About Absolute Galois Group

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Abstract

Absolute Galois Group defined as Galois group of algebraic numbers regarded as extension of rationals is very difficult concept to define. The goal of classical Langlands program is to understand the Galois group of algebraic numbers as algebraic extension of rationals - Absolute Galois Group (AGG) - through its representations. Invertible adeles -ideles - define Gl_1 which can be shown to be isomorphic with the Galois group of maximal Abelian extension of rationals (MAGG) and the Langlands conjecture is that the representations for algebraic groups with matrix elements replaced with adeles provide information about AGG and algebraic geometry.

I have asked already earlier whether AGG could act is symmetries of quantum TGD. The basis idea was that AGG could be identified as a permutation group for a braid having infinite number of strands. The notion of quantum adele leads to the interpretation of the analog of Galois group for quantum adeles in terms of permutation groups assignable to finite l braids. One can also assign to infinite primes braid structures and Galois groups have lift to braid groups.

Objects known as dessins d'enfant provide a geometric representation for AGG in terms of action on algebraic Riemann surfaces allowing interpretation also as algebraic surfaces in finite fields. This representation would make sense for algebraic partonic 2-surfaces, and could be important in the intersection of real and p-adic worlds assigned with living matter in TGD inspired quantum biology, and would allow to regard the quantum states of living matter as representations of AGG. Adeles would make these representations very concrete by bringing in cognition represented in terms of p-adics and there is also a generalization to Hilbert adeles.

1 Introduction

Langlands correspondence represents extremely abstract mathematics - perhaps too abstract for a simple minded physicist with rather mundane thinking habits. It takes years to get just a grasp about the basic motivations and notions, to say nothing about technicalities. Therefore I hope that my own prattlings about Langlands correspondence could be taken with a merciful understanding attitude. I cannot do anything for it: I just want desperately to understand what drives these mathematical physicists and somehow I am convinced that this exotic mathematics could be extremely useful for my attempts to develop the TGD view about Universe and everything. Writing is for me the only manner to possibly achieve understanding - or at least a momentary illusion of understanding - and I can only apologize if the reader has feeling of having wasted time by trying to understand these scribblings.

Ed Frenkel (see `http://tinyurl.com/y8sgk672`) lectured again about geometric Langlands correspondence and quantum field theories and this inspired a fresh attempt to understand what the underlying notions could mean in TGD framework. Frenkel has also article about the relationship between geometric Langlands program and conformal field theories [A15]. My own attempt might be regarded as hopeless but to my view it is worth of reporting.

The challenge of all challenges for a number theorist is to understand the Galois group of algebraic numbers regarded as extension of rationals - by its fundamental importance this group deserves to be called Absolute Galois Group (see `http://tinyurl.com/yaffmruw`) (AGG, [A1]). This group is monstrously big since it is in some sense union of all finite-D Galois groups. Another fundamental Galois group is the Maximal Abelian Galois Group (MAGG) associated with maximal Abelian extension of rationals (see `http://tinyurl.com/y8dosjut`) [A11]. This group is isomorphic with a subgroup assignable to thering of adeles (see `http://tinyurl.com/64pgerm`) [A3].

1.1 Could AGG Act As Permutation Group For Infinite Number Of Objects?

My own naive proposal for years ago is that AGG could be identified as infinite-dimensional permutation group S_∞ [K2]. What the subscript ∞ means is of course on non-trivial question. The set of all finite permutations for infinite sequence of objects at integer positions (to make this more concrete) or also of permutations which involve infinite number of objects? Do these object reside along integer points of half-line or the entire real line? In the latter case permutations acting as integer shifts along the real line are possible and bring in discrete translation group.

A good example is provided by 2-adic numbers. If only sequences consisting of a finite number of non-vanishing bits are allowed, one obtains ordinary integers - a discrete structure. If sequences having strictly infinite number of non-vanishing bits are allowed, one obtains 2-adic integers forming a continuum in 2-adic topology, and one can speak about differential calculus. Something very similar could take place in the case of AGG and already the example of maximal Abelian Galois group which has been shown to be essentially Cartesian product of real numbers and all p-adic number fields Q_p divided by rationals suggests that Cartesian product of all p-adic continuums is involved.

What made this proposal so interesting from TGD point of view is that the group algebra of S_∞ defined in proper manner is hyper-finite factor of II_1 (HFF) [K2]. HFFs are fundamental in TGD: WCW spinors form as a fermionic Fock spaces HFF. This would bring in the inclusions of HFFs, which could provide new kind understanding of AGG. Also the connection with physics might become more concrete. The basic problem is to identify how AGG acts on quantum states and the obvious guess is that they act on algebraic surfaces by affecting the algebraic number valued coefficients of the polynomials involved. How to formulate this with general coordinate invariant (GCI) manner is of course a challenge: one should be able to identify preferred coordinates or at least class of them related by linear algebraic transformations if possible. Symmetries make possible to consider candidates for this kind of coordinates but it is far from obvious that p-adic CP_2 makes sense - or is even needed!

In [K2] I proposed a realization of AGG or rather- its covering replacing elements of permutation group with flows - in terms of braids. Later I considered the possibility to interpret the mapping of the Galois groups assignable to infinite primes to symplectic flows on braids [K5]. This group is covering group of AGG with permutations being replaced with flows which in TGD framework could be realized as symplectic flows. Again GCI is the challenge. I have discussed the symplectic flow representation of generalized Galois groups assigned with infinite primes (allowing mapping to polynomial primes) in [K5] speculating in the framework provided by the TGD inspired physical picture. Here the notion of finite measurement resolution leading to finite Galois groups played a key role.

1.2 Dessins D'Enfant

Any algebraic surface defined as a common zero locus of rational (in special case polynomial) functions with algebraic coefficients defines a geometric representation of AGG. The action on algebraic coefficients is induced the action of AGG on algebraic numbers appearing as coefficients and in the roots of the polynomials involved. One can study many things: the subgroups of AGG leaving given algebraic surface invariant, the orbits of given algebraic surface under AGG, the subgroups leaving the elements at the orbit invariant, etc.... This looks simple but is extremely difficult to realize in practice.

One working geometric approach of this kind to AGG relies on so called dessins d'enfant (see `http://tinyurl.com/y927ebvd`) [A6] to be discussed later. These combinatorial objects provide an amazingly simple diagrammatic approach allowing to understand concretely what the action of AGG means geometrically at the level of algebraic Riemann surfaces. What is remarkable that every algebraic Riemann surface (with polynomials involved having algebraic coefficients) is compact by Belyi's theorem (see `http://tinyurl.com/ydxzerkr`) [A5] and bi-holomorphisms generate non-algebraic ones from these.

In TGD partonic 2-surfaces are the basic objects and necessarily compact. This puts bells ringing and suggests that the old idea about AGG as symmetry group of WCW might make sense in the algebraic intersection of real and p-adic worlds at the level of WCW identifies as the seat of life in TGD inspired quantum biology. Could this mean that AGG acts naturally on partonic 2-surfaces and its representations assign number theoretical quantum numbers to living systems? An intriguing additional result is that all compact Riemann surfaces can be representation as projective varities in CP_3 assigned to twistors. Could there be some connection?

1.3 Langlands Program

Another approach to AGG is algebraic and relies on finite-dimensional representations of AGG. If one manages to construct a matrix representation of AGG, one can identify AGG invariants as eigenvalues of the matrices characterizing their AGG conjugacy class. Langlands correspondence (see http://tinyurl.com/ybmcnqh8) [A15, A14] is a conjecture stating that the representations of adelic variants of algebraic matrix groups (see http://tinyurl.com/yde5mras) [A2].

Adelic representations are obtained by replacing the matrix elements with elements in the ring of rational adeles which is tensor product of rationals with Cartesian product of real numbers and all p-adic number fields with and they provide representations of AGG. Ideles represent elements of abelianization of AGG. Various completions of rationals are simply collected to form single super structure.

Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this manner an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

The conjecture is that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the complex analysis.

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at http://tgdtheory.fi/tgdglossary.pdf [?].

2 Langlands Program

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Adelic representations are obtained by replacing the matrix elements with elements in the ring of adeles and they provide representations of AGG. Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this manner an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

Langlands conjecture states that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the analysis. I hasten to admit that I have failed to understand intuitively the deeper motivations for this conjecture but there is support for it.

2.1 Adeles

This approach leads to adeles [A3].

1. AGG is extremely complex and the natural approach is to try something less ambitious first and construct representations of the Maximal Abelian Galois Group of rationals (MAGG)

[A11] assigned to an extension containing all possible roots of unity. One can show that MAGG is isomorphic to the group of invertible adeles divided by rationals. This is something concrete as compared to AGG albeit still something extremely complex.

2. The ring of rational adeles (see http://tinyurl.com/64pgerm) [A3] discovered by Chevalley is formed by the Cartesian product of all p-adic number fields and of reals and its non-vanishing elements have the property that only finite number of p-adic numbers in $(...., a_{p_n}, ...) \times a$ are not p-adic integers (that is possess norm > 1). Algebraic operations are purely local: multiplications in every completion of rationals involved. One can also understand this space as a tensor product of rationals with integer adeles defined by the cartesian product of reals and various p-adic integers. One can say that adeles organize reals and all p-adic number fields to infinite-dimensional Cartesian product and that identified rational numbers as common to all of them so that multiplication by rational acts just as it act in a finite dimensional Cartesian product. The idea that rationals are common to all completions of rationals is fundamental for quantum TGD so that adeles are expected to be important.

3. The ring property of adeles makes possible to talk about polynomials of adele valued argument having rational coefficients and one can extend algebraic geometry to adeles as long as one talks about varieties defined by polynomials. Existence of polynomials makes it possible to talk about matrices with adele valued elements. The notion of determinant is well-defined and one can also define the inverse of adele matrix so that classical algebraic groups have also adele counterpart. This is of utmost significance in Langlands program and means a breathtaking achievement in book keeping: all the p-adic number fields would be caught under single symbol "A" !

4. Ideles are rational adeles with inverse. Ideles form a group but sum of two ideles is not always idele so that ideles do not form a number field and one cannot dream of constructing genuine differential calculus of ideles or talking about rational functions of ideles. Also rational functions fail to make sense. This means quite a strong constraint: if one wants adelic generalization of physics the solutions of field equations must be representable in terms of polynomials or infinite Taylor series.

The conjecture of Langlands is that the algebraic groups with matrix elements replaced with adeles provide finite-dimensional representations of adeles in what can be loosely called group algebra of adelic algebraic group.

The construction of representation uses complex valued functions defined in the ring of adeles. This function algebra decomposes naturally to a tensor product of function algebras associated with reals and various p-adic number fields and one can speak about rational entanglement between these functions. From the TGD point of view this is very interesting since rational entanglement plays a key role in TGD inspired quantum biology.

2.2 Construction Of Representations Of Adelic Gl_2

I have explained some details about the construction of the representation of adelic Gl_2 in the Appendix and earlier in [K2].

1. The basic idea is to start from the tensor product of representations in various completions of rationals using the corresponding group algebras. It is natural to require that the functions are invariant under the *left* multiplication by $Gl_2(Q)$ and eigenstates of $Gl_2(R)$ Casimir operator C under the *right* multiplication. The functions are smooth in the sense that they are smooth in $Gl_2(R)$ and locally constant in $Gl_2(Q_p)$.

2. The diagonal subgroup $Z(A)$ consists of products of diagonal matrices in $Gl_2(A)$. Characters (see http://tinyurl.com/ybeheayk) are defined in $Z(A)$ as group homomorphisms to complex numbers. The maximal compact subgroup $K \subset Gl_2(A)$ is the Cartesian product of $Gl_2(Z_p)$ and $O_2(R)$ and finite-dimensionality under the action of these groups is also a natural condition.

3. The representations functions satisfy various constraints described in detail in the appendix and in the article of Frenkel (see `http://tinyurl.com/y7fhl75f`) [A15]. I just try to explain what I see as the basic ideas.

   (a) Functions f form a finite-dimensional vector space under the action of elements of the maximal compact subgroup K. Multiplication from left by diagonal elements reduces to a multiplication with character. The functions are eigenstates of the Casimir operator of $Gl_2(R)$ acting from left with a discrete spectrum of eigen values. they are bounded in $Gl_2(A)$. These conditions are rather obvious.

   (b) Besides this the functions satisfy also the so called cuspidality conditions, the content of which is not obvious for a novice like me. These conditions imply that the functions are invariant under the action for $Gl_2(Z_p)$ apart from finite number of primes called ramified. For these primes invariance holds true only under subgroup $\Gamma_0(p^{n_k})$ of $Sl_2(Z_p)$ consisting of 2×2-matrices for which the elements $a_{21}\equiv c$ vanish modulo p^n.

   (c) What is non-trivial and looks like a miracle to a physicist is that one can reduce everything to the study of so called automorphic functions (see `http://tinyurl.com/ybwzg73x`) [A4] defined in $\Gamma_0(N)/Sl(2,R)$, $N=\prod p^{n_k}$. Intuitively one might try to understand this from the idea that adeles for which elements in Z_p are powers of p represent rational numbers. That various p-adic physics somehow factorize the real physics would be the misty idea which in TGD inspired theory of consciousness translates to the idea that various p-adic physics make possible cognitive representations of real physics. Somehow the whole adele effectively reduces to a real number. Automorphic functions have a number theoretic interpretation and this is certainly one of the key motivations between Langlands program.

4. Automorphic functions reduce to complex analytic functions in the upper half plane $H=SL_2(R)/O(2)$ transforming in a simple manner under $\Gamma_0(N)$ (modular form of weight k). What one is left with are modular forms of weight k and level N in upper half plane.

   (a) The overall important cuspidality conditions characterized by integer N imply that the automorphic functions vanish at the cusp points of the algebraic Riemann surface defined as $H/\Gamma_0(N)$. The modular form can be expanded in Fourier series $f=\sum a_n q^n$ in powers of $q=exp(i2\pi\tau)$, where τ parameterizes upper half plane.

   (b) The Fourier coefficients a_n satisfy the condition $a_{mn}=a_m a_n$ and one ends up with the conclusion that for each elliptic curve (see `http://tinyurl.com/ybsdt65r`) [A7] $y^2=x^3+ax+b$ (a and b are rational numbers satisfying $4a^3+27b^2\neq 0$ and reduce to integer is the recent case) there should exist a modular form with the property that a_p codes for the numbers of points of this elliptic curve in finite field F_p for all but finite number of primes! This is really amazing and mysterious looking result.

   (c) τ can be interpreted as a complex coordinate parametrizing the conformal moduli of tori. Is this a pure accident or could this relate to the fact that the coefficients turn out to give numbers of roots for algebraic elliptic surfaces, which are indeed tori? Could cuspidality conditions have interpretation as vanishing of the modular forms for tori with moduli corresponding to cusps: could these be are somehow singular as elliptic surfaces? The objection is that the elliptic surfaces as sub-manifolds of C^2 have a unique induced metric and therefore correspond to a unique conformal modulus τ. But what about other Kähler metrics than the standard metric for C^2 and imbeddings to other complex spaces as algebraic surfaces? Could adelic Gl_2 representations generalize to adelic representations of Gl_{2g} acting on Teichmueller parameters of Riemann surface with genus g?

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

3 Compactness Is Guaranteed By Algebraicity: Dessins D'Enfant

This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focussed. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any childs drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.

This piece of text was written by Grothendieck. He described here the profound impact of the notion of dessins d'enfant (see `http://tinyurl.com/y927ebvd`) [A6] on him. The translation of the notion to english is "child's drawings". These drawings are graphical representations of Riemann surfaces (see `http://tinyurl.com/cgl2pj`) understood as pairs formed by an algebraic Riemann surface and its universal covering space from which Riemann surface is obtained as a projection which can be many-to-one one map. This diagram allows to construct the Riemann surface modulo bi-holomorphism. Algebraic Riemann surface means that the equations defining it involve only rational functions with coefficients which are algebraic numbers. This implies that the action of AGG on the algebraic Riemann surface is well defined as action on the coefficients. One can assign to the dessin d'enfant combinatorial invariants for the action of AGG.

3.1 Dessins D'Enfant

1. Dessin d'enfant is a bipartite graph (see `http://tinyurl.com/3x2cjf`) [A16] meaning that it is possible to label the nodes of the graphs by black and white points in such a manner that the black and white points alternate along edge paths. One can identify black and white nodes as sets U and V and every edge of the graph connects points of U and V. For instance, bipartite graph does not posses any odd edge cycles. Every tree is bipartite and every planar graphs with even number of edges is bipartite. The vertices of the bipartite graph are topologically characterized by the number of lines emerging to the vertex and also 2-vertices are possible. The surface and the embedding can be described combinatorially using rotation system assigned with each vertex of the graph and telling the order in which the edges would be crossed by a path that travels clockwise on the surface around the vertex.

2. The notions of dessin d'enfant and counterpart for Belyi function [A5] defining the projection from the covering of sphere to sphere dates back to the work of Felix Klein. A very deep and very surprising theorem by Belyi (`http://tinyurl.com/ydxzerkr`) states that all algebraic curves represent compact Riemann surfaces. These surfaces are ramified coverings of the Riemann sphere ramified at three points only which in suitable complex coordinates can be taken to be the rational points 0, 1, ∞ of real axis. Ramification means that the rational function f with algebraic number coefficients - known as Belyi's function - projecting the Riemann surface as covering of sphere to sphere has critical points which are pre-images of these three points. In the neighborhood of the critical points the projection map known as Belyi's function is characterized by degree telling how many points are mapped to single point of sphere. At the critical point itself these points coincide. A simplified example of criticality is z^n at origin.

   The Riemann surface in question can be taken to be H/Γ compactified by finite number of cusp points. Here H is upper half plane Γ a subgroup of modular group having finite index

3. Dessin d'enfant allows to code combinatorially the data about the Belyi function so that one can construct both the surface and its Belyi function from this data apart from bi-holomorhism. The interpretation as projection from covering allows to get grasp about the geometric meaning of dessin d'enfant. Physicist reader is probably familiar with the graphical representation of cusp catastrophe. The projection of the critical points and curves of cusp catastrophe as function of the two control parameters to the control parameter plane replaced in the recent case by complex plane is highly analogous to dessin d'enfant. The boundary

of cusp catastrophe in which cusp projection is three-to-one has V-shape and at the sides of V the covering of plane is 2-to-1 and and at the vertex and outside cusp region 1-to-1. The edges of V correspond to the edges of the dessin d'enfant and the vertex of V to a node of dessin d'enfant.

The number of edges entering given critical point tells the degree of the Belyi function at that critical point. Dessin d'enfant is imbedded on an oriented surface - plane in the simplest situation but also sphere and half plane can be considered. The lines of the graph correspond to curves at which two branches of the covering coincide.

The Wikipedia article (see `http://tinyurl.com/y927ebvd`) [A6] about dessin d'enfant discusses a nice example about the construction of dessin d'enfant and is recommended for the reader.

4. The Belyi function could be any holomorphic function from X to Riemann sphere having only 0, 1, and ∞ as critical values and the function f is determined only up to bi-holomorphism. If X is algebraic surface, f is rational function with algebraic coefficients.

5. What makes the dessin d'enfant so remarkable is that AGG has natural action on the algebraic coefficients of the rational functions defining algebraic Riemann surfaces and therefore on dessin d'enfant. For instance, the sequence of integers form by the degrees of the projection map at the critical points is geometric Galois invariant. One can identify the stabilize of dessin as the sub-group of AGG leaving dessin d'enfant invariant. One can identify the orbit of dessin d'enfant under AGG and the subgroup of AGG leaving the points of orbit invariant.

3.2 Could One Combine Quantum Adelic Representations With Dessin D'Enfant Representations?

As already noticed, dessin d'enfant representation of AGG allows to have representations of AGG at the orbits of dessins d'enfant. If the orbit consists of a finite number n of points, one obtains representations of AGG in the finite-dimensional discrete Hilbert space spanned by the points, and representation matrices are $n \times n$ matrices.

Suppose that the Galois group of quantum adeles is indeed isomorphic with the commutator group of AGG. If this is the case then quantum adele valued amplitudes defined in the discrete space formed by the orbits of dessins d'enfant would provide a representation of AGG with commutator group acting on the fiber analogous to spin degrees of freedom and AGG on the base space having role analogous to that of Minkowski space.

One can imagine an approach mimicking the construction of induced representations (see `http://tinyurl.com/y9nfp438`) [A8] of Mackey inspired by the representations of Poincare group. In this approach one identifies orbit of group G as a space carrying the fields with spin. The subgroup H of G leaving a given point of representation space invariant is same at all points of orbit apart from conjugation. The field would have values in H or group algebra of H or in space in which H acts linearly. In the recent case H could adelic Galois group of quantum adeles identified as AGG or the subgroup G_I of AGG leaving the dessins d'enfant invariant.

What can one say about G_I. How large it is? Can one identify it or its abelization A_{G_I} and assign it to the points of orbits to construct analogs of induced representations?

1. If the orbit of dessin d'enfant is finite as the fact that the number of its points is invariant under the action of AGG suggests, G_I must be infinite. This would suggests that also A_{G_I} is infinite. Does A_{G_I} possess adele representation? Is this adele representation identifiable as a sub-adele of A_{AGG} in some sense? Could it be obtained by dropping some quantum variants of Z_p: s from the decomposition of adele? What the interpretation of these lacking primes could be? Could these primes correspond to the primes which split in the extensions. If this is the case one could consider the representations in which A_{G_I} forms the fiber space at each point of dessin d'enfant.

2. One can consider also weaker option for which only so called ramified primes are dropped from the adele for rationals to obtain the adele for algebraic extension. In adele construction there are problematic primes p. For rational primes (or corresponding ideals) the representation

of p is as a product of primes of extension as $p = \prod P_i^{e_i}$ e_i are called degrees of ramification. For some $e_i > 1$ one has ramification analogous to the dependence of form $(z - z_0)^n$, $n > 1$ of holomorphic function around critical point have interpretation as ramified primes and corresponding factors Z_p are dropped from the adele. To eliminate the problems cause by number theoretic ramification one can drop ramified primes from the adele in the extensions of algebraic numbers associated with the roots of the polynomials appearing in the Belyi map. Could the resulting adele be the counterpart for the reduced MGGA?

3.3 Dessins D'Enfant And TGD

What might be the relevance of Belyi's theorem and dessins d'enfant for TGD?

1. In TGD framework effective 2-dimensionality implies that basic objects are partonic 2-surfaces together with their data related to the 4-D tangent space a them. I have already earlier proposed that Absolute Galois group could have a natural action in the world of the classical worlds (WCW). The horrible looking problem is how to achieve General Coordinate Invariance (GCI) for this action.

   Partonic 2-surfaces are compact so that they allow a representation as algebraic surfaces. The notion of dessin d'enfant suggests that partonic 2-surfaces could be described as simple combinatorial objects defined by dessin d'enfant as far as the action of Galois group is considered. This representation would be manifestly general coordinate invariant and would allow to construct representations as Galois group in terms of discrete wave functions at the orbits of dessin d'enfant. One can also expect that the representation reduces to those of finite Galois groups.

2. Second central problem is the notion of braid which is proposed to provide a realization for the notion of finite measurement resolution.The recent view is that time-like braids on light like surfaces and space-like braids at the 3-surfaces defining the ends of space-time surfaces contain braid strands as Legendrian knots for which the projection of Kähler gauge potential has vanishing inner product with the tangent vector of the braid strand. For light-like 3-surfaces this does not imply that the tangent vector of strand is orthogonal to the strand: if the tangent vector is light-like the condition is automatically satisfied and light-like braid strands define a good but - as it seems - not a unique guess for what the braid strands are. Note however that the condition that braid strands correspond to boundaries of string world sheets gives additional conditions. At space-like 3-surfaces orthogonality to induced Kähler gauge potential fixes the direction of the tangent vector field only partially.

   Suppose one manages to fix completely the equations for braid strands - say by the identification as light-like strands. What about the end points of strands? How uniquely their positions are determined? Number theoretical universality suggests that the end points are rational or algebraic points as points of imbedding space but again GCI poses a problem. Symmetry arguments suggest that one could use group theoretically preferred coordinates for M^4 and CP_2 and identify also the coordinates of partonic 2-surface as imbedding space coordinates for their projections to geodesic spheres of $\delta M^4_\pm$ and geodesic sphere of CP_2.

   A possible resolution of this problem comes from the fact that partonic 2-surface allows an interpretation as algebraic surface. Braid ends could correspond to the critical points of the Belyi function defining the projection from the covering so that they would be algebraic points in the complex coordinates of partonic 2-surfaces fixed apart from algebraic bi-holomorphism. One would a concrete topological interpretation for why the braid ends are so special. I have already earlier proposed that braid ends could correspond to singularities associated with coordinate patches.

3. Is it possible to have compact Riemann which cannot be represented as algebraic surfaces? Belyi's theorem does not deny this. For instance rational functions with real coefficients for polynomials are possible and must give rise to compact surfaces. Inherently non-algebraic partonic 2-surfaces are possible and for them one cannot define representations of AGG at the orbits of dessin d'denfant since the action of AGG on f is not well defined now.

This relates in an interesting manner to the conjecture [K3] that life resides in the in the intersection of real and p-adic worlds. At WCW level this would mean that the equations for the partonic 2-surfaces makes sense in any completion of rationals. For algebraic partonic 2-surfaces this is indeed the case if arbitrary high-dimensional algebraic extensions of p-adic numbers are allowed. Taking this seriously one can ask whether the existence of the representations of Galois group at the level of WCW is an essential aspect of what it is to be living. Could one assign Galois quantum numbers to the quantum states of living system? These would be realized in the discrete space provided by different quantum counterparts of a given integer and one would have discrete wave functions in these discrete spaces.

4. One also learns from Wikipedia (see `http://tinyurl.com/cgl2pj`) that any compact Riemann surface is a projective variety and thus representable using polynomial equations in projective space. It also allows an imbedding as as a surface n 3-dimensional complex projective space CP_3. Wikipedia states that if compactness condition is added the Riemann the surface is necessarily algebraic: here however algebraic means rational functions with arbitrary real or complex coefficients. Above it means algebraic coefficients. Whether this CP_3 could have anything to do with the twistor space appearing in Witten's twistor string model [B1] and also in the speculated twistorial formulation of TGD [K4] remains an open question.

5. Modular invariance plays central role in TGD [K1], and a natural additional condition on the representations of AGG would be that the quantum states in WCW are modular invariant. The action of AGG induces a well-defined action on the conformal moduli of the partonic 2-surfaces and therefore on Teichmueller parameters. This discrete action need not be simple - say linear- but it would be action in n-dimensional space. Modular invariance requires that the action of AGG transformation induces a conformal scaling of the induced metric and changes the conformal moduli by an action of modular group $Sl(2g, Z)$. For torus topology this group is Sl(2, Z) appearing in modular invariant functions assigned to the representations of AGG in the group algebra of adelic algebraic groups.

6. Could the combination of dessins d'enfant as a geometric representation and adelic matrix representations for the abelianizer of the isotropy group G_I of dessin d'enfant provide additional insights in to Langlands conjecture? The problem is that AGG elements do not leave MGGA invariant.

7. Bi-partite graphs (see `http://tinyurl.com/3x2cjf`) appear also in the construction of inclusions of hyper-finite factors of type II_1 (HFF). The TGD inspired proposal that AGG allows identification as S_∞ and the group algebra of permutation group S_∞ is HFF. In optimistic mood one might see dessins d'enfant as a piece of evidence for this identification of AGG and adele formed from the Galois group of quantum p-adic integers as its commutator group.

4 Appendix: Basic Concepts And Ideas Related To The Number Theoretic Langlands Program

The following representation of the basic ideas of Langlands program reflects my very limited understanding of the extremely refined conceptual framework involved. This pieces of text can be found almost as such also in [K2] and Ed Frenkel provides more detailed discussion in his article [A15, A14].

4.1 Langlands Correspondence And AGG

The representations of group carry information about the group and the natural question is how to represent the AGG and deduce invariants of AGG in this manner. Eigenvalues for the representation matrices are invariants characterizing conjugacy classes of the group. The generators of MAGG abelled by primes define so called Frobenius elements and the eigenvalues and traces for their representation matrics defined invariants of this kind. The big question is how to construct representations of the AGG. Langlands program is an attempt to answer this question.

1. 1-D representations of AGG corresponds those of maximal Abelian Galois group which is the factor group of AGG by its commutator group. The natural intuitive guess is that the n-dimensional representations of AGG in the group algebra of adelic algebraic group $Gl(n)$ could provide higher-dimensional representations of AGG. $Gl(n)$ would give rise to a kind of AGG spin. The action of AGG commutator group would be mapped to$GL_n(A)$ action. Does this mean that AGG is mapped homomorphically to adelic matrices in $Gl_n(A)$ as one might first think? I am not able to answer the question. From Wikipedia one learns that so called Langlands dual (see `http://tinyurl.com/yclcloaj`) [A9] extends AGG by the algebraic Lie group G_L so that one obtains semi-direct product of complex G_L with the AGG which acts on the algebraic root data of G_L. The adelic representations of G_L are said to control those of G. In this form the correspondence gives information about group representations rather than number theory.

   Remark: One naive guess would be that one could realize the representations of AGG by adjoint action $x \rightarrow gxg^{-1}$ in the commutator subgroup of AGG, which is maximal normal subgroup and closed with respect to this action. Also the adjoint action of the factor group defined my maximal Abelian group in this group could define representation? The guess of the outsider is that the practical problem is that the commutator group is not known.

2. Number theoretic Langlands program is however more than study of the relationships between representations of $Gl(F)$ and its adelic variant $Gl(A_F)$. The basic conjecture is the existence of duality between number theory and harmonic analysis. On number theoretical side one typically studies algebraic curves. Typical question concerns the number of rational points in modulo p approximation to the equations determining the algebraic curve. The conjecture about number theoretic Langlands correspondence was inspired by the observation that Fourier series expansions of automorphic forms code via their coefficients this kind of data and the proof of Fermat's theorem can be seen as application of this correspondence.

   There is support for the conjecture that adelic representations carry purely number theoretic information in the case of $Gl(n)$. The number theoretical invariants defined by the trace for the representation matrix for the Frobenius element generating the Abelian Galois group would corresponds to the trace of so called Hecke operator at the side of the harmonic analysis.

3. Intuitive motivations for the Langlands duality come from the fact the notion of algebraic surface defined by a polynomials with integer coefficients is number theoretically universal: the argument can belong to finite field, rational numbers or their extension, real numbers, or any p-adic number field and can represent even element of function field. Function fields defined algebraic functions at algebraic curves in finite fields are somehow between classical number fields and function fields associated with Riemann surfaces to which one can apply the tools of harmonic analysis.

4.2 Abelian Class Field Theory And TGD

The context leading to the discovery of adeles (`http://tinyurl.com/64pgerm`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \ mod \ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (see `http://tinyurl.com/y8aefmg2`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The

Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

4.2.1 Adeles and ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://tinyurl.com/y8aefmg2`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [A10, A15, A14]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where imbedding space is replaced with Cartesian product of real imbedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://tinyurl.com/64pgerm`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod_p' Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (http://tinyurl.com/yc3yrcxxIdele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

4.3 Langlands Correspondence And Modular Invariance

A strong motivation for Langlands correspondence is modular invariance - or rather its restricted form - which emerges in both number theory and in the automorphic representations of Gl_2 and relates directly to the ramification of primes for Galois extensions- now maximal Abelian extension. In TGD framework the restricted modular invariance could have interpretation in terms of concrete representations of AGG involving the action of AGG on the adelic variants of Teichmueller parameters characterizing the algebraic surfaces its variants in various number fields.

It is not necessary to know the explicit action of AGG to modular parameters. What is however needed is modular invariance in some sense. The first - and hard-to-realize - option is that allowed subgroup of AGG leaves the conformal equivalence class of Riemann surface invariant. Second option is that the action of both AGG and modular group $Sl(2g, Z)$ or its subgroup leave the states of representation invariant. This is the case if AGG induces Gl_{2g} transformations in each Cartesian factor of the adele and the states defined in the group algebra of Gl_{2g} are invariant. For ramified primes however modular invariance can break down to subgroup of Sl_{2g}. These conditions lead to automorphic modular forms.

These arguments are very heuristic and following arguments due to Frenkel give better view about the situation.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL_e(2, A)$ and more generally $GL_e(n, A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A12], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL_e(n, A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL_e(n, A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n, R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1, .., n$ acting in group algebra of $Gl(n, R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_e^L(F)$ of $G(F)$.

4.4 Correspondence Between N-Dimensional Representations Of $Gal(\overline{F}/F)$ And Representations Of $Gl_E(N,A_F)$ In The Space Of Functions In $Gl_E(N,F)\backslash Gl_E(N,A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^{\times}$, where $Z_p^{\times}$ consists of invertible p-adic integers [A15].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^{\times}\backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^{\times}\backslash A_F^{\times}$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL_e(1,A_F)$ in the space of functions defined in $GL_e(1,F)\backslash GL_e(1,A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL_e(n,A_F)$ in the space of functions defined in $GL_e(n,F)\backslash GL_e(n,A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F},F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F},F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n\to\infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q},Q)$ interpreted as HFF. Outer automorphisms result at the limit $n\to\infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n\to\infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n,F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p<n$ are more complex and actually not well-understood [A13]. In the case of elliptic curves [A15] (say $y^2 = x^3 + ax + b$, a,b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL_e(2,Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL_e(n,\overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL_e(n,E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative manners to state the same thing.

4.4.1 Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A15]. The representations can be also l-adic being defined in $GL_e(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A15] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p, 1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^\times$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^\times|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

4.4.2 Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A15] for the route from automorphic adelic representations of $GL_e(2, R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL_e(2, Q)$ are constructed in the space of smooth bounded functions $GL_e(2, Q)\backslash GL_e(2, A) \rightarrow C$ or equivalently in the space of $GL_e(2, Q)$ left-invariant functions in $GL_e(2, A)$. A denotes adeles and $GL_e(2, A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL_e(2, A)$. The motivating idea is the central role of double coset decompositions $G = K_1 A K_2$, where K_i are compact subgroups and A denotes the space of double cosets $K_1 g K_2$ in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL_e(2, Z_p)$. For ramified primes K_p consists of $SL_e(2, Z_p)$ matrices with $c \in p^{n_p} Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2, R)$ with eigenvalue ρ so that irreducible representations of $gl(2, R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$

where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) .$$

3. The center $A^{\times}$ of $GL_e(2, A)$ consists of $A^{\times}$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^{\times} \rightarrow C$ is a character providing a multiplicative representation of $A^{\times}$.

4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g)du = 0$$

   is satisfied [A15]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

   The groups $gl(2, R)$, $O(2)$ and $GL_e(2, Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL_e(2, A_F) \times gl(2, R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL_e(2, R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL_e(2, Q)\backslash GL_e(2, A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2, R)$, where N is conductor [A15]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Carteisian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL_e(2, Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) , \quad c \ mod \ N = 0.$$

   $_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL_e(2, Z)$ so that also $SL_e(2, Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL_e(2, R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL_e(2, R)$ to upper half-plane $H_u = SL_e(2, R)/SO(2)$

The representations of $(gl(2, C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A15]. For the discrete series representation π giving square integrable representation in $SL_e(2, R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ , \ X_+ v_\infty = 0 \ .$$

The latter module is in turn generated by the lowest weight vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\infty \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL_e(2, R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL_e(2, R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a + bi)/(c + di)$ determined by $SL_e(2, R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci + d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci + d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau + b)/(c\tau + d)) = (c\tau + d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n \ . \quad (4.1)$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

4.4.3 Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL_e(2, Z_p)$ bi-invariant functions on $GL_e(2, Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A15].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g, Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

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