Problems with Holography = Holomorphy Principle

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Abstract

The holography = holomorphy principle (H-H) has become a key principle of TGD. For the volume action H-H yields minimal surfaces as extremals and this encouraged the conjecture that H-H holds also for Kähler action.

Even stronger conjecture would be that even curvature scalar allows minimal surfaces as extremals. The strongest conjecture would be that the H-H principle holds for any general coordinate invariant action constructible in terms of the induced geometry and spinor structure.

Marko Manninen tested the conjectured universality by using NeuroPaja, a research setup combining AI systems with computer algebra, and found that it fails in two tested cases, namely for Kähler action and for the curvature scalar.

The curvature scalar does not appear in the basic action so that this is not a problem. The failure for Kähler action is however a problem. This is due to the failure of the Kähler current to be light-like. There is a very simple remedy to the problem. The condition that the Kähler function of CP_2 restricted to the space-time surface satisfies 2-D Laplace equation in transversal degrees of freedom implies that the transversal part of the Kähler current vanishes outside lower-dimensional singular surfaces, which could correspond to wormhole throats. This implies the Poincare - and general coordinate invariant condition that the Kähler current is light-like.

Hamilton-Jacobi (H-J) structure defines a generalization of Kähler structure and complex structure as a fusion of complex and hypercomplex structures. H-J structure can be identified as a slicing defined by 2-D Minkowskian string world sheets and Euclidean surfaces orthogonal to each other. The assumption that both these surfaces are minimal surfaces is not necessary but is very attractive. It is not always true.

1 Introduction

I have proposed that the holography = holomorphy principle (H-H) [L8, L11] is universal. For volume action H-H yields minimal surfaces as extremals and this encouraged the conjecture that it holds also for Kähler action. Even stronger conjecture is that even the curvature scalar allows minimal surfaces as extremals. The strongest conjecture would be that the H-H principle holds for any general coordinate invariant action constructible in terms of the induced geometry and spinor structure.

Marko Manninen tested the conjectured universality by using NeuroPaja, a research setup in which he works with the AI systems Claude and Codex and does the calculations with the computer algebra system Maxima and Python scripts, and found that it fails in two cases.

Universality fails for the curvature scalar.

1. Curvature scalar allows minimal surfaces as extremals if the Einstein tensor is of type (1,1). The naive assumption that this is the case turned out to be wrong: Einstein tensor also has terms of type (2,0) and (0,2). This would leave constant curvature spaces for which Einstein tensor is proportional to the metric as minimal surface extremals and surfaces for which the second fundamental form vanishes.

2. Why does the H-H principle not allow minimal surfaces as generic extremals in the case of the curvature scalar? The reason is that the curvature tensor contains second derivatives of the metric. The induced metric depends on both the generalized complex coordinates u, w and their conjugates $(\bar{u}, \bar{w})$.

   Consequently, the part of the curvature tensor involving second derivatives of the metric consists not only of terms of type (2,2) — which would be fine — but also of terms of types (3,1) and (1,3). As a result, the Einstein tensor comprises not just components of type (1,1) — which would be perfectly acceptable — but also components of types (2,0) and (0,2). The contraction of the Einstein tensor with the second fundamental form does not vanish and field equations fail.

3. This supports the view that the fundamental action principle does not involve curvature scalar. I considered this option during the first year of TGD. Curvature scalar allows as solutions surfaces with negative energy: in particular, string like objects with negative string tension. One could circumvent the problem by adding a cosmological term which makes the net string tension positive. Also the lensing effect due to these string like objects would be by a factor at least 10^7 too large.

4. A possible conclusion is that CP_2 radius R, rather than Planck l_P, is the fundamental length in TGD: in this case Newton's constant G as a coupling strength would be scaled down by $\hbar_0/\hbar$ from its value for $\hbar_0$. One can consider also the option that CP_2 radius is given by Planck length and CP_2 radius is "dark", being scaled up by $\sqrt{\hbar/\hbar_0}$ [L17].

The failure of the Universality for Kähler action is a more serious problem since the recent formulation of TGD based on twistor lift [L1, L2, L14] predicts that the action is sum of Kähler action and volume term generated in the dimensional reduction of 6-D Kähler action giving rise to the induced twistor structure. Twistor lift also makes TGD unique since only M^4 and CP_2 allow twistor space with Kähler structure [A4].

1. The reason is that the conjectured light-likeness of the Kähler current is not true in general and is true only for so-called "massless extremals" [K3, K2, K1].

2. There is a very simple remedy to the problem. The condition that the Kähler function of CP_2 restricted to the space-time surface satisfies 2-D Laplace equation in transversal degrees of freedom implies that the transversal part of the Kähler current vanishes outside lower-dimensional singular surfaces, which could correspond to wormhole throats. This implies the Poincare- and general coordinate invariant condition that the Kähler current is light-like.

Working with this problem led to considerable progress in the understanding of Hamilton-Jacobi structure [L5]. Hamilton-Jacobi (H-J) structure defines a generalization of Kähler structure

and complex structure as a fusion of complex and hypercomplex structures. H-J structure can be identified as a slicing defined by 2-D Minkowskian string world sheets and Euclidean surfaces orthogonal to each other. The assumption that both these surfaces are minimal surfaces is not necessary but is very attractive. It is not always true.

2 The universality of the Holography = Holomorphy Principle fails: What to do?

2.1 The H-H principle is not satisfied for Kähler action without additional assumptions

For volume action the field equations are solved by H-H principle implying holography based on holomorphy.

The conjecture has been that it is true also for the Kähler action [K1, K2]. For the so-called massless extremals this is true. Crucially, the Kähler current is light-like, and its contraction with the gradients of complex and hypercomplex coordinates vanishes.

Even stronger conjecture is that the independence from the specific form of the action appears to hold more generally for the H-H principle. Field equations would quite generally reduce to tensor contractions in such a way that the contracted tensors share no common components. In Kähler action a possible source of problems is the term coming from the variation with respect to the induced Kähler form. It gives terms proportional to Kähler current which might cause trouble. It will be found that this is the case the space-like part of the Kähler current can be non-vanishing and is not expected to give rise to vanishing contraction.

How to circumvent this constraint. What comes first in mind is that the Kähler current vanishes: this requires that the Kähler magnetic field has a vanishing curl so that it is a gradient of a magnetic potential. This condition could fail at singularities.

The combination of the Kähler term and the volume term is unique in that it is forced by 6-D Kähler action for twistor lift. It also yields Lagrangian minimal surfaces — a consequence of the fact that two-dimensionality is always associated with holomorphy [L17, L10]. In this case, however, the generalized complex structure in the transverse degrees of freedom is not induced from H but is inherent for 2-dimensional surfaces. Based on physical grounds, it seems that both types of extremals are required.

This fixes the action as the sum of the Kähler term and the volume term — a form that also emerges from a twistor lift.

2.2 How to solve the problem caused by non-lightlikeness of the Kähler current?

2.2.1 How do the current contractions emerge?

Let us examine the derivation of the equations of motion.

1. To deduce the field equations from the variation of the Kähler action, one must calculate the covariant divergence

   $D_\alpha L_k^\alpha$, where the canonical momentum current is given by

   $$L_k^\alpha = \frac{\partial L_K \sqrt{g}}{\partial_\alpha h^k} \quad . \tag{2.1}$$

   This yields one component from the variation of the Kähler form and another from the variation of the induced metric, in particular $\sqrt{g}$. The contribution from the Kähler form is proportional to the Kähler current $j^\alpha = D_\beta J^{\alpha\beta}$.

2. Besides the aforementioned term, L_k^α contains a term involving the variation of the induced metric. The variation of $\sqrt{g}$ gives rise to a term $J^{\mu\nu} J_{\mu\nu} g^{\alpha\beta} \partial_\beta h^k$.

3. The metric contribution manifests as the covariant divergence of the energy-momentum tensor. I have previously calculated these in my articles. The divergence $D_\alpha[T^{\alpha\beta}\partial_\beta h^k]$ yields a sum of two terms:

$$D_\alpha T^{\alpha\beta}\partial_\beta h^k + T^{\alpha\beta}D_\alpha\partial_\beta h^k \quad . \tag{2.2}$$

   The latter term vanishes due to generalized holomorphy.

4. The term $D_\alpha T^{\alpha\beta}\partial_\beta h^k$ reduces to $j_\alpha J^{\alpha\beta}\partial_\beta h^k$.

   This is proportional to the 4-force and would vanish for ordinary empty space Maxwell's equations. However, the 4-force does not necessarily have to vanish — rather, it is the 8-force that vanishes and guarantees the conservation laws associated with Poincaré and color symmetries.

   This gives the terms $j_\alpha J^{\alpha\beta}\partial_\beta h^k$ and the term $j_\alpha J^k_l\partial_\beta h^l$ arising from the variation of J.

   Based on general coordinate invariance, the sum of these terms acts as a vector field of H that is orthogonal to the 4-surface (by general coordinate invariance). This combination should vanish due to holomorphy.

5. Can this combination vanish even if j does not? The condition would be as follows:

$$j^\alpha X^k_\alpha = 0 \quad , \quad X^k_\alpha = [J^\beta_\alpha h^k_l - g^\beta_\alpha J^k_l]\partial_\beta h^l \quad . \tag{2.3}$$

   X^k_α should be orthogonal to the coordinate gradients with respect to the metric H. This yields 8 conditions. The holomorphy ansatz would, after all, lead to non-trivial equations of motion, which would be algebraic in nature.

   Associated with each isometry current j^α_A there would be a 4-vector $j_{Ak}X^k_\alpha$ that is orthogonal to j^α. The only interpretation that comes to mind is the projection of the isometry current onto the spacetime surface. Thus, the isometry currents would be orthogonal to j. Must these conditions be imposed, or could holomorphy somehow guarantee them automatically? This option is not plausible.

6. Naturally, there exist solutions for which j^α vanishes. Geodesic spheres in CP_2 are relevant here. Cosmic strings provide a prime example of this.

2.2.2 Could one replace Kähler action with instanton term?

How can we circumvent the problems due to the non-vanishing of Kähler current?

One could consider the possibility that instead of Kähler action only its Chern-Simons variant — is present only where it is strictly required: namely, on light-like parton surfaces. Elsewhere, the action would be purely geometric. H-H principle allows only volume action having interpretation in terms of cosmological constant.

1. The K-C-S (Kähler-Chern-Simons) term is required for the conservation of charges: charges flowing from the interior — associated with the volume term — would transform into KCS charges on the partonic orbits, thereby ensuring charge conservation.

2. This would also avoid a potential issue associated with modified gamma matrices: namely, that they do not anticommute with respect to the induced metric.

3. Also now the Lagrangian minimal surfaces [L10, L17] are possible.

4. Electroweak and color couplings arise from the Dirac action along the fermion lines on the partonic orbits; in this sense, nothing is lost. The results for [L15, L16, L17] remain valid.

5. This only affects the definition of the vacuum functional but does not have impact on calculations related to elementary particle physics [L15, L16, L17] that rely solely on the K-C-S term. This is because the fermion lines lie on the partonic orbits.

   If the WCW wave functions are restricted to surfaces of constant volume, the entire action exponent disappears from the scattering amplitudes — a phenomenon that occurs quite generally in field theories. One might therefore ask whether a number-theoretic identification for the WCW state functions could replace the action exponent.

2.2.3 Could the transversal part of the Kähler current vanish?

A possible solution of the problem is an additional constraint stating that the Kähler current j^α vanishes locally so that the Kähler magnetic field has a vanishing curl. This translates to the condition that the Kähler function of CP_2, restricted to the space-time surface, satisfies 2-D Laplace equation which is natural from the point of view of conformal invariance.

Since the solution ansatz for H-H principle involves only the coordinate u, also the 4-D d'Alembert equation is satisfied for the flat Hamilton-Jacobi (H-J) structure but not generally as NeuroPaja demonstrated. This reflects the fact that the notion of H-J structure means the analog of Kähler structure inside causal diamond $cd \subset M^4$ and breaks Lorentz invariance. Lorentz invariance and Poincare invariance are lifted to the level of moduli space of CDs [L6].

Kähler charge and currents would vanish only locally and would be non-vanishing at lower-dimensional singularities such as partonic orbits. The vision is that singularities can correspond to various multipoles for charges and currents. These singularities would be localized on wormhole contacts serving as bridges between different space-time sheets. To formulate this more precisely, one must discuss H-H principle and H-J structure in more detail.

This proposal has an interesting relation to the earlier proposal stating that for the solutions of field equations the 4-D Lorentz force vanishes [K2] [L12]. The outcome is a 4-dimensional generalization of a 3-D Beltrami flow [B2, B1, ?] for which the Lorentz 3-force vanishes: in 4-D the flow would not be static anymore and the time component of 4-force would state that the power dissipated in Kähler electric field vanishes. 3-D Beltrami flows are topologically highly complex; such flows have even been proposed as a general solution for volume-preserving hydrodynamic flow.

The u-component j^u of j is lightlike so that the proposed condition simply states that j is lightlike. This condition is Poincaré-invariant and generally coordinate-invariant. For flat H-J structures, the surfaces E^2 and M^2 undergo a Poincaré transformation. A Lorentz transformation leaving the vertex of the CD invariant leaves H^3 and δM^4_+ invariant, but the surface Y^2 undergoes a Lorentz transformation. General Poincaré transformations map δM^4_+ and H^3 onto another, equivalent structure of the same type.

An interesting question is whether the 3-D Lorentz force vanishes in the examples considered by NeuroPaja. Also, assuming that the 4-current vanishes outside singularities, one can ask whether the 4-D Lorentz forces assignable to the lower-dimensional singular surfaces vanishes.

2.3 Some comments on twistor lift

As found, H-H principle [L5] and Kähler action can be kept if Kähler functions of M^4 and CP_2 satisfy d'Alembert equation.

2.3.1 Could twistor lift give only the instanton action?

Kähler action plus volume term are obtained from twistor lift via dimensional reduction of 6-D Kähler action. If one gives up Kähler action, this victory is lost and cosmological constant should be brought in in an ad hoc way.

Twistor lift could however give KCS terms at partonic orbits.

1. The KCS (Chern-Simons-type) terms can also be interpreted as 4D instanton terms within the interior of X^4. Since this is purely topological, it reduces to 3D KCS terms. The gauge action theory would thus be topological in the 4D sense.

2. Could instanton effects be generated via twistor lift from a 6D topological action? This brings to mind a generalization of the instanton term involving a $J \wedge J \wedge J$ structure. The

coupling parameter is dimensionless, which is an elegant feature. This would incorporate contributions originating from both $T(M^4)$ and $T(CP_2)$.

Dimensional reduction could yield a 4D instanton term multiplied by the flux of J over the twistor sphere — a quantity that may well be non-zero.

3. For an induced twistor structure, the fiber S^2 corresponds to a sphere obtained by identifying the twistor spheres associated with M^4 and CP_2. This could perhaps be interpreted as follows: at each point on the sphere, the relevant direction vectors define momentum directions for the projections onto M^4 and CP_2 — implying that the momentum is massless in an 8D sense. This suggests a connection between twistor induction and warping.

2.3.2 Is there motivation for the twistor lift if Kähler action is given up

The existence of a Kähler structure for the twistor spaces $T(M^4)$ and $T(CP_2)$ renders TGD unique and selects 6-D Kähler action reducing in dimensional reduction to 4-D Kähler action plus cosmological term. If one gives up Kähler action the argument making H unique is lost.

Giving up the Kähler action forces challenges the notion of twistor lift. One could however invent other arguments for why the existence of twistor space with Kähler structure selects H.

1. Twistor spaces $T(M^4)$ and $T(CP_2)$ are required if one wishes to define wave functions over the choices of quantization axes: the direction of the spin axis in M^4, and the color quantization axes in CP_2.

   The Kähler structure could be necessitated by the requirement that the wave functions in these spaces be proportional to the exponential of their Kähler function — which is analogous to the Gaussian of a harmonic oscillator.

2. In a quantum measurement, the selection of quantization axes would constitute a reduction of the state function at a higher level — a concept that fits well within the framework provided by the theory of consciousness and its hierarchy.

3. I have applied the wave function in the twistor space $T(CP_2)$ to a highly surprising observation regarding the dance of honeybees: namely, that it reflects color symmetry. This was observed by the topologist Barbara Shipman. TGD explains this phenomenon through quantum coherence over long scales, as well as color interactions — at least at the cellular scale: [L3].

To sum up, twistor lift strongly supports the option in which Kähler functions satisfy d'Alembert equation.

3 Holography = Holomorphy Principle and H-J structures

3.1 Holography = Holomorphy Principle briefly

H-H principle applies in the Minkowski regions of the space-time surface with 4-D M^4 projection. One assumes that M^4 allows the H-J structure [L5] as a generalization of complex structure and Kähler structure. This involves generalized complex coordinates involving hyper-complex coordinates u, v and transversal complex coordinates $w, \overline{w}$. The H-J structure corresponds to a slicing of M^4 by pairs of Minkowskian string world sheets and orthogonal Euclidean 2-surfaces. They allow the required hyper-complex/complex structure without minimal surface property but it is interesting to see whether it is possible. In the sequel a detailed proposal for H-J structures will be considered.

One assumes a function pair $(f_1, f_2) : H \to C^2$ and space-time surfaces correspond to the roots $f_1 = f_2 = 0$ within Minkowskian regions. When f_1 and f_2 are — in a generalized sense — complex-analytic functions of their arguments, one can solve for the roots, and the task of solving the field equations becomes algebraic. In other words, there are no longer any partial differential equations. f_1 and f_2 can be polynomials, rational functions, or even general analytic functions. Tuomas Sorakivi and I have already investigated some simple cases [L13].

Cosmic strings having form $X^2 \times Y^2 \subset M^4 \times CP_2$ cannot be obtained in this way but they do satisfy H-H principle when Y^2 is a complex submanifold of CP_2 [L17]. Monopole flux tubes as their deformations having 4-D M^4 projection can be obtained from H-H principle

CP_2 type extremals, which can have 1-D M^4 projection are obtained from a different ansatz but satisfy the holomorphy. There are also Lagrangian minimal surfaces [L16, L10]. Lagrangian monopole flux tubes would carry Maxwellian Kähler fields and would not be stable against splitting.

3.2 On H-J Structures

In the H-H ansatz, holomorphy is induced by the generalized holomorphy of H. This requires an H-J structure in which the tangent spaces of M^4 are decomposed — locally, in the general case — into longitudinal and transverse components. In the longitudinal component, a hypercomplex structure replaces the standard complex structure.

Essentially, this involves an integrable decomposition of M^4 such that, at every point, there exists a 2D longitudinal tangent subspace and a 2D transverse tangent subspace. Integrability implies the existence of two light-like coordinate curves, u and v, as well as complex coordinate curves w and $\bar{w}$. These allow for the slicing of M^4 into two-dimensional Minkowskian string world sheets and orthogonal Euclidian surfaces.

In the case of the simplest H-J structure, hypercomplex structure implies that $z = x + iy$ (and $\bar{z} = x - iy$) is replaced by the coordinates $t + z$ and $t - z$. This is already familiar from solutions within the string model. One could say that associated with every point in M^4 is a 2-space defined by a longitudinal light-like momentum k, and an orthogonal 2-space defined by a polarization vector ϵ. In the general case, k and ϵ represent local directions.

There are two fundamental cases that I have investigated:

1. M^4 coordinates $t + z$, $t - z$, and w (where w is a complex coordinate on a plane), along with coordinate systems derived from these via generalized holomorphy. In this case the two orthogonal surfaces are minimal surfaces and geodesic sub-manifolds.

2. "Spherical coordinates" for M^4 — namely $t + r$ and $t - r$ — combined with complex coordinates on a sphere S^2 (a subset of δM^4_+). Here, the light cone is sliced by 3D light fronts $(t = r)$ that are parallel to its boundary.

   The H-J structure offers a rich subject for mathematical inquiry:

3. One question to resolve is the number of possible H-J structures and the nature of their moduli space. Furthermore, what is the most general integrable structure on M^4 such that at each point, there exists a longitudinal tangent space and a transverse tangent space, such that the structure is integrable — that is, one obtains two light-like coordinate curves, u and v, as well as complex coordinate curves, w and $\bar{w}$. At present, there are only educated guesses regarding this.

4. Based on physical grounds, it appears highly likely that the H-J structure generalizes in such a way that the degrees of freedom of M^4 and CP_2 can become mixed. Light-like geodesics in M^4 would be replaced by light-like geodesics in H, resulting in "warping" and a reduction in the local speed of light [L18, L15, L17]. Warping in the simplest situation means space-time surfaces with flat Minkowskian metric but reduced light velocity. Warping would also provide an analogue for the Higgs mechanism at the level of the geometry of the space-time surface.

One can consider an algebraic and geometric approach to the construct of H-J structures.

1. Construct mappings from $(u, v, w, \bar{w})$ to (m^0, m^1, m^2, m^3) could offer an algebraic approach to classifying H-J structures. The fundamental input would be the condition that $(\nabla u)^2 = 0$ and $(\nabla v)^2 = 0$.

2. Another approach would be based on geometry. The coordinates $(u, v, w, \bar{w})$ should yield an integrable, non-singular "slicing" of the 2+2 type. Integrability implies the existence of string-like trajectories X^2 (parametrized by w and $\bar{w}$) and Euclidian surfaces Y^2 (parameterized by u and v). The standard slicing as $M^2 \times E^2$ composition, the Euclidian surface is E^2.

3.3 H-J structures as slicings of M^4 involving minimal surfaces in E^3

The vision is that the basic H-J structures correspond to slicings of M^4 into string world sheets X^2 and orthogonal Euclidian surfaces Y^2.

The existence of hypercomplex/complex structure does not pose any additional condition but one can consider the possibility that either of these surfaces or both are minimal surfaces

Google Gemini tells that there are no space-like closed minimal surfaces in M^4. This restriction is governed by a definitive mathematical hurdle known as Calabi's Bernstein-type theorem [A3] (and its extensions by Shing-Tung Yau [A9, A6]). The induced metric of a space-like surface is Riemannian up to sign. With the positive induced metric and the mean curvature vector defined as the trace of the second fundamental form, the Laplacian of the linear Minkowski coordinate m^k on the surface is the component H^k of the mean curvature vector. On a minimal surface every m^k is therefore harmonic, and on a closed surface a harmonic function is constant by the maximum principle. The theorem of Calabi and its extension by Cheng and Yau concern a different question, namely that a complete maximal space-like hypersurface of M^4 is a hyperplane.

These slicings must satisfy Frobenius conditions guaranteeing integrability and orthogonality conditions pose additional conditions. Besides trivial slicings there are slicings such as non-oscillating string configurations of parallel (static) strings or certain stationary helicoids extended into time dimension. Their dynamics would break Frobenius integrability conditions.

Non-trivial twisting in a 2+2 orthogonal minimal slicing refers to configurations, where the 2-planes defining the tangent spaces of the sheets rotate, tilt, or "twist" as they move through spacetime but manage to remain integrable and minimal. Instead of cleanly separating into isolated coordinates (like the trivial Cartesian product $M^{1,1} \times \mathbb{R}^2$), the string world sheets X^2 and the spacelike surfaces Y^2 wrap around each other.

The phenomenon is governed by the interplay between the Frobenius integrability theorem and the normal bundle connection of the surfaces. Frobenius condition implies that the commutators of tangent vector fields remain inside X^2 and Y^2. One can say that the tangent planes of Y^2 rotate as one moves along the string world sheet, which itself is not dynamical.

Each slicing defines its own H-J structure. A flat slicing is a special case. String world sheets are static but the partonic minimal surface can rotate. Also a spherical slicing could be possible, corresponding to the slicing of CD by the time translates of the boundary of the light cone for which sphere corresponds to Y^2. The moduli space of slicings and therefore H-J structures is not large. This type of slicing yields non-flat u, v and $w, \bar{w}$ coordinates.

3.4 H-J structures for CD involving 2-surfaces in hyperbolic 3-space H^3?

Future light-cone is closely related to the causal diamond (CD), which is the basic geometric object in zero energy ontology (ZEO) and defines the analog of 4-D quantizaton volume. CD could be regarded as a flat expanding cosmology.

Causal diamond has a natural 1+3 slicing using hyperbolic 3-spaces labelled by the light-cone proper time a. From the TGD point of view these slicings are of special interest. Could the replacement of M^4 with future light-cone allow more slicings and H-J structures than the static situation?

The slicings should involve minimal surfaces in H^3 orthogonal to the string world sheets. They would be dynamical at least in the sense that the minimal surfaces expand like in cosmic expansion. Also string world sheets could be minimal.

Google Gemini provides the following information about slicings by 2-D minimal surfaces inside light-cone.

3.4.1 The general view of slicings

In Euclidean space $\mathbb{R}^3$, the maximum principle completely forbids compact, closed minimal surfaces. In H^3, the outcome depends entirely on whether the ambient space is the full, unconstrained space or a compact domain:

1. In complete H^3 compact, closed minimal surfaces are still forbidden. As in the case of M^4, the maximum principle remains elliptic, and any standalone closed surface would be forced to contract due to the absence of topology inside the ambient space.

NeuroPaja gives the following more precise argument. When H^3 is the hyperboloid $m \cdot m = a^2$ with $m^0 > 0$, the mean curvature vector of the minimal surface in M^4 is $(2/a)N$, where $N = m^k/a$ is the future-directed unit normal. It follows that $\Delta m^0 = 2m^0/a^2 > 0$. On a closed surface, one would expect $\Delta m^0 \leq 0$ at the point where m^0 attains its maximum; therefore, no closed minimal surface exists.

2. Hyperbolic 3-manifolds (H^3/Γ) are obtained by taking a quotient of H^3 by a discrete group Γ of isometry group (Lorentz group). This creates a compact, closed 3D manifold (like a compact hyperbolic universe). The rules change entirely. Closed minimal surfaces exist in abundance. According to the works of Schoen and Yau [A9], Sacks and Uhlenbeck [A6], Anderson [A8], and of Kahn and Markovic [A5], any incompressible surface in a compact hyperbolic 3-manifold can be deformed into a smooth, embedded, closed minimal surface that acts as a geometric backbone for the manifold.

3. Note that the formation of compact hyperbolic 3-manifold is analogous to the representation of torus as a lattice and an interesting question is whether the tessellations of H^3 playing a central role in TGD could give rise to a lattice like structures obtained by gluing compact minimal surfaces along their boundaries.

3.4.2 Construction recipe

The construction of the slicings is geometrically elegant. In fact, the future light-cone of Minkowski space M^4 provides the ideal setting for this exact 2+2 decomposition. The mechanism relies on the Milne Model (a geometric description of the interior of the future light-cone) and the properties of AdS/hyperbolic spaces. Google Gemini provides the following summary.

1. Slicing the future light-cone into hyperbolic spaces.

   Let the apex of the future light-cone be the origin $(0, 0, 0, 0)$ in M^4. The interior of the future light-cone can be foliated by the level sets of the proper time $\tau = \sqrt{t^2 - x^2 - y^2 - z^2} > 0$. Each individual slice of constant τ is a 3-dimensional hyperboloid. Equipped with the induced metric from Minkowski space, this 3-D slice is exactly hyperbolic 3-space (H^3) with a constant negative curvature $K = -1/\tau^2$. This is known cosmologically as the Milne universe.

2. Construction of the 2+2 slicing within the light-cone.

   Because the 4D interior splits cleanly into a 1-D time parameter (τ) and 3D hyperbolic spaces (H^3), it is possible to construct the desired orthogonal minimal slicing step-by-step.

   (a) Pick up minimal Euclidian 2-surfaces in H^3: According to the theorem of Michael Anderson [A7, A8], H^3 contains an abundance of stable, complete minimal 2-surfaces (Y^2), which can be anchored to loops at the boundary of the light-cone (the asymptotic sphere at infinity).

   Coskunuzer found a hole in Anderson's theorem [A1] and corrected it. Later he showed [A2] that in the space H^3, every Jordan curve on the sphere at infinity bounds a complete minimal-area disk; consequently, there exists a large number of stable minimal surfaces. Note that a Jordan curve is a non-self-intersecting curve on a sphere; thus, the shape of an expanding filament can be arbitrary, provided it does not intersect itself.

   (b) Propagate via homothetic scaling: Because the geometry of the light-cone expands linearly with τ, you can scale this surface out radially from the apex of the light-cone. As τ changes, the minimal 2-surface Y^2_τ sweeps out a 3-dimensional surface.

   Is it possible to slice $H^3(\tau)$ with surfaces Y^2_τ? This would require a family of Jordan curves slicing the sphere S^2 at infinity inducing the slicing of $H^3(\tau)$. One can imagine a slicing of S^2 by circles with varying polar angle θ. The disks of S^2 and also the minimal disks of H^3 spanned by the circles should define the slicing of H^3.

   (c) The orthogonal Euclidian timelike sheets (X^2): The trajectories orthogonal to these spatial slices are generated by the flow lines stretching from the apex. Because the scaling is purely radial (homothetic trajectories radiating from the origin), the timelike paths trace out 2-dimensional string worldsheets X^2 that are perpendicular to the spatial Y^2 minimal surfaces at every intersection point.

3. The light-cone construction ensures that both families simultaneously satisfy the zero mean curvature ($H = 0$) condition and are thus minimal surfaces.

   (a) The Euclidian slices (Y^2) Y^2 are intrinsically chosen to be a minimal surface within its hosting hyperbolic 3-space (H^3).

   (b) The orthogonal string worldsheets (X^2) surfaces are generated by the radial geodesics originating from the light-cone apex. Because these worldsheets are ruled by straight, unaccelerated inertial lines meeting the spatial profiles symmetrically, their extrinsic curvature tensors vanish identically along the time direction. They form a family of valid Nambu-Goto string worldsheets that are completely minimal.

   (c) Why does this work for light-cone but fails for M^4)? The dynamic, non-trivially twisted slicings in M^4 quickly "jam" due to the maximum principle and elliptic constraints.

   Slicing the interior of the light-cone bypasses this limitation because the ambient geometry is effectively warped into negative curvature. The exponential expansion of the H^3 sheets towards the boundary of the light-cone acts as an infinite geometric well. This allows the elliptic minimal equations of the 2-surfaces to find stable, non-collapsing solutions anchored to the null boundary of the cone, while maintaining completely smooth, orthogonal, timelike string expansions back to the singular apex.

3.4.3 Why a Jordan curve is needed?

Why the Jordan curve (no self intersections) spanning minimal disk is needed if one wants X^2 to be a minimal surface.

1. The essential point is to obtain a decomposition into u, v and $w, \bar{w}$ in an integrable manner; this is achievable, although singularities may arise. The minimal surface property for both X^2 and Y^2 is not strictly necessary, even if it would be elegant.

2. A complex structure — along with its hypercomplex generalization and the condition of being "locally straight" (or "rectilinear") — is sufficient. It is, of course, obvious that a sphere in H^3 is not a minimal surface.

3. The u-curves can be identified as null (light-like) geodesics U originating from the vertices of time-shifted light cones; the shift to these occurs along a timelike line T connecting the vertices of the CD (causal diamond), which corresponds to an infinitely thin, cylindrical "hole." The v-curves are the duals of these — null lines that begin at the edge of the light cone and, in a Euclidean metric, would be orthogonal to the u-curves.

4. A problem arises here: the v-curves V, starting from the boundary of the light-cone, intersect at T, at the vertex of the cone. This constitutes a singularity. The problem disappears if T corresponds to a cylindrical hole of CD, as has been assumed.

   Another issue is that the lines V originating from opposite sides possess different orientations. Specifically, curves V coming from the opposite sides of the sphere have different orientations. The assumption that the Jordan curve bounds a region contained within a hemisphere eliminates this problem.

   Thus, the (u, v) asymmetry necessitates a hypercomplexity in the H-H ansatz — this is a truly elegant result that supports the H-H hypothesis.

3.4.4 Can both X^2 and Y^2 be minimal surfaces?

Physicist's straightforward common sense suggests that minimal surface property is not necessary for the string world sheet. The mere presence of a Minkowski signature in the metric, combined with two-dimensionality, enables the existence of a hypercomplex structure. The same holds true for all 2D Euclidean metrics: a complex structure — and, by extension, its hypercomplex variant — is universal.

It would be however very nice if both X^2 and Y^2 were minimal surfaces.

1. To my understanding, the above summary informs that X^2 and Y^2 can be minimal surfaces and Y^2 is a minimal disk bounded by a Jordan curve. Jordan curve spans a piece of sphere, which is not a minimal surface. The disk (plane) bounded by it can be a minimal surface.

2. However, a second artificially intelligent source informs that both X^2 and Y^2 cannot be minimal surfaces. My interpretation is that this says that the entire 3-sphere cannot be a minimal surface: this is of course trivially true. The minimal disk restricted by the Jordan curve is a minimal surface.

3.4.5 What is interpretation of the infinite subgroup $\Gamma \subset SO(1,3)$?

What is the significance of the infinite subgroup $\Gamma \subset SO(1,3)$ and how is it visible in the structure of tessellations and minimal surfaces of H^3.

1. The subgroup Γ must contain a finite subgroup that leaves H^3/Γ as analog of the unit cell of Euclidian lattice invariant.

2. Given the spherical symmetry of H^3, this subgroup should be of the rotation group $SO(3)$. The Platonic solids — the tetrahedron, cube, octahedron, and icosahedron — come to mind, and are the basic building blocks of the H^3 tessellations. They are characterized by finite discrete subgroups of $SO(3)$.

3. One might consider, for instance, 2-surfaces in H^3/Γ, where the faces of a Platonic solid have been identified as minimal surfaces with geodesic boundaries as counterparts of planes for the ordinary Platonic solids? If so, one could visualize a Platonic solid as a 2-surface, whose faces are minimal surfaces — much like in the Euclidean case. These would constitute the analogue of a standard Euclidean lattice containing periodic 2-surface.

   The polygonal boundary curves of the faces — whether equilateral triangles, squares, or pentagons — would define a special cases of a Jordan curve. In the Euclidean case, these faces are minimal surfaces; the question is whether this holds true here as well, or if they can be chosen to be such.

3.5 H-J structures as slicings of the light-cone by the translates of the light-cone boundary

One can also consider the slicing of light-cone by the translates of the light-cone boundary along the line T assignable to the CD.

1. Spheres also appear in the slicing of the boundary of a light cone, a topic examined in the article [L17]. This slicing would, in effect, induce u and v curves, while T-translations would generate the slicing of the light cone. Do both X^2 and Y^2 have the minimal surface property?

2. Since the radial normal is light-like, it is natural to assume that for the induced metric the covariant component g_{rr} in the light-like radial direction vanishes: the light cone is metrically a 2-dimensional sphere.

   One can however worry about the contravariant component g^{rr} which is formally the inverse of g_{rr} and diverges: can one put g^{rr} to zero or does it effectively disappear totally. If so, the sphere as a geodesic submanifold of δM^4_+ is trivially a minimal surface.

   On the other hand, the radial component $H^r_{\alpha\beta}$ of the second fundamental form does not vanish if defined as the limit $a \to 0$ so that the sphere would *not* be a minimal surface. However, nothing requires this since this limit need not be continuous.

3. For $g^{rr} = 0$ option, the induced metric is effectively 2-dimensional and the light cone boundary is, in a metric sense, a sphere. The surface area associated with radial deformations of the sphere is identical to the area of its projection onto the sphere; consequently, the surface area remains invariant under such radial deformations. It is, therefore, a minimum or a maximum — or both, depending on how one chooses to view it.

   Physically, this corresponds to conformal invariance, and the radial Virasoro algebra defines the gauge invariance.

NeuroPaja stated that the sphere at the light-cone boundary cannot be regarded as a minimal surface. The underlying assumption is that the limit $a \to 0$ is continuous, which it need not be, and thus gives $g^{rr} = \infty$. This is a rather delicate question as the following argument shows. The above argument is geometric and the direct calculation gives a minimal surface if the contravariant counterpart of the radial component $g_{rr} = 0$ can be defined as $g^{rr} = 0$: the motivation would come from the nature of the radial coordinate as a gauge degree of freedom. If it is identified formally as $g^{rr} = 1/g_{rr} = \infty$, the situation changes.

1. Consider first the reason for $g^{rr} = 0$. At the limit $a \to 0$, the covariant metric $g_{\alpha\beta}$ becomes degenerate and maps 3-D contravariant radial vectors to zero so that the covariant image of the contravariant tangent space is 2-D and the inverse map is infinitely-many-valued since it must map 2-D to 3-D.

   Indeed, if the contravariant metric is defined as the limit and thus as the algebraic inverse, it maps all covariant radial vectors with a radial component $e_r \neq 0$ to infinity. To get image vectors with finite e^r, infinitesimals would be needed for e_r.

   Therefore the physically natural option would be $g^{rr} = 0$ so that the tangent space becomes 2-D and light-cone boundary metrically 2-D and there are no infinities. This implies that in the slicing of the light cone along its boundaries, both X^2 and Y^2 are minimal surfaces.

2. A central term in the calculation is denoted as h^r forming part of the H^r component of the trace H^k of the second fundamental form:

$$h^r = g^{\mu\nu}(\partial_{\mu\nu} r + C^r_{kl}\partial_\mu h^k \partial_\nu h^l) \quad . \tag{3.1}$$

   Here, μ and ν refer to the internal coordinates of S^2, while k and l refer to the corresponding coordinates on the boundary of the light cone. The two coordinates can be identified. $\partial_{\mu\nu} r = 0$ if r is constant.

   C^r_{kl} is a Christoffel symbol and proportional to g^{rr}. If g^{rr} is defined as $1/g_{rr}$, it diverges and is therefore problematic. The fundamental question is whether $g_{rr} = 0$ allows us to assume $g^{rr} = 0$ or $g^{rr} = \infty$. Or whether the light-cone boundary is effectively metrically 2-D in a strict sense or not.

3. H^k is proportional to the projector along the normal direction:

$$P^k_l = h^k_l - g^{\mu\nu}\partial_\mu h^k \partial_\nu h_l \quad . \tag{3.2}$$

   In the radial direction ($k = r$), the issue of whether $g^{rr} = 0$ arises again. If it is zero, then $h^r_r = 0$, and the latter term vanishes on a surface where r is constant. Thus, $P^r_l = 0$ holds. Furthermore, if $g^{rr} = 0$, then $h^r = 0$, and consequently $H^r = 0$.

   If g^{rr} is not zero, then $P^r_l = \delta^r_l$ and $H^r = h^r$ is true.

4. In h^k, $k \in \{\theta, \phi\}$, there appears terms proportional to Christoffel symbols $C^k_{kr}\partial_\mu r$, which vanish on a sphere of constant r. Here, $h^k = (-1/2)g^{\mu\nu}C^k_{mn}\delta^m_\mu \delta^n_\nu$, and is non-zero.

   Because r is constant, the component P^k_l of the projector in the normal direction vanishes, that is $\delta^k_l - g^{\mu\nu}\partial_\mu h^k \partial_\nu h^m h_{lm} = 0$ is true. Hence H^k is zero.

5. In summary: If it makes sense to define $g^{rr} = 0$ rather than $g^{rr} = 1/g_{rr} = \infty$, then a sphere of constant r can be regarded as a minimal surface. This assumption is very natural from the point of view of conformal invariance of the light-cone boundary.

Why the light-cone boundary is so interesting geometrically is that by its metric 2-dimensionality it allows an infinite group of isometries acting as conformal transformations. The boundary of a light cone is metrically a sphere. The metric is $r_M^2 d\Omega^2$, as there is no contribution from the radial direction.

1. What makes the boundary of a light cone unique (as is the case with null parton surfaces) is that its isometry group is infinite-dimensional and corresponds to conformal transformations of the cross-section of the light-cone boundary.

   The surface area can be defined as $S = r^2 d\Omega$. The isometries are mappings $r_M \to h(z, \bar{z}, r_M)$ and $z \to f(z, r_M)$ such that $h(z, \bar{z}, r_M)^2 |df/dz|^2 g_{z\bar{z}}(f(z), \overline{f(z)}) = R^2 g_{z\bar{z}}(z, \bar{z})$. Thus, the isometries correspond to standard conformal transformations, which is what makes the light cone boundary unique. A local radial deformation h can be transformed into an isometry if there exists a conformal transformation with the above property. I cannot prove this but I suspect that it is so.

   Notice that a constant radial scaling $r \to \lambda r$ would require a compensating scaling $z \to f(z)$ satisfying $\lambda^2 |df/dz|^2 g_{z\bar{z}}(f(z), \overline{f(z)}) = g_{z\bar{z}}(z, \bar{z})$: the radial scaling is compensated by the conformal scaling.

2. If it is possible to assign to a radial deformation a conformal transformation so that the surface area remains unchanged, this conformal transformation could define the metric for the deformed S^2. The local radial variations would not alter the surface area, and the variational problem would become trivial.

3. It is also possible to define the surface area directly as a solid angle modulo scale factor by regarding light-cone boundary effectively as a projective space. In this approach radial deformations have no effect on the surface area. A sphere defined by $r_M =$ constant would correspond to a specific choice of conformal gauge and metric.

   Personally, I find this way of defining the metric for deformed spheres to be physically more meaningful (due to conformal invariance). However, as previously noted, this has no impact whatsoever on the H-J formulation.

3.6 Some TGD inspired comments

Some comments from the TGD based view are in order.

1. An infinite number of compact quotient spaces can be constructed from H^3 and act as unit cells of tessellations. In the Euclidian 3-space E^3, the effective reduction to a 3-torus under periodic boundary conditions serves as a simpler example; the cube effectively becomes a torus, which is compact.

2. An infinite number of such quotient spaces H^3/Γ can be obtained from H^3 as tessellations associated with a symmetry group Γ. The minimal 2-surfaces are associated with the tesselations and satisfy periodic boundary conditions defined by the subgroup Γ of Lorentz group. This applies also to the space-time surfaces satisfying H-H principle. Also the minimal surfaces in E^3 can define periodic, lattice-like structures but these cannot be extended into slicings of M^4 by adding orthogonal string world sheets.

3. No boundaryless minimal surfaces exist in H^3. However, from the information theoretic point of view the single cell of tessellation corresponds to the entire tessellation so that in this sense compactness holds true.

   Within the CD (causal diamond), the boundary that defines the horizon renders the observable portion of the tessellation finite. Furthermore, the introduction of interactions via embedding to H can make boundaries possible via the generation of singularities and also closed space-time surfaces, which are many-sheeted with respect to M^4 become possible.

I have proposed applications of these tessellations in the TGD framework.

1. An icosa tetrahedral tessellation is unique in that it involves tetrahedron, octahedron and icosahedron having triangular faces as basic structural building blocks. The triangles would correspond to Jordan curves. It plays a fundamental role in TGD-based quantum biology and relates to a model of the genetic code [L4]. It would seem that the relevant finite subgroup $\Gamma \subset SO(1,3)$ in this case is the symmetry group of the triangle defining the Jordan curve?

2. I have also proposed, with inspiration coming from the recently discovered gravitational hum, that stars form an approximate tessellation on a scale of 5 light-years [L7] and that one can speak of diffraction in his structure amplifying the gravitational radiation in discrete directions.

3. The most recent proposal is that galaxies on a scale of one million light-years [L19]: this would solve the recent puzzle that the velocity curve of Milky Way starts to decline around 15 parsec and also explains the Axis of Evil.

I have earlier proposed that the tessellations of H^3 play a key role in TGD based physics. If they appear already at the level of H-J structures, causal diamonds and ZEO, this is indeed the case. This would also support the recent proposal that lattice like structures of H^3 formed by cosmic strings and formed in primordial string dominated cosmology make themselves visible in the structure of galaxies such as Milky Way [L19].

3.7 Modification of H-J structure by induced Kähler form and metric

The induced Kähler structure, based on the H-H principle, implies a deformation of a given H-J structure via a conformal mapping. This deformation preserves the H-J structure locally, so that the complex and hypercomplex structures are preserved. The deformation can however introduce singularities. The singularities introduce holes and in this sense also the H-J structure changes.

The requirement that the Kähler charge and current are localized at the singularities makes it possible to find solutions in the context of the Kähler action. The curl of the Kähler magnetic field vanishes outside the singularities. Thus, locally, it can be expressed as the gradient of a magnetic scalar potential.

This can be visualized: there are charges and dipoles at the 2D curved minimal surface Y^2. An attractive assumption is that the dipoles are orthogonal to the surface.

The Kähler function is annihilated by the Laplace operator. Non-trivial solutions in the Euclidean metric — whether induced or the original one provided by the fundamental H-J function — necessarily involve dipole and monopole singularities, pertaining to both charge and current. Otherwise, the solution is, in fact, trivial. This relates to the geometric analogy of a multipole expansion.

These singularities imply the existence of a surrounding region where the embedding, viewed as a Kähler field, breaks down. Consequently, the singularities are replaced by partonic surfaces, which enclose a Euclidean region.

Also other 3D singularities are likely to arise — in particular, those associated with changes in the "warping" — which physically correspond to interfaces where light undergoes refraction and reflection. In physical terms, this signifies replacing the vacuum H-J function with one in which particles correspond to singularities of the H-J function.

3.7.1 Classical electrodynamics as an intuitive guideline

It seems that H-J structure at for M^4 could reduces to orthogonal slicings by string world sheets and orthogonal minimal surfaces. The intuitive expecation is that the replacement of M^4 with space-time surfaces modifies the H-J structure. Consider now in more detail what might happen using intuition from classical electrodynamics.

1. Could the counterparts of minimal partonic 2-surface become closed and could their genus can be greater than genus $g = 0$ (a radial spherical wave at light-cone bounddary)? Could one identify the coordinate lines of the H-J structure with those of induced Kähler gauge potential? Could the modification of the M^4 metric with induced metric and the contribution of induced Kähler form allow deformation of the H-J structure by briging in singularities?

   If $g > 0$ is possible, could the conformal moduli determined by the induced metric of the partonic surfaces parameterize these partonic minimal 2 - surfaces and could the moduli be the same for all surfaces in the slicing? Singularities affect the field topology. Could the classification of deformed H-J structures reduce to the classification of 2D conformal structures for different genera, which is well understood?

2. A strong requirement is that the slicing must be non-singular except at singular points. The geometry of the CD (causal diamond) suggests that one could start with the assumption that u and v correspond to the light-cone proper time a and the radial coordinate r (or the hypercomplex angle η) of the mass shell H^3.

   In this case, for a given value of a, the slices of H^3 would correspond to spheres of radius r. They would be related by scaling and would be nested within one another. Given a spherically symmetric choice of u and v, a torus, for example, cannot serve as a slice; radial scaling of such slices would not produce nested tori, and the scaled slices would intersect.

3. If, for instance, a toroidal slicing were possible, the tori would necessarily have to be nested within one another. This brings to mind the realization of toroidal slicing in the form of a dipole field.

   In a dipole field, the field lines — whose tangents correspond to the field gradient — form surfaces with a toroidal topology. The equipotential surfaces of the potential form a dual slicing.

   A dipole line connects the south and north poles, resulting in a torus. The field lines loop around the outside and return along the dipole axis, effectively transforming a sphere into a torus. Under scaling, these tori remain distinct, thus yielding a valid slicing.

4. Does this generalize? Can we consider several dipoles to form multipoles? Here, n dipoles would correspond to n connections linking opposite points on a sphere, resulting in a topology with genus $g + 1$. Is it possible to identify coordinates u and v in such a case?

   In the simplest case, surfaces of constant r would correspond to a monopole charge. In the case of a dipole, the counterparts to these surfaces would be the equipotential surfaces of the dipole, and so on. Would multipole expansions have a geometric analogue in terms of H-J structures?

3.7.2 Connection of the H-J structure to the solutions of field equations

The construction of H-J structure must also have a direct connection to the solutions of field equations since they would depend on the H-J structure. If so, then the induced Kähler gauge potential in the interior would not be useless but would carry information about the H-J structure associated with the solution of field equations.

Could the induced Kähler gauge potential somehow define the four coordinate functions $(U + eV, U - eV, W, \overline{W})$ via a generalization of conformal map replacing the coordinate $(u + ev, u - ev, w, \overline{w})$ associated with the H-J structure.This is not necessary but might be convenient.

1. Suppose that one starts from standard H-J structure $(u, v, w, \overline{w})$ to construct more complex H-J structures by analytic maps bringing in singularities.

   The analytic functions would be functions $f(u, w)$ expressible in power series $u = t + ez$ and $w = x + iy$. New H-J structure would be generated if these functions have singularities, which could correspond to counterparts multipoles and possibly also cuts. Multipoles would correspond to line singularities (fermion lines) as analogs of and string world sheets as analogs of cuts and having fermion lines as boundaries. The edges of fermion lines would correspond to vertices.

2. Suppose that the induced Kähler gauge field has only multipoles as sources so that the charge densities and currents are located to lower-dimensional surfaces, such as partonic orbits. This allows only Kähler gauge potentials for which Kähler current is non-vanishing only at lower-dimensional singularities. The Laplace equation for the Kähler functions of M^4 and K guarantees this.

3. The field patterns involving only multipoles are special in that outside multipole singularities of electric and magnetic fields there exists electric gauge potential and also magnetic scalar potential associated with the magnetic field.

4. Magnetic field can be expressed either as a rotor or gradient in these regions and this is essentially what complex analyticity means (Cauchy-Riemann equations). There exists an analytic function of $W = X + iY$ whose real and imaginary parts would correspond to real and imaginary parts of W.

   These define 3 scalar functions that could define 3 coordinates as candidates for U (or V) and the real coordinates X and Y associated with $W = X + iY$.

5. What about identification of the function V if the field lines of V or us are identified as the field lines for the scalar potential associated with the induced electric field?

   The hypercomplex analytic functions can be expressed as power series of $t + ez$, where e satisfies $e^2 = 1$ (rather than -1) with coefficients which are hypercomplex numbers.

   The part of the power series proportional to e corresponds to hypercomplex coordinate U and the part proportional to 1 corresponds to V. If V is known from Φ there are hopes of constructing U as the hypercomplex imaginary part just as in the case of complex analytic function the knowledge of the real part at poles and cuts allows to construct the imaginary part by integral formula. Note that the functions $f(u = t + ez, w)$ generalize the notion of analytic function.

The motivation for studying H-J structure came from the observation made by NeuroPaja that the Kähler current does not vanish for the general H-J ansatz. If the only charges and currents reside at lower-D singularities, the situation changes except at the singularities. Since Kähler gauge potential defines induced Kähler field so that it is instructive to study in detail what it looks like.

1. The Kähler gauge potential of CP_2 is expressible as

$$(B_{\xi^k}, B_{\overline{\xi}^k}) = i(\partial_{\xi^k} K, -\partial_{\overline{\xi}^k} K) \quad . \tag{3.3}$$

   Without - sign it would be gradient. If Kähler magnetic field vanishes the projection of B reduces to gradient. This condition is completely analogous to the Cauchy-Riemann conditions and means that Kähler gauge potential is expressible also as gradient.

   B_u and B_v are gradients of the Kähler function of CP_2 playing the role of Coulomb potential:

$$\begin{aligned} B_u &= B_{\xi^k}\partial_u \xi^k + B_{\overline{\xi}^k}\partial_u \overline{\xi}^k \quad , \\ B_v &= B_{\xi^k}\partial_v \xi^k + B_{\overline{\xi}^k}\partial_v \overline{\xi}^k \quad . \end{aligned}$$

2. The transversal gauge potentials are

$$\begin{aligned} B_w &= i\partial_w K \quad , \\ B_{\overline{w}} &= -i\partial_{\overline{w}} K \quad . \end{aligned} \tag{3.4}$$

3. Transversal part reduces to gradient if the rotor B vanishes so that one has

$$\partial_w \partial_{\overline{w}} K = 0$$

   K therefore satisfies Laplace equation. This means that the only charges and currents correspond to singularities at which the analyticity and the Laplace equation fail. This implies that H-H principle works for the Kähler action.

4 Could one add curvature scalar to the action?

One can imagine two solutions to overcome the problem due to the non-vanishing Kähler current.

1. The first way to achieve this is to assume that currents and charges are concentrated on lower-dimensional singularities such as partonic orbits. This is actually the existing view.

   This leads to the additional condition that CP_2 and also M^4 Kähler function satisfy Laplace equation so that both electric and magnetic Kähler fields are derivable from a potential function outside the singularities. This option allows also the twistorialization since the 6-D Kähler action would reduce in dimensional reduction to the sum of 4-D Kähler action and volume action.

2. Another way way to get rid of the problems associated with the Kähler action would be to replace it with a mere instanton term, which reduces to boundary terms associated with the partonic orbits: only this term is directly visible in the proposed construction of scattering amplitudes. Gauge theory part would would reduce to topological QFT with fermions at partonic orbits. Both M^4 and CP_2 KCS terms would be present.

   The mere volume action, making possible cosmic strings and string like objects must be however postulated separately and would have interpretation in terms of cosmological constant, which should be length scale dependent. One does not lose the information coded by the induced Kähler gauge potentials, which could be used in the construction of H-J structure.

3. To overcome this aesthetic problem, one can consider the analog of gauge-gravitation duality: gauge interactions would be associated with the partonic orbits and gravitational interaction with the interior. A possible identification for the gravitational dynamics could be in terms of curvature scalar and volume term.

   It has already become clear that this option fails to be consistent with H-H principle but will be discussed since it might help to understand why the proposed solution of the problem is the most elegant.

4.1 Why curvature scalar option with cosmological term looks nice (at first)?

The analogy with Einstein's theory suggests the possibility of a curvature term. Also now minimal surfaces could solutions if the H-J structure is appropriately chosen.

1. The field equations would be derived from the curvature scalar plus volume action. This would simultaneously resolve the energy problem and yield Einstein's equations: the isometry currents would be determined by the Einstein tensor, which is expected to be of type [1,1] so that its contraction with second fundamental form would vanish.

2. This approach could avoid the already discussed problem plaguing Kähler action, as it involves only the variation of the metric and Einstein tensor is of type (1,1). It would however require a specific H-J structure since the solutions depend on its choice. This is the case quite generally and as it will be found there are very many H-J structures for CD. Gauge interactions would arise from the fermionic sector and this would conform with the gauge-gravitation duality.

3. Curvature scalar plus volume action — without invoking the Hilbert action outside the parton trajectories — is formally truly elegant and could be consistent with the expected field-theory limit. Moreover, this aligns perfectly with the original concept from long ago! Should one return to the roots?

4. The gauge/gravity duality would also be realized: gravity in the interior, and gauge interactions along the parton trajectories. The reduction to the partonic orbits could correspond to instanton term for Kähler action.

   The concept of monopole flux would still apply, but the magnetic contribution to the energy of cosmic strings would vanish, leaving only the volume contribution plus the contribution from the curvature scalar.

There are however problems. The first problem is that energy can be negative. The second problem is that the contribution of curvature scalar to the string tension is by a factor about 10^7 larger than the limit from lensing effect. One can find possible solutions to these problems but it seems mere cosmological term is the most elegant option. Volume term would yield a cosmological constant, but what about G?

4.2 Problems of the curvature scalar option

Suppose that space-time surface has boundary (for instance, at infinity). To obtain the Gauss-Bonnet term in effectively 2-D case without corrections, a boundary term can be added to the curvature scalar. This applies more generally.

This term is proportional to the trace of the second fundamental form of the boundary; it vanishes if the boundary is a 3D minimal surface within the spacetime manifold. However, the assumption of a minimal surface is questionable in M^4, which allows only minimal surfaces of infinite size as the boundary would be of infinite extent. As found, also for H^3 naturally associated with CD situation the situation is the same. Note however that for the space-time surfaces the dynamics of H-H principle leads to many-sheeted space-time surfaces which are closed and of finite size.

In a dynamic scenario involving at least dynamics due to the cosmic expansion associated with the CD, the situation might, of course, be different. Could the rotation, say that of the Earth, which stabilizes the system and prevents gravitational collapse, be an example of this? Dynamics would stabilize. Very mundane example is the moving bicycle which is stabilized by angular momentum conservation.

1. I am not certain whether the boundary term is chosen such that the boundary term arising from the Einstein equation compensates for the term resulting from the variation of the aforementioned boundary term. One interpretation would be that gravitational energy and momentum, which would otherwise escape, are instead converted into a form of gravitational energy associated with the boundary.

2. This point warrants critical scrutiny. One could argue that, since gravity is attractive, the boundary cannot remain stable; instead, a process analogous to the formation of a black hole would occur. Some interaction is required to stabilize the boundary. In a physical context, this manifests as interactions within condensed matter — for instance, the mechanism that stabilizes the Earth.

3. A cosmological volume term generates repulsion; it could stabilize the boundary and, at the very least, ensure that conservation laws are not violated.

   Kähler–Chern–Simons terms also exert an influence at the boundary, and gravitational energy and momentum can be transferred into these degrees of freedom.

4.2.1 Problems with the sign of energy for curvature scalar

One cannot avoid the problems inherent in this approach.

1. Surfaces of the form $M^1 \times X^2 \times S^1$ serve as vacuum solutions, where X^2 is a closed surface. The issue is that the energy is negative for a spherical topology.

   If a volume term — and thus a cosmological constant, Λ — is included, the solutions X^2 become minimal surfaces. X^2 cannot be a closed surface, as minimal surfaces in E^3 are always non-compact. Since E^3 is replete with minimal surfaces, this example could potentially be realistic.

2. The topological contribution to the energy depends on the number of holes in X^2. The energy is proportional to $4\pi(2-n)$, where n is the number of holes.

   (a) Inside the CD, there is an extremely thin cylindrical hole along the time-line T connecting the tips; this line passes through E^3 at a specific point — let us call it the origin. In the general case, X^2 does not contain the origin; thus, no hole is formed in the general scenario.

(b) There may also be a second hole, corresponding to a potential boundary of the manifold surface, in which case X^2 possesses the topology of a circular disk. If X^2 extends to infinity, there is an effective boundary at infinity. This results in a negative contribution, equal to half of the negative contribution associated with a sphere.

(c) If X^2 consists of two sheets joined together along their edges, it is possible to obtain the topology of a sphere, provided that T does not pass through X^2 — that is, the origin is not contained within X^2. In this case, the energy would generally be negative, corresponding to the case of a sphere. Adding a handle results in zero energy.

(d) In the case of two holes, the topology is that of a cylinder. According to the Gauss-Bonnet theorem, the invariant determined by the scalar curvature and the associated boundary term is the same as that of a torus — namely, zero. The problem of negative energy would thus disappear.
If X^2 contains only a single hole, the topology is that of a sphere with a hole; the energy would be negative, amounting to half of the value found in the spherical case.

The requirement for positive energy could provide an explanation for the mysterious cosmological constant Λ.

1. The volume term would contribute the following to the energy of a minimal surface in E^3 with surface area S:

$$E = (\Lambda S/4G)R - \pi R/G = (\Lambda S - 4\pi)R/(4G) \ . \tag{4.1}$$

E is positive for $S > 4\pi/\Lambda$. If S becomes very large, the situation converges to that found in standard theories. The probability of a configuration having such a large effect approaches zero exponentially.

2. TGD predicts that the cosmological constant Λ is proportional to $1/L_p^2$, where L_p is the p-adic length scale associated with the surface or the CD (causal diamond). This yields:

$$E = RS/(4GL_p^2) \ . \tag{4.2}$$

E is positive for $S > 4\pi/\Lambda$. If the area S is on the order of L_p^2, the total energy for a minimal surface with the topology of a sphere or a sphere with holes is positive. One could therefore hypothesize that the function of the mysterious cosmological constant is to ensure the energy remains positive. This would require p-adic physics.

4.2.2 On possible solution to the problem of negative energy for string like objects

It is also necessary to examine what happens to cosmic strings when the curvature scalar is included. If the cross-section of the flux tube is a sphere, the Gauss-Bonnet term contributes to the energy; this contribution is the integral of $-R/2$ over the sphere, multiplied by the length L of the string. The energy is negative and cosmological constant can compensate it.

1. For surfaces which are Cartesian products of flat string world sheets with 2-D sub-manifolds of type $E^2 \times CP_2$, the string tension associated with the curvature scalar plus boundary term is

$$T = \frac{1}{2G}(g - 1 + n/2) \ .$$

Here, n is the number of holes. When $g = 0$, we obtain $T = -1/2G$, which is negative.

2. An additional term related to the boundaries is associated with the curvature scalar; this term is proportional to the second fundamental form of the boundary. Essentially, it represents the acceleration of a point on the boundary — one could consider this as being generated by gravity. If this acceleration vanishes, the additional term also vanishes; in such a case, the path is a geodesic. This occurs, for example, at the boundary of a hemisphere.

3. The problem is that when $g = 0$ and $n \in \{0, 1\}$, string tension T is negative.

   A cosmological term can compensate for this. In the case of S^2, it would introduce an additional term, $(\Lambda/2G)R^2$, which must be greater than $1/2G$. This is achieved if the condition $\Lambda \geq 1/R^2$ holds.

   In the case of the hemisphere the situation depends on whether one adds the boundary contribution to the action or not. Its addition makes the Gauss-Bonnet integral a topological invariant when the sphere has holes. The geodesic property of the boundary circle implies that the added term vanishes. The integral over the hemisphere gives $\Lambda \geq 1/2R^2$ in accordance with the Gauss-Bonnet formula in the presence of a single hole. Non-negativity for the sphere without holes however forces $\Lambda \geq 1/R^2$.

4. This works for a sphere. However, for $g > 1$, the string tension would receive a very large contribution, which contradicts the observed lensing effect. If $\Lambda = 1/R^2$ is true (up to a numerical factor), the situation is resolved. It remains an open question whether the string tension in the case of a sphere is minimal — that is, equal to zero.

5. It would seem natural for the cross-section of cosmic strings to be a sphere; thus, $\Lambda > 1/R^2$ appears a natural condition. In this case, higher-genus configurations would be expected to be less probable due to their higher string tension — and, consequently, perhaps unstable.

6. Related to this is the hypothesis concerning cosmological constant: $\Lambda \propto 1/L_p^2$ which can be justified in twistor lift approach. One might propose the relation:

$$\Lambda = 1/R^2 + k/L_p^2 \ .$$

   For cosmic strings with a spherical topology, this yields the tension $T = k/L_p^2$. For $g > 0$, the tensions would remain large. For $g = 0$, small values of p correspond to cosmic strings, while large values of p could, for instance, correspond to hadronic strings.

   Phase transitions in which p increases result in a reduction of string tension. Here, p would correspond to powers of two or three (the p-adic length-scale hypothesis). This implies a sequence of transitions leading to lower string tensions — though only in the $g = 0$ case.

   This aligns with the emergence of a fractal hierarchy of Standard Model physics, corresponding to three types of Mersenne primes: ordinary, Gaussian, and Eisensteinian [L17]. This is also consistent with the concept of p-adic cooling [L9].

4.2.3 Problem arising from the lensing effect

The basic problem associated with the curvature scalar is that the string tension derived from R is $1/2G$, is with a factor of $x^2 = 1/4 \times 10^7$ larger than the empirical upper limit imposed by the lensing effect possibly caused by cosmic strings.

1. x^2 is rather near to the ratio of the square of the CP_2 radius R to the square of the Planck length l_P. This would be the case if the real Planck length were equal to the CP_2 radius R. How could this be possible?

2. The gravitational constant G would need to be at least x^2 times larger to prevent the string tension from being too high.

   Should we, then, make the following substitution at the fundamental level:

   $G = l_P^2/\hbar \to G_0 = l_P^2/\hbar_0 \approx x^2 G$?

   In this scenario, the Planck length would correspond to the CP_2 radius.

   This would be natural, as one might expect $\hbar_{\text{eff}} = \hbar_0$ to represent a minimum value in the context of cosmic strings. As $\hbar_0 \to \hbar_{\text{eff}}$, the coupling constant G transforms as $G \to G_0(\hbar_0/\hbar)$; this would facilitate the convergence of perturbation theory: note however that perturbation theory based on path integral is not needed in TGD framework.

3. Essentially, this is what I have already assumed [L17], albeit formulated differently. In this formulation the Planck length l_P would be the actual radius of CP_2 for $\hbar_{eff} = \hbar_0$, and it would scale to a "dark" radius of CP_2 for a given value of $\hbar_{eff}/\hbar_0$.

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