of measurement resolution would paradoxically reduce TGD to extremely simple theory.

2. Analog of Yang Mills theory is not enough since it describes massless particles. Massless states in 4-D sense are only a very small portion of the spectrum of states in TGD. Stringy mass squared spectrum characterizes these theories rather than massless spectrum. On the other hand, in TGD particles are massless in 8-D sense and this is crucial for the success of generalized twistor approach.

3. To define a generalization of gauge theory in WCW one needs homology and cohomology for differential forms and their duals. For infinite-dimensional WCW the notion of dual is difficult to define. The effective reduction of SSA and PSCA to SKMAs could however effectively replace WCW with a coset space of the Lie-group associated with SKMA and finite dimension would allow tod define dual.

4. The idea about non-Abelian counterparts of Kähler gauge potential A and J in WCW does not look promising and the TGD counterpart of the *fake flatness condition* does not however encourage this.

Just for curiosity one could however ask whether one could generalize the Kähler structure of WCW to n-Kähler structure to describe finite measurement resolution at the level of WCW and whether also now something analogous to the *fake flatness condition* emerges. The "fake flatness" condition has interesting analogy in TGD framework starting from totally different geometric vision.

1. SSA acts as dynamical symmetries on fermions at string world sheets. Gauge condition would make the situation effectively finite-dimensional and allow to define if the effectively finite-D variant of WCW n-structures using ordinary homotopies and homology and cohomology. Also n-gauge fields could be defined in this effectively finite-D WCW and they would allow a description in terms of string world sheets. The interpretation could be in terms of generalization of Bohm-Aharonov phase from space-time level to Berry phase in abstract configuration space defined now in reduced WCW.

2. The Kähler form of $H = M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) can be induced to space-time level. When limited to the string world sheet is both the curvature form of Kähler potential and the analog of flat 2-connection defining the 1-connection in the approaches of Schreiber's and Baez so that one would have $B = J$ and $dB = 0$.

3. 2-form J as it is interpreted in Screiber's approach is hwoever not enough to construct WCW geometry. The generalization of the geometry of $M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) to involve higher forms and its induction to the space-time level and level of WCW looks rather awkward idea and does not bring in anything new.

14.5.2 A little glossary

Topological Geometrodynamics (TGD): TGD can be regarded as a unified theory of fundamental interactions. In General Relativity space-time time is abstract pseudo-Riemannian manifold and the dynamical metric of the space-time describes gravitational interactions. In TGD space-time is a 4-dimensional surface of certain 8-dimensional space, which is non-dynamical and fixed by either physical or purely mathematical requirements. Hence space-time has shape besides metric properties. This identification solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity. Even more, sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry behind General Relativity, leads to a geometrization of known fundamental interactions and elementary particle quantum numbers.

Many-sheeted space-time, topological quantization, quantum classical correspondence (QCC): TGD forces the notion of many-sheeted space-time (see http://tinyurl.com/mf99gpv) with space-time sheets identified as geometric correlates of various physical objects (elementary particles, atoms, molecules, cells, ..., galaxies, ...). Quantum-classical correspondence (QCC) states that all quantum notions have topological correlates at the level of many-sheeted space-time.

Topological quantization: Topological field quantization is one of the basic distinctions between TGD and Maxwell's electrodynamics and GRT and means that various fields decompose to topological field quanta: radiation fields to "topological light rays"; magnetic fields to flux tube structures; and electric fields to electric flux quanta (electrets). Topological field quantization means that one can assign to every material system a field (magnetic) body, usually much larger than the material system itself, and providing a representation for various quantum aspects of the system.

Strong form of holography (SH): SH states that space-time surfaces as preferred extremals can be constructed from the data given at 2-D string world sheets and by a discrete set of points defining the cognitive representation of the space-time surface as points common to real and various p-adic variants of the space-time surface (intersection of realities and various p-adicities). Points of the cognitive representation have embedding space coordinates in the extension of rationals defining the adele in question. Effective 2-dimensionality is a direct analogy for the continuation of 2-D data to analytic function of two complex variables.

Zero energy ontology (ZEO): In ZEO quantum states are replaced by pairs of positive and negative energy states having opposite total quantum numbers. The pair corresponds to the pair of initial and final state for a physical event in classical sense. The members of the pair are at opposite boundaries of causal diamond (CD) (see http://tinyurl.com/mh9pbay), which is intersection of future and past directed light-cones with each point replaced with CP_2. Given CD can be regarded as a correlate for the perceptive field of conscious entity.

p-Adic physics, adelic physics, hierarchy of Planck constants, p-adic length scale hypothesis: p-Adic physics is a generalization of real number based physics to p-adic number fields and interpreted as a correlate for cognitive representations and imagination. Adelic physics fuses real physics with various p-adic physics ($p = 2, 3, 5, ...$) to adelic physics. Adele is structure formed by reals and extensions of various p-adic number fields induced by extensions of rationals forming an evolutionary hierarchy. Hierarchy of Planck constants corresponds to the hierarchy of orders of Galois groups for these extensions. Preferred p-adic primes satisfying p-adic length scale hypothesis $p \simeq 2^k$, are so called ramified primes for certain extension of rationals appearing as winners in algebraic evolution.

Cognitive representation: Cognitive representation corresponds to the intersection of the sensory and cognitive worlds - realities and p-adicities - defined by real and p-adic space-time surfaces. The points of the cognitive representation have H-coordinates which belong to an extension of rationals defining the adele. The choice of H-coordinates is in principle free but symmetries of H define preferred coordinates especially suitable for cognitive representations. The Galois group of the extension of rationals has natural action in the cognitive representation, and one can decompose it into orbits, whose points correspond the sheets of space-time surface as Galois covering. The number n of sheets equals to the dimension of the Galois group in the general case and is identified as the value $h_{eff}/h = n$ of effective Planck constant characterizing levels in the dark matter hierarchy. One can also consider replacing space-time surfaces as points of WCW with their cognitive representations defined by the cognitive representation of the space-time surface and defining the natural coordinates of WCW point.

Quantum entanglement, negentropic entanglement (NE), Negentropy maximization principle (NMP): Quantum entanglement does not allow any concretization in terms of everyday concepts. Schrödinger cat is the classical popularization of the notion (see http://tinyurl.com/lpjcjm9): the quantum state, which is a superposition of the living cat + the open bottle of poison and the dead cat + the closed bottle of poison represents quantum entangled state. Schrödinger cat has clearly no self identity in this state.

In adelic physics one can assign to the same entanglement both real entropy and various p-adic negentropies identified as measures of conscious information. p-Adic negentropy - unlike real - can be positive, and one can speak of negentropic entanglement (NE). Negentropy Maximization Principle (NMP) states that it tends to increase. In the adelic formulation NMP holding true only in statistical sense is a consequence rather than separate postulate.

Self, subself, self hierarchy: In ZEO self is generalized Zeno effect. At the passive boundary nothing happens to the members of state pairs and the boundary remains unaffected. At active boundary members of state pairs change and boundary itself moves farther away from the passive boundary reduction by reduction inducing localization of the active boundary in the moduli space of CDs after unitary evolution inducing delocalization in it. Self dies as the first reduction takes place at opposite boundary. A self hierarchy extending from elementary particle level to the level of the entire Universe is predicted. Selves can have sub-selves which they experience as mental images. Sub-selves of two separate selves can quantum entangle and this gives rise to fusion of the mental images and the fused mental image is shared by both selves.

Sensory representations: The separation of data processing and its representation is highly desirable. In computers processing of the data is performed inside CPU and representation is realized outside it (monitor screen, printer,...). In standard neuroscience it is however believed that both data processing and representations are realized inside brain. TGD leads the separation of data processing and representations: the "manual" of the material body provided by field (or magnetic) body serves as the counterpart of the computer screen at which the sensory and other representations of the data processed in brain are realized. Various attributes of the objects of the perceptive field processed by brain are quantum entangled with simple "something exists" mental images at the MB. The topological rays of EEG serve are the electromagnetic bridges serving as the topological correlates for this entanglement.

Chapter 15

Is Non-associative Physics and Language Possible only in Many-Sheeted Space-time?

15.1 Introduction

In Thinking Allowed Original (see https://www.facebook.com/groups/thinkallowed/) there was very interesting link added by Ulla about the possibility of non-associative quantum mechanics (see http://phys.org/news/2015-12-physicists-unusual-quantum-mechanics.html#jCp).

Also I have been forced to consider this possibility.

1. The 8-D embedding space of TGD has octonionic tangent space structure and octonions are non-associative. Octonionic quantum theory however has serious mathematical difficulties since the operators of Hilbert space are by definition associative. The representation of say octonionic multiplication table by matrices is possible but is not faithful since it misses the associativity. More concretely, so called associators associated with triplets of representation matrices vanish. One should somehow transcend the standard quantum theory if one wants non-associative physics.

2. Associativity seems to be fundamental in quantum theory as we understand it recently. Associativity is a fundamental and highly non-trivial constraint on the correlation functions of conformal field theories. It could be however broken in weak sense: as a matter of fact, Drinfeld's associator emerges in conformal field theory context. In TGD framework classical physics is an exact part of quantum theory so that quantum classical correspondence suggests that associativity could play a highly non-trivial role in classical TGD. The conjecture is that associativity requirement fixes the dynamics of space-time sheets - preferred extremals of Kähler action - more or less uniquely. One can endow the tangent space of 8-D imbedding $H = M^4 \times CP_2$ space at given point with octonionic structure: the 8 tangent vectors of the tangent space basis obey octonionic multiplication table.

   Space-time realized as n-D surface in 8-D H must be either associative or co-associative: this depending on whether the tangent space basis or normal space basis is associative. The maximal dimension of space-time surface is predicted to be the observed dimension $D = 4$ and tangent space or normal space allows a quaternionic basis.

3. There are also other conjectures [L10] about what the preferred extremals of Kähler action defining space-time surfaces are.

   (a) A very general conjecture states that strong form of holography allows to determine space-time surfaces from the knowledge of partonic 2-surfaces and 2-D string world sheets.

(b) Second conjecture involves quaternion analyticity and generalization of complex structure to quaternionic structure involving generalization of Cauchy-Riemann conditions.

(c) $M^8 - M^4 \times CP_2$ duality stating that space-time surfaces can be regarded as surfaces in either M^8 or $M^4 \times CP_2$ is a further conjecture.

(d) Twistorial considerations select $M^4 \times CP_2$ as a completely unique choice since M^4 and CP_2 are the only spaces allowing twistor space with Kähler structure. The conjecture is that preferred extremals can be identified as base spaces of 6-D sub-manifolds of the product $CP_3 \times SU(3)/U(1) \times U(1)$ of twistor spaces associated with M^4 and CP_2 having the property that it makes sense to speak about induced twistor structure.

The "super(optimistic)" conjecture is that all these conjectures are equivalent.

The motivation for what follows emerged from the observation that language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are however associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D embedding space $M^4 \times CP_2$ is one of the key conjectures of TGD.

But what about many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes-that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA.

Fortunately the mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity. With good luck this breaking of associativity is all that is needed. With not so good luck this breaking of associativity takes place already at the level of single space-time sheets and something else is needed in many-sheeted space-time.

15.2 Is Non-associative Physics Possible In Many-sheeted Space-time?

The key question in the sequel is whether non-associative physics could emerge in TGD via *many-sheeted* space-time as an outcome of many-sheetedness and therefore distinguishing TGD from GRT and various QFTs.

15.2.1 What Does Non-associativity Mean?

To answer this question one must first understand what non-associativity could mean.

1. In non-associative situation brackets matter. $A(BC)$ is different from $(AB)C$. Here AB need not be restricted to a product or sum: it can be anything depending on A and B. From schooldays or at least from the first year calculus course one recalls the algorithm: when calculating the expression involving brackets one first finds the innermost brackets and calculates what is inside them, then proceed to the next innermost brackets, etc... In

computer programs the realization of the command sequences involving brackets is called parsing and compilers perform it. Parsing involves decomposition of program to modules calling modules calling.... Quite generally, the analysis of linguistic expressions involves parsing. Bells start to ring as one realizes that parsings form a hierarchy as also do the space-time sheets!

2. More concretely, there is hierarchy of brackets and there is also a hierarchy of space-time sheets labelled by p-adic primes and perhaps also by Planck constants $h_{eff} = n \times h$. B and C inside brackets form (BC), something analogous to a bound state or chemical compound. In TGD this something could correspond to a "glueing" space-time sheets B and C at the same larger space-time sheet. More concretely, (BC) could correspond to braided pair of flux tubes B and C inside larger flux tube, whose presence is expressed as brackets (..). As one forms $A(BC)$ one puts flux tube A and flux tube (BC) containing braided flux tubes B and C inside larger flux tube. For $(AB)C$ flux one puts tube (AB) containing braided flux tubes A and B and tube C inside larger flux tube. The outcomes are obviously different.

3. Non-associativity in this sense would be a key signature of many-sheeted space-time. It could show itself in say molecular chemistry, where putting on same sheet could mean formation of chemical compound AB from A and B. Another highly interesting possibility is hierarchy of braids formed from flux tubes: braids can form braids, which in turn can form braids,... Flux tubes inside flux tubes inside... Maybe this more refined breaking of associativity could underly the possible non-associativity of biochemistry: biomolecules looking exactly the same would differ in subtle manner.

4. What about quantum theory level? Non-associativity at the level of quantum theory could correspond to the breaking of associativity for the correlation functions of n fields if the fields are not associated with the same space-time sheet but to space-time sheets labelled by different p-adic primes. At QFT limit of TGD giving standard model and GRT the sheets are lumped together to single piece of Minkowski space and all physical effects making possible non-associativity in the proposed sense are lost. Language would be thus possible only in TGD Universe!

15.2.2 Language And Many-sheeted Physics?

Non-associativity is an essentially linguistic phenomenon and relates therefore to cognition. p-Adic physics labelled by p-adic primes fusing with real physics to form adelic physics are identified as the physics of cognition in TGD framework.

1. Could many-sheeted space-time of TGD provides the geometric realization of language like structures? Could sentences and more complex structures have many-sheeted space-time structures as geometrical correlates? p-Adic physics as physics of cognition would suggest that p-adic primes label the sheets in the parsing hierarchy. Could bio-chemistry with the hierarchy of magnetic flux tubes added, realize the parsing hierarchies?

2. DNA is a language and might provide a key example about parsing hierarchy. The mystery is that human DNA and DNAs of most simplest creatures do not differ much. Our cousins have almost identical DNA with us. Why do we differ so much? Could the number of parsing levels be the reason- p-adic primes labelling space-time sheets? Could our DNA language be much more structured than that of our cousins. At the level of concrete language the linguistic expressions of our cousin are indeed simple signals rather than extremely complex sentences of old-fashioned German professor forming a single lecture each. Could these parsing hierarchies realize themselves as braiding hierarchies of magnetic flux tubes physically and - more abstractly - as analos of parsing hierarchies for social structures. Indeed, I have proposed that the presence of collective levels of consciousness having the hierarchy of magnetic bodies as a space-time correlates distinguishes us from our cousins so that this explanation is consistent with more quantitative one relying on language.

3. I have also proposed that intronic portion of DNA is crucial for understanding why we differ so much from our cousins [K4, K100]. How does this view relate to the above proposal? In

the simplest model for DNA as topological quantum computer introns would be connected by flux tubes to the lipids of nuclear and cell membranes. This would make possible topological quantum computations with the braiding of flux tubes defining the topological quantum computer program.

Ordinary computer programs rely on computer language. Same should be true about quantum computer programs realized as braidings. Now the hierarchical structure of parsings would correspond to that of braidings: one would have braids, braids of braids, etc... This kind of structure is also directly visible as the multiply coiled structure of DNA. The braids beginning from the intronic portion of DNA would form braided flux tubes inside larger braided flux tubes inside.... defining the parsing of the topological quantum computer program. The higher the number of parsing levels, the higher the position in the evolutionary hierarchy. Each braiding would define one particular fundamental program module and taking this kind of braided flux tubes and braiding them would give a program calling these programs as sub-programs.

4. The phonemes of language have no meaning to us (at our level of self hierarchy) but the words formed by phonemes and involving at basic level the braiding of "phoneme flux tubes" would have. Sentences and their substructures would in turn involve braiding of "word flux tubes". Spoken language would correspond to temporal sequences of braidings of flux tubes at various hierarchy levels.

5. The difference between us and our cousins (or other organisms) would not be at the level of visible DNA but at the level of magnetic body. Magnetic bodies would serve as correlates also for social structures and associated collective levels of consciousness. The degree of braiding would define the level in the evolutionary hierarchy. This is of course the basic vision of TGD inspired quantum biology and quantum bio-chemistry in which the double formed by organism and environment is completed to a triple by adding the magnetic body.

15.2.3 What About The Hierarchy Of Planck Constants?

p-Adic hierarchy is not the only hierarchy in TGD Universe: there is also the hierarchy of Planck constants $h_{eff} = n \times h$ giving rise to a hierarchy of intelligences. What is the relationship between these hierarchies?

1. I have proposed that speech and music are fundamental aspects of conscious intelligence and that DNA realizes what I call bio-harmonies in quite concrete sense [L13] [K78]: DNA codons would correspond to 3-chords. DNA would both talk and sing. Both language and music are highly structured. Could the relation of h_{eff} hierarchy to language be same as the relation of music to speech?

2. Are both musical and linguistic parsing hierarchies present? Are they somehow dual? What does parsing mean for music? How musical heard sounds could give rise to the analog of braided strands? Depending on the situation we hear music both as separate notes and as chords as separate notes fuse in our mind to a larger unit like phonemes fuse to a word. Could chords played by single instrument correspond to braidings of flux tubes at the same level? Could the duality between linguistic and musical intelligence (analogous to that between function and its Fourier transform) be very concrete and detailed and reflect itself also as the possibility to interpret DNA codons both as three letter words and as 3-chords [L13]?

15.3 Braiding Hierarchy Mathematically

More precise formulation of the braided flux tube hierarchy leads naturally to the notions of braid group and operad that I have considered earlier. They have a close relationship with quantum groups - more precisely, bialgebras and Hopf algebras and their generalizations quasi-bialgebras and quasi-Hopf algebras, which in turn allow to characterize what might be called minimal breaking of associativity in terms of Drinfeld associator. These notions are already familiar from conformal field theories and string theories them so that there are good hopes that no completely new mathematics is not needed.

It must be made clear that I am not a mathematician and the following is just a modest attempt to understand what the problem is. I try to identify the algebraic structure possibly allowing to realize the big vision and gather some results about these structures from Wikipedia: I confess that I do not understand the formulas at the deeper level and my goal is to find their physical interpretation in TGD framework.

15.3.1 How To Represent The Hierarchy Of Braids?

Before going to web to see how modern mathematics could help in the problem, try first to formulate the situation more concretely. One must consider a more detailed representation for braids and for their hierarchy.

Consider first rough physical geometric view about braids of braids represented in terms of flux tubes.

1. Braid strands have two ends: one can label them as "lower" and "upper". Flux tubes can be labelled by p-adic prime p and $h_{eff} = n \times h$. Magnetic flux tubes can carry monopole flux and this could be crucial for the breaking of associativity - at least it is so in the proposed model (see `http://tinyurl.com/y7oom5kh`). The possibility of apparent magnetic monopoles in TGD framework indeed involves many-sheetedness in an essential manner: monopole flux flows from space-time sheet to another one through wormhole contact. This can be taken as one possible hint about the concrete physics involved.

2. One can get more precise picture by using formulas. One has labelling of flux tubes by primes p and Planck constants h_{eff}: to be short call this label $a, b, c, ...$ Since the values of p and h_{eff} are graded one could also speak of grading. The states for given value of a assignable to braid strands are labelled by the quantum states $A, B, ...$ associated with them and analogous to algebra elements. One must however consider all possible situations so that has operators $A_a, B_a, ...$ analogous to algebra elements of a graded algebra about which Clifford algebras and super-algebras are familiar examples.

3. Consider now the physical interpretation for the breaking of associativity. For ordinary associative algebra one considers $A(BC) = (AB)C$. This condition as such make sense if $A(BC)$ and $(AB)C$ are inside same flux tube and perhaps also that the strands A, B, C are not braids. In the general case one must must add the labels a, b, c, d and a, b_1, c_1, d_1 and one obtains $((A_d B_d)_c C_b)_a$ and $(A_{b_1}(B_{d_1} C_{d_1}))_{c_1})_a$. Obviously, these two states need not identical unless one has $a = b = c = d = b_1 = c_1 = d_1$, which is also possible and means that all strands are at the same flux tube labelled by a. The challenge is to combine various almost copies of algebraic structure defined by braidings and labelled by $a, b, ..$ to larger algebraic structure and formulate the breaking of associativity for this structure.

15.3.2 Braid Groups As Coverings Of Permutation Groups

Consider next the definition of braid group.

1. The notion of braiding can be algebraized using the notion of braid group B_n of n strands, which is covering of the permutation group S_n. For ordinary permutations generating permutations are exchanges of P_i two neighboring elements in the ordered set $(a_1, ..., a_n)$: $(a_i, a_{i+1}) \rightarrow (a_{i+1}, a_i)$. Obviously one has P_i^2 so that permutation is analogous to reflection. For braid group permutation is replaced to twisting of neighboring braid strand. It looks like permutation if one looks at the ends of strands only. If one looks entire strands, there is no reason to have $P_i^2 = 1$ except possibly for the representation of braid group. For arbitrarily large n that one has $P_i^n \neq 1$. 2-D braid group B_n can be represented as a homotopies of 2-D plane with n punctures identifiable as ends of braid strands defined by their non-intersecting orbits.

2. At the level of quantum description one must allow quantum superpositions of different braidings and must describe the quantum state of braid as wave function in braid group: one has element of group algebra of braid group. To each element of braid group one can assign unitary matrix representing the braiding and this unitary matrix would define a "topological

time evolution" defined by braiding transforming the initial state at the lower end of braid to the state at upper end of braid. Hence it seems that braid group algebra is the proper mathematical notion. One has quantum superposition of topological time evolutions: something rather abstract.

15.3.3 Braid Having Braids As Strands

Many-sheeted space-time makes possible fractal hierarchy of braids. Braid group in above sense would act on flux tubes at the same space-time sheets or space-time of QFT and GRT. Braids can have as strands braids so that there is hierarchy of braiding levels. The hierarchy of coilings of DNA provides a simple example (very simple having not much to do with the hierarchy of braidings for flux tubes).

1. Suppose that one has only two levels in the hierarchy. One has n braid strands/flux tubes altogether and there are k larger flux tubes containing n_i, $i = 1,..,k$ flux tubes so that one has $\sum_{i=1}^{k} n_i = n$. One can imagine a coloring of the braid strands inside given flux tube characterizing it. Only braid strands inside same flux tube - with the same color - can be braided. The full braid group B_n braiding freely all n braid strands is restricted to a subgbroup $B_{n_1} \times .... \times B_{n_2}$. This group can be regarded as subgroup of B_n so that permutations of B_{n_i} have a well-defined outcome, which seems however to be trivial classically. In quantum situation the exchange of the factors B_{n_i} however corresponds to braiding and for non-trivial quantum deformations its action is non-trivial. One has braided commutativity instead of commutativity.

2. Besides this there are braidings for the k braids of braids and this gives braid group B_k acting at upper level of hierarchy. Clearly the higher level braids b_i, $i = 1,...,k$ and lower level braids b_{ij}, $j = 1,...,n_i$ form a two-levelled entity. The braid groups B_k and B_{n_i} form an algebraic entity such that B_k acts by permuting the entities. Same holds true for the braid group algebras. This structure generalizes to an entire hierarchy of braid groups and their group algebras.

The hierarchy of braid group algebras seems to closely relate to a very general notion known as operad (see http://tinyurl.com/yavyhcsk). The key motivation of the operad theory is to model the computational trees resulting from parsing. The action of permutations/braidings on the basic objects is central notion and one indeed has hierarchy of symmetric groups/braid groups such that the symmetric/braid group at $n+1$:th level permutes/braids the objects at n:th level. Now the objects would be braids whose strands are braided. The braids can be strands of higher level braids and these strands can be braided. The action of braidings extends to that on braid group algebras defining candidates for wave functions.

15.4 General Formulation For The Breaking Of Associativity In The Case Of Operads

The formulas characterizing weak form of associativity by Drinfeld and others look rather mysterious without understanding of their origins. This understanding emerges from very simple but general basic arguments. Instead of studying given algebra one transcends to a higher abstraction level and studies - not the results of algebraic expressions - but the very process how the algebraic expression is evaluated and what kind of rules one can pose on it. The rules can be abstracted to what is called algebraic coherence.

The evaluation process - parsing - starts from inner most brackets and proceeds outwards so that eventually all brackets have disappeared and one has the value for the expression. This process can be regarded as a tree which starts from n inputs which are algebra elements, in the recent case they could be braid group algebra elements.

For instance, $(AB)C$ corresponds to an tree in which A, B, C are the branches. As one comes downwards, A and B fuse in the upper node and AB and C in the lower node. One manner to see this is as particle reaction proceeding backwards in time. For $A(BC)$ B and C fuse to BC in the upper node and A and BC at the lower node. Associativity says that the two trees give the same result. "Braided associativity" would say that these trees give results differing by an isomorphism just as braided commutativity says that AB and BA give results differing by isomorphism.

One can formulate this more concretely by denoting algebra decomposition $A\otimes B \in V\otimes V \rightarrow AB \in V$ by θ. In associativity condition one has 3 inputs so that 3-linear map $V \otimes V \otimes V \rightarrow V$ is in question. $(AB)C$ corresponds to $\theta \circ (\theta, 1)$ applied to $(A \otimes B \otimes C)$. Indeed, $(\theta, 1)$ gives $(AB, C) \in V \otimes V$. Second step $\theta\circ$ applied to this gives $(AB)C$. In the same manner, $A(BC)$ corresponds to $(\theta \circ (1, \theta)$ and associativity condition can be expressed as

$$\theta \circ (\theta, 1) = \theta \circ (1, \theta) \ .$$

An important delicacy should be mentioned. Although operations can be non-associative, the composition of operations is assumed to be associative. One can imagine obtaining $((ab)c)d$ either by $\theta \circ (\theta, 1) \circ (\theta, 1, 1))$ or by $(\theta \circ (\theta, 1)) \circ (\theta, 1, 1))$. The condition that these expressions are identical is completely analogous to the associativity for the composition of functions $f \circ (g \circ h) = (f \circ g) \circ h$ and this axiom looks obvious becomes one is used to *define* $f \circ g$ using this formula (starting from rightmost brackets). One could however imagine starting the evaluation of the composition of operators also from leftmost brackets. This makes sense if the composition can be done without the substitution of the value of argument.

15.4.1 How Associativity Could Be Broken?

How to obtain the breaking of associativity? The first thing is to get some idea about what (weak) breaking of associativity could mean.

Breaking of associativity at the level of algebras

Basic examples about breaking of associativity might help in the attempts to understand how many-sheetedness could induce the breaking of associativity. The intuitive feeling is that the effect is not large and disappears at QFT limit of TGD.

In the case of algebras one has bilinear map $V\otimes V \rightarrow V$. Now this map is from $V\otimes V \rightarrow V\otimes V$ so that the two situations need not have much common. Despite this one can look the situation in the case of algebras.

Lie-algebras and Jordan algebras represent key examples about non-associative algebras. Associative algebras, Lie-algebras, and Jordan algebras can be unified by weakning the associativity condition $A(BC) = (AB)C$ to a condition obtained by cyclically symmetrizing this condition to get the condition

$$A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$$

plus the condition

$$(A^2B)A = A^2(BA)$$

defining together with commutativity condition $AB = BA$ Jordan algebra (`http://tinyurl.com/y8n9ol9p`). Note that Jordan algebra with multiplication $A \cdot B$ is realized in terms of associative algebra product as $A \cdot B = (AB + BA)/2$. A good guess is that the non-associative Malcev algebra formed by imaginary octonions with product $xy - yx$ satisfies these conditions.

Could the analog of the condition $A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$ make sense also for the braiding group algebra assignable to quantum states of braids? The condition would say that cyclic symmetrization by superposing different braiding topologies gives a quantum state, which is in well-defined sense associative. Cyclic symmetry looks attractive because it plays also a key role in twistor Grassmannian approach.

Bi-algebras and Hopf algebras

One must start from bi-algebra $(B, \nabla, \eta, \Delta, \epsilon)$. One has product ∇ and co-product Δ analogous to replication of algebra element: particle physicists has tendency to see it as "time reversal" of product analogous to particle decay as reversal of particle fusion. The key idea is that co-multiplication is algebra homomorphism for multiplication and multiplication algebra homomorphism for co-multiplication. This leads to four commutative diagrams essentially expressing this property (see `http://tinyurl.com/y897z3es`).

Instead of giving the general definitions it is easier to consider concrete example of bi-algebra defined by group algebra. Bi-algebra has product $\nabla : H\otimes H \rightarrow H$ and co-product $\Delta : H \rightarrow H\otimes H$, which intuitively corresponds to inverse or time reversal of product. In the case of group algebra this holds true in very precise sense since one has $\Delta(g) = g\otimes g$: Δ is clearly analogous to replication. Besides this one has map $\epsilon : H \rightarrow K$ assigning to the algebra element a scalar and inverse map taking the unit 1 of the field to unit element of H, called also 1 in the following. For group algebra one has $\epsilon(g) = 1$. Bi-algebras are associative and co-associative. Commutativity is however only braided commutativity.

Hopf algebra $(H, \nabla, \eta, \Delta, \epsilon, S)$ is special case of bi-algebra and often loosely called quantum group. The additional building brick is algebra anti-homomorphism $S : H \rightarrow H$ known as antipode. S is analogous to mapping element of h to its inverse (it need not exist always). For group algebra one indeed has $S(g) = g^{-1}$. Besides the four commuting diagrams for bi-algebra one has commutative diagrams $\nabla(S, 1)\Delta = \eta\epsilon$ and $\nabla(1, S)\Delta = \eta\epsilon$, where ϵ is co-unit. The right hand side gives a scalar depending on h multiplied by unit element of H. For group algebra this gives unit at both sides. In the general case the situation $\Delta(h) = h \otimes h$ is true for group like element only and one has more complex formula $\Delta(h) = \sum_i a_i \otimes b_i$. One also defines primitive elements as elements satisfying $\Delta(h) = h \otimes 1 + 1 \otimes h$. Also Hopf algebras are associative and co-associative.

Quasi-bialgebras and quasi-Hopf algebras

Quasi-bi-algebras giving as special case quasi-Hopf algebras were discovered by Russian mathematician Drinfeld (for technical definition, which does not say much to non-specialist see `http://tinyurl.com/y7b6lpop` and `http://tinyurl.com/y89cs5oy`). They are non-associative or associative modulo isomoprhism.

Consider first quasi-bi-algebra $(B, \Delta, \epsilon, \Phi, l, r)$. Δ and ϵ are as for bi-algebra. Besides this one has invertible elements Φ (Drinfeld associator) and r, l called right and lef unit constraints. The conditions satisfied are following

- $$(1 \otimes \Delta) \circ \Delta(a) = \Phi[((\Delta \otimes 1) \circ \Delta(a)]\Phi^{-1} \quad .$$

  For $\Phi = 1 \otimes 1 \otimes 1$ one obtains associativity.

- $$[(1 \otimes 1 \times \Delta)(\Phi)][(\Delta \otimes 1 \otimes 1)(\Phi)] = (1 \otimes \Phi)[1 \otimes \Delta \otimes 1)(\Phi)(\Phi \otimes 1) \quad .$$

- $$(\epsilon \otimes 1)(\Delta(a)) = l^{-1}al \quad , \quad (1 \otimes \epsilon)(\Delta(a)) = r^{-1}ar \quad .$$

- $$1 \otimes \epsilon \otimes 1)(\Phi) = 1 \otimes 1 \quad .$$

These mysterious looking conditions express the fact that Drinfeld associator is a bialgebra co-cycle.

Quasi-bialgebra is braided if it has universal R-matrix which is invertible element in $B \otimes B$ such that the following conditions hold true.

$$(\Delta^{op})(a) = R\Delta(a)R^{-1} \quad . \tag{15.4.1}$$

Note that for group algebra with $\Delta g = g \otimes g$ one has $\Delta^{op} = \Delta$ so that R must commute with Δ. Whether this forces R to be trivial is unclear to me. Certainly there are also other homomorphisms. A good candidate for a non-symmetric co-product is $\Delta g = g \times h(g)$ where h is a homomorpism of the braid group. This requires the replacement $S(g) \rightarrow S(h^{-1}g)$ in order to obtain unitarity for $\nabla(1, S)\Delta$ loop removing the braiding.

$$(1 \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{213}^{-1} \quad . \tag{15.4.2}$$

$$(\Delta \otimes 1)(R) = \Phi_{321}^{-1} R_{13} \Phi_{213}^{-1} R_{23} \Phi_{123} \quad . \tag{15.4.3}$$

This and second condition imply for trivial R that also Φ is trivial.
For $\Phi = 1 \otimes 1 \otimes 1$ the conditions reduces to those for ordinary braiding. The universal R-matrix satisfies the non-associative version of Yang-Baxter equation

$$R_{12}\Phi_{321}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123} = \Phi_{321}R_{23}(\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12} \quad . \tag{15.4.4}$$

Quasi-Hopf algebra is a special case of quasi-bialgebra. Also now one has product ∇, co-product Δ, antipode S not present in bialgebra, and maps ϵ and η. Besides this one has two special elements α and β of H such that the conditions $\nabla(S,\alpha) \cdot \Delta = \alpha$ and $\nabla(1, \beta S) \cdot \Delta = \alpha$. To my understanding these conditions generalize the conditions $\nabla(S,1)\Delta = \eta\epsilon$ and $\nabla(1,S)\Delta = \eta\epsilon$.

Associativity holds but only modulo a morphism in the same way as commutativity becomes braided commutativity in the case of quantum groups. The braided commutativity is characterized by R-matrix. The morphism defining "braided associativity" is characterized by the product $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ acting on triple tensor product $V \otimes V \otimes V$ and satisfying certain algebraic conditions. Φ has "inverse" $\Phi^{-1} = \sum_i P_i \otimes Q_i \otimes R_i$ The conditions $(1, \beta S, \alpha)\Phi = 1$ and $(S, \alpha, \beta S)\Phi = 1$. Here the action of S is that of algebra anti-homomorphism rather than algebra multiplication.

Drinfeld associator, which is a non-abelian bi-algebra 3-cocycle satisfying conditions analogous to the condition for weakened associativity holding true for Lie and Jordan algebras. These quasi-Hopf algebras are known in conformal field theory context and appear in Knizhnik-Zamolodchikov equations so that a lot of mathematical knowhow exists. According to Wikipedia, quasi-Hopf algebras are associated with finite-D irreps of quantum affine algebras in terms of F-matrices used to factorize R-matrix. The representations give rise to solutions of Quantum Yang-Baxter equation. The generalization of conformal invariance in TGD framework strongly suggests the relevance of Quasi-Hopf algebras in the realization of non-associativity in TGD framework.

Drinfeld double

Drinfeld double provides a concrete example about breaking of associativity. It can be formulated for finite groups as well as discrete groups. Drinfeld's approach is essentially algebraic: one works at the level of group algebra. In TGD framework the approach is geometric: algebraic constructs should emerge naturally from geometry. Braiding operations should induce algebras.

The basic notions involved are following.

1. One begins from a trivial tensor product of Hopf algebras and modified. In trivial case algebra product is tensor product of products, co-product is tensor product of co-products, antipode is tensor product of antipodes, map ϵ is product of the maps from the factors of the tensor product and delta maps unit element of field K to a product of unit elements. Drinfeld double represents a non-trivial tensor product of Hopf algebras.

2. One application of Drinfeld double construction is tensor product of group algebra and its dual. One can also interpret it as tensor product of braids as non-closed paths and closed braids (knots) as closed paths: in TGD framework this interpretation is suggestive and will be discussed later.

3. Drinfeld double allows breaking of associativity. It can be broken by introducing 3-cocycle (see `http://tinyurl.com/y9vcsmyg`) of group cohomology (see `http://tinyurl.com/y755gd36`). In the recent case group cohomology relies on homomorphism of group braid G to abelian group $U(1)$. n-cocycle is a map $G^n \rightarrow U(1)$ satisfying the condition that its derivation vanishes $d_n f = 0$. $d_n \circ d_{n-1} = 0$ holds true identically.

   The explicit definition of n-cocycle is in additive notion for U(1) product (usually multiplicative notation is used is) given by to illustrate that d_n acts like exterior derivative.

$$(d_nf)(g_1, g_2, g_n, g_{n+1}) = g_1f(g_1, ...g_n) - f(g_1g_2, g_2, ..., g_{n+1}) + f(g_1, g_2g_3, ..., g_{n+1}) \\ -... + (-1)^n f(g_1, g_2...g_ng_{n+1}) + (-1)^{n+1} f(g_1, g_2...g_n) \ . \tag{15.4.5}$$

This formula is easy to translate to multiplicative notion. The fact that group cohomology is universal concept strongly suggests that 3 co-cycle can be introduced quite generally to break associativity in the sense that different associations differ only by isomorphism.

The construction of quantum double of Hopf algebras is discussed in detail at `http://tinyurl.com/ybbvjaw5`. Here however non-associative option is not discussed. In `http://tinyurl.com/ya8n98o5` one finds explicit formula for Drinfeld double for the Drinfeld double formed by group algebra and its dual. Just to give some idea what is involved the following gives the formula for the product:

$$(h, y) \circ (g, x) = \frac{\omega(h, g, x)\omega(hgx((hg)^{-1}, h, g)}{\omega(h, gx(g)^{-1}, h, g)}(hg, x) \ . \tag{15.4.6}$$

Without background it does not tell much. What is essential however that the starting point is algebraic. The product is non-vanishing only between (g, x) and (h, gxg^{-1}). For gauge group like structure one would have x instead of $g^{-1}xg^{-1}$. ω is 3-cocycle: it it is non-trivial one as associativity modulo isomorphism.

I do not have any detailed understanding of quasi-Hopf algebras but to me they seem to provide a very promising approach in attempts to understand the character of non-associativity associated with the braiding hierarchy. The algebraic construction of Drinfeld double does not seem interesting from TGD point of view but the idea that group cocycle is behind the breaking of associativity is attractive. Also the generalization of construction of Drinfeld double to code what happens in braiding geometrically is attractive. One of the many difficult challenges is to understand the role of the varying parameters p, h_{eff}, q at the level of braid group algebras and their projective representations characterized by quantum phase q.

15.4.2 Construction Of Quantum Braid Algebra In TGD Framework

It seems that there is no hope that naïve application of existing formulas makes sense. The variety of different variants of quantum algebras is huge and one should have huge mathematical knowledge and understanding in order to find the correct option if it exists at all. Therefore I bravely take the approach of physicists. I try to identify the physical picture and then look whether I can identify the algebraic structure satisfying the axioms of Hopf algebra. In the following I first list various inputs which help to identify constraints on the algebraic structure, which should be simple if it is to be fundamental.

Trying to map out the situation

Usually physicists have enough trouble when dealing with single algebraic structure: say group and its representations. Unfortunately, this does not seem to be possible now. It seems that one must deal with entire collection of algebraic structures defined by braid groups B_n with varying value of n forming a hierarchy in which braid groups act on lower level braid groups.

1. What is clear that the algebraic operation $(A \otimes B) \rightarrow AB$ is somehow related to the braiding of flux tubes or fermionic strings connecting partonic 2-surfaces. One can also consider strings connecting the ends of light-like 3-surfaces so that one has both space-like and time-like braiding. One has flux tubes inside flux tubes.

   The challenge is to identify the natural algebra. It seems best to work with the braiding operations themselves - analogs of linguistic expressions - than the states to which they act. Braiding operations form discrete group, braid group. One must deal with the quantum superpositions of braidings so that one has wave functions in braid group identifiable as elements of discrete group algebra of braid group B_n. One can multiply group algebra elements and include the group algebra of B_m to that of B_n m a factor of n so that the desired product structure is obtained. The group algebras associated with various braid numbers can be organized to operad.

The operad formed by the braid group algebras has the desired hierarchical structure, and braid group algebra is one of the basic structures and quantum groups can be assigned with its projective representations.

2. For a given flux tube (and perhaps also for the fermionic string(s) assigned with it) one has degrees of freedom due different values of the quantum deformation parameter q for which roots of unity define preferred values in TGD framework. In TGD framework also hierarchy $h_{eff}/h = n$ of Planck constants brings in additional complexity. Also the p-adic prime p is expected to characterize the situation: preferred p-adic primes can be interpreted as so called ramified primes in the adelic vision about quantum TGD [K104] unifying real and various p-adic physics to a coherent whole. This brings in new elements. It is still unclear how closely n and $q = exp(i2\pi/m)$ are related and whether one might have $m = n$. Also the relationship of p to n is not well-understood. For instance, could p divide n.

3. Geometrically the association of braid strands means that they belong to the same flux tube. Moving the brackets in expression to transform say $(A(BC))$ to $((AB)C)$ means that strands are transferred from flux tube another one. Hence the breaking of associativity should take place at all hierarchy levels except the lowest one for which flux tube contains single irreducible braid strand - fermion line.

The general mechanism for a weak breaking of associativity is describable in terms of Drinfeld's associator for quasi-bialgebras and known in some cases explicitly - in particular, shown by Drinfeld to exists when the number field used is rational numbers - is the first guess for the mechanism of the breaking of associativity. Drinfeld's associator is determined completely by group cohomology, which encourages to think that it can be used as such as as a multipler in the definition of product in suitable tensor product algebra. How the Drinfeld's associator depends on the p,n, and q is the basic question.

4. Besides the geometric action of braidings it is important to understand how the braidings act on the fundamental fermions. An attractive idea is that the representation is as holonomies defined by the induced weak gauge potentials as non-integrable phase factors at the boundaries of string world sheets defining fermion lines. The vanishing of electroweak gauge fields at them implies that the non-Abelian part of holonomy is pure gauge as in topological gauge field theories for which the classical solutions have vanishing gauge field. The em part of the induce spinor curvature is however non-vanishing unless one poses the vanishing of electromagnetic field at the boundaries of string world sheets as boundary condition. This seems un-necessary. The outcome would be non-trivial holonomy and restriction to a particular representation of quantum group with quantum phase q coming as root of unity means conditions on the boundaries of string world sheets. Quantum phase would make itself visible also classically as properties of string world sheets which together with partonic 2-surfaces determined space-time surface by strong form of holography. An interesting question relates to the possibility of non-commutative statistics: it should come from the weak part of induced connection which is pure gauge and seems possible as it is possible also in topological QFTs based on Chern-Simons action.

Hints about the details of the braid structure

Concerning the details of the braid structure one has also strong hints.

1. There two are two basic types of braids: I have called them time-like and space-like braids. Time-like (or rather light-like) braids are associated with the 3-D light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes signature from Minkowskian to Euclidian. Braid strands correspond to fermionic lines identifiable as parts of boundaries of string world sheets. Space-like braids are associated with the space-like 3-surfaces at the

ends of causal diamond (CD). Also they consist of fermionic lines. These braids could be called fundamental.

If these braids are associated with magnetic flux tubes carrying monopole flux, the flux tubes are closed. Typically they connect wormhole throats at first space-time sheet, go to the second space-time sheet and return. Hence two-sheeted objects are in question. The braids in question can closed to knots and could correspond to closed loops assigned with the Drinfeld quantum double. The tensor product of the groupoid algebra associated with time-like braids and group algebra associated with space-like braids is highly suggestive as the analog of Drinfeld double.

Also magnetic flux tubes and light-like orbits of partonic 2-surfaces can become braided and one obtains the hierarchies of braids.

2. Since strong world sheets and partonic 2-surfaces have co-dimension 2 as sub-manifolds of space-time surface they can also get braided and knotted and give rise to 2-braids and 2-knots. This is something totally new. The unknotting of ordinary knots would take place via reconnections and the reconnections could correspond to the basic vertices for 2-knots analogous to the crossing of the plane projections of ordinary knot. Reconnections actually correspond to string vertices. A fascinating mathematical challenge is to generalize existing theories so that they apply to 2-braids and 2-knots.

3. Dance metaphor emerged in the model for DNA-lipid membrane system as topological quantum computer [K4, K100]. Dancers whose feet are connected to wall by threads define time-like braiding and also space-like braiding through the resulting entanglement of threads. The assumption was that DNA codons or nucleotides are connected by space-like flux tubes to the lipids of lipid layer of cell membrane or nuclear membrane.

   If they carry monopolo flux they make closed loops at the structure formed by two space-time sheets. The lipid layer of cell membrane is 2-dimensional and can be in liquid crystal state. The 2-D liquid flow of lipids induces braiding of both space-like braids if the DNA end is fixed and of time-like braids. This leads to the dance metaphor: the liquid flow is stored at space-time level to the topology of space-time as a space-like braiding of flux tubes induced by it. Space-like braiding would be like written text. Time-like braiding would be like spoken language.

4. If the space-like braids are closed, they form knots and the flow caused at the second end of braid by liquid flow must be compensated at the parallel flux tube by its reversal since braid strands cannot be cut. The isotopy equivalence class of knot remains unchanged since knots get gg^{-1} piece which can be deformed away. Second interpretation is that the braid X transforms to gXg^{-1}. This kind of transformation appears also in Drinfeld construction. This suggests that the purely algebraic tensor product of braid algebra and its dual corresponds in TGD framework semi-direct tensor product of the groupoid of time-like braids and space-like braids associated with closed knots. The semi-direct tensor product would define the fundamental topological interaction between braids.

5. One can also consider sequence of n tensor factors each consisting of time-like and space-like braids. This require a generalization of the product of two tensor factors to $2n$ tensor factors. Dance metaphor suggests that a kind of chain reaction occurs.

What the structure of the algebra could be?

With this background one can try to guess what the structure of the algebra in question is. Certainly the algebra is semi-direct product of above defined braid group algebras. The multiplication rule would have purely geometric interpretation.

1. The multiplication rule inspired by dance metaphor for 2 tensor factors would be

$$(a_1, a_2) \circ (b_1, b_2) = (a_1 a_2 b_1 a_2^{-1}, a_2 b_2) \quad . \tag{15.4.7}$$

Here a_1, b_1 correspond label elements of time-like braid groupoid and a_2, b_2 the elements of braid group associated with the space-like braid. This would replace the trivial product rule $(a_1, a_2)(b_1 g_)) = (a_1 b_1, a_2 b_2)$ for the trivial tensor product. The structure is same as for Poincare group as semi-direct product of Lorentz group and translation group: $(\Lambda_1, T_1)(\Lambda_2, T_2) = (\Lambda_1\Lambda_2, T_1 + \Lambda_1(T_2))$.

It is easy to check that this product is associative. One can however add exactly the same 3-cocycle factor

$$(h, y) \circ (g, x) = \frac{\omega(h, g, x)\omega(hgx((hg)^{-1}, h, g)}{\omega(h, gx(g)^{-1}, h, g)}(hg, x) \ . \tag{15.4.8}$$

Here (h, y) corresponds to (a_1, a_2) and (g, x) to (b_1, b_2). This should give breaking of non-associativity and third group cohomology of braid group B_n would characterize the non-equivalent associators.

2. The product rule generalizes to n factors. This generalization could be relevant for the understanding of braid hierarchy.

$$(a_1, a_2, ...a_n) \circ (b_1, b_2, ...b_n) \equiv (c_1, ..., c_n) \ , \tag{15.4.9}$$

where one has

$$\begin{array}{lll} c_n = a_n b_n \ , & c_{n-1} = a_{n-1} Ad_{a_n}(b_{n-1}) \ , & c_{n-2} = a_{n-2} Ad_{a_{n-1}a_n}(b_{n-2}) \ , \\ c_{n-3} = a_{n-3} Ad_{a_{n-2}a_{n-1}a_n}(b_{n-3}) & , ... & c_1 = a_1 Ad_{a_2.....a_n}(b_1) \ . \\ Ad_x(y) = xyx^{-1} \ . & & \end{array} \tag{15.4.10}$$

In this case a good guess for the breaking of associativity is that the associator is defined in terms of n-cocyle in group cohomology.

What is remarkable that this formula guarantees without any further assumptions the condition

$$\begin{array}{c} \nabla_{1\otimes 2}(\Delta_1(a), \Delta_2(b)) = \nabla_1(\Delta_1(a))\nabla_2(\Delta_2(b)) = \sum_{(a)} a_1 a_2 \sum_{(b)} b_1 b_2 \ , \\ \Delta_1(a) = \sum_{(a)} a_1 \otimes a_2 \quad , \quad \Delta_2(b) = \sum_{(b)} b_1 \otimes b_2 \end{array} \tag{15.4.11}$$

as a little calculation shows. For group algebra one has $\Delta(a) = g \otimes g$. $\nabla_{1\otimes 2}$ refers to the product defined above.

3. The formula for $\Delta_{1\otimes 2}$ is also needed. The simplest guess is that it corresponds to replication for both factors. This would mean $\Delta^{op} = \Delta$: non-symmetric form guaranteeing non-trivial braiding is however desirable. A candidate satisfying this condition in $n = 2$ case is asymmetric replication:

$$\begin{array}{l} \Delta_{1\otimes 2}(bab^{-1}, b) \otimes (a, b) \\ \Delta^{op}_{1\otimes 2}(a, b) \otimes (bab^{-1}, b) \ . \end{array} \tag{15.4.12}$$

4. In $n = 2$ case the formula for antipode would read as

$$S(a_1, a_2) = (a_2^{-1} a_1^{-1} a_2, a_2^{-1}) \tag{15.4.13}$$

instead of $S(a_1, a_2) = (a_1^{-1}, a_2^{-1})$. Again the semi-direct structure would be involved. One can check that the formula

$$\nabla_{1\otimes 2}(1, S)\Delta_{1\otimes 2} = 1 \otimes 1 \tag{15.4.14}$$

holds true.

15.4.3 Should One Quantize Complex Numbers?

The TGD inspired proposal for the concrete realization of quantum groups might help in attempts to understand the situation. The approach relies on what might be regarded as quantization of complex numbers appearing as matrix elements of ordinary matrices.

1. Quantum matrices are obtained by replacing complex number valued of matrix elements of ordinary matrices with operators. They are are products of hermitian non-negative matrix P analogous to modulus of complex number and unitary matrix S analogous to its phase. One can also consider the condition $[P, S] = iS$ inspired by the idea that radial momentum and phase angle define analog of phase space.

2. The notions of eigenvalue and eigenstate are generalized. Hermitian operator or equivalently the spectrum of its eigenvalues replaces real number. The condition that eigenvalue problem generalizes, demands that the symmetric functions formed from the elements of quantum matrix commute and can be diagonalized simultaneously. The commutativity of symmetric functions holds also for unitary matrices. These conditions is highly non-trivial, and consistent with quantum group conditions if quantum phases are roots of unity. In this framework also Planck constant is replaced by a hermitian operator having $h_{eff} = n \times h$ as its spectrum. Also $q = exp(in2\pi/m)$ generalizes to a unitary operator with these eigenvalues.

3. This leads to a possible concrete representation of quantum group in TGD framework allowing to realize the hierarchy of inclusions of hyperfinite factors obtained by repeatedly replacing the operators appearing as matrix elements with quantum matrices.

4. This procedure can be repeated. One might speak of a fractal quantization. At the first step one obtains what might be called 1-hermitian operators with eigenvalues replaced with hermitian operators. For 1-unitary matrices eigenvalues, which are phases are replaced with unitary operators. At the next step one considers what might be called 2-hermitian and 2-unitary operators. An abstraction hierarchy in which instance (localization to a point as member of class) is replaced with wave function in the class. This hierarchy is analogous to that formed by infinite primes and by the sheets of the many-sheeted space-time. Also braids of braids of ... form this kind of abstraction hierarchy as also the parsing hierarchy for linguistic expressions.

I have proposed that generalized Feynman diagrams or rather - TGD analogs of twistor diagrams - should have interpretation as sequences of arithmetic operators with each vertex representing product or co-product and having interpretation as time reversal of the product operation.

1. The arithmetic operations could be induced by the algebraic operations for Yangian algebra [A18] [B36, B26, B27] assignable to the super-symplectic algebra. I have also proposed that there TGD allows a very powerful symmetry generalizing the duality symmetry of old-fashioned string models relating s- and t-channel exchanges. This symmetry would state

that one can freely move the ends of the propagator lines around the diagrams and that one can remove loops by transforming the loop to tadpole and snipping it away. This symmetry would allow to consider only tree diagrams as shortest representations for computations: this would reduce enormously the calculational complexity. The TGD view about coupling constant evolution allows still to have discrete coupling constant evolution induced by the spectrum of critical values of Kähler coupling strength: an attractive conjecture is that the critical values can be expressed in terms of zeros of Riemann zeta [L17].

2. One can represent the tree representing a sequence of computations in algebra as an analog of twistor diagram and the proposed symmetry implies associativity since moving the line ends induces motion of brackets. If co-algebra operations are allowed also loops become possible and can be eliminated by this symmetry provided the loop acts as identity transformation. This would suggest strong form of associativity at the level of single sheet and weaker form at the level of many-sheeted space-time. One could however still hope that loops can be cancelled so that one would still have only tree diagrams in the simplest description. One would have however sum over amplitudes with different association structures.

3. Co-product could be associated with the basic vertices of TGD, which correspond to a fusion of light-like parton orbits along their ends having no counterpart in super-string models (tensor product vertex) or the decay of light-like parton orbit analogous to a splitting of closed string (direct sum vertex). For the direct sum vertex one has direct sum (unlike string models): one can say that the particle propagates along two path in the sense of superposition as photons in double slit experiment. For the tensor product vertex $D(g) = \Delta(g) = g \times g$ is the first guess. $D(g) = (1, S)\Delta(g) = g \otimes Sg$ or $D(g) = Sg \otimes g$ or their sum suitably normalized is natural second guess. Unitarity allows only the latter option since $\nabla\Delta$ does not conserve probability for probability amplitudes unlike $\nabla(1, S)\Delta$ although it does so for probability distributions. For the direct sum vertex $\Delta(g) = 1 \otimes g \oplus g \otimes 1$ suitably normalized is the natural first guess.

4. Co-product Δ might allow interpretation as annihilation vertex in particle physics context. Co-product might also allow interpretation in terms of replication - at least at the level of topological dynamics of braiding. The possible application of co-product to the replication occurring biology assumed to be induce by replication of magnetic flux tubes in TGD based vision is highly suggestive idea. Is the identification of co-product as replication consistent with its identification as particle annihilation?

   Second question relates to the antipode S, which is anti-homomorphism and brings in mind time reversal. Could one interpret also S as an operation, which should be included to the braid group algebra in the same way as the inclusion of complex conjugation to the algebra of complex numbers produces quaternions? Could one interpret the identity $\nabla(1 \otimes S)\Delta(g) = \eta\epsilon(g) = 1$ by saying that the annihilation to $g \otimes S(g)$ followed by fusion produces braid wave function concentrated on trivial braiding and destroying the information associated with braiding completely. The fusion would produce non-braided particle rather than destroying particles altogether.

5. The condition that loop involving product and annihilation does not affect braid group wave function would require that it takes g to g. For the standard realization of co-product Δ of group algebra $g \rightarrow g \otimes g \rightarrow g^2$ so that this is not the case. The condition defining Δ is not easy to modify since one loses homomorphism property of Δ. The repetitions of loops would give sequence of powers g^{2n}. For wave function $\sum D(g)g$ this would give the sequence $\sum D(g)g \rightarrow \sum D(g)g^2 \rightarrow .... \rightarrow \sum D(g)g^{2n}$: since given group element has typically several roots one expects that eventually the wave function becomes concentrated to unity with coefficient $\sum D(g)$! For wave functions one has $\sum D(g) = 0$ if they are orthogonal to $D(g) = constant$ as is natural to require. Almost all wave functions would approach to zero so that unitary would be lost. For probability distributions the evolution would make sense since the normalization condition would be respected.

   Also the irreversible behaviour looks strange from particle physics perspective unless $D(g)$ is concentrated on identity so that braiding is trivial. Topological dissipation might take care that this is the case. For elementary particles partonic 2-surfaces carry in the first approximation only single fermion so that braid group would be trivial. Braiding effects become interesting only for strand number larger than 2. The situations in which partonic surface carries large number of fermion lines would be more interesting. Anyonic systems to which TGD based model assigns large h_{eff} and parton surfaces of nanoscopic size could represent a condensed matter example of this situation.

6. Does the behavior of Δ force to regard generalized Feynman diagrams representing computations with different numbers of self-energy loops non-equivalent and to sum over self-energy loops in the construction of scattering amplitudes? The time evolution implied by topological self energy loops is not unitary which suggest that one must perform the sum. There are hopes that the sum converges since the contributions approaches to $\sum D(g) = 0$. This does not however look elegant and is in conflict with the general vision.

   Particle physics intuition tells that in pair annihilation second line has opposite time direction. Should one therefore identify annihilation $g \rightarrow g \otimes S(g)$. Antiparticles would differ from particles by conjugation in braid group. The self energy loop would give trivial braiding with coefficient $\sum D(g)D(g^{-1}) = \sum D(g)D(g)^* = 1$ so that unitarity would be respected and higher self energy loops would be trivial. The conservation of fermion number at fundamental level could also prevent the decays $g \rightarrow g \otimes g$.

One could also take biological replication as a guide line.

1. In biological scales replication by $g \rightarrow g \otimes g$ vertex might not be prevented by fermion number conservation but probability conservation favors $g \rightarrow g \otimes Sg$. Braid replication might be perhaps said to provide replicas of information: whether this conforms with no-cloning theorem remains to be seen. Braid replication followed by fusion means topological dissipation by a loss of braiding and loss of information. Could the fusion of reproduction cells corresponds to product and that replication to co-product possibly involving the action of S one the second line. Fusion followed by replication would lead to a loss of braiding: for $g \rightarrow g \otimes g$ perhaps making sense in probabilistic description gradually and for $g \rightarrow g \otimes Sg$ instantaneously: a reset for memory? Could these mechanisms serve as basic mechanisms of evolution?

2. There might be also a connection with the p-adic length scale hypothesis. The naïve expectation is that $g \rightarrow g^2$ in fusion followed by Δ means the increase of the length of braid by factor 2 - kind of ageing? Could the appearance of powers of two for the length of braid relate to the p-adic length scale hypothesis stating that primes p near powers of 2 are of special importance?

To summarize, the proposed framework gives hopes about description of braids of braids of .... Abstraction would mean transition from classical to quantum: from localized state to a de-localized one: from configuration space to the space of complex valued wave functions in configuration space. Now the configuration space would involve different braidings and corresponding evolutions, and various values of p, h_{eff} and q. If this general framework is to be useful it should be able to tell how the braiding matrices depend on p and h_{eff}: note that p and h_{eff} would be fixed only at the highest abstraction level - the largest flux tubes. This indeterminacy could be interpreted in terms of finite measurement resolution and inclusions of HFFs should help to describe the situation. Indeterminacy could also be interpreted in terms of abstraction in a way similar to the interpretation of negentropically entangled state as a rule for which the state pairs in the superposition represent instances of the rule.

Part III

MISCELLANEOUS TOPICS

Chapter 16

Does the QFT Limit of TGD Have Space-Time Super-Symmetry?

16.1 Introduction

Contrary to the original expectations, TGD seems to allow the analog of the space-time super-symmetry. This became clear with the increased understanding of both Kähler action and Kähler-Dirac action [K106, K23]. It is however far from clear whether SUSY type QFT can define the QFT limit of TGD and whether this kind of formulation is the optimal one.

16.1.1 Is The Analog Of Space-Time SUSY Possible In TGD?

The basic question is whether the huge algebras with super-conformal structure acting as symmetries of quantum TGD give rise to a SUSY algebra at space-time level (meaning super-Poincare symmetry). A more technical question is whether the QFT limit of TGD could be formulated as a generalization of SUSY QFT or whether one must generalize this approach just as it seems necessary to generalize the notion of twistor by replacing masslessness in 4-D sense with masslessness in 8-D sense.

1. From the beginning it was clear that super-conformal symmetry is realized in TGD but differs in many respects from the more standard realizations such as $\mathcal{N} = 1$ SUSY realized in MSSM [B4] involving Majorana spinors in an essential way.

   Note that the belief that Majorana spinors are somehow an intrinsic aspect of super-symmetry can be used as an objection against TGD. Besides Majorana spinors Weyl spinors meaning complex theta parameters are also possible. Theta parameters can also carry fermion number meaning only the supercharges carry fermion number and are non-hermitian. The general classification of super-symmetric theories indeed demonstrates that for $D = 8$ Weyl spinors and complex and non-hermitian super-charges are possible. The original motivation for Majorana spinors might come from MSSM assuming that right handed neutrino does not exist. This belief might have also led to string theories in D=10 and D=11 as the only possible candidates for TOE after it turned out that chiral anomalies cancel.

2. In TGD framework the covariantly constant right-handed neutrino generates the super-symmetry at the level of CP_2 geometry. The original idea was that the construction of super-partners would be more or less equivalent with the addition of *covariantly constant* right-handed neutrino and antineutrinos to the state. It was however not clear whether space-time supersymmetry is realized at all since one could argue that that by covariant constancy these states are just gauge degrees of freedom or that SUSY is only realized for the spinor harmonics of embedding space with 8-D notion of masslessness. Much later it became clear that covariantly constant right handed neutrino indeed represents gauge degree of freedom at *space-time level*.

3. A more general general SUSY algebra is generated by the modes of the Kähler-Dirac operator at partonic 2-surface being also Clifford algebra. This algebra can be associated with the ends of the boundaries of string world sheets and each string defines its own sub-algebra of oscillator operators.

   (a) At first it would seem that the value of $\mathcal{N}$ can be very large - even infinite as the fact that fermionic oscillator operators are labelled by conformal weight. It is however the number of *massless states* in M^4 sense, which determines the value of $\mathcal{N}$ for SUSY in M^4: for the full theory the analog of SUSY in H $\mathcal{N} = \infty$ could make sense. Indeed, super-symplectic generators bring in the analog of wave function of fermion at partonic 2-surfaces and constant wave functions and therefore massless states are expected to be favored by Uncertainty Principle. The dimension of SUSY algebra is expected to just the number of spinor components of the embedding space spinor possessing physical embedding space helicity.

   A more general situation is that the conformal gauge algebra is its sub-algebra isomorphic to the entire algebra having conformal weights coming as n-ples of those for the full algebra. The conformal gauge symmetry would be broken so that only the super-symplectic generators for which the conformal weight is proportional to fixed integer $n \in \{1, 2, ...\}$ annihilate the physical states. This increases the value of $\mathcal{N}$ and a possible interpretation is in terms of improved measurement resolution. N would also correspond to the value of Planck constant $h_{eff}/n = N$ and N would label phases of dark matter and also a hierarchy of criticalities. As N increases, super-conformal gauge degrees of freedom are transformed to physical ones. This kind of situation might be possible for quantum deformations of the oscillator operator algebra characterized by quantum phase as $q = exp(i2\pi/N)$ and possible by the 2-dimensionality of string world sheets.

   An alternative way to see the situation is as a fractionization of conformal weights due to the emergence of N-fold coverings of space-time surfaces analogous to coverings of complex plane defined by analytic function $z^{1/N}$. Only the states with integer conformal weights would be annihilated by the original conformal algebra and quantum group would describe the situation.

   The SUSY in standard sense is expected to be broken. First, the notion of masslessness is generalized: fermions associated with the boundaries of string world sheets have light-like 8-momentum and therefore can be massive in 4-D sense: this allows to generalize twistor description to massive case [L10]. The ordinary 4-D SUSY is expected to emerge only as an approximate description in massless sector (as it also appears in dimensional reduction). Secondly, standard SUSY characterizes the QFT description obtained by replacing many-sheeted space-time time with a slightly curved region of Minkowski space.

   (b) SUSY algebra is replaced with Clifford algebra at the level of partonic 2-surfaces and the generators can be identified as fermionic oscillator operatiors at the end points of fermionic lines, which are light-like geodesics. Light-like four-momenta in anti-commutation relations are replaced with 8-D light-like momenta demanding a generalization of twistor approach. The octonionic realization of twistors is a very attractive possibility in this framework and quaternionicity condition guaranteeing associativity leads to twistors which are almost equivalent with ordinary 4-D twistors.

   The space-time super-symmetry means addition of fermion to the state assign to a partonic surface and since the number of spinor modes is larger states with large spin and fermion numbers are obtained. This picture does not fit to the standard view about super-symmetry. In particular, the identification of theta parameters as Majorana spinors and super-charges as Hermitian operators is not possible. The non-hermitian character of super conformal generator $G \neq G^\dagger$ made impossible the naive generalization of stringy rules to TGD framework since they involve G as the analog of fermionic propagator. This problem disappears in the twistor Yangian approach [L10].

(c) The notion of super-field does not seem natural in the full TGD framework but would be replaced with a Yangian of the super-symplectic algebra and related conformal algebras with generators identified as Noether charges assignable to strings connecting partonic 2-surfaces. Multi-locality coded by Yangian in the scale of partonic surfaces is a new element. There is also the hierarchy of Planck constants interpreted in terms of dark matter and Zero Energy Ontology.

16.1.2 What Happens When Many-Sheeted Space-Time Is Approximated With Minkowski Space?

The question is what happens when one replaces many-sheeted space-time with a region of Minkowski space and identifies gauge potentials as sum of the induced gauge potentials?

1. It is plausible that gauge theory like description is a good approximation. But what happens to the SUSY? Can one replace 8-D light-likeness with 4-D light-likeness and describe massivation in terms of Higgs mechanism and analogous - not very successful - mechanisms for 4-D SUSY? It is quite possible that this is not possible: 4-D QFT approximation taken partonic 2-surfaces to points might miss too much of physics and too much elegance.

2. Should one try to find a generalization of ordinary 4-D SUSY allowing the description of massive particles in terms of 8-D light-likeness? This would allow also to understand baryons and lepton number conservation as 8-D chiral symmetry, to avoid Majorana spinors, and would force a new view about QCD color. Maybe the attempt to describe things by QFT or even ordinary string model is like an attempt to describe quantum physics using classical mechanics. To my opinion generalization of twistor approach from 4-D to 8-D context based on the notion of super-symplectic Yangian is a more promising approach than sticking to effective field theory thinking [L10].

The first guess - much before the understanding of the Kähler-Dirac equation and the role of right-handed neutrino - was that it might be possible to formulate even quantum TGD proper in terms of super-field defined in the world of classical worlds (WCW). Super-fields could provide in this framework an elegant book-keeping apparatus for the elements of local Clifford algebra of WCW extended to fields in the $M^4 \times CP_2$, whose points label the positions of the tips of the causal diamonds CDs). At this moment I feel skeptic about this approach.

16.1.3 What SUSY QFT Limit Could Mean?

What the actual construction of SUSY QFT limit means depends on how strong approximations one wants to make.

1. The minimal approach to SUSY QFT limit is based on an approximation assuming only the super-multiplets generated from fundamental fermions by right-handed neutrino or both right-handed neutrino and its antineutrino.

2. Elementary are particles are composed of fundamental fermions so that the super-multiplets are more complex for them. One of the key predictions of TGD is that elementary particles can be regarded as bound states of fermions and anti-fermions located at the throats of two wormhole contacts. As a special case this implies bosonic emergence meaning that it QFT limit can be defined in terms of Dirac action.

16.1.4 Scattering Amplitudes As Sequences Of Algebraic Operations

The attempts to generalize twistor Grassmannian approach in TGD framework led to a revival an old idea about scattering amplitudes as representations of sequences of algebraic operations connecting two sets of algebraic objects. Any two sequences connecting same sets would give rise to same scattering amplitudes. One might say that instead of mathematics representing physics physics represents mathematics.

1. In Yangian approach fundamental vertices correspond to product and co-product for the generators of Yangian of super-symplectic algebra with charges identified in terms of Noether charges assignable to strings connecting partonic 2-surfaces [L10]. Scattering amplitudes are obtained by the analog of Wick contraction procedure in which fermion lines connecting different vertices would be obtained. This also allows creation of fermion pairs from vacuum with members at opposite throats of wormhole contact defining the fundamental boson propagators. This picture about bosonic emergence is similar to the earlier one.

2. Yangian approach has huge symmetries since the duality symmetry of string models generalizes in the sense that one can freely move the ends of the lines and snip off loops in this way. The fact that all diagram representing computation connecting same initial and final states are equivalent implies huge number of constraints and it is clear that ordinary Feynman diagrammatics cannot satisfy these constraints. Twistor diagrammatics could however do so since it has turned out that twistor diagrams indeed have symmetries analogous to this kind of symmetry. It seems however that one must generalized 4-D twistors to 8-D ones so that the twistor Yangian approach looks like the most promising approach at this moment: if of course applies to full theory rather than only in massless sector of the theory.

The plan of the chapter reflects partially my own needs. I had to learn space-time supersymmetry at the level of the basic formalism and the best way to do it was to write it out. As the vision about fermions in TGD crystallized it became also clear that SUSY QFT in Feynman graph formulation does not catch the simplicity of what I identify as fundamental formulation of TGD. Therefore I dropped a lot of material in the original chapter.

1. The chapter begins with a brief summary of the basic concepts of SUSYs without doubt revealing my rather fragmentary knowledge about these theories. The original belief was that super-field formalism could be generalized to TGD framework. At this moment I however believe that Yangian approach is more realistic one for reasons already mentioned. Therefore I have dropped the section about the formalism proposed earlier. I have also dropped material about various attempts to understand the role right-handed neutrinos. The chapter in its recent form is about whether SUSY limit could emerge from TGD. Just general conditions are formulated since I do not have the expertise to formulate the theory in detail.

2. The Clifford algebra of fermionic oscillator operators assignable to the ends of strings connecting partonic 2-surfaces replaces SUSY algebra, and anti-commutation relations realize the analog of super Poincare symmetry. Since the number of conformal weights is infinite, one would naively expect $\mathcal{N} = \infty$ SUSY. States are however created by super-symplectic generators bringing in the analog of wave function of fermion at partonic 2-surface rather fermionic oscillator operators. Also conformal gauge invariance conditions are satisfied, and this is expected to change the situation. For ideal measurement resolution only the fermionic oscillator operators with vanishing conformal weight are expected to remain effective. The description of finite measurement resolution in terms of quantum variant of fermionic anti-commutation relations is expected to increase the number of conformal weights so that $\mathcal{N}$ increases for dark matter. Right-handed neutrino and its antineutrino would define the least broken sub-algebra of SUSY.

3. Twistors have become a part of the calculational arsenal of SUSY gauge theories, and TGD leads to a proposal how to avoid the problems caused by massive particles by using the notion of masslessness in 8-D sense and the notion of induced octo-twistor [L10]. The equivalence of octonionic spinor structure with the ordinary one leads also to the localization of spinors to string world sheets and fermions at light-like geodesics at their boundaries at partonic 2-surfaces. Already the fundamental formulation keeps just the knowledge that particle moves along light-like geodesic of $M^4 \times CP_2$ and strings connect partonic 2-surfaces. Could QFT limit could be formulated as SUSY in $M^4 \times S^1$ allowing to describe massive particles as massless particles in $M^4 \times S^1$? Or could simplified string model type description in $M^4 \times S^1$ make sense?

4. With the improved understanding of Kähler-Dirac equation one can develop arguments that $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SUSY generated by right-handed neutrino emerges naturally in TGD

framework and corresponds to the addition of a collinear right-handed neutrino and and antineutrino to the state representing massless particle.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at http://tgdtheory.fi/cmaphtml.html [L11]. Pdf representation of same files serving as a kind of glossary can be found at http://tgdtheory.fi/tgdglossary.pdf [L12].

16.2 SUSY Briefly

The Tasi 2008 lectures by Yuri Shirman [B66] provide a modern introduction to 4-dimensional $\mathcal{N}=1$ super-symmetry and super-symmetry breaking. In TGD framework the super-symmetry is 8-dimensional super-symmetry induced to 4-D space-time surface and one $\mathcal{N}=2N$ can be large so that this introduction is quite not enough for the recent purposes. This section provides only a brief summary of the basic concepts related to SUSY algebras and SUSY QFTs and the breaking of super-symmetry is mentioned only by passign. I have also listed the crucial basic facts about $\mathcal{N}>1$ super-symmetry [B1, B3] with emphasis in demonstrating that for 8-D super-gravity with one time-dimension super-charges are non-Hermitian and that Majorana spinors are absent as required by quantum TGD.

16.2.1 Weyl Fermions

Gamma matrices in chiral basis.

$$\begin{array}{llll}\gamma^\mu=\left(\begin{array}{cc}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{array}\right) \ , & \gamma_5=\left(\begin{array}{cc}\sigma_0 & 0\\ 0 & -\sigma_0\end{array}\right) \ , & & \\ \sigma^0=\left(\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right) \ , & \sigma^1=\left(\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right) \ , & \sigma^2=\left(\begin{array}{cc}0 & -i\\ i & 0\end{array}\right) \ , & \sigma^3=\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right) \ , \\ \overline{\sigma}^0=\sigma^0 \ , & \overline{\sigma}^i=-\sigma^i \ . & & \end{array} \quad (16.2.1)$$

Note that Pauli sigma matrices can be interpreted as matrix representation for hyper-quaternion units.

Dirac spinors can be expressed in terms of Weyl spinors as

$$\Psi = \left(\begin{array}{c}\eta^\alpha\\ \overline{\chi}^*_{\dot{\alpha}}\end{array}\right) \ . \quad (16.2.2)$$

Note that does not denote complex conjugation and that complex conjugation transforms non-dotted and dotted indices to each other. η and $\overline{\chi}$ are both left handed Weyl spinors and transform according to complex conjugate representations of Lorentz group and one can interpret $\overline{\chi}$ as representing that charge conjugate of right handed Dirac fermion.

Spinor indices can be lowered and raised using antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ and one has

$$\begin{array}{ll}\eta^\alpha\eta_\alpha=0 \ , & \overline{\chi}^*_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}*}=0 \ , \\ \eta\overline{\chi}=\overline{\chi}\eta=\epsilon^{\alpha\beta}\eta_\alpha\overline{\chi}_\beta \ , & \eta^*\overline{\chi}^*=\overline{\chi}^*\eta^*=\epsilon^{\alpha\beta}\eta^*_{\dot{\alpha}}\overline{\chi}^*_{\dot{\beta}} \ .\end{array} \quad (16.2.3)$$

Left-handed and right handed spinors can be combined to Lorentz vectors as

$$\eta^*_{\dot{\alpha}}\sigma^{\mu\dot{\alpha}\alpha}\eta_\alpha=-\eta^{*\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\eta^{*\dot{\alpha}} \ . \quad (16.2.4)$$

The SUSY algebra at QFT limit differs from the SUSY algebra defining the fundamental anti-commutators of the fermionic oscillator operators for the induced spinor fields since the Kähler-Dirac gamma matrices defined by the Kähler action are replaced with ordinary gamma matrices. This is quite a dramatic difference and raises two questions.

The Dirac action

$$L = i\overline{\Psi}\partial_\mu\gamma^\mu\Psi - m\overline{\Psi}\Psi \qquad (16.2.5)$$

for a massive particle reads in Weyl representation as

$$L = i\eta^*\partial_\mu\sigma^\mu\eta + i\overline{\chi}^*\partial_\mu\overline{\sigma}^\mu\overline{\chi} - m\overline{\chi}\eta - m\overline{\chi}^*\eta^* \ . \qquad (16.2.6)$$

16.2.2 SUSY Algebras

In the following 4-D SUSY algebras are discussed first following the representation of [B66]. After that basic results about higher-dimensional SUSY algebras are listed with emphasis on 8-D case.

$D = 4$ SUSY algebras

Poincare SUSY algebra contains as super-generators transforming as Weyl spinors transforming in complex conjugate representations of Lorentz group. The basic anti-commutation relations of Poincare SUSY algebra in Weyl fermion basis can be expressed as

$$\begin{aligned} \{Q_\alpha, Q_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu \ , \\ \{Q_\alpha, Q_\beta\} &= \{Q_{\dot\alpha}, Q_{\dot\beta}\} = 0 \ , \\ [Q_\alpha, P_\mu] &= [Q_{\dot\alpha}, P_\mu] = 0 \ . \end{aligned} \qquad (16.2.7)$$

By taking a trace over spinor indices one obtains expression for energy as $P^0 = \sum_i Q_i\overline{Q}_i + \overline{Q}_iQ_i$. Since super-generators must annihilated super-symmetric ground states, the energy must vanish for them.

This algebra corresponds to simplest $\mathcal{N} = 1$ SUSY in which only left-handed fermion appears. For $\mathcal{N} = 1$ SUSY the super-charges are are hermitian whereas in TGD framework super-charges carry fermion number. This implies that super-charges come in pairs of super charge so that $\mathcal{N} = 2N$ must hold true and its hermitian conjugate and only the second half of super-charges can annihilate vacuum state. Weyl spinors must also come as pairs of right- and left-handed spinors.

The construction generalizes in a straightforward manner to allow arbitrary number of fermionic generators. The most general anti-commutation relations in this case are

$$\begin{aligned} \{Q_{i\alpha}, Q^j_{\dot\beta}\} &= 2\delta^j_i\sigma^\mu_{\alpha\dot\beta}P_\mu \ , \\ \{Q_{i\alpha}, Q_{j\beta}\} &= \epsilon_{\alpha\beta}Z_{ij}, \\ \{Q_{\dot\alpha}, Q_{\dot\beta}\} &= \epsilon^{\dot\alpha\dot\beta}Z^*_{ij} \ . \end{aligned} \qquad (16.2.8)$$

The complex constants are called central charges because they commute with all generators of the super-Poincare group.

Higher-dimensional SUSY algebras

The character of supersymmetry is sensitive to the dimension D of space-time and to the signature of the space-time metric higher dimensions [B1]. The available spinor representations depend on k; the maximal compact subgroup of the little group of the Lorentz that preserves the momentum of a massless particle is $Spin(d-1)\times Spin(D-d-1)$, where d is the number of spatial dimensions $D-d$ is the number time dimensions and k is defined as $k = 2d - D$. Due to the mod 8 Bott periodicity of the homotopy groups of the Lorentz group, really we only need to consider $k = 2d - D$ modulo 8. In TGD framework one has $D = 8$, $d = 7$ and $k = 6$.

For any value of k there is a Dirac representation, which is always of real dimension $N = [2^{1+[(2d-k)/2]}$ where $[x]$ is the greatest integer less than or equal to x. For TGD this of course gives $2^5 = 32$ corresponding to complex 8-component quark and lepton like spinors. For $-2 \leq k \leq 2$ not realized in TGD there is a real Majorana spinor representation, whose dimension is $N/2$. When k is even (TGD) there is a Weyl spinor representation, whose real dimension is $N/2$. For $k\ mod\ 8 = 0$ (say in super-string models) there is a Majorana-Weyl spinor, whose real dimension is $N/4$. For $3 \leq k \leq 5$ so called symplectic Majorana spinor with dimension $D/2$ and for $k = 4$ symplectic Weyl-Majorana spinors with dimension $D/4$ is possible. The matrix Γ_{D+1} defined as the product of all gamma matrices has eigenvalues $\pm(-1)^{-k/2}$. The eigenvalue of Γ_{D+1} is the chirality of the spinor. CPT theorem implies that the for $D\ mod\ 4 = 0$ the numbers of left and right handed super-charges are same. For $D\ mod\ 4 = 2$ the numbers of left and right handed chiralities can be different and corresponding SUSYs are classified by $\mathcal{N} = (\mathcal{N}_L, \mathcal{N}_R)$, where $\mathcal{N}_L$ and $\mathcal{N}_R$ are the numbers of left and right handed super charges. Note that in TGD the chiralities are ± 1 and correspond to quark and leptons like spinors.

TGD does not allow super-symmetry with Majorana particles. It is indeed possible to have non-hermitian super-charges [B3] in dimension $D = 8$. In $D = 8$ SUGRA with one time dimension super-charges ar non-hermitian and Majorana particles are absent. Also in $D = 4$ SUGRA predicts super-charges are non-hermitian super-charges but Majorana particles are present.

1. $D = 8$ super-gravity corresponds to $\mathcal{N} = 2$ and allows complex super-charges $Q^i_\alpha \in 8$ and their hermitian conjugates $\overline{Q}^i_\alpha \in \overline{8}$. The group of R symmetries is $U(2)$. Bosonic fields consists the metric g_{mn}, seven real scalars, six vectors, three 2-form fields and one 3-form field. Fermionic fields consist of two Weyl (left) gravitini $\psi^{\alpha i}$, six Weyl (right) spinors plus their hermitian conjugates of opposite chirality. There are no Majorana fermions.

2. $D = 4, \mathcal{N} = 8$ SUGRA is second example allowing complex non-hermitian super-charges. The supercharges $Q^i_\alpha \in 2$ and their hermitian conjugates $\overline{Q}^i_{\dot{\alpha}} \in \overline{2}$. R-symmetry group is $U(8)$. Bosonic fields are metric g_{mn}, 70 real scalars and 28 vectors. Fermionic fields are 8 Majorana gravitini $\Psi^{a,i}_m$ and 56 Majorana spinors.

For $\mathcal{N} = 2N$ and at least $D = 8$ with one time dimension the super charges can be assumed to come in hermitian conjugate pairs and the non-vanishing anti-commutators can be expressed as

$$\begin{aligned} \{Q^\dagger_{i\alpha}, Q^j_{\dot{\beta}}\} &= 2\delta^j_i \sigma^\mu_{\alpha\dot{\beta}} P_\mu \ , \\ \{Q^\dagger_{i\alpha}, Q_{j\beta}\} &= \epsilon_{\alpha\beta} Z_{ij} , \end{aligned}$$
$$\{Q^\dagger_{\dot{\alpha}}, Q_{\dot{\beta}}\} = \epsilon^{\dot{\alpha}\ dot\beta} Z^*_{ij} \ . \qquad (16.2.9)$$

In this case Z_{ij} is anti-hermitian matrix. 8-D chiral invariance (separate conservation of lepton and quark numbers) suggests strongly that the condition $Z_{ij} = 0$ must hold holds true. A given pair of super-charges is analogous to creation and annihilation operators for a given fermionic chirality. In TGD framework opposite chiralities correspond to quark and lepton like spinors.

Representations of SUSY algebras in dimension $D = 4$

The physical components of super-fields correspond to states in the irreducible representations of SUSY algebras. The representations can be constructed by using the basic anti-commutation relations for $Q_{i\alpha}$ and $Q_{j\dot{\alpha}}$, $i, j \in \{1, ..., \mathcal{N}\}$, $\alpha, \dot{\alpha} \in \{1, 2\}$. The representations can be classified to massive and massless ones. Also the presence of central charges affects the situation. A given irreducible representation is characterized by its ground state and R-parity assignments distinguish between representations with the same spin content, say fermion and its scalar super-partner and Higgs with its fermionic super-partner.

1. In the massive case one obtains in the rest system just fermionic creation operators and $2^{\mathcal{N}}$ annihilation operators. The number of states created from a vacuum state with spin s_0 is $2\mathcal{N}$ and maximum spin is $s_0 + \mathcal{N}/2$. For instance, for $\mathcal{N} = 1$ and $s_0 = 0$ one obtains for 4 states with spins $J \leq 1/2$. Renormalizability requires massive matter to have $s \leq 1/2$ so that only $\mathcal{N} = 1$ is possible in this case. For particles massless at fundamental level and getting their masses by symmetry breaking this kind of restriction does not apply.

2. In the massless case only one half of fermionic oscillator operators have vanishing anti-commutators corresponding to the fact that for massless state only the second helicity is physical. This implies that the number of states is only $2^{\mathcal{N}}$ and the helicities vary from λ_0 to $\lambda_0 + \mathcal{N}/2$. For $\mathcal{N} = 1$ the representation is 2-dimensional.

3. In the presence of central charges $Z_{ij} = -Z_{ji}$ the representations are in general massive (Z_{ij} has dimensions of mass), $U(N)$ acts as symmetries of Z, and since Z^2 is symmetric its diagonalizability implies that Z matrix can be cast by a unitary transformation into a direct sum of 2-D antisymmetric real matrices multiplied by constants Z_i. Therefore the super-algebra can be cast in diagonal form with anti-commutators proportional to $M \pm Z_m$ with $M - Z_m \geq 0$ by unitarity. This implies the celebrated Bogomol'nyi bound $M \geq max\{Z_n\}$. For this value of varying mass parameter it is possible to have reduction of the dimension of the representation by one half. If the eigenvalues Z_n are identical the number of states is reduced to that for a massless representation. This multiplet is known as short BPS multiplet. Although BPS multiplets are massive (mass is expressible in terms of Higgs expectation value) they form multiplets shorter than the usual massive SUSY multiplets.

16.2.3 Super-Space

The heuristic view about super-space [B2] is as a manifold with D local bosonic coordinates x^μ and $\mathcal{N}D/2$ complex anti-commuting spinor coordinates θ_i^α and their complex conjugates $\overline{\theta}^i_{\dot{\alpha}} = (\theta_i^\alpha)^*$. For $\mathcal{N} = 1$, which is relevant to minimally super-symmetric standard model (MSSM), the spinors θ can also chosen to be real that is Majorana spinors, so that one has 4 bosonic and four real coordinates. In TGD framework one must however use Weyl spinors.

The anti-commutation relations for the super-coordinates are

$$\{\theta_\alpha, \theta_\beta\} = \{\theta_{\dot{\alpha}}, \theta_{\dot{\beta}}\} = \{\theta_\alpha, \theta_{\dot{\beta}}\} = 0 \ . \tag{16.2.10}$$

The integrals over super-space in 4-D $\mathcal{N} = 1$ case are defined by the following formal rules which actually state that super-integration is formally analogous to derivation.

$$\begin{gathered}
\int d\theta = \int d\overline{\theta} = \int d\theta\overline{\theta} = \int d\overline{\theta}\theta = 0 \ , \\
\int d\theta^\alpha d\theta_\beta = \delta^\alpha_\beta \ , \ \int d\overline{\theta}_{\dot{\alpha}} d\overline{\theta}_{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \ , \\
\int d^2\theta\theta^2 = \int d^2\overline{\theta}\overline{\theta}^2 \ , \ \int d^4\theta\theta^2\overline{\theta}^2 = 1 \ .
\end{gathered} \tag{16.2.11}$$

Here the shorthand notations

$$\begin{gathered}
d^2\theta \equiv -\frac{1}{4}\epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta \ , \\
d^2\overline{\theta} \equiv -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}} d\theta_{\dot{\alpha}} d\ theta_{\dot{\beta}} \ , \\
d^4\theta \equiv d^2\theta d^2\overline{\theta} \ .
\end{gathered} \tag{16.2.12}$$

are used.

The generalization of the formulas to $D > 4$ and $\mathcal{N} > 1$ cases is trivial. In infinite-dimensional case relevant for the super-symmetrization of the WCW geometry in terms of local Clifford algebra of WCW to be proposed later the infinite number of complex theta parameters poses technical problems unless one defines super-space functions properly.

Chiral super-fields

Super-multiplets can be expressed as single super-field define in super-space. Super-field can be expanded as a Taylor series with respect to the theta parameters. In 4-dimensional $\mathcal{N} = 1$ case one has

$$\Phi(x^\mu,\theta,\overline{\theta}) = \phi(x^\mu) + \theta\eta(x^\mu) + \overline{\theta}\eta^\dagger(x^\mu) + \overline{\theta\ sigma}^\alpha \theta V_\alpha(x^\mu) + \theta^2 F(x^\mu) + \overline{\theta^2}\overline{F}(x^\mu) ... + \theta^2\overline{\theta}^2 D(x^\mu) \quad (16.2.13)$$

The action of super-symmetries on super-fields can be expressed in terms of super-covariant derivatives defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\ dot\alpha}\frac{\partial}{\partial_\mu} \ , \ \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\ dot\alpha}\frac{\partial}{\partial_\mu} \ . \quad (16.2.14)$$

This allows very concise realization of super-symmetries.

General super-field defines a reducible representation of super-symmetry. One can construct irreducible representations of super-fields a pair of chiral and antichiral super-fields by posing the condition

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \ , \quad D_\alpha\Phi^\dagger = 0 \ . \quad (16.2.15)$$

The hermitian conjugate of chiral super-field is anti-chiral.

Chiral super-fields can be expressed in the form

$$\Phi = \Phi(\theta, y^\mu) \ , \ y^\mu = x^\mu + i\overline{\theta}\sigma^\mu\theta \ , y^{\mu\dagger} = x^\mu - i\overline{\theta}\sigma^\mu\theta \ . \quad (16.2.16)$$

These formulas generalize in a rather straightforward manner to $D > 4$ and $\mathcal{N} > 1$ case.

It is easy to check that any analytic function of a chiral super-field, call it $W(\Phi)$, is a chiral super-field. In super-symmetries its θ^2 component transforms by a total derivative so that the action defined by the super-space integral of $W(\phi)$ is invariant under super-symmetries. This allows to construct super-symmetric actions using $W(\Phi)$ and $W(\Phi^\dagger)$. The so called super-potential is defined using the sum of $W(\Phi) + W(\Phi^\dagger)$.

Analytic functions of does not give rise to kinetic terms in the action. The observation $\theta^2\overline{\theta}^2$ component of a real function of chiral super-fields transforms also as total derivative under super-symmetries allows to circumvent this problem by introducing the notion of Kähler potential $K(\Phi,\Phi^\dagger)$ as a real function of chiral super-field and its conjugate. In he simplest case one has

$$K = \sum_i \Phi_i^\dagger\Phi_i \ . \quad (16.2.17)$$

$L_K = \int K d^4\theta$ gives rise to simples super-symmetric action for left-handed fermion and its scalar super-partner.

Kähler potential allows an interpretation as a Kähler function defining the Kähler metric for the manifold defined by the scalars ϕ_i. This Kähler metric depends in the general case on ϕ_i and appears in the kinetic term of the super-symmetric action. Super-potential in turn can be interpreted as a counterpart of real part of a complex function which can be added to the Kähler function without affect the Kähler metric. This geometric interpretation suggests that in TGD framework every complex coordinate ϕ_i of WCW defines a chiral super-field whose bosonic part.

Wess-Zumino model as simple example

Wess-Zumino model without interaction term serves as a simple illustration of above formal considerations. The action density of Wess-Zumino Witten model can be deduced by integration Kähler potential $K = \Phi^\dagger\Phi$ for chiral super fields over theta parameters. The result is

$$L = \partial_u\phi^*\partial^\mu\phi + i\eta^*\partial^\mu\eta + F^*F \ . \tag{16.2.18}$$

The action of super-symmetry

$$\delta\Phi = \epsilon^\alpha D_\alpha\Phi \ , \quad \delta\Phi^\dagger = \overline{\epsilon}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}}\Phi \ , \quad \epsilon_{\dot{\alpha}} = \epsilon^{*\alpha} \tag{16.2.19}$$

gives the transformation formulas

$$\delta\phi = \epsilon^\alpha\eta_\alpha \ , \quad \delta\eta = -i\eta^{*\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi + \epsilon_\alpha F \ , \quad \delta F = -i\epsilon_{\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\alpha} \ partial_\mu\eta_\alpha \tag{16.2.20}$$

plus their hermitian conjugates. The corresponding Noether current is indeed hermitian since the transformation parameters ϵ^α and $\overline{\epsilon}_{\dot{\alpha}} = \epsilon^{*\alpha}$ appear in it and cannot be divided away. This conserved current has as such no meaning and the statement that ground state is annihilated by the corresponding super-charge means that vacuum field configuration rather than Fock vacuum remains invariant under supersymmetries. Rather, the breaking of super-symmetry by adding a super-potential implies that F develops vacuum expectation and the vacuum solution ($\phi = 0, \eta = 0, F = constant$) of field equations is not anymore invariant super super-symmetries.

The non-hermitian parts of the super current corresponding to different fermion numbers are separately conserved and corresponding super-charges are non-Hermitian and together with other charges define a super-algebra which to my best understanding is not equivalent with the super-algebra defined by allowing the presence of anti-commuting parameters ϵ. The situation is similar in TGD where one class of non-hermitian super-currents correspond to the modes of the induced spinor fields contracted with $\overline{\Psi}$ and their conjugates. The octonionic solution ansatz for the induced spinor field allows to express the solutions in terms of two complex scalar functions so that the super-currents in question would be analogous to those of $\mathcal{N} = 2$ SUSY and one might see the super-symmetry of quantum TGD extended super-symmetry obtained from the fundamental $\mathcal{N} = 2$ super-symmetry.

Vector super-fields and supersymmetric variant of YM action

Chiral super-fields allow only the super-symmetrization of Dirac action. The super-symmetrization of YM action requires the notion of a hermitian vector super field $V = V^\dagger$, whose components correspond to vector bosons, their super-counterparts and additional degrees of freedom which cannot be dynamical. These degrees of freedom correspond gauge degrees of freedom.

In the Abelian case the gauge symmetries are realized as $V \rightarrow V + \Lambda + \Lambda^\dagger$, where Λ is a chiral super-field. These symmetries induce gauge transformations of the vector potential. Their action on chiral super-fields is $\Phi \rightarrow exp(-q\Lambda)\Phi$, $\Phi^\dagger \rightarrow \Phi^\dagger exp(-\Lambda^\dagger)$. In non-Abelian case the realization is as $exp(V) \rightarrow exp(-\Lambda^\dagger)exp(V)exp(\Lambda)$ so that the modified Kähler potential $K(\Phi^\dagger, exp(qV)\Phi)$ remains invariant.

One can assign to V a gauge invariant chiral spinor super-field as

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{D}^2(e^V D_\alpha e^{-V}) \ , \\ \overline{D}^2 &= \epsilon^{\dot{\alpha}\dot{\beta}}\overline{D}_{\ al\dot{p}ha}\overline{D}_{\dot{\beta}} \end{aligned} \tag{16.2.21}$$

defining the analog of gauge field. $\overline{D}^2$ eliminates all terms the exponent of $\overline{\theta}$ is higher than that of θ since these would spoil the chiral super-field property (the anti-commutativity of super-covariant derivatives $\overline{D}_{\dot{\alpha}}$ makes this obvious). D_α in turn eliminates from the resulting scalar part so that one indeed has chiral spinor super-field. In higher dimensions and for larger value of $\mathcal{N}$ the definition of W_α must be modified in order to achieve this: what is needed is the product of all derivatives $\overline{D}_{i\dot{\alpha}}$.

The analytic functions of chiral spinor super-fields are chiral super-fields and θ^2 component of $W^\alpha W_\alpha$ transforms as a total derivatives. The super-symmetric Lagrangian of U(1) theory can be written as

$$L = \frac{1}{4g^2}\left(\int d^2\theta W^\alpha W_\alpha + \int d^2\overline{\theta} W^\dagger_{\dot{\alpha}} W^\dagger_{\dot{\alpha}}\right) . \quad (16.2.22)$$

Note that in standard form of YM action $1/2g^2$ appears.

R-symmetry

R-symmetry is an important concomitant of super-symmetry. In $\mathcal{N}=1$ case R-symmetry performs a phase rotation $\theta \rightarrow e^{i\alpha}\theta$ for the super-space coordinate θ and an opposite phase rotation for the differential $d\theta$. For $\mathcal{N}>1$ R-symmetries are $U(N)$ rotations. R-symmetry is an additional symmetry of the Lagrangian terms due to Kähler potential since both $d^4\theta$ (and its generalization) as well as Kähler potential are real. Also super-symmetric YM action is R-invariant. R-symmetry is a symmetry of if super-potential W only if it has super-charge $Q_R = 2$ ($Q_R = 2\mathcal{N}$) in order to compensate the super-charge of $d^{2\mathcal{N}}\theta$.

16.2.4 Non-Renormalization Theorems

Super-symmetry gives powerful constraints on the super-symmetric Lagrangians and leads to non-renormalization theorems.

The following general results about renormalization of supersymmetric gauge theories hold true (see [B66], where a heuristic justification of the non-renormalization theorems and explicit formulas are discussed).

1. Super-potential is not affected by the renormalization.
2. Kähler potential is subject to wave function renormalization in all orders. The renormalization depends on the parameters with dimensions of mass. In particular, quadratic divergences to masses cancel.
3. Gauge coupling suffers renormalization only by a constant which corresponds to one-loop renormalization. Any renormalization beyond one loop is due to wave function renormalization of the Kähler potential and it is possible to calculate the beta function exactly.

It is interesting to try to see these result from TGD perspective.

1. In TGD framework super-potential interpreted as defining the modification of WCW Kähler function, which does not affect Kähler metric and would reflect measurement interaction. The non-renormalization of W would mean that the measurement interaction is not subject to renormalization. The interpretation is in terms of quantum criticality which does not allow renormalization of the coefficients appearing in the measurement interaction term since otherwise Kähler metric of WCW would be affected.
2. The wave function renormalization of Kähler potential would correspond in TGD framework scaling of the WCW Kähler metric. Quantum criticality requires that Kähler function remains invariant. Also since no parameters with dimensions of mass are available, there is temptation to conclude that wave function renormalization is trivial.
3. Only the gauge coupling would be suffer renormalization. If one however believes in the generalization of bosonic emergence it is Kähler function which defines the SUSY QFT limit of TGD so that gauge couplings follow as predictions and their renormalization is a secondary -albeit real- effect having interpretation in terms of the dependence of the gauge coupling on the p-adic length scale. The conclusion would be that at the fundamental level the quantum TGD is RG invariant.

16.3 Does TGD Allow The Counterpart Of Space-Time Super-symmetry?

The question whether TGD allows space-time super-symmetry or something akin to it has been a longstanding problem. A considerable progress in the respect became possible with the better understanding of the Kähler-Dirac equation.

16.3.1 Kähler-Dirac Equation

Before continuing one must briefly summarize the recent view about Kähler-Dirac equation.

1. The localization of the induced spinor fields to 2-D string world sheets is crucial. It is demanded both by the well-definedness of em charge and by number theoretical constraints. Induced W boson fields must vanish, and the Frobenius integrability conditions guaranteeing that the K-D operator involves no covariant derivatives in directions normal to the string world sheet must be satisfied.

2. The Kähler-Dirac equation (or Kähler Dirac equation) reads as

$$D_K\Psi \;\; = \;\; 0 \; . \tag{16.3.1}$$

   in the interior of space-time surface. The boundary variation of K-D equation gives the term

$$\Gamma^n\Psi = 0 \tag{16.3.2}$$

   at the light-like orbits of partonic 2-surfaces. Clearly, Kähler-Dirac gamma matrix Γ^n in normal direction must be light-like or vanish.

3. To the boundaries of string world sheets at the orbits of partonic 2-surfaces one assigns 1-D Dirac action in induced metric line with length as bosonic counterpart. By field equations both actions vanish, and one obtains light-like geodesic carrying light-like 8-momentum. Algebraic variant of massless 8-D Dirac equation is satisfied for the 8-momentum parallel to 8-velocity.

   The boundaries of the string world sheets are thus pieces of light-like M^8 geodesics and different fermion lines should have more or less parallel M^4 momenta for the partonic 2-surface to preserve its size. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach encourages also by the very special twistorial properties of M^4 and CP_2.

   One can wonder how this relates to braiding which is one of the key ingredients of TGD. Is the braiding possible unless it is induced by particle exchanges so that the 8-momentum changes its direction and partonic 2-surface replicates. In principle it should be possible to construct the orbits of partonic 2-surfaces in such a way that braiding occurs. Situation is the reverse of the usual in which one has fixed 3-manifold in which one constructs braid.

4. One can construct preferred extremals by starting from string world sheets satisfying the vanishing of normal components of canonical momentum currents as analogs of boundary conditions. One can also fix 3-D space-like surfaces and partonic orbits and pose the vanishing of super-symplectic charges for a sub-algebra with conformal weights coming as multiples of fixed integer n as conditions selecting preferred extremals.

5. The quantum numbers characterizing zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse. Thermodynamics would naturally couple to the space-time geometry via the thermodynamical or quantum averages of the quantum numbers.

16.3.2 Development Of Ideas About Space-Time SUSY

Let us first summarize the recent overall view about space-time super-symmetry for TGD discussed in detail in chapter " WCW spinor structure" and also in [K106].

1. Right-handed covariantly constant neutrino spinor ν_R defines a super-symmetry in CP_2 degrees of freedom in the sense that CP_2 Dirac equation is satisfied by covariant constancy and there is no need for the usual ansatz $\Psi = D\Psi_0$ giving $D^2\Psi = 0$. This super-symmetry allows to construct solutions of Dirac equation in CP_2 [A43, A53, A31, A49].

2. In $M^4 \times CP_2$ this means the existence of massless modes $\Psi = \not{p}\Psi_0$, where Ψ_0 is the tensor product of M^4 and CP_2 spinors. For these solutions M^4 chiralities are not mixed unlike for all other modes which are massive and carry color quantum numbers depending on the CP_2 chirality and charge. As matter fact, massless right-handed neutrino covariantly constant in CP_2 spinor mode is the only color singlet. The mechanism leading to non-colored states for fermions is based on super-conformal representations for which the color is neutralized [K52, K52]. The negative conformal weight of the vacuum also cancels the enormous contribution to mass squared coming from mass in CP_2 degrees of freedom.

3. All spinor modes define conserved fermion super-currents and also the super-symplectic algebra has a fermion representation as Noether currents at string world sheets. WCW metric can be constructed as anti-commutators of super-symplectic Noether currents and one obtains a generalization of AdS/CFT duality to TGD framework from the possibility to express Kähler also in terms of Kähler function (and thus Kähler action). The fact that super-Poincare anti-commutator vanishes for oscillator operators associated with covariantly constant right-handed neutrino and anti-neutrino implies that it corresponds to a pure gauge degree of freedom.

4. The natural conjecture is that the TGD analog space-time SUSY is generated by the Clifford algebra of the second quantized fermionic oscillator operators at string world sheets. This algebra in turn generalizes to Yangian. The oscillator operators indeed allow the 8-D analog of super-Poincare anti-commutation relations at the ends of 1-D light-like geodesics defined by the boundaries of string world sheets belonging to the orbits of partonic 2-surfaces and carrying 8-D light-like momentum.

   For incoming on mass shell particles one can identify the M^4 part of 8-momentum as gravitational for momentum equal to the inertial four-momentum assignable to embedding space spinor harmonic for incoming on mass shell state. The square of E^4 momentum giving mass squared corresponds to the eigenvalue of CP_2 d'Alembertian.

   8-D light-like momentum forces an 8-D generalization of the twistor approach and M^4 and CP_2 are indeed unique in that they allow twistor space with Kähler structure [A54]. The conjecture is that integration over virtual momenta restricts virtual momenta to 8-D light-like momenta but the polarizations of virtual fermions are non-physical.

5. The 8-D generalization of SUSY describes also massive states and one has $\mathcal{N} = \infty$. Ordinary 4-D SUSY is obtained by restricting the states to the massless sector of the theory. The value of $\mathcal{N}$ is finite in this case and corresponds to the value of massless modes for fundamental fermions. Quark and lepton type spinor components with physical helicity for fermions and anti-fermions define the basis of the SUSY algebra as Clifford algebra of oscillator operators with anti-commutators analogous to those associated with super Poincare algebra. Therefore the generators of SUSY correspond to the 4+4 components of embedding space spinor modes (quarks and leptons) with vanishing conformal weight so that analogs of $\mathcal{N} = 4$ SUSY are obtained in quark and lepton sectors.

   The SUSY is broken due to the electro-weak and color interactions between the fundamental fermions. For right-handed neutrinos these interactions are not present but the mixing with left handed neutrino due to the mixing of M^4 and CP_2 gamma matrices in Kähler-Dirac gamma matrices at string world sheets implies SUSY breaking also now: also R-parity is broken.

Basically a small mixing with the states with CP_2 mass is responsible for the generation of mass and breaking of SUSY. p-Adic thermodynamics describes this mixing. SUSY is broken at QFT limit also due the replacement of the many-sheeted space-time with single slightly curved region of M^4.

6. The SUSY in question is not the conventional $\mathcal{N} = 1$ SUSY. Space-time (in the sense of Minkowski space M^4) $\mathcal{N} = 1$ SUSY in the conventional sense of the word is impossible in TGD framework since it would require require Majorana spinors. In 8-D space-time with Minkowski signature of metric Majorana spinors are definitely ruled out by the standard argument leading to super string model. Majorana spinors would also break the separate conservation of lepton and baryon numbers in TGD framework. What is remarkable is that in 8-D space-time one obtains naturally SUSY with Dirac spinors.

16.3.3 Summary About TGD Counterpart Of Space-Time SUSY

This picture allows to define more precisely what one means with the approximate super-symmetries in TGD framework.

1. One can in principle construct many-fermion states containing both fermions and anti-fermions at fermion lines located at given light-like parton orbit. The four-momenta of states related by super-symmetry need not be same. Super-symmetry breaking is present and has as the space-time correlate the deviation of the Kähler-Dirac gamma matrices from the ordinary M^4 gamma matrices. In particular, the fact that $\hat{\Gamma}^\alpha$ possesses CP_2 part in general means that different M^4 chiralities are mixed: a space-time correlate for the massivation of the elementary particles.

2. For right-handed neutrino super-symmetry breaking is expected to be smallest but also in the case of the right-handed neutrino mode mixing of M^4 chiralities takes place and breaks the TGD counterpart of super-symmetry. Maybe the correct manner to interpret the situation is to speak about 8-D massless states for which the counterpart of SUSY would not be broken but mass splittings are possible.

3. The fact that all helicities in the state are physical for a given light-like 3-surface has important implications. For instance, the addition of a right-handed antineutrino to right-handed (left-handed) electron state gives scalar (spin 1) state. Also states with fermion number two are obtained from fermions. For instance, for e_R one obtains the states $\{e_R, e_R\nu_R\overline{\nu}_R, e_R\overline{\nu}_R, e_R\nu_R\}$ with lepton numbers $(1,1,0,2)$ and spins $(1/2,1/2,0,1)$. For e_L one obtains the states $\{e_L, e_L\nu_R\overline{\nu}_R, e_L\overline{\nu}_R, e_L\nu_R\}$ with lepton numbers $(1,1,0,2)$ and spins $(1/2,1/2,1,0)$. In the case of gauge boson and Higgs type particles -allowed by TGD but not required by p-adic mass calculations- gauge boson has 15 super partners with fermion numbers $[2,1,0,-1,-2]$.

The cautious conclusion is that the recent view about quantum TGD allows the analog of super-symmetry, which is necessary broken and for which the multiplets are much more general than for the ordinary super-symmetry. Right-handed neutrinos might however define something resembling ordinary super-symmetry to a high extent. The question is how strong prediction one can deduce using quantum TGD and proposed super-symmetry.

1. For a minimal breaking of super-symmetry only the p-adic length scale characterizing the super-partner differs from that for partner but the mass of the state is same. This would allow only a discrete set of masses for various super-partners coming as half octaves of the mass of the particle in question. A highly predictive model results.

2. The quantum field theoretic description could be based on QFT limit of TGD, which I have formulated in terms of bosonic emergence. The idea was that his formulation allows to calculate the propagators of the super-partners in terms of fermionic loops. Similar description of exchanged boson as fermionic loop emerges also in the proposed identification of scattering amplitudes as representations of algebraic computations in Yangian using product and co-product as fundamental vertices assignable to partonic 2-surfaces at which 3-surfaces replicate.

3. This TGD variant of space-time super-symmetry resembles ordinary super-symmetry in the sense that selection rules due to the right-handed neutrino number conservation and analogous to the conservation of R-parity hold true (the mixing of right-handed neutrino with the left-handed one breaks R-parity). The states inside super-multiplets have identical electro-weak and color quantum numbers but their p-adic mass scales can be different. It should be possible to estimate reaction reaction rates using rules very similar to those of super-symmetric gauge theories.

4. It might be even possible to find some simple generalization of standard super-symmetric gauge theory to get rough estimates for the reaction rates. There are however problems. The fact that spins $J = 0, 1, 2, 3/2, 2$ are possible for super-partners of gauge bosons forces to ask whether these additional states define an analog of non-stringy strong gravitation. Note that graviton in TGD framework corresponds to a pair of wormhole throats connected by flux tube (counterpart of string) and for gravitons one obtains 2^8-fold degeneracy.

16.3.4 SUSY Algebra Of Fermionic Oscillator Operators And WCW Local Clifford Algebra Elements As Super-fields

Whether TGD allows space-time supersymmetry has been a long-standing question. Majorana spinors appear in $N = 1$ super-symmetric QFTs- in particular minimally super-symmetric standard model (MSSM). Majorana-Weyl spinors appear in M-theory and super string models. An undesirable consequence is chiral anomaly in the case that the numbers of left and right handed spinors are not same. For $D = 11$ and $D = 10$ these anomalies cancel, which led to the breakthrough of string models and later to M-theory. The probable reason for considering these dimensions is that standard model does not predict right-handed neutrino (although neutrino mass suggests that right handed neutrino exists) so that the numbers of left and right handed Weyl-spinors are not the same.

In TGD framework the situation is different. Covariantly constant right-handed neutrino spinor acts as a super-symmetry in CP_2. One might think that right-handed neutrino in a well-defined sense disappears from the spectrum as a zero mode so that the number of right and left handed chiralities in $M^4 \times CP_2$ would not be same. For light-like 3-surfaces covariantly constant right-handed neutrino does not however solve the counterpart of Dirac equation for a non-vanishing four-momentum and color quantum numbers of the physical state. Therefore it does not disappear from the spectrum anymore and one expects the same number of right and left handed chiralities.

In TGD framework the separate conservation of baryon and lepton numbers excludes Majorana spinors and also the the Minkowski signature of $M^4 \times CP_2$ makes them impossible. The conclusion that TGD does not allow super-symmetry is however wrong. For $\mathcal{N} = 2N$ Weyl spinors are indeed possible and if the number of right and left handed Weyl spinors is same super-symmetry is possible. In 8-D context right and left-handed fermions correspond to quarks and leptons and since color in TGD framework corresponds to CP_2 partial waves rather than spin like quantum number, also the numbers of quark and lepton-like spinors are same.

The physical picture suggest a new kind of approach to super-symmetry in the sense that the anti-commutations of fermionic oscillator operators associated with the modes of the induced spinor fields define a structure analogous to SUSY algebra in 8-D sense. Massless modes of spinors in 1-1 corresponds with embedding space spinors with physical helicity are in 1-1 correspondence with the generators of SUSY at space-time level giving $\mathcal{N} = 4 + 4$. Right handed neutrino modes define a sub-algebra for which the SUSY is only slightly broken by the absence of weak interactions and one could also consider a theory containing a large number of $\mathcal{N} = 2$ super-multiplets corresponding to the addition of right-handed neutrinos and antineutrinos at the wormhole throat.

Masslessness condition is essential if super-symmetric quantum field theories and at the fundamental level it can be generalized to masslessness in 8-D sense in terms of Kähler-Dirac gamma matrices using octonionic representation and assuming that they span local quaternionic sub-algebra at each point of the space-time sheet. SUSY algebra has standard interpretation with respect to spin and isospin indices only at the partonic 2-surfaces so that the basic algebra should be formulated at these surfaces: in fact, out that the formulation is needed only at the ends of fermion lines. Effective 2-dimensionality would require that partonic 2-surfaces can be taken to be ends of any light-like 3-surface Y_l^3 in the slicing of the region surrounding a given wormhole throat.

Super-algebra associated with the Kähler-Dirac action

Anti-commutation relations for fermionic oscillator operators associated with the induced spinor fields are naturally formulated in terms of the Kähler-Dirac gamma matrices. The canonical anti-commutation relations for the fermionic oscillator operators at light-like 3-surfaces or at their ends can be formulated as anti-commutation relations for SUSY algebra. The algebra creating physical states is super-symplectic algebra whose generators are expressed as Noether charges assignable to strings connecting partonic 2-surfaces.

Lepton and quark like spinors are now the counterparts of right and left handed Weyl spinors. Spinors with dotted and un-dotted indices correspond to conjugate representations of $SO(3,1) \times SU(4)_L \times SU(2)_R$. The anti-commutation relations make sense for sigma matrices identified as 6-dimensional matrices $1_6, \gamma_7, \gamma_1, ...\gamma_6$.

Consider first induced spinor fields at the boundaries of string world sheets at the orbits of wormhole throats. Dirac action for induced spinor fields and its bosonic counterpart defined by line-length are required by the condition that one obtains fermionic propagators massless in 8-D sense.

1. The localization of induced spinor fields to string world sheets and the addition of 1-D Dirac action at the boundaries of string world sheets at the orbits of partonic 2-surfaces reduces the quantization to that at the end of the fermion line at partonic 2-surface located at the boundary of CD. Therefore the situation reduces to that for point particle.

2. The boundary is by the extremization of line length a geodesic line of embedding space, which can be characterized by conserved four-momentum and conserved angular momentum like charge - call it hypercharge Y. The square of 8-velocity vanishes: $v_4^2 - (v^\phi)^2 = 0$ and one can choose $v_4^2 = 1$. 8-momentum is proportional to 8-velocity expressible as (v^k, v^ϕ).

3. Dirac equation gives $\Gamma^t \partial_t \Psi = (\gamma_k v^k + \gamma_\phi) v^\phi) \partial_t \Psi = 0$. The non-trivial solution corresponds to $\partial_t \Psi = i\omega\Psi$ and the light-likeness condition. The value of parameter ω defines the mass scale and quantum classical correspondences suggests that ω^2 gives the mass squared identifiable as the eigenvalue of CP_2 Laplacian for spinor modes.

4. Anti-commutation relations must be fixed at either end of fermion line for the oscillator operators associated with the modes of induced spinor field at string world sheet labelled by integer value conformal weight and spin and weak isospin for the H-spinor involved. These anti-commutation relations must be consistent with standard canonical quantization allowing in turn to assign Noether charges to super-symplectic algebra defined as integrals over string world sheet. The identification of WCW gamma matrices as these charges allows to calculate WCW metric as their anti-commutators.

5. The oscillator operators for the modes with different values of conformal weight vanish. Standard anti-commutation relations in massive case are completely fixed and correspond to just Kronecker delta for conformal weights, spin, and isospin.

Space-time supersymmetry and the need to generalize 4-D twistors to 8-D ones suggest the anti-commutation relations obeyed by 8-D analogs of massless Weyl spinors and thus proportional to $p_8^k \sigma_k$, where p_8^k is the 8-momentum associated with the end of the fermion line and σ_k are the 8-D analogs of 2×2 sigma matrices.

1. This requires the introduction of octonionic spinor structure with gamma matrices represented in terms of octonionic units and introducing octonionic gamma matrices. The natural condition is that the octonionic gamma matrices are equivalent with the ordinary one. This is true if fermions are localied at time-like or light-like geodesic lines of embedding space since they represent- not only quaternionic, but even hypercomplex sub-manifolds of embedding space. This allows ordinary matrix representations for the gamma matrices at fermion lines.

2. One can avoid the problems with the non-associativity also at string world sheets possible caused by the Kähler Dirac gamma matrices if the two Kähler Dirac gamma matrices span commutative subspace of complexified octonions. The sigma matrices appearing in induced gauge potentials could be second source of non-associativity. By assuming that the solutions are holomorphic spinors (just as in string models) and that in the gauge chosen only holomorphic or anti-holomorphic components of gauge boson fields are non-vanishing, one avoids these problems.

3. It must be admitted that the constraints on string world sheets are strong: vanishing W induced gauge fields, Frobenius integrability conditions, and the condition that K-D gamma matrices span a commutative sub-space of complexified octonions, and I have not really proven that they can be satisfied.

The super-generators of space-time SUSY are proportional to fermionic oscillator operators obeying the canonical anti-commutation relations. It is not quite clear to me whether the proportionality constant can be taken to be equal to one although intuition suggests this strongly. The anti-commutations can contain only the light-like 8-velocity at the right hand side carrying information about the direction of the fermion line.

One can wonder in how strong sense the strong form of holography is realized.

1. Is the only information about the presence of strings at the level of scattering amplitudes the information coded by the anti-commutation relations at their end points? This would be the case if the fermion super-conformal charges vanish or create zero norm states for non-vanishing conformal weights. It could however happen that also the super-conformal generators associated with a sub-algebra of conformal algebra with weights coming as integer multiples of the entire algebra do this. At least this should be the case for the super-symplectic algebra.

2. Certainly one must assume that the 8-velocities associated with the ends of the fermionic string are independent so that strings would imply bi-locality of the dynamics.

Summing up the anti-commutation relations

In leptonic sector one would have the anti-commutation relations

$$\begin{aligned} \{a^{\dagger}_{m\dot{\alpha}}, a^n_{\beta}\} &= 2\delta^n_m D_{\dot{\alpha}\beta} \ , \\ D &= (p_\mu + \sum_a Q^a_\mu)\sigma^\mu \ . \end{aligned} \tag{16.3.3}$$

In quark sector σ^μ is replaced with $\overline{\sigma}^\mu$ obtained by changing the signs of space-like sigma matrices in leptonic sector. p_μ and Q^a_μ are the projections of momentum and color charges in Cartan algebra to the space-time surface and their values correspond to those assignable to the fermion line and related by quantum classical correspondence to those associated with incoming spinor harmonic.

The anti-commutation relations define a generalization of the ordinary equal-time anti-commutation relations for fermionic oscillator operators to a manifestly covariant form. Extended SUSY algebra suggest that the anti-commutators could contain additional central charge term proportional to $\delta_{\alpha\beta}$ but the 8-D chiral invariance excludes this term.

In the octonionic representation of the sigma matrices matrix indices cannot be present at the right handed side without additional conditions. Octonionic units however allow a representation as matrices defined by the structure constants failing only when products of more than two octonions are considered. For the quaternionic sub-algebra this does not occur. Both spinor modes and and gamma matrices must belong to the local hyper-quaternionic sub-algebra and do trivially so for fermion lines and string. Octonionic representation reduces $SO(7,1)$ so G_2 as a tangent space group. Similar reduction for 7-dimensional compact space takes place also M-theory.

In standard SUSY local super-fields having values in the Grassmann algebra generated by theta parameters appear. In TGD framework this would mean allowance of many-fermion states at single space-time point and this is perhaps too heavy an idealization since partonic 2-surfaces are the fundamental objects. Multi-stringy generators in the extension of super-symplectic algebra to Yangian is a more natural concept in TGD framework since one expects that partonic 2-surfaces involve several strings connecting them to other partonic 2-surfaces. Super-symplectic charges would be Noether charges assignable to these strings and quantum states would be created by these charges from vacuum. Scattering amplitudes would be defined in terms of Yangian algebra [L10]. Only at QFT limit one can hope that super-field formalism works.

16.4 Understanding Of The Role Of Right-Handed Neutrino In Supersymmetry

The development of the TGD view about space-time SUSY has been like a sequence of questions loves -doesn't love- loves.... From the beginning it was clear that right-handed neutrino could generate super-conformal symmetry of some kind, and the natural question was whether it generates also space-time SUSY. Later it became clear that all fermion oscillator operators can be interpreted as super generators for the analog of space-time SUSY. After that the challenge was to understand whether all spin-isospin states of fermions correspond super generators.

$\mathcal{N} = 1$ SUSY was excluded by separate conservation of B and L but $\mathcal{N} = 2$ variant of this symmetry could be considered and could be generated by massless right-handed neutrino and antineutrino mode.

The new element in the picture was the physical realization of the SUSY by adding fermions - in special case right-handed neutrino - to the state associated with the orbit of partonic 2-surface. An important realization was the necessity to localized spinors to string world sheet and the assignment of fernionic oscillator operator with boundaries of string world sheets at them. Variational principles implies that the fermions have light-like 8-momenta and that the fermion lines are light-like geodesics in 8-D sense. This leads to a precise view about the quantization of induced spinor fields. Fermionic oscillator operator algebra would generate Clifford algebra replacing the SUSY algebra and one would obtain the analog of super Poincare algebra from anti-commutation relations.

16.4.1 Basic Vision

As already explained, the precise meaning of SUSY in TGD framework has been a long-standing head ache. In TGD framework SUSY is inherited from super-conformal symmetry at the level of WCW [K24, K23]. The SUSY differs from $\mathcal{N} = 1$ SUSY of the MSSM and from the SUSY predicted by its generalization and by string models. Allowing only right-handed neutrinos as SUSY generators, one obtains the analog of the $\mathcal{N} = 4$ SUSY in bosonic sector but there are profound differences in the physical interpretation. The most general view is that all fermion modes with vanishing conformal weights define super charges.

1. One could understand SUSY in very general sense as an algebra of fermionic oscillator operators acting on vacuum states at partonic 2-surfaces. Oscillator operators are assignable to braids ends and generate fermionic many particle states. SUSY in this sense is badly broken and the algebra corresponds to rather large $\mathcal{N}$. The restriction to covariantly constant right-handed neutrinos (in CP_2 degrees of freedom) gives rise to the counterpart of ordinary SUSY, which is more physically interesting at this moment.

2. Right handed neutrino and antineutrino are not Majorana fermions. This is necessary for separate conservation of lepton and baryon numbers. For fermions one obtains the analog $\mathcal{N} = 2$ SUSY.

3. Bosonic emergence means the construction of bosons as bound states of fermions and antifermions at opposite throats of wormhole contact. Later it became clear that all elementary particles emerge as bound states of fundamental fermions located at the wormhole throats of a pair of wormhole contacts. Two wormhole contacts are required by the assumption wormhole contacts carry monopole magnetic flux stabilizing them.

   This reduces TGD SUSY to that for fundamental fermions. This difference is fundamental and means deviation from the $\mathcal{N} = 4$ SUSY, where SUSY acts on gauge boson states. Bosonic representations are obtained as tensor products of representations assigned to the opposite throats of wormhole contacts. One can also have several fermion lines at given throat but these states are expected to be exotic.

Further tensor products with representations associated with the wormhole ends of magnetic flux tubes are needed to construct physical particles. This represents a crucial difference with respect to standard approach, where one introduces at the fundamental level both fermions and bosons or gauge bosons as in $\mathcal{N} = 4$ SUSY. Fermionic $\mathcal{N} = 2$ representations are analogous to "short" $\mathcal{N} = 4$ representations for which one half of super-generators annihilates the states.

4. If stringy super-conformal symmetries act as gauge transformations, the analog of $\mathcal{N} = 4$ SUSY is obtained in both quark and lepton sector. This extends to $\mathcal{N} = 8$ SUSY if parton orbits can carry both quarks and leptons. Lepto-quark is the simplest state of this kind.

5. The introduction of both fermions and gauge bosons as fundamental particles leads in quantum gravity theories and string models to $d = 10$ condition for the target space, spontaneous compactification, and eventually to the landscape catastrophe.

   For a supersymmetric gauge theory (SYM) in d-dimensional Minkowski space the condition that the number of transversal polarization for gauge bosons given by $d - 2$ equals to the number of fermionic states made of Majorana fermions gives $d - 2 = 2^k$, since the number of fermionic spinor components is always power of 2.

   This allows only $d = 3, 4, 6, 10, 16, ...$ Also the dimensions $d + 1$ are actually possible since the number of spinor components for d and $d + 1$ is same for d even. This is the standard argument leading to super-string models and M-theory. It it lost - or better to say, one gets rid of it - if the basic fields include only fermion fields and bosonic states are constructed as the tensor products of fermionic states. This is indeed the case in TGD, where spontaneous compactification plays no role and bosons are emergent.

6. Spontaneous compactification leads in string model picture from $\mathcal{N} = 1$ SUSY in say $d = 10$ to $\mathcal{N} > 1$ SUSY in $d = 4$ since the fermionic multiplet reduces to a direct sum of fermionic multiplets in $d = 4$. In TGD embedding space is not dynamical but fixed by internal consistency requirements, and also by the condition that the theory is consistent with the standard model symmetries. The identification of space-time as 4-surface makes the induced spinor field dynamical and the notion of many-sheeted space-time allows to circumvent the objections related to the fact that only 4 field like degrees of freedom are present.

16.4.2 What Is The Role Of The Right-Handed Neutrino?

Whether right-handed neutrinos generate a supersymmetry in TGD has been a long standing open question. $\mathcal{N} = 1$ SUSY is certainly excluded by fermion number conservation but already $\mathcal{N} = 2$ defining a "complexification" of $\mathcal{N} = 1$ SUSY is possible and could generate right-handed neutrino and its antiparticle. Right-handed neutrinos should however possess a non-vanishing light-like momentum since the fully covariantly constant right-handed neutrino generates zero norm states.

The general view about the preferred extremals of Kähler action and application of the conservation of em charge to the Kähler-Dirac equation have led to a rather detailed view about classical and TGD and allowed to build a bridge between general vision about super-conformal symmetries in TGD Universe and field equations. This vision is discussed in detail in [K106].

1. Many-sheeted space-time means that single space-time sheet need not be a good approximation for astrophysical systems. The GRT limit of TGD can be interpreted as obtained by lumping many-sheeted space-time time to Minkowski space with effective metric defined as sum M^4 metric and sum of deviations from M^4 metric for various space-time sheets involved [K99]. This effective metric should correspond to that of General Relativity and Einstein's equations would reflect the underlying Poincare invariance. Gravitational and cosmological constants follow as predictions and EP is satisfied.

2. The general structure of super-conformal representations can be understood: super-symplectic algebra is responsible for the non-perturbative aspects of QCD and determines also the

ground states of elementary particles determining their quantum numbers. The hierarchy of breakings of conformal symmetry as gauge gauge symmetry would explain dark matter. The sub-algebra for which super-conformal symmetry remains gauge symmetry would be isomorphic to the original algebra and generated by generators for which conformal weight is multiple of integer $n = h_{eff}/h$. This would would be true for super-symplectic algebra at least and possible for all other conformal algebras involved.

3. Super-Kac-Moody algebras associated with isometries and holonomies dictate standard model quantum numbers and lead to a massivation by p-adic thermodynamics: the crucial condition that the number of tensor factors in Super-Virasoro represention is 5 is satisfied.

4. One can understand how the Super-Kac-Moody currents assignable to stringy world sheets emerging naturally from the conservation of em charge defined as their string world sheet Hodge duals gauge potentials for standard model gauge group and also their analogs for gravitons. Also the conjecture Yangian algebra generated by Super-Kac-Moody charges emerges naturally.

5. One also finds that right handed neutrino is in a very special role because of its lacking couplings in electroweak sector and its role as a generator of the least broken SUSY. The most feasible option is that all modes of the induced spinor field are restricted to 2-D string world sheets. If covariantly constant right-handed neutrino could be de-localized completely it cannot generate ordinary kind of gauge super-symmetry. It is not yet completely clear whether the modes of the induced spinor field are localized at string world sheets also inside the Euclidian wormhole contacts defining the lines of the generalized Feynman diagrams.

   Intermediate gauge boson decay widths require that sparticles are either heavy enough or dark in the sense of having non-standard value of Planck constant. Darkness would provide an elegant explanation for their non-observability. It should be emphasized that TGD predicts that all fermions act as generators of badly broken super-symmetries at partonic 2-surfaces but these super-symmetries could correspond to much higher mass scale as that associated with the de-localized right-handed neutrino. The following piece of text summarizes the argument.

6. Ordinary SUSY means that apart from kinematical spin factors sparticles and particles behave identically with respect to standard model interactions. These spin factors would allow to distinguish between particles and sparticles. This requires strong correlations between fermion and right-handed neutrino: in fact, they should be at rest with respect to each other. Right-handed neutrinos have vanishing color and electro-weak quantum numbers. How it is possible to have sparticles as bound states with ordinary particle and right-handed neutrino?

   The localization of induced spinor fields to string world sheets suggests a solution to the problem.

   (a) The localization forces the fermions to move in parallel although they have no interactions. The 8-momenta and 8-velocities of fermion are light-like and they move along light-like 8-geodesics. Since the size of the partonic 2-surface should not change much. If all fundamental fermions involved are massive one can assume that they are at rest and in this manner geometrically stable state.

   (b) If one has massive fermion and massless right-handed neutrino, they should be at rest with respect to each other. What looks paradoxical that one cannot reduce the velocity to exactly zero in any coordinate system since covariantly constant right-handed neutrino represents a pure gauge degree of freedom. It is of course possible to assume that the relative velocity is some sufficiently low velocity. One can also argue that sparticles are unstable and that this is basically due to a geometric instability implied by the non-parallel 3-momenta of fundamental fermions.

   (c) If one assumes that the 4-momentum squared corresponds to that associated with the embedding space spinor harmonics, one can to estimate the mass of the sparticle once the energy of the right-handed neutrino is fixed. This argument applies also to n-fermion states at associated with the wormhole contact pairs.

(d) p-Adic mass calculations however give to mass squared also other contributions that coming from the spinor harmonic, in particular negative ground state contribution and that the mass squared of the fundamental fermion vanishes for lowest states which would therefore have vanishing CP_2 velocity. Why the light-like four-momentum of the resulting state should not characterize the fermion line? In this picture p-adic thermal excitations would make the state unstable. One could in fact turn this argument to an explanation for why the stable physical particles must parallel 4-momenta.

(e) What is still not well-understood is the tachyonic contribution to four-momentum. One possibility is that wormhole contact gives imaginary contribution to four-momentum. Second possibility is that the generating super-symplectic conformal weights are the negatives for the zeros of zeta. For non-trivial zeros the real part of the conformal would be -1/2.

So called massless extremals (MEs) define massless represent classical field pattern moving with light velocity and preserving its shape. This suggests that particle represented as a magnetic flux tube structure carrying monopole flux with two wormhole contacts and sliced between two MEs could serve as a starting point in attempts to understand the role of right handed neutrinos and how $\mathcal{N}=2$ or $\mathcal{N}=4$ type SYM emerges at the level of space-time geometry.

16.4.3 The Impact From LHC And Evolution Of TGD Itself

The missing energy predicted standard SUSY seems to be absent at LHC. The easy explanation would be that the mass scale of SUSY is unexpectedly high, of order 1-10 TeV. This would however destroy the original motivations for SUSY. The arguments developed in the following manner.

1. One must distinguish between embedding space spinor harmonics and the modes of the induced spinor field. Right-handed neutrino with vanishing color quantum numbers and thus covariantly constant in CP_2 is massless. All other modes of the induced spinor field are massive and in according to the p-adic mass calculations negative conformal weight of the ground state and the presence of Kac-Moody and super-symplectic generators make possible massless states having thermal excitations giving to the state a thermal mass. Right-handed neutrino can mix with left-handed neutrino ad can get mass. One can assign to any fermion a super-multiplet with 4 members.

   One cannot assign full super-4-plet also to non-colored right handed neutrino itself: the multiplet would contain only 3 states. The most natural possibility is that the ground state is now a color excitation of right-handed neutrino and massless non-colored right-handed neutrinos give rise to the 4-plet. The colored spinor mode at embedding space level is however a mixture or left- and right handed neutrinos.

2. In TGD framework the natural first guess is that right-handed neutrinos carrying four-momentum can give rise to missing energy. The assumption that fermions correspond to color partial waves in H implies that color excitations of the right handed neutrino that would appear in asymptotic states are necessarily colored. It could happen that these excitations are color neutralized by super-conformal generators. If this is not the case, these neutrinos would be like quarks and color confinement would explain why they cannot be observed as asymptotic states in macroscopic scales.

   Second possibility is that SUSY itself is generated by color partial waves of right-handed neutrino, octet most naturally. This option is however not consistent with the above model for one-fermion states and their super-partners.

16.4.4 Supersymmetry In Crisis

Supersymmetry is very beautiful generalization of the ordinary symmetry concept by generalizing Lie-algebra by allowing grading such that ordinary Lie algebra generators are accompanied by super-generators transforming in some representation of the Lie algebra for which Lie-algebra commutators are replaced with anti-commutators. In the case of Poincare group the super-generators would transform like spinors. Clifford algebras are actually super-algebras. Gamma matrices anti-commute to metric tensor and transform like vectors under the vielbein group (SO(n) in Euclidian signature). In supersymmetric gauge theories one introduced super translations anti-commuting to ordinary translations.

Supersymmetry algebras defined in this manner are characterized by the number of super-generators and in the simplest situation their number is one: one speaks about $\mathcal{N}=1$ SUSY and minimal super-symmetric extension of standard model (MSSM) in this case. These models are most studied because they are the simplest ones. They have however the strange property that the spinors generating SUSY are Majorana spinors- real in well-defined sense unlike Dirac spinors. This implies that fermion number is conserved only modulo two: this has not been observed experimentally. A second problem is that the proposed mechanisms for the breaking of SUSY do not look feasible.

LHC results suggest MSSM does not become visible at LHC energies. This does not exclude more complex scenarios hiding simplest $\mathcal{N}=1$ to higher energies but the number of real believers is decreasing. Something is definitely wrong and one must be ready to consider more complex options or totally new view abot SUSY.

What is the analog of SUSY in TGD framework? I must admit that I am still fighting to gain understanding of SUSY in TGD framework [K84]. That I can still imagine several scenarios shows that I have not yet completely understood the problem but I am working hardly to avoid falling to the sin of sloppying myself.

At the basic level one has super-conformal invariance generated in the fermion sector by the super-conformal charges assignable to the strings emanating from partonic 2-surfaces and connecting them to each other. For elementary particles one has 2 wormhole contacts and 4 wormhole throats. If the number of strings is just one, one has symplectic super-conformal symmetry, which is already huge. Several strings must be allowed and this leads to the Yangian variant of super-conformal symmetry, which is multi-local (multi-stringy).

One can also say that fermionic oscillator operators generate infinite-D super-algebra. One can restrict the consideration to lowest conformal weights if spinorial super-conformal invariance acts as gauge symmetry so that one obtains a finite-D algebra with generators labelled by electro-weak quantum numbers of quarks and leptons. This super-symmetry is badly broken but contains the algebra generated by right-handed neutrino and its conjugate as sub-algebra.

The basic question is whether covariantly constant right handed neutrino generators $\mathcal{N}=\in$ SUSY or whether the SUSY is generated as approximate symmetry by adding massless right-handed neutrino to the state thus changing its four-momentum. The problem with the first option is that it the standard norm of the state is naturally proportional to four-momentum and vanishes at the limit of vanishing four-momentum: is it possible to circumvent this problem somehow? In the following I summarize the situation as it seems just now.

1. In TGD framework $\mathcal{N}=1$ SUSY is excluded since B and L and conserved separately and embedding space spinors are not Majorana spinors. The possible analog of space-time SUSY should be a remnant of a much larger super-conformal symmetry in which the Clifford algebra generated by fermionic oscillator operators giving also rise to the Clifford algebra generated by the gamma matrices of the "world of classical worlds" (WCW) and assignable with string world sheets. This algebra is indeed part of infinite-D super-conformal algebra behind quantum TGD. One can construct explicitly the conserved super conformal charges accompanying ordinary charges and one obtains something analogous to $\mathcal{N}=\infty$ super algebra. This SUSY is however badly broken by electroweak interactions.

2. The localization of induced spinors to string world sheets emerges from the condition that electromagnetic charge is well-defined for the modes of induced spinor fields. There is however an exception: covariantly constant right handed neutrino spinor ν_R: it can be de-localized along entire space-time surface. Right-handed neutrino has no couplings to electroweak fields. It couples however to left handed neutrino by induced gamma matrices except when it is covariantly constant. Note that standard model does not predict ν_R but its existence is necessary if neutrinos develop Dirac mass. ν_R is indeed something which must be considered carefully in any generalization of standard model.

Could covariantly constant right handed neutrinos generate SUSY?

Could covariantly constant right-handed spinors generate exact $\mathcal{N} = 2$ SUSY? There are two spin directions for them meaning the analog $\mathcal{N} = 2$ Poincare SUSY. Could these spin directions correspond to right-handed neutrino and antineutrino. This SUSY would not look like Poincare SUSY for which anti-commutator of super generators would be proportional to four-momentum. The problem is that four-momentum vanishes for covariantly constant spinors! Does this mean that the sparticles generated by covariantly constant ν_R are zero norm states and represent super gauge degrees of freedom? This might well be the case although I have considered also alternative scenarios.

What about non-covariantly constant right-handed neutrinos?

Both embedding space spinor harmonics and the Kähler-Dirac equation have also right-handed neutrino spinor modes not constant in M^4 and localized to the partonic orbits. If these are responsible for SUSY then SUSY is broken.

1. Consider first the situation at space-time level. Both induced gamma matrices and their generalizations to Kähler-Dirac gamma matrices defined as contractions of embedding space gamma matrices with the canonical momentum currents for Kähler action are superpositions of M^4 and CP_2 parts. This gives rise to the mixing of right-handed and left-handed neutrinos. Note that non-covariantly constant right-handed neutrinos must be localized at string world sheets.

   This in turn leads neutrino massivation and SUSY breaking. Given particle would be accompanied by sparticles containing varying number of right-handed neutrinos and antineutrinos localized at partonic 2-surfaces.

2. One an consider also the SUSY breaking at embedding space level. The ground states of the representations of extended conformal algebras are constructed in terms of spinor harmonics of the embedding space and form the addition of right-handed neutrino with non-vanishing four-momentum would make sense. But the non-vanishing four-momentum means that the members of the super-multiplet cannot have same masses. This is one manner to state what SUSY breaking is.

What one can say about the masses of sparticles?

The simplest form of massivation would be that all members of the super-multiplet obey the same mass formula but that the p-adic length scales associated with them are different. This could allow very heavy sparticles. What fixes the p-adic mass scales of sparticles? If this scale is CP_2 mass scale SUSY would be experimentally unreachable. The estimate below does not support this option.

One can consider the possibility that SUSY breaking makes sparticles unstable against phase transition to their dark variants with $h_{eff} = n \times h$. Sparticles could have same mass but be non-observable as dark matter not appearing in same vertices as ordinary matter! Geometrically the addition of right-handed neutrino to the state would induce many-sheeted covering in this case with right handed neutrino perhaps associated with different space-time sheet of the covering.

This idea need not be so outlandish at it looks first.

1. The generation of many-sheeted covering has interpretation in terms of breaking of conformal invariance. The sub-algebra for which conformal weights are n-tuples of integers becomes the algebra of conformal transformations and the remaining conformal generators do note represent gauge degrees of freedom anymore. They could however represent conserved conformal charges still.

2. This generalization of conformal symmetry breaking gives rise to infinite number of fractal hierarchies formed by sub-algebras of conformal algebra and is also something new and a fruit of an attempt to avoid sloppy thinking. The breaking of conformal symmetry is indeed expected in massivation related to the SUSY breaking.

The following poor man's estimate supports the idea about dark sfermions and the view that sfermions cannot be very heavy.

1. Neutrino mixing rate should correspond to the mass scale of neutrinos known to be in eV range for ordinary value of Planck constant. For $h_{eff}/h = n$ it is reduced by factor $1/n$, when mass kept constant. Hence sfermions could be stabilized by making them dark.

2. A very rough order of magnitude estimate for sfermion mass scale is obtained from Uncertainty Principle: particle mass should be higher than its decay rate. Therefore an estimate for the decay rate of sfermion could give a lower bound for its mass scale.

3. Assume the transformation $\nu_R \rightarrow \nu_L$ makes sfermion unstable against the decay to fermion and ordinary neutrino. If so, the decay rate would be dictated by the mixing rate and therefore to neutrino mass scale for the ordinary value of Planck constant. Particles and sparticles would have the same p-adic mass scale. Large h_{eff} could however make sfermion dark, stable, and non-observable.

A rough model for the neutrino mixing in TGD framework

The mixing of neutrinos would be the basic mechanism in the decays of sfermions. The following argument tries to capture what is essential in this process.

1. Conformal invariance requires that the string ends at which fermions are localized at wormhole throats are light-like curves. In fact, light-likeness gives rise to Virasosoro conditions.

2. Mixing is described by a vertex residing at partonic surface at which two partonic orbits join. Localization of fermions to string boundaries reduces the problem to a problem completely analogous to the coupling of point particle coupled to external gauge field. What is new that orbit of the particle has edge at partonic 2-surface. Edge breaks conformal invariance since one cannot say that curve is light-like at the edge. At edge neutrino transforms from right-handed to left handed one.

3. In complete analogy with $\overline{\Psi}\gamma^t A_t \Psi$ vertex for the point-like particle with spin in external field, the amplitude describing $nu_R - \nu_L$ transition involves matrix elements of form $\overline{\nu}_R \Gamma^t(CP_2) Z_t \nu_L$ at the vertex of the CP_2 part of the Kähler-Dirac gamma matrix and classical Z^0 field.

   How Γ^t is identified? The Kähler-Dirac gamma matrices associated with the interior need not be well-defined at the light-like surface and light-like curve. One basis of weak form of electric magnetic duality the Kähler-Dirac gamma matrix corresponds to the canonical momentum density associated with the Chern-Simons term for Kähler action. This gamma matrix contains only the CP_2 part.

The following provides as more detailed view.

1. Let us denote by $\Gamma^t_{CP_2}(in/out)$ the CP_2 part of the Kähler-Dirac gamma matrix at string at at partonic 2-surface and by Z^0_t the value of Z^0 gauge potential along boundary of string world sheet. The direction of string line in embedding space changes at the partonic 2-surface. The question is what happens to the Kähler-Dirac action at the vertex.

2. For incoming and outgoing lines the equation

$$D(in/out)\Psi(in/out) = p^k(in, out)\gamma_k \Psi(in/out) \ ,$$

   where the Kähler-Dirac operator is $D(in/out) = \Gamma^t(in/out)D_t$, is assumed. ν_R corresponds to "in" and ν_R to "out". It implies that lines corresponds to massless M^4 Dirac propagator and one obtains something resembling ordinary perturbation theory.

   It also implies that the residue integration over fermionic internal momenta gives as a residue massless fermion lines with non-physical helicities as one can expect in twistor approach. For physical particles the four-momenta are massless but in complex sense and the imaginary part comes classical from four-momenta assignable to the lines of generalized Feynman diagram possessing Euclidian signature of induced metric so that the square root of the metric determinant differs by imaginary unit from that in Minkowskian regions.

3. In the vertex $D(in/out)$ could act in $\Psi(out/in)$ and the natural idea is that $\nu_R - \nu_L$ mixing is due to this so that it would be described the classical weak current couplings $\overline{\nu}_R \Gamma^t_{CP_2}(out) Z^0_t(in) \nu_L$ and $\overline{\nu}_R \Gamma^t_{CP_2}(out) Z^0_t(in) \nu_L$.

To get some idea about orders of magnitude assume that the CP_2 projection of string boundary is geodesic circle thus describable as $\Phi = \omega t$, where Φ is angle coordinate for the circle and t is Minkowski time coordinate. The contribution of CP_2 to the induced metric g_{tt} is $\Delta g_{tt} = -R^2\omega^2$.

1. In the first approximation string end is a light-like curve in Minkowski space meaning that CP_2 contribution to the induced metric vanishes. Neutrino mixing vanishes at this limit.

2. For a non-vanishing value of ωR the mixing and the order of magnitude for mixing rate and neutrino mass is expected to be $R \sim \omega$ and $m \sim \omega/h$. p-Adic length scale hypothesis and the experimental value of neutrino mass allows to estimate m to correspond to p-adic mass to be of order eV so that the corresponding p-adic prime p could be $p \simeq 2^{167}$. Note that $k = 127$ defines largest of the four Gaussian Mersennes $M_{G,k} = (1+i)^k - 1$ appearing in the length scale range 10 nm -2.5 μm. Hence the decay rate for ordinary Planck constant would be of order $R \sim 10^{14}$/s but large value of Planck constant could reduced it dramatically. In living matter reductions by a factor 10^{-12} can be considered.

To sum up, the space-time SUSY in TGD sense would differ crucially from SUSY in the standard sense. There would no Majorana spinors and sparticles could correspond to dark phase of matter with non-standard value of Planck constant. The signatures of the standard SUSY do not apply to TGD. Of course, a lot of professional work would be needed to derive the signatures of TGD SUSY.

16.4.5 Right-Handed Neutrino As Inert Neutrino?

There is a very interesting posting by Jester in Resonaances with title "How many neutrinos in the sky?" (see `http://tinyurl.com/y8scxzqr`) [C1]. Jester tells about the recent 9 years WMAP data [C3] and compares it with earlier 7 years data. In the earlier data the effective number of neutrino types was $N_{eff} = 4.34 \pm 0.87$ and in the recent data it is $N_{eff} = 3.26 \pm 0.35$. WMAP alone would give $N_{eff} = 3.89 \pm 0.67$ also in the recent data but also other data are used to pose constraints on N_{eff}.

To be precise, N_{eff} could include instead of fourth neutrino species also some other weakly interacting particle. The only criterion for contributing to N_{eff} is that the particle is in thermal equilibrium with other massless particles and thus contributes to the density of matter considerably during the radiation dominated epoch.

Jester also refers to the constraints on N_{eff} from nucleosynthesis (see `http://tinyurl.com/y8fkfn5y`) , which show that $N_{eff} \sim 4$ us slightly favored although the entire range $[3, 5]$ is consistent with data.

It seems that the effective number of neutrinos could be 4 instead of 3 although latest WMAP data combined with some other measurements favor 3. Later a corrected version e `http://tinyurl.com/y9er8szf`) of the eprint appeared [C3] telling that the original estimate of N_{eff} contained a mistake and the correct estimate is $N_{eff} = 3.84 \pm 0.40$.

An interesting question is what $N_{eff} = 4$ could mean in TGD framework?

1. One poses to the modes of the Kähler-Dirac equation the following condition: electric charge is conserved in the sense that the time evolution by Kähler-Dirac equation does not mix a mode with a well-defined em charge with those with different em charge. The implication is that all modes except pure right handed neutrino are restricted at string world sheets. The first guess is that string world sheets are minimal surfaces of space-time surface (rather than those of embedding space). One can also consider minimal surfaces of embedding space but

with effective metric defined by the anti-commutators of the Kähler-Dirac gamma matrices. This would give a direct physical meaning for this somewhat mysterious effective metric.

For the neutrino modes localized at string world sheets mixing of left and right handed modes takes place and they become massive. If only 3 lowest genera for partonic 2-surfaces are light, one has 3 neutrinos of this kind. The same applies to all other fermion species. The argument for why this could be the case relies on simple observation [K21]: the genera g=0, 1, 2 have the property that they allow for all values of conformal moduli Z_2 as a conformal symmetry (hyper-ellipticity). For $g > 2$ this is not the case. The guess is that this additional conformal symmetry is the reason for lightness of the three lowest genera.

2. Only purely right-handed neutrino is completely de-localized in 4-volume so that one cannot assign to it genus of the partonic 2-surfaces as a topological quantum number and it effectively gives rise to a fourth neutrino very much analogous to what is called sterile neutrino. De-localized right-handed neutrinos couple only to gravitation and in case of massless extremals this forces them to have four-momentum parallel to that of ME: only massless modes are possible. Very probably this holds true for all preferred extremals to which one can assign massless longitudinal momentum direction which can vary with spatial position.

3. The coupling of ν_R is to gravitation alone and all electroweak and color couplings are absent. According to standard wisdom de-localized right-handed neutrinos cannot be in thermal equilibrium with other particles. This according to standard wisdom. But what about TGD?

   One should be very careful here: de-localized right-handed neutrinos is proposed to give rise to SUSY (not $\mathcal{N} = 1$ requiring Majorana fermions) and their dynamics is that of passive spectator who follows the leader. The simplest guess is that the dynamics of right handed neutrinos at the level of amplitudes is completely trivial and thus trivially supersymmetric. There are however correlations between four-momenta.

   (a) The four-momentum of ν_R is parallel to the light-like momentum direction assignable to the massless extremal (or more general preferred extremal). This direct coupling to the geometry is a special feature of the Kähler-Dirac operator and thus of sub-manifold gravity.

   (b) On the other hand, the sum of massless four-momenta of two parallel pieces of preferred extremals is the - in general massive - four-momentum of the elementary particle defined by the wormhole contact structure connecting the space-time sheets (which are glued along their boundaries together since this is seems to be the only manner to get rid of boundary conditions requiring vacuum extremal property near the boundary). Could this direct coupling of the four-momentum direction of right-handed neutrino to geometry and four-momentum directions of other fermions be enough for the right handed neutrinos to be counted as a fourth neutrino species in thermal equilibrium? This might be the case!

   One cannot of course exclude the coupling of 2-D neutrino at string world sheets to 4-D purely right handed neutrinos analogous to the coupling inducing a mixing of sterile neutrino with ordinary neutrinos. Also this could help to achieve the thermal equilibrium with 2-D neutrino species.

16.4.6 Experimental Evidence For Sterile Neutrino?

Many physicists are somewhat disappointed to the results from LHC: the expected discovery of Higgs has been seen as the main achievement of LHC hitherto. Much more was expected. To my opinion there is no reason for disappointment. The exclusion of the standard SUSY at expected energy scale is very far reaching negative result. Also the fact that Higgs mass is too small to be stable without fine tuning is of great theoretical importance. The negative results concerning heavy dark matter candidates are precious guidelines for theoreticians. The non-QCD like behavior in heavy ion collisions and proton-ion collisions is bypassed my mentioning something about

AdS/CFT correspondence and non-perturbative QCD effects. I tend to see these effects as direct evidence for M_{89} hadron physics [K58].

In any case, something interesting has emerged quite recently. Resonaances tells that the recent analysis (see `http://tinyurl.com/ycf4vbkq`) [C2] of X-ray spectrum of galactic clusters claims the presence of monochromatic 3.5 keV photon line. The proposed interpretation is as a decay product of sterile 7 keV neutrino transforming first to a left-handed neutrino and then decaying to photon and neutrino via a loop involving W boson and electron. This is of course only one of the many interpretations. Even the existence of line is highly questionable.

One of the poorly understood aspects of TGD is right-handed neutrino, which is obviously the TGD counterpart of the inert neutrino.

1. The old idea is that covariantly constant right handed neutrino could generate $\mathcal{N}=2$ super-symmetry in TGD Universe. In fact, all modes of induced spinor field would generate superconformal symmetries but electroweak interactions would break these symmetries for the modes carrying non-vanishing electroweak quantum numbers: they vanish for ν_R. This picture is now well-established at the level of WCW geometry [K80]: super-conformal generators are labelled angular momentum and color representations plus two conformal weights: the conformal weight assignable to the light-like radial coordinate of light-cone boundary and the conformal weight assignable to string coordinate. It seems that these conformal weights are independent. The third integer labelling the states would label genuinely Yangian generators: it would tell the poly-locality of the generator with locus defined by partonic 2-surface: generators acting on single partonic 2-surface, 2 partonic 2-surfaces, ...

2. It would seem that even the SUSY generated by ν_R must be badly broken unless one is able to invent dramatically different interpretation of SUSY. The scale of SUSY breaking and thus the value of the mass of right-handed neutrino remains open also in TGD. In lack of better one could of course argue that the mass scale must be CP_2 mass scale because right-handed neutrino mixes considerably with the left-handed neutrino (and thus becomes massive) only in this scale. But why this argument does not apply also to left handed neutrino which must also mix with the right-handed one!

3. One can of course criticize the proposed notion of SUSY: wonder whether fermion + extremely weakly interacting ν_R at same wormhole throat (or interior of 3-surface) can behave as single coherent entity as far spin is considered [K84] ?

4. The condition that the modes of induced spinor field have a well-defined electromagnetic charge eigenvalue [K106] requires that they are localized at 2-D string world sheets or partonic 2-surfaces: without this condition classical W boson fields would mix the em charged and neutral modes with each other. Right-handed neutrino is an exception since it has no electroweak couplings. Unless right-handed neutrino is covariantly constant, the Kähler-Dirac gamma matrices can however mix the right-handed neutrino with the left handed one and this can induce transformation to charged mode. This does not happen if each Kähler-Dirac gamma matrix can be written as a linear combination of either M^4 or CP_2 gamma matrices and Kähler-Dirac equation is satisfied separately by M^4 and CP_2 parts of the Kähler-Dirac equation.

5. Is the localization of the modes other than covariantly constant neutrino to string world sheets a consequence of dynamics or should one assume this as a separate condition? If one wants similar localization in space-time regions of Euclidian signature - for which CP_2 type vacuum extremal is a good representative - one must assume it as a separate condition. In number theoretic formulation string world sheets/partonic 2-surfaces would be commutative/co-commutative sub-manifolds of space-time surfaces which in turn would be associative or co-associative sub-manifolds of embedding space possessing (hyper-)octonionic tangent space structure. For this option also right-handed neutrino would be localized to string world sheets. Right-handed neutrino would be covariantly constant only in 2-D sense.

   One can consider the possibility that ν_R is de-localized to the entire 4-D space-time sheet. This would certainly modify the interpretation of SUSY since the number of degrees of freedom would be reduced for ν_R.

6. Non-covariantly constant right-handed neutrinos could mix with left-handed neutrinos but not with charged leptons if the localization to string world sheets is assumed for modes carrying non-vanishing electroweak quantum numbers. This would make possible the decay of right-handed to neutrino plus photon, and one cannot exclude the possibility that ν_R has mass 7 keV.

   Could this imply that particles and their spartners differ by this mass only? Could it be possible that practically unbroken SUSY could be there and we would not have observed it? Could one imagine that sfermions have annihilated leaving only states consisting of fundamental fermions? But shouldn't the total rate for the annihilation of photons to hadrons be two times the observed one? This option does not sound plausible.

   What if one assumes that given sparticle is charactrized by the same p-adic prime as corresponding particle but is dark in the sense that it corresponds to non-standard value of Planck constant. In this case sfermions would not appear in the same vertex with fermions and one could escape the most obvious contradictions with experimental facts. This leads to the notion of shadron: shadrons would be [K84] obtained by replacing quarks with dark squarks with nearly identical masses. I have asked whether so called X and Y bosons having no natural place in standard model of hadron could be this kind of creatures.

The interpretation of 3.5 keV photons as decay products of right-handed neutrinos is of course totally ad hoc. Another TGD inspired interpretation would be as photons resulting from the decays of excited nuclei to their ground state.

1. Nuclear string model [K60] predicts that nuclei are string like objects formed from nucleons connected by color magnetic flux tubes having quark and antiquark at their ends. These flux tubes are long and define the "magnetic body" of nucleus. Quark and antiquark have opposite em charges for ordinary nuclei. When they have different charges one obtains exotic state: this predicts entire spectrum of exotic nuclei for which statistic is different from what proton and neutron numbers deduced from em charge and atomic weight would suggest. Exotic nuclei and large values of Planck constant could make also possible cold fusion [K32].

2. What the mass difference between these states is, is not of course obvious. There is however an experimental finding [C4] (see *Analysis of Gamma Radiation from a Radon Source: Indications of a Solar Influence* at `http://tinyurl.com/d9ymwm3`) that nuclear decay rates oscillate with a period of year and the rates correlate with the distance from Sun. A possible explanation is that the gamma rays from Sun in few keV range excite the exotic nuclear states with different decay rate so that the average decay rate oscillates [K60]. Note that nuclear excitation energies in keV range would also make possible interaction of nuclei with atoms and molecules.

3. This allows to consider the possibility that the decays of exotic nuclei in galactic clusters generates 3.5 keV photons. The obvious question is why the spectrum would be concentrated at 3.5 keV in this case (second question is whether the energy is really concentrated at 3.5 keV: a lot of theory is involved with the analysis of the experiments). Do the energies of excited states depend on the color bond only so that they would be essentially same for all nuclei? Or does single excitation dominate in the spectrum? Or is this due to the fact that the thermal radiation leaking from the core of stars excites predominantly single state? Could $E = 3.5$ keV correspond to the maximum intensity for thermal radiation in stellar core? If so, the temperature of the exciting radiation would be about $T \simeq E/3 \simeq 1.2 \times 10^7$ K. This in the temperature around which formation of Helium by nuclear fusion has begun: the temperature at solar core is around 1.57×10^7 K.

16.4.7 Delicacies of the induced spinor structure and SUSY mystery

The discussion of induced spinor structure leads to a modification of an earlier idea (one of the many) about how SUSY could be realized in TGD in such a way that experiments at LHC energies could not discover it and one should perform experiments at the other end of energy spectrum at energies which correspond to the thermal energy about .025 eV at room temperature. I have the feeling that this observation could be of crucial importance for understanding of SUSY.

Induced spinor structure

The notion of induced spinor field deserves a more detailed discussion. Consider first induced spinor structures.

1. Induced spinor field are spinors of $M^4 \times CP_2$ for which modes are characterized by chirality (quark or lepton like) and em charge and weak isospin.
2. Induced spinor spinor structure involves the projection of gamma matrices defining induced gamma matrices. This gives rise to superconformal symmetry if the action contains only volume term.

   When Kähler action is present, superconformal symmetry requires that the modified gamma matrices are contractions of canonical momentum currents with embedding space gamma matrices. Modified gammas appear in the modified Dirac equation and action, whose solution at string world sheets trivializes by super-conformal invariance to same procedure as in the case of string models.
3. Induced spinor fields correspond to two chiralities carrying quark number and lepton number. Quark chirality does not carry color as spin-like quantum number but it corresponds to a color partial wave in CP_2 degrees of freedom: color is analogous to angular momentum. This reduces to spinor harmonics of CP_2 describing the ground states of the representations of super-symplectic algebra.

   The harmonics do not satisfy correct correlation between color and electroweak quantum numbers although the triality t=0 for leptonic waves and t=1 for quark waves. There are two ways to solve the problem.

   (a) Super-symplectic generators applied to the ground state to get vanishing ground states weight instead of the tachyonic one carry color and would give for the physical states correct correlation: leptons/quarks correspond to the same triality zero(one partial wave irrespective of charge state. This option is assumed in p-adic mass calculations [K52].

   (b) Since in TGD elementary particles correspond to pairs of wormhole contacts with weak isospin vanishing for the entire pair, one must have pair of left and right-handed neutrinos at the second wormhole throat. It is possible that the anomalous color quantum numbers for the entire state vanish and one obtains the experimental correlation between color and weak quantum numbers. This option is less plausible since the cancellation of anomalous color is not local as assume in p-adic mass calculations.

The understanding of the details of the fermionic and actually also geometric dynamics has taken a long time. Super-conformal symmetry assigning to the geometric action of an object with given dimension an analog of Dirac action allows however to fix the dynamics uniquely and there is indeed dimensional hierarchy resembling brane hierarchy.

1. The basic observation was following. The condition that the spinor modes have well-defined em charge implies that they are localized to 2-D string world sheets with vanishing W boson gauge fields which would mix different charge states. At string boundaries classical induced W boson gauge potentials guarantee this. Super-conformal symmetry requires that this 2-surface gives rise to 2-D action which is area term plus topological term defined by the flux of Kähler form.
2. The most plausible assumption is that induced spinor fields have also interior component but that the contribution from these 2-surfaces gives additional delta function like contribution: this would be analogous to the situation for branes. Fermionic action would be accompanied by an area term by supersymmetry fixing modified Dirac action completely once the bosonic actions for geometric object is known. This is nothing but super-conformal symmetry.

   One would actually have the analog of brane-hierarchy consisting of surfaces with dimension D= 4,3,2,1 carrying induced spinor fields which can be regarded as independent dynamical variables and characterized by geometric action which is D-dimensional analog of the action for Kähler charged point particle. This fermionic hierarchy would accompany the hierarchy of geometric objects with these dimensions and the modified Dirac action would be uniquely determined by the corresponding geometric action principle (Kähler charged point like particle, string world sheet with area term plus Kähler flux, light-like 3-surface with Chern-Simons term, 4-D space-time surface with Kähler action).

3. This hierarchy of dynamics is consistent with SH only if the dynamics for higher dimensional objects is induced from that for lower dimensional objects - string world sheets or maybe even their boundaries orbits of point like fermions. Number theoretic vision [K104] suggests that this induction relies algebraic continuation for preferred extremals. Note that quaternion analyticity [L22] means that quaternion analytic function is determined by its values at 1-D curves.

4. Quantum-classical correspondences (QCI) requires that the classical Noether charges are equal to the eigenvalues of the fermionic charges for surfaces of dimension $D = 0, 1, 2, 3$ at the ends of the CDs. These charges would not be separately conserved. Charges could flow between objects of dimension $D + 1$ and D - from interior to boundary and vice versa. Four-momenta and also other charges would be complex as in twistor approach: could complex values relate somehow to the finite life-time of the state?

   If quantum theory is square root of thermodynamics as zero energy ontology suggests, the idea that particle state would carry information also about its life-time or the time scale of CD to which is associated could make sense. For complex values of α_K there would be also flow of canonical and super-canonical momentum currents between Euclidian and Minkowskian regions crucial for understand gravitational interaction as momentum exchange at embedding space level.

5. What could be the physical interpretation of the bosonic and fermionic charges associated with objects of given dimension? Condensed matter physicists assign routinely physical states to objects of various dimensions: is this assignment much more than a practical approximation or could condensed matter physics already be probing many-sheeted physics?

SUSY and TGD

From this one ends up to the possibility of identifying the counterpart of SUSY in TGD framework [K84, **?**].

1. In TGD the generalization of much larger super-conformal symmetry emerges from the super-symplectic symmetries of WCW. The mathematically questionable notion of super-space is not needed: only the realization of super-algebra in terms of WCW gamma matrices defining super-symplectic generators is necessary to construct quantum states. As a matter of fact, also in QFT approach one could use only the Clifford algebra structure for super-multiplets. No Majorana condition on fermions is needed as for $\mathcal{N} = 1$ space-time SUSY and one avoids problems with fermion number non-conservation.

2. In TGD the construction of sparticles means quite concretely adding fermions to the state. In QFT it corresponds to transformation of states of integer and half-odd integer spin to each other. This difference comes from the fact that in TGD particles are replaced with point like particles.

3. The analog of $\mathcal{N} = 2$ space-time SUSY could be generated by covariantly constant right handed neutrino and antineutrino. Quite generally the mixing of fermionic chiralities implied by the mixing of M^4 and CP_2 gamma matrices implies SUSY breaking at the level of particle masses (particles are massless in 8-D sense). This breaking is purely geometrical unlike the analog of Higgs mechanism proposed in standard SUSY.

There are several options to consider.

1. The analog of brane hierarchy is realized also in TGD. Geometric action has parts assignable to 4-surface, 3-D light like regions between Minkowskian and Euclidian regions, 2-D string world sheets, and their 1-D boundaries. They are fixed uniquely. Also their fermionic counterparts - analogs of Dirac action - are fixed by super-conformal symmetry. Elementary particles reduce so composites consisting of point-like fermions at boundaries of wormhole throats of a pair of wormhole contacts.

   This forces to consider 3 kinds of SUSYs! The SUSYs associated with string world sheets and space-time interiors would certainly be broken since there is a mixing between M^4 chiralities in the modified Dirac action. The mass scale of the broken SUSY would correspond to the length scale of these geometric objects and one might argue that the decoupling between the degrees of freedom considered occurs at high energies and explains why no evidence for SUSY has been observed at LHC. Also the fact that the addition of massive fermions at these dimensions can be interpreted differently. 3-D light-like 3-surfaces could be however an exception.

2. For 3-D light-like surfaces the modified Dirac action associated with the Chern-Simons term does not mix M^4 chiralities (signature of massivation) at all since modified gamma matrices have only CP_2 part in this case. All fermions can have well-defined chirality. Even more: the modified gamma matrices have no M^4 part in this case so that these modes carry no four-momentum - only electroweak quantum numbers and spin. Obviously, the excitation of these fermionic modes would be an ideal manner to create spartners of ordinary particles consting of fermion at the fermion lines. SUSY would be present if the spin of these excitations couples - to various interactions and would be exact.

   What would be these excitations? Chern-Simons action and its fermionic counterpart are non-vanishing only if the CP_2 projection is 3-D so that one can use CP_2 coordinates. This strongly suggests that the modified Dirac equation demands that the spinor modes are covariantly constant and correspond to covariantly constant right-handed neutrino providing only spin.

   If the spin of the right-handed neutrino adds to the spin of the particle and the net spin couples to dynamics, $\mathcal{N} = 2$ SUSY is in question. One would have just action with unbroken SUSY at QFT limit? But why also right-handed neutrino spin would couple to dynamics if only CP_2 gamma matrices appear in Chern-Simons-Dirac action? It would seem that it is independent degree of freedom having no electroweak and color nor even gravitational couplings by its covariant constancy. I have ended up with just the same SUSY-or-no-SUSY that I have had earlier.

3. Can the geometric action for light-like 3-surfaces contain Chern-Simons term?

   (a) Since the volume term vanishes identically in this case, one could indeed argue that also the counterpart of Kähler action is excluded. Moreover, for so called massless extremals of Kähler action reduces to Chern-Simons terms in Minkowskian regions and this could happen quite generally: TGD with only Kähler action would be almost topological QFT as I have proposed. Volume term however changes the situation via the cosmological constant. Kähler-Dirac action in the interior does not reduce to its Chern-Simons analog at light-like 3-surface.

   (b) The problem is that the Chern-Simons term at the two sides of the light-like 3-surface differs by factor $\sqrt{-1}$ coming from the ratio of $\sqrt{g_4}$ factors which themselves approach to zero: oOne would have the analog of dipole layer. This strongly suggests that one should not include Chern-Simons term at all.

   Suppose however that Chern-Simons terms are present at the two sides and α_K is real so that nothing goes through the horizon forming the analog of dipole layer. Both bosonic and fermionic degrees of freedom for Euclidian and Minkowskian regions would decouple completely but currents would flow to the analog of dipole layer. This is not physically attractive.

   The canonical momentum current and its super counterpart would give fermionic source term $\Gamma^n \Psi_{int,\pm}$ in the modified Dirac equation defined by Chern-Simons term at given side ±: ± refers to Minkowskian/Euclidian part of the interior. The source term is proportional to $\Gamma^n \Psi_{int,\pm}$ and Γ^n is in principle mixture of M^4 and CP_2 gamma matrices and therefore induces mixing of M^4 chiralities and therefore also 3-D SUSY breaking. It must be however emphasized that Γ^n is singular and one must be consider the limit carefully also in the case that one has only continuity conditions. The limit is not completely understood.

(c) If α_K is complex there is coupling between the two regions and the simplest assumption has been that there is no Chern-Simons term as action and one has just continuity conditions for canonical momentum current and hits super counterpart.

The cautious conclusion is that 3-D Chern-Simons term and its fermionic counterpart are absent.

4. What about the addition of fermions at string world sheets and interior of space-time surface ($D = 2$ and $D = 4$). For instance, in the case of hadrons $D = 2$ excitations could correspond to addition of quark in the interior of hadronic string implying additional states besides the states obtained assuming only quarks at string ends. Let us consider the interior ($D = 4$). For instance, inn the case of hadrons $D = 2$ excitations could correspond to addition of quark in the interior of hadronic string implying additional states besides the states obtained assuming only quarks at string ends. The smallness of cosmological constant implies that the contribution to the four-momentum from interior should be rather small so that an interpretation in terms of broken SUSY might make sense. There would be mass $m \sim .03$ eV per volume with size defined by the Compton scale $\hbar/m$. Note however that cosmological constant has spectrum coming as inverse powers of prime so that also higher mass scales are possible.

   This interpretation might allow to understand the failure to find SUSY at LHC. Sparticles could be obtained by adding interior right-handed neutrinos and antineutrinos to the particle state. They could be also associated with the magnetic body of the particle. Since they do not have color and weak interactions, SUSY is not badly broken. If the mass difference between particle and sparticle is of order $m = .03$ eV characterizing dark energy density ρ_{vac}, particle and sparticle could not be distinguished in higher energy physics at LHC since it probes much shorter scales and sees only the particle. I have already earlier proposed a variant of this mechanism but without SUSY breaking.

   To discover SUSY one should do very low energy physics in the energy range $m \sim .03$ eV having same order of magnitude as thermal energy $kT = 2.6 \times 10^{-2}$ eV at room temperature 25 °C. One should be able to demonstrate experimentally the existence of sparticle with mass differing by about $m \sim .03$ eV from the mass of the particle (one cannot exclude higher mass scales since Λ is expected to have spectrum). An interesting question is whether the sfermions associated with standard fermions could give rise to Bose-Einstein condensates whose existence in the length scale of large neutron is strongly suggested by TGD view about living matter.

16.4.8 Conclusions

The conclusion that the standard SUSY ($\mathcal{N} = 1$ SUSY with Majorana spinors) is absent in TGD Universe and also in the real one looks rather feasible in light of various arguments discussed in this chapter and also conforms with the LHC data. A more general SUSY with baryon and lepton conservation and Dirac spinors is however possible in TGD framework.

During the attempts to understand SUSY several ideas have emerged and the original discussions are retained as such in this chapter. It is interesting to see that their fate is if standard SUSY has no TGD counterpart.

1. One of the craziest ideas was that spartners indeed exists and even with the same p-adic mass scale but might be realized as dark matter. Same mass scale is indeed a natural prediction if right-handed neutrino and particle have same mass scale. Therefore even the mesons of ordinary hadron physics would be accompanied by smesons - pairs of squark and anti-squark.

In fact, this is what the most recent form of the theory predicts: unfortunately there is no manner to experimentally distinguish between fermion and pseudo-sfermion if ν_R is zero momentum state lacking even gravitational interactions.

2. There are indications that charmonium as exotic states christened as X and Y mesons and the question was that they could correspond to mesons built either from colored excitations of charged quark and antiquark or from squark and anti-squark. The recent view leaves only the option based on colored excitations alive. The states in question would be analogous to pairs of color excitations of leptons introduced to explain various anomalies in leptonic sector [K97]. The question was whether lepto-hadrons could correspond to bound states of colored sleptons and have same p-adic mass scale as leptons have [K97]. The original form of lepto-hadron hypothesis remains intact.

3. Evidence that pion and also other hadrons have what could be called infrared Regge trajectories has been reported, and one could ask whether these trajectories could include spion identified as a bound state of squarks. Also this identification is excluded and the proposed identification in terms of stringy states assignable to long color magnetic flux tubes accompanying hadron remains under consideration. IR Regge trajectories would serve as a signature for the non-perturbative aspects of hadron physics.

4. The latest idea along these lines is that spartners are obtained by adding right-handed neutrinos to the interior of space-time surface assignable to the particle. SUSY would not be detectable at high energies, which would explain the negative findings at LHC. Spartners could be discovered at low energy physics perhaps assignable to the magnetic bodies of particles: the mass scale could be as low .03 eV determined by cosmological constant in the scale of cosmology. Note however that cosmological constant has spectrum coming as inverse powers of prime.

16.5 SUSY Algebra At QFT Limit

The first expectation is that QFT limit TGD corresponds to a situation in which given space-time surface is representable as a graph for some map $M^4 \rightarrow CP_2$. This assumption is essential for the understanding of how the QFT limit of TGD emerges when many-sheeted space-time is replaced with a piece of Minkowski space in macroscopic scales and how gauge potentials of standard model relate to the induced gauge potentials. Already at elementary particle scales this assumption fails if they are regarded as pairs of wormhole contacts at distance characterized by Compton length: two sheetedness is involved in an essential manner.

This assumption is not actually needed in zero energy ontology if M^4 is assumed to label the positions of either tip of CD rather than points of the space-time sheet. The position of the other tip of CD relative to the first one could be interpreted in terms of Robertson-Walker coordinates for quantum cosmology [K86].

An intuitively plausible idea is that particle space-time sheets with Euclidian signature of the induced metric are replaced with world-lines. Fermions can be said to propagate along the boundaries of string world sheets so that this approximation would force all fermion lines of the parton orbit to form single line. Intuitively this might correspond to the replacement of multi-stringy Yangian [L10] with a super-field.

Strings bring in bi-locality at fundamental level and the hierarchy of Planck constants implies this non-locality in arbitrarily long length scales. The formation of gravitational bound states would involve gigantic values of Planck constant $h_{eff} = n \times h$ and macroscopic quantum coherence in astrophysical scales [K35, **?**, K85]. This requires a generalization of quantum theory itself and of course challenges the idea that SUSY limit of TGD could make sense except in special situations.

What is essential for QFT limit is that only perturbations around single maximum of Kähler function are considered. If several maxima are important, one must include a weighting defined by the values of the exponent of Kähler function. The huge symmetries of WCW geometry are expected to make the functional integral over perturbations calculable.

16.5.1 Minimum Information About Space-Time Sheet And Particle Quantum Numbers Needed To Formulate SUSY Algebra

The basic problem is how to feed just the essential information about quantum states and space-time surfaces to the definition of the QFT limit.

1. The information about quantum numbers of particles must be fed also to the QFT approximation. It is natural to start from the classical description of point like fermions in H in terms of light-like geodesics of H at the light-like parton orbits carrying light-like 8-momentum: action principle indeed leads to this picture. Momentum and color charges serve as natural quantum numbers besides electroweak quantum numbers. The conserved color charges associated with CP_2 geodesics need not correspond to the usual color charges since they correspond to center of mass rotational motion in CP_2 degrees of freedom. Ordinary color charges correspond to the spinorial partial wavs assignable to CP_2 type extremals.

   The propagators of fundamental fermions massless in 8-D sense are the basic building bricks of the scattering amplitudes in the fundamental formulation of TGD. Elementary particles emerge as bound states of fundamental fermions, and one might of hope that the scattering amplitudes might allow also at the QFT limit a formulation involving only fundamental fermions. The basic vertices would correspond to product and co-product for super-symplectic Yangian and these 3-vertices should correspond to gauge theory vertices. The basic building brick of gauge boson would be wormhole contact with throats carrying fermion and antifermion. It might be that the QFT limit requires the introduction of boson fields. Both fermions and bosons consist of at least two wormhole contacts.

2. Should one interpret QFT limit as a QFT in X^4 representable as a graph for a map $M^4 \rightarrow CP_2$, or in M^4, or perhaps in $M^4 \times CP_2$? In zero energy ontology the proper interpretation is in terms of QFT in M^4 defining the coordinates of the M^4 projection of space-time point. Minimal Kaluza-Klein type extension to $M^4 \times S^1$ might be required in order to take into account the geodesic motion of fundamental fermions in CP_2 degrees of freedom.

3. What information about space-time surface is needed?

   (a) One can in principle feed all information about space-time sheet without losing Poincare invariance since momentum operators do not act on space-time coordinates. The description becomes however in-practical even if one restricts the consideration to the maxima of Kähler function.

   (b) Partonic two-surfaces X^2 are identified as intersections of 3-D light-like wormhole throats with the boundary of CD characterizes basic building bricks of elementary particles and elementary particle itself corresponds to space-like 3-surface at the boundary of CD. The minimal approach would use only cm degrees of freedom for the 3-surface characterizing the particle. A better accuracy would be obtained by using cm coordinates for the partonic 2-surfaces. Even better approximation would be obtained by using the positions fermions associated with given partonic 2-surface.

   (c) The ends of fermion lines defined by the boundaries of string world sheets represent necessary information but correspond to single point of M^4 in QFT approximation. The conformal moduli of the partonic 2-surface are very relevant and the elementary particle vacuum functional in the moduli space [K21] depending on the genus of the partonic 2-surface codes for a relevant information. This information could be compressed to genus its genus characterizing fermion generations plus a rule stating that the particles in the same 3-vertex have same genus and that bosons are superpositions over different genera. Only the three lowest genera have been observed and this can be understood in terms of hyper-ellipticity [K21].

   (d) Some information about zero modes characterized by the induced Kähler form invariant under quantum fluctuations assignable to Hamiltonians of $\delta M^4_\pm \times CP_2$ at boundaries of CD is certainly needed: here the identification of Kähler potential as the Kähler function of WCW is highly attractive hypothesis.

16.5.2 The Physical Picture Behind The Realization Of SUSY Algebra At Point Like Limit

The challenge is to deduce SUSY algebra in the approximation that particle like 3-surfaces are replaced by points. The basic physical constraint on the realization of the SUSY algebra come from the condition that one must be able to describe also massive particles as members of SUSY multiplets. This should make possible also 8-D counterpart of twistorialization in terms of octonionic gamma matrices reducing to quaternionic ones using representation of octonion units in terms of the structure constants of the octonionic algebra. The general structure of Kähler-Dirac action suggests how to proceed. $p^k\gamma_k$ should be replaced with a simplified version of its 8-D variant in $M^4\times CP_2$ and the CP_2 part of this operator should describe the massivation.

1. Fermion lines correspond to light-like geodesics of embedding space. For particles which are massless in M^4, the geodesic circle defining CP_2 projection must contract to a point.

2. The generalization of the Dirac operator appearing in commutation relations reads as

$$p^k\gamma_k \rightarrow D = p^k\gamma_k + Q\gamma_k\frac{ds^k}{ds} ,$$
$$s_{kl}\frac{ds^k}{dt}\frac{ds^l}{dt} = 1 . \qquad (16.5.1)$$

Mass shell condition fixes the value of Q

$$Q = \pm m . \qquad (16.5.2)$$

For geodesic circle the angle coordinate to be angle parameterizing the geodesic circle is the natural variable and the gamma matrices can be taken to be just single constant gamma matrix along the geodesic circle.

3. Embedding space spinors have anomalous color charge equal to -1 unit for lepton and 1/3 units for quarks. Mass shell condition is satisfied if Q is proportional to anomalous hypercharge and mass of the particle in turn determined by p-adic thermodynamics. Quantum classical correspondence suggests that the square of CP_2 part of 8-momentum equals to the eigenvalue of CP_2 spinor Laplacian given the mass square of the spinor mode for an incoming particle.

4. Particle mass m should relate closely to the frequencies characterizing general extremals. Quite generally, one can write in cylindrical coordinates the general expressions of CP_2 angle variables Ψ and Φ as $(\Psi,\Phi) = (\omega_1 t + k_1 z + n_1\phi ..., \omega_2 t + k_2 z + n_2\phi ...)$. Here... denotes Fourier expansion [L4], [L4]: this corresponds to Cartan algebra of Poincare group with energy, one momentum component and angular momentum defining the quantum numbers. One can say that the frequencies define a warping of M^4 for $(\Psi,\Phi) = (\omega_1 t, \omega_2 t)$. The frequencies characterizing the warping of the canonically imbedded M^4 should closely relate to the mass of the particle. This raises the question whether the replacement of S^1 with $S^1\times S^1$ is appropriate.

5. Twistor description is also required. Generalization of ordinary twistors to octotwistor with quaternionicity condition as constraint allows to describe massive particles using almost-twistors. For massive particle the unit octonion corresponding to momentum in rest frame, the octonion defined by the polarization vector $\epsilon_k\gamma_k$, and the tangent vector $\gamma_k ds^k/ds$ (analog of polarization vector in CP_2) generate quaternionic sub-algebra. For massless particle momentum and polarization generate quaternionic sub-algebra as M^4 tangent space.

The SUSY algebra at QFT limit differs from the SUSY algebra defining the fundamental anti-commutators of the fermionic oscillator operators for the induced spinor fields since the Kähler-Dirac gamma matrices defined by the Kähler action are replaced with ordinary gamma matrices. The canonical commutation relations are however those between Ψ and its canonical momentum density $\overline{\Psi}\Gamma^t_{K-D}$ with the same right-hand side as usually (for quantum variant quantum phase appears in the anti-commutation relations). Hence the general form of anti-commutation relations are not changed in the transition and SUSY character is preserved if present in the fundamental formulation.

16.5.3 Explicit Form Of The SUSY Algebra At QFT Limit

The explicit form of the SUSY algebra follows from the proposed picture.

1. Spinor modes at X^2 correspond to the generators of the algebra. Effective 2-D property implies that spinor modes at partonic 2-surface can be assumed to have well-defined weak isospin and spin and be proportional to constant spinors.

2. The anti-commutators of oscillator operators define SUSY algebra. In leptonic sector one has

$$\begin{aligned} \{a^{\dagger}_{m\dot{\alpha}}, a^n_{\beta}\} &= \delta^n_m D_{\dot{\alpha}\beta} \ , \\ D &= (p^k\sigma_k + Q^a\sigma_a) \ . \end{aligned} \tag{16.5.3}$$

   Q^a denote color charges. The notions are same as in the case of WCW Clifford algebra. In quark sector one has opposite chirality and σ is replaced with $\hat{\sigma}$. Both the ordinary and octonionic representations of sigma matrices are possible.

16.5.4 How The Representations Of SUSY In TGD Differ From The Standard Representations?

The minimal super-sub-algebra generated by right-handed neutrino and antineutrino are the most interesting at low energies, and it is interesting to compare the naturally emerging representations of SUSY to the standard representations appearing in super-symmetric YM theories.

The basic new element is that it is possible to have short representations of SUSY algebra for massive states since particles are massless in 8-D sense. The mechanism causing the massivation remains open and p-adic thermodynamics can be responsible for it. Higgs mechanism could however induce small corrections to the masses.

The SUSY representations of SYM theories are constructed from $J = 0$ ground state (chiral multiplet for $\mathcal{N} = 1$ hyper-multiplet for $\mathcal{N} = 2$: more logical naming convention would be just scalar multiplet) and $J = 1/2$ ground state for vector multiplet in both cases. $\mathcal{N} = 2$ multiplet decomposes to vector and chiral multiplets of $\mathcal{N} = 1$ SUSY. Hyper-multiplet decomposes into two chiral multiplets which are hermitian conjugates of each other. The group of R-symmetries is $SU(2)_R \times U(1)_R$. In TGD framework the situation is different for two reasons.

1. The counterparts of ordinary fermions are constructed from $J = 1/2$ ground state with standard electro-weak quantum numbers associated with wormhole throat rather than $J = 0$ ground state.

2. The counterparts of ordinary bosons are constructed from $J = 0$ and $J = 1$ ground states assigned to wormhole contacts with the electroweak quantum numbers of Higgs and electroweak gauge bosons. If one poses no restrictions on bound states, the value of $\mathcal{N}$ is effectively doubled from that for representation associated with single wormhole throat.

These differences are allowed by general SUSY symmetry which allow the ground state to have arbitrary quantum numbers. Standard SYM theories however correspond to different representations so that the formalism used does not apply as such.

Consider first the states associated with single wormhole throat. The addition of right-handed neutrinos and their antineutrinos to a state with the constraint that $p^k\gamma^k$ annihilates the state at partonic 2-surface X^2 would mean that the helicities of the two super-symmetry generators are opposite. In this respect the situation is same as in the case of ordinary SUSY.

1. If one starts from $J = 0$ ground state, which could correspond to a bosonic state generated by WCW Hamiltonian and carrying $SO(2) \times SU(3)_c$ quantum numbers one obtains the counterparts of chiral/hyper- multiplets. These states have however vanishing electro-weak quantum numbers and do not couple to ordinary quarks neither.

2. If one starts $J = 1/2$ ground state one obtains the analog of the vector multiplet as in SYM but but belonging to a fundamental representation of rotation group and weak isospin group rather than to adjoint representation. For $\mathcal{N} = 1$ one obtains the analog of vector chiral multiplet but containing spins $J = 1/2$ and $J = 1$. For $\mathcal{N} = 2$ on obtains two chiral multiplets with $(J, F, R) = (1, 2, 1)$ and $(J, F, R) = (1/2, 1, 0)$ and $(J, F, R) = (0, 0, -1)$ and $(-1/2, 1, 0) = (0, 0, 0)$.

3. It is possible to have standard SUSY multiplet if one assumes that the added neutrino has always fermionic number opposite that the fermion in question. In this case on obtains $\mathcal{N} = 1$ scalar multiplet. This option could be defended by stability arguments and by the fact that it does not put right-handed neutrino itself to a special role.

For the states associated with wormhole contact zero energy ontology allows to consider two non-equivalent options. The following argument supports the view that gauge bosons are obtained as wormhole throats only if the throats correspond to different signs of energy.

1. For the first option the both throats correspond to positive energies so that spin 1 bosons are obtained only if the fermion and anti-fermion associated with throats have opposite M^4 chirality in the case that they are massless (this is important!). This looks somewhat strange but reflects the fact that $J = 1$ states constructed from fermion and anti-fermion with same chirality and parallel 4-momenta have longitudinal polarization. If the ground state has longitudinal polarization the spin of the state is due to right-handed neutrinos alone: in this case however spin 1 states would have fermion number 2 and -2.

2. If the throats correspond to positive and negative energies the momenta are related by time reflection and physical polarizations for the negative energy anti-fermion corespond to non-physical polarizations of positive energy anti-fermion. In this case physical polarizations are obtained.

If one assumes that the signs of the energy are opposite for the wormhole throats, the following picture emerges.

1. If fermion and anti-fermion correspond to $\mathcal{N} = 2$-dimensional representation of super-symmetry, one expects $2\mathcal{N} = 4$ gauge boson states obtained as a tensor product of two hyper-multiplets if bound states with all possible quantum number combinations are possible. Taking seriously the idea that only the bound states of fermion and anti-fermion are possible, one is led to consider the idea that the wormhole throats carry representations of $\mathcal{N} = 1$ super-symmetry generated by M^4 Weyl spinors with opposite chiralities at the two wormhole throats (right-handed neutrino and its antineutrino). This would give rise to a vector representation and eliminate a large number of exotic quantum number combinations such as the states with fermion number equal to two and also spin two states. This idea makes sense a also for a general value of $\mathcal{N}$. Bosonic representation could be also seen as the analog of short representation for $\mathcal{N} = 2N$ super-algebra reducing to a long representation $\mathcal{N} = N$. Short representations occur quite generally for the massive representations of SUSY and super-conformal algebras when 2^r generators annihilate the states [B61].

   Note that in TGD framework the fermionic states of vector and hyper multiplets related by $U(2)_R$ R-symmetry differ by a $\nu_R\overline{\nu}_R$ pair whose members are located at the opposite throats of the wormhole contact.

2. If no restrictions on the quantum numbers of the boson like representation are posed, zero energy ontology allows to consider also an alternative interpretation. $\mathcal{N}=4$ (or more generally, $\mathcal{N}=2N$-) super-algebra could be interpreted as a direct sum of positive and negative energy super-algebras assigned to the opposite wormhole throats. Boson like multiplets could be interpreted as a long representation of the full algebra and fermionic representations as short representations with states annihilated either by the positive or negative energy part of the super-algebra. The central charges Z_{ij} must vanish in order to have a trivial representations with $p^k=0$. This is expected since the representations are massless in the generalized sense.

3. Standard $\mathcal{N}=2$ multiplets are obtained if one assume that right-handed neutrino has always opposite fermion number than the fermion at the throat. The arguments in favor of this option have been already given.

16.5.5 SUSY after LHC

As we now know, SUSY was not found at LHC and the basic motivation for SUSY at LHC energies has disappeared. The popular article "Where Are All the 'Sparticles' That Could Explain What's Wrong with the Universe?" (see `http://tinyurl.com/y6n5cjhv`) tells about the situation. The title is however strange. There is nothing wrong with the Universe. Theoreticians stubbornly sticking to a wrong theory are the problem.

Could it be that the interpretation of SUSY has been wrong? For instance, the minimal $\mathcal{N}=1$ SUSY predicts typically Majorana neutrinos and non-conservation of fermion number. This does not conform with my own physical intuition. Perhaps we should seriously reconsider the notion of supersymmetry itself and ask what goes wrong with it.

Can TGD framework provide any new insights?

1. TGD can be seen as a generalization of superstring models, which emerged years before superstring models came in fashion. In superstring models supersymmetry is extended to super-conformal invariance and could give badly broken SUSY as space-time symmetry. SUSY in standard QFT framework requires massless particles and this requires generalization of the Higgs mechanism. The proposals are not beautiful - this is most diplomatic manner to state it.

   In TGD framework super-conformal symmetries generalize dramatically since light-like 3-D surfaces - in particular light-cone boundary and boundaries of causal diamond (CD) have one light-like direction and are metrically 2-D albeit topologically 3-D. One outcome is modification of AdS/CFT duality - which turned out to be a disappointment - to a more realistic duality in which 2-D surfaces of space-time regarded itself as surface in $H=M^4\times CP_2$ are basic objects. The holography in question is very much like strong form of ordinary holography and is akin to the holography assigned with blackhole horizons.

2. The generators of supersymmetries are fermionic oscillator operators and the Fock states can be regarded as members of SUSY multiplets but having totally different physical interpretation. At elementary particle level these many fermion states are realized at partonic 2-surfaces carrying point-like fermions assignable to lepton and quark like spinors associated with single fermion generations. There is infinite number of modes and most of them are massive.

   This gives rise to infinite super-conformal multiplets in TGD sense. Ordinary light elementary particles could correspond to partonic 2-surfaces carrying only fermion number at most ± 1.

3. By looking the situation from the perspective of 8-D embedding space $M^4\times CP_2$ situation gets really elegant and simple.

   8-D twistorialization [L58] requires massless states in 8-D sense and these can be massive in 4-D sense. Super-conformal invariance for 8-D masslessness is infinite-D variant of SUSY: all modes of fundamental fermions generate supersymmetries. The counterpart SUSY algebra is generated by the fermionic oscillator operators for induced spinor fields. All modes independently of their 4-D mass are generators of supersymmetries. M^4 chirality conservation of 4-D SUSY requiring 4-D masslessness is replaced by 8-D chirality conservation implying a separate conservation of baryon and lepton numbers. Quark-lepton symmetry is possible since color quantum numbers are not spin-like but realized as color partial waves in cm degrees of freedom of particle like geometric object.

No breaking of superconformal symmetry in the sense of ordinary SUSYs is needed. p-Adic thermodynamics causes massivation of massless (in 4-D sense) states of spectrum via mixing with very heavy excitations having mass scale determined by CP_2 mass.

One could say that the basic mistake of colleagues - who have been receiving prizes for impressively many breakthroughs during last years - is the failure to realize that 4-D spinors must be replaced with 8-D ones. This however requires 8-D embedding space and space-time surfaces and one ends up to TGD by requiring standard models symmetries or just the existence of twistor lift of TGD. All attempts to overcome the problems lead to TGD. Colleagues do not seem like this at all so that they prefer to continue as hitherto. And certainly this strategy has been an amazing professional success.

What about the counterpart of space-time supersymmetry - SUSY - in TGD framework? The question whether TGD allows space-time SUSY or not has bothered me for a long time, and I have considered SUSY from TGD point of view in [**?**, K84, K1]. In the following I summarize my recent views, which reflect the increased understanding of twistor lift and cosmological constant and of preferred extremals as minimal surfaces having 2-D string world sheets as singularities analogous to edges [L57, L63, L67] [L58].

1. The analog of SUSY would be generated by massless or light modes of induced spinor fields. Space-time SUSY would correspond to the lightest slowly varying modes for the induced spinor fields being in 1-1-correspondence with the components of H-spinors. The number $\mathcal{N}$ associated with SUSY is quite large as the number of components of H-spinors. The corresponding fermionic oscillator operators generate repsesentations of Clifford algebra and SUSY multiplets are indeed such.

   If space-time surface is canonically imbedded Minkowski space M^4, no SUSY breaking occurs. This is however an unrealistic situation. For general preferred extremal right- and left handed components of spinors mix, which causes in turn massivation and breaking of SUSY in 4-D sense.

   Could right-handed neutrino be an exception. It does not couple to electroweak and color gauge potentials. Does this mean that ν_R and its antiparticle generate exact $\mathcal{N}=2$ SUSY? No: ν_R has small coupling to CP_2 parts of induced gamma matrices mixing neutrino chiralities and this coupling causes also SUSY breaking. This coupling is completely new and not present in standard QFTs since they do not introduce induced spinor structure forced by the notion of sub-manifold gemetry.

   Even worse, one can argue that right-handed neutrino is "eaten" as right- and left-handed massless neutrinos combine to massive neutrino unless one has canonically imbedded M^4. There fate resembles that of charge Higgs components. One could still however say that one has an analog of broken SUSY generated by massive lepton and quark modes. But it would be better to talk about 8-D supersymmetry.

2. The situation is now however so simple as this. TGD space-time is many-sheeted and one has a hierarchy of space-time sheets in various scales labelled by p-adic primes labelling also particles and by the value of Planck constant $h_{eff} = n \times h_0$.

   Furthermore, spinors can be assigned to 4-D space-time interiors, to 2-D string world sheets, to their light-like 1-D boundaries at 3-D light-like orbits of partonic 2-surfaces, or even with the partonic orbits. 2-D string world sheets are analogous to edges of 3-D object and action receives "stringy" singular contribution from them because of edge property. Same applies to the boundaries of string world sheets location at the light-like orbits of partonic 2-surfaces. Think of a cloth, which has folds which move along it as an analog. Space-time interior is a minimal surface in 4-D sense except at 2-D folds and string world sheets and their boundaries are also minimal surfaces.

   Therefore one has many kinds of fermions: 4-D space-time fermions, 2-D string world sheet fermions possibly associated with hadrons (there presence might provide new insights to the spin puzzle of proton), and 1-D boundary fermions for these as point-like particles and naturally identifiable as basic building bricks of ordinary elementary particles. Perhaps even 3-D fermions associated with light-like partonic orbits can be considered. All these belong to the spectrum and the situation is very much like in condensed matter physics, where people talk fluently about edge states.

3. In TGD framework ordinary elementary particles are assigned with the light-like boundaries of string world sheets. Right-handed neutrino and antineutrino generate $\mathcal{N} = 2$ SUSY for massless states assignable as light-like curves at light-like orbits of partonic 2-surfaces. This implies badly broken SUSY and it seems that one cannot talk about SUSY at all in the conventional sense. These states are however massless in 8-D sense, not in 4-D sense!

   In TGD framework one can however consider an analogy of SUSY for which massless ν_R modes in 4-D space-time interior - rather than at orbits of partonic 2-surfaces - generate supersymmetry. One could say that the many particle state, rather than particle has a spartner. Think of any system - it can contain larger number of ordinary particles forming a single quantum coherent entity to which one an assign space-time sheet. One can assign to this system space-time shet a right-handed neutrino, antineutrino, or both. This gives the superpartner of the system. The presence of ν_R is not seen in the same manner in interactions as in SUSY theories.

This picture [L57, L63, L67] is an outcome of a work lasted for decades, not any ad hoc model. One can say that classical aspects of TGD (exact part of quantum theory in TGD framework) are now well understood. To sum up, the simplest realizations of SUSY in TGD sense are following and the best manner to look at them is from the perspective 8-D masslessness.

1. Massless 4-D supersymmetry generated by ν_R. Other fermions which are massive because of their electroweak and color interactions not possessed by ν_R. Also ν_R generates small mass. These spartners are not however visible in elementary particle physics but belong to condensed matter physics.

2. Massive neutrino and other fermions but no supersymmetry generatig ν_R anymore since it is "eaten". This would be realized as very badly broken SUSY in 4-D sense and the spartners would be very massive. At the partonic 2-surfaces, this option forced by Uncertainty Principle.

Chapter 17

Could $\mathcal{N} = 2$ Super-conformal Theories Be Relevant For TGD?

17.1 Introduction

The concrete realization of the super-conformal symmetry (SCS) in TGD framework has remained poorly understood. In particular, the question how SCS relates to super-conformal field theories (SCFTs) has remained an open question. The most general super-conformal algebra assignable to string world sheets by strong form of holography has $\mathcal{N}$ equal to the number of 4+4 =8 spin states of leptonic and quark type fundamental spinors but the space-time SUSY is badly broken for it. Covariant constancy of the generating spinor modes is replaced with holomorphy - kind of "half covariant constancy". I have considered earlier a proposal that $\mathcal{N} = 4$ SCA could be realized in TGD framework but given up this idea. Right-handed neutrino and antineutrino are excellent candidates for generating $\mathcal{N} = 2$ SCS with a minimal breaking of the corresponding space-time SUSY. Covariant constant neutrino is an excellent candidate for the generator of $\mathcal{N} = 2$ SCS. The possibility of this SCS in TGD framework will be considered in the sequel.

17.1.1 Questions about SCS in TGD framework

This work was inspired by questions not related to $\mathcal{N} = 2$ SCS, and it is good to consider first these questions.

Could the super-conformal generators have conformal weights given by poles of fermionic zeta?

The conjecture [L17] is that the conformal weights for the generators super-symplectic representation correspond to the negatives of $h = -ks_k$ of the poles s_k fermionic partition function $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ defining fermionic partition function. Here k is constant, whose value must be fixed from the condition that the spectrum is physical. $\zeta(ks)$ defines bosonic partition function for particles whos energies are given by $log(p)$, p prime. These partition functions require complex temperature but is completely sensible in Zero Energy Ontology (ZEO), where thermodynamics is replaced with its complex square root.

For non-trivial zeros $2ks = 1/2 + iy$ of $\zeta(2ks)$ s would correspond pole $s = (1/2 + iy)/2k$ of $z_F(ks)$. The corresponding conformal weights would be $h = (-1/2 - iy)/2k$. For trivial zeros $2ks = -2n$, $n = 1, 2, ..$ $s = -n/k$ would correspond to conformal weights $h = n/k > 0$. Conformal confinement is assumed meaning that the sum of imaginary parts of of generators creating the state vanishes.

What can one say about the value of k? The pole of $\zeta(ks)$ at $s = 1/k$ would correspond to pole and conformal weight $h = -1/k$. For $k = 1$ the trivial conformal weights would be positive integers $h = 1, 2, ...$: this certainly makes sense. This gives for the real part for non-trivial conformal weights $h = -1/4$. By conformal confinement both pole and its conjugate belong to the state so that this contribution to conformal weight is negative half integers: this is consistent with the facts about super-conformal representations. For the ground state of super-conformal representation the conformal weight for conformally confined state would be $h = -K/2$. In p-adic mass calculations one would have $K = 6$ [K52].

The negative ground state conformal weights of particles look strange but p-adic mass calculations require that the ground state conformal weights of particles are negative: $h = -3$ is required.

What could be the origin of negative ground state conformal weights?

Super-symplectic conformal symmetries are realized at light-cone boundary and various Hamiltonians defined analogs of Kac-Moody generators are proportional functions $f(r_M)H_{J,m}H_A$, where $H_{J,m}$ correspond to spherical harmonics at the 2-sphere $R_M = constant$ and H_A is color partial wave in CP_2, $f(r_M)$ is a partial wave in radial light-like coordinate which is eigenstate of scaling operator $L_0 = r_M d/dR_M$ and has the form $(r_M/r_0)^{-h}$, where h is conformal weight which must be of form $h = -1/2 + iy$.

To get plane wave normalization for the amplitudes

$$(\frac{r_M}{r_0})^{-h} = (\frac{r_M}{r_0})^{-1/2}exp(iyx) \ \ , \ \ x = log(\frac{r_M}{r_0}) \ \ ,$$

one must assume $h = -1/2+iy$. Together with the invariant integration measure dr_M this gives for the inner product of two conformal plane waves $exp(iy_ix)$, $x = log(r_M/r_0)$ the desired expression $\int exp[iy_1 - y_2)x]dx = \delta(y_1 - y_2)$, where $dx = dr_M/r_M$ is scaling invariance integration measure. This is just the usual inner product of plane waves labelled by momenta y_i.

If r_M/r_0 can be identified as a coordinate along fermionic string (this need not be always the case) one can interpret it as real or imaginary part of a hypercomplex coordinate at string world sheet and continue these wave functions to the entire string world sheets. This would be very elegant realization of conformal invariance.

How to relate degenerate representations with $h > 0$ to the massless states constructed from tachyonic ground states with negative conformal weight?

This realization would however suggest that there must be also an interpretation in which ground states with negative conformal weight $h_{vac} = -k/2$ are replaced with ground states having vanishing conformal weights $h_{vac} = 0$ as in minimal SCAs and what is regarded as massless states have conformal weights $h = -h_{vac} > 0$ of the lowest physical state in minimal SCAs.

One could indeed start directly from the scaling invariant measure dr_M/r_M rather than allowing it to emerge from dr_M. This would require in the case of p-adic mass calculations that has representations satisfying Virasoro conditions for weight $h = -h_{vac} > 0$. p-Adic mass squared would be now shifted downwards and proportional to $L_0 + h_{vac}$. There seems to be no fundamental reason preventing this interpretation. One can also modify scaling generator L_0 by an additive constant term and this does not affect the value of c. This operation corresponds to replacing basis $\{z^n\}$ with basis $\{z^{n+1/2}\}$.

What makes this interpretation worth of discussing is that the entire machinery of conformal field theories with non-vanishing central charge and non-vanishing but positive ground state conformal weight becomes accessible allowing to determine not only the spectrum for these theories but also to determine the partition functions and even to construct n-point functions in turn serving as basic building bricks of S-matrix elements [L22].

ADE classification of these CFTs in turn suggests at connection with the inclusions of hyperfinite factors and hierarchy of Planck constants. The fractal hierarchy of broken conformal symmetries with sub-algebra defining gauge algebra isomorphic to entire algebra would give rise to dynamic symmetries and inclusions for HFFs suggest that ADE groups define Kac-Moody type symmetry algebras for the non-gauge part of the symmetry algebra.

17.1.2 Questions about $\mathcal{N} = 2$ SCS

$\mathcal{N} = 2$ SCFTs has some inherent problems. For instance, it has been claimed that they reduce to topological QFTs. Whether $\mathcal{N} = 2$ can be applied in TGD framework is questionable: they have critical space-time dimension $D = 4$ but since the required metric signature of space-time is wrong.

Inherent problems of $\mathcal{N}=2$ SCS

$\mathcal{N}=2$ SCS has some severe inherent problems.

1. $\mathcal{N}=2$ SCS has critical space-time dimension $D=4$, which is extremely nice. On the other, $\mathcal{N}=2$ requires that space-time should have complex structure and thus metric signature (4,0), (0,4) or (2,2) rather than Minkowski signature. Similar problem is encountered in twistorialization and TGD proposal is Hamilton-Jacobi structure (se the appendix of [K8]), which is hybrid of hypercomplex structure and Kähler structure. There is also an old proposal by Pope *et al* [B54]that one can obtain by a procedure analogous to dimensional reduction $\mathcal{N}=2$ SCS from a 6-D theory with signature (3,3). The lifting of Kähler action to twistor space level allows the twistor space of M^4 to have this signature and the degrees of freedom of the sphere S^2 are indeed frozen.

2. There is also an argument by Eguchi that $\mathcal{N}=2$ SCFTs reduce under some conditions to mere topological QFTs [B31]. Thi looks bad but there is a more refined argument that $\mathcal{N}=2$ SCFT transforms to a topological CFT only by a suitable twist [B28, B52]. This is a highly attractive feature since TGD can be indeed regarded as almost topological QF. For instance, Kähler action in Minkowskian regions could reduce to Chern-Simons term for a very general solution ansatz. Only the volume term having interpretation in terms of cosmological constant [L22] (extremely small in recent cosmology) would not allow this kind of reduction. The topological description of particle reactions based on generalized Feynman diagrams identifiable in terms of space-time regions with Euclidian signature of the induced metric would allow to build n-point functions in the fermionic sector as those of a free field theory. Topological QFT in bosonic degrees of freedom would correspond naturally to the braiding of fermion lines.

Can one really apply $\mathcal{N}=2$ SCFTs to TGD?

TGD version of SCA is gigantic as compared to the ordinary SCA. This SCA involves super-symplectic algebra associated with metrically 2-dimensional light-cone boundary (light-like boundaries of causal diamonds) and the corresponding extended conformal algebra (light-like boundary is metrically sphere S^2). Both these algebras have conformal structure with respect to the light-like radial coordinate r_M and conformal algebra also with respect to the complex coordinate of S^2. Symplectic algebra replaces finite-dimensional Lie algebra as the analog of Kac-Moody algebra. Also light-like orbits of partonic 2-surfaces possess this SCA but now Kac-Moody algebra is defined by isometries of embedding space. String world sheets possess an ordinary SCA assignable to isometries of the embedding space. An attractive interpretation is that r_M at light-cone boundary corresponds to a coordinate along fermionic string extendable to a hypercomplex coordinate at string world sheet.

$\mathcal{N}=8$ SCS seems to be the most natural candidate for SCS behind TGD: all fermion spin states would correspond to generators of this symmetry. Since the modes generating the symmetry are however only half-covariantly constant (holomorphic) this SUSY is badly broken at space-time level and the minimal breaking occurs for $\mathcal{N}=2$ SCS generated by right-handed neutrino and antineutrino.

The key motivation for the application of minimal $\mathcal{N}=2$ SCFTs to TGD is that SCAs for them have a non-vanishing central charge c and vacuum weight $h \geq 0$ and the degenerate character of ground state allows to deduce differential equations for n-point functions so that these theories are exactly solvable. It would be extremely nice is scattering amplitudes were basically determined by n-point functions for minimal SCFTs.

A further motivation comes from the following insight. ADE classification of $\mathcal{N}=2$ SCFTs is extremely powerful result and there is connection with the hierarchy of inclusions of hyperfinite factors of type II_1, which is central for quantum TGD. The hierarchy of Planck constants assignable to the hierarchy of isomorphic sub-algebras of the super-symplectic and related algebras suggest interpretation in terms of ADE hierarchy a rather detailed view about a hierarchy of conformal field theories and even the identification of primary fields in terms of critical deformations.

The application $\mathcal{N} = 2$ SCFTs in TGD framework can be however challenged. The problem caused by the negative value of vacuum conformal weight has been already discussed but there are also other problems.

1. One can argue that covariantly constant right-handed neutrino - call it ν_R - defines a pure gauge super-symmetry and it has taken along time to decide whether this is the case or not. Taking at face value the lacking evidence for space-time SUSY from LHC would be easy but too light-hearted way to get rid of the problem.

   Could it be that at space-time level covariantly constant right-handed neutrino (ν_R) and its antiparticle ($\overline{\nu_R}$) generates pure gauge symmetry so that the resulting sfermions correspond to zero norm states? The oscillator operators for ν_R at embedding space level have commutator proportional to $p^k\gamma_k$ vanishing at the limit of vanishing massless four-momentum. This would imply that they generate sfermions as zero norm states. This argument is however formulated at the level of embedding space: induced spinor modes reside at string world sheets and covariant constancy is replaced by holomorphy!

   At the level of induced spinor modes located at string world sheets the situation is indeed different. The anti-commutators are not proportional to $p^k\gamma_k$ but in Zero Energy Ontology (ZEO) can be taken to be proportional to $n^k\gamma_k$ where n_k is light-like vector dual to the light-like radial vector of the point of the light-like boundary of causal diamond CD (part of light-one boundary) considered. Therefore also constant ν_R and $\overline{\nu_R}$ are allowed as non-zero norm states and the 3 sfermions are physical particles. Both ZEO and strong form of holography (SH) would play crucial role in making the SCS dynamical symmetry.

2. Second objection is that LHC has failed to detect sparticles. In TGD framework this objection cannot be taken seriously. The breaking of $\mathcal{N} = 2$ SUSY would be most naturally realized as different p-adic length scales for particle and sparticle. The mass formula would be the same apart from different p-adic mass scale. Sparticles could emerge at short p-adic length scale than those studied at LHC (labelled by Mersenne primes M_{89} and $M_{G,79} = (1+i)^79)$.

   One the other hand, one could argue that since covariantly constant right-handed neutrino has no electroweak-, color- nor gravitational interactions, its addition to the state should not change its mass. Again the point is however that one considers only neutrinos at string world sheet so that covariant constancy is replaced with holomorphy and all modes of right-handed neutrino are involved. Kähler Dirac equation brings in mixing of left and right-handed neutrinos serving as signature for massivation in turn leading to SUSY breaking. One can of course ask whether the p-adic mass scales could be identical after all. Could the sparticles be dark having non-standard value of Planck constant $h_{eff} = n \times h$ and be created only at quantum criticality [**?**].

This is a brief overall view about the most obvious problems and proposed solution of them in TGD framework and in the following I will discuss the details. I am of course not a SCFT professional. I however dare to trust my physical intuition since experience has taught to me that it is better to concentrate on physics rather than get drowned in poorly understood mathematical technicalities.

17.2 Some CFT backround

The construction of CFTs involves as the first step construction of irreducible unitary representations of conformal algebras. They are completely known for the central charge $0 \leq c \leq 1$. One can also construct modular invariant partition functions for tensor products possibly serving as partition functions of CFTs. Already Belavin, Polyakov and Zhamolodzhikov [B17] discovered in their pioneering paper so called minimal models with the defining property that the state space realizes only finite number of irreducible representations.

17.2.1 Modular invariant partition functions

The classification of modular functions leads to the ADE scheme [B23] (http://tinyurl.com/h9val5g). The physical picture is that the primary fields of minimal CFT correspond to deformations of a critical system in some configuration space. One can construct all minimal orbifold CFTs in orbifolds $G\backslash C^2$ of C^2 in which the discrete subgroup G of $SU(2)$ acts linearly [B57]. This is a minimal realization. ADE scheme enters via the ADE classification for the discrete subgroups of $SU(2)$ (see http://tinyurl.com/jyjplzc).

ADE classification gives an amazingly detailed view about the spectrum of minimal models and also about their partition functions [B23] (see http://tinyurl.com/zlhk3wu). More general rational CFTs can possess infinite families of Virasoro representations, which an be however organized to representations of W-algebra. So called WZW models provide an important example constructible for any semi-simple Lie algebra.

The decomposition of RCFT Hilbert space to sum over tensor products of spaces carrying irreducible unitary representation conformal algebra and is conjugate can be written as

$$H = \oplus_{j\overline{j}} N_{j\overline{j}} H_j \otimes H_{\overline{j}} \quad . \tag{17.2.1}$$

There are consistency conditions on the coefficients due to the conditions that the CFT must exist on any Riemann surface. Verlinde algebra (see http://tinyurl.com/y8p9muj6) expresses the fusion rules. The associative Verlinde algebra is finite-dimensional and has as its elements primary fields and its structure constants code for the fusion rules. Especially interesting primary fields are those which are simple in the sense that the product of two primary fields contains only one prime field.

It is good to understand how one ends up with the expression of partition function in conformal field theories.

1. Start from the fact that conformal invariance fixes the complex function by data at 1-dimensional curve and one can speak about analog of time evolution in direction orthogonal to this curve. Introduce Hamiltonian for the Euclidian "time" evolution in finite "time" interval defining an annulus at 2-D surface with boundaries identified as initial and final times. Assume periodic boundary conditions in Euclidian "time" direction so that the annulus effectively closes to a torus. The outcome is a conformal field theory at torus although one starts from conformal invariance at sphere or even Riemann surface with higher genus.

2. Torus has several conformally inequivalent variants since it can be obtained from complex plane by identifying the points differing by a translation generated by real unit 1 and complex number τ. The possible values of τ defines the moduli space for conformal equivalence classes of torus since the angle angle between the sides of this elementary cell and the ratio of the lengths of homologically non-trivial geodesics of torus are conformal invariants. Modular invariance however implies that the values of τ differing by PSL(2,Z) transformation are equivalent.

3. What happens if one applies this procedure at higher genus surface? If the annulus is around the handle of this kind of surface, one might have a problem since it is not clear whether periodic boundary conditions can be identified in terms of a compactification to torus - this kind of annulus cannot be physically compactified to a torus. One can also consider a Hamiltonian evolution associated with any curve characterized by homology class telling how many times the curve winds around various handles. Can one just use the parameter τ or should one take into account the homology class of the annulus.

   One can challenge the idea about Hamiltonian time evolution as a formal trick and consider the possibility that partition functions is defined for the entire 2-surface in moduli space. In this kind of situation it would be trivial for sphere.

4. One can write explicitly the expression for the Euclidian "time" evolution operator between the ends of annulus as an exponential:

$$exp(-H_{cycl}L) = exp\left[2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\overline{tau}(\overline{L}_0 - \frac{c}{24}\right] \quad . \qquad (17.2.2)$$

Partition function is defined as the trace

$$Z(\tau) = Tr\left[exp(-H_{cycl}L)\right] \quad . \qquad (17.2.3)$$

$$\chi_j(q) = Tr\left[exp\left[2\pi i\tau(L_0 - \tfrac{c}{24})\right]\right] = q^{h_j - \frac{c}{24}}\sum m_n q^n \quad , \quad q = exp(i2\pi\tau) \quad , \quad \overline{q} = exp(-i2\pi\overline{\tau}) \qquad (17.2.4)$$

5. The decomposition of Hilbert space translates to a decomposition of the partition function as

$$Z(\tau) = \sum_{j\overline{j}} N_{j\overline{j}}\chi_j(q) \times \chi_{\overline{j}}(\overline{q}) \quad . \qquad (17.2.5)$$

Here one can wonder whether one could give up the interpretation in terms of Hamiltonian time evolution and consider just partition function in the moduli space of torus (or higher genus surface).

Modular invariance poses strong conditions of the expression of partition function of system as sum over products $chi_j\overline{\chi}_{\overline{j}}$ of characters assignable to irreducible unitary representations of Virasoro algebra. In the case of torus moduli correspond to complex plane whose points differing by a transformations by the discrete group SL(2,Z) are identified. The resulting moduli space has topology of torus. The generators of modular transformations are unit shift T: $\tau \to \tau + 1$ and inversion S: $\tau \to -1/\tau$ and it is enough to demand that the partition function is invariant under these transformations. The action of these transformations on characters induce an unitary automorphisms of the matrix $N_{j\overline{j}}$ and the condition is that the actions of S and T are trivial

$$TNT^{\dagger} = SNS^{\dagger} = N \quad . \qquad (17.2.6)$$

It is interesting to relate this picture to TGD framework where one has string world sheets and partonic 2-surfaces.

1. The annulus picture applies to string world sheets. At the ends space-time surface at boundaries of CD one has fermionic strings connecting wormhole throat to another one along the first space-time sheet and returning back along second space-time sheet and forming thus a closed string, whose time evolution defines string space-time sheet as a cylindrical object. The strings at the ends of CD can get knotted and braided. They can also reconnect - the interpretation is in terms of standard stringy vertices. In fact this gives rise to 2-braiding possible because space-time dimension is 4.

   One can also consider loops as handles attached to these annuli: since the induced metric is allowed to have Euclidian signature, they are in principle possible but involve always Euclidian regions around points, where the time direction of closed homologically trivial time loop defined by the time coordinate of Minkowski space changes. Preferred extremal property might forbid loop corrections in Minkowskian space-time regions but allow them inside Euclidian regions representing lines of scattering diagrams.

2. The moduli space for the conformal equivalence classes of partonic 2-surfaces is important in the TGD based model for family replication phenomenon [K21]. In TGD context one must construct modular invariant partition functions in these higher-dimensional moduli spaces - I call them elementary particle vacuum functionals. These partition functions do not allow interpretation in terms of Hamiltonian time evolution.

17.2.2 Degenerate conformal representations and minimal models

So called degenerate representations allow to construct minimal models with finite number of primary fields and derive also differential equations for their correlation functions. Degeneracy condition fixes the spectrum of so called minimal conformal field theories.

1. The conformal weight the ground state is fixed to $h \geq 0$. Virasoro conditions must be satisfied: it is enough that the generators L_1 and L_2 annihilate the ground state. The defining feature of degenerate representations is that they possess states with zero norm created by generates with negative conformal weights from the ground state.

2. Degenerate states are obtained as linear combinations of states constructible using products $\prod_k L_{-k}^{-n_k}$, $N=\sum_k n_k k$ of generators with total conformal weight $-N$ operating on ground state with weight h. Degeneracy means that some combination of the generators with total weight $-N$ annihilates the state. Besides this ordinary Virasoro conditions for generators with positive weight are satisfied. The existence of the degenerate state means that the metric of this sub-state space is degenerate so that its determinant - so called Kac determinant vanishes. This brings strongly in mind criticality: at criticality sub-representation is isolated from the larger representation and defines zero norm states. These would correspond to zero modes appearing at criticality and not contributing to the potential function.

3. Vanishing of Kac determinant gives a condition allowing to deduce a general formula for the allowed values of the central charge c defining the central extension of conformal algebra. One can factorize Kac determinant to a product form $\prod_n (h-h_n)$ and the eigenvalues h_n defined the ground state weights allowing the degeneracy. Unitarity gives a further condition on the representation and for $c<1$ this dictates the spectrum of vacuum conformal weights completely.

One can deduce an explicit expression for the Kac determinant as function of c and h and this gives rise to the following fundamental formulas [B23] (see `http://tinyurl.com/h9val5g`) for the values of central charge c and ground state conformal weight h for which the determinant vanishes. For $c>1$ the determinant does not vanish and is positive. For $c<1$ situation is different.

$$\begin{array}{llll} c=c_{p,q}=1-\frac{6(p-q)^2}{pq} \ , & p \ and \ q \ coprime \ , & p,q=1,2,3,... \\ h=h_{r,s}(p,q)=\frac{[pr-qs]^2-(p-q)^2}{4pq} \ , & 1\leq r\leq q-1, & 1\leq s\leq p-1 \ . \end{array} \tag{17.2.7}$$

For these values of c and h the representation defined by dividing away zero norm states is irreducible and unitary. So called minimal models forming a special case of them and possessing finite number of primary fields correspond to these representations.

Why the degeneracy is so important? Suppose that primary conformal fields Φ_k have conformal weight h and satisfy the degeneracy condition. Then n-point functions satisfy also the appropriate form of the degeneracy condition being annihilated by the combination of Virasoro generators with total weight $-N$. This gives rise to n partial differential equations of order N for $\langle\Phi(z_1)...\Phi(z_n)\rangle$ allowing to solve the conformal field theory exactly. In TGD this generalizes would give a powerful tool to determine the correlation functions at string world sheets.

The standard example is provided by the $N=2$ case. The operator $O=L_{-2}-\frac{3}{2h+1}L_{-1}^2$ generates from the ground state with conformal weight h zero norm state provided the condition $c=2h(5-8h)/(2h+1)$ is satisfied. For $h=1/2$ this gives $c=1/2$. Primary fields of the CFT are annihilated by this operator as also n-point functions and this gives second order differential equations for the n-point functions.

If the proposed interpretation of negative conformal weights in TGD framework is correct then one can add the condition $h=K/2$ to the conditions fixing c and h. Although SCFT rather than CFT is expected to be interesting from TGD point of view, one can just for fun see the above conditions for c and h allow $h=K/2$. Direct calculation for $p=m, q=m+1$ shows that for $m=4$ $(c=1/2)$, $x=1$ and $x=1/2$ are realized for $(r=3, s=1)$ and $(r=3, s=2)$ respectively. For $m=5$ one obtains $x=3$ corresponding to $r=4$ and $s=1$. For $m=6$ one obtains $x=5$. It is not clear $(p,q)=(m,m+1)$ allows to realize $h=K/2$ or even $h=5/2$ and $h=2$.

17.2.3 Minimal $\mathcal{N}=2$ SCFTs

$\mathcal{N}=2$ SCA

$\mathcal{N}=2$ SCA is spanned by Virasoro generators L_n and their super counterparts G_r, where r is either integer (Ramond) or half-odd integer (Neveu-Schwartz) plus generators of conserved U(1) current J (se `http://tinyurl.com/yblzbovb`). Ramond and Neveu-Scwartz and these representations can be mapped to each other by spectral automorphism.

The commutation/anticommutation relations for $\mathcal{N}=2$ algebra are given by

$$\begin{array}{l} [L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0} \ , \\ [L_m,J_n]=-nJ_{m+n} \ , \\ [J_m,J_n]=\frac{c}{3}m\delta_{m+n,0} \ , \\ \{G_r^+,G_s^-\}=L_{r+s}+\frac{1}{2}(r-s)J_{r+s}+\frac{c}{6}(r^2-\frac{1}{4})\delta_{r+s,0} \ , \\ \{G_r^+,G_s^+\}=0=\{G_r^-,G_s^-\} \ , \\ [L_m,G_r^{\pm}]=(\frac{m}{2}-r)G^{\pm}_{r+m} \ , \\ [J_m,G_r^{\pm}]=\pm G^{\pm}_{m+r} \ . \end{array} \tag{17.2.8}$$

Also in the case of SCFTs one it is natural to search for sub-representations with ground state weight h and annihilated by some generator of conformal weight $-N$. In this case the operators would be monomials of Virasoro generators and their super counterparts and also now the vanishing of Kac-determinant [B19], whose expression was deduced by Boucher, Friedan and Kent, would allow to deduce information about allowed values of c and h. Also in this case the n-point functions $\langle\Phi(z_1)...\Phi(z_n)$ satisfy N:th order the differential equations implied by the condition that the generator in question annihilates the primary fields.

Spectral automorphism mapping Ramond and N-S representations to each other

Spectral automorphism maps both the algebra and its representations to new ones. The spectral automorphism mapping Ramond representation to N-S representation is given by

$$\begin{array}{l} \alpha(L_n)=L_n+\theta J_n+\frac{\theta^2}{6}\delta_{n,0} \ , \\ \alpha(J_n)=J_n+\frac{\theta}{3}\delta_{n,0} \ , \\ \alpha(G_r^{\pm})=G^{\pm}_{r\pm\theta} \ . \end{array} \tag{17.2.9}$$

The inverse of the automorphism is given by

$$\begin{array}{l} \alpha^{-1}(L_n)=L_n-\theta J_n+\frac{\theta^2}{6}\delta_{n,0} \ , \\ \alpha^{-1}(J_n)=J_n-\frac{c}{3}\theta\delta_{n,0} \ , \\ \alpha^{-1}(G_r^{\pm})=G^{\pm}_{r\mp\theta} \ . \end{array} \tag{17.2.10}$$

For $\theta=1/2$ one obtains Ramond-NS spectral mapping.

Central extension term contains par linear in m. This is changed as one finds by calculating the commutators of the transformed Virasoro generators and expressing it in in terms of transformed generators. This does not affect the value of c. No change occurs for $k=2$ minimal representations with $Q=k/2(k+2)-1/4=0$. Also the term linear in m remains unaffected if the $\theta=1/2$ flow is modified to

$$\alpha(L_n) \quad = \quad L_n+\frac{1}{2}J_n+(\frac{1}{24}-\frac{Q_{N-S}}{2})\delta_{n,0} \ . \tag{17.2.11}$$

Also the ground state is changed in the spectral flow and Q_{N-S} labels the ground state charge for the resulting N-S representation. For minimal SCAs the flow must label $(h,Q)_R$ to Ramond state to $(h,Q)_{N-S}$.

If the linear term of central extension is unaffected in the flow, the values of h and Q change as follows:

$$\begin{array}{l} h_R \rightarrow h_{new,R} + \frac{c}{24} = h_{N-S} \ , \\ Q \rightarrow Q_{new,R} + \frac{c}{6} = Q_{N-S} \ . \end{array} \quad (17.2.12)$$

The simplest guess is that the change leaves (a,b) unchanged and just drops the $1/8$ term from h and Q. This condition determines the values of $h_{new,R}$ and $Q_{new,R}$ for minimal representations to

$$\begin{array}{l} h_{new,R} = \frac{1}{8} - \frac{c}{24} = \frac{1}{8} - \frac{k}{8(k+2)} \ , \\ Q_{new,R} = \frac{1}{4} - \frac{1}{2k(k+2)} \ . \end{array} \quad (17.2.13)$$

Degenerate representations

The classification of unitary minimal super-conformal field theories is surprisingly well-understood [B60] (see `http://tinyurl.com/yctvyk2o`). ADE patterns are involved also in the classification of minimal SCFTs. The good news is that $\mathcal{N}=2$ superstrings have critical dimension $D=4$. The bad news is that the signature of the space-time metric is either (0,4), (2,2) or (4,0) rather than Minkowkian (1,3). This problem will be considered later in more detail.

I am not specialist and can only list the results. It is to be emphasized that not only the spectrum of basic parameters but also the partition functions are known, and correlation functions can be constructed.

1. The values of the central charge are given by

$$c = \frac{3k}{k+2} \ , \quad k = 0,1,2... \quad (17.2.14)$$

   Central charge has values $c = 0, 1, 3/2, 9/5, ...$ and approaches $c = 3$ for large values of k.

2. The vacuum conformal weights and U(1) charges depend on two integer valued parameters a,b besides k

$$\begin{aligned} h_{ab} &= \frac{a(a+2)-b^2}{4(k+2)} + \frac{(a+b)_2^2}{8} \ , \\ Q_{ab} &= \frac{b}{2(k+2)} - \frac{(a+b)_2^2}{4} \ . \end{aligned} \quad (17.2.15)$$

   Here the conditions

$$a = 0,...,k \ , \quad |b-(a+b)_2| \leq a \ , \quad (a+b)_2 \equiv a+b \mod 2 \quad (17.2.16)$$

   are satisfied. For Ramond type representations $(a+b)_2 = 1$ ($a+b$ is odd) is satisfied and for N-S type representations $(a+b)_2 = 0$ ($a+b$ is even) is satisfied. Note that $(h,Q) = (0,0)$ is possible only for $(a,b) = 0$ in the case of $N-S$ representation. For Ramond representation this would give $(h,Q) = (1/8, -1/4)$.

17.3 Could $\mathcal{N} = 2$ super-conformal algebra be relevant for TGD?

Despite various objections already discussed in the introduction there are good reasons to pose the question of the title.

17.3.1 How does the ADE picture about SCFTs and criticality emerge in TGD?

The crucial question in TGD framework is how the ADE picture relates to criticality and SCFTs in 2 dimensions. That the SCFT would be defined in 2 dimensions follows from SH.

1. The connection of ADE with inclusions of hyperfinite factors and with the hierarchy of Planck constants defining a hierarchy of dark matters are basic conjectures of TGD.

2. Finite number of degrees of freedom is left when a H_+ sub-algebra of super-symplectic or some other conformal algebra isomorphic to the entire algebra G_+ and the commutator $[H_+, G_+]$ ("+" refers to non-negative conformal weights) annihilate the states. The conjecture is that this gives rise to a finite-dimensional ADE type algebra defining Kac-Moody algebra or gauge algebra whose constant generators however act non-trivially. Denote the resulting finite-D ADE group by A_+. The Kac-Moody algebra might act on fermionic strings whereas the super-symplectic algebra would at at the boundary of CD.

3. At criticality a phase transition changing the value of Planck constant and thus H_+ and A_+ take place. These phase transitions would have a natural description in ZEO: the group ADE group A_+ would be smaller or larger an the other end of space-time surface at the opposite boundary of CD.

4. If the groups $A_{+,i}$ and $A_{+,f}$ satisfy $A_{+,i} \subset A_{+,f}$, new degrees of freedom appear. They correspond to the coset space $A_{+,f}/A_{+,i}$. Coset spaces typically form orbifolds: in fact the term orbi-fold comes from the identification of orbifold as the space of orbits, now those of $A_{+,i}$ in $A_{+,f}$. One would have orbifolds of ADE groups belonging associated with the hierarchy of inclusions labelled perhaps by Planck constants.

5. The orbifolds $O = A_{+,f}/A_{+,i}$ are however orbifolds of ADE groups, which are in 1-1 correspondence with the finite ADE subgroups G of SU(2). Does this mean that the orbifold $O = A_{+,f}/A_{+,i}$ is somehow determined by orbifold $G\backslash SU(2)$? As far as orbifold property is considered, $A_{+,i}$ would be effectively finite-D $G \subset SU(2)$. Mathematician could probably answer this question immediately.

   This kind of reduction of relevant degrees of freedom takes place in catastrophe theory, where only very few degrees of freedom determine the type of catastrophe: also in this case criticality is involved and catastrophes correspond to a hierarchy of criticalities.

6. The hierarchy of Planck constants corresponds to a hierarchy of coverings of space-time surface determined by strong form of holography by those for string world sheets. Could the discrete ADE groups G act in both the fibers and bases of these coverings?

Orbifoldings correspond to pairs of ADE groups appearing in the tensor product of representations. The first guess is that this is due to pairing of Ramond and N-S representations but ADE pairs appear also for conformal minimal models without super-symmetry. Second guess is that the tensor product pairing in TGD framework reflects the fact that one has always a pair of wormhole throats associated with the wormhole contact.

Concluding, it would be very natural to identify the orbifold degrees of in $O = A_{+,f}/A_{+,i}$ primary fields of minimal SCFT. This makes sense if the orbifolding reduces effectively to that for SU(2) by finite discrete subgroup.

17.3.2 Degrees of freedom and dynamics

$N = 2$ SCA or should be generated by the addition of right-handed neutrino or antineutrino to one-fermion state. The interpretation as a pure gauge symmetry seems plausible. Instead of trying to make ad hoc guesses by searching the enormous highly technical literature on the subject, it is better to try to build the physical picture first and hope that professionals could get motivated to perform detailed constructions.

Consider first the degrees of freedom involved.

1. In bosonic sector one has at the fundamental level deformations of string world sheets (possibly of partonic 2-surfaces too). There are also deformations of string world sheets in CP_2 degrees of freedom: the latter could be assigned with electroweak gauge bosons and SU(3) Killing vectors related to color gauge potentials defining representation spaces for Kac-Moody algebras involved. $\mathcal{N} = 2$ SCA should determine correlation functions for these. At higher abstraction level the dynamical variables would correspond to representations of ADE groups assignable to inclusions of HFFs and primary fields would correspond to orbifolds of groups assignable to the hierarchy of Planck constants.

2. In M^4 degrees of freedom there are 2 degrees of freedom orthogonal to string world sheets which correspond to complex coordinate. They would give rise to 2 additional tensor factors to the super Virasoro algebra, which should have 5 tensor factors if p-adic mass calculations are taken at face value. $N = 2$ SCA should have this number of tensor factors.

3. There are also fermionic degrees of freedom associated with the induced spinors at string world sheets and they would contribute to SCA too.

What one can say about the dynamics?

1. The dynamics at the level of physical particles would be essentially due to the non-trivial topological vertex in which 3 light-like 3-surfaces join along their ends. This dynamics would have huge symmetry generalizing the duality symmetry of hadronic string models: scattering diagram would be analogous to a computation with vertices having identification as algebraic operations and all computations connecting given sets of objects in initial and final state would be equivalent. This symmetry would allow to move the ends of internal lines so that loops could be transformed to tadpoles and snipped away giving a braided tree diagram as minimal scattering diagram. Something analogous to this happens for twistor Grassmann diagrams.

2. To the lines meeting at vertices defined by partonic 2-surfaces one can assign the fundamental four-fermion vertex [L22] defining second dynamics. This vertex does not however correspond to ordinary fermion vertex involving quartic term in fermion fields but corresponds to redistribution of fermion lines between the 3-legs. Therefore fermion dynamics would be free and this would allow to avoid divergences. The tensor net construction [L22] suggests for a very elegant description of these computations in terms of so called perfect tensors defining the nodes of the net and defining isometries between any leg and its complement with each leg involving unitary braiding operation.

3. The third dynamics would be at the level of Kähler action defined by the functional integral for the exponent of Kähler action. Quantum criticality motivates the proposal is that it is RG invariant in the sense that loop corrections vanish since Kähler coupling strength is analogous to critical temperature and is piecewise constant so that coupling constant evolution is discrete and the values fo α_K are labelled by a subset of p-adic primes.

17.3.3 Covariantly constant right-handed neutrinos as generators of super-conformal symmetries

As explained in the introduction, holomorphic right-handed neutrinos could generate the super-conformal symmetries with minimal breaking. Also other fermionic spin states (at embedding base level) would generate super-conformal symmetries but they would be badly broken.

1. At embedding space level massless modes of right-handed neutrino are covariantly constant in CP_2 and do not mix with left handed neutrinos. On the other hand, *induced* (as opposed to embedding space -) right-handed neutrino spinors, which are not constant, mix with the left handed neutrino spinor modes and they are physical degrees of freedom. This follows from the mixing of the M^4 and CP_2 contributions to modified gamma matrices determined by the Kähler action and are essentially contractions of canonical momentum currents with embedding space gamma matrices.

2. Induced spinor modes at string world sheets must carry vanishing weak W and possibly also Z fields to guarantee that em charge is well-defined. SH implies that the data at string world sheets are enough to construct the quantum theory. The assumption about localization is thus natural but not actually necessary, and it is not even clear whether Kähler-Dirac equation is really consistent with the localization at string world sheets although the special properties of Kähler Dirac gamma matrices (in particular, the degenerate character of the effective space-time metric defined by their anti-commutators) suggests this.

3. One must not forget that the conformal structure of solutions is extremely powerful and makes the situation almost independent of the Dirac action used. Dirac equation reduces essentially to holomorphy and to the condition that other half of the modified gamma matrices annihilate the spinor mode. One can therefore ask whether string world sheets could be minimal surfaces and whether Dirac equation in the induced metric could be satisfied at string world sheets. The trace of the second fundamental form giving rise to a term mixing M^4 chiralities vanishes in this case but there is still the mixing of gamma matrices inducing mixing of M^4 chiralities serving as a signal for massivation in M^4 sense.

4. The interpretation of $\mathcal{N} = 2$ supersymmetry possibly generated by right-handed neutrino has remained unresolved. As explained in the introduction, this problem disappears in ZEO since the boundary of CD allows anti-commutators of holomorphic ν_R oscillator operators to be non-vanishing also for constant mode and one obtains constant modes with non-vanishing norm to which space-time $\mathcal{N} = 2$ SUSY can be assigned.

5. A further complication is brought by the recent progress in twistorialization of Kähler action [L22]. It adds to the Kähler action extremely small volume term, and this term could spoil the idea about localization of the modes at string world sheets. Again the conformal structure of the solutions would save the situation if one does not require localization to string world sheets. The picture would be in accordance with SH.

17.3.4 Is $\mathcal{N} = 2$ SCS possible?

Could one assign $\mathcal{N} = 2$ SCA with these degrees of freedom?

1. $\mathcal{N} = 2$ SCA can be associated with any Super-Kac Moody algebra defined by simple Lie group by coset construction (see `http://tinyurl.com/yd2zqjvz`), in particular for $CP_2 = SU(3)/SU(2) \times U(1)$. The Kac-Moody algebra defined by the product of color group and electroweak group is not simple, but the fact that electroweak group holonomy group of CP_2 strongly suggests that $\mathcal{N} = 2$ SCA is possible. This would take care of color and electroweak degrees of freedom.

2. There are also 2 degrees of freedom corresponding to M^4 deformations of string world sheet orthogonal to the sheet. Free field construction would assign $\mathcal{N} = 2$ to the degrees of freedom orthogonal to the string world sheet but the central charge is $c = 3 > 3k/(k + 2)$ for the unitary $\mathcal{N} = 2$ SCFTs. Personally I do not see any reason why one could not have tensor product of several $\mathcal{N} = 2$ SCAs with different central charges.

There are some objections against the idea of understanding the correlation functions of this dynamics in terms of $\mathcal{N} = 2$ SCA.

1. $\mathcal{N} = 2$ SCA is claimed to require (2,2) signature for the metric of the target space in stringy realization: in Minkowskian resp. Euclidian space-time regions the induced metric

has signature (1,-1,-1,-1) resp.(-1,-1,-1,-1). To my best understanding the target space is associated with one particular realization so that this objection need not be crucial. Note that also in twistor Grassmann approach (2,2) signature plays also special role making things well-defined whereas in other signature one must apply Wick-rotation.

2. There is also an argument that $\mathcal{N} = 2$ SCFTs reduce to topological QFTs. TGD is indeed almost topological QFT and inside the string world sheets one expects the S-matrix to reduce to braiding S-matrix. The non-triviality of the scattering amplitudes would come from topology: one could assign the points of n-points functions to the ends of different legs of the diagrams.

The minimal models seem however to have the same symmetries as TGD and could therefore give some idea about what might be expected. $h = K/2$ condition for the representations of degenerate representations of $\mathcal{N} = 2$ SCA follows if h corresponds to the actual conformal weight of a massless state shifted to zero by redefinition of the scaling generator L_0 by shift $L_0 \rightarrow L_0 - h$. In the alternative picture this shift would map vacuum state with vanishing conformal weight to that with negative conformal weight $-h$. If $-h$ is sum over conformal weights $-1/2$ for the "wave functions" at light-cone boundary are proportional to $r_M^{-1/2}$ factor then it must be negative half integer and one has $h = K/2$.

This picture conforms also with the hypothesis that the poles of fermionic zeta determine the conformal weights for the generators of super-conformal symmetry with physical states assumed to satisfy conformal confinement implying that the imaginary parts of generators of SCA remain hidden. Note that the number of generators for the SCAs would be infinite unlike for ordinary SCAs: this would be also due to the fact that symplectic group is infinite-dimensional. Conformal confinement allows how the reduction of the conformal algebra at string world sheets to the ordinary super-conformal algebra. Also thermalization would occur only for this algebra.

For these reasons it is interesting to look what one obtains now by applying $h = K/2$ condition

1. $\mathcal{N} = 2$ super-conformal symmetry algebra (see `http://tinyurl.com/yd2zqjvz`) involving besides Virasoro generators also generators for U(1) current and their super-counterparts is a reasonable candidate in TGD framework where classical Kähler current is conserved. The addition of right-handed neutrino or its antiparticle is an excellent candidate for generating exact $\mathcal{N} = 2$ space-time supersymmetry as super-gauge symmetry as already explained. The conservation of quark and lepton numbers however allows to consider badly broken conformal SUSY algebra with larger value of $\mathcal{N}$.

2. The infinite-D symplectic algebra replaces the Kac-Moody algebra at light-cone boundary. At the light-like orbits of partons one obtains the counterpart of Kac-Moody algebra associated with the isometries of H and holonomies of CP_2. One might hope that p-adic thermodynamics involving only super-Virasoro generators is not affected at all by these complications. The states of additional algebras would only define the ground states of the Kac-Moody typ Super-Virasoro representations assignable to string world sheets (no thermalization in super-symplectic nor Kac-Moody degrees of freedom would occur), and the quantum numbers in question would correspond to quantum numbers of massless particles with massive excitations having mass scale defined by CP_2 mass scale.

17.3.5 How to circumvent the signature objection against $\mathcal{N} = 2$ SCFT?

As already noticed $\mathcal{N} = 2$ SCA is claimed to require (2,2) signature for the metric of the target space in the stringy realization. The problem is that $\mathcal{N} = 2$ super-conformal symmetry requires space-time to have complex structure. Could one circumvent this objection?

The first attempt is based on the observation that the notion of Kähler structure generalizes in TGD framework to what I have called Hamilton-Jacobi structure. This means that the complex structure is hybrid of hypercomplex structure in longitudinal tangent space M^2 and of ordinary complex structure in transversal space E^2. The signature poses also problem in the definition of twistor structure and is circumvented using this construction.

The second attempt is based on the twistor lift of Kähler action.

1. Pope *et al* [B54] (see `http://tinyurl.com/jnon4fh`) propose that one might start from 6-D theory space-time signature (1,1,1-1,1,-1) with $\mathcal{N}=2$ supersymmetry and perform kind of dimensional reduction freezing 2 time coordinates of a 6-D space to obtain $\mathcal{N}=2$ superstrings in the resulting effectively 4-dimensional space-time with signature (1,-1,-1,-1).

2. The twistor lift of TGD replaces space-time surface with its 6-D twistor space. One can choose the metric signature of the sphere S^2 having radius of order Planck constant defining the fiber of twistor space $M^4\times S^2$ to be (1,1) or (-1,-1). For (1,1) one obtains signature (1,1,1,-1,-1,-1). Dimensional reduction is involved and the analog for the freezing of S^2 time dimensions takes place. This suggests that one could have $\mathcal{N}=2$ symmetry at the level of twistor spaces of space-time surfaces.

3. These two approaches seem to be very closely related in TGD framework.

Third trial would be based on the idea that the signature of the effective metric defined by the anticommutators of the modified gamma matrices appearing in modified Dirac action takes care of the problem by giving signature (1,1,-1,-1) for the effective metric. The following argument does not support this option.

1. In Kähler-Dirac action the modified gamma matrices define effective space-time metric $G^{\alpha\beta}$ via their anticommutators. The physical role of $G^{\alpha\beta}$ has remained obscure. One has $G^{\alpha\beta}=T^{\alpha}_kT^{beta}_lh^{kl}$, where T^{α}_k is the canonical momentum current.

2. There are two contributions to T^{α}_k corresponding to Kähler action and extremely small volume term suggested by the twistor lift of Kähler action having interpretation in terms of cosmological constant. Let us write Kähler action density as $L_K=kJ^{\mu\nu}J_{mu\nu}\sqrt{g}/2$ and volume action density as $L_{vol}=K\sqrt{g}$. One can write T^{α}_k as

$$T^{\alpha}_k=[T^{\alpha\beta}[g]g_{k\beta}+T^{\alpha\beta}[J]J_{k,\beta} \ ,$$
$$g_{k\beta}=h_{kl}\partial_\beta h^l \ , \ J_{k,\beta}=J_{kl}\partial_\beta h^l \ , \quad (17.3.1)$$

The tensors appearing in this formula can be expressed in a concise notation as

$$T[g]=T[K,g]+T[vol,g] \ ,$$
$$T[K,g]=\frac{\partial L_K}{\partial g}\equiv k[J\circ J-\frac{1}{4}Tr(J\circ J)\sqrt{g}\equiv T_{K,1}+T_{K,2} \ ,$$
$$T[vol,g]=\frac{\partial L_{vol}}{\partial g}=\frac{K}{2}g \ ,$$
$$T[J]=\frac{\partial L_K}{\partial J}=kJ\sqrt{g} \ , \quad (17.3.2)$$

$\circ$ denotes product of tensors defined by contraction. $T^{\alpha\beta}[g]$ is energy momentum tensor and $T^{\alpha\beta}[J]=kJ^{\alpha\beta}$ is its analog coming from variations with respect to induced Kähler form. The following formulas will be used.

$$g_{k\mu}g^k_{\ \nu}=g_{\mu\nu} \ , \ g_{k\mu}J^k_{\ \nu}=J_{\mu\nu} \ , \ J_{k\mu}J^k_{\ \nu}=-s_{\mu\nu} \quad (17.3.3)$$

Here s refers to CP_2 metric. G can be written in compact notation as

$$
\begin{aligned}
&G = G[g,g]+G[J,J]+2G[g,J] \ , \\
&G[g,g] = T\circ T \ , \\
&G[J,J] = -T[J]\circ s\circ T[J] = -k^2 J\circ s\circ J\times det(g) \ , \\
&G[g,J] = T\circ J\circ T[J] = kT\circ J\circ J\times\sqrt{g} = T\circ T_{K,1} \ .
\end{aligned} \tag{17.3.4}
$$

The expression for G boils down to

$$
\begin{aligned}
G &= 4T_{K,1}\circ T_{K,1}+4T_{K,1}\circ T_{K,2}+T_{K,2}\circ T_{K,2} \\
&- kJ\circ s\circ J+KT_{K,1}+\frac{kK}{2}T_{1K} \\
&+ \frac{K^2}{4}g \ .
\end{aligned} \tag{17.3.5}
$$

The terms are quartic, quadratic, and zeroth order in J. One should disentangle these terms and be able to see whether the signature of G could be (1,1,-1,-1) in the vicinity of string world sheets. I have not been able to identify any obvious mechanism.

17.3.6 The necessity of Kac-Moody algebra of $SU(2)\times U(1)$

An interesting observation [B57] (see `http://tinyurl.com/hdy661t`) is that the central charge $c=3k/(k+2)$ emerges by Sugawara construction of the (Super-)Virasoro algebra for $SU(2)$ for (Super-)Kac-Moody algebra with central charge k.

1. In the general case one has following expressions for the central charge c and ground state weight h of the Super Virasoro algebra associated with Super-Kac-Moody algebra

$$
\begin{aligned}
c &= \frac{kdim(G)}{k+g} \ , \\
h(\lambda) &= \frac{C(\lambda)}{2(k+g)} \ .
\end{aligned} \tag{17.3.6}
$$

   C is Casimir operator in representation λ of G and g is the dual Coxeter number (half of the value of Casimir in fundamental representation).

2. If one accepts these formulas for c and h, the $\mathcal{N}=2$ SUSY fixes Kac-Moody group to be $SU(2)$ or possibly electroweak $SU(2)\times U(1)$ as physical intuition suggests. The value $c=3k_1/(k_1+1)$ requires $k=2k_1$ and $h=K/2$ gives $C(\lambda)=j(j+1)=2K(k_1+1)$.

3. What is the interpretation of $SU(2)$? Electroweak $SU(2)$ operating in fermionic electro-weak spin degrees of freedom is a natural candidate and would require and also allow the inclusion of also U(1) factor naturally identifiable as the U(1) charge of the $\mathcal{N}=2$ SCFT. In fact, the detailed study of Ramond representations show that U(1) factor must contribute to the ground state conformal weight in order to satisfy $h=K/2$ condition.

17.3.7 $h = K/2$ condition for Ramond representations

The question is whether $h = K/2$ suggested by the conformal invariance for the radial coordinate at light-like boundary can be achieved for these representations. Consider first Ramond type representations.

1. The condition on the allowed values $h = K/2$ of the ground state conformal weight gives

$$\begin{aligned} h_{ab} &= \tfrac{a(a+2)-b^2}{4(k+2)} + \tfrac{1}{8} = \tfrac{K}{2} \ , \quad 0 \leq a \leq k \ , \quad b \leq a+1 \ , \\ Q_{ab} &= \tfrac{b}{2(k+2)} - \tfrac{1}{4} \ . \end{aligned} \tag{17.3.7}$$

   Also the value of U(1) charge is given.

2. A possible manner to get rid of the problematic 1/8 term is to assume

$$-\frac{b^2}{4(k+2)} + \frac{1}{8} = 0 \tag{17.3.8}$$

   satisfied under the conditions

$$k = 2k_1 \ , \quad b^2 = k_1 + 1 \ . \tag{17.3.9}$$

   This fixes the spectrum of k_1 to values $0, 3, 8, 15, 24, 35, ...$ and non-negative integer b satisfying $|b-1| < a$ determines the value of k_1.

3. As a consequence, one obtains the condition

$$\frac{a(a+2)}{4(k+2)} = \frac{K}{2} \ . \tag{17.3.10}$$

   This condition can be satisfied if one has

$$a = k = K \ . \tag{17.3.11}$$

   Second option $a = k+2 = K-2$ does not satisfy the condition $a \leq k$.

4. Altogether one obtains

$$\begin{aligned} & k = 2k_1 \ , \quad k_1 = b^2 - 1 \ , \quad a = k = K \leq k \ , \\ & c = \tfrac{3k_1}{k_1+1} \ , \quad Q = \tfrac{1}{4}(\tfrac{1}{b} - 1) \ . \end{aligned} \tag{17.3.12}$$

   U(1) charge is quantized unless one as $b = 1$ giving $k_1 = 0$ so that one has also $k = 0$. One can ask whether the fractionization of U(1) charge could relate to the charge fractionization possibly related to the hierarchy of Planck constants and/or to the braid statistics. Should one require that physical states have integer charge? Could conformal confinement imply vanishing of ground state U(1) charge automatically? This is is true if complex conjugate conformal weights correspond to opposite U(1) charges.

It is interesting to see whether this picture is consistent with the predictions of the $SU(2)\times U(1)$ Kac-Moody algebra option.

1. Ramond option corresponds naturally to the half-odd integers spin for the Super-Kac-Moody associated with SU(2) as will be found. For physical reasons one can expect that also U(1) tensor factor is present and adds to the vacuum conformal weight. From the general expression of the conformal weight one expects that the term 1/8 is this contribution.

   This would suggests the condition in SU(2) degrees of freedom in terms of half odd integer spin $j=(2r+1)/2$

$$\frac{a(a+2)-b^2}{4(k+2)} = \frac{a(a+2)}{4(k+2)}-\frac{1}{8} = \frac{(2r+1)(2r+3)}{8(k+2)} = \frac{K}{2}-\frac{1}{8} \ . \tag{17.3.13}$$

   This gives the conditions

$$2a(a+2)-k+2 = (2r+1)(2r+3) \ , \quad \frac{(2r+1)(2r+3)}{k+2} = 4K-1 \ . \tag{17.3.14}$$

   This condition can be satisfied if $k+2$ divides the numerator - say $(2r+1)$ or $(2r+3)$. The conclusion is that the U(1) factor must be present, which in turn supports the interpretation in terms of gauge group of electroweak interactions and extended holonomy group of CP_2 needed to obtain respectable spinor structure.

17.3.8 $h=K/2$ condition for N-S type representations

One can look the situation also for the N-S type representations. In this case one expects that spin is even. It is rather clear that the interpretation is in terms of sfermions is not correct. Spin for N-S states is even, which encourages the interpretation as bosonic states involving fermion and antifermion at same or opposite throats of wormhole contact.

1. The values of ground state conformal weight and U(1) charge are assumed to be given by

$$\begin{aligned} h_{ab} &= \frac{a(a+2)-b^2}{4(k+2)} = \frac{K}{4} \ , \\ Q_{ab} &= \frac{b}{2(k+2)} \ . \end{aligned} \tag{17.3.15}$$

2. In the case of SU(2) Kac-Moody algebra one would have $h_{ab}=j(j+1)/2(k+2)$, which would give

$$a(a+2)-b^2 = 2j(j+1) \ , \quad \frac{j(j+1)}{k+2} = K \ . \tag{17.3.16}$$

   Two solutions of the latter equation are

   - $j=k+2$ giving $k=K-3$ and $j=K-1$
   - $j+1=k+2$ given $k=K-1$ and $j=K$.

The values of j are integers as expected.

3. The condition $a(a+2)-b^2=j(j+1)$ gives a further number theoretic constraint. Special solutions are $a=j-1, b=0$ and $a=j=b^2$.

To sum up, $\mathcal{N}=2$ superconformal theories provide an attractive approach in attempts to gain a more detailed understanding of the super-conformal invariance at string world sheets. The fermionic n-point functions as restricted to string world sheets in turn could correspond to n-point functions for a CFT assignable to partonic 2-surfaces and one should understand the relationship between these two CFTs. More generally, strong form of holography allows to except CFT description for both the spin and orbital degrees of freedom of WCW and one should understand their relationship. It must be however emphasized that the actual SCA in TGD corresponds to the number $\mathcal{N}=\forall$ of spin states for H-spinors. The corresponding space-time SUSY is expected to be badly broken.

Chapter 18

Does Riemann Zeta Code for Generic Coupling Constant Evolution?

18.1 Introduction

During years I have made several attempts to understand coupling evolution in TGD framework.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K45]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.

   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K108] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking

2. p-Adic mass calculations for 2 decades ago [K52] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity [L57]. Needless to say these attempts were premature and a hoc.

3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [**?**] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated

with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

Quite recently the number theoretic interpretation of coupling constant evolution [K104] [L16] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K99]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.

   Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.

   The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s = 2$. The trivial poles for $s = 2n$, $n = 1, 2, ..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff} = n \times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s = 2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?

   It turns out that at p-adic length scale $k = 131$ ($p \simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K104]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k = 127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w = w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `https://en.wikipedia.org/wiki/M\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{o\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\`

relax\let\ignorespaces\relax\accent127o\egroup\spacefactor\accent@spacefactorbius_transformation) with real coefficients (element of $GL(2,R)$) so that one as $\zeta_F((as+b)/(cs+d))$. Rather general arguments force it to be and element of $GL(2,Q)$, $GL(2,Z)$ or maybe even $SL(2,Z)$ $(ad-bc=1)$ satisfying additional constraints. Since TGD predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2,Z)$ and by a scaling factor K.

Could one understand the general qualitative features of color and weak coupling contant evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs+d$ and color confinement with the zero of $as+b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as+b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as+b/)(cs+d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p\simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a,b,c,d). In the sequel this vision is discussed in more detail.

18.2 Fermionic Zeta As Partition Function And Quantum Criticality

Riemann zeta has formal interpretation as a partition function $\zeta = Z_B = \prod 1/(1-p^s)$ for a gas of bosons with energies coming as integer multiples of $log(p)$, for given mode labelled by prime p. I have proposed different interpretation based on the fermionic zeta ζ_F based on its representation as a product

$$\zeta_F = \prod_p (1+p^s)$$

of single fermion partition functions associated with fermions with energy $log(p)$ (by Fermi statistics the fermion number is 0 or 1). In this framework the *poles* (not zeros!) of the fermionic zeta $\zeta_F(ks)=\zeta(ks)/\zeta(2ks)$ (the value of k turns out to be $k=1/2$) (this identity is trivial to deduce) correspond to $s/2$, where s is either trivial or non-trivial zero of zeta (denominator), or the pole of zeta at $s=1$ (numerator). Trivial poles are negative integers $s=-1-2,-3...$ suggesting an interpretation as conformal weights. This interpretation is proposed also for the nontrivial poles.

ζ_F emerges naturally in TGD, where the only fundamental (to be distinguished from elementary) particles are fermions. The assignment of physics to *poles* rather than zeros of ζ_F is also natural. The interpretation inspired by the structure of super-symplectic algebra is as conformal weights associated with the representations of extended super-conformal symmetry associated with super-symplectic algebra defining symmetries of TGD at the level of "World of Classical Worlds" (WCW).

"Conformal confinement" states that the sum of conformal weights of particles in given state is real. I discovered the idea for decade ago but gave it up to end up with it again. The fractal structure of superconformal algebra conforms with quantum criticality: infinite hierarchy of symmetry breakings to sub-symmetry isomorphic to original one! The conformal structure is infinitely richer than the ordinary one since the algebra in question has infinite number of generating elements labelled by all zeros of zeta rather than a handful of conformal weights ($n=-2,...+2$ for Virasoro algebra). Kind of Mandelbrot fractal is in question. There is however deviation from the ordinary conformal symmetry since real conformal weights can have only one sign (for generating elements all negative conformal weights $n = -1, -2, -.$ are realized as poles of $1/\zeta(2s)$ but $n = 1$ realized as pole of $\zeta(s)$ is the only positive conformal weight). Situation is therefore not quite identical with that in conformal field theories although also conformal field theories realizes only positive conformal weights (positivity is a convention) and have also some tachyonic conformal weights which are negative.

The problem of all attempts to interpret zeros of zeta relates to the fact that zeros are *not* purely imaginary but possess the troublesome real part $Re(s) = 1/2$. This led me to consider coherent states instead of eigenstates of Hamiltonian in my proposal, which I christened a strategy for proving Riemann hypothesis [K81], [L3]. Zeta has phase at the critical line so the interpretation as a partition function can be only formal. So called Z function defined at critical line and obtained by extracting the phase of zeta out, is real at critical line.

In TGD framework the solution of these problems is provided by zero energy ontology (ZEO). Quantum theory is "complex square root" of thermodynamics and means that partition function becomes a complex entity having also a phase. The well-known function

$$\xi(s) = \frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(s/2)(\zeta(s)$$

assignable to Riemann zeta having same zeros and basic symmetries has at critical line phase equal ± 1 except at zeros where the phase can be defined only as a limit depending the direction from which the zero is approached. Fermionic partion function $\zeta_F(s)$ has a complex phase and it is not clear whether it makes sense to assign with it the analog of $\xi(s)$. Ordinary partition function is modulus squared for the generalized partition function.

Why does the partition function interpretation does demand poles?

1. In ordinary thermodynamics the vanishing of partition function makes sense only at the limit of zero temperature when all Boltzmann weights approach to zero. By subtracting the energy of the lowest energy state from the energies the partition function becomes non-vanishing also in this case. Hence the idea that partition function vanishes does not look very attractive. The varying sign is even worse problem.

2. Since the temperature interpreted as $1/s$ in the partition function is not infinite could mean that one has analog of Hagedorn temperature (see `http://tinyurl.com/pvkbrum`): the degeneracy of states increases exponentially with temperature and at Hagedorn temperature compensates the s exponential decreases of Boltzmann weights so that partition function is sum of infinite number of terms approaching to unity. Hagedorn temperature relates by strong form of holography to magnetic flux tubes behaving as strings with infinite number of degrees of freedom. One would have quantum critical system possessing supersymplectic symmetry and other superconformal symmetries predicted by TGD [K24, K23, L10].

3. The temperature is complex for non-trivial zeros. This requires a generalization of thermodynamics by making partition function complex. Modulus squared of this function takes the role of an ordinary partition function. One can allow in the case of Kähler action the replacement of argument s with $ks + b$ without giving up the basic features of U(1) coupling constant evolution. Here one can allow rational numbers k and b. The inverse temperature for $\zeta_F(ks+b)$ is identified as $\beta = 1/T = k(s+b)$. It turns out that in the model for coupling constant evolution the scaling factor $k = 1/2$ is required. b is not completely fixed.

   Complex temperature is indeed the natural quantity to consider in ZEO. The real part of temperature at critical line equals to $Re(\beta) = (s + b)/4k$, with b rational or integer for $\zeta_F(w = k(s+b))$ at poles assignable with the zeros of $\zeta(2k(s+b))$ in denominator. Imaginary part

$$Im\,[\beta] = \frac{1}{T} = \frac{1}{2k}(b + \ frac12 + iy) \qquad (18.2.1)$$

   of the inverse temperature does not depend on b. Infinite number of critical temperatures is predicted and a discrete coupling constant evolution takes place already at the level of basic quantum TGD rather than emerging only at the QFT limit - I have also considered the possibility that coupling constant evolution emerges at the QFT limit only [L57]. One could even allow Möbius transformation with real coefficients in the argument of ζ_F and that this could allow the understanding of the evolutions of weak and colour coupling constants.

$\zeta_F(w)$ at $s = -(n-b)/k$ are also present. For $s = 1/T$ they would correspond to negative temperatures $\beta = (-n+b)/k$? In the real context and for Hamiltonian with a fixed sign this looks weird. Preferred extremals can be however dominated by either electric or magnetic fields and the sign of the action density depends on this.

4. Interestingly, in p-adic thermodynamics p-adic temperatures has just the values $T = -1/n$ if one defines p-adic Boltzmann weight as $exp(-E/T) \rightarrow p^{-E/T}$, with $E = n \geq 0$ conformal weight. The condition that weight approaches zero requires that T identified in this is as real integer negative for p-adic thermodynamics! Trivial poles would correspond to p-adic thermodynamics and non-trivial poles to ordinary real thermodynamics! Note that the earlier convention is that $T = 1/n$ is positive: the change of the sign is just a convention. Could the hierarchy of p-adic thermodynamics labelled by p-adic primes corresponds to the sequence of critical zeros of zeta? Number theoretic vision indeed leads to this proposal [L16], [K104].

   The factor $1/(1-p^n)$ at the real poles $s = -2n$ would exist p-adically in p-adic number field Q_p so that the factors of zeta would correspond to adelic decomposition of the partition function. At critical line in turn $1/1+p^{1/2+iy}$ would exist for zeros y for which p^{iy} is root of unity (note that $p^{1/2}$ is somewhat problematic for Q_p: does it make sense to speak about an extension of Q_p containing *sqrtp* or is the extension just the same p-adic number field but with different definition of norm?). That p^{iy} is root of unity for some set $C(p)$ of zeros y associated with p was proposed in [L16], [K104]. Now $C(p)$ would consist of single zero $y = y(p)$.

18.2.1 Could The Spectrum Of Kähler Couplings Strength Correspond To Poles Of $\zeta_F(s/2)$?

The idea that the spectrum of conformal weights for supersymplectic algebra is given by the poles of ζ_F is not new [L16].

Poles of $\zeta_F(ks)$ ($k = /2$ turns out to be the correct choice) have also interpretation as complexified temperatures. Kähler action can be interpreted as a complexified partition function and the inverse $1/\alpha_K$ of Kähler coupling appears in the role of critical inverse temperature β. The original hypothesis was that Kähler coupling strength has only single value. The hierarchy of quantum criticalities and its assignment with number theoretical hierarchy of algebraic extensions of rationals led to consider the possibility that Kähler coupling strength has a spectrum corresponding to a hierarchy of critical temperatures. Quantum criticality and Hagedorn temperature for magnetic flux tubes as string like objects are indeed key elements of TGD.

The hypothesis to be studied is that the values $1/\alpha_K$ correspond to poles of

$$\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$$

with the identification $1/\alpha_K = ks$. The model for coupling constant evolution however favors $k = 1/2$ predicting that poles correspond to zeros of zeta in the denominator of ζ_F and $s = 2$ in its numerator. For $k = 1/2$ only even negative integers would appear in the spectrum and there would be pole at $s = 2$. Here one onr also allow the sift $ks \rightarrow ks + b$, b integer without shifting the imaginary parts of poles crucial for the coupling constant evolution. This induces a shift $Re[s] \rightarrow kRe[s] + b$ for the real parts of poles.

For nontrivial poles this requires the replacement of temperature with a complex temperature. Therefore also $1/\alpha_K$ becomes complex. This is just what the ZEO inspired idea about quantum theory as complex square root of thermodynamics suggests. Kähler action is also complex already for real values of $1/\alpha_K$ since Euclidian *resp.* Minkowskian regions give real/imaginary contribution to the Kähler action.

The poles of ζ_F would appear both as spectrum of complex critical temperatures $\beta = 1/T = 1/\alpha_K$ and as spectrum of supersymplectic conformal weights. ζ_F is complex along the critical line containing the complex poles. This makes sense only in ZEO. ξ function associated with ζ is real at critical line but the problems are vanishing at finite temperature, indefinite sign, and also the fact that partition function interpretation fails at positive real axis. This does not conform with the intuitive picture about partition function defined in terms of Boltzmann weights.

18.2.2 The Identification Of $1/\alpha_K$ As Inverse Temperature Identified As Pole Of ζ_F

Let us list the general assumptions of the model based on the identification of $1/\alpha_K$ as a complexified inverse temperature in turn identified as zero of ζ_F.

1. I have earlier considered the number theoretical vision based on the assumption that vacuum functional identified as exponent of Kähler action receiving real/imaginary contributions from Euclidian/Minkowskian space-time regions exists simultaneously in all number fields. This is in spirit with the idea of integrability meaning that functional integral reduces to a sum over exponents of Kähler action associated with stationary points. What is nice that by the Kähler property of WCW metric Gaussian and metric determinants cancel [K45, K104] and one indeed obtains a discrete sum over exponentials making sense also in p-adic sectors, where ordinary integration does not make sense. Number theoretic universality is realized if one allows the extension of rationals containing also some roots of e if the exponent reduces to a product of root of unity and product of rational powers of e (e^p is ordinary p-adic number) and integer powers of primes p. It is perhaps needless to emphasize the importance of this result.

   The criticism is obvious: how does one know, which preferred extremals have a number theoretically universal action exponent? For calculational purposes it might not be necessary to know this. The easy option would be that all preferred extremals are number theoretically universal: this cannot be however the case if the values of $1/\alpha_K$ correspond to zeros of ζ. Second option is that in the sum over preferred extremals those which do not have a number theoretically universal exponent give a vanishing net contribution and are effectively absent. The situation brings in mind the reduction of momentum spectrum of a particle in a box to momenta equal to $k = n2\pi/L$, L the length of the box. The contributions of other plane waves in integrals vanish since they are dropped away by boundary conditions.

   Strong form of number theoretic universality requires that the exponent of Kähler action reduces to a product of rational power of some prime p or $e^{m/n}$ and a root of unity [K104], [L16]. This might be too strong a condition and weaker condition allows also powers of p mapped to real sector and vice versa by canonical identification. One could pose root of unity condition for the phase of $exp(S_K)$ as a boundary condition at the ends of causal diamond (CD) stating that some integer power of the exponent of Kähler action for the given value of α_K is real. If $exp(K)$ contains $e^{m/n}$ factor but no p^n factors, the reality of the n^{th} power of $exp(i\pi K)$ would reveal this. Single p^n factor in absence of $e^{m/n}$ factor could be detected by requiring that the exponent $exp(iyK)$ is real for some y (imaginary part of zero of zeta with p^{iy} a root of unity).

2. The assumption that $1/\alpha_K$ corresponds to a nontrivial zero of zeta has strong constraints on the values of the reduced Kähler action $S_{K,red} = \alpha_K S_K$ for which the classical field equations do not depend on α_K at all. The reason is that the S_K must be proposal to $1/\alpha_K$ to achieve number theoretical universality. Number theoretical universality thus implies that preferred extremals depend on $1/\alpha_K$ - this is something very quantal. The proportionality $1/\alpha_K$ to $h_{eff} = n \times h$ is highly suggestive. It does not destroy number theoretical universality for given preferred extremal.

3. $1/\alpha_K$ has form $1/\alpha_K = s = a+ib = (1/2k)(1/2+iy/2)$ for nontrivial poles, $1/\alpha_K = s = -n/k$ for trivial poles of $1/\zeta(2s)$, and $1/\alpha_K = s = 1/k$ for the pole of ζ. $k = 1/2$ is the physically preferred choice.

   Kähler action can be written as a sum of Euclidian and Minkowskian contributions: $K = K_E + iK_M$. For non-trivial poles in the case of $1/\alpha_K = ks$ one has

$$K = s\times (K_E+iK_M) = \frac{1}{k}\times \left[\frac{K_E}{2} - yK_M + i(\frac{K_M}{2}+yK_E)\right] \quad . \tag{18.2.2}$$

Here $K_{red} = K_E+iK_M$ is *reduced* Kähler action. This option generalizes directly the original proposal.

4. For trivial poles $s = -n/k$ and $s = 1/k$ one has

$$K = \frac{s}{k}\times K_{red} = \frac{s}{k}\times (K_E+iK_M) \quad . \tag{18.2.3}$$

5. For real poles universality holds true without additional conditions since the multiplication of $1/\alpha_K$ by the scaling factor $-n_2/n_1$ does not spoil number theoretical universality. One can of course consider this condition. It predicts that the K_{red} is scaled by n_1/n_2 in the transition $n_2\rightarrow n_1$. For nontrivial poles K_{red} is scaled by the complex ratio s_2/s_1.

   An attractive possibility is that the hierarchy of Planck constants corresponds to this RG evolution. n would correspond to the number of sheets in the n-sheeted covering for which sheets co-incide at the ends of space-time at the boundaries of CD. Therefore p-adic and $h_{eff} = n\times h$ hierarchies would find a natural interpretation in terms of zeros of ζ_F. To avoid confusion let us make clear that the values of $n = h_{eff}/h$ would not correspond to trivial poles.

Number theoretical universality could be realized in terms of RG invariance leaving the vacuum functional invariant but deforming the vacuum extremal. The hierarchy of Planck constants and p-adic length scale hierarchy could be interpreted as RG flows along real axis and critical line.

1. The grouping of poles to 4 RG orbits corresponding to non-trivial poles $y>0$ and $y<0$, to poles $s=-n/k<0$, and $s=1/k$ looks natural. The differential equations for RG evolution of Kähler action would be replaced with a difference equation relating the values of Kähler action for two subsequent critical poles of ζ_F.

2. Number theoretical universality allows to relate Minkowskian and Euclidian contributions K_M and K_E to each other. Earlier I have not even tried to deduce any correlation between them although the boundary conditions at light-like wormhole throats at which the signature of the induced metric changes, probably give strong constraints.

   The strongest form of the number theoretical universality condition assumes

$$K_{red} = K_{red,E} + iK_{red,M} = \alpha_K K_1 = \frac{K_1}{s} = K(\alpha_K=1) \ \ , \ \ s=\frac{1}{\alpha_K} \quad . \tag{18.2.4}$$

   K_1 satisfies the number theoretic universality meaning that $exp(K_1) = expK(\alpha_K = 1)$ reduces to a product of powers primes, root of e and root of unity.

   This ansatz has the very remarkable property that α_K disappears from the vacuum functional completely so that the RG action can be regarded as a symmetry leaving vacuum function invariant. This operation however changes the preferred extremal and reduced Kähler action so that the situation is non-classical. RG orbit would start from the pole $s=1$ and contain complex poles.

3. The large CP breaking suggested by complexity of α_K would disappear at the level of vacuum functional and appears only at the level of preferred extremals. If this is to conform with the quantum classical correspondence, correlation functions, which must break CP symmetry receive this breaking from preferred extremals. $s=1/2k$ and complex poles belong to the same orbit. This ansatz is not necessary for poles $s=1/k$ and $s=-n/k$ for which number theoretic universality conditions are satisfied irrespective of the value of s.

4. A more realistic looking solution is obtained by assuming that complex poles correspond to separate orbit or even that positive and negative values of y correspond to separate orbits. RG flow would begin from the lowest zero of zeta at either side of real axis. This gives

$$K_{red} = \frac{\alpha_K}{\alpha_{K,0}} \times K_{red}(\alpha_{K,0}) \quad . \tag{18.2.5}$$

Also now the vacuum functional is invariant and preferred extremal changes in RG evolution. In accordance with quantum classical correspondence one has however a breaking of CP symmetry also at the level of vacuum functional due to the complexity of $\alpha_{K,0}$ unless $K_{red}(\alpha_{K,0})$ is proportional to $\alpha_{K,0}$.

Remark: The above arguments must be modified if one includes to the action cosmological volume term strongly suggested by twistor lift of TGD.

18.3 About Coupling Constant Evolution

p-Adic mass calculations inspired the hypothesis that the continuous coupling constant evolution in QFTs reduces in TGD framework to a discrete p-adic coupling constant evolution but assuming that α_K is absolute RG invariant. Therefore the hypothesis that the evolution of $1/\alpha_K$ defined by the non-trivial poles of ζ_F corresponds to the p-adic coupling constant evolution deserves a serious consideration.

1. p-Adic length scale hypothesis in the strong form states that primes $p \simeq 2^k$, k prime, correspond to physically preferred p-adic length scales. This would suggest that non-trivial zeros $s_1, s_2, s_3, ..$ taken in increasing order for magnitude correspond to primes $k = 2, 3, 5, 7...$ as suggested also in [L16], [K104]. This allows to assign to each zero s_n a unique prime: $p \leftrightarrow y(p)$ and this suggests more precise of the earlier hypothesis to state that $p^{iy(p)}$ is root of unity. The class of zeros associated with p would contain single zero.

   Discrete p-adic length scale evolution would thus correspond to the evolution of non-trivial zeros. The evolution associated with the hierarchy of Planck constants could only multiple Kähler action with integer. To make this more concrete one must consider detailed physical interpretation.

2. $1/\alpha_K$ corresponds to $U(1)$ coupling of standard model: $\alpha_K = \alpha(U(1)) \equiv 1/\alpha_1$. Kähler action could be seen as analogous to a Hamiltonian associated with electroweak U(1) symmetry. $U(1)$ gauge theory is not asymptotically free and this correspond to the fact that $Im(1/\alpha_K) = y$ approaches in UV to the lowest zero $y = 14.12...$ In IR y diverges, which conforms with $U(1)$ gauge theory symmetry.

   Electromagnetic coupling corresponds to

$$\frac{1}{\alpha_{em}} = \frac{1}{\alpha_K cos^2(\theta_W)} \quad . \tag{18.3.1}$$

   The challenge is to understand also the evolution of $cos^2(\theta_W)$ allowing in turn to understand the entire electroweak evolution.

3. The values of electroweak couplings at the length scale of electron ($k = 127$ or at 4 times longer length scale $k = 131$ ($L(131) = .1$ Angstrom) are well-known and this provides a killer test for the model. Depending on whether one assumes fine structure constant to correspond to $L(127)$ associated with electron or to 4 times long length scale $L(131)$ one has too options. $L(131)$ allows to reproduce fine structure constant with a value of $p = sin^2(\theta_W)$ deviating only .7 per cent from its measured value in this length scale! If this is not a mere nasty

accident, Riemann zeta might code the entire electroweak physics and perhaps even strong interactions!

The first guess is that UV asymptotia for the Weinberg angle is same as for GUTS: $p \rightarrow 3/8$ for $p = 2$ giving $1/\alpha_{em} \rightarrow 22.61556016$. IR asymptotia corresponds to $p \rightarrow 0$ implying $1/\alpha_{em} = 1/\alpha_K$. Notice that the evolution is rather fast in extreme UV. In extreme IR it becomes slow. It turns out that the UV behavior of Weinberg angle does not conform with this naïve expectation.

4. Since p-adic length scale is proportional to $1/p^{1/2}$ it is enough to obtain RG evolution for coupling constnt as function of p. One obtains reasonably accurate understanding about the evolution by deducing an estimate for pdy/dp . This is obtained as $pdy/dp = (dy/dN)(dN/dk)p(dk/dp)$.

   - $p \simeq 2^k$ implies $k \simeq log(p)/log(2)$ and $pdk/dp \simeq 1/log(2)$.
   - The approximate formula for the number $N(y)$ of zeros smaller than y is given by

   $$N(y) \sim u \times log(u) \ \ , \ \ u = \frac{y}{2\pi}$$

   giving

   $$\frac{dN}{dy} \sim \frac{1}{2\pi} \times (log(u) - 1), \ \ u = \frac{y}{2\pi} \ \ .$$

   - The number $\pi(k)$ of primes smaller than k is given by

   $$N(k) \sim \frac{k}{log(k)}$$

   giving

   $$\frac{dN(y)}{dk} \sim \frac{1}{log(k)} - \frac{1}{log(k)^2} \ \ .$$

   By combining the formulas, one obtains

   $$p\frac{dy}{dp} = \beta = \frac{2\pi}{log(2)} \times (\frac{1}{log(y/2\pi)} - 1) \times (\frac{1}{log(k)} - \frac{1}{log(k)^2}) \ \ , \ \ k = \frac{log(p)}{log(2)} \ \ . \tag{18.3.2}$$

   The beta function for the evolution as function of p-adic length scale differs by factor 2 from this one. Note that also double logarithms appear in the formula. Note that beta function depends on y logarithmically making the equation rather nonlinear. This dependence can be shifted to the left hand side and by replacing y which appropriation chosen function of it one obtains

   $$p\frac{dN(y)}{dp} = \beta_1 = \frac{1}{log(k)} - \frac{1}{log(k)^2} \ \ , \ \ k = \frac{log(p)}{log(2)} \ \ . \tag{18.3.3}$$

5. Coupling constant evolution would take place at the level of single space-time sheet. Observations involve averaging over space-time sheet sizes characterized by p-adic length scales so that a direct comparison with experimental facts is not quite easy and requires a concrete statistical model.

The entire electroweak $U(1)$ coupling constant evolution would be predicted exactly from number theory. Physics would represent mathematics rather than vice versa. Concerning experimental testing a couple of remarks are in order.

1. An open question is how much many-sheetedness of space-time affects situation: one expects kind of statistical average of say Weinberg angles over p-adic length scales coming from a superposition over space-time sheets of many-sheeted space-time. Space-time with single sheet is not easy to construct experimentally although mathematically it is extremely simple system as compared to the space-time of GRT.

2. The discreteness of the coupling constant evolution at fundamental level is one testable prediction. There is no continuous flow but sequence of phases with fixed point behavior with discrete phase transitions between them. At QFT limit one expects that continuous coupling constant evolution emerges is statistical average.

3. Later it will be found that the entire electroweak evolution can be predicted and this prediction is certainly testable.

18.3.1 General Description Of Coupling Strengths In Terms Of Complex Square Root Of Thermodynamics

The above picture is unsatisfactory in the sense that it says nothing about the evolution of other electroweak couplings and of color coupling strength. Does number theory fix also them rather than only $U(1)$ coupling? And what about color coupling strength α_s?

Here quantum TGD as a complex square root of thermodynamics vision helps.

1. Kähler action reduces for preferred extremals to Abelian Chern-Simons action localized at the ends of space-time surfaces at boundaries of causal diamond (CD) and possibly contains terms also at light-like orbits of partonic 2-surfaces. This corresponds to almost topological QFT property of TGD.

2. Kähler action contains additional boundary terms which serve as analogs for Lagrangian multipler terms fixing the numbers of various particles in thermodynamics. Now they fix the values of isometry charges for instance, or force the symplectic charges for a sub-algebra to vanish.

   Lagrangian multiplies can be written in the form μ_i/T in ordinary thermodynamics: μ_i denotes the chemical potentials assignable to particle of type i. Number theoretical universality strongly favors similar representation now. For instance, this would give

$$\frac{1}{\alpha_{em}} = \frac{\mu_{em}}{\alpha_K} \ , \ \mu_{em} = \frac{1}{cos^2(\theta_W)} \ . \tag{18.3.4}$$

   In the same manner SU(2) coupling strength given by

$$\frac{1}{\alpha_W} = \frac{\mu_W}{\alpha_K} = \frac{cot^2(\theta_W)}{\alpha_K} \tag{18.3.5}$$

   would define $cot^2(\theta_W)$ as analog of chemical potential.

3. In the case of weak interactions Chern-Simons term for induced SU(2) gauge potentials as a boundary term would be the analog of Kähler action having interpretation as Lagrangian multiplier term. In color degrees of freedom also an analog of Chern-Simons term would be in question and would be associated with the classical color gauge field defined by H_AJ, where H_A is Hamiltonian of color isometry in CP_2 and J is induced Kähler form.

4. The conditions for number theoretical universality would become more complex as also RG invariance interpreted in terms of number theoretical universality.

This picture assuming a linear relationship between generic coupling strength α and α_K in terms of chemical potential is not yet general enough.

18.3.2 Does ζ_F With $GL(2,Q)$ Transformed Argument Dictate The Evolution Of Other Couplings?

It seems that one cannot avoid dynamics totally. The dynamics at (quantum) criticality is however universal. This raises the hope that the evolution of coupling constant is universal and does not depend on the details of the dynamics at all. This could also explain the marvellous successes of QED and standard model

At criticality the dynamics reduces to conformal invariance by quantum criticality, and this inspires the idea about the values of coupling constant strength as poles of a meromorphic function obtained from ζ_F by a conformal transformation of the argument. After all, what one must understand is the relationship between $1/\alpha_W$ and $1/\alpha_K$, and the linear relationship between them can be seen as a simplifying assumption and an approximation.

The values of generic coupling strength - call it just α (to be not confused with α_{em}) without specifying the interaction - would still correspond to poles of $\zeta_F(s)$ but with a transformed argument s. Conformal transformation would relate various coupling constant evolutions to each other and allow to combine them together in a unique manner. Discreteness is of course absolutely essential. The analysis of the situation leads to a surprisingly simple picture about the coupling constant evolutions for weak and color coupling strengths.

1. By the symmetry of ζ_F under the reflection with respect to x-axis one can restrict the consideration to globally defined conformal transformations of the upper half plane identifiable as Möbius tranformations $w = (as+b)/(cs+d)$ with the real matrix coefficients (a,b,c,d). One can express the transformation as a product of an overall scaling by factor k and $GL(2,R)$ transformation with $ad-bc=1$. Number theoretical universality demands that k and the coefficients a,b,c,d of $GL(2,R)$ matrix are real rationals. Number theoretically $GL(2,Q)$ is attractive and one can consider also the possibility that the transformation matrix $GL(2,Z)$ matrix with a,b,c,d integers. $SL(2,Z)$ is probably too restrictive option.

2. The Möbius transformation $w = (as+b)/(cs+d)$ acts on zeros of ζ mapping the discrete coupling constant evolution for $1/\alpha_K$ to that for $1/\alpha_W$ or $1/\alpha_s$. The transformed coupling constant depends logarithmically on p-adic length scale via $1/\alpha_K$ supporting the interpretation in terms of RG flow induced by that for $1/\alpha_K$ - something very natural since Kähler action is in special role in TGD framework since it determines the dynamics of preferred extremals.

3. Asymptotically (long length scales) one has $w \to a/c$ for $a \neq 0$ so that both at critical line and real axis one has accumulation of critical points to $w = a/c$! Thus for the option allowing only very large value of coupling strength in IR one has

$$w = K \times \frac{as+b}{cs+d} \quad , \quad ad-bc = 1 \quad \text{(Option 1)} \quad . \tag{18.3.6}$$

$a/c = 0$ ($a = 0$) corresponds to a diverging coupling strength (for color interactions and for weak interactions for vanishing Weinberg angle) and corresponds to $w = K \times b/cs + d$. $ad - bc = 1$ gives $b = -c = 1$ and if one accepts the IR divergence of coupling constant, one has

$$w = \frac{K}{-s+d} \quad \text{(Option 2)} \quad . \tag{18.3.7}$$

The only free parameters are the rational $K>0$ and integer d. w has pole at $s=d$ mapped to 1 by ζ_F.

To gain physical insight consider the situation at real axes.

1. The real poles $s=-n/k$ and $s=1/k$ are mapped to poles on real axes and the reflection symmetry with respect to x-axis is respected. Negative poles would be thus mapped to negative poles for $d \in 0,1$ and $k<0$. One could also require that the pole $s=1$ is mapped to positive pole. For option 2 it is mapped to $w=+\infty$.

2. For option 1 this is true if one has $cs+d<0$ and $as+b>0$. The other manner to satisfy the conditions is $cs+d>0$ and $as+b<0$ for $s=-1,-2,\ldots$. By replacing the (a,b,c,d) with $(-a,-b,-c,-d)$ these conditions can be transformed to each other so that it is enough to consider the first conditions. The first form of the condition requires $c>0$ and $a<0$.

   The condition that $s=1/k$ goes to a positive pole gives $c/k+d>0$ and $a/k+b>0$. Altogether this gives for the two Options the conditions

$$\begin{aligned} w &= K\times\frac{as+b}{cs+d}<0 \ , \\ k &> 0 \ , a<0 \ , \ c>0 \ , \ \frac{c}{k}+d>0 \ , \ \frac{a}{k}+b>0 \ . \ \text{(Option 1)} \ , \end{aligned} \tag{18.3.8}$$

   and

$$w = \frac{K}{-s+\frac{1}{k}}<0 \ , \ k>0 \ . \ \text{(Option 2)} \tag{18.3.9}$$

3. For option 2 $s=1/k$ phase is mapped to $w=+\infty$. Coupling strength vanishes in this phase: this brings in mind the asymptotic freedom for QCD realized at extreme UV? In long scales α would behave like $1/\alpha_K$ and diverge suggesting that Option 2 provides at least an idealized description of QCD. The scaling parameter K would remain the only free parameter.

   For option 1 α can become arbitrary large in long scales but remains finite. The analog of asymptotically free phase is replaced with that having non-vanishing inverse coupling strength $w=(a+b)/(c+d)$. The interpretation could be in terms of weak coupling constant evolution. The non-vanishing of the parameter a would distinguish between weak and strong coupling constant evolution.

   By feeding in information about the evolution of weak and color coupling strengths, one can deduce information about the values of K and a.

Whether the analogs of weak and Chern-Simons actions can satisfy the number theoretical universality, when the transformation is non-linear is far from obvious since the induced gauge fields are not independent.

18.3.3 Questions About Coupling Constant Evolution

The simplest hypothesis conforming with the general form of Yang-Mills action is $1/\alpha_K=s$, where s is zero of zeta. With the identification $1/\alpha_K=1/\alpha_{U(1)}$ this predicts the evolution of U(1) coupling and one obtains excellent prediction in p-adic length scale $k=131$ ($L(131\simeq 10^{-11}$ meters).

How general is the formula for $1/\alpha_K$?

Is the simplest linear form for $1/\alpha_K$ general enough? Consider first the most general form of $2\pi/\alpha_K$ taking as input the fact that its imaginary is equal to $1/\alpha_{U(1)}$ and corresponds to imaginary part y of zero of zeta at critical line.

Linear Möbius transformations $w = (as+b)/d$ with real coefficients do not affect $Im[s]$ and therfore the inverse of the imaginary part of the Kähler coupling strength which corresponds to th inverse of the measured U(1) coupling strength. The general formula for complex Kähler coupling strength would be

$$w = s + \frac{b}{d} \tag{18.3.10}$$

in the case of $SL(2,Q)$ giving $Re[1/\alpha_K] = 1/2 + b/d$. This would correspond to the analog of the inverse temperature appearing in the real exponent of Kähler function. For $SL(2,Z)$ on obtains

$$w = s + b \ \ , \ \ b \in Z \ \ . \tag{18.3.11}$$

This gives $Re[1/\alpha_K] = 1/2 + b$.

Does the reduction to Chern-Simons term give constraints

The coefficient of non-Abelian Chern-Simons action is quantized to integer and one can wonder whether this has any implications in TGD framework.

1. The Minkowskian term in Kähler action reduces to Abelian Chern-Simons term for Kähler action. In non-Abelian case the coefficient of Chern-Simons action (see `http://tinyurl.com/y7nfaj67`) is $k_1/4\pi$, where k_1 is integer.

   In Abelian case the triviality of gauge transformations does not give any condition on the phase factor so that in principle no conditions are obtained. One can however look what this condition gives. The coefficient of Chern-Simons term coming from in Kähler action is $1/(8\pi\alpha_K)$. For non-Abelian Chern-Simons theory with n fermions one obtains action $k \rightarrow k - n/2$. Depending on gauge group k_1 can vanish modulo 2 or 4. For the zeros at the real axes this would give the condition

   $$\frac{s}{2} = s + \frac{b}{d} = Re[\frac{1}{\alpha_K}] = 2k_1 \ \ , \ \ s = -2n < 0 \ \text{ or } \ s = 2 \ \ , \tag{18.3.12}$$

   which is identically satisfied for integer valued b/d. Thus it seems that $SL(2,Z)$ is forced by the Chern-Simons argument in the case of Kähler action, which is however not too convincing for U(1).

   For non-trivial zeros it is not at all clear whether one certainly cannot apply the condition since there is also a contribution yS_E to the imaginary part. In any case, the condition would be

   $$\frac{Re[s]}{2} = 1/2 + \frac{b}{d} = Re[\frac{1}{\alpha_K}] = 2k_1 \ \ . \tag{18.3.13}$$

   b/d must be half odd integer to satisfy the condition so that one would have $SL(2,Z)$ instead of $SL(2,Q)$. This is however in conflict with the Chern-Simons condition at real axis.

2. $w = s + b/d$ implies that the trivial poles $s = -2n$, $n > 0$, at the real axes are shifted to $s = -2n + b/d$ and become fractional. The poles at $s = 2$ is shifted to $2 + b/d$.

In the non-Abelian case one expects also Chern-Simons term but now emerging as an analog of Lagrange multiplier term rather than fundamental action reducing to Chern-Simons term. For $w = (as+b)(cs+d)$ the poles at real axis are mapped to rational numbers $w = (am+b)/(cm+d)$, $m = -2n$ or $m = 2$. Chern-Simons action would suggest integers. Gauge transformations would transform the action by a phase which is a root of unity. Vacuum functional is ZEO an analog of wave function as a square root of action exponential. Can one allow the wave function to be a finitely-many valued section in bundle?

Does the evolution along real axis corresponds to a confining or topological phase?

At real axis the imaginary part of s vanishes. Since it corresponds to the inverse of the gauge coupling strength, one can ask whether the proper interpretation is in terms of confining phase in which gauge coupling is literally infinite and it does not make sense to speak of perturbation theory. Instead one would have a phase in which Minkowski part of the Kähler action contributes only to the imaginary Chern-Simons term but not to the real part of the action. Topological QFT also based on Chern-Simons action also suggests itself.

The vanishing of gauge coupling strength is not a catastrophe now since the real part is non-vanishing. What looks strange that this phase is obtained also for Kähler coupling strength. Could this interpreted in terms of the fact that induced gauge potentials are not independent dynamical degrees of freedom but expressible in terms of CP_2 coordinates.

The spectrum of $1/\alpha_K$ at real axis has the $-2n + \frac{b}{d}$ and $2 + \frac{b}{d}$ and is integer or half-odd integer valued by the conditions on Chern-Simons action. One could make the entire spectrum integer value by a proper choice of b/d.

Integer valuedness forced by Chern-Simons condition leads to ask whether the situation could relate to hierarchy of Planck constants. This cannot be the case. One can assign to each value of y p-adic coupling constant labelled by prime k ($p \simeq 2^k$) a hierarchy of Planck constants $h_{eff} = n \times h$. If number theoretical universality is realized for $n = 1$, it is realized for all values of n and one can say that one has $1/\alpha = n/\alpha$ fora generic coupling strength α.

p-Adic temperature $T = 1/n$ using $log(p)$ as a unit correspond to the temperature parameter defined by α_K: the values of both are inverse integers. p-Adic thermodynamics might therefore provide a proper description for the confining phase as also the success of p-adic mass calculations encourages to think.

The sign of $1/\alpha_K$ is not fixed for the simplest option. The shift by $\frac{b}{d}$ could fix the sign to be negative. There is however no absolute need for a fixed sign since in Minkowskian regions the sign of Kähler action density depends on whether magnetic or electric fields dominate. In Euclidian regions the sign is always positive. Since the real part of Kähler action receives contributions from both Euclidian and Minkowskian regions it can can well have both signs so that for preferred extremals the signs of the real part of Kähler coupling strength and proper Kähler action compensate each other.

18.4 A Model For Electroweak Coupling Constant Evolution

In the following a model for electroweak coupling constant evolution using as inputs Weinberg angle at p-adic length scale $k = 127$ of electron or at four times longer scale $k = 131$ and in weak length scale $k = 89$ is developed.

18.4.1 Evolution Of Weinberg Angle

Concerning the electroweak theory, a key question is whether the notion of Weinberg angle still makes sense or whether one must somehow generalize the notion. Experimental data plus the prediction for $1/\alpha_{U(1)}$ as zero of zeta suggest that Weinberg angle varies. For instance, the value of$1/\alpha_{U(1)}$ for $k = 89$ corresponds to weak length scale and is 87.4 whereas fine structure constant is around 127. This gives $sin^2(\theta_W) \sim .312$, which is larger than standard model value.

1. Assume that the coupling constant evolutions for $1/\alpha_{em}$ and $1/\alpha_W$ correspond to different Möbius transformations acting in a nonlinear manner to s. Tangent of Weinberg angle is

defined as the ratio of weak and U(1) coupling constants: $tan(\theta_W) = g_W/g_{U(1)}$ and it expresses the vectorial character of electromagnetic coupling. One can write

$$sin^2(\theta_W) = \frac{1}{1+X} \quad , \quad X = \frac{\alpha_{U(1)}}{\alpha_W} \quad . \tag{18.4.1}$$

One can write the ansätze for for the coupling strengths as imaginary parts of complexified ones:

$$\begin{aligned} \frac{1}{\alpha_{U(1)}} &= Im[s+b] = y \quad , \quad s = \frac{1}{2} + iy \\ \frac{1}{\alpha_W} &= Im[\frac{a_W s + b_W}{c_W s + d_W}] = \frac{Dy}{c^2(\frac{1}{4}+y^2)+cd+d^2)} \quad , \\ D &= ad - bc \quad . \end{aligned} \tag{18.4.2}$$

Here $GL(2,Q)$ matrices are assumed and determinant $D = ad - bc$ is allowed to differ from unity. From this one obtains for the Weinberg angle the expression

$$sin^2(\theta_W(y)) = \; = \frac{1}{1+[\frac{c^2}{D}(y^2+\frac{1}{4})+\frac{d}{c}+(\frac{d}{c})^2]} \quad , \quad D = ad - bc \quad .$$

As the physical intuition suggests, Weinberg angle approaches zero at long length scales ($y \to \infty$). The value at short distance limit (the lowest zero $y_0 = 14.13$ at critical line) assignable to $p = 2$ is given by

$$sin^2(\theta_W(y_1)) \quad = \quad \frac{1}{1+\frac{c^2}{D}[(y_1^2+\frac{1}{4}+\frac{d}{c}+\frac{d}{c})^2]} \quad .$$

Note that Weinberg angle decreases monotonically with y. The choices for which c^2/D are equivalent but the parameters (a, b, c, d) can be chosen nearer to integers for large enough D.

2. How to fix the parameters D, c, d?

   (a) The first guess $D = ad - bc = 1$ would reduces the unknown parameters to c, d. This does not however allow even approximately integer valued parameters a, b, cd.

   (b) The GUT value of Weinberg angle at this limit is $sin^2(\theta_W) = 3/8$. TGD suggests that the values of Weinberg angle correspond to Pythagorean triangles (see `http://tinyurl.com/o7c4pkt`). The lowest primitive Pythagorean triangle (side lengths are coprimes, (see `http://tinyurl.com/j6ojlko`) corresponds to the triplet (3,4,9) forming the trunk of the 3-tree formed by the primitive Pythagorean triangles with 3 triangles emanating at each node) and to $sin^2(\theta_W) = 9/25$ slightly smaller than the GUT value. The problem is that y_0 is not a rational number and for rational values of c, d the equation for Weinberg angule cannot be satisfied.

   (c) An alternative more reliable option is to use as input Weinberg angle at intermediate boson length scale $k = 89$ which corresponds to $y(24) = 87.4252746$. The value of fine structure constant at Z^0 boson length scale is about $1/\alpha_{em}(89) \simeq 127$. From this one would obtain

$$sin^2(\theta_W(k=89)) = 1 - \frac{y_{24}}{\alpha_{em}(89)} = 1 - \frac{\alpha_{U(1)}(24)}{\alpha_{em}(89)} \simeq 0.3116, \quad . \tag{18.4.3}$$

One can obviously criticize the rather large value of the Weinberg angle forced by the value of $y(24)$ as being smaller than the experimental value. Experiments however suggests that Weinberg angle starts to increase after Z^0 pole. Gauge theory limit corresponds to a limit at which the sheets of many-sheeted are lumped together and one obtains a statistical average and the contributions of longer scale might increase the value of $1/\alpha_{U(1)}(24)$ and therefore reduce the value of the effective Weinberg angle.

(d) Another input is the value of fine structure constant either at $k = 127$ corresponding to electron's p-adic length scale or at $k = 131$ ($L(131) = 10^{-11}$ meters and four times the p-adic length scale of electron) fixed by the condition that fine structure constant $\alpha_{em} = \alpha_{(U(1)}cos^2(\theta_W)$ corresponds its low energy value $1/\alpha_{em} = 137.035999139$ assigned often to electron length scale. From $y(32(= 1/\alpha_{U(1)} = 105.446623$ or $y(31) = 103.725538$ and $1/\alpha_{em}(131) = 137.035999139$ one can estimate the value of Weinberg angle as

$$\begin{aligned} sin^2(\theta_W(k=131)) &= 1 - \frac{y_{32}}{\alpha_{em}(131)} \simeq 0.23052 \text{ or} \\ sin^2(\theta_W(k=130)) &= 1 - \frac{y_{32}}{\alpha_{em}(127)} \quad . \end{aligned} \tag{18.4.4}$$

It turns out that the first option does not work unless one assumes $1/alpha_{em}(k = 89) \leq 125.5263$ rather than $1/alpha_{em}(k = 89) \simeq 127$. The deviation is about 1-2 per cent. Second option works with a minimal modification for $1/alpha_{em}(k = 89) \simeq 127$.

(e) The value of $y(1)$ is $y_1 = 14.13472$. The two latter conditions give rise to the following series of equations

$$\begin{aligned} X(k) &= cot^2(\theta_W)(k) = \frac{c^2}{D}(y^2(k) + A) \ , \ A = \frac{1}{4} + \frac{d}{c} + (\frac{d}{c})^2 \ , \\ \frac{X(24)}{X(K)} &\equiv Y = \frac{cot^2(\theta_W)(24)}{cot^2(\theta_W)(K)} = \frac{y^2(24) + A}{y^2(K) + A} \ , \\ A &= \frac{Y(y^2(K) - y^2(24))}{1 - Y} \quad . \end{aligned} \tag{18.4.5}$$

Here K is either $K = 31$ or $K = 32$ corresponding to the p-adic length scale $k = 127$ or 131. It turns out that only $K = 31$ works fo $1/\alpha_{em}(89) = 127$.

Also following parameters can be expressed in terms of the data.

$$\begin{aligned} \frac{c^2}{D} &= \frac{cot^2(\theta_W)(K)}{y^2(K) + A} \ , \\ \frac{d}{c} &= \frac{1}{2}\left(-1 + \sqrt{A}\right) \ , \\ sin^2(\theta_W)(1) &= \frac{1}{1 + X(1)} \ , \ X(1) = \frac{c^2}{D}\left(y^2(1) + A\right) \quad . \end{aligned} \tag{18.4.6}$$

If the parameters a, b, c, d are integers, the equations cannot be satisfied exactly. For $K = 32$ it turns out that parameter A is negative for $1/alpha_{em}(k = 89) \leq 125.5263$. For $K = 31$ still negative and small so that $A = 0$ is the natural choice breaking slightly the conditions. **Table 18.1** represent both options.

(f) For $D = 1$ one has $c^2 \simeq 0.0002894$, which is very near to zero and not an integer. c must be non-vanishing to obtain a running Weinberg angle. For the general value of D the role c is taken by c^2D as an invariant fixed by the input data. $c \rightarrow c = 2$ requires $D = 1 \rightarrow int(4/c^2) = 138$. $D = 139$ almost equally good. One has $d/c = -0.5$ for $A = 0$ so that one would have $d = -1, c = 2$ for mimimum option. The condition $ad - bc = -a - 2b = D$ allows to estimate the values of the integer valued parameters a and b and get additional constraint on integer D. The values are not completely unique without additional conditions, say $b = 1$. This would give $a = -D + 2 = -137$ for $D = 139$ (one cannot avoid association with the famous "137"!).

3. Consider now the physical predictions. The evolution of Weinberg angle is depicted in the tables **18.1** and **18.2** for $k = 127$ model whereas tables **18.3** and **18.4** give the predictions of $k = 131$ model. The value of Weinberg angle at electron scale $k = 127$ is predicted to be $sin^2(\theta_w) \simeq 0.2430$ deviating from its measured value by 5 per cent. For $k = 131$ the Weinberg angle deviates .7 per cent from the measured value but the value of $1/\alpha_{em}(k = 89)$ is about 1 per cent too small.

   The expression for the predicted value of Weinberg angle at p-adic length scale $p = 2$ is $sin^2(\theta_W)_{p=2} \simeq 0.9453368487$, which is near to its maximal value and much larger than the $sin^2(\theta_W)_{p=2} \simeq 0.375$ of GUTs. This prediction was a total surprise but could be consistent with the new physics predicted by TGD predicting several scaled up copies of hadron physics above weak scale.

   A related surprise at the high energy end was that $1/\alpha_{em}$ begins to increase again at $k = 13$ and is near to fine structure constant at $k = 11$! As if asymptotic freedom would apply to all couplings except $U(1)$ coupling. This behavior is due to the approach of $cos^2(\theta_W)$ to zero. One can of course ask whether $sin^2(\theta_W) = 1$ could be obtained for a suitable choice of the parameters. This can be achieved only for $y(1) = 0$ which is not possible since A the parameter A cannot be negative.

To sum up, experimental input allows to fix electroweak coupling constant evolution completely. The problematic feature of $k = 127$ model is the possibly too large value of Weinberg theta at low energies. The predicted scaled up copies of hadron physics could explain why Weinberg angle must increase at high energies. At electron length scale the 5 per cent too high value is somewhat disturbing. The many-sheeted space-time requiring lumping together of sheets to get space-time of General Relativity might help to understand why measured Weinberg angle is smaller than predicted. Average over sheets of different sizes could be in question.

18.4.2 Test For The Model Of Electroweak Coupling Constant Evolution

One can check whether the values of 100 lowest non-trivial zeros are consistent with their assignment with primes k in $p \simeq 2^k$ and whether the model is consistent with the value of fine structure constant $1/\alpha_{em} = 137.035999139$ and experimental value $P = .2312$ of Weinberg angle assigned either with electron's p-adic length scale $k = 127$ or $k = 131$ (0.1 Angstroms).

The tables below summarize the values of $1/\alpha_K$ identified as imaginary part of Riemann zero and $\alpha_{em} = \alpha_K(1 - P)$ for the model already discussed. P is .7 per cent smaller than the experimental value $P = .2312$ for $k = 131$. This agreement is excellent but it turns out that the model works only if fine structure constant corresponds to $\alpha_{em}(k)$ in electron length scale $k = 127$.

For $k = 127$ one obtains fine structure constant correctly for $P = 0.243078179077$ about 10 per cent larger than the experimental value. The predicted value of α_K at scale $k = 127$ changes from $\alpha_K = \alpha_{em}$ to $\alpha(U(1))$ due the presence of $cos^2(\theta_W) = .77$. One can wonder whether this is consistent with the p-adic mass calculations and the condition on CP_2 coming from the string tension of cosmic strings.

The predicted value of α_K changes at electron length scale by the introduction of $cos(\theta_W)$ factor. The formula for the p-adic mass squared involves second order contribution which cannot be predicted with certainty. This contribution is 20 per cent at maximum so that the change of α_K by 10 per cent can be tolerated.

Galactic rotation velocity spectrum gives also constraint on the string tension of cosmic strings and in this manner also to the value of the inverse $1/R$ of CP_2 radius to which p-adic mass scales are proportional. The size scale or large voids corresponds roughly to $k = 293$. From **Table 18.2** one has $1/\alpha_K = 167.2$. If the condition $\alpha_K \simeq \alpha_{em}$ holds true in long length scales, the scaling of $1/\alpha_K = 1/\alpha_{em}$ used earlier would be given by $r \simeq 167/137$ and would increase the string tension of cosmic strings by factor 1.2. This could be compensated by scaling $R^2_{CP_2}$ by the same factor. CP_2 mass scale would be scaled by factor $1/\sqrt{1.2} \simeq .9$. Also this can be tolerated. Note that maximal value cosmic string tension is assumed making sense only for the ideal cosmic strings with 2-D M^4 projection. Thickening of cosmic strings reduces their tension since magnetic energy per length is reduced.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 1	14.1347251	2	0.945336	258.5784
2	21.0220396	3	0.886600	185.3802
3	25.0108575	5	0.846706	163.1566
4	30.4248761	7	0.788698	143.9880
5	32.9350615	11	0.761068	137.8428
6	37.5861781	13	0.709786	129.5121
7	40.9187190	17	0.673584	125.3579
8	43.3270732	19	0.647955	123.0727
9	48.0051508	23	0.599889	119.9796
10	49.7738324	29	0.582401	119.1907
11	52.9703214	31	0.551851	118.1982
12	56.4462476	37	0.520249	117.6574
13	59.3470440	41	0.495203	117.5663
14	60.8317785	43	0.482855	117.6301
15	65.1125440	47	0.449024	118.1767
16	67.0798105	53	0.434344	118.5877
17	69.5464017	59	0.416691	119.2275
18	72.0671576	61	0.399493	120.0105
19	75.7046906	67	0.376117	121.3444
20	77.1448400	71	0.367315	121.9326
21	79.3373750	73	0.354389	122.8874
22	82.9103808	79	0.334500	124.5836
23	84.7354929	83	0.324876	125.5111
24	87.4252746	89	0.311321	126.9464
25	88.8091112	97	0.304627	127.7144
26	92.4918992	101	0.287691	129.8480
27	94.6513440	103	0.278326	131.1552
28	95.8706342	107	0.273213	131.9102
29	98.8311942	109	0.261303	133.7912
30	101.317851	113	0.251824	135.4198
31	103.725538	127	0.243078	137.0359
32	105.446623	131	0.237073	138.2133
33	107.168611	137	0.231264	139.4088
34	111.029535	139	0.218919	142.1486
35	111.874659	149	0.216337	142.7587

Table 18.1: Table represents the first 35 zeros of zeta identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the proposed model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 36	114.320220	151	0.209095	144.5436
37	116.226680	157	0.203677	145.9543
38	118.790782	163	0.196690	147.8767
39	121.370125	167	0.189990	149.8379
40	122.946829	173	0.186049	151.0495
41	124.256818	179	0.182861	152.0633
42	127.516683	181	0.175248	154.6123
43	129.578704	191	0.170659	156.2431
44	131.087688	193	0.167407	157.4452
45	133.497737	197	0.162390	159.3794
46	134.756509	199	0.159853	160.3964
47	138.116042	211	0.153349	163.1322
48	139.736208	223	0.150345	164.4624
49	141.123707	227	0.147838	165.6068
50	143.111845	229	0.144348	167.2548
51	146.000982	233	0.139481	169.6662
52	147.422765	239	0.137170	170.8597
53	150.053520	241	0.133037	173.0796
54	150.925257	251	0.131706	173.8183
55	153.024693	257	0.128579	175.6036
56	156.112909	263	0.124167	178.2452
57	157.597591	269	0.122123	179.5214
58	158.849988	271	0.120436	180.6009
59	161.188964	277	0.117374	182.6243
60	163.030709	281	0.115040	184.2239
61	165.537069	283	0.111970	186.4094
62	167.184439	293	0.110016	187.8511
63	169.094515	307	0.107811	189.5277
64	169.911976	311	0.106886	190.2468
65	173.411536	313	0.103056	193.3360
66	174.754191	317	0.101639	194.5256
67	176.441434	331	0.099898	196.0238
68	178.377407	337	0.097952	197.7472
69	179.916484	347	0.096444	199.1206
70	182.207078	349	0.094262	201.1698

Table 18.2: Table represents the zeros y_n of zeta in the range $n \in [35,70]$ identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the proposed model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 1	14.1347251	2	0.943414	249.7949
2	21.0220396	3	0.882868	179.4744
3	25.0108575	5	0.841896	158.1927
4	30.4248761	7	0.782535	139.9074
5	32.9350615	11	0.754350	134.0732
6	37.5861781	13	0.702190	126.2089
7	40.9187190	17	0.665488	122.3238
8	43.3270732	19	0.639563	120.2072
9	48.0051508	23	0.591074	117.3933
10	49.7738324	29	0.573475	116.6964
11	52.9703214	31	0.542785	115.8544
12	56.4462476	37	0.511110	115.4580
13	59.3470440	41	0.486058	115.4744
14	60.8317785	43	0.473724	115.5892
15	65.1125440	47	0.439988	116.2700
16	67.0798105	53	0.425376	116.7369
17	69.5464017	59	0.407825	117.4423
18	72.0671576	61	0.390747	118.2878
19	75.7046906	67	0.367570	119.7045
20	77.1448400	71	0.358853	120.3232
21	79.3373750	73	0.346062	121.3225
22	82.9103808	79	0.326403	123.0862
23	84.7354929	83	0.316902	124.0459
24	87.4252746	89	0.303530	125.5263
25	88.8091112	97	0.296931	126.3164
26	92.4918992	101	0.280251	128.5057
27	94.6513440	103	0.271035	129.8435
28	95.8706342	107	0.266007	130.6152
29	98.8311942	109	0.254301	132.5350
30	101.317851	113	0.244992	134.1945
31	103.725538	127	0.236408	135.8390
32	105.446623	131	0.230518	137.0359
33	107.168611	137	0.224822	138.2504
34	111.029535	139	0.212726	141.0304
35	111.874659	149	0.210197	141.6489

Table 18.3: Table represents the first 35 zeros of zeta identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the $k = 131$ model for $sin^2(\theta_W)$.

n	y	k	$sin^2(\theta_W)$	$1/\alpha_{em}$
hline 36	114.320220	151	0.203108	143.4576
37	116.226680	157	0.197806	144.8861
38	118.790782	163	0.190972	146.8316
39	121.370125	167	0.184423	148.8150
40	122.946829	173	0.180571	150.0397
41	124.256818	179	0.177456	151.0641
42	127.516683	181	0.170022	153.6387
43	129.578704	191	0.165542	155.2850
44	131.087688	193	0.162368	156.4981
45	133.497737	197	0.157474	158.4494
46	134.756509	199	0.154999	159.4751
47	138.116042	211	0.148658	162.2333
48	139.736208	223	0.145730	163.5739
49	141.123707	227	0.143287	164.7270
50	143.111845	229	0.139887	166.3872
51	146.000982	233	0.135146	168.8158
52	147.422765	239	0.132897	170.0175
53	150.053520	241	0.128873	172.2522
54	150.925257	251	0.127578	172.9957
55	153.024693	257	0.124534	174.7923
56	156.112909	263	0.120242	177.4499
57	157.597591	269	0.118254	178.7336
58	158.849988	271	0.116613	179.8194
59	161.188964	277	0.113635	181.8541
60	163.030709	281	0.111367	183.4623
61	165.537069	283	0.108383	185.6594
62	167.184439	293	0.106483	187.1085
63	169.094515	307	0.104341	188.7935
64	169.911976	311	0.103443	189.5162
65	173.411536	313	0.099722	192.6201
66	174.754191	317	0.098346	193.8152
67	176.441434	331	0.096655	195.3201
68	178.377407	337	0.094766	197.0512
69	179.916484	347	0.093302	198.4305
70	182.207078	349	0.091184	200.4884

Table 18.4: Table represents the zeros y_n of zeta in the range $n \in [35, 70]$ identified as values of $\alpha_K = \alpha(U(1))$, the corresponding primes k ($p \simeq 2^k$), the predicted values of both Weinberg angle and of $\alpha_{em} = \alpha(U(1))cos^2(\theta_W)$ assuming the $k = 131$ model for $sin^2(\theta_W)$.

Chapter i

Appendix

A-1 Introduction

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of embedding space and related spaces are discussed and the relationship of CP_2 to the standard model is summarized. The basic vision is simple: the geometry of the embedding space $H = M^4 \times CP_2$ geometrizes standard model symmetries and quantum numbers. The assumption that space-time surfaces are basic objects, brings in dynamics as dynamics of 3-D surfaces based on the induced geometry. Second quantization of free spinor fields of H induces quantization at the level of H, which means a dramatic simplification.

The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Also the relationship to string models is discussed briefly.

Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure analogous to adele [L43, L42]. In the recent view of quantum TGD [L116], both notions reduce to physics as number theory vision, which relies on $M^8 - H$ duality [L82, L83] and is complementary to the physics as geometry vision.

Zero energy ontology (ZEO) [L73] [K109] has become a central part of quantum TGD and leads to a TGD inspired theory of consciousness as a generalization of quantum measurement theory having quantum biology as an application. Also these aspects of TGD are briefly discussed.

A-2 Embedding space $M^4 \times CP_2$

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that embedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Embedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) [L73, L102] [K109] causal diamond (CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure [A54] so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2.1 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrate that it codes for the symmetries of standard models via its isometries and holonomies.

CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A43] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.2}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi + \Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi - \Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.3}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://tgdtheory.fi/appfigures/cp2.jpg`

Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i\bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-2.5}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.6}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r, \phi) = (tan(\theta/2), \phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-2.7}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-2.8}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-2.9}$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , & e^1 &= \frac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} \ , & e^3 &= \frac{r\sigma_3}{F} \ . \end{aligned} \tag{A-2.10}$$

The explicit representations of vierbein vectors are given by

$$\begin{array}{lllllll} e^0 & = & \frac{dr}{F} \ , & e^1 & = & \frac{r(sin\Theta cos\Psi d\Phi+sin\Psi d\Theta)}{2\sqrt{F}} & , \\ e^2 & = & \frac{r(sin\Theta sin\Psi d\Phi-cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 & = & \frac{r(d\Psi+cos\Theta d\Phi)}{2F} & . \end{array} \quad \text{(A-2.11)}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 \ = \ \frac{dr^2}{F^2}+\frac{r^2}{4F^2}(d\Psi+cos\Theta d\Phi)^2+\frac{r^2}{4F}(d\Theta^2+sin^2\Theta d\Phi^2) \ . \quad \text{(A-2.12)}$$

From this expression one finds that at coordinate infinity $r=\infty$ line element reduces to $\frac{r^2}{4F}(d\Theta^2+sin^2\Theta d\Phi^2)$ of S^2 meaning that 3-sphere degenerates metrically to 2-sphere and one can say that CP_2 is obtained by adding to R^4 a 2-sphere at infinity.

The vierbein connection satisfying the defining relation

$$de^A \ = \ -V^A_B\wedge e^B \ , \quad \text{(A-2.13)}$$

is given by

$$\begin{array}{llllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ . \end{array} \quad \text{(A-2.14)}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{llllll} R_{01} & = & e^0\wedge e^1-e^2\wedge e^3 \ , & R_{23} & = & e^0\wedge e^1-e^2\wedge e^3 \ , \\ R_{02} & = & e^0\wedge e^2-e^3\wedge e^1 \ , & R_{31} & = & -e^0\wedge e^2+e^3\wedge e^1 \ , \\ R_{03} & = & 4e^0\wedge e^3+2e^1\wedge e^2 \ , & R_{12} & = & 2e^0\wedge e^3+4e^1\wedge e^2 \ . \end{array} \quad \text{(A-2.15)}$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J \ = \ -is_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \quad \text{(A-2.16)}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k_{\ r}J^{rl} \ = \ -s^{kl} \ . \quad \text{(A-2.17)}$$

The condition states that J and g give representations of real unit and imaginary units related by the formula $i^2=-1$.

Kähler form is expressible locally in terms of Kähler gauge potential

$$J \ = \ dB \ , \quad \text{(A-2.18)}$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

$dJ=ddB=0$ gives the topological half of Maxwell equations (vanishing of magnetic charges and Faraday's induction law) and self-duality $*J=J$ reduces the remaining equations to $dJ=0$. Hence the Kähler form can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling).

The magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{aligned} B &= 2re^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2}dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta \wedge d\Phi \ . \end{aligned} \tag{A-2.19}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1, 1).

Useful coordinates for CP_2 are the so called canonical (or symplectic or Darboux) coordinates in which the Kähler potential and Kähler form have very simple expressions

$$\begin{aligned} B &= \sum_{k=1,2} P_k dQ_k \ , \\ J &= \sum_{k=1,2} dP_k \wedge dQ_k \ . \end{aligned} \tag{A-2.20}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned} P_1 &= -\frac{1}{1+r^2} \ , \\ P_2 &= -\frac{r^2 cos\Theta}{2(1+r^2)} \ , \\ Q_1 &= \Psi \ , \\ Q_2 &= \Phi \ . \end{aligned} \tag{A-2.21}$$

Spinors In CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A31]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1-factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential

$exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the embedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A83] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \ \text{ for } \ X,Y,Z \in t \ . \tag{A-2.22}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2.2 CP_2 geometry and Standard Model symmetries

Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B53] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-2.23}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \otimes \gamma_5$, $1 \otimes \gamma_5$ and $\gamma_5 \otimes 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4)$ having as its covering group $SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \tag{A-2.24}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ , \end{array} \tag{A-2.25}$$

and

$$B = 2re^3 \ , \tag{A-2.26}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \tag{A-2.27}$$

where one have defined

$$\begin{array}{lll} I_L^1 & = & \frac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\ I_L^2 & = & \frac{(\Sigma_{02} - \Sigma_{13})}{2} \ . \end{array} \tag{A-2.28}$$

A_{ch} is clearly left handed so that one can perform the identification of the gauge potential as

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} \ , \tag{A-2.29}$$

where $W^\pm$ denotes the charged intermediate vector boson.

The covariantly constant curvature tensor is given by

$$\begin{array}{l} R_{01} = -R_{23} = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} = -R_{31} = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \\ R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , \\ R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \tag{A-2.30}$$

The charged part of the curvature tensor is left handed.

This is to be compared with the Weyl tensor, which defines a representation of quaternionic imaginary units.

$$W_{03} = W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ ,$$
$$W_{01} = W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ ,$$
$$W_{02} = W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \quad \text{(A-2.31)}$$

The charged part of the Weyl tensor is right-handed and that the relative sign of the two terms in the curvature tensor and Weyl tensor are opposite.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$X = re^3 \ ,$$
$$Y = \frac{e^3}{r} \ , \quad \text{(A-2.32)}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\bar{\gamma} = aX + bY \ ,$$
$$\bar{Z}^0 = cX + dY \ , \quad \text{(A-2.33)}$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$A_{nc} = [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma}$$
$$+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \quad \text{(A-2.34)}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \ . \quad \text{(A-2.35)}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \quad \text{(A-2.36)}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$Q_{em} = \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ ,$$
$$I_L^3 = \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \quad \text{(A-2.37)}$$

The fields γ and Z^0 are defined via the relations

$$\gamma = 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ ,$$
$$Z^0 = 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \quad \text{(A-2.38)}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \qquad \text{(A-2.39)}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of the Weinberg angle is a dynamical problem. The original approach was based on the assumption that it makes sense to talk about electroweak action defined at fundamental level and introduce a symmetry breaking by adding an additional term proportional to Kähler action. The recent view is that Kähler action plus volume term defines the fundamental action.

The Weinberg angle is completely fixed if one requires that the electroweak action contains no cross term of type γZ^0. This leads to a definite value for the Weinberg angle.

One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle. As a matter fact, color gauge action identifying color gauge field as proportional to $H^A J_{\alpha\beta}$ is proportional to Kähler action. A possible interpretation would be as a sum of electroweak and color gauge interactions.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \qquad \text{(A-2.40)}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0\wedge e^3 + e^1\wedge e^2) \ , \\ R_{12} &= 2(e^0\wedge e^3 + 2e^1\wedge e^2) \ , \\ J &= 2(e^0\wedge e^3 + e^1\wedge e^2) \ , \end{aligned} \qquad \text{(A-2.41)}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \qquad \text{(A-2.42)}$$

Evaluating the expressions above, one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \qquad \text{(A-2.43)}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \qquad \text{(A-2.44)}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \qquad \text{(A-2.45)}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$X = -\frac{K}{2g^2} + \frac{fp}{18} ,$$
$$K = Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] , \quad \text{(A-2.46)}$$

This parameter can be calculated by substituting the values of quark and lepton charges and weak isospins.

In the general case the value of the coefficient K is given by

$$K = \sum_i \left[-\frac{(18+2n_i^2)sin^2\theta_W}{9}\right] , \quad \text{(A-2.47)}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i (18+n_i^2))} . \quad \text{(A-2.48)}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} . \quad \text{(A-2.49)}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is not far from the typical value 9/24 of GUTs at high energies [B12]. The experimental value at the scale length scale of the electron can be deduced from the ratio of W and Z boson masses as $sin^2\theta_W = 1-(m_W/m_Z)^2 \simeq .22290$. This ratio and also the weak boson masses depend on the length scale.

If one interprets the additional term proportional to J as color action, one could perhaps interpret the value of Weinberg angle as expressing a connection between strong and weak coupling constant evolution. The limit $f \rightarrow 0$ should correspond to an infinite value of color coupling strength and at this limit one would have $sin^2\theta_W = \frac{9}{28}$ for $f/g^2 \rightarrow 0$. This does not make sense since the Weinberg angle is in the standard model much smaller in QCD scale Λ corresponding roughly to pion mass scale. The Weinberg angle is in principle predicted by the p-adic coupling constant evolution fixed by the number theoretical vision of TGD.

One could however have a sum of electroweak action, correction terms changing the value of Weinberg angle, and color action and coupling constant evolution could be understood in terms of the coupling parameters involved.

Electroweak symmetry breaking

One of the hardest challenges in the development of the TGD based view of weak symmetry breaking was the fact that classical field equations allow space-time surfaces with finite but arbitrarily large size. For a fixed space-time surface, the induced gauge fields, including classical weak fields, are long ranged. On the other hand, the large mass for weak bosons would require a short correlation length. How can one understand this together with the fact that a photon has a long correlation length?

In zero energy ontology quantum states are superpositions of space-time surfaces as analogs of almost unique Bohr orbits of particles identified as 3-D surfaces. For some reason the superposition should be such that the quantum averages of weak gauge boson fields vanish below the weak scale whereas the quantum average of electromagnetic fields is non-vanishing.

This is indeed the case.

1. The supersymplectic symmetries form isometries of the world of classical worlds (WCW) and they act in CP_2 degrees of freedom as symplectic transformations leaving the CP_2 symplectic form J invariant and therefore also its contribution to the electromagnetic field since this part is the same for all space-time surfaces in the superposition of space-time surfaces as a representation of supersymplectic isometry group (as a special case a representation of color group).

2. In TGD, color and electroweak symmetries acting as holonomies are not independent and for the $SU(2)_L$ part of induced spinor connection the symplectic transformations induces $SU(2)_L \times U(1)_R$ gauge transformation. This suggests that the quantum expectations of the induced weak fields over the space-time surfaces vanish above the quantum coherence scale. The averages of W and of the left handed part of Z^0 should therefore vanish.

3. $\langle Z^0 \rangle$ should vanish. For $U(1)_R$ part of Z^0, the action of gauge transformation is trivial in gauge theory. Now however the space-time surface changes under symplectic transformations and this could make the average of the right-handed part of Z^0 vanishing. The vanishing of the average of the axial part of the Z^0 is suggested by the partially conserved axial current hypothesis.

One can formulate this picture quantitatively.

1. The electromagnetic field [L118] contains, besides the induced Kähler form, also the induced curvature form R_{12}, which couples vectorially. Conserved vector current hypothesis suggests that the average of R_{12} is non-vanishing. One can express the neutral part of the induced gauge field in terms of induced spinor curvature and Kähler form J as

$$
\begin{aligned}
R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) = J + 2e^0 \wedge e^3 \ , \\
J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\
R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) = 3J - 2e^0 \wedge e^3 \quad , \qquad \text{(A-2.50)}
\end{aligned}
$$

2. The induced fields γ and Z^0 (photon and Z- boson) can be expressed as

$$
\begin{aligned}
\gamma &= 3J - sin^2\theta_W R_{12} \ , \\
Z^0 &= 2R_{03} = 2(J + 2e^0 \wedge e^3) \qquad \text{(A-2.51)} \\
per. & \qquad \text{(A-2.52)}
\end{aligned}
$$

The condition $\langle Z^0 \rangle = 0$ gives $2\langle e^0 \wedge e^3 \rangle = -2J$ and this in turn gives $\langle R_{12} \rangle = 4J$. The average over γ would be

$$
\langle \gamma \rangle = (3 - 4sin^2\theta_W)J \ .
$$

For $sin^2\theta_W = 3/4 \ langle \gamma \rangle$ would vanish.

The quantum averages of classical weak fields quite generally vanish. What about correlation functions?

1. One expects that the correlators of classical weak fields as color invariants, and perhaps even symplectic invariants, are non-vanishing below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as hadron.

2. The intuitive picture is that in longer length scales one has disjoint 3-surfaces with a size scale of Compton length. If the states associated with two disjoint 3-surfaces are separately color invariant there are no correlations in color degrees of freedom and correlators reduce to the products of expectations of classical weak fields and vanish. This could also hold when the 3-surfaces are connected by flux tube bonds.

   Below the Compton length weak bosons would thus behave as correlated massless fields. The Compton lengths of weak bosons are proportional to the value of effective Planck constant h_{eff} and in living systems the Compton lengths are proposed to be even of the order of cell size. This would explain the mysterious chiral selection in living systems requiring large parity violation.

3. What about the averages and correlators of color gauge fields? Classical color gauge fields are proportional to the products of Hamiltonians of color isometries induced Kähler form and the expectations of color Hamiltonians give vanishing average above Compton length and therefore vanishing average. Correlators are non-vanishing below the hadron scale. Gluons do not propagate in long scales for the same reason as weak bosons. This is implied by color confinement, which has also classical description in the sense that 3-surfaces have necessarily a finite size.

   A large value of h_{eff} allows colored states even in biological scales below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as dark hadron.

Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

1. Symmetries must be realized as purely geometric transformations.

2. Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B18] .

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \qquad \text{(A-2.53)}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \qquad \text{(A-2.54)}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \qquad \text{(A-2.55)}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

A-3.1 Induction procedure for gauge fields and spinor connection

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has embedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as a base space the imbedded manifold, whose points have as fiber the fiber if embedding space at their image points. In the recent case the embedding of space-time surface to embedding space defines the induction procedure. The induced gauge potentials and gauge fields are projections of the spinor connection of the embedding space to the space-time surface (see `http://tgdtheory.fi/appfigures/induct.jpg`).

Induction procedure makes sense also for the spinor fields of embedding space and one obtains geometrization of both electroweak gauge potentials and of spinors. The new element is induction of gamma matrices which gives their projections at space-time surface.

As a matter fact, the induced gamma matrices cannot appear in the counterpart of massless Dirac equation. To achieve super-symmetry, Dirac action must be replaced with Kähler-Dirac action for which gamma matrices are contractions of the canonical momentum currents of Kähler action with embedding space gamma matrices. Induced gamma matrices in Dirac action would correspond to 4-volume as action.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://tgdtheory.fi/appfigures/induct.jpg`.

A-3.2 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP$_2$ projection, only vacuum extremals and space-time surfaces for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

A-3.3 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four embedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. `http://tgdtheory.fi/appfigures/manysheeted.jpg`

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. `http://tgdtheory.fi/appfigures/wormholecontact.jpg`

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg`

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of embedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)") as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-3.4 Embedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

1. Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

2. Spinor harmonics of embedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of embedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

1. Although the embedding space spinor connection carries W gauge potentials one can say that the embedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

2. The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the

spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

3. Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

4. One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

5. This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-3.5 About induced gauge fields

In the following the induced gauge fields are studied for general space-time surface without assuming the preferred extremal property (Bohr orbit property). Therefore the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2}dr \wedge (d\Psi + cos(\Theta)d\Phi) + \frac{r^2}{2F}sin(\Theta)d\Theta \wedge d\Phi \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-3.1}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2}dr \wedge (d\Psi + cos(\Theta)d\Phi) + (3+p)\frac{r^2}{2F}sin(\Theta)d\Theta \wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.2}$$

where Θ_W denotes Weinberg angle.

1. The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p)sin(\Theta) = 0 \ , \end{aligned} \tag{A-3.3}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C = k + cos(\Theta_0) \ , \ D = \frac{r_0^2}{1+r_0^2} \ , \ \epsilon = \frac{3+p}{3+2p} \ , \end{aligned} \quad \text{(A-3.4)}$$

where C and D are integration constants. $0 \le X \le 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \le 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1 + r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \le k+1 \ ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \quad \text{(A-3.5)}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

2. The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

3. The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\ \gamma &= -\frac{p}{2} Z^0 \ . \end{aligned} \quad \text{(A-3.6)}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \tag{A-3.7}$$

and is useful in the construction of vacuum embedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$\begin{aligned} \Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\ \Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ . \end{aligned} \tag{A-3.8}$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global embedding for, say a constant magnetic field. Although global embedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \tag{A-3.9}$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-4 The relationship of TGD to QFT and string models

The recent view of the relationship of TGD to QFT and string models has developed slowly during years and it seems that in a certain sense TGD means a return to roots: instead of QFT like description involving path integral one would have wave mechanics for 3-surfaces.

A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces

The first vision of TGD was as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

The later work has revealed that TGD could be seen as a generalization of the wave mechanism based on the replacement of a point-like particle with 3-D surface. This is due to holography implied by general coordinate invariance. The definition of the metric of the "world of classical worlds" (WCW) must assign a unique or at least almost unique space-time surface to a given 3-surface. This 4-surface is analogous to Bohr orbit so that also Bohr orbitology becomes an exact part of quantum physics. The failure of strict determinism forces to replace 3-surfaces with 4-surfaces and this leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K45, K24, K80] [L104, L116].

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://tgdtheory.fi/appfigures/particletgd.jpg`

A-4.2 Extension of superconformal invariance

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in a completely unique position as far as symmetries are considered.

In fact, this leads to a generalization of the Kac-Moody type symmetries of string models. $\delta M^4_+ \times CP_2$ allows huge supersymplectic symmetries for which the radial light-like coordinate of δM^4_+ plays the role of complex string coordinate in string models. These symmetries are assumed to act as isometries of WCW.

A-4.3 String-like objects and strings

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of the TGD Universe and the inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string-like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situations in Minkowskian regions of space-time surface. `http://tgdtheory.fi/appfigures/fermistring.jpg`

A-4.4 TGD view of elementary particles

The TGD based view about elementary particles has two key aspects.

1. The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidean signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

2. At the level of H Fermion number is carried by the modes of the induced spinor field. In space-time regions with Minkowski signature the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle orbit corresponds to a 4-D generalization of a world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidean signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacts carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompanied by fermionic strings connecting the throats at the same sheet: the strings do not extend inside the wormhole contacts. `http://tgdtheory.fi/appfigures/elparticletgd.jpg`

Particle interactions involve both stringy and QFT aspects.

1. The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having a hadronic string as a physical counterpart. Long strings should be distinguished from wormhole contacts which due to their super-conformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

2. Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats behave like virtual bosons so that one has BFF type vertices typically.

3. In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particles along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. `http://tgdtheory.fi/appfigures/tgdgraphs.jpg`

A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K45, K80].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

A-5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [L45] [L104, L108, L109] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and

$T(CP_2)$ of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A54] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

1. Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

2. Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

3. The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

   For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

1. The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

2. Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

   For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

A-5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

1. If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

2. WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L116]. There are also other important algebras but these will not be discussed now.

1. The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

   The super symplectic algebra A has an infinite hierarchy of sub-algebras [L116] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

   This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

2. The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

   The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on M^8-H duality [L116] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

1. The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by M^8-H duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

2. This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L116] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P)=n(braid)=n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

3. The actions related to different values of $n(P)=n(braid)=n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

   WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

4. Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

   A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L111] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i=2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r=1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

   A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

5. A subset of p-adic primes allowed by the p-adic length scale hypothesis $p\simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K58, K59]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

1. Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

   P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L111], and the forward scattering amplitude corresponds to the propagator.

   This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

2. According to [L111], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

   A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L111].

3. p-Adic length scale hypothesis [L117] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

   For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

   Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

1. The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

   In [L116] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

2. The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K45, K24]. As isometries they would naturally permute the maxima with each other.

A-6 Number theoretic vision of TGD

Physics as number theory vision is complementary to the physics as geometry vision and has developed gradually since 1993. Langlands program is the counterpart of this vision in mathematics [L114].

The notion of p-adic number fields emerged with the motivation coming from the observation that elementary particle mass scales and mass ratios could be understood in terms of the so-called p-adic length scale hypothesis [K62, K52, K21]. The fusion of the various p-adic physics leads to what I call adelic physics [L43, L42]. Later the hypothesis about hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter emerged [K27, K28, K29, K29].

Eventually this led to that the values of effective Planck constant could be identified as the dimension of an algebraic extension of rationals assignable to polynomials with rational coefficients. This led to the number theoretic vision in which so-called $M^8 - H$ duality [L82, L83] plays a key role. M^8 (actually a complexification of real M^8) is analogous to momentum space so that the duality generalizes momentum position duality for point-like particles. M^8 has an interpretation as complexified octonions.

The dynamics of 4-surfaces in M^8 is coded by polynomials with rational coefficients, whose roots define mass shells H^3 of $M^4 \subset M^8$. It has turned out that the polynomials satisfy stringent additional conditions and one can speak of number theoretic holography [L111, L114]. Also the ordinary 3→ 4 holography is needed to assign 4-surfaces with these 3-D mass shells. The number theoretic dynamics is based on the condition that the normal space of the 4-surface in M^8 is associative (quaternionic) and contains a commutative complex sub-space. This makes it possible to assign to this surface space-time surface in $H = M^4 \times CP_2$.

At the level of H the space-time surfaces are by holography preferred extremals and are assumed to be determined by the twistor lift of TGD [L45] giving rise to an action which is sum of the Kähler action and volume term. The preferred extremals would be minimal surfaces analogous to soap films spanned by frames. Outside frames they would be simultaneous extremals of the Kähler action, which requires a generalization of the holomorphy characterizing string world sheets.

In the following only p-adic numbers and hierarchy of Planck constants will be discussed.

A-6.1 p-Adic numbers and TGD

p-Adic number fields

p-Adic numbers (p is prime: 2, 3, 5, ...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A29]. p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k , \; x(k) = 0, ..., p-1 \; . \tag{A-6.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \tag{A-6.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \tag{A-6.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} \ . \tag{A-6.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D \ . \tag{A-6.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.
2. Distances of points x and y inside single class are smaller than distances between different classes.
3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B46]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

1. Basic form of the canonical identification

There exists a natural continuous map $I: R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0,1,..,p-1\} \ . \end{aligned} \tag{A-6.6}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ .
\end{aligned}
\tag{A-6.7}
$$

The p-adic images associated with these expansions are different

$$
\begin{aligned}
y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\
y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \ ,
\end{aligned}
\tag{A-6.8}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

2. The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{A-6.9}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{A-6.10}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.11}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3. Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.12}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4. Generalization of number concept and notion of embedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic embedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real embedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \ mod \ p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real embedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles".

Fig. 16. The basic idea between p-adic manifold. `http://tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

1. Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

2. Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

3. Canonical identification violates general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic embedding space with chart maps to real embedding space and assuming preferred coordinates made possible by isometries of embedding space: one however obtains several inequivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-6.2 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the embedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of embedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of embedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-6.3 $M^8 - H$ duality as it is towards the end of 2021

The view of $M^8 - H$ duality (see Appendix **??**) has changed considerably towards the end 2021 [L104] after the realization that this duality is the TGD counterpart of momentum position duality of wave mechanics, which is lost in QFTs. Therefore M^8 and also space-time surface is analogous to momentum space. This forced us to give up the original simple identification of the points $M^4 \subset M^4 \times E^4 = M^8$ and of $M^4 \times CP_2$ so that it respects Uncertainty Principle (UP).

The first improved guess for the duality map was the replacement with the inversion $p^k \to m^k = \hbar_{eff} p^k/p^2$ conforming in spirit with UP but turned out to be too naive.

The improved form [L104] of the $M^8 - H$ duality map takes mass shells $p^2 = m^2$ of $M^4 \subset M^8$ to cds with size $L(m) = \hbar_{eff}/m$ with a common center. The slicing by mass shells is mapped to a Russian doll like slicing by cds. Therefore would be no CDs in M^8 contrary to what I believed first.

Quantum classical correspondence (QCC) inspires the proposal that the point $p^k \in M^8$ is mapped to a geodesic line corresponding to momentum p^k starting from the common center of cds. Its intersection with the opposite boundary of cd with size $L(m)$ defines the image point. This is not yet quite enough to satisfy UP but the additional details [L104] are not needed in the sequel.

The 6-D brane-like special solutions in M^8 are of special interest in the TGD inspired theory of consciousness. They have an M^4 projection which is $E = E_n$ 3-ball. Here E_n is a root of the real polynomial P defining $X^4 \subset M^8_c$ (M^8 is complexified to M^8_c) as a "root" of its octonionic continuation [L82, L83]. E_n has an interpretation as energy, which can be complex. The original interpretation was as moment of time. For this interpretation, $M^8 - H$ duality would be a linear identification and these hyper planes would be mapped to hyperplanes in $M^4 \subset H$.

This motivated the term "very special moment in the life of self" for the image of the $E = E_n$ section of $X^4 \subset M^8$ [L65]. This notion does not make sense at the level M^8 anymore.

The modified $M^8 - H$ duality forces us to modify the original interpretation [L104]. The point $(E_n, p = 0)$ is mapped $(t_n = \hbar_{eff}/E_n, 0)$. The momenta (E_n, p) in $E = E_n$ plane are mapped to the boundary of cd and correspond to a continuous time interval at the boundary of CD: "very special moment" becomes a "very special time interval".

The quantum state however corresponds to a set of points corresponding to quark momenta, which belong to a cognitive representation and are therefore algebraic integers in the extension determined by the polynomial. These active points in E_n are mapped to a discrete set at the boundary of cd(m). A "very special moment" is replaced with a sequence of "very special moments".

So called Galois confinement [L93] forces the total momenta for bound states of quarks and antiquarks to be rational integers invariant under Galois group of extension of rationals determined by the polynomial P [L104]. These states correspond to states at boundaries of sub-CDs so that one obtains a hierarchy. Galois confinement provides a universal number theoretic mechanism for the formation of bound states.

A-7 Zero energy ontology (ZEO)

ZEO is implied by the holography forced in the TGD framework by general coordinate invariance.

A-7.1 Basic motivations and ideas of ZEO

The following gives a brief summary of ZEO [L73] [K109].

1. In ZEO quantum states are not 3-dimensional but superpositions of 4-dimensional deterministic time evolutions connecting ordinary initial 3-dimensional states. By holography they are equivalent to pairs of ordinary 3-D states identified as initial and final states of time evolution. One can say that in the TGD framework general coordinate invariance implies holography and the slight failure of its determinism in turn forces ZEO.

   Quantum jumps replace this state with a new one: a superposition of deterministic time evolutions is replaced with a new superposition. Classical determinism of individual time evolution is not violated and this solves the basic paradox of quantum measurement theory. There are two kinds of quantum jumps: ordinary ("big") state function reductions (BSFRs) changing the arrow of time and "small" state function reductions (SSFRs) (weak measurements) preserving it and giving rise to the analog of Zeno effect [L73].

2. To avoid getting totally confused it is good to emphasize some aspects of ZEO.

   (a) ZEO does not mean that physical states in the usual 3-D sense as snapshots of time evolution would have zero energy state pairs defining zero energy states as initial and final states have same conserved quantities such as energy. Conservation implies that one can adopt the conventions that the values of conserved quantities are opposite for these states so that their sum vanishes: one can think that incoming and outgoing particles come from geometric past and future is the picture used in quantum field theories.

   (b) ZEO means two times: subjective time as sequence of quantum jumps and geometric time as space-time coordinate. These times are identifiable but are strongly correlated.

3. In BSFRs the arrow of time is changed and the time evolution in the final state occurs backwards with respect to the time of the external observer. BSFRs can occur in all scales since TGD predicts a hierarchy of effective Planck constants with arbitrarily large values. There is empirical support for BSFRs.

   (a) The findings of Minev et al [L62] in atomic scale can be explained by the same mechanism [L62]. In BSFR a final zero energy state as a superposition of classical deterministic time evolutions emerges and for an observer with a standard arrow of time looks like a superposition of deterministic smooth time evolutions leading to the final state. Interestingly, once this evolution has started, it cannot be stopped unless one changes the stimulus signal inducing the evolution in which case the process does not lead to anywhere: the interpretation would be that BSFR back to the initial state occurs!

(b) Libets' experiments about active aspects of consciousness [J1] can be understood. Subject person raises his finger and neural activity starts before the conscious decision to do so. In the physicalistic framework it is thought to lead to raising of the finger. The problem with the explanation is that the activity beginning .5 seconds earlier seems to be dissipation with a reversed arrow of time: from chaotic and disordered to ordered at around .15 seconds. ZEO explanation is that macroscopic quantum jump occurred and generated a signal proceeding backwards in time and generated neural activity and dissipated to randomness.

(c) Earthquakes involve a strange anomaly: they are preceded by ELF radiation. One would expect that they generate ELF radiation. The identification as BSFR would explain the anomaly [L64]. In biology the reversal of the arrow of time would occur routinely and be a central element of biological self-organization, in particular self-organized quantum criticality (see [L69, L119].

A-7.2 Some implications of ZEO

ZEO has profound implications for understanding self-organization and self-organized quantum criticality in terms of dissipation with non-standard arrow of time looking like generation of structures [L69, L119]. ZEO could also allow understanding of what planned actions - like realizing the experiment under consideration - could be.

1. Second law in the standard sense does not favor - perhaps even not allow - realization of planned actions. ZEO forces a generalization of thermodynamics: dissipation with a non-standard arrow of time for a subsystem would look like self-organization and planned action and its realization.

   Could most if not all planned action be like this - induced by BSFR in the geometric future and only apparently planned? There would be however the experience of planning and realizing induced by the signals from geometric future by a higher level in the hierarchy of conscious entities predicted by TGD! In long time scales we would be realizing our fates or wishes of higher level conscious entities rather than agents with completely free will.

2. The notion of magnetic body (MB) serving as a boss of ordinary matter would be central. MB carries dark matter as $h_{eff} = nh_0$ phases of ordinary matter with n serving as a measure for algebraic complexity of extension of rationals as its dimension and defining a kind of universal IQ. There is a hierarchy of these phases and MBs labelled by extension of rationals and the value of n.

   MBs would form a hierarchy of bosses - a realization for master slave hierarchy. Ordinary matter would be at the bottom and its coherent behavior would be induced from quantum coherence at higher levels. BSFR for higher level MB would give rise to what looks like planned actions and experienced as planned action at the lower levels of hierarchy. One could speak of planned actions inducing a cascade of planned actions in shorter time scales and eventually proceeding to atomic level.

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred embedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes. There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. `http://tgdtheory.fi/appfigures/padictoreal.jpg`

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. `http://tgdtheory.fi/appfigures/sharing.jpg`

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see **Fig.** `http://tgdtheory.fi/appfigures/timemirror.jpg` or **Fig.** 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. `http://tgdtheory.fi/appfigures/timemirror.jpg`

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REFERENCES

Mathematics

[A1] Calabi-Yau manifold. Available at: https://en.wikipedia.org/wiki/Calabi\OT1\textendashYau_manifold.

[A2] Category theory. Available at: https://en.wikipedia.org/wiki/Category_theory.

[A3] Farey sequence. Available at: https://en.wikipedia.org/wiki/Farey_sequence.

[A4] Groupoid. Available at: https://en.wikipedia.org/wiki/Groupoid.

[A5] Kac-Moody algebra. Available at: https://en.wikipedia.org/wiki/Kac\OT1\textendashMoody_algebra.

[A6] Langlands program. Available at: https://en.wikipedia.org/wiki/Langlands_program.

[A7] Little disk operad. Available at: https://en.wikipedia.org/wiki/Little_disks_operad#Examples.

[A8] Multicategories. Available at: https://en.wikipedia.org/wiki/Multicategory.

[A9] Operad theory. Available at: https://en.wikipedia.org/wiki/Operad.

[A10] Planar algebra. Available at: https://en.wikipedia.org/wiki/Planar_algebra.

[A11] Planar Algebras, TFTs with Defects. Available at: https://golem.ph.utexas.edu/category/2008/09/planar_algebras_tfts_with_defe.html.

[A12] Quadratic resiprocity theorem. Available at: https://en.wikipedia.org/wiki/Quadratic_reciprocity.

[A13] Set theory. Available at: https://en.wikipedia.org/wiki/Set_theory.

[A14] Super Virasoro algebra. Available at: https://en.wikipedia.org/wiki/Super_Virasoro_algebra.

[A15] Symplectification of science. Available at: https://www.pims.math.ca/~gotay/Symplectization(E).pdf.

[A16] Video about Edward Witten's talk relating to his work with knot invariants. Available at: https://video.ias.edu/webfm_send/1787.

[A17] Weekly Finds (Week 102). Available at: https://math.ucr.edu/home/baez/week202.html.

[A18] Yangian symmetry. Available at: https://en.wikipedia.org/wiki/Yangian.

[A19] *Objects of categories as complex numbers*, 2007. Available at: https://www.arxiv.org/abs/math.CT/0212377.

[A20] Connes A. *Non-commutative Geometry*. Academic Press, San Diego, 1994.

[A21] Goncharov A. Volumes of hyperbolic manifolds and mixed Tate motives, 1996. Available at: https://arxiv.org/abs/alg-geom/9601021.

[A22] Song A. Existence of infinitely many minimal hypersurfaces in closed manifolds, 2018. Available at: https://arxiv.org/pdf/1806.08816.pdf.

[A23] Casselman B. Strange associations, 1998. Montly Essays on Mathematical Topics (AMS). https://www.ams.org/samplings/feature-column/fcarc-associahedra.

[A24] Mitchell B. *Theory of Categories*. Academic Press, 1965.

[A25] Shipman B. The geometry of momentum mappings on generalized flag manifolds, connections with a dynamical system, quantum mechanics and the dance of honeybee, 1998. Available at: https://math.cornell.edu/~oliver/Shipman.gif.

[A26] Stay M Baez J. Physics, topology, logic and computation: a Rosetta Stone, 2008. Available at: https://math.ucr.edu/home/baez/rosetta.pdf.

[A27] Urbano F Barros MM, Chen B-Y. Quaternion CR-submanifolds of quaternion manifolds. *Kodai Mathematical Journal*, 4(3):399–417, 2020. Available at: https://www.researchgate.net/publication/38325032_Quaternion_CR-submanifolds_of_quaternion_manifolds.

[A28] Gosh SK Bisch D, Das P. Planar algebra of group-type subfactors, 2008.Available at: https://arxiv4.library.cornell.edu/abs/0807.4134.

[A29] Shafarevich IR Borevich ZI. *Number Theory*. Academic Press, 1966.

[A30] Ershov M Carbone L. Abstract simplicity of complete Kac-Moody groups over finite fields, 2010. Available at: https://arxiv.org/abs/math/0612772.

[A31] Pope CN. Eigenfunctions and $Spin^c$ Structures on CP_2, 1980.

[A32] Rovelli C Connes A. Von Neumann algebra automorphisms and time-thermodynamics relation in general covariant quantum theories, 1994. Available at: https://arxiv.org/PS_cache/gr-qc/pdf/9406/9406019v1.pdf.

[A33] Martellini M Cotta-Rasmussino P. BF theories and 2-knots, 1994. Available at: https://arxiv.org/pdf/hep-th/9407097.

[A34] Ruberman D. Comment in discussion about unitary cobordisms. Available at: https://math.ucr.edu/home/baez/quantum/ruberman.html.

[A35] Evans DE and Pugh M. Spectral measures for G_2, 2014. Available at: https://www.researchgate.net/publication/261437422_Spectral_Measures_for_G_2_II_finite_subgroups.

[A36] Hofstadter DR. *Gödel, Escher, Bach: an Eternal Braid*. Penguin Books, 1980.

[A37] Freed DS. The Geometry of Loop Groups, 1985.

[A38] Frenkel E. Free Field Realizations in Representation Theory and Conformal Field Theory. Available at: https://www.mathunion.org/ICM/ICM1994.2/Main/icm1994.2.1256.1269.ocr.pdf.

[A39] Frenkel E. Recent Advances in Langlands program. *AMS*, 41(2):151–184, 2004.

[A40] Frenkel E. Lectures on Langlands program and conformal field theory, 2005. Available at: https://arxiv.org/abs/hep-th/0512172.

[A41] Whitten E. Resolution of singularities in algebraic varieties, 2008. Available at:https://tinyurl.com/yb7cuwkf.

[A42] Witten E. Quantum field theory and the Jones polynomial. *Comm Math Phys*, 121:351–399, 1989.

[A43] Hanson J Eguchi T, Gilkey B. *Phys Rep*, 66, 1980.

[A44] Eisenhart. *Riemannian Geometry.* Princeton University Press, 1964.

[A45] Butin F. Branching Law for the Finite Subgroups of $SL(4, C)$. *Ukrainian Mathematical Journal*, 67:1484–1497, 2016. Available at: https://link.springer.com/article/10.1007/s11253-016-1167-8.

[A46] Butin F and Perets GS. Branching law for finite subgroups of $SL(3, C)$ and McKay correspondence. *Journal of Group Theory, De Gruyter*, 17(2):191–251, 2013. Available at: https://hal.archives-ouvertes.fr/hal-00412643/document.

[A47] Kato G. Sheaf Cohomology of Conscious Entity. *Int J Comp Antic Systems*, 2002.

[A48] Witten E Gaiotto D. Knot Invariants from Four-Dimensional Gauge Theory, 2011. Available at: https://arxiv.org/abs/1106.4789v1.

[A49] Pope CN Gibbons GW. CP_2 as gravitational instanton. *Comm Math Phys*, 55, 1977.

[A50] Browne H. Connectivity Properties of McKay Quivers, 2020. Available at: https://arxiv.org/pdf/2003.09502.pdf.

[A51] Schubert H. *Categories.* Springer Verlag, New York, 1972.

[A52] Sugawara H. A field theory of currents. *Phys Rev*, 176, 1968.

[A53] Pope CN Hawking SW. Generalized Spin Structures in Quantum Gravity. *Phys Lett*, (1), 1978.

[A54] N. Hitchin. Kählerian twistor spaces. *Proc London Math Soc*, 8(43):133–151, 1981.. Available at: https://tinyurl.com/pb8zpqo.

[A55] Nakamura I. McKay Correspondence, 2002. Available at: https://www.math.sci.hokudai.ac.jp/~nakamura/McKay071002.pdf.

[A56] Marques FC Irie K and Neves A. Density of minimal hypersurfaces for generic metrics, 2018. Available at: https://arxiv.org/pdf/1710.10752.pdf.

[A57] Butterfield J Isham CJ. Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity, 1999. Available at: https://arxiv.org/abs/gr-gc/9910005.

[A58] Zuber J-B Iztykson C. *Quantum Field Theory*, volume 549. Mc Graw-Hill, New York, 1980.

[A59] Baez J. Categorification, 1998. Available at: https://arxiv.org/pdf/math/9802029v1.

[A60] Baez J. Quantum Quandaries, 2007. Available at: https://math.ucr.edu/home/baez/quantum/node1.html.

[A61] Baez J. Quasicrystals and the Riemann Hypothesis. The n-Category Cafe, 2013. Available at: https://golem.ph.utexas.edu/category/2013/06/quasicrystals_and_the_riemann.html..

[A62] Coates J and Kim M. Selmer varieties for curves with CM Jacobians. *Kioto J Math.*, 50(4):827–852, 2010.Available at: https://people.maths.ox.ac.uk/kimm/papers/cm3.pdf.

[A63] McKay J. Cartan matrices, finite groups of quaternions, and kleinian singularities. *Proc AMS*, 1981. Available at: https://tinyurl.com/ydygjgge.

[A64] Mickelson J. Gerbes, (Twisted) K-Theory, and the Supersymmetric WZW Model, 2002. Available at: https://arxiv.org/abs/hep-th/0206139.

[A65] Milnor J. *Topology form Differential Point of View.* The University Press of Virginia, Virginia, 1965.

[A66] Baez JC. The Octonions. *Bull Amer Math Soc*, 39(2002), 2001. Available at: https://math.ucr.edu/home/baez/Octonions/octonions.html.

[A67] Grabowski JE and Launois S. Graded quantum cluster algebras and an application to quantum Grassmannians, 2018. Available at: https://arxiv.org/pdf/1301.2133.pdf.

[A68] Struppa DC Kato G. Category Theory and Consciousness. 2001.

[A69] Lambropoulou S Kauffman LH. Hard Unknots and Collapsing Tangles, 2006. Available at: https://arxiv.org/abs/math/0601525.

[A70] Marques FC Liokumovich Ye and Neves A. Weyl law for the volume spectrum, 2018. Available at: https://arxiv.org/pdf/1607.08721.pdf.

[A71] Khovanov M. MacKay correspondence, Available at: https://www.math.columbia.edu/~khovanov/finite/QiYou1.pdf.

[A72] Kim M. Arithmetic Gauge Theory: A Brief Introduction. *Mod Phys Lett A*, 33(29), 2018. Available at: https://www.worldscientific.com/doi/10.1142/S0217732318300124.

[A73] Reid M. The du val singularities an, dn, e6, e7, e8. Available at:https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf.

[A74] Reid M. Chapters on algebraic surfaces, 1996. Available at:https://arxiv.org/abs/alg-geom/9602006.

[A75] Spivak M. *Differential Geometry I,II,III,IV*. Publish or Perish, Boston, 1970.

[A76] Sheppeard MD. Gluon Phenomenology and a Linear Topos, 2007. Available at: https://arcadianfunctor.files.wordpress.com/2007/11/thesismds.pdf.

[A77] MacKay NJ. Introduction to Yangian symmetry in integrable field theory, 2009. Available at: https://wenku.baidu.com/view/01e4ebdbad51f01dc281f180.html.

[A78] Goldblatt R. *The Categorial Analysis of Logic*. North-Holland, Amsterdam, 1984.

[A79] Stefanik R. *Structuralism, Category Theory and Philosophy of Mathematics*. MSG Press, Washington, 1994.

[A80] Thom R. *Comm Math Helvet*, 28, 1954.

[A81] Hoffnung A Rogers C. Categorified symplectic geometry and the classical string, 2008. Available at: https://math.ucr.edu/home/baez/2plectic_1.pdf.

[A82] Gelbart S. An elementary introduction to the langlands program. *Bulletin of the AMS*, 10(2), 1984. Available at:https://tinyurl.com/mkqhp5n.

[A83] Helgason S. *Differential Geometry and Symmetric Spaces*. Academic Press, New York, 1962.

[A84] L-Sheng Tseng and Shing-Tung Yan. Non-Kähler Calabi-Yau submanifolds. *Proceedings in Symposia in Pure Mathematics*, 85:241–253, 2012. Available at: https://www.math.uci.edu/~lstseng/pdf/TsengYau201216.pdf.

[A85] Meyer U. Quantum determinants, 1994. Available at: https://arxiv.org/pdf/hep-th/9406172.pdf.

[A86] Bouchard V. Lectures on complex geometry, Calabi-Yau manifolds and toric geometry, 2005. Available at: https://www.ulb.ac.be/sciences/ptm/pmif/Rencontres/ModaveI/CGL.ps.

[A87] Jones V. In and around the origin of quantum groups, 2003. Available at: https://arxiv.org/abs/math/0309199.

[A88] Duke W and Imamoglu Ö. The zeros of the Weierstrass $\mathcal{P}$-function and hypergeometric series. *Mathematische Annalen*, 340(4):897–905, 2008. Available at: https://cutt.ly/uIZSK4T.

[A89] Wallace. *Differential Topology.* W. A. Benjamin, New York, 1968.

[A90] Vandoren S Wit de B, Rocek M. Hypermultiplets, Hyperkähler Cones and Quaternion-Kähler Geometry, 2001. Available at: https://arxiv.org/pdf/hep-th/0101161.pdf.

[A91] Yen Y-H and Song JS. Of McKay Correspondence, Non-linear Sigma-model and Conformal Field Theory. *Advances in Theoretical and Mathematical Physics*, 4(4), 1999. Available at: https://cutt.ly/jLQPgkQ.

Theoretical Physics

[B1] Higher-dimensional super-gravity. Available at: https://wapedia.mobi/en/Higher-dimensional_supergravity?t=3.

[B2] Super-space. Available at: https://en.wikipedia.org/wiki/Superspace.

[B3] Susy Summary. Available at: https://bolvan.ph.utexas.edu/~vadim/Classes/01f/396T/table.pdf.

[B4] The Minimally Super-Symmetric Standard Model. Available at: https://en.wikipedia.org/wiki/Minimal_Supersymmetric_Standard_Model.

[B5] Video talk of Nima Arkani-Hamed about the notion of amplituhedron. Available at: https://video.ias.edu/PiTP-ArkaniHamed.

[B6] The quantum Super Yangian and Casimir operators of $U_q(gl(m|n))$, 1994. Available at:https://tinyurl.com/l8asm33.

[B7] Bernewig A and Neupert T. Topological Superconductors and Category Theory, 2015. Available at:https://tinyurl.com/l2jsrzc.

[B8] Hodges A. Twistor diagram recursion for all gauge-theoretic tree amplitudes, 2005. Available at:https://arxiv.org/pdf/hep-th/0503060.pdf.

[B9] Jadczyk A. Conformally compactified minkowski space: myths and facts, 2011. Available at: https://arxiv.org/abs/1803.00545.

[B10] Kitaev A. Quantum computations: algorithms and error correction. *Russ Math Survey*, pages 52–61, 1997.

[B11] Kitaev A. Fault tolerant quantum computation by anyons, 1997. Available at: https://arxiv.org/abs/quant-ph/9707021.

[B12] Zee A. *The Unity of Forces in the Universe.* World Sci Press, Singapore, 1982.

[B13] He S Yan G Arkani-Hamed N, Bai Y. Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet, 2017. Available at:https://arxiv.org/abs/1711.09102.

[B14] Trnka J Arkani-Hamed N, Hodges A. Positive Amplitudes In The Amplituhedron, 2014. Available at:https://arxiv.org/abs/1412.8478.

[B15] Trnka Y Arkani-Hamed N. The Amplituhedron, 2013. Available at: https://arxiv.org/abs/1312.2007.

[B16] Morrison DR Aspinwall PS, Greene BR. Calabi-Yau Moduli Space, Mirror Manifolds, and Space-time Topology Change in String Theory, 1993. Available at: `https://arxiv.org/abs/hep-th/9309097`.

[B17] Zamolodchikov AB Belavin AA, Polyakov AM. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl Phys B*, 241:333–380, 1984. Available at: `https://users.physik.fu-berlin.de/~kamecke/ps/BPZ.pdf`.

[B18] Drell S Björken J. *Relativistic Quantum Fields*. Mc Graw-Hill, New York, 1965.

[B19] Kent A Boucher W, Friedan D. Determinant formulae and unitarity for the $\mathcal{N} = 2$ superconformal algebras in two-dimensions or exact results on string compactification. *Phys Lett B*, 172(316), 1986.

[B20] Feng B Witten E Britto R, Cachazo F. Direct Proof of Tree-Level Recursion Relation in Yang- Mills Theory. *Phys Rev*, 94:181602, 2005. Available at: `https://arxiv.org/abs/hep-th/0501052`.

[B21] Skinner D Bullimore M, Mason L. Twistor-Strings, Grassmannians and Leading Singularities, 2009. Available at: `https://arxiv.org/abs/0912.0539`.

[B22] Witten E Cachazo F, Svrcek P. MHV Vertices and Tree Amplitudes In Gauge Theory, 2004. Available at: `https://arxiv.org/abs/hep-th/0403047`.

[B23] Zuber J-B Cappelli A. A-D-E Classification of Conformal Field Theories, 2010. Available at:`https://tinyurl.com/h9val5g`.

[B24] White CD. The double copy: gravity from gluons, 2017. Available at: `https://arxiv.org/abs/1807.06581`.

[B25] Rapoport D. Stochastic processes in conformal Riemann-Cartan-Weyl gravitation, 1991. Available at: `https://link.springer.com/article/10.1007/BF00675614`.

[B26] Witten E Dolan L, Nappi CR. Yangian Symmetry in $D = 4$ superconformal Yang-Mills theory, 2004. Available at: `https://arxiv.org/abs/hep-th/0401243`.

[B27] Plefka J Drummond J, Henn J. Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, 2009. Available at: `https://cdsweb.cern.ch/record/1162372/files/jhep052009046.pdf`.

[B28] Witten E. Topological Quantum Field Theory. *Comm Math Phys*, 117(353), 1988.

[B29] Witten E. Perturbative Gauge Theory As a String Theory In Twistor Space, 2003. Available at: `https://arxiv.org/abs/hep-th/0312171`.

[B30] Witten E. Perturbative Gauge Theory As A String Theory In Twistor Space. *Communications in Mathematical Physics*, 252:189–258, 2003. Available at: `https://arxiv.org/abs/hep-th/0312171`.

[B31] Yang S-K Eguchi T. $\mathcal{N} = \in$ superconformal models as topological field theories. *Mod Phys Lett A*, 5(1693), 1990. Available at:`https://arxiv.org/pdf/hep-th/9611018.pdf`.

[B32] Huang Y-T Elvang H. Scattering amplitudes, 2013. Available at: `https://arxiv.org/pdf/1308.1697v1.pdf`.

[B33] Arkani-Hamed N et al. Scattering amplitides and the positive Grassmannian. Available at: `https://arxiv.org/pdf/1212.5605v1.pdf`.

[B34] Arkani-Hamed N et al. A duality for the S-matrix, 2009. Available at: `https://arxiv.org/abs/0907.5418`.

[B35] Arkani-Hamed N et al. The S-Matrix in Twistor Space, 2009. Available at:`https://arxiv.org/abs/0903.2110`.

[B36] Arkani-Hamed N et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM, 2010. Available at: https://arxiv.org/abs/1008.2958.

[B37] Arkani-Hamed N et al. Unification of Residues and Grassmannian Dualities, 2010. Available at: https://arxiv.org/pdf/0912.4912.pdf.

[B38] Arkani-Hamed N et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM, 2011. Available at: https://arxiv.org/abs/1008.2958..

[B39] Arkani-Hamed N et al. On-Shell Structures of MHV Amplitudes Beyond the Planar Limit, 2014. Available at:https://arxiv.org/abs/1412.8475.

[B40] Dixon LJ et al. Folding Amplitudes into Form Factors: An Antipodal Duality, 2022. Available at: https://arxiv.org/pdf/2112.06243.pdf.

[B41] Drummond JM et al. Magic identities for conformal four-point integrals, 2006. Available at:https://tinyurl.com/gsy5u6y.

[B42] Goncharov AB et al. Classical Polylogarithms for Amplitudes and Wilson Loops, 2010. Available at: https://arxiv.org/abs/1006.5703.

[B43] Minev ZK et al. To catch and reverse a quantum jump mid-flight, 2019. Available at: https://arxiv.org/abs/1803.00545.

[B44] Preskill J et al. Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence, 2015. Available at: https://arxiv.org/pdf/1503.06237.pdf.

[B45] Loebbert F. Lectures on Yangian Symmetry, 2016. Available at: https://arxiv.org/pdf/1606.02947.pdf.

[B46] Parisi G. *Field Theory, Disorder and Simulations*. World Scientific, 1992.

[B47] Remmen GN. Amplitudes and the riemann zeta function. *Phys Rev Lett*, 2021. Available at: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.127.241602.

[B48] Vergu C Goncharov AB, Spradlin M and Volovich A. Classical Polylogarithms for Amplitudes and Wilson Loops. *Phys Rev Lett*, 105(151605), 2010. Available at: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.105.151605.

[B49] Yang-Hui He. On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective, 2002. Available at: https://arxiv.org/pdf/hep-th/0209230.pdf..

[B50] Trnka J. Scattering amplitudes and the positive Grassmannian, 2013. Available at: https://tinyurl.com/pqjzffj.

[B51] Baez JC. Higher Yang-Mills Theory, 2002. Available at: https://xxx.lanl.gov/abs/hep-th/0206130.

[B52] Figueroa-O′Farrill JM. Untwisting topological field theories, 1996. Available at:https://arxiv.org/pdf/hep-th/9611018.pdf.

[B53] Huang K. *Quarks,Leptons* & *Gauge Fields*. World Scientific, 1982.

[B54] Wang XJ Xu KW Lu H, Pope CN. $\mathcal{N} = 2$ superstrings with $(1, 2m)$ space-time signature. *Phys Lett B*, 284(3-4), 1992. Available at:https://tinyurl.com/jnon4fh.

[B55] Bullimore M. Inverse Soft Factors and Grassmannian Residues, 2010. Available at: https://arxiv.org/abs/1008.3110.

[B56] Jimbo M. Introduction to the Yang-Baxter equation. *Int J Mod Phys. A*, 4(15):3759–3777, 1989. Available at: https://tinyurl.com/l4z6zyr.

[B57] Maio M. Permutation Orbifolds in Conformal Field Theories and String Theory, 2011. Available at: https://tinyurl.com/hdy66lt.

[B58] Skinner D Mason L. Scattering Amplitudes and BCFW Recursion in Twistor Space, 2009. Available at: https://arxiv.org/pdf/0903.2083v3.pdf.

[B59] Skinner D Mason L. Dual Superconformal Invariance, Momentum Twistors and Grassmannians, 2009. Available at: https://arxiv.org/pdf/0909.0250v2.

[B60] Gray O. On the complete classification of unitary $\mathcal{N} = 2$ minimal superconformal field theories, 2011. Available at:https://arxiv.org/pdf/0812.1318v3.pdf.

[B61] Binetruy P. *Supersymmetry: Theory, Experiment and Cosmology.* Oxford University Press, Oxford, 2006. Available at: https://tinyurl.com/pr4mjqe.

[B62] Boels R. On BCFW shifts of integrands and integrals, 2010. Available at: https://arxiv.org/abs/1008.3101.

[B63] Penrose R. The Central Programme of Twistor Theory. *Chaos, Solitons & Fractals*, 10, 1999.

[B64] Adamo T. Twistor actions for gauge theory and gravity, 2013. Available at: https://arxiv.org/pdf/1308.2820.pdf.

[B65] Schreiber U. Nonabelian 2-forms and loop space connections from SCFT deformations, 2004. Available at:https://arxiv.org/abs/hep-th/0407122.

[B66] Shriman Y. TASI 2008 Lectures: Introduction to Supersymmetry and Supersymmetry Breaking, 2009.Available at: https://arxiv.org/abs/0907.0039.

[B67] Trnka Y. Grassmannian Origin of Scattering Amplitudes, 2013. Available at: https://www.princeton.edu/physics/graduate-program/theses/theses-from-2013/Trnka-Thesis.pdf.

Particle and Nuclear Physics

[C1] How many neutrinos in the sky? Available at: https://resonaances.blogspot.fi/2013/01/how-many-neutrinos-in-sky.html.

[C2] Bulbul E et al. Detection of an unidentified emission line in the stacked x-ray spectrum of galaxy clusters, 2014. Available at: https://arxiv.org/abs/1402.2301.

[C3] Hinshaw G et al. Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, 2013. Available at: https://arxiv.org/abs/1212.5226.

[C4] Sturrock PA et al. Analysis of Gamma Radiation from a Radon Source: Indications of a Solar Influence, 2012. Available at: https://arxiv.org/abs/1205.0205.

Condensed Matter Physics

[D1] Fried AD. Relationship of time reversal symmetry breaking with optical kerr rotation, 2014.Available at:https://arxiv.org/pdf/1406.2019.pdf.

[D2] Keimer B and Moore JE. The physics of quantum materials. *Nature*, 13:1045,1055, 2017. Available at: https://www.nature.com/articles/nphys4302.

Cosmology and Astro-Physics

[E1] Nottale L Da Rocha D. Gravitational Structure Formation in Scale Relativity, 2003. Available at: https://arxiv.org/abs/astro-ph/0310036.

Neuroscience and Consciousness

[J1] Libet B. Readiness potentials preceding unrestricted spontaneous and preplanned voluntary acts, 1982. Available at: https://tinyurl.com/jqp1. See also the article *Libet's Research on Timing of Conscious Intention to Act: A Commentary* of Stanley Klein at https://tinyurl.com/jqp1.

[J2] Blackman CF. *Effect of Electrical and Magnetic Fields on the Nervous System*, pages 331–355. Plenum, New York, 1994.

[J3] Fantappie L. *Teoria Unitaria del Mondo Fisico e Biologico.* Di Renzo Editore, Roma, 1942.

[J4] Sacks O. *The man who mistook his wife for a hat.* 1998. Available at: First edition 1985.

[J5] Suzuki S. *Zen Mind, Beginner's Mind.* Water hill,, New York, 1988.

Books related to TGD

[K1] Pitkänen M. Could $\mathcal{N} = 2$ Super-conformal Theories Be Relevant For TGD? In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/N2sconf.pdf.

[K2] Pitkänen M. TGD Inspired Theory of Consciousness. In *Topological Geometrodynamics: Overview: Part II. https://tgdtheory.fi/tgdhtml/Btgdview1.html*.

[K3] Pitkänen M. *Topological Geometrodynamics.* 1983. Thesis in Helsinki University 1983.

[K4] Pitkänen M. DNA as Topological Quantum Computer. In *Quantum - and Classical Computation in TGD Universe. https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: https://tgdtheory.fi/pdfpool/dnatqc.pdf, 2015.

[K5] Pitkänen M. Topological Quantum Computation in TGD Universe. In *Quantum - and Classical Computation in TGD Universe. https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: https://tgdtheory.fi/pdfpool/tqc.pdf, 2015.

[K6] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory: Part I. https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/adelephysics.pdf, 2017.

[K7] Pitkänen M. About Nature of Time. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timenature.pdf, 2023.

[K8] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I. https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/prext.pdf, 2023.

[K9] Pitkänen M. About the Nottale's formula for h_{gr} and the possibility that Planck length l_P and CP_2 length R are related. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/vzerovariableG.pdf, 2023.

[K10] Pitkänen M. About twistor lift of TGD? In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/hgrtwistor.pdf, 2023.

[K11] Pitkänen M. Appendix A: Quantum Groups and Related Structures. In *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Available at: https://tgdtheory.fi/pdfpool/bialgebra.pdf, 2023.

[K12] Pitkänen M. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I. https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/class.pdf, 2023.

[K13] Pitkänen M. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://www.tgdtheory.fi/public_html/pdfpool/append.pdf, 2023.

[K14] Pitkänen M. *Bio-Systems as Conscious Holograms.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/holography.html., 2023.

[K15] Pitkänen M. *Bio-Systems as Self-Organizing Quantum Systems.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioselforg.html., 2023.

[K16] Pitkänen M. Breakthrough in understanding of $M^8 - H$ duality. Available at: https://tgdtheory.fi/pdfpool/M8H.pdf., 2023.

[K17] Pitkänen M. Can one apply Occam's razor as a general purpose debunking argument to TGD? In *Topological Geometrodynamics: Overview: Part I. https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/simplicity.pdf, 2023.

[K18] Pitkänen M. Category Theory and Quantum TGD. In *TGD and Hyper-finite Factors. https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/categorynew.pdf, 2023.

[K19] Pitkänen M. Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness. In *TGD and Hyper-finite Factors. https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/categoryc.pdf, 2023.

[K20] Pitkänen M. Classical TGD. In *Topological Geometrodynamics: Overview: Part I:https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdclass.pdf, 2023.

[K21] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K22] Pitkänen M. Construction of Quantum Theory: M-matrix. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/towards.pdf, 2023.

[K23] Pitkänen M. Construction of Quantum Theory: Symmetries. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/quthe.pdf, 2023.

[K24] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K25] Pitkänen M. Cosmic Strings. In *Physics in Many-Sheeted Space-Time: Part II. https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cstrings.pdf, 2023.

[K26] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? In *TGD and Hyper-finite Factors. https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/Yangianagain.pdf, 2023.

[K27] Pitkänen M. Criticality and dark matter: part I. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark1.pdf, 2023.

[K28] Pitkänen M. Criticality and dark matter: part II. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark2.pdf, 2023.

[K29] Pitkänen M. Criticality and dark matter: part III. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark3.pdf, 2023.

[K30] Pitkänen M. Criticality and dark matter: part IV. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark4.pdf, 2023.

[K31] Pitkänen M. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG: Part I. https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/eegdark.pdf, 2023.

[K32] Pitkänen M. Dark Nuclear Physics and Condensed Matter. In *TGD and Nuclear Physics. https://tgdtheory.fi/tgdhtml/Bnucl.html*. Available at: https://tgdtheory.fi/pdfpool/exonuclear.pdf, 2023.

[K33] Pitkänen M. Dark valence electrons and color vision. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/colorvision.pdf, 2023.

[K34] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? In *TGD as a Generalized Number Theory: Part III. https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/fermizeta.pdf, 2023.

[K35] Pitkänen M. Does TGD Predict a Spectrum of Planck Constants? In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/Planck, 2023.

[K36] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Topological Geometrodynamics: Overview: Part II. https://tgdtheory.fi/tgdhtml/Btgdoverview2*. Available at: https://tgdtheory.fi/pdfpool/vNeumannnew, 2023.

[K37] Pitkänen M. Expanding Earth Model and Pre-Cambrian Evolution of Continents, Climate, and Life. In *Physics in Many-Sheeted Space-Time: Part I. https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/expearth.pdf., 2023.

[K38] Pitkänen M. From Principles to Diagrams. In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/diagrams.pdf, 2023.

[K39] Pitkänen M. General Theory of Qualia. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/qualia.pdf, 2023.

[K40] Pitkänen M. *Genes and Memes.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/genememe.html., 2023.

[K41] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/tensornet.pdf, 2023.

[K42] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck.html., 2023.

[K43] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck1.html., 2023.

[K44] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part II.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck2.html., 2023.

[K45] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/kahler.pdf., 2023.

[K46] Pitkänen M. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/knotstgd.pdf, 2023.

[K47] Pitkänen M. Langlands Program and TGD. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/Langlands.pdf, 2023.

[K48] Pitkänen M. Langlands Program and TGD: Years Later. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/langlandsnew.pdf, 2023.

[K49] Pitkänen M. Life and Death, and Consciousness. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lifedeath.pdf, 2023.

[K50] Pitkänen M. Magnetic Sensory Canvas Hypothesis. In *TGD and Quantum Biology: Part I. https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/mec.pdf, 2023.

[K51] Pitkänen M. *Magnetospheric Consciousness*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/magnconsc.html., 2023.

[K52] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K53] Pitkänen M. *Mathematical Aspects of Consciousness Theory*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/mathconsc.html, 2023.

[K54] Pitkänen M. Matter, Mind, Quantum. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/conscic.pdf, 2023.

[K55] Pitkänen M. More Precise TGD View about Quantum Biology and Prebiotic Evolution. In *Evolution in TGD Universe. https://tgdtheory.fi/tgdhtml/Btgdevolution.html*. Available at: https://tgdtheory.fi/pdfpool/geesink.pdf, 2023.

[K56] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory: Part III. https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infmotives.pdf, 2023.

[K57] Pitkänen M. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/nmpc.pdf, 2023.

[K58] Pitkänen M. New Physics Predicted by TGD: Part I. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys1.pdf, 2023.

[K59] Pitkänen M. New Physics Predicted by TGD: Part II. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys2.pdf, 2023.

[K60] Pitkänen M. Nuclear String Hypothesis. In *TGD and Nuclear Physics. https://tgdtheory.fi/tgdhtml/Bnucl.html*. Available at: https://tgdtheory.fi/pdfpool/nuclstring.pdf, 2023.

[K61] Pitkänen M. Number theoretic vision, Hyper-finite Factors and S-matrix. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/UandM.pdf, 2023.

[K62] Pitkänen M. *p-Adic length Scale Hypothesis*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/padphys.html., 2023.

[K63] Pitkänen M. p-Adic Numbers and Generalization of Number Concept. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/padmat.pdf, 2023.

[K64] Pitkänen M. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mass3.pdf, 2023.

[K65] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K66] Pitkänen M. p-Adic Physics: Physical Ideas. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/phblocks.pdf, 2023.

[K67] Pitkänen M. Physics as a Generalized Number Theory. In *Topological Geometrodynamics: Overview: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdnumber.pdf, 2023.

[K68] Pitkänen M. Possible Role of p-Adic Numbers in Bio-Systems. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/biopadc.pdf, 2023.

[K69] Pitkänen M. Quantum Adeles. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/galois.pdf, 2023.

[K70] Pitkänen M. Quantum Astrophysics. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/qastro.pdf, 2023.

[K71] Pitkänen M. Quantum gravity, dark matter, and prebiotic evolution. In *Evolution in TGD Universe.* *https://tgdtheory.fi/tgdhtml/Btgdevolution.html*. Available at: https://tgdtheory.fi/pdfpool/hgrprebio.pdf, 2023.

[K72] Pitkänen M. Quantum Hall effect and Hierarchy of Planck Constants. In *TGD and Condensed Matter.* *https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/anyontgd.pdf, 2023.

[K73] Pitkänen M. *Quantum Hardware of Living Matter.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioware.html., 2023.

[K74] Pitkänen M. Quantum Mind and Neuroscience. In *TGD and EEG: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPN.pdf, 2023.

[K75] Pitkänen M. Quantum Mind, Magnetic Body, and Biological Body. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPB.pdf, 2023.

[K76] Pitkänen M. Quantum Model for Bio-Superconductivity: I. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondI.pdf, 2023.

[K77] Pitkänen M. Quantum Model for Bio-Superconductivity: II. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondII.pdf, 2023.

[K78] Pitkänen M. Quantum Model for Hearing. In *TGD and EEG: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdeeg2.html*. Available at: https://tgdtheory.fi/pdfpool/hearing.pdf, 2023.

[K79] Pitkänen M. *Quantum TGD.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdquantum.html., 2023.

[K80] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K81] Pitkänen M. Riemann Hypothesis and Physics. In *TGD as a Generalized Number Theory: Part III. https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/riema.pdf, 2023.

[K82] Pitkänen M. Self and Binding: Part I. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/selfbindc.pdf, 2023.

[K83] Pitkänen M. Some questions related to the twistor lift of TGD. In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistquestions.pdf, 2023.

[K84] Pitkänen M. SUSY in TGD Universe. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/susychap.pdf, 2023.

[K85] Pitkänen M. TGD and Astrophysics. In *Physics in Many-Sheeted Space-Time: Part II. https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/astro.pdf, 2023.

[K86] Pitkänen M. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time: Part II. https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cosmo.pdf, 2023.

[K87] Pitkänen M. *TGD and EEG.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdeeg.html, 2023.

[K88] Pitkänen M. *TGD and Fringe Physics.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/freenergy.html., 2023.

[K89] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I. https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K90] Pitkänen M. TGD as a Generalized Number Theory: p-Adicization Program. In *Quantum Physics as Number Theory: Part I. https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visiona.pdf, 2023.

[K91] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory: Part I. https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionb.pdf, 2023.

[K92] Pitkänen M. *TGD Based View About Living Matter and Remote Mental Interactions.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdlian.html, 2023.

[K93] Pitkänen M. *TGD Inspired Theory of Consciousness.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdconsc.html., 2023.

[K94] Pitkänen M. TGD View about Coupling Constant Evolution. In *Quantum TGD: Part II. https://tgdtheory.fi/tgdhtml/Btgdquantum2.html*. Available at: https://tgdtheory.fi/pdfpool/ccevolution.pdf, 2023.

[K95] Pitkänen M. The classical part of the twistor story. In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistorstory.pdf, 2023.

[K96] Pitkänen M. The Geometry of the World of the Classical Worlds. In *Topological Geometrodynamics: Overview: Part I*:*https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgeom.pdf, 2023.

[K97] Pitkänen M. The Recent Status of Lepto-hadron Hypothesis. In *p-Adic Physics*. *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/leptc.pdf, 2023.

[K98] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III*. *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

[K99] Pitkänen M. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time: Part I*. *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgrt.pdf., 2023.

[K100] Pitkänen M. Three new physics realizations of the genetic code and the role of dark matter in bio-systems. In *Genes and Memes: Part II*. *https://tgdtheory.fi/tgdhtml/Bgenememe2.html*. Available at: https://tgdtheory.fi/pdfpool/dnatqccodes.pdf, 2023.

[K101] Pitkänen M. *Topological Geometrodynamics: an Overview*. Online book.Available at: https://www.tgdtheory.fi/tgdhtml/tgdview.html., 2023.

[K102] Pitkänen M. Topological Geometrodynamics: Basic Visions. In *TGD Inspired Theory of Consciousness: Part I*. *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPTGD.pdf, 2023.

[K103] Pitkänen M. Topological Geometrodynamics: Three Visions. In *Topological Geometrodynamics: Overview: Part I*. *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgd2010.pdf, 2023.

[K104] Pitkänen M. Unified Number Theoretical Vision. In *TGD as a Generalized Number Theory: Part I*. *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/numbervision.pdf, 2023.

[K105] Pitkänen M. Was von Neumann Right After All? In *TGD and Hyper-finite Factors*. *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/vNeumann.pdf, 2023.

[K106] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry*. *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

[K107] Pitkänen M. What p-Adic Icosahedron Could Mean? And What about p-Adic Manifold? In *TGD as a Generalized Number Theory: Part III*. *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/picosahedron.pdf, 2023.

[K108] Pitkänen M. Why TGD and What TGD is? In *Topological Geometrodynamics: an Overview*. *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/WhyTGD.pdf, 2023.

[K109] Pitkänen M. Zero Energy Ontology. In *Quantum TGD: Part I*. *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/ZEO.pdf, 2023.

Articles about TGD

[L1] Geometric Langlands program. Available at: https://matpitka.blogspot.com/2007/11/geometric-langlands.html.

[L2] Pitkänen M. The Geometry of CP_2 and its Relationship to Standard Model. 2010. Available at: https://tgdtheory.fi/pdfpool/appendb.pdf.

[L3] Pitkänen M. A Strategy for Proving Riemann Hypothesis. Available at: https://www.emis.math.ca/EMIS/journals/AMUC/, 2003.

[L4] Pitkänen M. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. Available at: https://tgdtheory.fi/pdfpool/append.pdf., 2006.

[L5] Pitkänen M. TGD Inspired Theory of Consciousness. Available at: https://tgdtheory.fi/public_html/articles/tgdconsc.cpdf., 2008.

[L6] Pitkänen M. Topological Geometrodynamics: What Might Be The Basic Principles. Available at: https://tgdtheory.fi/public_html/articles/tgd2008.pdf., 2008.

[L7] Pitkänen M. Algebraic braids, sub-manifold braid theory, and generalized Feynman diagrams. Available at: https://tgdtheory.fi/public_html/articles/algknots.pdf., 2011.

[L8] Pitkänen M. Langlands conjectures in TGD framework. Available at: https://tgdtheory.fi/public_html/articles/Langlandsagain.pdf., 2011.

[L9] Pitkänen M. Hypnosis as Remote Mental Interaction. Available at: https://tgdtheory.fi/public_html/articles/hypnosisarticle.pdf., 2013.

[L10] Pitkänen M. Classical part of the twistor story. Available at: https://tgdtheory.fi/public_html/articles/twistorstory.pdf., 2014.

[L11] Pitkänen M. CMAP representations about TGD. Available at: https://www.tgdtheory.fi/cmaphtml.html, 2014.

[L12] Pitkänen M. CMAP representations about TGD, and TGD inspired theory of consciousness and quantum biology. Available at: https://www.tgdtheory.fi/tgdglossary.pdf., 2014.

[L13] Pitkänen M. Geometric theory of harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[L14] Pitkänen M. Jeremy England's vision about life and evolution: comparison with TGD approach . Available at: https://tgdtheory.fi/public_html/articles/englandtgd.pdf., 2015.

[L15] Pitkänen M. Could one Define Dynamical Homotopy Groups in WCW? Available at: https://tgdtheory.fi/public_html/articles/dynatopo.pdf., 2015.

[L16] Pitkänen M. Could one realize number theoretical universality for functional integral? Available at: https://tgdtheory.fi/public_html/articles/ntu.pdf., 2015.

[L17] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? . Available at: https://tgdtheory.fi/public_html/articles/fermizeta.pdf., 2015.

[L18] Pitkänen M. Is Non-Associative Physics and Language Possible Only in Many-Sheeted Space-time? . Available at: https://tgdtheory.fi/public_html/articles/braidparse.pdf., 2015.

[L19] Pitkänen M. Why the non-trivial zeros of Riemann zeta should reside at critical line? . Available at: https://tgdtheory.fi/public_html/articles/rhagain.pdf., 2015.

[L20] Pitkänen M. About minimal surface extremals of Kähler action. Available at: https://tgdtheory.fi/public_html/articles/minimalkahler.pdf., 2016.

[L21] Pitkänen M. Cyclic cosmology from TGD perspective. Available at: https://tgdtheory.fi/public_html/articles/turok.pdf., 2016.

[L22] Pitkänen M. From Principles to Diagrams. Available at: https://tgdtheory.fi/public_html/articles/diagrams.pdf., 2016.

[L23] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. Available at: https://tgdtheory.fi/public_html/articles/tensornet.pdf., 2016.

[L24] Pitkänen M. How the hierarchy of Planck constants might relate to the almost vacuum degeneracy for twistor lift of TGD? Available at: https://tgdtheory.fi/public_html/articles/hgrtwistor.pdf., 2016.

[L25] Pitkänen M. Hydrinos again. Available at: https://tgdtheory.fi/public_html/articles/Millsagain.pdf., 2016.

[L26] Pitkänen M. Langlands Program and TGD: Years Later. Available at: https://tgdtheory.fi/public_html/articles/langlandsnew.pdf., 2016.

[L27] Pitkänen M. p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework. Available at: https://tgdtheory.fi/public_html/articles/padicgeom.pdf., 2016.

[L28] Pitkänen M. The anomalies in rotating magnetic systems as a key to the understanding of morphogenesis? Available at: https://tgdtheory.fi/public_html/articles/godin.pdf., 2016.

[L29] Pitkänen M. What could be the physical origin of Pythagorean scale? Available at: https://tgdtheory.fi/public_html/articles/geesinkscale.pdf., 2016.

[L30] Pitkänen M. Twistorial Lift of Kahler Action. *Pre-Space-Time Journal*, 7(7), 2016. See also https://tgtheory.fi/public_html/articles/twistquestions.pdf.

[L31] Pitkänen M. About some unclear issues of TGD. Available at: https://tgdtheory.fi/public_html/articles/fermionsector.pdf., 2017.

[L32] Pitkänen M. Are higher structures needed in the categorification of TGD? Available at: https://tgdtheory.fi/public_html/articles/nstructures.pdf., 2017.

[L33] Pitkänen M. Breaking of CP, P, and T in cosmological scales in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/rotuniverse.pdf., 2017.

[L34] Pitkänen M. Cold fusion, low energy nuclear reactions, or dark nuclear synthesis? Available at: https://tgdtheory.fi/public_html/articles/krivit.pdf., 2017.

[L35] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? Available at: https://tgdtheory.fi/public_html/articles/Yangianagain.pdf., 2017.

[L36] Pitkänen M. Could McKay correspondence generalize in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2017.

[L37] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? Available at: https://tgdtheory.fi/public_html/articles/ratpoints.pdf., 2017.

[L38] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: https://tgdtheory.fi/public_html/articles/ratpoints1.pdf., 2017.

[L39] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: https://tgdtheory.fi/public_html/articles/ratpoints2.pdf., 2017.

[L40] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part III. Available at: https://tgdtheory.fi/public_html/articles/ratpoints3.pdf., 2017.

[L41] Pitkänen M. p-Adicization and adelic physics. Available at: https://tgdtheory.fi/public_html/articles/adelicphysics.pdf., 2017.

[L42] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L43] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L44] Pitkänen M. Quantum computations without definite causal structure: TGD view. Available at: https://tgdtheory.fi/public_html/articles/causalstructure.pdf., 2017.

[L45] Pitkänen M. Questions about twistor lift of TGD. Available at: https://tgdtheory.fi/public_html/articles/twistquestions.pdf., 2017.

[L46] Pitkänen M. Re-examination of the basic notions of TGD inspired theory of consciousness. Available at: https://tgdtheory.fi/public_html/articles/conscrit.pdf., 2017.

[L47] Pitkänen M. Some questions related to the twistor lift of TGD. Available at: https://tgdtheory.fi/public_html/articles/graviconst.pdf., 2017.

[L48] Pitkänen M. What could be the role of complexity theory in TGD? Available at: https://tgdtheory.fi/public_html/articles/numberchaos.pdf., 2017.

[L49] Pitkänen M. p-Adicization and Adelic Physics. *Pre-Space-Time Journal.*, 8(3), 2017. See also https://tgtheory.fi/public_html/articles/adelicphysics.pdf.

[L50] Pitkänen M. About the physical interpretation of the velocity parameter in the formula for the gravitational Planck constant . Available at: https://tgdtheory.fi/public_html/articles/vzero.pdf., 2018.

[L51] Pitkänen M. Dance of the honeybee and New Physics. Available at: https://tgdtheory.fi/public_html/articles/Shipmanagain.pdf., 2018.

[L52] Pitkänen M. Dark valence electrons and color vision. Available at: https://tgdtheory.fi/public_html/articles/colorvision.pdf., 2018.

[L53] Pitkänen M. Further comments about classical field equations in TGD framework . Available at: https://tgdtheory.fi/public_html/articles/morecomments.pdf., 2018.

[L54] Pitkänen M. Maxwell's lever rule and expansion of water in freezing: two poorly understood phenomena. Available at: https://tgdtheory.fi/public_html/articles/leverule.pdf., 2018.

[L55] Pitkänen M. Morphogenesis in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/morphoTGD.pdf., 2018.

[L56] Pitkänen M. New insights about quantum criticality for twistor lift inspired by analogy with ordinary criticality. Available at: https://tgdtheory.fi/public_html/articles/zeocriticality.pdf., 2018.

[L57] Pitkänen M. TGD view about coupling constant evolution. Available at: https://tgdtheory.fi/public_html/articles/ccevolution.pdf., 2018.

[L58] Pitkänen M. The Recent View about Twistorialization in TGD Framework. Available at: https://tgdtheory.fi/public_html/articles/smatrix.pdf., 2018.

[L59] Pitkänen M. Recent Views About Twistorialization in TGD Framework (Part I). *Pre-Space-Time Journal*, 9(3), 2018. See also https://tgtheory.fi/public_html/articles/smatrix.pdf.

[L60] Pitkänen M. Recent Views About Twistorialization in TGD Framework (Part II). *Pre-Space-Time Journal*, 9(3), 2018. See also https://tgtheory.fi/public_html/articles/smatrix.pdf.

[L61] Pitkänen M. An overall view about models of genetic code and bio-harmony. Available at: https://tgdtheory.fi/public_html/articles/gcharm.pdf., 2019.

[L62] Pitkänen M. Copenhagen interpretation dead: long live ZEO based quantum measurement theory! Available at: https://tgdtheory.fi/public_html/articles/Bohrdead.pdf., 2019.

[L63] Pitkänen M. Does coupling constant evolution reduce to that of cosmological constant? Available at: https://tgdtheory.fi/public_html/articles/ccevoTGD.pdf., 2019.

[L64] Pitkänen M. Earhquakes and volcanic eruptions as macroscopic quantum jumps in zero energy ontology. Available at: https://tgdtheory.fi/public_html/articles/earthquakes.pdf., 2019.

[L65] Pitkänen M. $M^8 - H$ duality and consciousness. Available at: https://tgdtheory.fi/public_html/articles/M8Hconsc.pdf., 2019.

[L66] Pitkänen M. Minimal surfaces: comparison of the perspectives of mathematician and physicist. Available at: https://tgdtheory.fi/public_html/articles/minimalsurfaces.pdf., 2019.

[L67] Pitkänen M. More about the construction of scattering amplitudes in TGD framework. Available at: https://tgdtheory.fi/public_html/articles/scattampl.pdf., 2019.

[L68] Pitkänen M. New results related to $M^8 - H$ duality. Available at: https://tgdtheory.fi/public_html/articles/M8Hduality.pdf., 2019.

[L69] Pitkänen M. Quantum self-organization by h_{eff} changing phase transitions. Available at: https://tgdtheory.fi/public_html/articles/heffselforg.pdf., 2019.

[L70] Pitkänen M. Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals . Available at: https://tgdtheory.fi/public_html/articles/symplorbsm.pdf., 2019.

[L71] Pitkänen M. Secret Link Uncovered Between Pure Math and Physics. Available at: https://tgdtheory.fi/public_html/articles/KimTGD.pdf., 2019.

[L72] Pitkänen M. Solar Metallicity Problem from TGD Perspective. Available at: https://tgdtheory.fi/public_html/articles/darkcore.pdf., 2019.

[L73] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L74] Pitkänen M. SUSY in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/susyTGD.pdf., 2019.

[L75] Pitkänen M. Tesla still inspires. Available at: https://tgdtheory.fi/public_html/articles/teslastill.pdf., 2019.

[L76] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, and Inclusions of Hyperfinite Factors. Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2019.

[L77] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors. Available at: https://tgdtheory.fi/public_html/articles/McKaygeneral.pdf., 2019.

[L78] Pitkänen M. Twistors in TGD. Available at: https://tgdtheory.fi/public_html/articles/twistorTGD.pdf., 2019.

[L79] Pitkänen M. Two anomalies of hadron physics from TGD perspective. Available at: https://tgdtheory.fi/public_html/articles/twoanomalies.pdf., 2019.

[L80] Pitkänen M. Twistors in TGD (Part I). *Pre-Space-Time Journal*, 10(7), 2019. See also https://tgtheory.fi/public_html/articles/twistorTGD.pdf.

[L81] Pitkänen M. Twistors in TGD (Part II) . *Pre-Space-Time Journal*, 10(7), 2019. See also https://tgtheory.fi/public_html/articles/twistorTGD.pdf.

[L82] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L83] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L84] Pitkänen M. Could quantum randomness have something to do with classical chaos? Available at: https://tgdtheory.fi/public_html/articles/chaostgd.pdf., 2020.

[L85] Pitkänen M. Could TGD provide new solutions to the energy problem? Available at: https://tgdtheory.fi/public_html/articles/proposal.pdf., 2020.

[L86] Pitkänen M. Could ZEO provide a new approach to the quantization of fermions? Available at: https://tgdtheory.fi/public_html/articles/secondquant.pdf., 2020.

[L87] Pitkänen M. How to compose beautiful music of light in bio-harmony? https://tgdtheory.fi/public_html/articles/bioharmony2020.pdf., 2020.

[L88] Pitkänen M. MIP*= RE: What Could This Mean Physically? Available at: https://tgtheory.fi/public_html/articles/deeptheorem.pdf., 2020.

[L89] Pitkänen M. Summary of Topological Geometrodynamics. https://tgdtheory.fi/public_html/articles/tgdarticle.pdf., 2020.

[L90] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: https://tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[L91] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory (short version). Available at: https://tgdtheory.fi/public_html/articles/kahlersm.pdf., 2020.

[L92] Pitkänen M. About TGD counterparts of twistor amplitudes. https://tgdtheory.fi/public_html/articles/twisttgd.pdf., 2021.

[L93] Pitkänen M. About the role of Galois groups in TGD framework. https://tgdtheory.fi/public_html/articles/GaloisTGD.pdf., 2021.

[L94] Pitkänen M. Can one regard leptons as effectively local 3-quark composites? https://tgdtheory.fi/public_html/articles/leptoDelta.pdf., 2021.

[L95] Pitkänen M. Comparing the Berry phase model of super-conductivity with the TGD based model. https://tgdtheory.fi/public_html/articles/SCBerryTGD.pdf., 2021.

[L96] Pitkänen M. Does the notion of polynomial of infinite order make sense? https://tgdtheory.fi/public_html/articles/transcendGalois.pdf., 2021.

[L97] Pitkänen M. Galois code and genes. https://tgdtheory.fi/public_html/articles/Galoiscode.pdf., 2021.

[L98] Pitkänen M. Is genetic code part of fundamental physics in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/TIH.pdf., 2021.

[L99] Pitkänen M. Negentropy Maximization Principle and Second Law. Available at: https://tgdtheory.fi/public_html/articles/nmpsecondlaw.pdf., 2021.

[L100] Pitkänen M. Neutrinos and TGD. https://tgdtheory.fi/public_html/articles/TGDneutrino.pdf., 2021.

[L101] Pitkänen M. Questions about coupling constant evolution. https://tgdtheory.fi/public_html/articles/ccheff.pdf., 2021.

[L102] Pitkänen M. Some questions concerning zero energy ontology. https://tgdtheory.fi/public_html/articles/zeonew.pdf., 2021.

[L103] Pitkänen M. Spin Glasses, Complexity, and TGD. https://tgdtheory.fi/public_html/articles/sg.pdf., 2021.

[L104] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L105] Pitkänen M. TGD based interpretation for the strange findings of Eric Reiner. https://tgdtheory.fi/public_html/articles/unquantum.pdf., 2021.

[L106] Pitkänen M. Three alternative generalizations of Nottale's hypothesis in TGD framework. https://tgdtheory.fi/public_html/articles/MDMdistance.pdf., 2021.

[L107] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.

[L108] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[L109] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[L110] Pitkänen M. Can quasi-time crystal be created by a Fibonacci process?: TGD point of view. https://tgdtheory.fi/public_html/articles/qtc.pdf., 2022.

[L111] Pitkänen M. Finite Fields and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L112] Pitkänen M. McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product? . https://tgdtheory.fi/public_html/articles/McKayGal.pdf., 2022.

[L113] Pitkänen M. Quantum Gravitation and Topological Quantum Computation. https://tgdtheory.fi/public_html/articles/TQCTGD.pdf., 2022.

[L114] Pitkänen M. Some New Ideas Related to Langlands Program *viz.* TGD. https://tgdtheory.fi/public_html/articles/Langlands2022.pdf., 2022.

[L115] Pitkänen M. TGD inspired model for freezing in nano scales. https://tgdtheory.fi/public_html/articles/freezing.pdf., 2022.

[L116] Pitkänen M. Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole. https://tgdtheory.fi/public_html/articles/fusionTGD.pdf., 2022.

[L117] Pitkänen M. Two objections against p-adic thermodynamics and their resolution. https://tgdtheory.fi/public_html/articles/padmass2022.pdf., 2022.

[L118] Pitkänen M. Reduction of standard model structure to CP_2 geometry and other key ideas of TGD. https://tgdtheory.fi/public_html/articles/cp2etc.pdf., 2023.

[L119] Pitkänen M and Rastmanesh R. Homeostasis as self-organized quantum criticality. Available at: https://tgdtheory.fi/public_html/articles/SP.pdf., 2020.

[L120] Pitkänen M and Rastmanesh R. The based view about dark matter at the level of molecular biology. Available at: https://tgdtheory.fi/public_html/articles/darkchemi.pdf., 2020.

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