

# RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE

Matti Pitkänen

ABSTRACT. A strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis is suggested. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator  $D^+$  having the zeros of Riemann Zeta as its eigenvalues. The construction of  $D^+$  is inspired by the conviction that Riemann Zeta is associated with a physical system allowing superconformal transformations as its symmetries and second quantization in terms of the representations of superconformal algebra. The eigenfunctions of  $D^+$  are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows by reductio ad absurdum from the hypothesis that ordinary superconformal algebra acts as gauge symmetries for all coherent states orthogonal to the vacuum state, including also the non-physical might-be coherent states off from the critical line.

## CONTENTS

1. Introduction	2
2. Modified form of the Hilbert-Polya conjecture	3
3. Formal solution of the eigenvalue equation for operator $D^+$	3
4. $D^+ = D^\dagger$ condition and Hermitian form	4
5. How to choose the function $F$ ?	6
6. Study of the Hermiticity condition	7
7. Does the Hermitian form define inner product?	9
8. Superconformal symmetry	12

---

*Mathematics Subject Classification.* Number Theory.

*Key words and phrases.* Number theory, Riemann hypothesis.

I am grateful for my son Timo for stimulating discussions relating to p-adic numbers in the context of the Riemann Zeta function. I would like to thank Carlos Castro for inspiring email conversations relating to Riemann Zeta and for turning my attention to Hilbert-Polya hypothesis and the possible physics involved with it. I also want to express my deep gratitude to Matthew Watkins for providing me with information about Riemann Zeta and for generous help, in particular for reading the manuscript and pointing out several inaccuracies and errors.

9. Is the proof of the Riemann hypothesis by reductio ad absurdum possible using superconformal invariance?	16
References	18

## 1. Introduction

The Riemann hypothesis [Rie, Tit86] states that the non-trivial zeros (as opposed to zeros at  $s = -2n$ ,  $n \geq 1$  integer) of Riemann Zeta function obtained by analytically continuing the function

$$(1) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

from the region  $Re[s] > 1$  to the entire complex plane, lie on the line  $Re[s] = 1/2$ . Hilbert and Polya [Edw74] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator  $H$  such that the eigenvalues of this operator correspond to the imaginary parts of the nontrivial zeros  $z = x + iy$  of  $\zeta$ . One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-Hermitian operator  $D^+$  contains the non-trivial zeros of  $\zeta$ . The eigenstates in question are eigenstates of an annihilation operator type operator  $D^+$  and analogous to the so called coherent states encountered in quantum physics [IZ80]. In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of  $\zeta$  is constructed.

a) The construction relies crucially on the interpretation of the vanishing of  $\zeta$  as an orthogonality condition in a Hermitian metric which is a priori more general than Hilbert space inner product.

b) Second basic element is the scaling invariance motivated by the belief that  $\zeta$  is associated with a physical system which has superconformal transformations [ISZ88] as its symmetries. This vision was inspired by the generalization of  $\zeta$  and the Riemann hypothesis to a  $p$ -adic context forcing the sharpening of the Riemann hypothesis to the conjecture that  $p^{iy}$  defines a rational phase factor for all non-trivial zeros  $x + iy$  of  $\zeta$  and for all primes  $p$  [Pit95]. Here however only the Riemann hypothesis is discussed.

The core elements of the construction are following.

a) All complex numbers are candidates for the eigenvalues of  $D^+$  and genuine eigenvalues are selected by the requirement that the condition  $D^\dagger = D^+$  holds true in the set of the genuine eigenfunctions. This condition is equivalent with the Hermiticity of the Hermitian metric defined by a function proportional to  $\zeta$ .

b) The eigenvalues turn out to consist of  $z = 0$  and the non-trivial zeros of  $\zeta$  and only the zeros with  $Re[z] = 1/2$  correspond to the eigenfunctions having real norm. The vanishing of  $\zeta$  tells that the 'physical' positive norm eigenfunctions, which are *not* orthogonal to each other, are orthogonal to the the 'unphysical' negative norm eigenfunction associated with the eigenvalue  $z = 0$ . The requirement that the Hermitian form in question defines an inner product implies that the the sums

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

$z_{12} = 1 + i(y_1 + y_2)$  of the zeros  $z = 1/2 + y_i$ ,  $i = 1, 2$ , of  $\zeta$  correspond to almost-zeros of  $\zeta$  for large values of  $y_1 + y_2$ .

c) The theory allows supersymmetrization and second quantization in terms of the representations of a superconformal algebra associated with the operator  $D^+$  and containing the ordinary superconformal algebra [ISZ88] as its subalgebra. The states on the critical line correspond to the representations of the ordinary superconformal algebra acting as gauge symmetries. If one requires that this is also the case for the might-exist unphysical coherent states orthogonal to the vacuum state but off from the critical line, Riemann hypothesis follows by a reductio ad absurdum argument.

### **2. Modified form of the Hilbert-Polya conjecture**

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the Hermiticity of the Hilbert-Polya operator. This means introduction of the non-Hermitian operators  $D^+$  and  $D$  which are Hermitian conjugates of each other such that  $D^+$  has the nontrivial zeros of  $\zeta$  as its complex eigenvalues

$$(2) \quad D^+ \Psi = z \Psi \quad .$$

The counterparts of the so called coherent states [IZ80] are in question and the eigenfunctions of  $D^+$  are not expected to be orthogonal in general. The following construction is based on the idea that  $D^+$  also allows the eigenvalue  $z = 0$  and that the vanishing of  $\zeta$  at  $z$  expresses the orthogonality of the states with eigenvalue  $z = x + iy \neq 0$  and the state with eigenvalue  $z = 0$  which turns out to have a negative norm.

The trial

$$(3) \quad \begin{aligned} D &= L_0 + V \quad , \quad D^+ = -L_0 + V \\ L_0 &= t \frac{d}{dt} \quad , \quad V = \frac{d \log(F)}{d(\log(t))} = t \frac{dF}{dt} \frac{1}{F} \end{aligned}$$

is motivated by the requirement of invariance with respect to scalings  $t \rightarrow \lambda t$  and  $F \rightarrow \lambda F$ . The range of variation for the variable  $t$  consists of non-negative real numbers  $t \geq 0$ . The scaling invariance implying conformal invariance (Virasoro generator  $L_0$  represents scaling which plays a fundamental role in the superconformal theories [ISZ88]) is motivated by the belief that  $\zeta$  codes for the physics of a quantum critical system having, not only supersymmetries [BK99], but also superconformal transformations as its basic symmetries [Pit95, Cas01].

### **3. Formal solution of the eigenvalue equation for operator $D^+$**

One can formally solve the eigenvalue equation

$$(4) \quad D^+ \Psi_z = \left[ -t \frac{d}{dt} + t \frac{dF}{dt} \frac{1}{F} \right] \Psi_z = z \Psi_z \quad .$$

for  $D^+$  by factoring the eigenfunction to a product:

$$(5) \quad \Psi_z = f_z F \ .$$

The substitution into the eigenvalue equation gives

$$(6) \quad L_0 f_z = t \frac{d}{dt} f_z = -z f_z$$

allowing as its solution the functions

$$(7) \quad f_z(t) = t^z \ .$$

These functions are nothing but eigenfunctions of the scaling operator  $L_0$  of the superconformal algebra analogous to the eigenstates of a translation operator. A priori all complex numbers  $z$  are candidates for the eigenvalues of  $D^+$  and one must select the genuine eigenvalues by applying the requirement  $D^\dagger = D^+$  in the space spanned by the genuine eigenfunctions.

It must be emphasized that  $\Psi_z$  is *not* an eigenfunction of  $D$ . Indeed, one has

$$(8) \quad D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z \ .$$

This is in accordance with the analogy with the coherent states which are eigenstates of annihilation operator but not those of creation operator.

#### 4. $D^+ = D^\dagger$ condition and Hermitian form

The requirement that  $D^+$  is indeed the Hermitian conjugate of  $D$  implies that the Hermitian form satisfies

$$(9) \quad \langle f | D^+ g \rangle = \langle Df | g \rangle \ .$$

This condition implies

$$(10) \quad \langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \langle D\Psi_{z_1} | \Psi_{z_2} \rangle \ .$$

The first (not quite correct) guess is that the Hermitian form is defined as an integral of the product  $\bar{\Psi}_{z_1} \Psi_{z_2}$  of the eigenfunctions of the operator  $D$  over the non-negative real axis using a suitable integration measure. The Hermitian form can be defined by continuing the integrand from the non-negative real axis to the entire complex  $t$ -plane and noticing that it has a cut along the non-negative real axis. This suggests the definition of the Hermitian form, not as a mere integral over the non-negative real axis, but as a contour integral along curve  $C$  defined so that it encloses the non-negative real axis, that is  $C$

a) traverses the non-negative real axis along the line  $Im[t] = 0_-$  from  $t = \infty + i0_-$  to  $t = 0_+ + i0_-$ ,

b) encircles the origin around a small circle from  $t = 0_+ + i0_-$  to  $t = 0_+ + i0_+$ ,

c) traverses the non-negative real axis along the line  $Im[t] = 0_+$  from  $t = 0_+ + i0_+$  to  $t = \infty + i0_+$  .

Here  $0_\pm$  signifies taking the limit  $x = \pm\epsilon$ ,  $\epsilon > 0$ ,  $\epsilon \rightarrow 0$ .

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

$C$  is the correct choice if the integrand defining the inner product approaches zero sufficiently fast at the limit  $Re[t] \rightarrow \infty$ . Otherwise one must assume that the integration contour continues along the circle  $S_R$  of radius  $R \rightarrow \infty$  back to  $t = \infty + i0_-$  to form a closed contour. It however turns out that this is not necessary. One can deform the integration contour rather freely: the only constraint is that the deformed integration contour does not cross over any cut or pole associated with the analytic continuation of the integrand from the non-negative real axis to the entire complex plane.

Scaling invariance dictates the form of the integration measure appearing in the Hermitian form uniquely to be  $dt/t$ . The Hermitian form thus obtained also makes possible to satisfy the crucial  $D^+ = D^\dagger$  condition. The Hermitian form is thus defined as

$$(11) \quad \langle f|g \rangle = -\frac{K}{2\pi i} \int_C \bar{f} g \frac{dt}{t} .$$

$K$  is a numerical constant to be determined later. The possibility to deform the shape of  $C$  in wide limits realizes conformal invariance stating that the change of the shape of the integration contour induced by a conformal transformation, which is nonsingular inside the integration contour, leaves the value of the contour integral of an analytic function unchanged. This scaling invariant Hermitian form is indeed a correct guess. By applying partial integration one can write

$$(12) \quad \langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \langle D \Psi_{z_1} | \Psi_{z_2} \rangle - \frac{K}{2\pi i} \int_C dt \frac{d}{dt} [\bar{\Psi}_{z_1}(t) \Psi_{z_2}(t)] .$$

The integral of a total differential comes from the operator  $L_0 = td/dt$  and must vanish. For a non-closed integration contour  $C$  the boundary terms from the partial integration could spoil the  $D^+ = D^\dagger$  condition unless the eigenfunctions vanish at the end points of the integration contour ( $t = \infty + i0_\pm$ ).

The explicit expression of the Hermitian form is given by

$$(13) \quad \begin{aligned} \langle \Psi_{z_1} | \Psi_{z_2} \rangle &= -\frac{K}{2\pi i} \int_C \frac{dt}{t} F^2(t) t^{z_{12}} , \\ z_{12} &= \bar{z}_1 + z_2 . \end{aligned}$$

It must be emphasized that it is  $\bar{\Psi}_{z_1} \Psi_{z_2}$  rather than eigenfunctions which is continued from the non-negative real axis to the complex  $t$ -plane: therefore one indeed obtains an analytic function as a result.

An essential role in the argument claimed to prove the Riemann hypothesis is played by the crossing symmetry

$$(14) \quad \langle \Psi_{z_1} | \Psi_{z_2} \rangle = \langle \Psi_0 | \Psi_{\bar{z}_1 + z_2} \rangle$$

of the Hermitian form. This symmetry is analogous to the crossing symmetry of particle physics stating that the S-matrix is symmetric with respect to the replacement of the particles in the initial state with their antiparticles in the final state or vice versa [IZ80].

The Hermiticity of the Hermitian form implies

$$(15) \quad \langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle} .$$

This condition, which is *not* trivially satisfied, in fact determines the eigenvalue spectrum.

### 5. How to choose the function $F$ ?

The remaining task is to choose the function  $F$  in such a manner that the orthogonality conditions for the solutions  $\Psi_0$  and  $\Psi_z$  reduce to the condition that  $\zeta$  or some function proportional to  $\zeta$  vanishes at the point  $-z$ . The definition of  $\zeta$  based on analytical continuation performed by Riemann suggests how to proceed. Recall that the expression of  $\zeta$  converging in the region  $Re[s] > 1$  reads [Tit86] as

$$(16) \quad \Gamma(s)\zeta(s) = \int_0^\infty \frac{dt}{t} \frac{\exp(-t)}{[1 - \exp(-t)]} t^s .$$

One can analytically continue this expression to a function defined in the entire complex plane by noticing that the integrand is discontinuous along the cut extending from  $t = 0$  to  $t = \infty$ . Following Riemann it is however more convenient to consider the discontinuity for a function obtained by multiplying the integrand with the factor

$$(-1)^s \equiv \exp(-i\pi s) .$$

The discontinuity  $Disc(f) \equiv f(t) - f(\text{te}^{i2\pi})$  of the resulting function is given by

$$(17) \quad Disc \left[ \frac{\exp(-t)}{[1 - \exp(-t)]} (-t)^{s-1} \right] = -2i \sin(i\pi s) \frac{\exp(-t)}{[1 - \exp(-t)]} t^{s-1} .$$

The discontinuity vanishes at the limit  $t \rightarrow 0$  for  $Re[s] > 1$ . Hence one can define  $\zeta$  by modifying the integration contour from the non-negative real axis to an integration contour  $C$  enclosing non-negative real axis defined in the previous section.

This amounts to writing the analytical continuation of  $\zeta(s)$  in the form

$$(18) \quad -2i\Gamma(s)\zeta(s)\sin(i\pi s) = \int_C \frac{dt}{t} \frac{\exp(-t)}{[1 - \exp(-t)]} (-t)^{s-1} .$$

This expression equals to  $\zeta(s)$  for  $Re[s] > 1$  and defines  $\zeta(s)$  in the entire complex plane since the integral around the origin eliminates the singularity.

The crucial observation is that the integrand on the righthand side of Eq. 18 has precisely the same general form as that appearing in the Hermitian form defined in Eq. 13 defined using the same integration contour  $C$ . The integration measure is  $dt/t$ , the factor  $t^s$  is of the same form as the factor  $t^{\bar{z}_1+z_2}$  appearing in the Hermitian form, and the function  $F^2(t)$  is given by

$$F^2(t) = \frac{\exp(-t)}{1 - \exp(-t)} .$$

Therefore one can make the identification

## RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE

$$(19) \quad F(t) = \left[ \frac{\exp(-t)}{1 - \exp(-t)} \right]^{1/2} .$$

Note that the argument of the square root is non-negative on the non-negative real axis and that  $F(t)$  decays exponentially on the non-negative real axis and has  $1/\sqrt{t}$  type singularity at origin. From this it follows that the eigenfunctions  $\Psi_z(t)$  approach zero exponentially at the limit  $Re[t] \rightarrow \infty$  so that one can use the non-closed integration contour  $C$ .

With this assumption, the Hermitian form reduces to the expression

$$(20) \quad \begin{aligned} \langle \Psi_{z_1} | \Psi_{z_2} \rangle &= -\frac{K}{2\pi i} \int_C \frac{dt}{t} \frac{\exp(-t)}{[1 - \exp(-t)]} (-t)^{z_{12}} \\ &= \frac{K}{\pi} \sin(i\pi z_{12}) \Gamma(z_{12}) \zeta(z_{12}) . \end{aligned}$$

Recall that the definition  $z_{12} = \bar{z}_1 + z_2$  is adopted. Thus the orthogonality of the eigenfunctions is equivalent to the vanishing of  $\zeta(z_{12})$ .

### 6. Study of the Hermiticity condition

In order to derive information about the spectrum one must explicitly study what the statement that  $D^\dagger$  is Hermitian conjugate of  $D$  means. The defining equation is just the generalization of the equation

$$(21) \quad A_{mn}^\dagger = \bar{A}_{nm} .$$

defining the notion of Hermiticity for matrices. Now indices  $m$  and  $n$  correspond to the eigenfunctions  $\Psi_{z_i}$ , and one obtains

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = z_2 \langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | D \Psi_{z_1} \rangle} = \overline{\langle D^+ \Psi_{z_2} | \Psi_{z_1} \rangle} = z_2 \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle} .$$

Thus one has

$$(22) \quad \begin{aligned} G(z_{12}) &= \overline{G(z_{21})} = \overline{G(\bar{z}_{12})} \\ G(z_{12}) &\equiv \langle \Psi_{z_1} | \Psi_{z_2} \rangle . \end{aligned}$$

The condition states that the Hermitian form defined by the contour integral is indeed Hermitian. This is *not* trivially true. Hermiticity condition obviously determines the spectrum of the eigenvalues of  $D^+$ .

To see the implications of the Hermiticity condition, one must study the behaviour of the function  $G(z_{12})$  under complex conjugation of both the argument and the value of the function itself. To achieve this one must write the integral

$$G(z_{12}) = -\frac{K}{2\pi i} \int_C \frac{dt}{t} \frac{\exp(-t)}{[1 - \exp(-t)]} (-t)^{z_{12}}$$

in a form from which one can easily deduce the behaviour of this function under complex conjugation. To achieve this, one must perform the change  $t \rightarrow u = \log(\exp(-i\pi)t)$  of the integration variable giving

$$G(z_{12}) = -\frac{K}{2\pi i} \int_D du \frac{\exp(-\exp(u))}{[1 - \exp(-(\exp(u)))]} \exp(z_{12}u) . \quad (23)$$

Here  $D$  denotes the image of the integration contour  $C$  under  $t \rightarrow u = \log(-t)$ .  $D$  is a fork-like contour which

- a) traverses the line  $Im[u] = i\pi$  from  $u = \infty + i\pi$  to  $u = -\infty + i\pi$ ,
- b) continues from  $-\infty + i\pi$  to  $-\infty - i\pi$  along the imaginary  $u$ -axis (it is easy to see that the contribution from this part of the contour vanishes),
- c) traverses the real  $u$ -axis from  $u = -\infty - i\pi$  to  $u = \infty - i\pi$ ,

The integrand differs on the line  $Im[u] = \pm i\pi$  from that on the line  $Im[u] = 0$  by the factor  $\exp(\mp i\pi z_{12})$  so that one can write  $G(z_{12})$  as integral over real  $u$ -axis

$$G(z_{12}) = -\frac{2K}{\pi} \sin(i\pi z_{12}) \int_{-\infty}^{\infty} du \frac{\exp(-\exp(u))}{[1 - \exp(-(\exp(u)))]} \exp(z_{12}u) . \quad (24)$$

From this form the effect of the transformation  $G(z) \rightarrow \overline{G(\bar{z})}$  can be deduced. Since the integral is along the real  $u$ -axis, complex conjugation amounts only to the replacement  $z_{21} \rightarrow z_{12}$ , and one has

$$\begin{aligned} \overline{G(\bar{z}_{12})} &= -\frac{2K}{\pi} \overline{\sin(i\pi z_{12})} \int_{-\infty}^{\infty} du \frac{\exp(-\exp(u))}{[1 - \exp(-(\exp(u)))]} \exp(z_{12}u) \\ (25) \quad &= -\frac{\overline{\sin(\pi z_{12})}}{\sin(\pi z_{12})} G(z_{12}) . \end{aligned}$$

The substitution of this result to the Hermiticity condition gives

$$(26) \quad G(z_{12}) = -\frac{\overline{\sin(i\pi z_{12})}}{\sin(i\pi z_{12})} G(z_{12}) \text{ for } x_1 + x_2 \leq 1 .$$

There are two manners to satisfy the Hermiticity condition.

- a) The condition

$$(27) \quad G(z_{12}) = 0$$

is the only manner to satisfy the Hermiticity condition for  $x_1 + x_2 < 1$  and  $y_2 - y_1 \neq 0$ . This implies the vanishing of  $\zeta$ :

$$(28) \quad \zeta(z_{12}) = 0 \text{ for } 0 < x_1 + x_2 < 1 , y_1 \neq y_2 .$$

In particular, this condition must be true for  $z_1 = 0$  and  $z_2 = 1/2 + iy$ . Hence the eigenfunctions with the eigenvalue  $z = 1/2 + iy$  correspond to the zeros of  $\zeta$ .

- b) The condition

$$(29) \quad \frac{\overline{\sin(i\pi z_{12})}}{\sin(i\pi z_{12})} = -1 ,$$

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

implying

$$(30) \quad \exp(-\pi i(x_1 + x_2)) = 1 \quad ,$$

is satisfied. This condition is satisfied for  $x_1 + x_2 = n$ . The highly non-trivial implication is that the states  $\Psi_z$  having *real norm* and  $0 < \text{Re}[z] < 1$  correspond to the zeros of  $\zeta$  on the line  $\text{Re}[s] = 1/2$ . Thus the study of mere Hermiticity conditions almost proves the Riemann hypothesis.

### **7. Does the Hermitian form define inner product?**

Before considering the question whether the Hermitian form defines a positive definite Hilbert space inner product, a couple of comments concerning the general properties of the Hermitian form are in order.

a) The Hermitian form is proportional to the factor

$$\sin(i\pi(y_2 - y_1)) \quad ,$$

which vanishes for  $y_1 = y_2$ . For  $y_1 = y_2$  and  $x_1 + x_2 = 1$  ( $x_1 + x_2 = 0$ ) the diverging factor  $\zeta(1)$  ( $\zeta(0)$ ) compensates the vanishing of this factor. Therefore the norms of the eigenfunctions  $\Psi_z$  with  $z = 1/2 + iy$  must be calculated explicitly from the defining integral. Since the contribution from the cut vanishes in this case, one obtains only an integral along a small circle around the origin. This gives the result

$$(31) \quad \langle \Psi_{z_1} | \Psi_{z_1} \rangle = K \quad \text{for} \quad z_1 = \frac{1}{2} + iy \quad , \quad \langle \Psi_0 | \Psi_0 \rangle = -\frac{K}{2} \quad .$$

Thus the norms of the eigenfunctions are finite. For  $K = 1$  the norms of  $z = 1/2 + iy$  eigenfunctions are equal to one.  $\Psi_0$  has however negative norm  $-1/2$  so that the Hermitian form in question is not a genuine inner product in the space containing  $\Psi_0$ .

b) For  $x_1 = x_2 = 1/2$  and  $y_1 \neq y_2$  the factor is nonvanishing and one has

$$(32) \quad \langle \Psi_{z_1} | \Psi_{z_2} \rangle = -\frac{1}{\pi} \zeta(1 + i(y_2 - y_1)) \Gamma(1 + i(y_2 - y_1)) \sinh(\pi(y_2 - y_1)) \quad .$$

The nontrivial zeros of  $\zeta$  are known to belong to the critical strip defined by  $0 < \text{Re}[s] < 1$ . Indeed, the theorem of Hadamard and de la Vallee Poussin [Var99] states the non-vanishing of  $\zeta$  on the line  $\text{Re}[s] = 1$ . Since the non-trivial zeros of  $\zeta$  are located symmetrically with respect to the line  $\text{Re}[s] = 1/2$ , this implies that the line  $\text{Re}[s] = 0$  cannot contain zeros of  $\zeta$ . This result implies that the states  $\Psi_{z=1/2+iy}$  are non-orthogonal unless  $\Gamma(1 + i(y_2 - y_1))$  vanishes for some pair of eigenfunctions.

It is quite possible that the Hermitian form in question defines an inner product in the space spanned by the states  $\Psi_z$ ,  $z = 1/2 + iy$  having real and positive norm. Besides Hermiticity, a necessary condition for this is

$$|\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \leq 1$$

and gives

$$(33) \quad -\frac{1}{\pi}|\zeta(1 + iy_{12})| \times |\Gamma(1 + iy_{12}) \times |\sin(i\pi y_{12})| \leq 1 \quad ,$$

where the shorthand notation  $y_{12} = y_2 - y_1$  has been used. The diagonalized metric is positive definite if  $G(1/2 + iy_{12})$  approaches zero sufficiently fast for large values of argument  $y_2 - y_1$  so that the nondiagonal part of the metric can be regarded as a small perturbation. On physical grounds this is to be expected since coherent states should have overlap which is essentially Gaussian function of the distance  $y_2 - y_1$ .  $\sin(i\pi y_{12})$  however increases exponentially and this growth must be compensated by the behaviour of the the remaining terms.

To get some grasp on the behaviour of the Hermitian metric, one can use the integral formula

$$(34) \quad \begin{aligned} \zeta(s) &= 1 + \frac{1}{s-1} + \frac{1}{\Gamma(s)}I(s) \quad , \\ I(s) &= \int_0^\infty \frac{dt}{t} \left[ \frac{1}{[1 - \exp(-t)]} - \frac{1}{t} \right] \exp(-t)t^s \quad , \\ \Gamma(s) &= \int_0^\infty \frac{dt}{t} \exp(-t)t^s \end{aligned}$$

proved already by Riemann. Applying the formula in present case, one has

$$(35) \quad \begin{aligned} G(1 + iy_{12}) &= -\frac{1}{\pi i} \sinh(\pi y_{12}) \Gamma(1 + iy_{12}) \left(1 + \frac{1}{iy_{12}}\right) \\ &\quad - \frac{1}{\pi i} \sinh(\pi y_{12}) I(1 + iy_{12}) \quad . \end{aligned}$$

Hyperbolic sine increases exponentially as a function of  $y_{12}$  but cannot spoil the Gaussian decay suggested by the analogy with the coherent states. One can try to demonstrate the Gaussian behaviour by an approximate evaluation of the integrals appearing on the left hand side by changing the integration variable to  $t = \exp(u)$ . This gives

$$(36) \quad \begin{aligned} I(1 + iy_{12}) &= \int_{-\infty}^\infty du \left[ \frac{1}{[1 - \exp(-e^u)]} - \frac{1}{e^u} \right] \exp(-e^u + u + iy_{12}u) \quad , \\ \Gamma(1 + iy_{12}) &= \int_{-\infty}^\infty du \exp(-e^u + u + iy_{12}u) \quad . \end{aligned}$$

The exponential term has a maximum at  $u = 0$  and vanishes extremely rapidly as a function of  $u$  for  $u > 0$ . The troublesome feature is that for  $u < 0$  the integrand decays only exponentially and is expected to give a slowly decreasing contribution which oscillates as a function of  $y$ . Thus it seems that a Gaussian, and even an exponential, overall decay is excluded.

The analogy with the coherent states however requires that the integral decomposes to a Gaussian term plus an oscillating remainder which becomes very small for  $y = y_{12}$ . Even the inner product property requires that the oscillating term decays faster than  $\exp(-\pi y_{12})$  as a function of  $y_{12}$ . The needed faster than  $\exp(-\pi y_{12})$  decay requires that that the points  $y = y_2 - y_1$  are approximate zeros of  $G(1 + iy)$ ,

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

that is approximate zeros of  $\zeta(1 + iy)$  or, less probably, those of  $\Gamma(1 + iy)$ . The mechanism giving rise to an approximate zero would be a cancellation of the terms proportional to  $\Gamma(1 + iy)$  and  $I(1 + iy)$  in the expression of Eq. 34 for  $\zeta$ . An extremely intricate organization of the apparently chaotically located zeros and almost-zeros of  $\zeta$  is required to guarantee that the Hermitian form defines an inner product. Whether the differences  $y = y_2 - y_1$  represent approximate zeros of  $\zeta(1 + iy)$  on the line  $Re[s] = 1$ , can be tested numerically.

That the behaviour of  $\zeta(1 + iy)$  as a function of  $y$  can be regarded as a superposition of a Gaussian term and an oscillating term, is suggested by the following argument. If the Gaussian approximation around the origin

$$-exp(u) + u \simeq -\frac{u^2}{2}$$

were a good approximation, the integrals in question would reduce to Gaussian integrals

$$\begin{aligned} I(1 + iy) &\simeq \frac{1}{e-1} J(y) \ , \\ \Gamma(1 + iy) &\simeq J(y) \ , \\ (37) \quad J(y) &= \int_{-\infty}^{\infty} du \exp\left(-\frac{u^2}{2} + iyu\right) = \sqrt{2\pi} \exp\left(-\frac{y^2}{2}\right) \ . \end{aligned}$$

Thus one would have

$$(38) \quad G(1 + iy) \simeq -\frac{\sqrt{2}}{\sqrt{\pi i}} \sinh(\pi y) \exp\left(-\frac{y^2}{2}\right) \left[ \frac{1}{iy} + \frac{e}{e-1} \right] \ .$$

The behaviour would be indeed Gaussian for large values of  $y$ .

Possible problems are also caused by the small values of  $y_{12}$  for which one might have  $|G(1 + iy_{12})| > 1$  implying the failure of the Schwartz inequality

$$(39) \quad |\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \leq |\Psi_{z_1}| |\Psi_{z_2}|$$

characterizing positive definite metric. In the Gaussian approximation the value of  $|G(1 + iy_{12})|$  at the limit  $y_{12} = 0$  is  $\sqrt{2\pi} \simeq 2.5066$  so that the danger is real. The direct calculation of  $G(1 + iy)$  at the limit  $y \rightarrow 0$  by using  $\zeta(1 + iy) \simeq 1/iy$  however gives

$$(40) \quad G(1) = 1 \ .$$

By a straightforward calculation one can also verify that  $z = 1$  is a local maximum of  $|G(z)|$ .

Intuitively it seems obvious that Schwartz inequality must hold true quite generally. The point is that the might-be inner product for the superpositions  $\sum_y f(y) \Psi_{1/2+iy}$  and  $\sum_y g(y) \Psi_{1/2+iy}$  of  $\Psi_z$  describes net correlation for the functions  $\bar{f}(y)$  and  $g(y)$ . This correlation can be written as

$$(41) \quad \langle f | g \rangle = \sum_{y_1, y_2} \bar{f}(y_1) G(1 + i(y_1 - y_2)) g(y_2) \ .$$

Since  $G(1 + i(y_1 - y_2))$  decays like Gaussian, the correlation of the functions  $\bar{f}$  and  $g$  is determined mainly by the correlation  $\bar{f}$  and  $g$  at very small distances  $y_1 - y_2$ . It is obvious that correlation is largest when  $f$  and  $g$  resemble each other maximally, that is when one has  $f = g$ .

It is easy to see that arbitrary small values of  $y_{12}$  are unavoidable. The estimate of Riemann for the number of the zeros of  $\zeta$  in the interval  $Im[s] \in [0, T]$  along the line  $Re[s] = 1/2$  reads as

$$(42) \quad N(T) \simeq \frac{T}{2\pi} \left[ \log\left(\frac{T}{2\pi}\right) - 1 \right] ,$$

and allows to estimate the average density  $dN_T/dy$  of the zeros and to deduce an upper limit for the minimum distance  $y_{12}^{min}$  between two zeros in the interval  $T$ :

$$(43) \quad \begin{aligned} \frac{dN_T}{dy} &\simeq \frac{1}{2\pi} \left[ \log\left(\frac{T}{2\pi}\right) - 1 \right] , \\ y_{12}^{min} &\leq \frac{1}{\frac{dN_T}{dy}} = \frac{2\pi}{\left[ \log\left(\frac{T}{2\pi}\right) - 1 \right]} \rightarrow 0 \text{ for } T \rightarrow \infty . \end{aligned}$$

This implies that arbitrary small values of  $y_{12}$  are unavoidable. Thus a rigorous proof for  $|G(1 + i(y_1 + y_2))| < 1$  for  $y_1 + y_2 \neq 0$  is required.

## 8. Superconformal symmetry

The reduction ad absurdum argument to be discussed below relies on the assumption that the orthogonality of  $\Psi_w$  with  $\Psi_0$  for  $Re[w] < 1/2$  implies the orthogonality of  $\Psi_w$  with all eigenfunctions  $\Psi_z$ ,  $z = 1/2 + iy$  zero of  $\zeta$ . In other words, the vanishing of  $\zeta(w)$  implies the vanishing of  $\zeta(w + z)$  for any zero  $z$  of  $\zeta$ , and one has an infinite number of zeros on the line  $Re[s] = Re[w] + 1/2$ .

This means the decomposition of the space of the eigenfunctions orthogonal with respect to  $\Psi_0$  to a direct sum  $V = \oplus_{x < 1/2} V_x \oplus H_{1/2}$ , such that  $V_x$  (for which Hermitian form is not inner product) contains the non-orthogonal eigenfunctions  $\Psi_{x+iy}$  and  $\Psi_{1-x-iy}$  and the spaces  $H_x$  and  $H_{1/2}$  are orthogonal to each other for each value of  $x$ . The requirement that the eigenfunctions having a positive norm are orthogonal to the eigenfunctions with complex norm and orthogonal to the state  $\Psi_0$ , looks very natural but it is not easy to justify rigorously this assumption without assuming some kind of a symmetry.

Here superconformal symmetry, which stimulated the idea behind the proposed proof of the Riemann hypothesis, could come in rescue. First of all, one can 'understand' the restriction of the non-trivial zeros to the line  $Re[s] = 1/2$  by noticing that  $x$  can be interpreted as the real part of conformal weight defined as eigenvalue of the scaling operator  $L_0 = td/dt$  in superconformal field theories [ISZ88, Pit90, Pit95]. For the generators of the superconformal algebra, conformal weights are indeed half-integer valued. The following construction is essentially a construction of a second-quantized superconformal quantum field theory for the system described by  $D^+$ .

One can indeed identify a conformal algebra naturally associated with the proposed dynamical system. The generators

**RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

$$(44) \quad L_z = \Psi_z D^+$$

generate conformal algebra with commutation relations ( $[A, B] \equiv AB - BA$ )

$$(45) \quad [L_{z_1}, L_{z_2}] = (z_2 - z_1)L_{z_1+z_2} .$$

The extension of this algebra to superconformal algebra requires the introduction of the fermionic generators  $G_z$  and  $G_z^\dagger$ . To avoid confusions it must be emphasized that following convention concerning Hermitian conjugation is adopted to make notation more fluent:

$$(46) \quad (O_w)^\dagger = O_{\bar{w}}^\dagger .$$

Fermionic generators  $G_z$  and  $G_z^\dagger$  satisfy the following anticommutation and commutation relations:

$$(47) \quad \{G_{z_1}, G_{z_2}^\dagger\} = L_{z_1+z_2} , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} , \quad [L_{z_1}, G_{z_2}^\dagger] = -z_2 G_{z_1+z_2}^\dagger .$$

This definition differs from that used in the standard approach [ISZ88] in that generators  $G_z$  and  $G_z^\dagger$  are introduced separately. Usually one introduces only the generators  $G_n$  and assumes Hermiticity condition  $G_{-n} = G_n^\dagger$ . The anticommutation relations of  $G_z$  contain usually also central extension term. Now this term is not present as will be found.

Conformal algebras are accompanied by Kac Moody algebra which results as a central extension of the algebra of the local gauge transformations for some Lie group on circle or line [ISZ88]. In the standard approach Kac Moody generators are Hermitian in the sense that one has  $T_{-n} = T_n^\dagger$  [ISZ88]. Now this condition is dropped and one introduces also the generators  $T_z^\dagger$ . In present case the counterparts for the generators  $T_z^\dagger$  of the local gauge transformations act as translations  $z_1 \rightarrow z_1 + z$  in the index space labelling eigenfunctions and geometrically correspond to the multiplication of  $\Psi_{z_1}$  with the function  $t^z$

$$(48) \quad T_{z_1}^\dagger \Psi_{z_2} = t^{z_1} \Psi_{z_2} = \Psi_{z_1+z_2} .$$

These transformations correspond to the isometries of the Hermitian form defined by  $G(z_{12})$  and are therefore natural symmetries at the level of the entire space of the eigenfunctions.

The commutation relations with the conformal generators follow from this definition and are given by

$$(49) \quad [L_{z_1}, T_{z_2}] = z_2 T_{z_1+z_2} , \quad [L_{z_1}, T_{z_2}^\dagger] = -z_2 T_{z_1+z_2}^\dagger ,$$

The central extension making this commutative algebra to Kac-Moody algebra is proportional to the Hermitian metric

$$(50) \quad [T_{z_1}, T_{z_2}] = 0 , \quad [T_{z_1}^\dagger, T_{z_2}^\dagger] = 0 , \quad [T_{z_1}^\dagger, T_{z_2}] = (z_1 - z_2)G(z_1 + z_2) .$$

One could also consider the central extension  $[T_{z_1}^\dagger, T_{z_2}] = G(z_1 + z_2)$ , which is however not the standard Kac-Moody central extension.

One can extend Kac Moody algebra to a super Kac Moody algebra by adding the fermionic generators  $Q_z$  and  $Q_z^\dagger$  obeying the anticommutation relations ( $\{A, B\} \equiv AB + BA$ )

$$(51) \quad \{Q_{z_1}, Q_{z_2}\} = 0 \quad , \quad \{Q_{z_1}^\dagger, Q_{z_2}^\dagger\} = 0 \quad , \quad \{Q_{z_1}, Q_{z_2}^\dagger\} = G(z_1 + z_2) \quad .$$

Note that also  $Q_0$  has a Hermitian conjugate  $Q_0^\dagger$ , and one has

$$(52) \quad \{Q_0, Q_0^\dagger\} = G(0) = -\frac{1}{2}$$

implying that also the fermionic counterpart of  $\Psi_0$  has negative norm. One can identify the fermionic generators as the gamma matrices of the infinite-dimensional Hermitian space spanned by the eigenfunctions  $\Psi_z$ . By their very definition, the complexified gamma matrices  $\Gamma_{\bar{z}_1}$  and  $\Gamma_{z_2}$  anticommute to the Hermitian metric  $\langle \Psi_{z_1} | \Psi_{z_2} \rangle = G(\bar{z}_1 + z_2)$ .

The commutation relations of the conformal and Kac Moody generators with the fermionic generators are given by

$$(53) \quad \begin{aligned} [L_{z_1}, Q_{z_2}] &= z_2 Q_{z_1+z_2} \quad , \quad [L_{z_1}, Q_{z_2}^\dagger] = -z_2 Q_{z_1+z_2}^\dagger \quad , \\ [T_{z_1}, Q_{z_2}] &= 0 \quad , \quad [T_{z_1}, Q_{z_2}^\dagger] = 0 \quad . \end{aligned}$$

The nonvanishing commutation relations of  $T_z$  with  $G_z$  and nonvanishing anticommutation relations of  $Q_z$  with  $G_z$  are given by

$$(54) \quad \begin{aligned} [G_{z_1}, T_{z_2}^\dagger] &= Q_{z_1+z_2} \quad , \quad [G_{z_1}^\dagger, T_{z_2}] = -Q_{z_1+z_2}^\dagger \quad , \\ \{G_{z_1}, Q_{z_2}^\dagger\} &= T_{z_1+z_2} \quad , \quad \{G_{z_1}^\dagger, Q_{z_2}\} = T_{z_1+z_2}^\dagger \quad . \end{aligned}$$

Superconformal generators clearly transform bosonic and fermionic Super Kac-Moody generators to each other.

The final step is to construct an explicit representation for the generators  $G_z$  and  $L_z$  in terms of the Super Kac Moody algebra generators as a generalization of the Sugawara representation [ISZ88]. To achieve this, one must introduce the inverse  $G^{-1}(z_a z_b)$  of the metric tensor  $G(z_a z_b) \equiv \langle \Psi_{z_a} | \Psi_{z_b} \rangle$ , which geometrically corresponds to the contravariant form of the Hermitian metric defined by  $G$ . Adopting these notations, one can write the generalization for the Sugawara representation of the superconformal generators as

$$(55) \quad \begin{aligned} G_z &= \sum_{z_a} T_{z+z_a} G^{z_a z_b} Q_{z_b}^\dagger \quad , \\ G_z^\dagger &= \sum_{z_a} T_{z+z_a}^\dagger G^{z_a z_b} Q_{z_b} \quad . \end{aligned}$$

One can easily verify that the commutation and anticommutation relations with the super Kac-Moody generators are indeed correct. The generators  $L_z$  are obtained as the anticommutators of the generators  $G_z$  and  $G_z^\dagger$ . Due to the introduction of the generators  $T_z, T_z^\dagger$  and  $G_z, G_z^\dagger$ , the anticommutators  $\{G_{z_1}, G_{z_2}^\dagger\}$  do not contain any

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

central extension terms. The expressions for the anticommutators however contains terms of form  $T^\dagger T Q^\dagger Q$  whereas the generators in the usual Sugawara representation contain only bilinears of type  $T^\dagger T$  and  $Q^\dagger Q$ . The inspiration for introducing the generators  $T_z, G_z$  and  $T_z^\dagger, G_z^\dagger$  separately comes from the construction of the physical states as generalized superconformal representations in quantum TGD [Pit90]. The proposed algebra differs from the standard superconformal algebra [ISZ88] also in that the indices  $z$  are now complex numbers rather than half-integers or integers as in the case of the ordinary superconformal algebras [ISZ88]. It must be emphasized that one could also consider the commutation relations  $[T_{z_1}^\dagger, T_{z_2}] = iG(z_1 + z_2)$  and they might be more the physical choice since  $z_2 - z_1$  is now a complex number unlike for ordinary superconformal representations. It is not however clear how and whether one could construct the counterpart of the Sugawara representation in this case.

Imitating the standard procedure used in the construction of the representations of the superconformal algebras [ISZ88], one can assume that the vacuum state is annihilated by *all* generators  $L_z$  irrespective of the value of  $z$ :

$$(56) \quad L_z|0\rangle = 0 \quad , \quad G_z|0\rangle = 0 \quad .$$

That all generators  $L_z$  annihilate the vacuum state follows from the representation  $L_z = \Psi_z D_+$  because  $D_+$  annihilates  $\Psi_0$ . If  $G_0$  annihilates vacuum then also  $G_z \propto [L_z, G_0]$  does the same.

The action of  $T_z^\dagger$  on an eigenfunction is simply a multiplication by  $t^z$ : therefore one cannot require that  $T_z$  annihilates the vacuum state as is usually done [ISZ88]. The action of  $T_0$  is multiplication by  $t^0 = 1$  so that  $T^0$  and  $T_0^\dagger$  act as unit operators in the space of the physical states. In particular,

$$(57) \quad T_0|0\rangle = T_0^\dagger|0\rangle = |0\rangle \quad .$$

This implies the condition

$$(58) \quad [T_0, T_z^\dagger] = izG(z) = 0$$

in the space of the physical states so that physical states must correspond to the zeros of  $\zeta$  and possibly to  $z = 0$ . Thus one can generate the physical states from vacuum by acting using operators  $Q_z^\dagger$  and  $T_z^\dagger$  with  $\zeta(z) = 0$ . If one requires that the physical states also have real and positive norm squared, only the zeros of  $\zeta$  on the line  $Re[s] = 1/2$  are allowed. Hence the requirement that a unitary representation of the superconformal algebra is in question, forces Riemann hypothesis.

It is important to notice that  $T_z^\dagger$  and  $Q_z^\dagger$  cannot annihilate the vacuum: this would lead to the condition  $G(z_1 + z_2) = 0$  implying the vanishing of  $\zeta(z_1 + z_2)$  for any pair  $z_1 + z_2$ . One can however assume that  $Q_z$  annihilates the vacuum state

$$(59) \quad Q_z|0\rangle = 0 \quad .$$

This inspires the hypothesis that only the generators with conformal weights  $z = 1/2 + iy$  generate physical states from vacuum realizable in the space of the eigenfunctions  $\Psi_z$  and their fermionic counterparts. This means that the action of

the bosonic generators  $T_{1/2+iy}^\dagger$  and fermionic generators  $Q_0^\dagger$  and  $Q_{1/2+iy}^\dagger$ , as well as the action of the corresponding superconformal generators  $G_{1/2+iy}^\dagger$ , generates bosonic and fermionic states with conformal weight  $z = 1/2 + iy$  from the vacuum state:

$$(60) \quad |1/2 + iy\rangle_B \equiv T_{1/2+iy}^\dagger |0\rangle, \quad |1/2 + iy\rangle_F \equiv Q_{1/2+iy}^\dagger |0\rangle.$$

One can identify the states generated by the Kac Moody generators  $T_z^\dagger$  from the vacuum as the eigenfunctions  $\Psi_z$ . The system as a whole represents a second quantized supersymmetric version of the bosonic system defined by the eigenvalue equation for  $D^+$  obtained by assigning to each eigenfunction a fermionic counterpart and performing second quantization as a free quantum field theory.

### 9. Is the proof of the Riemann hypothesis by reductio ad absurdum possible using superconformal invariance?

Riemann hypothesis is proven if all eigenfunctions for which the Riemann Zeta function vanishes, correspond to the states having a real and positive norm squared. The expectation is that superconformal invariance realized in some sense excludes all zeros of  $\zeta$  except those on the line  $Re[s] = 1/2$ . The problem is to define precisely what one means with superconformal invariance and one can generate large number of reduction ad absurdum type proofs depending on how superconformal invariance is assumed to be realized.

The most conservative option is that superconformal invariance is realized in the standard sense. The action of the ordinary superconformal generators  $L_n$ , and  $G_n$ ,  $n \neq 0$  on the vacuum states  $|0\rangle_{B/F}$  or on any state  $|1/2 + iy\rangle_{B/F}$  indeed creates zero norm states as is obvious from the vanishing of the factor  $\sin(i\pi z_{12}) = \sin(\pi(x_1 + x_2))$  associated with the inner inner products of these states. Thus the zeros of  $\zeta$  define an infinite family of ground states for the representations of the ordinary superconformal algebra. A generalization of this hypothesis is that the action of  $L_n$  and  $G_n$ ,  $n \neq 0$ , on any state  $|w\rangle_{B/F}$ ,  $\zeta(w) = 0$ , creates states which are mutually orthogonal zero norm states. This implies  $\zeta(n + 2Re[w]) = 0$  for all values of  $n \neq 0$  and, since the real axis contains zeros of  $\zeta$  only at the points  $Re[s] = -2n$ ,  $n > 0$ , leads to a reductio ad absurdum unless one has  $Re[w] = 1/2$ . Thus the proof of the Riemann hypothesis would reduce to showing that the action of the ordinary superconformal algebra generates mutually orthogonal zero norm states from any state  $|w\rangle_{B/F}$  with  $\zeta(w) = 0$ . The proof of this physically plausible hypothesis is not obvious.

One can imagine also other strategies. The minimal requirement is certainly that some subalgebra of the superconformal algebra generates a space of states satisfying the Hermiticity condition. The quantity

$$(61) \quad \Delta(\bar{w}_1 + w_2) \equiv \langle w_1 | w_2 \rangle - \overline{\langle w_2 | w_1 \rangle} = G(\bar{w}_1 + w_2) - \overline{G(\bar{w}_2 + w_1)}$$

must define the conformal invariant in question since this quantity must vanish in the space of the physical states for which the metric is Hermitian. This requirement does not however imply anything nontrivial for the ordinary conformal algebra

## **RIEMANN HYPOTHESIS AND SUPERCONFORMAL INVARIANCE**

having generators  $L_n$  and  $G_n$ : for  $Re[w] \neq 1/2$  the condition is indeed satisfied because  $G(n + 2Re[w])$  does *not* satisfy the Hermiticity condition for any value of  $n$ .

One can try to abstract some property of the states associated with the zeros of  $\zeta$  on the line  $Re[s] = 1/2$ . The generators  $L_{1/2-iy}$  and  $G_{1/2-iy}$  generate zero norm states from the states  $|1/2 + iy\rangle_{B/F}$ , when  $1/2 + iy$  corresponds to the zero of  $\zeta$  on the line  $Re[s] = 1/2$ . One can try to generalize this observation so that it applies to an arbitrary state  $|w\rangle_{B/F}$ ,  $\zeta(w) = 0$ . The generators  $L_{1-\bar{w}}$  and  $G_{1-\bar{w}}$  certainly generate zero norm states from the states  $|w\rangle_{B/F}$ . Also the Hermiticity condition holds true identically and does not have nontrivial implications. One can however consider alternative generalizations by assuming that

- a) either the generators  $L_{\bar{w}}$  and  $G_{\bar{w}}$  or
- b)  $L_{1/2+iy}$  and  $G_{1/2+iy}$  generate from the states  $|w\rangle_{B/F}$ ,  $\zeta(w) = 0$  states satisfying the Hermiticity condition.

These two hypothesis lead to two versions of a reductio ad absurdum argument. Suppose that  $w$  is a zero of  $\zeta$ . This means that the inner product of the states  $Q_0^\dagger|0\rangle$  and  $Q_w^\dagger|0\rangle$  and thus also  $\Delta(w)$  vanishes:

$$(62) \quad \langle 0|Q_0Q_w^\dagger|0\rangle = 0 \quad , \quad \Delta(w) = 0 \quad .$$

a) By acting on this matrix element by the conformal algebra generator  $L_{\bar{w}}$  (which acts like derivative operator on the arguments of the should-be Hermitian form), and using the fact that  $L_{\bar{w}}$  annihilates the vacuum state, one obtains

$$(63) \quad \langle 0|Q_0Q_{\bar{w}+w}^\dagger|0\rangle = G(w + \bar{w}) \quad .$$

The requirement  $\Delta(w + \bar{w}) = 0$  implies the reality of  $G(w + \bar{w})$  and thus the condition  $Re[w] = 1/2$  leading to the Riemann hypothesis. Note that the argument implying the reality of  $G(w + \bar{w})$  assumes only that  $L_w$  annihilates vacuum.

If this line of approach is correct, the basic challenge would be to show on the basis of the superconformal invariance alone that the condition  $\zeta(w) = 0$  implies that the generators  $L_{\bar{w}}$  and  $G_{\bar{w}}$  generate new ground states satisfying the Hermiticity condition.

b) An alternative line of argument uses only the invariance under the generators  $L_{1/2+iy}$  associated with the zeros of  $\zeta$ , and thus certainly belonging to the conformal algebra associated with the physical states. By applying the generators  $L_{1/2+iy_i}$  to the the matrix element  $\langle 0|Q_0Q_w^\dagger|0\rangle = 0$  and requiring that Hermiticity is respected, one can deduce that  $G(w + 1/2 + iy_i)$  satisfies the Hermiticity condition. Hence the line  $Re[s] = Re[w] + 1/2$ , and by the reflection symmetry also the line  $Re[s] = 1/2 - Re[w]$ , contain an infinite number of zeros of  $\zeta$  if one has  $Re[w] \neq 1/2$ . By repeating this process once for the zeros on the line  $Re[s] = 1/2 - Re[w]$ , one finds that the lines  $Re[s] = 1 - Re[w]$  and  $Re[s] = Re[w]$  contain infinite number of the zeros of  $\zeta$  of form  $w_{ij} = w + i(y_i + y_j)$ , where  $y_i$  and  $y_j$  are associated with the zeros of  $\zeta$  on the line  $Re[s] = 1/2$ . By applying this two-step procedure repeatedly, one can fill the lines  $Re[s] = Re[w], 1 - Re[w], 1/2 - Re[w], 1/2 + Re[w]$  with the zeros of  $\zeta$ .

To sum up, contrary to the original over-optimistic beliefs inspired by the beauty of the proposed quantum model, Riemann hypothesis demonstrates once again that

it is equally resistible against proof as it is capable of stimulating new mathematical ideas. One might however hope that superconformal invariance could in one of the proposed forms or in some other form be used to rigorously prove Riemann hypothesis.

## References

- [BK99] M. V. Berry and J. P. Keating (1999), "H= $x\pi$  and the Riemann Zeros." In *Supersymmetry and Trace Formulae: Chaos and Disorder* (Ed. I. V. Lerner, J. P. Keating, and D. E. Khmel'nitskii). New York: Kluwer, pp. 355-367.
- [Cas01] C. Castro (2001), *On p-adic Stochastic Dynamics, Supersymmetry, and the Riemann Conjecture*, arXiv:physics/0101104. This paper led to the realization that generalization of Hilbert-Polya operator, when assumed to possess 'number theoretic' conformal symmetry strongly suggested by the previous p-adic considerations, might provide the route to the proof of Riemann hypothesis.
- [Edw74] H. M. Edwards (1974), *Riemann's Zeta Function*, Academic Press, New York, London.
- [ISZ88] C. Itzykson, H. Saleur, J-B. Zuber (Editors)(1988): *Conformal Invariance and Applications to Statistical Mechanics*, Word Scientific.
- [IZ80] C. Itzykson and J-B. Zuber (1980),"Quantum Field Theory", 549, New York: Mc Graw-Hill Inc.
- [Pit90] M. Pitkänen (1995) *Topological Geometrodynamics* Internal Report HU-TFT-IR-95-4 (Helsinki University). Summary of Topological Geometrodynamics in book form. Book can be found as .pdf files on my homepage. <http://www.physics.helsinki.fi/~matpitka/tgd.html>.
- [Pit95] M. Pitkänen (1995) *Topological Geometrodynamics and p-Adic Numbers*. Internal Report HU-TFT-IR-95-5 (Helsinki University). The chapter "Number theory and quantum TGD" contains sharpened form of Riemann hypothesis motivated by the generalization of Riemann hypothesis to p-adic context. Book can be found as .pdf files on my homepage. <http://www.physics.helsinki.fi/~matpitka/padtgd.html>.
- [Rie] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859* (1860), 671-680; also , *Gesammelte mat. Werke und wissenshc. Nachlass*, 2. Aufl. 1892, 145-155.
- [Var99] I. Vardi (1999), *An Introduction to Analytic Number Theory*, <http://algo.inria.fr/banderier/Seminar/Vardi/index.htm> .
- [Tit86] E. C. Titchmarsh (1986), *The Theory of the Riemann Zeta Function*, 2nd ed. revised by R. D. Heath-Brown, Oxford Univ. Press.

DEPT. OF PHYSICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND.

matpitka@rock.helsinki.fi, URL: <http://www.physics.helsinki.fi/~matpitka/>