%\begin{abstract}

Number theoretic Langlands program can be seen as an attempt to unify
number theory on one hand and theory of representations of reductive Lie
groups on one hand. So called automorphic functions to which various zeta
functions are closely related define the common denominator. Geometric
Langlands program tries to achieve a similar conceptual unification in the
case of function fields. This program has caught the interest of
physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional
K\"ahler geometry and spinor structure of the \blockquote{world of classical worlds}
(WCW). If TGD can be regarded also as a generalized number theory, it is
difficult to escape the idea that the interaction of Langlands program with
TGD could be fruitful. I of course hasten to confess that I am not number
theorists nor group theorists and that the following considerations are
just speculations inspired by TGD.

More concretely, TGD leads to a generalization of number concept based on
the fusion of reals and various p-adic number fields and their extensions
implying also a generalization of manifold concept, which inspires the
notion of number theoretic braid crucial for the formulation of quantum
TGD. TGD leads also naturally to the notion of infinite primes and
rationals. The identification of Clifford algebra of WCW in terms of
hyper-finite factors of type II$_1$ in turn inspires further generalization
of the notion of imbedding space and the idea that quantum TGD as a whole
emerges from number theory. The ensuing generalization of the notion of
imbedding space predicts a hierarchy of macroscopic quantum phases
characterized by finite subgroups of $SU(2)$ and by quantized Planck
constant. All these new elements serve as potential sources of fresh
insights.

\vm {\it 1. The Galois group for the algebraic closure of rationals

as
infinite symmetric group?}\vm

The naive identification of the Galois groups for the algebraic closure of
rationals would be as infinite symmetric group S_{∞} consisting of
finite permutations of the roots of a polynomial of infinite degree having
infinite number of roots. What puts bells ringing is that the corresponding
group algebra is nothing but the hyper-finite factor of type II$_1$ (HFF).
One of the many avatars of this algebra is infinite-dimensional Clifford
algebra playing key role in Quantum TGD. The projective representations of
this algebra can be interpreted as representations of braid algebra
B_{∞} meaning a connection with the notion of number theoretical
braid.

\vm {\it 2. Representations of finite subgroups of S_{∞} as outer
automorphisms of HFFs}\vm

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial
for Langlands program. Apart from one-dimensional representations complex
finite-dimensional representations are not possible if S_{∞}
identification is accepted (there might exist finite-dimensional l-adic
representations). This suggests that the finite-dimensional
representations correspond to those for finite Galois groups and result
through some kind of spontaneous breaking of S_{∞} symmetry.

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\item Sub-factors determined by finite groups G can be interpreted as
representations of Galois groups or, rather infinite diagonal imbeddings of
Galois groups to an infinite Cartesian power of S_n acting as outer
automorphisms in HFF. These transformations are counterparts of global
gauge transformations and determine the measured quantum numbers of gauge
multiplets and thus measurement resolution. All the finite approximations
of the representations are inner automorphisms but the limit does not

belong to S_{∞} and is therefore outer. An analogous picture
applies
in the case of infinite-dimensional Clifford algebra.

\item The physical interpretation is as a spontaneous breaking of
S_{∞} to a finite Galois group. One decomposes infinite
braid to a
series of n-braids such that finite Galois group acts in each n-
braid in
identical manner. Finite value of n corresponds to IR cutoff in
physics
in the sense that longer wave length quantum fluctuations are cut
off.
Finite measurement resolution is crucial. Now it applies to   braid
and
corresponds in the language of new quantum measurement theory to a
sub-factor ${\cal N}\subset {\cal M}$ determined by the finite
Galois
group G implying non-commutative physics with complex rays
replaced by
${\cal N}$ rays. Braids give a connection to topological quantum
field
theories, conformal field theories (TGD is almost topological
quantum field
theory at parton level), knots, etc..

\item TGD based space-time correlate for the action of finite
Galois
groups on braids and for the cutoff is in terms of the number
theoretic
braids obtained as the intersection of real partonic 2-surface and
its
p-adic counterpart. The value of the p-adic prime p associated
with the
parton is fixed by the scaling of the eigenvalue spectrum of the
modified
Dirac operator (note that renormalization group evolution of
coupling
constants is characterized at the level free theory since p-adic
prime
characterizes the p-adic length scale). The roots of the polynomial
would
determine the positions of braid strands so that Galois group
emerges
naturally. As a matter fact, partonic 2-surface decomposes into
regions,
one for each braid transforming independently under its own Galois
group.
Entire quantum state is modular invariant, which brings in
additional
constraints.

\item Braiding brings in homotopy group aspect crucial for geometric
Langlands program. Another global aspect is related to the modular

degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g,R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

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\vm{\it 3. Correspondence between finite groups and Lie groups}\vm

The correspondence between finite and Lie group is a basic aspect of Langlands.

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\item Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_{∞} rather than being induced from S_{∞} outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_{∞}, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

\item There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G\subset SU(2)$ to ADE type groups.

Similar correspondence is found for Jones inclusions with index ${\cal M}:{\cal N}\leq 4$. The challenge is to understand this correspondence.

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\item The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G\subset SU(2)$ to those of $SU(2)$.

\item The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with ${\cal N}$ rays, where ${\cal N}$ is the sub-factor.

\item The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H=M^4_{\pm}\times CP_2\rightarrow H/G_a\times G_b$, $G_a\times G_b \subset SU(2)\times SU(2)\subset SL(2,C)\times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

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\vm{\it 4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?}\vm

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_{∞} would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s))\times (s-1)/s$, where $\xi(s)= \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

\vm{\it 5. What does one mean with S_{∞}?}\vm

There is also the question about the meaning of S_{∞}. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_{∞} and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

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