TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of K\"ahler action. So called $M^8-H$ duality is one of these approaches. The beauty of $M^8-H$ duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

The construction  and interpretation of the octonionic geometry involves several challenges.

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\item   The fundamental challenge is to prove that the  octonionic polynomials with real coefficients can give rise to  associative (co-associative)  surfaces as the zero loci of their  real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis,  the identification $M^4 \subset O$  as   as a co-associative region turns out to be the correct choice making  light-cone  boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions.  This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex   or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for  which  real/imaginary part are quaternionic/co-quaternionic,  or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and  $M^8-H$ correspondence could generalize.

\item It turns out that in the generic case   associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i=RE(Y)^i+IM(Y)^iI_4$ of the gradient of $RE(P)= Y=0$ with respect to the complex coordinates $z_i^k$, $k=1,2$, of $O$ vanishes that is critical as function of quaternionic components  $z_1^k$ or $z_2^k$ associated with $q_1$ and $q_2$ in the decomposition $o= q_1+q_2I_4$, call this component $X_i$. In the generic case this gives 3-D surface.

In this generic case $M^8-H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to $H$, and only determines the boundary conditions of the dynamics in $H$ determined by the twistor lift of K\"ahler action. $M^8-H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial $P$ so that the criticality conditions do not reduce the dimension: $X_i$ would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components $X_i$. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of $X_i$ conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of K\"ahler action in $H$ in regions, where K\"ahler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8-H$ duality to preferred critical extremals for the twistor lift of K\"ahler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of K\"ahler action and volume term: these represent external particles. $M^8-H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which K\"ahler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8-H$ duality determines boundary conditions.

\item This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

\item The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

\end{enumerate}

Also a sketchy proposal for the description of interactions is discussed.

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\item The surprise that $RE(P)=0$ and $IM(P)=0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii $t_n$ given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety $X^4$. The intersections $X^2= X^4\cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices.

The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties $X^2$ are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

\item CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis
the condition $IM(\prod P_i)=0$ reduces to $IM(P_i)=0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i)=0$ does not reduce to $RE(\prod P_i)=0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

\item The possibility of super octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in ${\cal N}=4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

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Scattering diagrams would be determined by points of space-time
variety, which are in extension of rationals. In adelic physics the
interpretation is as  cognitive representations.

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\item   Cognitive representations are  identified as sets of
rational points for algebraic varieties with "active" points
containing fermion. The representations are discussed  at both
$M^8$- and $H$ level. General conjectures from algebraic geometry
support the vision that these sets are concentrated at lower-
dimensional algebraic varieties such as string world sheets and
partonic 2-surfaces and their 3-D orbits  identifiable also  as
singularities of these surfaces.   For the earlier work related to
adelic TGD and cognitive representations see
\cite{adelicphysics,adelephysics,nstructures,McKay}.

\item Some aspects related to homology charge (K\"ahler magnetic
charge) and genus-generation correspondence are discussed. Both
topological quantum numbers are central in the proposed model of
elementary particles and it is interesting to see whether  the
picture is internally consistent and how algebraic variety property
affects the situation. Also possible problems related to $h_{eff}/
h=n$ hierarchy \cite{Planck, qcritdark} \cite{adelicphysics}
realized  in terms of $n$-fold coverings  of space-time surfaces
are discussed from this perspective.

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